

Operator log-convex functions and operator means

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Abstract We study operator log-convex functions on $(0, \infty)$, and prove that a continuous nonnegative function on $(0, \infty)$ is operator log-convex if and only if it is operator monotone decreasing. Several equivalent conditions related to operator means are given for such functions. Operator log-concave functions are also discussed.

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Introduction

In 1930s the theory of matrix/operator monotone functions was initiated by Löwner [14], soon followed by the theory of matrix/operator convex functions due to Kraus [12]. Nearly half a century later, a modern treatment of operator monotone and convex functions was established by a seminal paper [11] of Hansen and Pedersen. Comprehensive expositions on the subject are found in [1, 5, 8] for example.

Our first motivation to the present paper is the question to determine $\alpha \in \mathbb{R}$ for which the functional $\log \omega(A^\alpha)$ is convex in positive operators A for any positive linear functional ω . In the course of settling the question, we arrived at the idea to characterize continuous nonnegative functions f on $(0, \infty)$ for which the operator inequality $f(A \nabla B) \leq f(A) \# f(B)$ holds for positive operators A and B , where $A \nabla B := (A + B)/2$ is the arithmetic mean and $A \# B$ is the geometric mean [1, 15].

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This inequality was indeed considered by Aujla, Rawla and Vasudeva [4] as a matrix/operator version of log-convex functions. In fact, a function f satisfying the above inequality may be said to be operator log-convex because the numerical inequality $f((a+b)/2) \leq \sqrt{f(a)f(b)}$ for $a, b > 0$ means the convexity of $\log f$ and the geometric mean $\#$ is the most standard operator version of geometric mean. Moreover, it is worth noting that some matrix eigenvalue inequalities involving log-convex functions were shown in [3].

In this paper we show that a continuous nonnegative function f on $(0, \infty)$ is operator log-convex if and only if it is operator monotone decreasing, and furthermore present several equivalent conditions related to operator means for the operator log-convexity. The operator log-concavity counterpart is also considered, and we show that f is operator log-concave, i.e., f satisfies $f(A \nabla B) \geq f(A) \# f(B)$ for positive operators A, B if and only if it is operator monotone (or equivalently, operator concave).

The paper is organized as follows. In Sect. 1, after preliminaries on basic notions, the convexity of $\log \omega(f(A))$ in positive operators A is proved when f is operator monotone decreasing on $(0, \infty)$. Sects. 2 and 3 are the main parts of the paper, where a number of equivalent conditions are provided for a continuous nonnegative function on $(0, \infty)$ to be operator log-convex (equivalently, operator monotone decreasing), or to be operator log-concave (equivalently, operator monotone). In Sect. 4 another characterization in terms of operator means is provided for a function on $(0, \infty)$ to be operator monotone.

1 Operator log-convex functions: motivation

In this paper we consider operator monotone and convex functions defined on the half real line $(0, \infty)$. Let \mathcal{H} be an infinite-dimensional (separable) Hilbert space. Let $B(\mathcal{H})^+$ denote the set of all positive operators in $B(\mathcal{H})$, and $B(\mathcal{H})^{++}$ the set of all invertible $A \in B(\mathcal{H})^+$. A continuous real function f on $(0, \infty)$ is said to be *operator monotone* (more precisely, *operator monotone increasing*) if $A \geq B$ implies $f(A) \geq f(B)$ for $A, B \in B(\mathcal{H})^{++}$, and *operator monotone decreasing* if $-f$ is operator monotone or $A \geq B$ implies $f(A) \leq f(B)$, where $f(A)$ and $f(B)$ are defined via functional calculus as usual. Also, f is said to be *operator convex* if $f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda)f(B)$ for all $A, B \in B(\mathcal{H})^{++}$ and $\lambda \in (0, 1)$, and *operator concave* if $-f$ is operator convex. In fact, as easily seen from continuity, the mid-point operator convexity (when $\lambda = 1/2$) is enough for f to be operator convex.

As well known (see [1, Examples III.2], [5, Chapter V] for example), a power function x^α on $(0, \infty)$ is operator monotone (equivalently, operator concave) if and only if $\alpha \in [0, 1]$, operator monotone decreasing if and only if $\alpha \in [-1, 0]$, and operator convex if and only if $\alpha \in [-1, 0] \cup [1, 2]$.

An axiomatic theory on operator means for operators in $B(\mathcal{H})^+$ was developed by Kubo and Ando [13] related to operator monotone functions. Corresponding to each nonnegative operator monotone function h on $[0, \infty)$ with $h(1) = 1$ the *operator mean* $\sigma = \sigma_h$ is introduced by

$$A \sigma B := A^{1/2}h(A^{-1/2}BA^{-1/2})A^{1/2}, \quad A, B \in B(\mathcal{H})^{++},$$

which is further extended to $A, B \in B(\mathcal{H})^+$ as

$$A \sigma B := \lim_{\varepsilon \searrow 0} (A + \varepsilon I) \sigma (B + \varepsilon I) \tag{1.1}$$

in the strong operator topology, where I is the identity operator on \mathcal{H} . The function h is conversely determined by σ as $h(x) = 1 \sigma x$ (more precisely, $h(x)I = I \sigma xI$) for $x > 0$. The following property of operator means is useful:

$$X^*(A \sigma B)X = (X^*AX) \sigma (X^*BX)$$

for all invertible $X \in B(\mathcal{H})$ [13].

The most familiar operator means are

$$\begin{aligned} A \nabla B &:= \frac{A + B}{2} \quad (\text{arithmetic mean}), \\ A \# B &:= A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2} \quad (\text{geometric mean}), \\ A ! B &:= \left(\frac{A^{-1} + B^{-1}}{2} \right)^{-1} = 2(A : B) \quad (\text{harmonic mean}) \end{aligned}$$

for $A, B \in B(\mathcal{H})^{++}$ (also for $A, B \in B(\mathcal{H})^+$ via (1.1)), where $A : B$ is the so-called *parallel sum*, that is, $A : B := (A^{-1} + B^{-1})^{-1}$. The geometric mean was first introduced by Pusz and Woronowicz [15] in a more general setting for positive forms. Basic properties of the geometric and the harmonic means for operators are found in [1]. Note that the operator version of the *arithmetic–geometric–harmonic mean inequality* holds:

$$A \nabla B \geq A \# B \geq A ! B.$$

The original motivation to discuss an operator version of log-convex functions came from the question whether the functional

$$A \in B(\mathcal{H})^{++} \mapsto \log \omega(A^\alpha)$$

is convex for any $\alpha \in [-1, 0]$ and for any positive linear functional ω on $B(\mathcal{H})$. This is settled by the following:

Proposition 1.1 *Let f be a nonnegative operator monotone decreasing function on $(0, \infty)$, and ω be a positive linear functional on $B(\mathcal{H})$. Then the functional*

$$A \in B(\mathcal{H})^{++} \mapsto \log \omega(f(A)) \in [-\infty, \infty)$$

is convex.

