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Multiple blow-up for a porous medium equation with reaction

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Abstract The present paper is concerned with the Cauchy problem

$$\begin{cases} \partial_t u = \Delta u^m + u^p & \text{in } \mathbb{R}^N \times (0, \infty), \\ u(x,0) = u_0(x) \geq 0 & \text{in } \mathbb{R}^N, \end{cases}$$

with $p, m > 1$. A solution u with bounded initial data is said to blow up at a finite time T if $\limsup_{t \nearrow T} \|u(t)\|_{L^\infty(\mathbb{R}^N)} = \infty$. For $N \geq 3$ we obtain, in a certain range of values of p, weak solutions which blow up at several times and become bounded in intervals between these blow-up times. We also prove a result of a more technical nature: proper solutions are weak solutions up to the complete blow-up time.

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1 Introduction and main results

This paper is concerned with the existence of solutions having multiple blow-up times for the Cauchy problem

$$\begin{cases} \partial_t u = \Delta u^m + u^p & \text{in } \mathbb{R}^N \times (0, \infty), \\ u(x, 0) = u_0(x) \geq 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $p, m > 1$. The equation is chosen as a simple and representative example of quasilinear reaction-diffusion equation where the diffusion operator is nonlinear and degenerate parabolic [14]. We assume bounded, integrable and nonnegative initial data, $u_0 \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$, $u_0 \geq 0$. There is no lack of generality in assuming that u_0 is also continuous [7,43].

The semilinear case, $\partial_t u = \Delta u + u^p$, $p > 1$, is a typical blow-up prototype that has been treated in a large number of publications, see for example [39] and the references therein. One of the aspects which has received attention only more recently is the possibility of having multiple blow-up [32–34]. However, the techniques do not extend naturally to quasilinear equations. Nevertheless, the property of finite propagation associated with porous medium diffusion (i.e., the exponent $m > 1$) implies the existence of solutions with compact support with respect to the space variable. This offers interesting possibilities for the construction of different types of blow-up solutions. Such constructions are not easy in the more standard case of linear diffusion, $m = 1$.

Before stating our results, we introduce the relevant definitions and review some of the known main results. The porous medium operator is degenerate parabolic at the level $u = 0$. Hence, there exists in general no classical solution of the equation for just nonnegative initial data. This means that a suitable concept of weak or generalized solution has to be introduced to ensure existence and uniqueness of solutions of the Cauchy problem [43]. The same happens for reaction-diffusion equations of the form

$$\partial_t u = \Delta u^m + f(u), \tag{1.2}$$

where f is a Lipschitz continuous real function [14,41]. The presence of a locally Lipschitz but not globally Lipschitz f creates the problem of blow-up, and this adds further difficulties in the suitable definition of solution.

Weak solutions A standard way of starting the study of these questions is introducing a class of data and solutions with reasonable regularity properties. We define a *weak solution*[1] of Eq. 1.2 in $Q_T = \mathbb{R}^N \times (0, T)$ as a function $u \in L^1_{\text{loc}}(Q_T)$ such that $u^m, f(u) \in L^1_{\text{loc}}(Q_T)$, and

$$\iint_{Q_T} (u \partial_t \phi + u^m \Delta \phi + f(u)\phi)\, dx\, dt = 0$$

[1] In order to be more precise we could call this an L^1_{loc}-weak solution.

is satisfied for any $\phi \in C^2(Q_T)$ compactly supported in Q_T. Given $u_0 \in L^1_{\rm loc}(\mathbb{R}^N)$, a *weak solution of the Cauchy problem* for Eq. 1.2 with initial data u_0 will be a weak solution in the previous sense that satisfies

$$u \in C([0,T) : L^1_{\rm loc}(\mathbb{R}^N)), \quad \lim_{t\to 0^+} u(t) = u_0 \quad \text{in } L^1_{\rm loc}(\mathbb{R}^N).$$

For initial data u_0 in $L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ and Lipschitz continuous functions f there exists a unique solution of the Cauchy problem in Q_T for all $T > 0$ [4,40] (see also [37] for initial data which are not in $L^1(\mathbb{R}^N)$, but do not grow too much at infinity). This solution turns out to be bounded for all $t > 0$. Moreover, the regularity theory for quasilinear parabolic equations, which includes these degenerate models, implies that bounded solutions are locally Hölder continuous with a Hölder uniform exponent [7,43]. Finally, when u_0 is positive and f is smooth for $u > 0$, the weak solution becomes a smooth classical solution.

When f is superlinear the situation is quite different. If the initial data belong to $L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, a weak solution exists and is bounded at least for a small time interval $0 < t < t_1$. However, it may become unbounded in a finite time, T,

$$\limsup_{t \nearrow T} \|u(t)\|_{L^\infty(\mathbb{R}^N)} = \infty.$$

In this case we say that u *blows up* at $t = T$.

Once solutions become unbounded, things become more involved. A first possible difficulty arises when $\lim_{t\to T^-} u(t)$ does not belong to $L^1_{\rm loc}(\mathbb{R}^N)$. If this happens (this is expected to be the case when $f(u) = u^p$ with $p \in (1, m]$, though a proof is only available with some restrictions on the initial data [35]), there is no weak solution in Q_{T_1} for $T_1 > T$. Even when the previous limit is in $L^1_{\rm loc}(\mathbb{R}^N)$, we have to face serious problems, since uniqueness is not guaranteed for unbounded weak solutions [14]. Therefore, in order to tackle blow-up problems after the blow-up time we need a different concept of solution.

Proper solutions We concentrate on Eq. 1.1. We recall the concept of *proper solution*, which was introduced as a suitable generalized solution in [14], based on the pioneering work [3]. Given $n > 0$, let u_n be the weak solution of

$$\begin{cases} \partial_t u = \Delta u^m + \min\{u^p, n^p\} & \text{in } \mathbb{R}^N \times (0,\infty), \\ u(x,0) = \chi_{B_n(0)}(x) \min\{u_0(x), n\} & \text{in } \mathbb{R}^N. \end{cases} \tag{1.3}$$

Then $u(x,t) \equiv \lim_{n\to\infty} u_n(x,t)$ is the proper solution of (1.1). In the sequel we will work by default with this concept of solution.

Remark In the definition of u_n we may take as initial data $P_n u_0$, where $\{P_n\}_{n>0}$ is any family of *ordered* approximation operators,

$$\begin{aligned} &P_n : L^1_{\rm loc}(\mathbb{R}^N) \mapsto L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), \\ &P_n u \ge P_m u \quad \text{if } n \ge m, \\ &\lim_{n\to\infty} P_n u_0 = u_0 \quad \text{in } L^1_{\rm loc}(\mathbb{R}^N). \end{aligned}$$

The limit (the proper solution) does not depend on the particular family of ordered approximation operators being used [14].

Complete blow-up As long as there exists a weak solution, the proper solution is a weak solution, which is *minimal* in the sense that it is below any other weak solution. Moreover, as long as there is a unique weak solution, as is the case while solutions remain bounded, the concepts of proper and weak solution coincide. Thus, the proper solution provides a way of defining a unique continuation of a weak solution past the blow-up time.

Let u be the proper solution of (1.1) and let $\mathcal{C} = \{t \geq 0 : u(x,t) = \infty \text{ for all } x \in \mathbb{R}^N\}$. We define

$$T^c = T^c(u_0) = \begin{cases} \inf \mathcal{C} & \text{if } \mathcal{C} \neq \emptyset, \\ \infty & \text{if } \mathcal{C} = \emptyset. \end{cases}$$

Hence T^c is the maximal time for which there is a nontrivial continuation. If $T^c(u_0) < \infty$, we say that u blows up *completely* at $T^c(u_0)$. If u blows up at a finite time T and $T < T^c(u_0)$, blow-up is said to be *incomplete*.

Let moreover

$$p_S = \begin{cases} +\infty & \text{if } N \leq 2, \\ m(N+2)/(N-2) & \text{if } N \geq 3, \end{cases}$$

denote the Sobolev exponent. Baras and Cohen [3] proved, in the case of a bounded domain with homogeneous Dirichlet boundary data, that blow-up is always complete if $m = 1$ and $p \in (1, p_S)$. Further results with more general reaction non-linearities can be found, for example, in [23,26]. As for the problem posed in the whole space, we have [14], where the authors prove that blow-up is always complete if $m \geq 1$ and $p \in (1, p_S]$ (the fast diffusion case, $m < 1$, is also considered). However, this result is restricted to radially symmetric solutions. Non-radial solutions have been treated in [42], for $m > 1$ and a special class of initial data that include the important case of measurable, bounded, compactly supported functions: blow-up is complete if $p \in (m, p_S)$ (the critical case $p = p_S$ is not covered). For the same class of initial data and $p \in (1, m]$, the *blow-up set* (the set where the solution becomes infinite as t approaches the blow-up time T) is known to have positive Lebesgue measure [35]. Hence, $\lim_{t \to T^-} u(t)$ does not belong to $L^1_{\text{loc}}(\mathbb{R}^N)$. We will see in Sect. 2 that this implies that blow-up is complete. Summarizing, it seems that in order to look for incomplete blow-up phenomena we should take $N \geq 3$, $p > p_S$ (an excellent review of recent results in the critical and supercritical cases, $p \geq p_S$, when $m = 1$ can be found in [8]).

