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On microlocal analyticity and smoothness of solutions of first-order nonlinear PDEs

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Abstract We study the microlocal analyticity and smoothness of solutions u of of the nonlinear PDE $u_t = f(x, t, u, u_x)$ under some assumptions on the repeated brackets of the linearized operator and its conjugate.

Résumé Nous étudions l'analyticité microlocale et la régularité des solutions u de l'EDP non linéaire $u_t = f(x, t, u, u_x)$ sous certaines conditions portant sur les crochets itérés de l'opérateur linéarisé et de son conjugué.

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1 Introduction

This article is inspired by a recent paper [13] in which the authors studied the microlocal analyticity and strong instability (with respect to a C^∞ perturbation) of the Cauchy problem for quasi-linear equations of the type

$$\begin{cases} \frac{\partial u}{\partial t} + \sum_{j=1}^{N} a_j(x, t, u)\frac{\partial u}{\partial x_j} = b(x, t, u), & 0 < t < T, \ \ x \in \Omega, \\ u(x, 0) = \omega(x), & x \in \Omega \end{cases} \tag{1}$$

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where $\Omega \subseteq \mathbb{R}^N$ is an open subset, $T > 0$. The functions $a_j, b,\ j = 1, \dots, N$ are the restrictions to $\Omega \times [0, T] \times V_3$ of some holomorphic functions defined on a domain $V = V_1 \times V_2 \times V_3 \subseteq \mathbb{C}^{N+2}$. Let

$$\mathcal{L} = \frac{\partial}{\partial t} + \sum_{j=1}^{N} a_j(x, t, v)\frac{\partial}{\partial x_j} + b(x, t, v)\frac{\partial}{\partial v}.$$

We recall from [13] the vectors ν_k for $k \in \mathbb{N}$ defined by

$$\nu_0 = (a_1, \dots, a_N),\ \nu_1 = (\mathcal{L}(a_1), \dots, \mathcal{L}(a_N)) = \mathcal{L}(\nu_0), \dots, \nu_k = \mathcal{L}(\nu_{k-1}) = \mathcal{L}^k(\nu_0).$$

The main result in [13] is as follows:

Theorem 1 *Let $k \in \mathbb{N}$. If the Cauchy Problem (1) has a C^{k+1} solution for $t \geq 0$ on a neighborhood of $(x_0, 0)$, and $\forall x \in \Omega$, $\forall j$ with $0 \leq j < k$, $\Im \nu_j(x, 0, \omega(x)) = 0$, $\Im \nu_k(x_0, 0, \omega(x_0)) \neq 0$, then $\forall \xi^0 \in \mathbb{R}^N$ such that $\Im \nu_k(x_0, 0, \omega(x_0)) \cdot \xi^0 > 0$, the point $(x_0, \xi^0) \notin WF_a(\omega)$.*

Here $WF_a(\omega)$ denotes the analytic wave-front set of $\omega(x)$ (see [14,15] for the definition of microlocal analyticity). In this work we extend the preceding theorem to solutions of the Cauchy Problem for fully nonlinear equations of the form

$$\begin{cases} u_t = f(x, t, u, u_x), & 0 < t < T,\ x \in \Omega, \\ u(x, 0) = \omega(x), & x \in \Omega \end{cases} \tag{2}$$

where $f = f(x, t, \zeta_0, \zeta)$ is the restriction of a holomorphic function.

To extend the preceding theorem to the fully nonlinear case, we first generalize the vectors $\Im \nu_j(x, 0, \omega(x))$. We achieve this in Sect. 2 by expressing $\Im \nu_j(x, 0, \omega(x))$ in terms of the repeated brackets of $\mathcal{L}^{\upsilon}$ and its complex conjugate $\overline{\mathcal{L}^{\upsilon}}$ where $\mathcal{L}^{\upsilon}$ is the linearization of the equation $u_t = f(x, t, u(x, t), u_x(x, t))$ at u given by

$$\mathcal{L}^{\upsilon} = \frac{\partial}{\partial t} - \sum_{j=1}^{N} f^{\upsilon}_{\zeta_j}(x, t)\frac{\partial}{\partial x_j}$$

and $f^{\upsilon}_{\zeta_j}(x, t) = f_{\zeta_j}(x, t, u(x, t), u_x(x, t))$ for $1 \leq j \leq N$. Section 3 contains two applications to microlocal analyticity and smoothness of the trace $u(x, 0)$ of a solution to the nonlinear equation. For the result on smoothness, $f(x, t, \zeta_0, \zeta)$ is assumed to be C^∞ in all variables, and holomorphic in (ζ_0, ζ) in an appropriate domain.

When the initial datum $\omega(x)$ is real-analytic, as in [13], our results imply the strong instability of the Cauchy–Kovalevskaya solution of the Cauchy problem for (2) with respect to a C^∞ perturbation. Results on microlocal analyticity for brackets up to order 3 were proved in [6,7] under assumptions on brackets made just at a point. In the linear case, analyticity results were proved under repeated brackets assumptions in [10,11]. Microlocal smoothness results for nonlinear PDEs were obtained in [3,9]. For results on Gevrey/Denjoy–Carleman regularity we refer the reader to [1,2,5]. The approach to the fully nonlinear case by using the Holomorphic Hamiltonian is motivated by [4].

2 A bracket condition

In this section we show how to express $\Im \upsilon_j(x, 0, \omega(x))$ in terms of the repeated brackets of $\mathcal{L}^\upsilon$ and its complex conjugate $\overline{\mathcal{L}^\upsilon}$. Let u be a sufficiently smooth solution of the non-linear equation

$$u_t = f(x, t, u, u_x) \tag{3}$$

where $f(x, t, \zeta_0, \zeta)$ is a C^∞ function on $\Omega \times [0, T) \times \mathbb{C} \times \mathbb{C}^N$ (Ω is an open subset of $\mathbb{R}^N$), and f is holomorphic in the variables (ζ_0, ζ). Let

$$\mathcal{L} = \frac{\partial}{\partial t} - \sum_{j=1}^{N} f_{\zeta_j}(x, t, \zeta_0, \zeta) \frac{\partial}{\partial x_j}.$$

Set $\upsilon = (u, u_x)$. If $\psi = \psi(x, t, \zeta_0, \zeta)$ is a smooth function, holomorphic in (ζ_0, ζ), we will use the notation

$$\psi^\upsilon(x, t) = \psi(x, t, u, u_x).$$

With this notation, the linearized operator of $u_t = f(x, t, u, u_x)$ can be written as

$$\mathcal{L}^\upsilon = \frac{\partial}{\partial t} - \sum_{j=1}^{N} f_{\zeta_j}^\upsilon(x, t) \frac{\partial}{\partial x_j}.$$

It follows that

$$\mathcal{L}^\upsilon \upsilon = g^\upsilon(x, t), \tag{4}$$

where $g = (g_0, \ldots, g_N)$,

$$g_0(x, t, \zeta_0, \zeta) = f(x, t, \zeta_0, \zeta) - \sum_{j=1}^{N} \zeta_j f_{\zeta_j}(x, t, \zeta_0, \zeta), \quad \text{and}$$
$$g_i(x, t, \zeta_0, \zeta) = f_{x_i}(x, t, \zeta_0, \zeta) + \zeta_i f_{\zeta_0}(x, t, \zeta_0, \zeta) \quad (1 \le i \le N).$$

Consider now the principal part of the holomorphic Hamiltonian of (4):

$$\mathcal{H} = \mathcal{L} + g_0 \partial_{\zeta_0} + \sum_{j=1}^{N} g_j \partial_{\zeta_j}. \tag{5}$$

Lemma 2 *Let $\psi = \psi(x, t, \zeta_0, \zeta)$ be a smooth function, holomorphic in (ζ_0, ζ). Then*

(i) *For all $n \in \mathbb{N}$, we have*

$$\left(\mathcal{L}^\upsilon\right)^n \psi^\upsilon = \left(\mathcal{H}^n \psi\right)^\upsilon. \tag{6}$$

(ii) *For all $n \in \mathbb{N}$, we have*

$$\left(\mathcal{L}^{\upsilon}\right)^{n} \psi^{\upsilon} = \partial_t^{n}\left(\psi^{\upsilon}\right) - \sum_{j=1}^{N}\sum_{l=0}^{n-1}\sum_{s=0}^{l}\binom{l}{s} \times \left(\left(\mathcal{L}^{\upsilon}\right)^{s} f_{\zeta_j}^{\upsilon}\right)\left(\left(\mathcal{L}^{\upsilon}\right)^{l-s} \partial_{x_j}\partial_t^{n-1-l}\psi^{\upsilon}\right). \tag{7}$$