Proof The first part of the proof below is same as the proof of [4, Proposition 2.1] while we include it for the convenience of the reader. If $f(x) = 0$ for some $x \in (0, \infty)$, then f is identically zero due to analyticity of f (see [5, V.4.7]) and the conclusion follows trivially. So we assume that $f(x) > 0$ for all $x \in (0, \infty)$. Since $1/f$ is positive and operator monotone on $(0, \infty)$, it follows (see [5, V.2.5], [11, Theorem 2.5]) that $1/f$ is operator concave on $(0, \infty)$. Hence

$$f(A \nabla B)^{-1} \geq f(A)^{-1} \nabla f(B)^{-1}$$

so that

$$f(A \nabla B) \leq f(A) ! f(B), \quad A, B \in B(\mathcal{H})^{++}. \tag{1.2}$$

For each $\lambda > 0$, since

$$f(A) ! f(B) \leq f(A) \# f(B) = (\lambda f(A)) \# (\lambda^{-1} f(B)) \leq \frac{\lambda f(A) + \lambda^{-1} f(B)}{2},$$

we have

$$\omega(f(A \nabla B)) \leq \frac{\lambda \omega(f(A)) + \lambda^{-1} \omega(f(B))}{2}, \quad A, B \in B(\mathcal{H})^{++}.$$

Minimizing the above right-hand side over $\lambda > 0$ yields that

$$\omega(f(A \nabla B)) \leq \sqrt{\omega(f(A))\omega(f(B))},$$

and hence

$$\log \omega(f(A \nabla B)) \leq \frac{\log \omega(f(A)) + \log \omega(f(B))}{2}.$$

Since $A \in B(\mathcal{H})^{++} \mapsto \log \omega(f(A)) \in [-\infty, \infty)$ is continuous in the operator norm, the convexity follows from the mid-point convexity. □

In the following we state, for convenience, the concave counterpart of Proposition 1.1. This is immediately seen from the operator concavity of f and the concavity of $\log x$.

Proposition 1.2 *Let f be a nonnegative operator monotone function on $(0, \infty)$, and ω be a positive linear functional on $B(\mathcal{H})$. Then the functional $A \in B(\mathcal{H})^{++} \mapsto \log \omega(f(A))$ is concave.*

Let f be a continuous nonnegative function on $(0, \infty)$. An essential point in the proof of Proposition 1.1 is the following operator inequality considered in [4]:

$$f(A \nabla B) \leq f(A) \# f(B), \quad A, B \in B(\mathcal{H})^{++}. \tag{1.3}$$

When f satisfies (1.3), we say that f is *operator log-convex*. The term seems natural because the numerical inequality $f((a + b)/2) \leq \sqrt{f(a)f(b)}$, $a, b > 0$, means the convexity of $\log f$. On the other hand, it is said that f is *operator log-concave* if it satisfies

$$f(A \nabla B) \geq f(A) \# f(B), \quad A, B \in B(\mathcal{H})^{++}.$$

Indeed, another operator inequality

$$\log f(A \nabla B) \leq \{\log f(A)\} \nabla \{\log f(B)\}, \quad A, B \in B(\mathcal{H})^{++}, \tag{1.4}$$

was also considered in [4] for a continuous function $f > 0$ on $(0, \infty)$, where the term “log matrix convex functions” was referred to (1.4) while “multiplicatively matrix convex functions” to (1.3). But we prefer to use operator log-convexity for (1.3) and we say simply that $\log f$ is operator convex if f satisfies (1.4) (see Remark 3.4 in Sect. 3 in this connection).

In the rest of the paper we will prove:

- (1°) f is operator monotone decreasing if and only if f is operator log-convex,
- (2°) f is operator monotone (increasing) if and only if f is operator log-concave.

We will indeed prove results much sharper than (1°) and (2°), and moreover present several conditions which are equivalent to those in (1°) and (2°), respectively.

2 Operator monotony, operator log-convexity, and operator means

When f is a continuous nonnegative function on $(0, \infty)$, the operator convexity of f is expressed as

$$f(A \nabla B) \leq f(A) \nabla f(B), \quad A, B \in B(\mathcal{H})^{++}. \tag{2.1}$$

Recall that an operator mean σ is said to be *symmetric* if $A \sigma B = B \sigma A$ for all $A, B \in B(\mathcal{H})^{++}$. Note that the arithmetic mean ∇ and the harmonic mean $!$ are the maximum and the minimum symmetric means, respectively:

$$A \nabla B \geq A \sigma B \geq A ! B, \quad A, B \in B(\mathcal{H})^{++}, \tag{2.2}$$

for every symmetric operator mean σ , or equivalently,

$$\frac{x + 1}{2} \geq h(x) \geq \frac{2x}{x + 1}, \quad x \geq 0, \tag{2.3}$$

for every nonnegative operator monotone function h on $[0, \infty)$ satisfying $h(1) = 1$ and the symmetry condition $h(x) = xh(x^{-1})$ for $x > 0$ [13].

The next theorem characterizes the class of functions f that satisfy the variant of (2.1) where ∇ in the right-hand side is replaced with a different symmetric operator mean. The statement (1°) in Sect. 1 is included in the theorem.

Theorem 2.1 *Let f be a continuous nonnegative function on $(0, \infty)$. Then the following conditions are equivalent:*

- (a1) f is operator monotone decreasing;
- (a2) $f(A \nabla B) \leq f(A) \sigma f(B)$ for all $A, B \in B(\mathcal{H})^{++}$ and for all symmetric operator means σ ;
- (a3) f is operator log-convex, i.e., $f(A \nabla B) \leq f(A) \# f(B)$ for all $A, B \in B(\mathcal{H})^{++}$;
- (a4) $f(A \nabla B) \leq f(A) \sigma f(B)$ for all $A, B \in B(\mathcal{H})^{++}$ and for some symmetric operator mean $\sigma \neq \nabla$.

The following lemma will play a crucial role in proving the theorem.

Lemma 2.2 *Let φ be a continuous non-decreasing function on $[0, \infty)$ such that $\varphi(0) = 0$ and $\varphi(1) = 1$. If a symmetric operator mean σ satisfies*

$$\varphi(A \nabla B) \leq \varphi(A) \sigma \varphi(B), \quad A, B \in B(\mathcal{H})^{++},$$

then $\sigma = \nabla$. (Indeed, it is enough to assume that the above inequality holds for all positive definite 2×2 matrices A, B .)

Proof Let P and Q be two orthogonal projections in $B(\mathcal{H})^+$ such that $P \wedge Q = 0$. By the assumption of the lemma applied to $A_\varepsilon := P + \varepsilon I$ and $B_\varepsilon := Q + \varepsilon I$ for $\varepsilon > 0$, we have

$$\varphi(A_\varepsilon \nabla B_\varepsilon) \leq \varphi(A_\varepsilon) \sigma \varphi(B_\varepsilon).$$

Since $A_\varepsilon \nabla B_\varepsilon = P \nabla Q + \varepsilon I \rightarrow P \nabla Q$ in the operator norm, $\varphi(A_\varepsilon \nabla B_\varepsilon) \rightarrow \varphi(P \nabla Q)$ as $\varepsilon \searrow 0$ in the operator norm. Furthermore, since $\varphi(A_\varepsilon) \searrow \varphi(P) = P$, $\varphi(B_\varepsilon) \searrow \varphi(Q) = Q$ as $\varepsilon \searrow 0$ and the operator mean is continuous in the strong operator topology under the downward convergence, we have

$$\varphi(P \nabla Q) \leq P \sigma Q. \tag{2.4}$$

It follows from [13, Theorem 3.7] that $P \sigma Q = h(0)(P + Q)$, where h is a symmetric operator monotone function corresponding to σ . Now choose two orthogonal projections

$$P := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad Q := \begin{bmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{bmatrix} \quad \text{for } 0 < \theta < \pi/2$$

in the realization of the 2×2 matrix algebra in $B(\mathcal{H})$. Then $P \wedge Q = 0$, and the diagonalization of $P \nabla Q$ is

$$P \nabla Q = \begin{bmatrix} \cos \frac{\theta}{2} & \sin \frac{\theta}{2} \\ \sin \frac{\theta}{2} & -\cos \frac{\theta}{2} \end{bmatrix} \begin{bmatrix} \frac{1+\cos \theta}{2} & 0 \\ 0 & \frac{1-\cos \theta}{2} \end{bmatrix} \begin{bmatrix} \cos \frac{\theta}{2} & \sin \frac{\theta}{2} \\ \sin \frac{\theta}{2} & -\cos \frac{\theta}{2} \end{bmatrix}.$$