Proper solutions are weak solutions up to T^c A natural question is whether the proper solution is a weak solution up to the *complete blow-up* time, T^c (after T^c there is no possibility of having a weak solution). The answer is obviously yes in those cases for which blow-up is always complete, for example, when $1 < p \leq m$ if the initial data satisfy certain conditions (see the previous paragraph). On the contrary, if $1 < p < 2 - m$, which is only possible if $m < 1$, there are proper solutions which are not weak solutions up to T^c [14]. In our first theorem, which will be proved in Sect. 2, we extend the study of this question to the range $1 \leq m < p$, including all nonnegative initial data in $L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$.

Theorem 1 *Let $1 \leq m < p$. Let u be the proper solution of (1.1) and T^c its complete blow-up time. Then u is a weak solution in Q_{T^c}.*

This result is in some sense independent of the rest of the paper. However, it will be useful to simplify some other proofs.

Peaking and multiple blow-up solutions In the case of incomplete blow-up, it may happen that the proper solution becomes bounded and continuous immediately after blow-up for some time. As time proceeds, it may continue to be bounded for all later times, or it may blow up again. In the former case, we have a *peaking solution*. In the latter, we have *multiple blow-up*. Our main aim is to construct solutions of both kinds. Notice that, since proper solutions are "small" (they are minimal), this seems to be the right class in which to look for solutions with incomplete blow-up.

A peaking solution for (1.1) was constructed in [14] for all $m \geq 1$ and $p \in (p_S, p_L)$ by connecting a backward self-similar blow-up solution with a forward self-similar solution (the idea of using such connection stems from Lacey and Tzanetis [24], who followed this approach to construct formally a peaking solution for the semilinear equation $u_t = \Delta u + e^u$ in $\mathbb{R}^3$). This solution goes to zero as $t \to \infty$. The value of the new critical exponent (due to Lepin [25] in the semilinear case) is given by

$$p_L = \begin{cases} +\infty & \text{if } N \leq 10, \\ 1 + \dfrac{3m + \alpha(m, N)^{1/2}}{N - 10} & \text{if } N \geq 11, \end{cases}$$

where $\alpha(m, N) = \{(m-1)^2(N-10)^2 + 2(m-1)(5m-4)(N-10) + 9m^2\}$. There seems to be no other known results concerning this topic for (1.1) when $m > 1$. For $m = 1$, peaking solutions with various behaviors as $t \to \infty$ have been obtained in [31].

Up to now, multiple blow-up solutions have only been obtained for the semilinear case. The first example, a proper solution blowing up twice, was constructed in [32]. This result was improved in [33], where *global* weak solutions (that is, defined in Q_T for all $T > 0$) with an arbitrary finite number of blow-up times were given. These two papers are restricted to $p > p_{JL}$, where the critical exponent p_{JL} (introduced by Joseph and Lundgren in the semilinear case [20]) stands for

$$p_{JL} = \begin{cases} +\infty & \text{if } N \leq 10, \\ m\left(1 + \dfrac{4}{N - 4 - 2(N-1)^{1/2}}\right) & \text{if } N \geq 11. \end{cases}$$

As for the whole range $p > p_S$, we have [34], where the authors construct proper solutions blowing up twice, with blow-up times as close as desired to any two prescribed times. The second blow-up is complete.

Peaking and multiple blow-up new results While the self-similar peaking solution constructed in [14] is positive everywhere, we construct here a peaking solution which has compact support for all times. However, we are not able to go as far as [14] in the range of exponents (notice that $p_S < p_{JL} < p_L$).

Theorem 2 *Let $p \in (p_S, p_{JL})$. Given $T > 0$, there exists a radially symmetric peaking solution of* (1.1) *that blows up at the origin at $t = T$ and has compact support in x for all t. It goes to zero as $t \to \infty$.*

We will construct multiple blow-up solutions gluing peaking solutions with compact support centered at different points. We just have to control the supports of the different peaking solutions so that they do not intersect before the last blow-up time. In order to achieve this, the blow-up points have to fall apart enough one from each other.

Theorem 3 *Let $p \in (p_S, p_{JL})$. Given any bounded sequence of times $\{T_i\}_{i=0}^k$, $k \in \mathbb{N}$ or $k = \infty$, such that $T_0 = 0$, $T_{i-1} < T_i$ for $1 \le i \le k$ ($i \ge 1$ if $k = \infty$), there exists a proper solution u of* (1.1) *such that u blows up incompletely at $t = T_i$ and is bounded for $t \in (T_{i-1}, T_i)$, $1 \le i \le k$ ($i \ge 1$ if $k = \infty$).*

Notice that we are able to choose the blow-up times. This is in contrast with what is known for the semilinear case, where, up to now, the blow-up times can only be prescribed in an approximate manner, and only for solutions blowing up twice [34].

Observe also that we are able to construct solutions with infinitely many blow-up times. Fila, Matano and Poláčik proved recently [9] that this is not possible in the case of the heat equation posed in a ball with radial initial data, both for power-like and for exponential reaction nonlinearities. In this respect let us notice that our construction does not work in a ball: the blow-up points escape to infinity. Let us also mention that there is a previous example of a solution having infinitely many blow-up times due to Michel Pierre.[2] He observed that

$$u(x,t) = \frac{1}{|x|^2 + \psi(t)}$$

is an L^1-weak solution of the equation

$$\partial_t u = \Delta u + g(|x|, t)u^2, \quad g(r,t) = 2N - \psi'(t) - \frac{8r^2}{r^2 + \psi(t)},$$

provided ψ is a nonnegative C^1 function and $N > 4$. Obviously, u blows up at each time such that $\psi(t) = 0$. However, in contrast with our problem, Pierre's equation is not autonomous.

If we choose the blow-up times as in Theorem 3, we are not able to fix the blow-up points. However, we have a kind of dual result. We may choose the blow-up points at the cost of not being able to decide completely the blow-up times: they have to be small enough.

[2] It appears, cited as a private communication, in [9].

Theorem 4 *Let $p \in (p_S, p_{JL})$ and let $\{x_i\}_{i=0}^k$, $k \in \mathbb{N}$ or $k = \infty$, be any sequence of points such that $|x_i - x_j| \geq \delta$, $i \neq j$, for some $\delta > 0$. Then, there exists a constant $\Theta = \Theta(\delta) > 0$ such that for any sequence of times $\{T_i\}_{i=0}^k$, satisfying $T_0 = 0$, $T_{i-1} \leq T_i$, $0 < T_i < \Theta(\delta)$, for $1 \leq i \leq k$ ($i \geq 1$ if $k = \infty$), there is a proper solution u of (1.1) that blows up incompletely at the point x_i at time $t = T_i$. This solution is bounded for $t \in (T_{i-1}, T_i)$ for all $1 \leq i \leq k$ ($i \geq 1$ if $k = \infty$) such that $T_{i-1} \neq T_i$.*

Difference with the semilinear case Let us explain a bit the difference between the theory for Eq. 1.1 and the semilinear case when trying to get multiple blow-up solutions. Let $N \geq 3$. Given any $m \geq 1$ and $p > p_{ST} = mN/(N-2)$, there is a radially symmetric singular steady state of (1.1) given by

$$\varphi_\infty(r) = c_\infty r^{-\frac{2}{p-m}} \quad \text{for } r = |x| > 0, \tag{1.4}$$

with $c_\infty = \left\{ \frac{2m}{p-m} \left(N - 2 - \frac{2m}{p-m} \right) \right\}^{\frac{1}{p-m}}$.

Remark The restriction $p > p_{ST}$ guarantees that c_∞ is well defined and also that $\varphi_\infty, \varphi_\infty^m, \varphi_\infty^p \in L^1_{\mathrm{loc}}(Q_T)$. Hence φ_∞ is a weak solution. This is an example of incomplete blow-up of a special kind: it has a single-point, stationary blow-up set. Notice that $m < p_{ST} < p_S$. In particular, the range of existence of such incomplete blow-up solutions goes down beyond Sobolev exponent.

Since the pioneering work of Giga and Kohn [15–17], many features of blow-up are known to be best seen by passing to self-similar variables. Thus, given a solution u of (1.1) blowing up at time $T > 0$ at the point $a \in \mathbb{R}^N$, we put

$$w(y, s) = (T - t)^{1/(p-1)} u(a + (T - t)^\beta y, t)$$

with $\beta = (p - m)/2(p - 1)$, and $s = -\log(T - t)$, so that $s \to \infty$ as $t \to T^-$. Then w satisfies

$$\begin{cases} w_s = \Delta(w^m) - \beta y \cdot \nabla w - \frac{1}{p-1} w + w^p & \text{in } \mathbb{R}^N \times (s_T, \infty), \\ w(y, s_T) = T^{1/(p-1)} u_0(T^\beta y) & \text{in } \mathbb{R}^N, \end{cases} \tag{1.5}$$

where $s_T = -\log T$. It is immediate that φ_∞ is also the singular steady state of (1.5).

A solution that blows up at time T is said to have type I blow-up if

$$(T - t)^{\frac{1}{p-1}} \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \quad \text{remains bounded as } t \nearrow T.$$

Note that $(T - t)^{\frac{1}{p-1}}$ is the blow-up rate of solutions of the ordinary differential equation $u_t = u^p$. If u has type I blow-up, the corresponding w will approach a bounded stationary state of (1.5). Blow-up solutions which are not of type I, which were first considered by Herrero and Velázquez in [19][3] for $m = 1$, are said to be of type II. A very complete characterization of Type I and II blow-up is done by Matano and Merle [29].

The method followed in [32] and [33] to construct multiple blow-up solutions for the semilinear case makes use of an argument to construct a type II blow-up solution which is based on the linearization of (1.5) around φ_∞. However, to apply the manner of the above mentioned papers for $m > 1$, we should linearize the diffusion term Δw^m in addition to w^p. Surely, this would need extremely heavy calculations.