Proof (i) We use induction on n. The case $n = 1$ can be easily checked. Now, suppose that the result holds for all $1 \leq j \leq n$. Then

$$\left(\mathcal{L}^{\upsilon}\right)^{n+1} \psi^{\upsilon} = \mathcal{L}^{\upsilon}\left(\left(\mathcal{L}^{\upsilon}\right)^{n} \psi^{\upsilon}\right) = \mathcal{L}^{\upsilon}\left(\left(\mathcal{H}^{n}\psi\right)^{\upsilon}\right) = \left(\mathcal{H}^{n+1}\psi\right)^{\upsilon},$$

where the second equality above follows from the induction hypothesis when $j = n$, and the third equality follows from the case $n = 1$ but taking $\mathcal{H}^{n}\psi$ instead of ψ. Hence, by induction, we obtain the equation in (6).
(ii) We use induction on n. The case $n = 1$ follows by definition of $\mathcal{L}^{\upsilon}$. Now, suppose that the result holds for some $n \geq 1$. Then

$$\begin{aligned}
\left(\mathcal{L}^{\upsilon}\right)^{n+1}\left(\psi^{\upsilon}\right) &= \mathcal{L}^{\upsilon}\left(\left(\mathcal{L}^{\upsilon}\right)^{n} \psi^{\upsilon}\right) \\
&= \mathcal{L}^{\upsilon}\left(\partial_t^{n}\left(\psi^{\upsilon}\right) - \sum_{j=1}^{N}\sum_{l=0}^{n-1}\sum_{s=0}^{l}\binom{l}{s}\left(\left(\mathcal{L}^{\upsilon}\right)^{s} f_{\zeta_j}^{\upsilon}\right)\right. \\
&\qquad \left.\times\left(\left(\mathcal{L}^{\upsilon}\right)^{l-s}\left(\partial_{x_j}\partial_t^{n-1-l}\psi^{\upsilon}\right)\right)\right) \\
&= \mathcal{L}^{\upsilon}\left(\partial_t^{n}\left(\psi^{\upsilon}\right)\right) - \sum_{j=1}^{N}\sum_{l=0}^{n-1}\sum_{s=0}^{l}\binom{l}{s}\left(\left(\mathcal{L}^{\upsilon}\right)^{s+1} f_{\zeta_j}^{\upsilon}\right) \\
&\qquad \times\left(\left(\mathcal{L}^{\upsilon}\right)^{l-s}\left(\partial_{x_j}\partial_t^{n-1-l}\psi^{\upsilon}\right)\right) \\
&\qquad - \sum_{j=1}^{N}\sum_{l=0}^{n-1}\sum_{s=0}^{l}\binom{l}{s}\left(\left(\mathcal{L}^{\upsilon}\right)^{s} f_{\zeta_j}^{\upsilon}\right)\left(\left(\mathcal{L}^{\upsilon}\right)^{l-s+1}\left(\partial_{x_j}\partial_t^{n-1-l}\psi^{\upsilon}\right)\right) \\
&= I_1 - I_2 - I_3.
\end{aligned}$$

We have, by definition of $\mathcal{L}^{\upsilon}$,

$$I_1 = \partial_t^{n+1}\left(\psi^{\upsilon}\right) - \sum_{j=1}^{N} f_{\zeta_j}^{\upsilon}\partial_{x_j}\partial_t^{n}\left(\psi^{\upsilon}\right).$$

The second term, after rearrangement, can be written as

$$
\begin{aligned}
I_2 &= \sum_{j=1}^{N}\sum_{l=0}^{n-1}\sum_{s=1}^{l+1}\binom{l}{s-1}\left(\left(\mathcal{L}^{\upsilon}\right)^{s} f_{\zeta_j}^{\upsilon}\right)\left(\left(\mathcal{L}^{\upsilon}\right)^{l-s+1}\left(\partial_{x_j}\partial_t^{n-1-l}\psi^{\upsilon}\right)\right)\\
&= \sum_{j=1}^{N}\sum_{l=0}^{n-1}\sum_{s=1}^{l}\binom{l}{s-1}\left(\left(\mathcal{L}^{\upsilon}\right)^{s} f_{\zeta_j}^{\upsilon}\right)\left(\left(\mathcal{L}^{\upsilon}\right)^{l-s+1}\left(\partial_{x_j}\partial_t^{n-1-l}\psi^{\upsilon}\right)\right)\\
&\quad+\sum_{j=1}^{N}\sum_{l=0}^{n-1}\left(\left(\mathcal{L}^{\upsilon}\right)^{l+1} f_{\zeta_j}^{\upsilon}\right)\left(\partial_{x_j}\partial_t^{n-1-l}\psi^{\upsilon}\right)
\end{aligned}
$$

The third term can be written as

$$
\begin{aligned}
I_3 &= \sum_{j=1}^{N}\sum_{l=0}^{n-1}\sum_{s=1}^{l}\binom{l}{s}\left(\left(\mathcal{L}^{\upsilon}\right)^{s} f_{\zeta_j}^{\upsilon}\right)\left(\left(\mathcal{L}^{\upsilon}\right)^{l-s+1}\left(\partial_{x_j}\partial_t^{n-1-l}\psi^{\upsilon}\right)\right)\\
&\quad+\sum_{j=1}^{N}\sum_{l=0}^{n-1} f_{\zeta_j}^{\upsilon}\left(\left(\mathcal{L}^{\upsilon}\right)^{l+1}\left(\partial_{x_j}\partial_t^{n-1-l}\psi^{\upsilon}\right)\right).
\end{aligned}
$$

Using the fact that for all $1 \le s \le l$,

$$
\binom{l}{s}+\binom{l}{s-1}=\binom{l+1}{s},
$$

we get

$$
\begin{aligned}
I_2+I_3 &= \sum_{j=1}^{N}\sum_{l=0}^{n-1}\sum_{s=0}^{l+1}\binom{l+1}{s}\left(\left(\mathcal{L}^{\upsilon}\right)^{s} f_{\zeta_j}^{\upsilon}\right)\left(\left(\mathcal{L}^{\upsilon}\right)^{(l+1)-s}\left(\partial_{x_j}\partial_t^{n-(l+1)}\psi^{\upsilon}\right)\right)\\
&= \sum_{j=1}^{N}\sum_{l=1}^{n}\sum_{s=0}^{l}\binom{l}{s}\left(\left(\mathcal{L}^{\upsilon}\right)^{s} f_{\zeta_j}^{\upsilon}\right)\left(\left(\mathcal{L}^{\upsilon}\right)^{l-s}\left(\partial_{x_j}\partial_t^{n-l}\psi^{\upsilon}\right)\right).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
I_1-I_2-I_3 &= \left(\partial_t^{n+1}\left(\psi^{\upsilon}\right)-\sum_{j=1}^{N} f_{\zeta_j}^{\upsilon}\partial_{x_j}\partial_t^{n}\left(\psi^{\upsilon}\right)\right)\\
&\quad-\sum_{j=1}^{N}\sum_{l=1}^{n}\sum_{s=0}^{l}\binom{l}{s}\left(\left(\mathcal{L}^{\upsilon}\right)^{s} f_{\zeta_j}^{\upsilon}\right)\left(\left(\mathcal{L}^{\upsilon}\right)^{l-s}\left(\partial_{x_j}\partial_t^{n-l}\psi^{\upsilon}\right)\right)\\
&= \partial_t^{n+1}\left(\psi^{\upsilon}\right)-\sum_{j=1}^{N}\sum_{l=0}^{n}\sum_{s=0}^{l}\binom{l}{s}\left(\left(\mathcal{L}^{\upsilon}\right)^{s} f_{\zeta_j}^{\upsilon}\right)\left(\left(\mathcal{L}^{\upsilon}\right)^{l-s}\left(\partial_{x_j}\partial_t^{n-l}\psi^{\upsilon}\right)\right).
\end{aligned}
$$