Therefore,

$$\varphi(P \nabla Q) = \begin{bmatrix} \cos \frac{\theta}{2} & \sin \frac{\theta}{2} \\ \sin \frac{\theta}{2} & -\cos \frac{\theta}{2} \end{bmatrix} \begin{bmatrix} \varphi\left(\frac{1+\cos\theta}{2}\right) & 0 \\ 0 & \varphi\left(\frac{1-\cos\theta}{2}\right) \end{bmatrix} \begin{bmatrix} \cos \frac{\theta}{2} & \sin \frac{\theta}{2} \\ \sin \frac{\theta}{2} & -\cos \frac{\theta}{2} \end{bmatrix}.$$

Comparing the (1, 1)-entries of both sides of (2.4) we have

$$\cos^2 \frac{\theta}{2} \varphi\left(\frac{1+\cos\theta}{2}\right) + \sin^2 \frac{\theta}{2} \varphi\left(\frac{1-\cos\theta}{2}\right) \leq h(0)(1 + \cos^2 \theta)$$

so that

$$h(0) \geq \frac{\cos^2 \frac{\theta}{2} \varphi\left(\frac{1+\cos\theta}{2}\right) + \sin^2 \frac{\theta}{2} \varphi\left(\frac{1-\cos\theta}{2}\right)}{1 + \cos^2 \theta}.$$

Letting $\theta \rightarrow 0$ gives $h(0) \geq 1/2$. Since $h(1) = 1$ and h is concave, it follows that $h(x) \geq (x + 1)/2$ and so by (2.3) $h(x) = (x + 1)/2$ on $[0, 1]$, implying $\sigma = \nabla$ by analyticity of h . The last statement in the parentheses is obvious from the above proof. □

Proof of Theorem 2.1 As shown in the proof of Proposition 1.1, (a1) implies the inequality (1.2). Hence (a1) \Rightarrow (a2) holds since the harmonic mean ! is the smallest among the symmetric operator means. It is clear that (a2) \Rightarrow (a3) \Rightarrow (a4). Now let us prove that (a4) \Rightarrow (a1).

Assume (a4). Since

$$f(A \nabla B) \leq f(A) \sigma f(B) \leq f(A) \nabla f(B), \quad A, B \in B(\mathcal{H})^{++},$$

f is operator convex (hence analytic) on $(0, \infty)$. Hence we may assume that $f(x) > 0$ for all sufficiently large $x > 0$; otherwise f is identically zero. Since $f(\varepsilon + x)$ obviously satisfies (a4) for any $\varepsilon > 0$, we may further assume that the finite limits $f(+0) := \lim_{x \searrow 0} f(x)$ and $f'(+0) := \lim_{x \searrow 0} f'(x)$ exist. Then f admits an integral representation

$$f(x) = \alpha + \beta x + \gamma x^2 + \int_{(0, \infty)} \frac{(\lambda + 1)x^2}{\lambda + x} d\mu(\lambda), \tag{2.5}$$

where $\alpha, \beta \in \mathbb{R}$ (indeed, $\alpha = f(+0), \beta = f'(+0)$), $\gamma \geq 0$, and μ is a finite positive measure on $(0, \infty)$ (see [5, V.5.5]). In the following we divide the proof into three steps; each step consists of a proof by contradiction.

Step 1 For $c > 0$ large enough so that $f(c) > 0$, we write

$$\frac{f(cx)}{f(c)} = \frac{\frac{\alpha}{c^2} + \frac{\beta}{c}x + \gamma x^2 + \int_{(0, \infty)} \frac{(\lambda+1)x^2}{\lambda+cx} d\mu(\lambda)}{\frac{\alpha}{c^2} + \frac{\beta}{c} + \gamma + \int_{(0, \infty)} \frac{\lambda+1}{\lambda+c} d\mu(\lambda)},$$

and notice that for any fixed $x > 0$,

$$\lim_{c \rightarrow \infty} \int_{(0, \infty)} \frac{(\lambda + 1)x^2}{\lambda + cx} d\mu(\lambda) = 0$$

by the bounded convergence theorem. Suppose, by contradiction, that $\gamma > 0$; then we have

$$\lim_{c \rightarrow \infty} \frac{f(cx)}{f(c)} = x^2, \quad x > 0.$$

Note that $f_c(x) := f(cx)/f(c)$ satisfies (a4) as well as f . Since the operator mean σ is continuous when restricted on the pairs of positive definite matrices, for every positive definite 2×2 matrices A, B (realized in $B(\mathcal{H})$) we can take the limit of $f_c(A \nabla B) \leq f_c(A) \sigma f_c(B)$ as $c \rightarrow \infty$ to obtain $(A \nabla B)^2 \leq A^2 \sigma B^2$. By Lemma 2.2 for $\varphi(x) = x^2$, this yields a contradiction with the assumption $\sigma \neq \nabla$. Hence we must have $\gamma = 0$ so that

$$f(x) = \alpha + \beta x + \int_{(0, \infty)} \frac{(\lambda + 1)x^2}{\lambda + x} d\mu(\lambda).$$

Step 2 For $c > 0$ large enough, we write

$$\frac{f(cx)}{f(c)} = \frac{\frac{\alpha}{c} + \beta x + \int_{(0, \infty)} \frac{(\lambda+1)cx^2}{\lambda+cx} d\mu(\lambda)}{\frac{\alpha}{c} + \beta + \int_{(0, \infty)} \frac{(\lambda+1)c}{\lambda+c} d\mu(\lambda)}. \tag{2.6}$$

For each fixed $x > 0$, since $(\lambda + 1)cx/(\lambda + cx) \nearrow \lambda + 1$ as $c \nearrow \infty$, we notice by the monotone convergence theorem that

$$\lim_{c \rightarrow \infty} \int_{(0, \infty)} \frac{(\lambda + 1)cx^2}{\lambda + cx} d\mu(\lambda) = \left(\int_{(0, \infty)} (\lambda + 1) d\mu(\lambda) \right) x.$$

Suppose, by contradiction, that $\int_{(0, \infty)} (\lambda + 1) d\mu(\lambda) = +\infty$. For each $c, x \in (0, \infty)$ we set

$$\rho(c, x) := \frac{\int_{(0, \infty)} \frac{(\lambda+1)cx}{\lambda+cx} d\mu(\lambda)}{\int_{(0, \infty)} \frac{(\lambda+1)c}{\lambda+c} d\mu(\lambda)}. \tag{2.7}$$