A different approach, based on the use of an energy functional has been recently used in [34]. If $m = 1$, then $\beta = 1/2$ and, after multiplication of (1.5) by the integrating factor $e^{-|y|^2/4}$, we may get rid of the term involving $y \cdot \nabla w$. It turns out that there is an *energy* functional

$$E[w] = \int\limits_{\mathbb{R}^N} \left\{ \frac{1}{2}|\nabla w|^2 + \frac{1}{2(p-1)} w^2 - \frac{1}{p+1} w^{p+1} \right\} \exp\left(-\frac{|y|^2}{4} \right) dy$$

which is nonincreasing with respect to s and has other useful properties. It seems difficult to construct the energy functional in a suitable function space as above for $m > 1$, and we have been unable to overcome the technical difficulties. On the other hand, it is not clear whether the known Lyapunov functional is enough or not to get our desired properties.

Sketch of the proof of Theorem 2 We follow an idea that was used in [30] to obtain a solution with incomplete blow-up for the semilinear case.

Given any compactly supported bounded function $h \not\equiv 0$, and any $\mu > 0$, we consider the proper solution u_μ to (1.1) with initial data μh. Since $p > p_S$, we are above the so-called Fujita exponent, $p_F = m + \frac{2}{N}$. Therefore, there is a critical value μ^* such that u_μ blows up in finite time if $\mu > \mu^*$ and remains bounded for all times if $\mu < \mu^*$. Indeed, for $p > p_F$ solutions with large initial data blow up, while they are bounded for all times if they are initially small. On the contrary, if $p \in (1, p_F]$ all notrivial solutions blow up in finite time, see [10,18,22] for the semilinear problem, $m = 1$, and [11,13] for $m > 1$.

Incomplete blow-up solutions are expected to belong to the threshold case, $\mu = \mu^*$. However, to ensure that $u^* = u_{\mu^*}$ is a peaking solution, we will need to impose additional assumptions on the function h.

Thus, if h is radially symmetric and non-increasing in $r = |x|$, then u^* is a global weak solution. This is an immediate consequence of a pointwise estimate for radially symmetric, non-increasing in $r = |x|$, proper solutions of (1.1) with $T^c = \infty$, that will be proved in Sect. 3.

It is not very difficult to show that u^* has to blow up in a finite time T^*. Otherwise, a continuous dependence argument would allow to push μ^* a bit further, contradicting its definition.

[3] The unpublished preprint *A blow up result for semilinear heat equations in the supercritical case*, by the same authors, which contains a detailed proof of the result, has widely circulated among the specialists in the field of blow-up.

If, moreover, h is such that μh has at most two intersections (see the beginning of Sect. 4 for a precise definition) with φ_∞ for all $\mu > 0$, then u^* is a peaking solution. This follows from a result of immediate regularization for solutions of (1.1) which will be proved in Sect. 4. Unfortunately, it is restricted to $p \in (p_S, p_{JL})$. A similar (and, in fact, more general) regularization result for $m = 1$ was given in [9], see also [29]. In the course of the proof, which is based on the theory of dynamical systems, the authors used in an essential way the convergence along a subsequence $\{s_n\}$ of the rescaled variable $w(\cdot, s)$ to a stationary solution. However, as mentioned above, the transformation through backward self-similar variables is not useful for $m > 1$. Therefore, the proof of our regularization result, Theorem 6, is different, though it also applies to the semilinear case, see [34].

Since h is compactly supported, a comparison argument will show that u^* has compact support in x for all times, Finally, to fix the blow-up time at a prescribed value, we just use a scaling argument.

Remark The first authors that noticed (for $m = 1$, in a bounded domain with homogeneous Dirichlet boundary data) that threshold solutions like u^* have special features were Ni, Sacks, and Tavantzis [36], who showed that u^* is a global unbounded solution for any $p > p_S$. At that time it was not known whether those unbounded solutions blow up in finite time or only tend to infinity as $t \to \infty$. Galaktionov and Vázquez [14] later showed that those solutions indeed blow up in finite time if $p \in (p_S, p_L)$, provided the domain is a ball and the solutions are radially symmetric.

2 Proper solutions are weak solutions before the complete blow-up

We start with two technical results and some notation. We denote by γ_R and ϕ_R the first eigenvalue and eigenfunction of the problem

$$\begin{cases} -\Delta\phi = \gamma\phi & \text{in } B_R(0) = \{x \in \mathbb{R}^N : |x| < R\}, \\ \phi = 0 & \text{on } \partial B_R(0), \end{cases}$$

normalized by $\|\phi_R\|_{L^1(B_R(0))} = 1$. It is immediate that

$$\gamma_R = \gamma_1 R^{-2}, \quad \phi_R(r) = R^{-N}\phi_1(R^{-1}r) \quad \text{for } 0 \le r \le R.$$

Lemma 1 *Let u be a weak solution of (1.2) in Q_T, $R > 0$ and $I = [\tau, t]$, $0 < \tau < t < T$. Then,*

$$\int\limits_{B_R(0)} u(x,t)\phi_R(|x|)\,dx - \int\limits_{B_R(0)} u(x,\tau)\phi_R(|x|)\,dx$$
$$\ge -\gamma_R \iint\limits_{B_R(0)\times I} u^m(x,t)\phi_R(|x|)\,dxdt + \iint\limits_{B_R(0)\times I} f(u(x,t))\phi_R(|x|)\,dxdt. \quad (2.1)$$

Proof For $\varepsilon > 0$ we define

$$\zeta_\varepsilon(x) = \begin{cases} e^{-\frac{\varepsilon}{\phi_R(|x|)}}, & |x| < R, \\ 0, & |x| \geq R, \end{cases}$$

Notice that

$$\Delta(\zeta_\varepsilon \phi_R) = \begin{cases} e^{-\frac{\varepsilon}{\phi_R}} \left(\Delta\phi_R - \varepsilon\gamma_R + \varepsilon^2 \frac{|\nabla\phi_R|^2}{\phi_R^3} \right), & |x| < R, \\ 0, & |x| \geq R, \end{cases}$$

so that for $|x| < R$ we have $\Delta(\zeta_\varepsilon\phi_R) \geq e^{-\frac{\varepsilon}{\phi_R}} \left(\Delta\phi_R - \varepsilon\gamma_R\right)$.

Let now $\theta_\varepsilon \in C^\infty([0,\infty))$ be such that $0 \leq \theta_\varepsilon(s) \leq 1$ for all $s \geq 0$, $\theta_\varepsilon(s) = 1$ for $s \in [\tau, t]$, $\theta_\varepsilon(s) = 0$ if $s \notin (\tau - \varepsilon, t + \varepsilon)$. Using the weak formulation for (1.2) with test function $\phi_R(x)\zeta_\varepsilon(x)\theta_\varepsilon(s)$, we have

$$\begin{aligned} &- \iint\limits_{B_R(0)\times(\tau-\varepsilon,t+\varepsilon)} u\phi_R\zeta_\varepsilon\theta_\varepsilon'\, dxds \\ &\geq \iint\limits_{B_R(0)\times(\tau-\varepsilon,t+\varepsilon)} \left(u^m\theta_\varepsilon e^{-\frac{\varepsilon}{\phi_R}} \left(\Delta\phi_R - \varepsilon\gamma_R\right) + f(u)\phi_R\zeta_\varepsilon\theta_\varepsilon \right) dxds, \end{aligned}$$

from where we get, after letting $\varepsilon \to 0$, the inequality

$$\begin{aligned} &\int\limits_{B_R(0)} u(x,t)\phi_R(|x|)\, dx - \int\limits_{B_R(0)} u(x,\tau)\phi_R(|x|)\, dx \\ &\geq \iint\limits_{B_R(0)\times I} u^m(x,t)\Delta\phi_R(|x|)\, dxdt + \iint\limits_{B_R(0)\times I} f(u(x,t))\phi_R(|x|)\, dxdt. \end{aligned}$$

Using the equation $\Delta\phi_R = -\gamma_R\phi_R$, we get the result. □

Proposition 1 *Let $p, m > 1$ and let u be a proper solution of (1.2). If $u(\cdot, T) \notin L^1_{\mathrm{loc}}(\mathbb{R}^N)$, then u is identically infinite for all $t > T$.*

Proof This follows immediately from the following inequality due to Aronson and Caffarelli [2], valid for weak solutions of the porous medium equation,

$$\int\limits_{B_r(x_0)} u(x,T)\, dx \leq C \left(r^{\lambda/(m-1)}(t-T)^{-1/(m-1)} + (t-T)^{N/2} u^{\lambda/2}(x_0,(t-T)) \right),$$

with $\lambda = N(m-1)+2$, $x_0 \in \mathbb{R}^N$, $r, t > T$ arbitrary. Indeed, the inequality implies that solutions u of the porous medium equation such that $u(\cdot, T) \notin L^1_{\mathrm{loc}}(\mathbb{R}^N)$ are identically infinite for all $t > T$. Since solutions to (1.1) are supersolutions to the porous medium equation, the same is true for them by the Maximum Principle. □

We now have the tools to prove that, for our type of equation, the proper solution is a weak solution up to the complete blow-up time T^c.

Proof of Theorem 1 Let $\mathcal{W} = \{T > 0 : u \text{ is a weak solution in } Q_T\}$, and

$$T^w = T^w(u_0) = \begin{cases} \max \mathcal{W} & \text{if } \mathcal{W} \neq (0, \infty), \\ \infty & \text{if } \mathcal{W} = (0, \infty), \end{cases}$$

Hence, T^w is the maximal time for which the proper solution is a weak solution. It is obvious that $T^w(u_0) \leq T^c(u_0)$.