Hence, by induction, we obtain the equation in (7). □

Notation 3 Let $\psi = \psi(x, t, \zeta_0, \zeta)$ be a smooth function on $\Omega \times (-T, T) \times \mathbb{C} \times \mathbb{C}^N$, holomorphic in (ζ_0, ζ). For $1 \le j \le N$, we set

$$B_j(\psi) = \overline{\partial_{x_j}(\psi^\upsilon)}.$$

Also, for $1 \le j \le N$, $n \in \mathbb{N}$, and $0 \le k \le n-1$, we define the functions $A_j^{k,n} = A_j^{k,n}(x, t)$ recursively as follows:

$$\begin{aligned}
A_j^{0,1} &= B_j(\psi) \quad (1 \le j \le N),\\
A_j^{0,n} &= \mathcal{L}^\upsilon A_j^{0,n-1} + B_j\left(\mathcal{H}^{n-1}\psi\right) + \sum_{s=0}^{n-2}\sum_{l=1}^{N} B_j\left(\mathcal{H}^s f_{\zeta_l}\right) A_l^{s,n-1} \quad (n \ge 2,\ 1 \le j \le N),\\
A_j^{k,n} &= A_j^{k-1,n-1} + \mathcal{L}^\upsilon A_j^{k,n-1} \quad (n \ge 3,\ 1 \le k \le n-2,\ 1 \le j \le N), \quad \textit{and}\\
A_j^{n-1,n} &= A_j^{n-2,n-1} \quad (n \ge 2,\ 1 \le j \le N).
\end{aligned} \tag{8}$$

Lemma 4 *Let $\psi = \psi(x, t, \zeta_0, \zeta)$ be a smooth function, holomorphic in (ζ_0, ζ) and let $A_j^{k,n}$ be as in (8). Then for all $n \in \mathbb{N}$,*

$$\begin{aligned}
\left(\mathcal{L}^\upsilon\right)^n \Im\psi^\upsilon &= \Im\left(\left(\mathcal{L}^\upsilon\right)^n \psi^\upsilon\right) + \sum_{j=1}^{N}\sum_{k=0}^{n-1}\left(\Im\left(\left(\mathcal{L}^\upsilon\right)^k f_{\zeta_j}^\upsilon\right) \times A_j^{k,n}\right)\\
&= \Im\left(\mathcal{H}^n\psi\right)^\upsilon + \sum_{j=1}^{N}\sum_{k=0}^{n-1}\Im\left(\mathcal{H}^k f_{\zeta_j}\right)^\upsilon \times A_j^{k,n},
\end{aligned} \tag{9}$$

where the second equation follows from Lemma 2 above.

Proof We use induction on n. The case $n = 1$ follows since

$$\begin{aligned}
\mathcal{L}^\upsilon\left(\Im\psi^\upsilon\right) &= \left(\partial_t - \sum_{j=1}^{N} f_{\zeta_j}^\upsilon \partial_{x_j}\right)\left(\Im\psi^\upsilon\right)\\
&= \Im\left(\psi_t^\upsilon + \psi_{\zeta_0}^\upsilon u_t + \sum_{j=1}^{N}\psi_{\zeta_j}^\upsilon u_{tx_j}\right) - \sum_{j=1}^{N} f_{\zeta_j}^\upsilon \partial_{x_j}\left(\Im\psi^\upsilon\right).
\end{aligned}$$

Using Eq. (3) we see that the first term above equals

$$\Im\left(\psi_t^{\upsilon}+\psi_{\zeta_0}^{\upsilon}f^{\upsilon}+\sum_{j=1}^{N}\psi_{\zeta_j}^{\upsilon}\partial_{x_j}\left(f^{\upsilon}\right)\right)$$
$$=\Im\left(\psi_t^{\upsilon}+f^{\upsilon}\psi_{\zeta_0}^{\upsilon}+\sum_{j=1}^{N}\left(f_{x_j}^{\upsilon}+f_{\zeta_0}^{\upsilon}u_{x_j}+\sum_{k=1}^{N}f_{\zeta_k}^{\upsilon}u_{x_jx_k}\right)\psi_{\zeta_j}^{\upsilon}\right).$$

We next note that the last quantity above is equal to

$$\Im\left(\mathcal{H}\psi\right)^{\upsilon}+\Im\sum_{j=1}^{N}f_{\zeta_j}^{\upsilon}\left(\partial_{x_j}\left(\psi^{\upsilon}\right)\right).$$

Hence,

$$\mathcal{L}^{\upsilon}\left(\Im\psi^{\upsilon}\right)=\Im\left(\mathcal{H}\psi\right)^{\upsilon}+\Im\sum_{j=1}^{N}f_{\zeta_j}^{\upsilon}\left(\partial_{x_j}\left(\psi^{\upsilon}\right)\right)-\sum_{j=1}^{N}f_{\zeta_j}^{\upsilon}\partial_{x_j}\left(\Im\psi^{\upsilon}\right).$$

Using the fact that

$$\Im\left(zw\right)-z\left(\Im w\right)=\left(\Im z\right)\left(\overline{w}\right),$$

we see that

$$\mathcal{L}^{\upsilon}\left(\Im\psi^{\upsilon}\right)=\Im\left(\mathcal{H}\psi\right)^{\upsilon}+\sum_{j=1}^{N}\Im\left(f_{\zeta_j}^{\upsilon}\right)\times\overline{\partial_{x_j}\left(\psi^{\upsilon}\right)}$$
$$=\Im\left(\mathcal{H}\psi\right)^{\upsilon}+\sum_{j=1}^{N}\Im\left(f_{\zeta_j}^{\upsilon}\right)\times A_j^{0,1}.$$

and this is exactly Eq. (9) with $n=1$. Suppose now that the result is true for all $1\le j\le n$. Then

$$\left(\mathcal{L}^{\upsilon}\right)^{n+1}\Im\psi^{\upsilon}=\mathcal{L}^{\upsilon}\left(\Im\left(\mathcal{L}^{\upsilon}\right)^{n}\psi^{\upsilon}+\sum_{j=1}^{N}\sum_{k=0}^{n-1}\left(\Im\left(\left(\mathcal{L}^{\upsilon}\right)^{k}\left(f_{\zeta_j}^{\upsilon}\right)\right)\times A_j^{k,n}\right)\right)$$
$$=\mathcal{L}^{\upsilon}\left(\Im\left(\mathcal{L}^{\upsilon}\right)^{n}\psi^{\upsilon}\right)+\sum_{j=1}^{N}\sum_{k=0}^{n-1}\left(\mathcal{L}^{\upsilon}\Im\left(\left(\mathcal{L}^{\upsilon}\right)^{k}\left(f_{\zeta_j}^{\upsilon}\right)\right)\right)\times A_j^{k,n}$$
$$+\sum_{j=1}^{N}\sum_{k=0}^{n-1}\Im\left(\left(\mathcal{L}^{\upsilon}\right)^{k}\left(f_{\zeta_j}^{\upsilon}\right)\right)\times\left(\mathcal{L}^{\upsilon}A_j^{k,n}\right).$$

We now compute each term separately. We have by the induction hypothesis (but replacing ψ^{υ} with $(\mathcal{H}^n\psi)^{\upsilon}$)

$$\mathcal{L}^{\upsilon}\Im\left(\left(\mathcal{L}^{\upsilon}\right)^{n}\psi^{\upsilon}\right)=\Im\left(\left(\mathcal{L}^{\upsilon}\right)^{n+1}\psi^{\upsilon}\right)+\sum_{j=1}^{N}\Im\left(f_{\zeta_j}^{\upsilon}\right)\times B_j\left(\mathcal{H}^n\psi\right).$$

Similarly, we also have

$$\mathcal{L}^{\upsilon}\Im\left(\left(\mathcal{L}^{\upsilon}\right)^{k}\left(f_{\zeta_j}^{\upsilon}\right)\right)=\Im\left(\left(\mathcal{L}^{\upsilon}\right)^{k+1}\left(f_{\zeta_j}^{\upsilon}\right)\right)+\sum_{l=1}^{N}\Im\left(f_{\zeta_l}^{\upsilon}\right)\times B_l\left(\mathcal{H}^k f_{\zeta_j}\right).$$