Since

$$\begin{aligned} \frac{(\lambda + 1)c}{\lambda + c} x &\leq \frac{(\lambda + 1)cx}{\lambda + cx} \leq \frac{(\lambda + 1)c}{\lambda + c} \quad \text{if } 0 < x \leq 1, \\ \frac{(\lambda + 1)c}{\lambda + c} &\leq \frac{(\lambda + 1)cx}{\lambda + cx} \leq \frac{(\lambda + 1)c}{\lambda + c} x \quad \text{if } x \geq 1, \end{aligned}$$

we notice that for every $c > 0$,

$$\begin{cases} x \leq \rho(c, x) \leq 1 & \text{if } 0 < x \leq 1, \\ 1 \leq \rho(c, x) \leq x & \text{if } x \geq 1, \end{cases} \tag{2.8}$$

and furthermore $\rho(c, x)$ is non-decreasing in $x > 0$ for each fixed $c > 0$. Let D denote the countable set of all positive algebraic numbers. Since $\{\rho(c, x) : c > 0\}$ is bounded for each fixed $x > 0$, one can choose a sequence $\{c_n\}$ with $0 < c_n \nearrow \infty$ such that the limit

$$\kappa(x) := \lim_{n \rightarrow \infty} \rho(c_n, x) \tag{2.9}$$

exists for all $x \in D$. Then from (2.6) we obtain

$$\varphi(x) := x\kappa(x) = \lim_{n \rightarrow \infty} \frac{f(c_n x)}{f(c_n)}, \quad x \in D.$$

Moreover, for each n large enough, since $f_n(x) := f(c_n x)/f(c_n)$ satisfies (a4) and so f_n is operator convex on $(0, \infty)$, it follows that $\varphi(x)$ is convex on D . Hence φ can be extended to a continuous non-decreasing function on $[0, \infty)$, and it follows from (2.8) that

$$\begin{cases} x^2 \leq \varphi(x) \leq x & \text{if } 0 < x \leq 1, \\ x \leq \varphi(x) \leq x^2 & \text{if } x \geq 1. \end{cases}$$

In particular, $\varphi(0) = 0$ and $\varphi(1) = 1$. Now let A, B be positive definite 2×2 matrices (realized in $B(\mathcal{H})$) whose entries are all rational complex numbers. Since the eigenvalues of A, B , and $A \nabla B$ are in D , we can take the limit of $f_n(A \nabla B) \leq f_n(A) \sigma f_n(B)$ to obtain

$$\varphi(A \nabla B) \leq \varphi(A) \sigma \varphi(B). \tag{2.10}$$

Furthermore, we approximate arbitrary positive definite 2×2 matrices by those of rational complex entries and take the limit of (2.10) for approximating matrices to see that (2.10) holds for all positive definite 2×2 matrices A, B . Then Lemma 2.2 implies that $\sigma = \nabla$, a contradiction, so it must follow that $\int_{(0, \infty)} (\lambda + 1) d\mu(\lambda) < +\infty$.

Step 3 Finally, suppose, by contradiction, that $\beta + \int_{(0,\infty)} (\lambda + 1) d\mu(\lambda) \neq 0$. Then it is immediately seen from (2.6) again that

$$\lim_{c \rightarrow \infty} \frac{f(cx)}{f(c)} = x, \quad x > 0.$$

By Lemma 2.2 for $\varphi(x) = x$, this yields a contradiction again, so we must have $\beta + \int_{(0,\infty)} (\lambda + 1) d\mu(\lambda) = 0$ so that

$$f(x) = \alpha + \int_{(0,\infty)} \left\{ \frac{(\lambda + 1)x^2}{\lambda + x} - (\lambda + 1)x \right\} d\mu(\lambda) = \alpha - \int_{(0,\infty)} \frac{\lambda(\lambda + 1)x}{\lambda + x} d\mu(\lambda).$$

Since

$$-\frac{x}{\lambda + x} = \frac{\lambda}{\lambda + x} - 1$$

is operator monotone decreasing on $(0, \infty)$, so is f and (a1) follows. □

The next theorem is the counterpart of Theorem 2.1 for operator log-concave functions, including the statement (2°) in Sect. 1.

Theorem 2.3 *Let f be a continuous nonnegative function on $(0, \infty)$. Then the following conditions are equivalent:*

- (b1) f is operator monotone;
- (b2) $f(A \nabla B) \geq f(A) \sigma f(B)$ for all $A, B \in B(\mathcal{H})^{++}$ and for all symmetric means σ ;
- (b3) f is operator log-concave, i.e., $f(A \nabla B) \geq f(A) \# f(B)$ for all $A, B \in B(\mathcal{H})^{++}$;
- (b4) $f(A \nabla B) \geq f(A) \sigma f(B)$ for all $A, B \in B(\mathcal{H})^{++}$ and for some symmetric operator mean $\sigma \neq !$.

We need the following lemma to prove the theorem.

Lemma 2.4 *Let f be a continuous nonnegative function on $(0, \infty)$, and assume that*

$$f(A \nabla B) \geq f(A) ! f(B), \quad A, B \in B(\mathcal{H})^{++}. \tag{2.11}$$

Then, either $f(x) > 0$ for all $x > 0$ or f is identically zero. (Indeed, it is enough to assume that the above inequality holds for all positive definite 2×2 matrices A, B .)

Proof Assume that $f(x) = 0$ for some $x > 0$ but f is not identically zero. The assumption (2.11) applied to $A = aI$ and $B = bI$ gives $f(a \nabla b) \geq f(a) ! f(b)$ for every scalars $a, b > 0$. By induction on $n \in \mathbb{N}$ one can easily see that

$$f((1 - \lambda)a + \lambda b) \geq f(a) !_\lambda f(b) \tag{2.12}$$

for all $a, b > 0$ and all $\lambda = k/2^n, k = 0, 1, \dots, 2^n, n \in \mathbb{N}$, where $u !_\lambda v$ with $0 \leq \lambda \leq 1$ is the λ -harmonic mean for scalars $u, v \geq 0$ defined as

$$u !_\lambda v := \lim_{\varepsilon \searrow 0} \left((1 - \lambda)(u + \varepsilon)^{-1} + \lambda(v + \varepsilon)^{-1} \right)^{-1}.$$

Furthermore, thanks to the continuity of f , (2.12) holds for all $a, b > 0$ and all $\lambda \in [0, 1]$. So we notice that $f(x) > 0$ for all x between a, b whenever $f(a) > 0$ and $f(b) > 0$. Thus it follows from the assumption on f that there is an $\alpha \in (0, \infty)$ such that the following (i) or (ii) holds:

- (i) $f(x) = 0$ for all $x \in (0, \alpha]$ and $f(x) > 0$ for all $x \in (\alpha, \alpha + \delta]$ for some $\delta > 0$,
- (ii) $f(x) > 0$ for all $x \in (0, \alpha)$ and $f(x) = 0$ for all $x \in [\alpha, \infty)$.

Let H and K be 2×2 Hermitian matrices in the realization of the 2×2 matrix algebra in $B(\mathcal{H})$. For every $\gamma \in \mathbb{R}$ such that $\alpha I + \gamma H, \alpha I + \gamma K > 0$ (i.e., positive definite), one can apply (2.11) to $A := \alpha I + \gamma H$ and $B := \alpha I + \gamma K$ to obtain

$$f \left(\alpha I + \gamma \frac{H + K}{2} \right) \geq f(\alpha I + \gamma H) ! f(\alpha I + \gamma K). \tag{2.13}$$

Write for short

$$X := f(\alpha I + \gamma H), \quad Y := f(\alpha I + \gamma K), \quad Z := f \left(\alpha I + \gamma \frac{H + K}{2} \right),$$

and let $s(X), s(Y)$, and $s(Z)$ denote the support projections of X, Y , and Z , respectively, that is, the orthogonal projections onto the ranges of X, Y , and Z (in \mathbb{C}^2), respectively. Since $X \geq \varepsilon s(X)$ and $Y \geq \varepsilon s(Y)$ for a sufficiently small $\varepsilon > 0$, (2.13) implies that

$$Z \geq \{\varepsilon s(X)\} ! \{\varepsilon s(Y)\} = \varepsilon \{s(X) \wedge s(Y)\}.$$