In order to prove the converse inequality we first observe that if $u \in L^p_{\text{loc}}(Q_T)$, then $u \in L^m_{\text{loc}}(Q_T)$, $u \in L^1_{\text{loc}}(Q_T)$. Hence, we can pass to the limit in the Eq. (1.3) satisfied by the approximate solutions to obtain that u is a weak solution in Q_T. Therefore, if $T^w < T^c$, then $u \notin L^p_{\text{loc}}(Q_{T^c})$.

Suppose next that $T^w < T^c$. In this case we claim that there is a value $T \in (T^w, T^c)$, such that $u(\cdot, T) \notin L^1_{\text{loc}}(\mathbb{R}^N)$. By the previous proposition, this would imply that u is identically infinite after T. Hence $T^c \leq T$, a contradiction which would end the proof.

Let us prove our claim. We consider two cases.

(i) If $u \in L^m_{\text{loc}}(Q_{T^c})$, we consider Eq. 2.1 with reaction term $f(u) = \min\{u^p, n^p\}$. For any $T \in (T^w, T^c)$ we have

$$\gamma_R \iint\limits_{B_R(0)\times I} u_n^m(x,t)\phi_R(|x|)\,dxdt + \int\limits_{B_R(0)} u_n(x,T)\phi_R(|x|)\,dx \geq \iint\limits_{B_R(0)\times I} \min\{u_n^p(x,t), n^p\}\phi_R(|x|)\,dxdt.$$

Since $u_n \leq u < \infty$ almost everywhere, $\lim_{n\to\infty} \min\{u_n^p, n^p\} = u^p$ almost everywhere. Passing to the limit as $n \to \infty$ (remember that u_n converges monotonically), we get

$$\gamma_R \iint\limits_{B_R(0)\times I} u^m(x,t)\phi_R(|x|)\,dxdt + \int\limits_{B_R(0)} u(x,T)\phi_R(|x|)\,dx \geq \iint\limits_{B_R(0)\times I} u^p(x,t)\phi_R(|x|)\,dxdt.$$

Hence, if $u \notin L^p_{\text{loc}}(Q_{T^c})$, we have that $u(\cdot, T) \notin L^1_{\text{loc}}(\mathbb{R}^N)$.

(ii) Let us assume now that $u \notin L^m_{\text{loc}}(Q_{T^c})$. In this case the argument is not so direct, and a double limit is involved. Given $k, n > 0$, let $v_{k,n}$ be the weak solution of

$$v_t = \Delta v^m + f_{k,n}(v), \quad v(x,0) = \chi_{B_k(0)}(x)\min\{u_0(x), k\},$$

where $f_{k,n}(v) = \min\{v^p, kv^m, n^p\}$. In the weak formulation for this problem we use as test function $\zeta\theta_\varepsilon$, where ζ and θ_ε, $\varepsilon > 0$, are defined as follows:

- ζ is a C^∞, nonnegative function such that for some constants $0 < R_1 < R$ (R as large as desired), $c > 0$:
  - (i) $\zeta = 0$ for $x \in \mathbb{R}^N \setminus B_R(0)$;
  - (ii) $-c \le \Delta\zeta \le 0$, $\zeta \ge 1$, for $x \in B_{R_1}(0)$;
  - (iii) $\Delta\zeta \ge 0$ for $x \in \mathbb{R}^N \setminus B_{R_1}(0)$.
- θ_ε is a C^∞ function such that:
  - (i) $0 \le \theta_\varepsilon(s) \le 1$ for all $s \ge 0$;
  - (ii) $\theta_\varepsilon(s) = 1$ for $s \in [\tau, T]$; $\theta_\varepsilon(s) = 0$ if $s \notin (\tau - \varepsilon, T + \varepsilon)$.

We get

$$\begin{aligned}
-\iint\limits_{B_R(0)\times(\tau-\varepsilon,T+\varepsilon)} v_{k,n}\zeta\theta'_\varepsilon &= \iint\limits_{B_R(0)\times(\tau-\varepsilon,T+\varepsilon)} \left(v_{k,n}^m\theta_\varepsilon\Delta\zeta + f_{k,n}(v_{k,n})\zeta\theta_\varepsilon\right)\\
&\ge -c\iint\limits_{B_{R_1}(0)\times(\tau-\varepsilon,T+\varepsilon)} v_{k,n}^m\theta_\varepsilon\\
&\quad + \iint\limits_{B_R(0)\times(\tau-\varepsilon,T+\varepsilon)} f_{k,n}(u)\zeta\theta_\varepsilon,
\end{aligned}$$

which, letting $\varepsilon \to 0$, yields

$$\begin{aligned}
&\int\limits_{B_R(0)} v_{k,n}(x,T)\zeta(x)\,dx - \int\limits_{B_R(0)} v_{k,n}(x,\tau)\zeta(x)\,dx\\
&\quad\ge -c\iint\limits_{B_{R_1}(0)\times(\tau,T)} v_{k,n}^m(x,t)\,dxdt\\
&\qquad + \iint\limits_{B_R(0)\times(\tau,T)} f_{k,n}(v_{k,n}(x,t))\zeta(x)\,dx.
\end{aligned}$$

Moreover, $f_{k,n}(v) \ge f_{k_0,n}(v) \ge \min\{k_0v^m, n^p\} - C$, for all $k \ge k_0$ for some $C = C(k_0, m, p)$. Hence,

$$\begin{aligned}
&\int\limits_{B_R(0)\times(\tau,T)} v_{k,n}(x,T)\zeta(x)\,dx + C\iint\limits_{B_R(0)\times(\tau,T)} dxdt\\
&\quad\ge \iint\limits_{B_{R_1}(0)\times(\tau,T)} \left(\min\{k_0v_{k,n}^m(x,t), n^p\} - cv_{k,n}^m(x,t)\right)\,dxdt. \qquad (2.2)
\end{aligned}$$

Comparison yields that $v_{k,n} \le u_n$ if $n \ge k$. Hence, $v_{k,n} \le u$ if $n \ge k$. Therefore, the proper solution of

$$v_t = \Delta v^m + \min\{v^p, kv^m\}, \quad v(x,0) = \chi_{B_k(0)}(x)\min\{u_0(x), k\}, \tag{2.3}$$

which is given by $v_k = \lim_{n\to\infty} v_{k,n}$, satisfies $v_k \le u$. Thus, $v_k(\cdot,t) \not\equiv \infty$ for $t < T^c(u_0)$. On the other hand, it is known that solutions of (2.3) cannot blow up incompletely. Indeed, if $m = 1$ they do not blow up at all, while if $m > 1$ the blow-up set has positive Lebesgue measure according to [35] (see also [6] for a different approach). In any case, v_k is bounded in $\mathbb{R}^N \times [0,T]$ for all $T < T^c(u_0)$. This provides a uniform bound for the integrals

$$\iint\limits_{B_{R_1}(0)\times(\tau,T)} \min\{k_0 v_{k,n}^m(x,t), n^p\}\,dxdt, \qquad \iint\limits_{B_{R_1}(0)\times(\tau,T)} v_{k,n}^m(x,t)\,dxdt.$$

Hence, we may pass to the limit $n \to \infty$, to obtain

$$\int\limits_{B_R(0)} v_k(x,T)(x,t)\zeta(x)\,dx + \widehat{C} \ge (k_0 - c) \iint\limits_{B_{R_1}(0)\times(\tau,T)} v_k^m(x,t)\,dxdt.$$

It is easily checked that $v_k \ge u_{k^{1/(p-m)}}$. Hence, $\lim_{k\to\infty} v_k = u$. Since the limit is monotone, we get

$$\int\limits_{B_R(0)} u(x,T)\zeta(x)\,dx + \widehat{C} \ge (k_0 - c) \iint\limits_{B_{R_1}(0)\times(\tau,T)} u^m(x,t)\,dxdt.$$

Taking $k_0 \ge c$, the claim is proved also in this case. □

3 A pointwise Kaplan-type estimate

The aim of this section is to obtain an upper bound for global proper solutions of (1.1). The upper bound will follow from an integral estimate that will be proved using a technique which can be traced back to Kaplan's classical paper [21]. An analogous upper bound for the semilinear case was obtained in [30]; it was later improved in [28], where the authors remove the monotonicity restriction (see also [9] for more general reaction non-linearities).

Lemma 2 *Let $p > m > 1$ and let u be the proper solution of (1.1) and $T^c > 0$ its complete blow-up time. If there is a value $\underline{t} \in [0, T^c)$ such that*

$$\int\limits_{B_R(0)} u(\underline{t})\phi_R\,dx \ge \{(1+\delta)\gamma_R\}^{\frac{1}{p-m}} \tag{3.1}$$

for some $\delta, R > 0$, *then*

$$T^c - \underline{t} \leq \frac{\gamma_R^{-\frac{p-1}{p-m}}}{(p-1)\delta(1+\delta)^{\frac{m-1}{p-m}}}. \tag{3.2}$$

Proof The idea is to obtain a differential inequality for $g(t) = \int_{B_R(0)} u(t)\phi_R dx$. Since $m > 1$, we need to take care of the lack of regularity of weak solutions.