Hence, we have

$$\begin{aligned}
\left(\mathcal{L}^{\upsilon}\right)^{n+1}\left(\Im\psi^{\upsilon}\right) &= \Im\left(\left(\mathcal{L}^{\upsilon}\right)^{n+1}\left(\psi^{\upsilon}\right)\right)+\sum_{j=1}^{N}\Im\left(f_{\zeta_j}^{\upsilon}\right)\times B_j\left(\mathcal{H}^n\psi\right)\\
&\quad+\sum_{j=1}^{N}\sum_{k=0}^{n-1}\left(\Im\left(\left(\mathcal{L}^{\upsilon}\right)^{k+1}\left(f_{\zeta_j}^{\upsilon}\right)\right)+\sum_{l=1}^{N}\Im\left(f_{\zeta_l}^{\upsilon}\right)\times B_l\left(\mathcal{H}^k f_{\zeta_j}\right)\right)\\
&\quad\times A_j^{k,n}+\sum_{j=1}^{N}\sum_{k=0}^{n-1}\Im\left(\left(\mathcal{L}^{\upsilon}\right)^{k}\left(f_{\zeta_j}^{\upsilon}\right)\right)\times\left(\mathcal{L}^{\upsilon}A_j^{k,n}\right)\\
&= \Im\left(\left(\mathcal{L}^{\upsilon}\right)^{n+1}\left(\psi^{\upsilon}\right)\right)\\
&\quad+\sum_{j=1}^{N}\Im\left(f_{\zeta_j}^{\upsilon}\right)\times\left(\mathcal{L}^{\upsilon}A_j^{0,n}+B_j\left(\mathcal{H}^n\psi\right)+\sum_{k=0}^{n-1}\sum_{l=1}^{N}B_j\left(\mathcal{H}^k f_{\zeta_l}\right)\times A_l^{k,n}\right)\\
&\quad+\sum_{j=1}^{N}\sum_{k=1}^{n-1}\Im\left(\left(\mathcal{L}^{\upsilon}\right)^{k}\left(f_{\zeta_j}^{\upsilon}\right)\right)\times\left(A_j^{k-1,n}+\mathcal{L}^{\upsilon}A_j^{k,n}\right)\\
&\quad+\sum_{j=1}^{N}\Im\left(\left(\mathcal{L}^{\upsilon}\right)^{n}\left(f_{\zeta_j}^{\upsilon}\right)\right)\times A_j^{n-1,n}\\
&= \Im\left(\left(\mathcal{L}^{\upsilon}\right)^{n+1}\left(\psi^{\upsilon}\right)\right)+\sum_{j=1}^{N}\sum_{k=0}^{n}\left(\Im\left(\left(\mathcal{L}^{\upsilon}\right)^{k}\left(f_{\zeta_j}^{\upsilon}\right)\right)\times A_j^{k,n+1}\right)
\end{aligned}$$

where the last equality follows from the way we defined the $A_j^{k,n}$'s. Hence, by induction and Lemma 2, we obtain Eq. (9). □

Corollary 5 (i) $\Im\left(\mathcal{H}^k f_{\zeta}\right)^{\upsilon}(x_0,0)=0$, *for all* $0\le k\le n-1$, *if and only if*

$$\left(\mathcal{L}^{\upsilon}\right)^{k}\left(\Im f_{\zeta_j}^{\upsilon}\right)(x_0,0)=0,$$

for all $0\le k\le n-1$, *and all* $1\le j\le N$.

(ii) *If* $\Im\left(\mathcal{H}^k f_\zeta\right)^\upsilon(x_0,0)=0$ *for all* $0\le k\le n-1$, *and* $\Im\left(\mathcal{H}^n f_\zeta\right)^\upsilon(x_0,0)\ne 0$, *then*

$$\left(\mathcal{L}^\upsilon\right)^n\left(\Im f^\upsilon_{\zeta_j}\right)(x_0,0)=\Im\left(\mathcal{H}^n f_{\zeta_j}\right)^\upsilon(x_0,0)\ \ \forall j.$$

Proof The two claims follow immediately from Eq. (9) in Lemma (4). □

Notation 6 Let $k\in\mathbb{N}$, $k\ge 1$ be fixed and let $X_0, X_1, X_2, \ldots, X_k$ be complex vector fields. We shall use the notation

$$\left[X_k,\left[X_{k-1},\ldots,[X_1,X_0]\right]\right]$$

to denote the k-bracket Y_k, where

$$\begin{aligned}Y_1&=[X_1,X_0],\\ Y_n&=\left[X_n,Y_{n-1}\right],\quad (2\le n\le k).\end{aligned}$$

Lemma 7 *Define* $M^\upsilon=\sum_{j=1}^N \Im f^\upsilon_{\zeta_j}(x,t)\,\partial_{x_j}$. *Let* $X_0=M^\upsilon$, *and for* $k\ge 1$, *let* $X_1, X_2,\ldots,X_k$ *be vector fields where each* X_j, $1\le j\le k$, *is either* M^υ *or* $\mathcal{L}^\upsilon$. *Then the* k-*bracket*

$$\begin{aligned}\left[X_k,\left[X_{k-1},\ldots,[X_1,X_0]\right]\right]=&\sum_{j=1}^N\sum_{\sigma_1,\ldots,\sigma_k}\left(X_k^{\sigma_k}X_{k-1}^{\sigma_{k-1}}\cdots X_1^{\sigma_1}\Im f^\upsilon_{\zeta_j}\right)\\ &\times\left[X_k^{1-\sigma_k},\left[X_{k-1}^{1-\sigma_{k-1}},\ldots,\left[X_1^{1-\sigma_1},\partial_{x_j}\right]\right]\right],\end{aligned}\tag{10}$$

where each σ_j *is either* 0 *or* 1 *and if a power of zero appears in a bracket, we simply ignore the term raised to zero and delete its brackets. (e.g.,* $\left[X_1^0,\partial_{x_j}\right]=\partial_{x_j}$ *and* $\left[X_3^1,\left[X_2^0,\left[X_1^1,\partial_{x_j}\right]\right]\right]=\left[X_3,\left[X_1,\partial_{x_j}\right]\right]$).

Proof We use induction on $k\ge 1$. The case $k=1$ follows because if X_1 is either M^υ or $\mathcal{L}^\upsilon$, then

$$\begin{aligned}\left[X_1,M^\upsilon\right]&=\left[X_1,\sum_{j=1}^N\Im f^\upsilon_{\zeta_j}\partial_{x_j}\right]\\ &=\sum_{j=1}^N X_1\left(\Im f^\upsilon_{\zeta_j}\right)\partial_{x_j}+\sum_{j=1}^N\left(\Im f^\upsilon_{\zeta_j}\right)\left[X_1,\partial_{x_j}\right]\\ &=\sum_{j=1}^N (X_1)^1\left(\Im f^\upsilon_{\zeta_j}\right)\left[(X_1)^0,\partial_{x_j}\right]+\sum_{j=1}^N (X_1)^0\left(\Im f^\upsilon_{\zeta_j}\right)\left[(X_1)^1,\partial_{x_j}\right]\\ &=\sum_{j=1}^N\sum_{\sigma_1}(X_1)^{\sigma_1}\left(\Im f^\upsilon_{\zeta_j}\right)\left[(X_1)^{1-\sigma_1},\partial_{x_j}\right],\end{aligned}$$

where σ_1 is either 0 or 1. Now suppose that the result is true for $k \geq 1$. Then with X_{k+1} either M^υ or $\mathcal{L}^\upsilon$,

$$\begin{aligned}
&\left[X_{k+1},[X_k,\ldots,[X_1,X_0]]\right]\\
&=\sum_{j=1}^{N}\sum_{\sigma_1,\ldots,\sigma_k}\left(X_{k+1}X_k^{\sigma_k}\cdots X_1^{\sigma_1}\Im f_{\zeta_j}^{\upsilon}\right)\left[X_k^{1-\sigma_k},\left[X_{k-1}^{1-\sigma_{k-1}},\ldots,\left[X_1^{1-\sigma_1},\partial_{x_j}\right]\right]\right]\\
&\quad+\sum_{j=1}^{N}\sum_{\sigma_1,\ldots,\sigma_k}\left(X_k^{\sigma_k}X_{k-1}^{\sigma_{k-1}}\cdots X_1^{\sigma_1}\Im f_{\zeta_j}^{\upsilon}\right)\left[X_{k+1},\left[X_k^{1-\sigma_k},\ldots,\left[X_1^{1-\sigma_1},\partial_{x_j}\right]\right]\right]\\
&=\sum_{j=1}^{N}\sum_{\sigma_1,\ldots,\sigma_k}\left(X_{k+1}^{1}X_k^{\sigma_k}\cdots X_1^{\sigma_1}\Im f_{\zeta_j}^{\upsilon}\right)\left[X_{k+1}^{0},\left[X_k^{1-\sigma_k},\ldots,\left[X_1^{1-\sigma_1},\partial_{x_j}\right]\right]\right]\\
&\quad+\sum_{j=1}^{N}\sum_{\sigma_1,\ldots,\sigma_k}\left(X_{k+1}^{0}X_k^{\sigma_k}\cdots X_1^{\sigma_1}\Im f_{\zeta_j}^{\upsilon}\right)\left[X_{k+1}^{1},\left[X_k^{1-\sigma_k},\ldots,\left[X_1^{1-\sigma_1},\partial_{x_j}\right]\right]\right]\\
&=\sum_{j=1}^{N}\sum_{\sigma_1,\ldots,\sigma_{k+1}}\left(X_{k+1}^{\sigma_{k+1}}X_k^{\sigma_k}\cdots X_1^{\sigma_1}\Im f_{\zeta_j}^{\upsilon}\right)\left[X_{k+1}^{1-\sigma_{k+1}},\left[X_k^{1-\sigma_k},\ldots,\left[X_1^{1-\sigma_1},\partial_{x_j}\right]\right]\right],
\end{aligned}$$