Letting $P := s(X) \wedge s(Y)$ we have

$$0 = (I - s(Z))Z(I - s(Z)) \geq \varepsilon(I - s(Z))P(I - s(Z))$$

so that $P(I - s(Z)) = 0$ or equivalently $P \leq s(Z)$. Therefore,

$$s(Z) \geq s(X) \wedge s(Y).$$

For each Hermitian matrix S let $S = S_+ - S_-$ be the Jordan decomposition of S . In the case (i) choose a $\gamma > 0$ small enough so that $\alpha I + \gamma H, \alpha I + \gamma K \leq (\alpha + \delta)I$, and in the case (ii) choose a $\gamma < 0$ so that $\alpha I + \gamma H, \alpha I + \gamma K > 0$. Then we have

$$s(X) = s(H_+), \quad s(Y) = s(K_+), \quad s(Z) = s((H + K)_+)$$

and so

$$s((H + K)_+) \geq s(H_+) \wedge s(K_+). \tag{2.14}$$

Thus, to prove the lemma by contradiction, it suffices to show that (2.14) is not true in general. We notice that (2.14) yields

$$s(H_+) \geq s(K_+) \text{ whenever } H > K. \tag{2.15}$$

In fact, letting $G := H - K > 0$ (hence $s(G_+) = s(G) = I$) we have

$$s(H_+) = s((G + K)_+) \geq s(G_+) \wedge s(K_+) = s(K_+).$$

Hence it suffices to show that (2.15) is not true in general. Now let $P := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $Q := \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$, and define $H := P$ and $K := \varepsilon Q - (I - Q)$ for $\varepsilon > 0$. Then $s(H_+) = P \not\geq Q = s(K_+)$. But since

$$H - K = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \varepsilon \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} + \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} \frac{3-\varepsilon}{2} & -\frac{1+\varepsilon}{2} \\ -\frac{1+\varepsilon}{2} & \frac{1-\varepsilon}{2} \end{bmatrix}$$

and

$$\det(H - K) = \left(\frac{3-\varepsilon}{2}\right)\left(\frac{1-\varepsilon}{2}\right) - \left(\frac{1+\varepsilon}{2}\right)^2 = \frac{1-3\varepsilon}{2},$$

we have $H > K$ for small $\varepsilon > 0$. Hence (2.15) is not true. The last statement in the parentheses is obvious from the above proof. □

Proof of Theorem 2.3 Assume (b1); then f is operator concave [11, Theorem 2.5], so (b2) follows. It is obvious that (b2) \Rightarrow (b3) \Rightarrow (b4). Finally, let us prove that (b4) \Rightarrow (b1). Since (b4) implies the assumption of Lemma 2.4, we may assume by Lemma 2.4 that $f(x) > 0$ for all $x > 0$. Then (b4) implies that

$$f(A \nabla B)^{-1} \leq (f(A) \sigma f(B))^{-1} = f(A)^{-1} \sigma^* f(B)^{-1}, \quad A, B \in B(\mathcal{H})^{++},$$

where σ^* is the adjoint of σ , the symmetric operator mean defined by $A \sigma^* B := (A^{-1} \sigma B^{-1})^{-1}$ [13]. Since $\sigma \neq \nabla$ means that $\sigma^* \neq \nabla$, Theorem 2.1 implies that $1/f$ is operator monotone decreasing, so (b1) follows. □

Remark 2.5 By Lemma 2.4 it is also seen that a continuous nonnegative function f on $(0, \infty)$ satisfies (2.11) if and only if f is identically zero, or $f > 0$ and $1/f$ is operator convex.

Remark 2.6 For each $\lambda \in [0, 1]$ the λ -arithmetic and the λ -harmonic means are $A \nabla_\lambda B := (1-\lambda)A + \lambda B$ and $A !_\lambda B := ((1-\lambda)A^{-1} + \lambda B^{-1})^{-1}$ for $A, B \in B(\mathcal{H})^{++}$. Let σ be an operator mean corresponding to an operator monotone function h on $[0, \infty)$ such that $h'(1) = \lambda$. Then we have $A \nabla_\lambda B \geq A \sigma B \geq A !_\lambda B$ extending (2.2). As in the proof of Proposition 1.1,

$$f(A \nabla_\lambda B) \leq f(A) !_\lambda f(B) \leq f(A) \sigma f(B), \quad A, B \in B(\mathcal{H})^{++},$$

whenever $f \geq 0$ is operator monotone decreasing on $(0, \infty)$. Consequently, for such a function f ,

$$f(A \nabla_\lambda B) \leq f(A) \#_\lambda f(B), \quad A, B \in B(\mathcal{H})^{++}, \tag{2.16}$$

where $\#_\lambda$ is the λ -power mean corresponding to the power function x^λ . The reversed inequality of (2.16) holds if f is operator monotone. We may adopt (2.16) for the definition of operator log-convexity. Indeed, if f is a nonnegative function (not assumed to be continuous) on $(0, \infty)$ and satisfies (2.16) for all positive definite $n \times n$ matrices A, B of every n , then f is continuous and a standard convergence argument shows that f is operator log-convex.

Remark 2.7 The arithmetic and the harmonic means of n operators A_1, \dots, A_n in $B(\mathcal{H})^{++}$ are

$$\mathbf{A}(A_1, \dots, A_n) := \frac{A_1 + \dots + A_n}{n}, \quad \mathbf{H}(A_1, \dots, A_n) := \left(\frac{A_1^{-1} + \dots + A_n^{-1}}{n} \right)^{-1}.$$

The geometric mean $\mathbf{G}(A_1, \dots, A_n)$ for $n \geq 3$ was rather recently introduced in [2] in a recursive way. (A different notion of geometric means for n operators is in [7].) From the arithmetic–geometric–harmonic mean inequality for n operators in [2], we have

$$f(\mathbf{A}(A_1, \dots, A_n)) \leq \mathbf{H}(f(A_1), \dots, f(A_n)) \leq \mathbf{G}(f(A_1), \dots, f(A_n))$$

if $f \geq 0$ is operator monotone decreasing on $(0, \infty)$, and if f is operator monotone,

$$f(\mathbf{A}(A_1, \dots, A_n)) \geq \mathbf{A}(f(A_1), \dots, f(A_n)) \geq \mathbf{G}(f(A_1), \dots, f(A_n)).$$

3 Further characterizations

In this section we present further conditions equivalent to those of Theorems 2.1 and 2.3, respectively. To exclude the singular case of identically zero function and thus make statements simpler, we assume throughout the section that f is a continuous positive (i.e., $f(x) > 0$ for all $x > 0$) function on $(0, \infty)$.