Using Jensen's inequality in Eq. 2.1, we get

$$\int_{B_R(0)} u(t)\phi_R dx - \int_{B_R(0)} u(\tau)\phi_R dx \geq \int_\tau^t F(s(t))\, dt, \tag{3.3}$$

where $F(s) = -\gamma_R s + s^{p/m}$ and $s(t) = \int_{B_R(0)} u^m(t)\phi_R\, dx$. Using again Jensen's inequality, together with (3.1), we get

$$s(\underline{t}) \geq \left(\int_{B_R(0)} u(\underline{t})\phi_R\, dx \right)^m \geq \{(1+\delta)\gamma_R\}^{\frac{m}{p-m}}.$$

Hence, $F(s(\underline{t})) > 0$. Let us show that $F(s(t)) > 0$ for all $t \geq \underline{t}$.

Notice that $g(t)$ is monotone increasing while $F(s(t))$ stays positive, see (3.3). Assume that $F(s(t))$ is not positive for all $t \geq \underline{t}$. Let $\bar{t}$ be the first time such that $F(s(t)) = 0$. Thanks to the monotonicity of $g(t)$ up to $\bar{t}$ we have that

$$s(\bar{t}) \geq g^m(\bar{t}) \geq g^m(\underline{t}) = \{(1+\delta)\gamma_R\}^{\frac{m}{p-m}},$$

which implies that $F(s(\bar{t})) > 0$, a contradiction.

As a consequence, g is monotone increasing, and hence $g(t) \geq \{(1+\delta)\gamma_R\}^{\frac{1}{p-m}}$, for all $t \geq \underline{t}$. Moreover, since $s(t) \geq g^m(t) \geq \{(1+\delta)\gamma_R\}^{\frac{m}{p-m}}$, we have, using that F is monotone increasing above the level $\left(\frac{m}{p}\gamma_R\right)^{\frac{m}{p-m}}$, that

$$F(s(t)) \geq F(g^m(t)) = g^p(t)(1 - \gamma_R g^{m-p}(t)) \geq \frac{\delta}{1+\delta} g^p(t).$$

Thus, we arrive to

$$g(t) \geq g(\underline{t}) + \frac{\delta}{1+\delta} \int_{\underline{t}}^t g^p(t)\, dt \equiv \psi(t).$$

Therefore,

$$\psi'(t) \geq \frac{\delta}{1+\delta}\psi^p(t).$$

Hence, ψ approaches infinity monotonically in a finite time $T_1 \geq T^c$. Integrating this differential inequality in $(\underline{t}, T_1)$, we get

$$\frac{\delta}{1+\delta}(T_1 - \underline{t}) \leq \frac{1}{p-1}\psi^{1-p}(\underline{t}) = \frac{1}{p-1}g^{1-p}(\underline{t}),$$

from where the result follows. □

Remark This lemma shows that all nontrivial solutions blow up completely in finite time if $p \in (m, p_F)$. Notice that $p_F < p_{ST} < p_S$.

As a by-product we obtain a result about the stability of complete blow-up. A similar result for the case $m = 1$ has been obtained in [34].

Proposition 2 *Let $p > m > 1$ and let u and $\widetilde{u}$ be proper solutions of* (1.1) *with initial data u_0 and $\widetilde{u}_0$ respectively. If $T^c(u_0) < \infty$, then, for each $\varepsilon > 0$ there exists a value $\delta > 0$ such that $|T^c(\widetilde{u}_0) - T^c(u_0)| < \varepsilon$ if $\|\widetilde{u}_0 - u_0\|_\infty < \delta$.*

Proof Let u_n be a solution of (1.3). Since u blows up completely at $t = T^c(u_0)$,

$$\lim_{n\to\infty} u_n(x,t) = +\infty \quad \text{for } (x,t) \in \mathbb{R}^N \times (T^c(u_0), \infty). \tag{3.4}$$

Hence, given values $\varepsilon, M > 0$ and $x \in \overline{B_1(0)}$, there exists $n_x = n_x(\varepsilon, M)$ such that $u_{n_x}(x, T^c(u_0)+\varepsilon) > 3M$. Since the functions u_n are continuous, there exists $\delta_x > 0$ such that $u_{n_x}(y, T^c(u_0)+\varepsilon) > 2M$ if $|y-x| < \delta_x$. Thanks to the compactness of $\overline{B_1(0)}$, we can take $x_1, x_2, \ldots, x_k \in \overline{B_1(0)}$ such that $\overline{B_1(0)} \subset \cup_{i=1}^k B_{\delta_{x_i}}(x_i)$. Putting $\bar{n} = \max\{n_{x_1}, n_{x_2}, \ldots, n_{x_k}\}$, we have

$$u_{\bar{n}}(x, T^c(u_0)+\varepsilon) > 2M \quad \text{for } x \in \overline{B_1(0)},$$

since the functions u_n are nondecreasing with respect to n in $\mathbb{R}^N \times (0,\infty)$. Now we use the continuous dependence of solutions of (1.3) with respect to the initial data, to obtain that there exists $\delta > 0$ such that if $\|\widetilde{u}_0 - u_0\|_\infty < \delta$, then

$$\widetilde{u}_{\bar{n}}(x, T^c(u_0)+\varepsilon) > M \quad \text{for } x \in \overline{B_1(0)},$$

where $\widetilde{u}_n$ is a solution of (1.3) with initial data $\widetilde{u}_0$. By comparison, we get that

$$\widetilde{u}(x, T^c(u_0)+\varepsilon) \geq \widetilde{u}_{\bar{n}}(x, T^c(u_0)+\varepsilon) > M \quad \text{for } x \in \overline{B_1(0)}.$$

Then, if $M^{p-m} \geq \gamma_1$,

$$\int\limits_{B_1(0)} \widetilde{u}(T^c(u_0)+\varepsilon)\phi_1 dx \geq M = \left\{(1+\delta)\gamma_1\right\}^{\frac{1}{p-m}}.$$

for $\delta = \left(\frac{M^{p-m}}{\gamma_1} - 1\right)$. Assume that $T^c(\widetilde{u}_0) - (T^c(u_0)+\varepsilon) = \nu > 0$. Then Lemma 2 implies that

$$\nu = T^c(\widetilde{u}_0) - (T^c(u_0)+\varepsilon) \leq \frac{\gamma_1^{-\frac{p-1}{p-m}}}{(p-1)\delta(1+\delta)^{\frac{m-1}{p-m}}}.$$

But the right hand side can be made smaller than ν just be taking M large enough, a contradiction. □

Remark The time of incomplete blow-up is not stable under perturbation of the data. It may even jump to infinity.

We can now give a local version of the pointwise Kaplan-type estimate.

Proposition 3 *Let u be a radially symmetric proper solution of (1.1) and $T^c > 0$ its complete blow-up time. If $u(r,t)$ is nonincreasing with respect to r for $r \leq R_*$, $t \in [\widehat{T}, T_*]$, for some $R_* > 0$, $0 \leq \widehat{T} < T_* < T^c$, then, there exist values $C_1, C_2 > 0$ depending only on N, m and p such that*

$$u(r,t) \leq C_1 r^{-\frac{2}{p-m}} \quad \text{for } 0 < r < \frac{1}{2}\min\{R_*, C_2(T^c - T_*)^{\frac{p-m}{2(p-1)}}\} \quad \text{and} \quad t \in [\widehat{T}, T_*]. \tag{3.5}$$

Proof Suppose that there exist $\underline{t} \leq T_*$ and $R > 0$ such that

$$\int\limits_{B_R(0)} u(\underline{t})\phi_R dx \geq (2\gamma_R)^{\frac{1}{p-m}}.$$

From Lemma 2 with $\delta = 1$, we have

$$T^c - T_* \leq T^c - \underline{t} \leq \frac{2\gamma_1^{-\frac{p-1}{p-m}}}{(p-1)2^{\frac{m-1}{p-m}}} R^{\frac{2(p-1)}{p-m}},$$

which is a contradiction if $R < C_2(T^c - T_*)^{\frac{p-m}{2(p-1)}}$, $C_2 = \gamma_1^{\frac{1}{2}}\left((p-1)2^{\frac{m-1}{p-m}-1}\right)^{\frac{p-m}{2(p-1)}}$.

Thus, for $R < C_2(T^c - T_*)^{\frac{p-m}{2(p-1)}}$, $t \leq T_*$, we have

$$\int_0^R u(r,t)\phi_1(R^{-1}r)r^{N-1}\,dr < \frac{(2\gamma_1)^{\frac{1}{p-m}}}{\omega_N} R^{N-\frac{2}{p-m}},$$

where ω_N is the surface area of $(N-1)$-dimensional unit sphere. Since $u(r,t)$ is nonincreasing with respect to $r \leq R_*$ for $t \in [\widehat{T}, T_*]$, and $\phi_1(r)$ is nonincreasing with respect to $r \leq 1$, it follows that

$$\int_0^R u(r,t)\phi_1(R^{-1}r)r^{N-1}\,dr \geq u(R/2,t)\phi_1(1/2)\int_0^{R/2} r^{N-1}\,dr,$$

for $0 < R < \min\{R_*, C_2(T^c - T_*)^{\frac{p-m}{2(p-1)}}\}$ and $t \in [\widehat{T}, T_*]$. Putting $r = R/2$, we get (3.5) with $C_1 = \frac{Nk^{N-\frac{1}{p-m}}\gamma_1^{\frac{1}{p-m}}}{\omega_N\phi_1(1/2)}$. □

The global version of the pointwise Kaplan-type estimate is just a corollary of the local one.

Corollary 1 *Let u be a radially symmetric, nonincreasing with respect to $r = |x|$, proper solution of* (1.1) *with $T^c = \infty$. There exists $C > 0$ depending only on N, m and p such that*

$$u(r,t) \leq Cr^{-\frac{2}{p-m}} \quad \textit{for } r > 0 \quad \textit{and} \quad t \geq 0. \tag{3.6}$$

Remark If (3.6) holds for a proper solution u and $p > p_{ST}$, then $u \in L^p_{\text{loc}}(Q_T)$ for all T. Hence, u is a global weak solution.