with σ_{k+1} either 0 or 1. Our result now follows by induction. □

Corollary 8 *If all of the X_j's $(1 \leq j \leq k)$ in Lemma 7 are equal to $\mathcal{L}^\upsilon$, then*

$$\begin{aligned}
[\mathcal{L}^\upsilon,[\mathcal{L}^\upsilon,\cdots,[\mathcal{L}^\upsilon,M^\upsilon]]] &= \sum_{j=1}^{N}\sum_{l=0}^{k}\binom{k}{l}\left(\left(\mathcal{L}^\upsilon\right)^l \Im f_{\zeta_j}^{\upsilon}\right)\\
&\quad\times\underbrace{[\mathcal{L}^\upsilon,[\mathcal{L}^\upsilon,\cdots,[\mathcal{L}^\upsilon,\partial_{x_j}]]]}_{k-l \text{ brackets}}.
\end{aligned} \tag{11}$$

Proof Follows immediately from Lemma 7. □

Corollary 9 *Let $k \geq 1$. If all the brackets $[\mathcal{L}^\upsilon,[\mathcal{L}^\upsilon,\cdots,[\mathcal{L}^\upsilon,M^\upsilon]]]$ of length $< k$ vanish at $(x,0)$ for some $x \in \Omega$ (where we define the bracket of length 0 to be M^υ), then, at the point $(x,0)$, the $k-$ bracket*

$$[\mathcal{L}^\upsilon,[\mathcal{L}^\upsilon,\cdots,[\mathcal{L}^\upsilon,M^\upsilon]]] = \sum_{j=1}^{N}\left(\left(\mathcal{L}^\upsilon\right)^k \Im f_{\zeta_j}^{\upsilon}\right)\partial_{x_j}. \tag{12}$$

Proof For $k \geq 1$, let $P(k)$ denote the statement "if all the brackets of length $< k$ vanish at $(x,0)$, then the k-bracket is given by Eq. (12) at $(x,0)$". We will prove that $P(k)$ holds for all k using induction. If $k = 1$, then we are assuming that M^υ vanishes at $(x, 0)$, and so $\Im f^{\upsilon}_{\zeta_j}(x, 0) = 0$ for all $1 \leq j \leq N$. Since

$$\left[\mathcal{L}^{\upsilon}, M^{\upsilon}\right] = \sum_{j=1}^{N} \left(\Im f^{\upsilon}_{\zeta_j}\right)\left[\mathcal{L}^{\upsilon}, \partial_{x_j}\right] + \left(\mathcal{L}^{\upsilon}\Im f^{\upsilon}_{\zeta_j}\right)\partial_{x_j},$$

at the point $(x, 0)$, we have

$$\left[\mathcal{L}^{\upsilon}, M^{\upsilon}\right] = \sum_{j=1}^{N} \left(\mathcal{L}^{\upsilon}\Im f^{\upsilon}_{\zeta_j}\right)\partial_{x_j}.$$

Hence, $P(1)$ is true. Suppose now that $P(l)$ is true for all $1 \leq l \leq k$. We would like to show that $P(k+1)$ is true. So suppose that all the brackets $[\mathcal{L}^{\upsilon}, [\mathcal{L}^{\upsilon}, \ldots, [\mathcal{L}^{\upsilon}, M^{\upsilon}]]]$ of length $< k+1$ vanish at $(x, 0)$. Then the induction hypothesis implies that for all $0 \leq l \leq k$, all $1 \leq j \leq N$,

$$\left(\left(\mathcal{L}^{\upsilon}\right)^{l}\Im f^{\upsilon}_{\zeta_j}\right)(x, 0) = 0.$$

Hence, Eq. (11) implies that, at the point $(x, 0)$, the $k+1$-bracket

$$\left[\mathcal{L}^{\upsilon}, \left[\mathcal{L}^{\upsilon}, \ldots, \left[\mathcal{L}^{\upsilon}, M^{\upsilon}\right]\right]\right] = \sum_{j=1}^{N} \left(\left(\mathcal{L}^{\upsilon}\right)^{k+1}\Im f^{\upsilon}_{\zeta_j}\right)\partial_{x_j},$$

as desired. Corollary 9 now follows. □

Lemma 10 *Suppose that for some $x_0 \in \Omega$, and*

$$\forall\, 0 \leq j \leq k-1, \quad \Im\left(\mathcal{H}^{j} f_{\zeta}\right)^{\upsilon}(x_0, 0) = 0, \ \Im\left(\mathcal{H}^{k} f_{\zeta}\right)^{\upsilon}(x_0, 0) \neq 0. \tag{13}$$

With the same notation as in Lemma 7, if at least one of the $X'_j s$ is M^{υ}, then for all possible choices of σ_j, we have:

$$\left(X_k^{\sigma_k} X_{k-1}^{\sigma_{k-1}} \cdots X_1^{\sigma_1}\Im f^{\upsilon}_{\zeta_j}\right)(x_0, 0) = 0, \tag{14}$$

and if all of the X_j's are $\mathcal{L}^{\upsilon}$, then at the point $(x_0, 0)$, we have

$$\left[\mathcal{L}^{\upsilon}, \left[\mathcal{L}^{\upsilon}, \ldots, \left[\mathcal{L}^{\upsilon}, M^{\upsilon}\right]\right]\right](x_0, 0) = \sum_{j=1}^{N} \Im\left(\mathcal{H}^{k} f_{\zeta_j}\right)^{\upsilon}(x_0, 0)\,\frac{\partial}{\partial x_j}. \tag{15}$$

Proof We prove (14) first. Assume that at least one of the X_j's $(1 \leq j \leq k)$ is M^{υ}. For $1 \leq l \leq k$, let $Y_l Y_{l-1} \cdots Y_1$ denote the elements that have their $\sigma_j = 1$ in

$X_k^{\sigma_k} X_{k-1}^{\sigma_{k-1}} \cdots X_1^{\sigma_1}$, keeping the order the same (e.g., we denote $X_4^1 X_3^0 X_2^1 X_1^0$ by $Y_2 Y_1$ with $Y_2 = X_4$ and $Y_1 = X_2$). We would like to show that

$$\left(Y_l Y_{l-1} \cdots Y_1 \Im f_{\zeta_j}^{\upsilon}\right)(x_0, 0) = 0.$$

To do so, note first that if $Y_l = M^{\upsilon}$, then

$$\begin{aligned}\left(Y_l Y_{l-1} \cdots Y_1 \Im f_{\zeta_j}^{\upsilon}\right) &= M^{\upsilon}\left(Y_{l-1} \cdots Y_1 \Im f_{\zeta_j}^{\upsilon}\right)\\ &= \sum_{j=1}^{N}\left(\Im f_{\zeta_j}^{\upsilon}\right)\partial_{x_j}\left(Y_{l-1} \cdots Y_1 \Im f_{\zeta_j}^{\upsilon}\right),\end{aligned}$$

and this vanishes at $(x_0, 0)$ by assumption. Hence, we are left with the case in which $l \geq 2$, $Y_l = \mathcal{L}^{\upsilon}$ and some $Y_s = M^{\upsilon}$ for some $1 \leq s \leq l-1$. Let s be the biggest such. Then

$$\begin{aligned}\left(Y_l Y_{l-1} \cdots Y_1 \Im f_{\zeta_j}^{\upsilon}\right) &= \left(\mathcal{L}^{\upsilon}\right)^{l-s} M^{\upsilon} Y_{s-1} \cdots Y_1 \Im f_{\zeta_j}^{\upsilon}\\ &= \left(\mathcal{L}^{\upsilon}\right)^{l-s} \sum_{p=1}^{N}\left(\Im f_{\zeta_p}^{\upsilon}\right)\partial_{x_p}\left(Y_{s-1} \cdots Y_1 \Im f_{\zeta_j}^{\upsilon}\right)\\ &= \sum_{p=1}^{N}\sum_{m=0}^{l-s}\binom{l-s}{m}\left(\left(\mathcal{L}^{\upsilon}\right)^{m} \Im f_{\zeta_p}^{\upsilon}\right)\\ &\quad \times\left(\left(\mathcal{L}^{\upsilon}\right)^{l-s-m}\partial_{x_p} Y_{s-1} \cdots Y_1 \Im f_{\zeta_j}^{\upsilon}\right),\end{aligned}$$

and this vanishes at $(x_0, 0)$ by assumption and Corollary 5. Hence, we have proved (14). It remains to prove (15). Notice that (15) follows immediately from Corollary 5 and Corollary 8. □