Theorem 3.1 *For a continuous positive function f on $(0, \infty)$, each of the following conditions (a5)–(a13) is equivalent to (a1)–(a4) of Theorem 2.1:*

- (a5) $\begin{bmatrix} f(A) & f(A \nabla B) \\ f(A \nabla B) & f(B) \end{bmatrix} \geq 0$ for all $A, B \in B(\mathcal{H})^{++}$, where $\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$ for $\bar{X}_{ij} \in B(\mathcal{H})$ is considered as an operator in $B(\mathcal{H} \oplus \mathcal{H})$ as usual;
- (a6) $f(A \nabla B) f(B)^{-1} f(A \nabla B) \leq f(A)$ for all $A, B \in B(\mathcal{H})^{++}$;
- (a7) $f(A \nabla B) \leq \frac{1}{2} \{ \lambda f(A) + \lambda^{-1} f(B) \}$ for all $A, B \in B(\mathcal{H})^{++}$ and all $\lambda > 0$;
- (a8) $A \in B(\mathcal{H})^{++} \mapsto \log \langle \xi, f(A) \xi \rangle$ is convex for every $\xi \in \mathcal{H}$;
- (a9) $(A, \xi) \mapsto \langle \xi, f(A) \xi \rangle$ is jointly convex for $A \in B(\mathcal{H})^{++}$ and $\xi \in \mathcal{H}$;
- (a10) f is operator convex and the numerical function $\log f(x)$ is convex;
- (a11) both f and $\log f$ are operator convex;
- (a12) f is operator convex and the numerical function $f(x)$ is non-increasing;
- (a13) f admits a representation

$$f(x) = \alpha + \int_{[0, \infty)} \frac{\lambda + 1}{\lambda + x} d\mu(\lambda), \tag{3.1}$$

where $\alpha \geq 0$ and μ is a finite positive measure on $[0, \infty)$.

Before proving the theorem we give the next lemma, which may be of independent interest.

Lemma 3.2 *Let $\varphi(x)$ be a continuous non-decreasing function on $(0, \infty)$ such that $\varphi(0) = 0$ and $\varphi(1) = 1$. Then $(A, \xi) \mapsto \langle \xi, \varphi(A) \xi \rangle$ for $A \in B(\mathcal{H})^{++}$ and $\xi \in \mathcal{H}$ cannot be jointly convex. (Indeed, this functional cannot be jointly convex even when A is restricted to positive definite 2×2 matrices and ξ to vectors in \mathbb{C}^2 .)*

Proof First, recall the well-known expression for the parallel sum:

$$\langle \xi, (A : B) \xi \rangle = \inf \{ \langle \xi_1, A \xi_1 \rangle + \langle \xi_2, B \xi_2 \rangle : \xi = \xi_1 + \xi_2, \xi_1, \xi_2 \in \mathcal{H} \} \tag{3.2}$$

for any $A, B \in B(\mathcal{H})^{++}$ and $\xi \in \mathcal{H}$ (see [1, Theorem I.3] for example). Suppose, by contradiction, that the functional in question is jointly convex. Let us show that

$$\varphi(A \nabla B) \leq \varphi(A) ! \varphi(B), \quad A, B \in B(\mathcal{H})^{++}. \tag{3.3}$$

For any decomposition $\xi = \xi_1 + \xi_2$ of $\xi \in \mathcal{H}$ we have

$$\begin{aligned} \langle \xi, \varphi(A \nabla B) \xi \rangle &= 4 \left\langle \frac{\xi_1 + \xi_2}{2}, \varphi \left(\frac{A + B}{2} \right) \left(\frac{\xi_1 + \xi_2}{2} \right) \right\rangle \\ &\leq 2 \{ \langle \xi_1, \varphi(A) \xi_1 \rangle + \langle \xi_2, \varphi(B) \xi_2 \rangle \}, \end{aligned}$$

which implies by (3.2) that

$$\langle \xi, \varphi(A \nabla B) \xi \rangle \leq \langle \xi, (\varphi(A) ! \varphi(B)) \xi \rangle.$$

Hence (3.3) follows, which yields a contradiction by Lemma 2.2. □

Proof of Theorem 3.1 (a5) \Leftrightarrow (a6) is well known (see [1, Theorem I.1], [6, 1.3.3]). (a5) \Rightarrow (a3) follows from the following characterization of the geometric mean given in [1, Theorem I.2 and its proof]:

$$X \# Y = \max \left\{ Z \in B(\mathcal{H})^+ : \begin{bmatrix} X & Z \\ Z & Y \end{bmatrix} \geq 0 \right\} \text{ for } X, Y \in B(\mathcal{H})^+.$$

The implications (a3) \Rightarrow (a7) \Rightarrow (a8) were already shown in the proof of Proposition 1.1.

(a8) \Rightarrow (a10). The operator convexity of f is immediate because f is operator convex if (and only if) $A \in B(\mathcal{H})^{++} \mapsto \langle \xi, f(A)\xi \rangle$ is convex for every $\xi \in \mathcal{H}$. The convexity of $\log f(x)$ is also obvious by taking $A = aI$ in (a8).

(a10) \Rightarrow (a1). This can be shown in a manner similar to the three-stepped proof of (a4) \Rightarrow (a1) of Theorem 2.1. By considering $f(\varepsilon + x)$ for each $\varepsilon > 0$, we may assume that f admits the representation (2.5). For Step 1, suppose that $\gamma > 0$; then we have $\lim_{c \rightarrow \infty} f(cx)/f(c) = x^2$ for all $x > 0$. Since $\log f(cx)$ is convex by assumption, the limit function $2 \log x$ is convex as well, which is absurd. Hence $\gamma = 0$.

For Step 2, suppose that $\int_{(0,\infty)} (\lambda + 1) d\mu(\lambda) = +\infty$. One can choose a sequence $\{c_n\}$ with $0 < c_n \nearrow \infty$ such that the limit $\kappa(x)$ in (2.9), with $\rho(c, x)$ in (2.7), exists for all rational numbers $x > 0$. From (2.8) and (2.6) we have $1 \leq \kappa(x) \leq x$ for all rational $x \geq 1$ and $\varphi(x) := x\kappa(x) = \lim_{n \rightarrow \infty} f(c_n x)/f(c_n)$ for all rational $x > 0$. Since $\log f(c_n x)$ is convex on $(0, \infty)$, it follows that $\log \varphi(x)$ is convex on the rational numbers $x \geq 1$. Hence φ can be extended to a continuous function on $[1, \infty)$ so that $\psi(x) := \log \varphi(x)$ is convex on $[1, \infty)$ and

$$\log x \leq \psi(x) \leq 2 \log x, \quad x \geq 1. \tag{3.4}$$

For any $a \geq 1$, by convexity of ψ we have

$$\frac{\psi(a)}{a} \leq \lim_{x \rightarrow \infty} \frac{\psi(x)}{x} \leq 2 \lim_{x \rightarrow \infty} \frac{\log x}{x} = 0.$$

Hence $\psi(a) = 0$ for all $a \geq 1$, which contradicts the first inequality in (3.4). Hence $\int_{(0,\infty)} (\lambda + 1) d\mu(\lambda) < +\infty$.

Step 3 here is the same as that in the proof of (a4) \Rightarrow (a1) of Theorem 2.1 by considering the limit function $\log x$ of $\log(f(cx)/f(c))$ as $c \rightarrow \infty$.

(a1) \Rightarrow (a13). This implication was shown in the proof of the main theorem of [9], and the converse is obvious. We state (a13) since it is useful to derive (a5) from (a1). The following proof is slightly simpler than that in [9]. Since (a1) is equivalent to $f(x^{-1})$ being operator monotone, we have a representation

$$f(x^{-1}) = \alpha + \beta x + \int_{(0,\infty)} \frac{(\lambda + 1)x}{\lambda + x} d\nu(\lambda), \tag{3.5}$$

where $\alpha, \beta \geq 0$ and ν is a positive finite measure on $(0, \infty)$ [5, pp. 144-145]. By taking $d\mu(\lambda) := d\nu(\lambda^{-1})$ on $(0, \infty)$ and by extending it to a measure on $[0, \infty)$ with $\mu(\{0\}) = \beta$, the representation (3.5) is transformed into (3.1).