4 Immediate regularization after blow-up

For a radially symmetric function h with $h \not\equiv 0$, the number of sign changes of h, $z(h)$, is defined as the supremum over all j such that there exist $0 \leq r_1 < r_2 < \cdots < r_{j+1} < +\infty$ with

$$h(r_i) \cdot h(r_{i+1}) < 0 \quad \text{for } i = 1, 2, \ldots, j.$$

Let u_1, u_2 be bounded radially symmetric solutions of (1.1). Then $z(u_1(\cdot,t) - u_2(\cdot,t))$ is nonincreasing, see, for example, [1,5,12,27].

The purpose of the present section is to prove, for $p \in (p_S, p_{JL})$, that if u is a radially symmetric proper solution of (1.1) with $z(u_0 - \varphi_\infty) \leq 2$ that blows up incompletely at some time T, then $u(\cdot,t)$ is bounded for all $t > T$. The key point is to show that u has to lose at least two intersections with φ_∞ at a time of incomplete blow-up: if it does not lose any, then it will not blow up, while if it loses exactly one, then it will blow up completely. Hence, if $z(u_0 - \varphi_\infty) \leq 2$, we have that $u(x,T) \leq \varphi_\infty(x)$, from where a comparison argument with a self-similar solution will give the result.

The proofs depends strongly on the behaviour of bounded stationary radial solutions of $\Delta u^m + u^p = 0$. Let us review the facts that we need, which were first analyzed in [20] (the change of variable $w = u^m$ reduces our elliptic problem to the semilinear one considered in that paper).

For $a > 0$, let φ_a be the solution of

$$(\varphi^m)'' + \frac{N-1}{r}(\varphi^m)' + \varphi^p = 0, \quad r > 0, \quad \varphi(0) = a, \quad \varphi'(0) = 0. \tag{4.1}$$

These functions satisfy $\varphi_a(r) = a\varphi_1(a^{(p-m)/2}r)$ for $r \geq 0$.

For $p > p_S$ the functions φ_a are positive and

$$\frac{\varphi_a(r)}{\varphi_\infty(r)} \to 1, \quad \frac{\varphi_a'(r)}{\varphi_\infty'(r)} \to 1 \quad \text{as } a \to \infty$$

uniformly on $[\nu, \infty)$, $\nu > 0$. Moreover, for $p_S < p < p_{JL}$ all the solutions φ_a intersect each other and also intersect φ_∞ infinitely many times. For $p \geq p_{JL}$ the functions φ_a are strictly monotone increasing in a and $\varphi_a(r) \to \varphi_\infty(r)$ as $a \to \infty$ from below uniformly for $r \in [\nu, \infty)$, $\nu > 0$.

In all the cases there exists the envelope, $\Phi(r) = \sup_{a>0} \varphi_a(r)$ for $r \geq 0$, which is given by

$$\Phi(r) = Kr^{-2/(p-m)}, \quad K = K(p, m, N) > 0.$$

It is immediate that

$$K > c_\infty \quad \text{if } p \in (p_S, p_{JL}), \quad K = c_\infty \quad \text{if } p \geq p_{JL}.$$

We start by proving that at least one intersection has to be lost at any time where there is incomplete blow-up. Otherwise, the solution would not blow up.

Proposition 4 *Let $p \in (p_S, p_{JL})$. Let u be a radially symmetric proper solution of (1.1). If u blows up incompletely at $t = T$, then it loses at least one intersection with φ_∞ at $r = 0$ at $t = T$.*

Proof Since u is radial and blow-up is incomplete, u only blows up at the origin [14]. Suppose that no intersection between $u(t)$ and φ_∞ disappears at $r = 0$ and $t = T$, that is, there exist $R, \delta > 0$ such that

$$u(r, t) < \varphi_\infty(r) \quad \text{for } 0 < r < R \quad \text{and} \quad T - \delta \leq t \leq T.$$

Hence, there is a value a such that the solution φ_a of (4.1) satisfies $u(r, T-\delta) \leq \varphi_a(r)$ for $0 \leq r \leq R$ and $\varphi_a(R) > u(R, t)$ for $T - \delta \leq t \leq T$. Therefore, by comparison we get

$$u(r,t) \le \varphi_a(r) \quad \text{for } 0 \le r \le R \quad \text{and} \quad T-\delta \le t \le T,$$

and u would not blow up, a contradiction. □

To prove that at least two intersections with φ_∞ are lost at any incomplete blow-up time, we first have to show that if u loses exactly one intersection with φ_∞, then there are constants $K > c_\infty, C, r_*, \delta > 0$, such that

$$Kr^{-\frac{2}{p-m}} \le u(r,t) \le Cr^{-\frac{2}{p-m}}, \quad 0 < r \le r_*, \quad T \le t \le T+\delta. \tag{4.2}$$

The lower bound, which is only valid for $p \in (p_S, p_{JL})$, will follow from an intersection comparison argument which requires that $z(u_0 - \varphi_\infty) < \infty$.

Lemma 3 *Let $p \in (p_S, p_{JL})$. Let u be a radially symmetric proper solution of (1.1) with initial data satisfying $z(u_0 - \varphi_\infty) < \infty$ and such that u blows up incompletely at time T. If u loses exactly one intersection with φ_∞ at the origin at $t = T$, then there exist constants $\bar{r}, \bar{\delta} > 0$, $K > c_\infty$ such that*

$$u(r,t) \ge Kr^{-\frac{2}{p-m}}, \quad 0 < r \le \bar{r}, \quad T \le t \le T+\bar{\delta}. \tag{4.3}$$

Proof As u loses exactly one intersection at the origin with φ_∞ at $t = T$, there exists a small value $\delta > 0$ such that $u(x, T-\delta)$ has at least one intersection with φ_∞, the first of them moving to the origin separately from the rest. Since $\varphi_a \to \varphi_\infty$ locally uniformly as $a \to \infty$, we can take $0 < r_1 < r_2 \le R_1$, $a_1 > 0$ so that for all $a \ge a_1$ there exist $r_a \in [r_1, r_2]$ and $\bar{\delta} > 0$ such that:

(i) $u(T-\delta)$ and φ_a have exactly one intersection in $[0, r_a]$;
(ii) $u(r_a, t) > \varphi_\infty(r_a) > \varphi_a(r_a)$ for $T-\delta \le t \le T+\bar{\delta}$;
(iii) $u(0, T-\delta) < \varphi_a(0)$.

As u blows up at $t = T$, there exists a time $t_a \in (T-\delta, T)$ such that $a = \varphi_a(0) < u(0, t_a)$. Since the number of intersections in $[0, r_a]$ does not increase with time, we have

$$u(r,t) > \varphi_a(r), \quad 0 \le r \le r_a, \quad T \le t \le T+\bar{\delta},$$

from where we get the result just observing that for some radius $\bar{r} = \bar{r}(a_1)$ we have

$$\sup_{a \ge a_1} \varphi_a(r) = \sup_{a > 0} \varphi_a(r) = \Phi(r) \quad \text{for } 0 \le r \le \bar{r}.$$

□

The upper bound will now follow from the local version of the pointwise Kaplan-type estimate given in Proposition 3 together with the following size estimate at local (spatial) minima.

Lemma 4 *Let u be a radially symmetric proper solution of* (1.1)*, and let $T^c > 0$ be its complete blow-up time. If there exists a curve $\underline{r}(t)$ of local minimizers of $u(t)$ for all $t \in [0, \overline{T})$, $\overline{T} \leq T^c$, then,*

$$u(\underline{r}(t), t) \leq \kappa(\overline{T} - t)^{-\frac{1}{p-1}} \quad \text{for } t \in [0, \overline{T}), \tag{4.4}$$

where $\kappa = (p-1)^{-1/(p-1)}$.

Proof We assume without loss of generality that $u > 0$ at the curve of local minimizers. Let $m(t) = u(\underline{r}(t), t)$. Suppose that there is a time $\hat{t}$ such that $m(\hat{t}) > \kappa(\overline{T} - \hat{t})^{-\frac{1}{p-1}}$. For $\varepsilon > 0$ sufficiently small, let $\underline{m}_\varepsilon$ be the solution of the initial value problem

$$\underline{m}'(t) = \underline{m}^p(t) - \varepsilon, \quad \underline{m}(\hat{t}) = m(\hat{t}) - \varepsilon \geq 0.$$

At the first time $\tau > \hat{t}$ for which $m(\tau) = \underline{m}_\varepsilon(\tau)$ we have

$$0 \geq \partial_t u(\underline{r}(\tau), \tau) - \underline{m}_\varepsilon'(\tau) = \Delta u^m(\underline{r}(\tau), \tau) + u^p(\underline{r}(\tau), \tau) - \underline{m}_\varepsilon^p(\tau) + \varepsilon \geq \varepsilon,$$

a contradiction. Therefore, $m(t) \geq \underline{m}_\varepsilon(t)$ for all $t \geq \hat{t}$. Letting ε to 0, we have in particular that $m(t) \geq \underline{m}_0(t)$ for $t \geq \hat{t}$. But $\underline{m}_0(t) = \kappa(\hat{T} - t)^{-\frac{1}{p-1}}$ for some blow-up time $\hat{T}$. Since $\underline{m}_0(\hat{t}) = m(\hat{t})$, we conclude that $\hat{T} < \overline{T}$, a contradiction. □