Proposition 11 *Condition* (13) *in Lemma* 10 *holds if and only if the following two conditions hold:*

(a) *All the brackets of* $\mathcal{L}^{\upsilon}$ *and* $\overline{\mathcal{L}^{\upsilon}}$ *of order* $< k$ *vanish at* $(x_0, 0)$*,*
(b) *the* $k-$*bracket*

$$\frac{1}{2i}\left[\mathcal{L}^{\upsilon}, \left[\mathcal{L}^{\upsilon}, \ldots, \left[\mathcal{L}^{\upsilon}, \overline{\mathcal{L}^{\upsilon}}\right]\right]\right](x_0, 0) \neq 0. \tag{16}$$

Proof (⇒) Suppose that condition (13) in Lemma 10 holds. Then

(**a**) Notice that

$$\overline{\mathcal{L}^{\upsilon}} = \left(\partial_t - \sum_{j=1}^{N} f_{\zeta_j}^{\upsilon}\partial_{x_j}\right) + \sum_{j=1}^{N}\left(f_{\zeta_j}^{\upsilon} - \overline{f_{\zeta_j}^{\upsilon}}\right)\partial_{x_j} = \mathcal{L}^{\upsilon} + 2iM^{\upsilon}. \tag{17}$$

Hence,

$$\frac{1}{2i}\left[\mathcal{L}^{\upsilon}, \overline{\mathcal{L}^{\upsilon}}\right] = \left[\mathcal{L}^{\upsilon}, M^{\upsilon}\right]. \tag{18}$$

If we take any bracket of $\mathcal{L}^{\upsilon}$ and $\overline{\mathcal{L}^{\upsilon}}$ of order $< k$, then we can use Eq. (17) to replace each $\overline{\mathcal{L}^{\upsilon}}$ by $\mathcal{L}^{\upsilon} + 2iM^{\upsilon}$. This way we can write our original bracket in terms of $\mathcal{L}^{\upsilon}$ and M^{υ} only. Then the result follows from Lemma 10.
(b) Using Eq. (18), we have $\frac{1}{2i}[\mathcal{L}^{\upsilon}, [\mathcal{L}^{\upsilon}, \ldots, [\mathcal{L}^{\upsilon}, \overline{\mathcal{L}^{\upsilon}}]]] = [\mathcal{L}^{\upsilon}, [\mathcal{L}^{\upsilon}, \ldots, [\mathcal{L}^{\upsilon}, M^{\upsilon}]]]$, and Eq. (16) follows from Lemma 7 and Lemma 10.
($\Leftarrow$) This follows from Corollary 9. □

We end this section with the following lemma that will be used in the proof of the main theorem.

Lemma 12 *Fix $k \in \mathbb{N}$. If*

$$\left(\partial_t^l \Im f_{\zeta_i}^{\upsilon}\right)(x, 0) = 0 \quad \forall x \in \Omega,\ \ 1 \le i \le N,\ \ 0 \le l \le k-1, \tag{19}$$

then

$$\left(\left(\mathcal{L}^{\upsilon}\right)^k \Im f_{\zeta_i}^{\upsilon}\right)(x, 0) = \partial_t^k \left(\Im f_{\zeta_i}^{\upsilon}\right)(x, 0) \quad \forall x \in \Omega,\ \ 1 \le i \le N. \tag{20}$$

Proof Condition (19) implies that for all $x \in \Omega$,

$$\left(\mathcal{L}^{\upsilon}\right)^l \partial_t^p \Im f_{\zeta_i}^{\upsilon}(x, 0) \equiv 0, \quad \text{for } 0 \le l + p \le k-1,$$

and hence the lemma follows from Eq. (7) in Lemma 2, namely,

$$\left(\mathcal{L}^{\upsilon}\right)^k \left(\Im f_{\zeta_i}^{\upsilon}\right) = \partial_t^k \left(\Im f_{\zeta_i}^{\upsilon}\right) - \sum_{j=1}^{N} \sum_{l=0}^{k-1} \sum_{s=0}^{l} \binom{l}{s} \left(\left(\mathcal{L}^{\upsilon}\right)^s f_{\zeta_j}^{\upsilon}\right) \left(\partial_{x_j} \left(\mathcal{L}^{\upsilon}\right)^{l-s} \partial_t^{k-1-l} \Im f_{\zeta_i}^{\upsilon}\right).$$

□

3 Applications to solutions of first order nonlinear PDE

3.1 The smooth case

In the following Theorem, Ω is an open subset of $\mathbb{R}^N$ and $f = f(x, t, \zeta_0, \zeta)$ is a C^∞ function on $\Omega \times [0, T] \times \mathbb{C} \times \mathbb{C}^N$, holomorphic in the variables (ζ_0, ζ). Also, recall the notation $f^{\upsilon}(x, t) = f(x, t, u(x, t), u_x(x, t))$. For the definition and basic properties of the concept of the C^∞ wave-front set of a distribution, we refer the reader to chapter 8 in the book [12].

Theorem 13 *Let $f(x,t,\zeta_0,\zeta)$ be a C^∞ function that is holomorphic in (ζ_0,ζ). Let $k \in \mathbb{N}$. If the nonlinear first order equation*

$$\partial_t u = f(x,t,u(x,t),u_x(x,t)), \quad 0<t<T, \ x\in\Omega, \tag{21}$$

has a C^{k+1} solution for $t \geq 0$ on a neighborhood of $(x_0,0)$, and

$$\forall x\in\Omega, \quad \forall 0\leq j<k, \quad \Im\left(\mathcal{H}^j f_\zeta\right)^\upsilon(x,0)=0, \ \Im\left(\mathcal{H}^k f_\zeta\right)^\upsilon(x_0,0)\neq 0, \tag{22}$$

then for all $\xi^0\in\mathbb{S}^{N-1}$ such that

$$\Im\left(\mathcal{H}^k f_\zeta\right)^\upsilon(x_0,0)\cdot\xi^0<0, \tag{23}$$

the point $\left(x_0,\xi^0\right)$ does not belong to the C^∞ wave-front set of the trace $u(x,0)$.

Proof We consider the (holomorphic) Hamiltonian given by (see the beginning of Sect. 2 for the definition of $\mathcal{L}$ and g_j):

$$\mathcal{H}=\mathcal{L}+g_0\partial_{\zeta_0}+\sum_{j=1}^N g_j\partial_{\zeta_j}, \tag{24}$$

on a neighborhood $V=V_1\times V_2\times V_3\times V_4\subset\mathbb{C}_x^N\times\mathbb{C}_t\times\mathbb{C}_{\zeta_0}\times\mathbb{C}_\zeta^N$ of the point $(x_0,0,u(x_0,0),u_x(x_0,0))$, and we assume, with $r_0, T_0, \rho_0, \rho>0$, that

$$\begin{aligned} V_1 &= \left\{x\in\mathbb{C}^N : |x-x_0|<r_0\right\},\\ V_2 &= \{t\in\mathbb{C} : |t|<T_0\},\\ V_3 &= \{\zeta_0\in\mathbb{C} : |\zeta_0-u(x_0,0)|<\rho_0\}, \quad \text{and}\\ V_4 &= \left\{\zeta\in\mathbb{C}^N : |\zeta-u_x(x_0,0)|<\rho\right\}. \end{aligned}$$

For $1\leq j\leq N$, $0\leq l\leq N$, let $Z_j(x,t,\zeta_0,\zeta)$, and $\Xi_l(x,t,\zeta_0,\zeta)$ be smooth functions in V, holomorphic in (ζ_0,ζ), such that (see [A])

$$\begin{aligned} \mathcal{H}Z_j &= O\left(t^n\right), \ n=1,2,\ldots, \quad \textit{and} \ Z_j(x,0,\zeta_0,\zeta)=x_j \ \ (1\leq j\leq N),\\ \mathcal{H}\Xi_l &= O\left(t^n\right), \ n=1,2,\ldots, \quad \textit{and} \ \Xi_l(x,0,\zeta_0,\zeta)=\zeta_l \ \ (0\leq l\leq N). \end{aligned}$$

Let

$$\mathcal{L}^\upsilon=\partial_t-\sum_{j=1}^N f_{\zeta_j}^\upsilon(x,t)\,\partial_{x_j}.$$

Recall from Lemma 2 that $\mathcal{L}^\upsilon f^\upsilon=(\mathcal{H}f)^\upsilon$. This implies that the Z_j's and Ξ_l's (when we restrict them to $(x,t,u(x,t),u_x(x,t))$) are approximate solutions of $\mathcal{L}^\upsilon$.