(a13) \Rightarrow (a5). Thanks to (a5) \Leftrightarrow (a6) as mentioned above, it suffices to show that the component functions $f_1(x) := \alpha$, $f_2(x) := 1/x$, and $f_3(x) := 1/(x + \lambda)$ for $\lambda > 0$ in the expression (3.1) satisfy the inequality in (a6). It is trivial for f_1 . For f_2 we have to show that

$$\left(\frac{A+B}{2}\right)^{-1} B \left(\frac{A+B}{2}\right)^{-1} \leq A^{-1},$$

or equivalently,

$$\left(\frac{A+B}{2}\right) B^{-1} \left(\frac{A+B}{2}\right) \geq A. \tag{3.6}$$

With $C := B^{-1/2} A B^{-1/2}$, (3.6) is further reduced to $\frac{1}{4}(C+I)^2 \geq C$, which obviously holds. The assertion for f_3 follows from that for f_2 by taking $A + \lambda I$ and $B + \lambda I$ in place of A and B .

Now, conditions (a9), (a11), and (a12) are outside the above proved circle of equivalence, whose equivalence to (a1) is proved below.

(a1) \Leftrightarrow (a11). Since (a1) implies that $1/f$ is operator monotone and since $\log x$ is operator monotone on $(0, \infty)$, it is immediate to see that $\log(1/f) = -\log f$ is operator monotone. This implies that $-\log f$ is operator concave or $\log f$ is operator convex. For the converse, (a11) \Rightarrow (a10) is trivial.

(a1) \Leftrightarrow (a9). The implication (a13) \Rightarrow (a9) was shown in [10, Remark 4.6]. The proof of (a9) \Rightarrow (a1) can be done similarly to (a4) \Rightarrow (a1) of Theorem 2.1 by dividing into three steps. First, from the fact mentioned in the proof of (a8) \Rightarrow (a10), we may assume that f admits the representation (2.5). Then for Steps 1 and 3, we may only notice that the functions x^2 and x do not satisfy (a9) as particular cases of Lemma 3.2. For Step 2, suppose that $\int_{(0,\infty)} (\lambda + 1) d\mu(\lambda) = +\infty$; then as in the proof of (a4) \Rightarrow (a1) there is a sequence $c_n \nearrow \infty$ such that $\varphi(x) := \lim_{n \rightarrow \infty} f(c_n x)/f(c_n)$ exists for all algebraic numbers $x > 0$, and φ can be extended to a continuous non-decreasing function on $[0, \infty)$ with $\varphi(0) = 0$ and $\varphi(1) = 1$. Furthermore, since $f(c_n x)$ satisfies (a9), it follows as in the proof of (a4) \Rightarrow (a1) that φ satisfies (a9) as well when A is restricted to positive definite 2×2 matrices. This yields a contradiction by Lemma 3.2, which shows that $\int_{(0,\infty)} (\lambda + 1) d\mu(\lambda) < +\infty$.

(a1) \Leftrightarrow (a12). The implication (a1) \Rightarrow (a12) is immediate since (a1) implies the operator convexity of f . The converse can be proved once again similarly to (a10) \Rightarrow (a1); just use the non-increasingness of $f(cx)$ instead of the convexity of $\log f(cx)$. In fact, for Step 2, if we suppose that $\int_{(0,\infty)} (\lambda + 1) d\mu(\lambda) = +\infty$, then the function $\varphi(x)$ defined and extended as above is non-increasing by the assumption (a12) as well as non-decreasing with $\varphi(x) \geq x$ for $x \geq 1$ (by the definition of φ). This is a contradiction. □

Remark 3.3 Let $\Phi : B(\mathcal{H}) \rightarrow B(\mathcal{K})$ be a positive linear map, where \mathcal{K} is another Hilbert space. If f is operator log-convex on $(0, \infty)$, then we have

$$\Phi(f(A \nabla B)) \leq \Phi(f(A) \# f(B)) \leq \Phi(f(A)) \# \Phi(f(B))$$

for all $A, B \in B(\mathcal{H})^+$ thanks to [1, Corollary IV.1.3]. This in particular gives another proof of (a3) \Rightarrow (a8) by taking a positive linear functional as Φ .

Remark 3.4 The implication (a3) \Rightarrow (a11) says that (1.3) implies (1.4), that is, the operator log-convexity of f implies that $\log f$ is operator convex. This may also justify our term operator log-convexity.

Remark 3.5 In [10, Remark 4.6] Hansen posed the question to characterize functions f on $(0, \infty)$ for which condition (a9) holds. By taking $A = aI$ in $\langle \xi, f(A)\xi \rangle$ for any fixed $a \in (0, \infty)$, it is clear that f must be nonnegative whenever it satisfies (a9). Consequently, Theorem 3.1 settles the above question as follows: A continuous function f on $(0, \infty)$ satisfies (a9) if and only if f is nonnegative and operator monotone decreasing, or equivalently, f admits a representation in (a13).

Remark 3.6 In [16] Uchiyama recently proved that a continuous (not necessarily positive) function f on $(0, \infty)$ is operator monotone decreasing if and only if it is operator convex and $f(\infty) := \lim_{x \rightarrow \infty} f(x) < +\infty$. This implies that (a1) \Leftrightarrow (a12), because the non-increasingness of a convex function f on $(0, \infty)$ is equivalent to $f(\infty) < +\infty$.

The following is the concave counterpart of Theorem 3.1, which is easily shown by converting corresponding conditions of Theorem 3.1.

Theorem 3.7 *For a continuous positive function f on $(0, \infty)$, each of the following conditions (b5)–(b10) is equivalent to (b1)–(b4) of Theorem 2.3:*

- (b5) $\begin{bmatrix} f(A) & f(A \! B) \\ f(A \! B) & f(B) \end{bmatrix} \geq 0$ for all $A, B \in B(\mathcal{H})^{++}$;
- (b6) $f(A \nabla B)f(B)^{-1}f(A \nabla B) \geq f(A)$ for all $A, B \in B(\mathcal{H})^{++}$;
- (b7) $f(A \! B) \leq \frac{1}{2}\{\lambda f(A) + \lambda^{-1} f(B)\}$ for all $A, B \in B(\mathcal{H})^{++}$ and all $\lambda > 0$;
- (b8) $A \in B(\mathcal{H})^{++} \mapsto \log \langle \xi, f(A)\xi \rangle$ is concave for every $\xi \in \mathcal{H}$;
- (b9) f is operator concave;
- (b10) f admits a representation

$$f(x) = \alpha + \beta x + \int_{(0, \infty)} \frac{(\lambda + 1)x}{\lambda + x} d\mu(\lambda),$$

where $\alpha, \beta \geq 0$ and μ is a finite positive measure on $(0, \infty)$.

Proof Since f satisfies (b1) if and only if $1/f$ (or $f(x^{-1})$) satisfies (a1), each condition of Theorem 3.1 for $1/f$ (or $f(x^{-1})$) instead of f is equivalent to (b1). (b5) and (b7) are (a5) and (a7) for $f(x^{-1})$, respectively. Also, (b6) is (a6) for $1/f$.