Lemma 5 *Let $p \in (p_S, p_{JL})$. Let u be a radially symmetric proper solution of* (1.1) *with initial data satisfying $z(u_0 - \varphi_\infty) < \infty$ and such that u blows up incompletely at time T. If u loses exactly one intersection with φ_∞ at the origin at $t = T$, then there exist constants $\underline{r}, \underline{\delta}, C > 0$, such that*

$$u(r, t) \leq C r^{-\frac{2}{p-m}}, \quad 0 < r \leq \underline{r}, \quad T \leq t \leq T + \underline{\delta}.$$

Proof Notice that under our assumptions (4.3) holds. Assume that u has still some spatial local minimum at time T. If no spatial minima are lost in the time interval $[T, T + \bar{\delta}]$, set $\overline{T} = T + \bar{\delta}$. Otherwise, let $\overline{T} > T$ be the first time when one of these minima is lost. We know from Lemma 4 that at any local minimum point we have $u \leq \kappa(\overline{T} - t)^{-\frac{1}{p-1}}$ for $t \in [0, \overline{T})$. Combining this with (4.3) we get that there is no minimum for

$$r < \min\left\{\bar{r}, \left(\frac{K}{\kappa}\right)^{\frac{p-m}{2}} \left(\frac{\overline{T} - T}{2}\right)^{\frac{p-m}{2(p-1)}}\right\}, \quad t \in \left[T, \frac{T + \overline{T}}{2}\right].$$

The result now follows from Proposition 3. □

Now we are ready to prove that at least two intersections are lost at any time where there is incomplete blow-up. The proof uses the following scaling property: if u is a weak solution of the equation in (1.1), so is $\mathcal{T}_\lambda u$ given by

$$(\mathcal{T}_\lambda u)(x,t) = \lambda^\alpha u(\lambda^\beta x, \lambda t), \quad \alpha = \frac{1}{p-1}, \quad \beta = \frac{p-m}{2(p-1)}, \tag{4.5}$$

for all $\lambda > 0$.

Proposition 5 *Let $p \in (p_S, p_{JL})$. Let u be a radially symmetric proper solution of* (1.1) *with $z(u_0 - \varphi_\infty) < \infty$. If u blows up incompletely at $t = T$, then it loses at least two intersections with φ_∞ at $r = 0$ at $t = T$.*

Proof We have already seen that u has to lose at least one intersection with φ_∞ at the origin at $t = T$. If it loses exactly one intersection, then (4.2) holds for some $C, r_*, \delta > 0$, $K > c_\infty$. Hence, if $u_T(r,t) = u(r, t+T)$, the rescaled function $\mathcal{T}_\lambda u_T$, which solves (1.1) for some time interval, satisfies

$$Kr^{-\frac{2}{p-m}} \le (\mathcal{T}_\lambda u_T)(r,t) \le Cr^{-\frac{2}{p-m}} \quad \text{for } 0 < r < \lambda^{-1} r_*, \ 0 \le t \le \delta \lambda^{-\frac{2(p-1)}{p-m}}.$$

Passing to the limit in the weak formulation we get that $\lim_{\lambda \to 0} \mathcal{T}_\lambda u_T$ is a weak solution of (1.1) defined for all $t > 0$, which satisfies the estimate

$$Kr^{-\frac{2}{p-m}} \le \lim_{\lambda \to 0} (\mathcal{T}_\lambda u_T)(r,t) \le Cr^{-\frac{2}{p-m}} \quad \text{for all } r > 0, \ t \ge 0.$$

Such a solution does not exist, a contradiction. Indeed, if $p > p_{ST}$, the only solution to (1.1) in the class $\{u \ge \varphi_\infty\}$ is φ_∞ [14]. □

Proposition 6 *Let $p \in (p_S, p_{JL})$. Let u be a proper solution of* (1.1) *with $z(u_0 - \varphi_\infty) < \infty$. If u blows up incompletely at $t = T$ losing exactly two intersections with φ_∞ at $r = 0$ at that time, then there is a value $\overline{T} > T$ such that $u(t)$ is bounded for $t \in (T, \overline{T})$. If, moreover, $z(u_0 - \varphi_\infty) = 2$, then we may take $\overline{T} = \infty$.*

Proof We recall once more that, since u is radial and blow-up is incomplete, u only blows up at the origin [14]. Since u loses exactly two intersections with φ_∞ at the origin a $t = T$, there exist values $R_1, \delta_1 > 0$ such that

$$u(r,t) < \varphi_\infty(r) \quad \text{for } 0 < r < R_1 \quad \text{and} \quad T \le t \le T + \delta_1,$$

Let $\overline{u}$ be the proper solution for $t > T$ with $\overline{u}(r,T) = c_\infty r^{-2/(p-m)}$. Such solution is forward self-similar, and, in the range of values of p that we are considering, $\overline{u}(t) \in L^\infty(\mathbb{R}^N)$ for all $t > T$, see [14]. Then, there exists values $R_2, \delta_2 > 0$, $R_2 \le R_1$, $\delta_2 \le \delta_1$, such that $u(R_2, t) \le \overline{u}(R_2, t)$ for $t \in [T, T + \delta_2]$. Hence, by the maximum principle,

$$u(r,t) \le \overline{u}(r,t) \quad \text{for } 0 \le r \le R_2 \quad \text{and} \quad t \in [T, T+\delta_2],$$

which means that $u(t)$ is bounded for $t \in (T, T + \delta_2]$.

If $z(u_0 - \varphi_\infty) = 2$, then $u(r,T) < \varphi_\infty(r)$ for $r > 0$. Hence, comparison with $\overline{u}$ gives that $u(t)$ is bounded for all $t > T$. □

Remark The proper forward self-similar solution $\overline{u}$ that we have used in the proof is different from φ_∞. This shows that there is not uniqueness of L^1_{loc}-weak solutions in the range $p_S < p < p_{JL}$. On the contrary, if $p \geq p_{JL}$, then $\overline{u} \equiv \varphi_\infty$. These facts were proved in [14].

5 Peaking and multiple blow-up solutions

We devote this section to the proofs of Theorems 2, 3 and 4, dealing with the construction of peaking solutions which have compact support in space for all time, and of solutions with multiple blow-up.

Proof of Theorem 2 We arrange the proof in three steps. First we construct a peaking solution with compactly supported initial data. Next we show that this solution remains compactly supported in space for all times. Finally, using a scaling argument, we show that we can choose the blow-up time.

(i) Fix a radially symmetric nonnegative function $h \not\equiv 0$ which is nonincreasing with respect to r, has compact support, and is such that μh intersects φ_∞ at most twice for all $\mu > 0$. For $\mu > 0$, let u_μ be the proper solution of (1.1) with initial data μh. Since $p > p_F$ in the range of exponents we are considering, it is immediate that u_μ exists globally in time as a bounded weak solution for sufficiently small μ and that u_μ blows up in finite time for sufficiently large μ. Putting

$$\mu^* = \sup\{\mu > 0 : u_\mu \text{is bounded for all } t > 0\},$$

we have $0 < \mu^* < \infty$. Corollary 1 implies that the functions u_μ, $\mu < \mu_*$, satisfy (3.6) for some constant $C > 0$ depending only on N, m and p. Hence, passing to the limit as $\mu \nearrow \mu^*$, we get that $u^* \equiv u_{\mu^*}$ satisfies

$$u^*(r,t) \leq Cr^{-\frac{2}{p-m}} \quad \text{for } r > 0 \quad \text{and } t > 0. \tag{5.1}$$

Therefore u^* is a weak solution for all times, $T^w = \infty$.

We next assume that u^* is bounded for each $t > 0$. If $p_S < p < p_L$, then for all $c < c_\infty$ and every $T > 0$ there exists a radially symmetric backward self-similar solution

$$\widetilde{u}(r,t) = (T-t)^{-\frac{1}{p-1}} f(\xi), \quad \xi = r(T-t)^{-\frac{p-m}{2(p-1)}},$$

of (1.1) blowing up at $t = T$ whose blow-up profile is precisely $cr^{-2/(p-m)}$, see [14]. The profile $f = f(\xi)$, which does not depend on T, is nonincreasing with ξ. Hence, we can choose T large enough such that the initial data $\widetilde{u}_0$ is so flat that it has exactly one intersection with μh for $\mu^* \leq \mu \leq \mu^* + \delta_1$ for some $\delta_1 > 0$. Then, using the continuous dependence on μ of the solutions in bounded time intervals, and the assumed boundedness of u^*, we can conclude that there exist $0 < t_1 < T$ and $0 < \delta_2 < \delta_1$ such that $u_\mu(0,t_1) < \widetilde{u}(0,t_1)$ and $T_\mu > 2T$ for $\mu^* \leq \mu \leq \mu^* + \delta_2$, where T_μ is the first blow-up time of u_μ. Since the number of intersections cannot increase,

we see that $u_\mu(r,t_1) < \widetilde{u}(r,t_1)$ for $r \geq 0$ and $\mu^* \leq \mu \leq \mu^* + \delta_2$. We extend $\widetilde{u}$ as a forward self-similar solution for $t > T$. Then $T_\mu = \infty$ for $\mu^* \leq \mu \leq \mu^* + \delta_2$, which contradicts the definition of μ^*. Consequently u^* blows up incompletely at some finite time $t = T^*$. Now Proposition 6 guarantees that u^* is bounded for $t > T^*$.
(ii) We get from estimate (5.1) that u^* is a subsolution of

$$u_t = \Delta u^m + Mu \tag{5.2}$$

in the complement of the ball of radius $R > 0$ if we take $M = C^{p-1}R^{-2(p-1)/(p-m)}$. If we perform the change of variables

$$w(r,\tau) = e^{-Mt}u^*(r,t), \quad \tau = \frac{e^{M(m-1)t}}{M(m-1)},$$

we get that w is a subsolution of the following problem,

$$\begin{cases} w_\tau = \Delta w^m, & R > 0,\ \tau > \frac{1}{M(m-1)}, \\ w(R,\tau) = \tau^{-1/(m-1)}(M(m-1))^{-1/(m-1)}CR^{-\frac{2}{p-m}}, & \tau > \frac{1}{M(m-1)}, \\ w(r, \frac{1}{M(m-1)}) = u^*(r,0), & 0 < r < R. \end{cases} \tag{5.3}$$