Observe that for any i, $\partial_t^j \Im f_{\zeta_i}^{\upsilon}(x,0) = 0$ for $0 \le j \le k-1$. Indeed, the case $j = 0$ follows from (22). Assume it holds for all $0 \le i \le j$, for some $j < k-1$. Then using (22), Corollary 5, and Lemma 12, we have:

$$0 = \Im \left(\mathcal{H}^{j+1} f_{\zeta_i}\right)^{\upsilon}(x,0) = \left(\mathcal{L}^{\upsilon}\right)^{j+1} \Im f_{\zeta_i}^{\upsilon}(x,0) = \partial_t^{j+1} \Im f_{\zeta_i}^{\upsilon}(x,0).$$

Thus

$$\partial_t^j \Im f_{\zeta_i}^{\upsilon}(x,0) = 0 \quad for\ 0 \le j \le k-1. \tag{25}$$

Equation (25), Lemma 12, and Corollary 5 lead to:

$$\Im \left(\mathcal{H}^{k} f_{\zeta_i}\right)^{\upsilon}(x_0,0) = \left(\mathcal{L}^{\upsilon}\right)^{k} \Im f_{\zeta_i}^{\upsilon}(x_0,0) = \partial_t^{k} \Im f_{\zeta_i}^{\upsilon}(x_0,0). \tag{26}$$

We can write the approximate solution

$$Z^{\upsilon}(x,t) = x + t\psi(x,t) = x + t\psi^{(1)}(x,t) + it\psi^{(2)}(x,t),$$

where $\psi^{(1)}$ and $\psi^{(2)}$ are real-valued. Since $\mathcal{L}^{\upsilon} Z_j^{\upsilon}(x,t) = O(t^n),\ n \in \mathbb{N}$, we have:

$$t\frac{\partial \psi_j}{\partial t} + \psi_j - \sum_{i=1}^{N} f_{\zeta_i}^{\upsilon}\left(\delta_{ij} + t\frac{\partial \psi_j}{\partial x_i}\right) = O(t^n) \quad n \in \mathbb{N}.$$

Differentiating this latter equation repeatedly with respect to t and using (25) and (26), we get

$$\begin{aligned}\partial_t^j \psi^{(2)}(x,0) = 0 \quad for\ 0 \le j \le k-1, \quad \partial_t^k \psi^{(2)}(x_0,0) &= \partial_t^k \Im f_{\zeta_i}^{\upsilon}(x_0,0)\\ &= \Im\left(\mathcal{H}^k f_{\zeta_i}\right)^{\upsilon}(x_0,0)\end{aligned} \tag{27}$$

Let $M_i = \sum_{j=1}^{N} b_{ij}(x,t)\frac{\partial}{\partial x_j}$ be vector fields for $i = 1,\dots,N$ that satisfy $M_i Z_l^{\upsilon} = \delta_{il}$ for $1 \le i,l \le N$. For any C^1 function $h = h(x,t)$,

$$dh = \sum_{i=1}^{N} M_i(h)\, dZ_i^{\upsilon} + \left(\mathcal{L}^{\upsilon} h - \sum_{j=1}^{N} M_j(h)\mathcal{L}^{\upsilon}(Z_j^{\upsilon})\right) dt$$

as can be seen by applying both sides of the equation to the basis of vector fields $\{\mathcal{L}^{\upsilon}, M_1,\dots,M_N\}$. The latter implies that

$$d(h\, dZ_1^{\upsilon} \wedge \dots \wedge dZ_N^{\upsilon}) = \left(\mathcal{L}^{\upsilon} h - \sum_{j=1}^{N} M_j(h)\mathcal{L}^{\upsilon} Z_j^{\upsilon}\right) dt \wedge dZ_1^{\upsilon} \wedge \dots \wedge dZ_N^{\upsilon}. \tag{28}$$

For $(y, \xi) \in \mathbb{R}^N$, let

$$Q(x, t, y, \xi) = i\xi \cdot \big(y - Z^\upsilon(x, t)\big) - |\xi|\,(y - Z^\upsilon(x, t))^2$$

where for $z \in \mathbb{C}^N$, we write $z^2 = z_1^2 + \cdots + z_N^2$. Let $\eta(x) \in C_0^\infty(\mathbb{R}^N)$, $\eta \equiv 1$ for $|x - x_0| < r_0$ and $\eta \equiv 0$ when $|x - x_0| > 2r_0$, r_0 to be chosen later. Let

$$g(x, t, y, \xi) = \eta(x)\Xi_0^\upsilon(x, t)e^{Q(x,t,y,\xi)}$$

where y and ξ are parameters. Denoting $dZ_1^\upsilon \wedge \cdots \wedge dZ_N^\upsilon$ by dZ and using (28), we get:

$$d(g\,dZ) = \left(\mathcal{L}^\upsilon(\eta\Xi_0^\upsilon) + (\eta\Xi_0^\upsilon)\mathcal{L}^\upsilon(Q) - \sum_{j=1}^{N}(M_j(\eta\Xi_0^\upsilon) + \eta\Xi_0^\upsilon(M_j Q))\mathcal{L}^\upsilon Z_j^\upsilon\right) \times e^Q\,dt \wedge dZ. \tag{29}$$

By Stokes theorem we have, after decreasing $T_0 > 0$:

$$\int_{\mathbb{R}^N} g(x, 0, y, \xi)\,dx = \int_{\mathbb{R}^N} g(x, T_0, y, \xi)\,dZ(x, T_0) + \int_0^{T_0}\int_{\mathbb{R}^N} d(gdZ). \tag{30}$$

We will estimate the two integrals on the right in (30). Note that

$$\Re Q\,(x, t, y, \xi) = t\xi \cdot \psi^{(2)} - |\xi|\left(|y - x|^2 - 2t\left[(y - x)\cdot\psi^{(1)}\right] + t^2\left|\psi^{(1)}\right|^2 - t^2\left|\psi^{(2)}\right|^2\right).$$

Observe that using (27),

$$\begin{aligned}\psi^{(2)}(x, t) &= \frac{\partial_t^k\psi^{(2)}(x, 0)}{k!}t^k + O(t^{k+1})\\ &= \frac{\partial_t^k\psi^{(2)}(x_0, 0)}{k!}t^k + O(|x - x_0|t^k) + O(t^{k+1})\\ &= \frac{\Im\left(\mathcal{H}^k f_\zeta\right)^\upsilon(x_0, 0)}{k!}t^k + O(|x - x_0|t^k) + O(t^{k+1}).\end{aligned}$$

Hence if $\xi^0 \in \mathbb{S}^{N-1}$ satisfies condition (23), then there exists a conic neighborhood $B_{2r_0}(x_0) \times \Gamma$ of (x_0, ξ^0) in $\mathbb{R}^N \times (\mathbb{R}^N\backslash\{0\})$ and a constant $a > 0$ such that for T_0 small enough, and for $(x, \xi) \in B_{2r_0}(x_0) \times \Gamma$,

$$t\psi^{(2)}(x, t)\cdot\xi \le -a|\xi|t^{k+1}, \quad 0 \le t \le T_0.$$

Since $\psi^{(2)}(x,t) = O(t^k)$, after decreasing r_0 and T_0, for $a > 0$ small enough,

$$t\psi^{(2)}(x,t)\cdot\xi + |\xi|t^2|\psi^{(2)}(x,t)|^2 \leq -a|\xi|t^{k+1}, \quad 0 \leq t \leq T_0. \tag{31}$$

Using the inequality

$$2|t(y-x)\cdot\psi^{(1)}(x,t)| \leq |y-x|^2 + t^2|\psi^{(1)}(x,t)|^2$$

and (31), we get:

$$\Re Q(x,t,y,\xi) \leq -a|\xi|t^{k+1}, \quad 0 \leq t \leq T_0 \tag{32}$$

whenever $(x,\xi) \in B_{2r_0}(x_0) \times \Gamma$. Next observe that for T_0 sufficiently small, we can find $\delta > 0$ such that

$$\Re Q(x,t,y,\xi) \leq -\delta|\xi| \quad \text{whenever } r_0 \leq |x-x_0| \leq 2r_0 \quad \text{and } |y-x_0| \leq \frac{r_0}{2} \tag{33}$$