The implication (b1) \Rightarrow (b8) is a particular case of Proposition 1.2. Conversely, assume (b8). For every $A \in B(\mathcal{H})^{++}$ and $\xi \in \mathcal{H}$ notice that

$$\langle \xi, f(A)^{-1}\xi \rangle = \sup_{\eta \neq 0} \frac{|\langle \xi, \eta \rangle|^2}{\langle \eta, f(A)\eta \rangle}$$

and so

$$\log \langle \xi, f(A)^{-1}\xi \rangle = \sup_{\eta \neq 0} \{2 \log |\langle \xi, \eta \rangle| - \log \langle \eta, f(A)\eta \rangle\}.$$

Since (b8) implies that $A \in B(\mathcal{H})^{++} \mapsto 2 \log |\langle \xi, \eta \rangle| - \log \langle \eta, f(A)\eta \rangle$ is convex, it follows that $1/f$ satisfies (a8). Hence (b8) \Rightarrow (b1).

Finally, (b1) \Leftrightarrow (b9) and (b1) \Leftrightarrow (b10) are well known [5, 11], which were indeed used in the proofs of Theorems 2.3 and 3.1. We state (b9) and (b10) just for the sake of completeness. □

4 More about operator monotony and operator means

When f is an operator monotone (not necessarily nonnegative) function on $(0, \infty)$, it is obvious that

$$f(A \nabla B) \geq f(A \# B) \geq f(A ! B), \quad A, B \in B(\mathcal{H})^{++}.$$

In the next proposition we show that an inequality such as $f(A \nabla B) \geq f(A \# B)$ for all $A, B \in B(\mathcal{H})^{++}$ conversely implies the operator monotony of f , thus giving yet another characterization of operator monotone functions on $(0, \infty)$ in terms of operator means.

Proposition 4.1 *A continuous function f on $(0, \infty)$ is operator monotone if and only if one of the following conditions holds:*

- (1) $f(A \nabla B) \geq f(A \sigma B)$ for all $A, B \in B(\mathcal{H})^{++}$ and for some symmetric operator mean $\sigma \neq \nabla$;
- (2) $f(A ! B) \leq f(A \sigma B)$ for all $A, B \in B(\mathcal{H})^{++}$ and for some symmetric operator mean $\sigma \neq !$.

The operator monotone decreasingness of f is equivalent to each of (1) and (2) with the reversed inequality.

Note by (2.2) that the inequalities in (1) and (2) actually hold for all symmetric operator means if f is operator monotone. We first prove the next lemma.

Lemma 4.2 *Let σ be a symmetric operator mean such that $\sigma \neq \nabla$, and let $\gamma_0 := 2\sigma 0$. If $X, Y \in B(\mathcal{H})^{++}$ and $X \geq Y \geq \gamma X$ with $\gamma \in (\gamma_0, 1]$, then there exist $A, B \in B(\mathcal{H})^{++}$ such that $X = A \nabla B$ and $Y = A \sigma B$.*

Proof Let h be the operator monotone function on $[0, \infty)$ corresponding to σ , i.e., $h(x) := 1 \sigma x$ for $x \geq 0$. Since $\gamma_0 = 2h(0)$, we have $0 \leq \gamma_0 \leq 1$ by (2.3). Note that $h(0) = 1/2$ implies $\sigma = \nabla$ (see the last part of the proof of Lemma 2.2). Hence we have $0 \leq \gamma_0 < 1$.

Note that $X \geq Y \geq \gamma X$ is equivalent to $I \geq X^{-1/2} Y X^{-1/2} \geq \gamma I$. When we have $A, B \in B(\mathcal{H})^{++}$ such that $I = A \nabla B$ and $X^{-1/2} Y X^{-1/2} = A \sigma B$, it follows that $X = (X^{1/2} A X^{1/2}) \nabla (X^{1/2} B X^{1/2})$ and $Y = (X^{1/2} A X^{1/2}) \sigma (X^{1/2} B X^{1/2})$. Thus we may assume that $I \geq Y \geq \gamma I$ with $\gamma \in (\gamma_0, 1]$ and find $A, B \in B(\mathcal{H})^{++}$ such that $I = A \nabla B$ and $Y = A \sigma B$. For this, it suffices to find an $A \in B(\mathcal{H})^{++}$ such that $A \leq I$ and $A \sigma (2I - A) = Y$. Define $\varphi(t) := t \sigma (2 - t)$ for $0 \leq t \leq 1$; then for $0 < t \leq 1$ we have $\varphi(t) = t h(2t^{-1} - 1)$ and so

$$\varphi'(t) = h(2t^{-1} - 1) - 2t^{-1} h'(2t^{-1} - 1).$$

Letting $a := 2t^{-1} - 1 \in (1, \infty)$ for any $t \in (0, 1)$, one can see that $h'(a) < (h(a) - 1)/(a - 1)$. In fact, suppose on the contrary that $h'(a) \geq (h(a) - 1)/(a - 1)$; then by concavity h must be linear on $[1, a]$. Furthermore, $h'(1) = 1/2$ since σ is symmetric, that is, $h(x) = x h(x^{-1})$ for $x > 0$. Hence it follows that $h(x) = (x + 1)/2$ on $[1, a]$, implying $\sigma = \nabla$. Therefore we have

$$h'(a) < \frac{h(a) - 1}{a - 1} \leq \frac{h(a)}{a + 1}$$

thanks to $h(a) \leq (a + 1)/2$. This yields that $\varphi'(t) = h(a) - (a + 1)h'(a) > 0$, so φ is strictly increasing on $[0, 1]$. Since $\varphi(t) = (2 - t) \sigma t$ by symmetry of σ , $\varphi(0) = 2 \sigma 0 = \gamma_0$. Also $\varphi(1) = 1$. Hence one can define $A := \varphi^{-1}(Y)$ so that $A \in B(\mathcal{H})^{++}$, $A \leq I$, and $Y = \varphi(A) = A \sigma (2I - A)$. □

When $\gamma_0 = 0$, for every $X, Y \in B(\mathcal{H})^{++}$ with $X \geq Y$ we have $A, B \in B(\mathcal{H})^{++}$ such that $X = A \nabla B$ and $Y = A \sigma B$. For example, when $\sigma = !$ and $\#$, A and B can be chosen, respectively, as follows:

$$\begin{cases} A = X - X \# (X - Y), \\ B = X + X \# (X - Y), \end{cases} \quad \begin{cases} A = X - X \# (X - Y X^{-1} Y), \\ B = X + X \# (X - Y X^{-1} Y). \end{cases}$$

Proof of Proposition 4.1 The necessity of (1) and (2) for f to be operator monotone is obvious. Assume (1) and let $X, Y \in B(\mathcal{H})^{++}$ with $X \geq Y$. Choose a $\gamma \in (\gamma_0, 1)$, where $\gamma_0 \in [0, 1)$ be as in Lemma 4.2, and define for $k = 0, 1, 2, \dots$

$$X_k := \gamma^k X + (1 - \gamma^k) Y.$$

Then $X_0 = X$, and we have $X_k \geq X_{k+1} \geq \gamma X_k$ for each $k \geq 0$ because

$$X_k - X_{k+1} = (\gamma^k - \gamma^{k+1})(X - Y) \geq 0, \quad X_{k+1} - \gamma X_k = (1 - \gamma) Y \geq 0.$$

Hence by Lemma 4.2, (1) implies that

$$f(X) \geq f(X_1) \geq \cdots \geq f(X_k) \geq \cdots, \quad k \geq 1.$$

Since $X_k - Y = \gamma^k(X - Y) \rightarrow 0$ so that $f(X_k) \rightarrow f(Y)$ in the operator norm, we have $f(X) \geq f(Y)$.

In the same way it follows that f is operator monotone decreasing if and only if the reversed inequality of (1) holds. Moreover, conditions (1) and (2) are transformed into each other when f is replaced by $f(x^{-1})$ and σ by the adjoint σ^* . Hence the assertions for (2) are immediate from those for (1). \square

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