As a supersolution for this problem we will use a solution $\mathcal{U}_K$ of

$$\begin{cases} \mathcal{U}_\tau = \Delta\mathcal{U}^m + K\delta(x) & \text{in } \mathcal{D}'(\mathbb{R}^N \times (0,\infty)), \\ \mathcal{U}(x,0) = 0 & \text{for } x \in \mathbb{R}^N,\ x \neq 0. \end{cases}$$

Such solutions, which were studied in [38], exist for every $K > 0$, and are known to have a self-similar structure,

$$\mathcal{U}_K(x,\tau) = \tau^{-\frac{n-2}{n(m-1)+2}}\Phi_K(|\xi|), \quad \xi = x\tau^{-\frac{m}{n(m-1)+2}},$$

where Φ_K is a compactly supported function which has a singular behaviour at the origin, $\Phi_K(x) \sim c|x|^{(2-N)/m}$, where c is related to K through

$$K = \begin{cases} N(N-2)\omega_n c^m & \text{if } N \geq 3, \\ 2\pi c^m & \text{if } N = 2. \end{cases}$$

Moreover, $\mathcal{U}_K(x,\tau) \to cR^{(2-N)/m}$ as $\tau \to \infty$ if $|x| = R$. Thus, there are big enough values $K > 0$, $L > 0$, such that $\hat{w}(x,\tau) = \mathcal{U}_K(x,t+L)$ is a supersolution to (5.3). Since $\hat{w}$ has compact support for all times, so does w, and hence u^*.

Notice, for future reference, that the above comparison argument provides a bound for the radius of the support at time t, $R^*(t)$, of the peaking solution u^*: there are constants $\xi_0, L > 0$ such that

$$R^*(t) \leq \xi_0(t+L)^{m/(n(m-1)+2)}. \tag{5.4}$$

(iii) If u^* blows up at $t = T^*$, then $\mathcal{T}_\lambda u^*$ given by (4.5) blows up at $T_\lambda = T^*/\lambda$. Choosing $\lambda = T^*/T$, we get that $\mathcal{T}_\lambda u^*$ is a peaking solution that blows up at time T which is compactly supported in space for all times. □

Proof of Theorem 3 Let u^*, R^*, T^* have the same meaning as in the proof of Theorem 2. For each i, $1 \le i \le k$ ($i \ge 1$ if $k = \infty$), we consider the scaled function $u_i^* = \mathcal{T}_{T^*/T_i} u^*$. Then u_i^* is a peaking solution with compact support in space for all times that blows up at time T_i. The radius of its support at time t is $R_i(t) = \left(\frac{T_i}{T^*}\right)^\beta R^*\left(\frac{T^*}{T_i}t\right)$. Let Θ be a bound for the sequence $\{T_i\}_{i=0}^k$. Then $|R_i(\Theta)| \le \mathcal{R} = \left(\frac{\Theta}{T^*}\right)^\beta R^*\left(\frac{T^*}{T_1}\Theta\right)$, $1 \le i \le k$ ($i \ge 1$ if $k = \infty$). Take a sequence of points $\{x_i\}_{i=0}^k$, such that $|x_i - x_j| > 2\mathcal{R}$, $i \ne j$. Then,

$$u(x,t) = \sum_{j=1}^{k} u_j^*(x - x_j, t)$$

is the required solution. □

Proof of Theorem 4 Let u^*, R^*, T^* have the same meaning as in the proof of Theorem 2. The scaled function $\mathcal{T}_{T^*/T} u^*$ is a peaking solution that blows up at time T which has compact support of radius $R_T(t) = \left(\frac{T}{T^*}\right)^\beta R^*\left(\frac{T^*}{T}t\right)$ at time t. Using the upper bound (5.4), we get that $R_T(1) \to 0$ as $T \to 0$ if $p > p_{ST}$ (as is our case). Hence, there is a certain time $\Theta = \Theta(\delta)$ such that the radius of the support or the function $\mathcal{T}_{T^*/T} u^*$ at time Θ is smaller than $\delta/2$ if $T < \Theta$. Therefore, for any sequence of times $\{T_i\}_{i=0}^k$ satisfying $T_0 = 0$, $T_{i-1} < T_i$, $T_i < \Theta$, for $1 \le i \le k$ ($i \ge 1$ if $k = \infty$), the supports of the functions $u_i^* = \mathcal{T}_{T^*/T_i} u^*$ do no intersect for $t \le \Theta$. Again,

$$u(x,t) = \sum_{i=1}^{k} u_i^*(x - x_i, t)$$

is the required solution. □

6 Open problems

Fast diffusion The main idea of this paper is to take profit of the finite speed of propagation of solutions of the equation. This does not apply to the fast diffusion case $m < 1$. Hence, a new idea is needed if we want to construct multiple blow-up solutions in this range of values of m.

Big exponents, $p \ge p_{JL}$ A peaking solution which is positive everywhere exists for all $p \in (p_S, p_L)$ [14]. We have constructed a peaking solution with compact support for $p \in (p_S, p_{JL})$. Does a compactly supported peaking solution exist for $p \in [p_{JL}, p_L)$? The existence of such solutions would allow to construct multiple blow-up solutions in this extended range. Is there any peaking solution (with or without compact support) for $p \geq p_L$?

Blow-up type The peaking solutions constructed in [14] for $p \in (p_S, p_L)$ have type I blow-up. For the heat equation and $p \in (p_S, p_{JL})$, blow-up is always of type I for radial solutions which have a finite number of intersections with φ_∞ [28]. Such a result is not known for the porous medium case, $m > 1$. If it were true, it would imply that the solution we have constructed also has type I blow-up.

On the other hand, the multiple blow-up solutions constructed in [33] in the case of the heat equation for $p \geq p_{JL}$ are of type II. No blow-up solution of any kind with type II blow-up has been constructed yet for $m \neq 1$.

Exponential Instead of u^p, one may consider other reaction non-linearities, for example e^u. Peaking solutions have been constructed for this kind of reaction non-linearity when $m = 1$, both for the problem posed in the whole space, with $N = 3$ [24], and for the problem posed in a ball with homogenous Dirichlet boundary data, $3 \leq N \leq 9$ [9]. Multiple blow-up solutions have not been constructed yet for this kind of reaction nonlinearity, not even for $m = 1$. If $m \neq 1$, no results are available.

Other diffusion operators Very little is known about solutions with incomplete blow-up if we substitute the diffusion operator Δu^m by other diffusion operators like, for example, the p-laplacian, $\Delta_p u = \nabla \cdot (|\nabla u|^{p-2} \nabla u)$, or the fractional laplacian, $(-\Delta u)^{\alpha/2}$, $0 < \alpha < 2$.

Kaplan-type estimate without monotonicity An estimate like (3.6) has been obtained for radial global, proper solutions in the semilinear case, without any monotonicity assumption [28]. On the contrary, in the proof that we give for the case $m > 1$ we impose that the solutions are non-increasing in $r = |x|$. Is it possible to remove this restriction when $m \neq 1$?

Behaviour as $t \to \infty$ Both the peaking solution with compact support that we have constructed and the positive one given in [14] go to zero as $t \to \infty$. Peaking solutions with other behaviours as $t \to \infty$ are available in the semilinear case [31]. Are such different behaviours also possible when $m \neq 1$?

Regarding the large time behaviour of multiple blow-up solutions, let us mention that the ones that we have constructed are expected to blow up completely in finite time. However, a proof is still needed. Another interesting question is whether there are multiple blow-up solutions with $T^c = \infty$, and, if this is the case, whether there are such solutions that go to zero as $t \to \infty$.

Threshold solutions Let $h \geq 0$, $h \not\equiv 0$, with some decay at infinity. For all $\mu > 0$, let u_μ be the proper solution of (1.1) with initial data μh. If $p > p_F$, there is a critical value μ^* such that u_μ blows up in finite time when $\mu > \mu^*$ and remains bounded for all times when $\mu < \mu^*$. If $p \in (p_S, p_{JL})$ and h is as in the proof of Theorem 2, then we have seen that u_{μ^*} is a peaking solution.

Moreover, if $\mu > \mu^*$, then u_μ blows up completely at its first blow-up time. Indeed, if $\mu > \mu^*$, then there is $\lambda > 1$ such that $\mathcal{T}_\lambda u_{\mu^*}(r, 0) \leq \mu h(r)$ for all $r \geq 0$. By the comparison theorem, $\lambda^{-1} T_{\mu^*} \geq T_\mu$ and hence $T_\mu < T_{\mu^*}$. If blow-up were not complete, then by our immediate regularization result, Proposition 6, $u_\mu(\cdot, t)$ would be finite for all $t > T_\mu$. But then u_{μ^*} would not blow up, a contradiction.

A similar dependence on μ of the blow-up behaviour of u_μ is expected to be true for more general initial data and all $p > p_F$. However, a proof is only available for $m = 1$, $p \in (p_S, p_L)$ [30].

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