For y near x_0 and $\xi \in \Gamma$, inequality (32) leads to

$$\left|\int_{\mathbb{R}^N} g(x,T_0,y,\xi)\,dZ(x,T_0)\right| \leq c_1 e^{-c_2|\xi|}$$

for some $c_1, c_2 > 0$. To estimate the second integral on the right in (30), we will use the expression (29) which has two kinds of terms. The first type consists of terms which can be bounded by constant multiples of

$$t^n e^{\Re Q(x,t,y,\xi)} \quad \text{for } n = 1, 2, \ldots$$

and hence using (32), the integrals of these terms decay rapidly in ξ. The second type of terms involve derivatives of $\eta(x)$ and hence (33) can be used to get an exponential decay in ξ for their integrals. Since $\Xi_0(x,0) = u(x,0) = \omega(x)$, it follows that the FBI transform of the trace

$$\mathcal{F}_{\eta\omega}(y,\xi) = \int_{\mathbb{R}^N} e^{i\xi\cdot(y-x)-|\xi|(y-x)^2}\,\eta(x)\omega(x)\,dx$$

decays rapidly in a conic neighborhood of (x_0,ξ^0). By a result in [8] (see the last part of the proof of Theorem 2.1 in [8]), we conclude that (x_0,ξ^0) is not in the C^∞ wave-front set of $\omega(x) = u(x,0)$. □

Example 14 Let $u(x,t)$ be a C^{k+1} solution of the semilinear equation

$$\frac{\partial u}{\partial t} + \sqrt{-1}t^k\frac{\partial u}{\partial x} = g(x,t,u), \quad 0 \leq t < T, \;\; x \in (a,b)$$

where $g(x, t, \zeta_0)$ is a C^∞ function that is holomorphic in ζ_0. Then for any $x_0 \in (a, b)$ and $\xi > 0$, the point (x_0, ξ) is not in the C^∞ wave-front set of the trace $u(x, 0)$.

Note also that if u is a solution in a full neighborhood of a point $(x_0, 0)$ and k is an even integer, then the trace $u(x, 0)$ is a C^∞ function near x_0. When the function $g(x, t, \zeta_0)$ is real analytic, the microlocal analyticity of $u(x, 0)$ was discussed in [13].

Example 15 Let $u(x, t)$ be a C^2 solution of the Cauchy problem

$$\begin{cases} \frac{\partial u}{\partial t} + \sqrt{-1}tu\frac{\partial u}{\partial x} = g(x, t, u), \quad 0 \le t < T, \ \ x \in (a, b) \\ u(x, 0) = \omega(x) \end{cases}$$

where $g(x, t, \zeta_0)$ is a C^∞ function that is holomorphic in ζ_0. For any $x_0 \in (a, b)$ and ξ, if $\xi\, \Re\omega(x_0) > 0$, then the point (x_0, ξ) is not in the C^∞ wave-front set of the trace $u(x, 0)$. Again when the function $g(x, t, \zeta_0)$ is real analytic, the microlocal analyticity of $u(x, 0)$ was treated in [13].

3.2 The analytic case

In the paper [13], the authors studied microlocal analyticity of solutions of a class of quasi-linear first order PDE. Their main result (see Theorem 2.3 in [13]) can be generalized to the fully nonlinear case as we show in this subsection. For the definition and basic introduction to the concept of microlocal analyticity, we refer the reader to chapter 9 in [15] and to [14]. We consider the Cauchy problem:

$$\begin{cases} u_t = f\,(x, t, u\,(x, t), u_x\,(x, t)), \quad 0 < t < T, \ \ x \in \Omega, \\ u|_{t=0} = \omega\,(x), \quad x \in \Omega, \end{cases} \tag{34}$$

where $\Omega \subseteq \mathbb{R}^N$ is an open set, $T > 0$, and the function f is the restriction on $\Omega \times [0, T) \times V_3 \times V_4$ of some holomorphic function defined on a complex open domain $V = V_1 \times V_2 \times V_3 \times V_4 \subset \mathbb{C}^N \times \mathbb{C}^1 \times \mathbb{C}^1 \times \mathbb{C}^N$. We have the following theorem:

Theorem 16 *Let $f(x, t, \zeta_0, \zeta)$ be be real analytic in all the variables, and holomorphic in (ζ_0, ζ). Let $k \in \mathbb{N}$. If the nonlinear first order equation*

$$\partial_t u = f\,(x, t, u\,(x, t), u_x\,(x, t)), \quad 0 < t < T, \ \ x \in \Omega, \tag{35}$$

has a C^{k+1} solution for $t \ge 0$ on a neighborhood of $(x_0, 0)$, and

$$\forall x \in \Omega, \quad \forall 0 \le j < k, \quad \Im\left(\mathcal{H}^j f_\zeta\right)^\upsilon (x, 0) = 0, \ \Im\left(\mathcal{H}^k f_\zeta\right)^\upsilon (x_0, 0) \ne 0, \tag{36}$$

then for all $\xi^0 \in \mathbb{S}^{N-1}$ such that

$$\Im\left(\mathcal{H}^k f_\zeta\right)^\upsilon (x_0, 0) \cdot \xi^0 < 0, \tag{37}$$

the point $\left(x_0, \xi^0\right)$ *does not belong to the analytic wave-front set of the trace* $u\,(x, 0)$.

Proof For $1 \leq j \leq N, 0 \leq l \leq N$, let $Z_j\,(x, t, \zeta_0, \zeta)$, and $\Xi_l\,(x, t, \zeta_0, \zeta)$ be the holomorphic first integrals satisfying

$$\mathcal{H} Z_j = 0, \quad and \;\; Z_j\,(x, 0, \zeta_0, \zeta) = x_j \quad (1 \leq j \leq N),$$
$$\mathcal{H} \Xi_l = 0, \quad and \;\; \Xi_l\,(x, 0, \zeta_0, \zeta) = \zeta_l \quad (0 \leq l \leq N).$$

We have

$$\mathcal{L}^{\upsilon}(Z_j^{\upsilon}) = (\mathcal{H} Z_j)^{\upsilon} = 0 \quad and \;\; \mathcal{L}^{\upsilon}(\Xi_l^{\upsilon}) = (\mathcal{H} \Xi_l)^{\upsilon} = 0 \quad \forall j, l.$$

Therefore, with

$$g(x, t, y, \xi) = \eta(x)\,\Xi_0^{\upsilon}(x, t) e^{Q(x,t,y,\xi)}$$

as before, this time (29) becomes

$$d(g\,dZ) = \left(\mathcal{L}^{\upsilon}(\eta)\,\Xi_0^{\upsilon}\right) e^{Q}\,dt \wedge dZ.$$

Hence each of the two integrals on the right in (30) decay exponentially leading to

$$\left|\mathcal{F}_{\eta\omega}\,(y, \xi)\right| \leq c_1 e^{-c_2|\xi|}$$

for y near x_0 and ξ in an open cone Γ containing ξ^0. It follows that the point $\left(x_0, \xi^0\right)$ does not belong to the analytic wave-front set of $\omega(x) = u(x, 0)$. □

As in [13], the preceding theorem together with Lemma 4.3 in [13] lead to the following instability result with respect to a non-analytic perturbation of an analytic initial datum.

Corollary 17 *Suppose that for some* (x_0, v_0), *there exists* $k \in \mathbb{N}$ *such that*

$$\Im\left(\mathcal{H}^k f_{\zeta}\right)^{\upsilon}(x_0, 0) \neq 0.$$

Then for any analytic function ω_0 *such that* $\omega_0(x_0) = v_0$, *the Cauchy–Kovalevskaya solution of the Cauchy problem* (2) *with Cauchy datum* ω_0 *is strongly instable with respect to a* C^{∞} *perturbation, in the sense that, for any neighborhood* W *of* x_0 *and any neighborhood* $\mathcal{W}$ *of* ω_0 *in* $C^{\infty}(W)$, *there exists* $\omega \in \mathcal{W}$ *such that the Cauchy problem* (2) *with initial datum* ω *does not have a* C^{k+1} *solution. Moreover, for any analytic function* ω_0 *such that* $\omega_0(x_0) = v_0$, *there exists a* C^{∞} *function* ω *with the same Taylor expansion at* x_0 *as* ω_0 *such that the Cauchy problem* (2) *with initial datum* ω *does not have a* C^{k+1} *solution.*

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