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Homological properties of the perfect and absolute integral closures of Noetherian domains

Mohsen Asgharzadeh

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Abstract For a Noetherian local domain R let R^+ be the absolute integral closure of R and let R_∞ be the perfect closure of R, when R has prime characteristic. In this paper we investigate the projective dimension of residue rings of certain ideals of R^+ and R_∞. In particular, we show that any prime ideal of R_∞ has a bounded free resolution of countably generated free R_∞-modules. Also, we show that the analogue of this result is true for the maximal ideals of R^+, when R has residue prime characteristic. We compute global dimensions of R^+ and R_∞ in some cases. Some applications of these results are given.

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1 Introduction

For an arbitrary domain R, the absolute integral closure R^+ is defined as the integral closure of R inside an algebraic closure of the field of fractions of R. The notion of absolute integral closure was first studied by Artin in [2], where among other things, he proved R^+ has only one maximal ideal $\mathfrak{m}_{R^+}$, when $(R, \mathfrak{m})$ is local and Henselian.

Dedicated to Paul Roberts.

M. Asgharzadeh
Department of Mathematics, Shahid Beheshti University, Tehran, Iran

M. Asgharzadeh (✉)
School of Mathematics, Institute for Research in Fundamental Sciences (IPM),
P.O. Box 19395-5746, Tehran, Iran
e-mail: asgharzadeh@ipm.ir

In that paper Artin proved that the sum of any collection of prime ideals of R^+ is either a prime ideal or else equal to R^+.

As were shown by the works [11,13,15,17,28], the absolute integral closure of Noetherian domains has a great significance in commutative algebra. One important result of Hochster and Huneke [15, Theorem 5.5] asserts that R^+ is a balanced big Cohen–Macaulay algebra in the case R is an excellent local domain of prime characteristic, i.e., every system of parameters in R is a regular sequence on R^+. As shown by Hochster, the flat dimension of the residue field of R^+ is bounded by dimension of R, when R is a Henselian local domain and has residue prime characteristic, see [13, Proposition 2.15]. Furthermore, the bound is achieved by all mixed characteristic local domains if and only if the Direct Summand Conjecture holds in mixed characteristic, see [1, Theorem 3.5]. These observations along with [24] motivate us to investigate the projective dimension of residue rings of certain ideals of the absolute integral closure and the perfect closure of Noetherian domains. One of our main results is:

Theorem 1.1 *Let $(R, \mathfrak{m})$ be a Noetherian local Henselian domain of residue prime characteristic.*

(i) *The R^+-module $R^+/\mathfrak{m}_{R^+}$ has a free resolution of countably generated free R^+-modules of length bounded by* $2\dim R$. *In particular,* $\mathrm{pd}_{R^+}(R^+/\mathfrak{m}_{R^+}) \leq 2\dim R$.
(ii) *If R is one-dimensional and complete, then* $\mathrm{gldim}\, R^+ = \mathrm{pd}_{R^+}(R^+/\mathfrak{m}_{R^+}) = 2$.

Let R be a ring of prime characteristic p. Recall that the perfect closure of R is defined by

$$R_\infty := \{x \in R^+ : x^{p^n} \in R \text{ for some } n \in \mathbb{N} \cup \{0\}\}.$$

The analogue of Theorem 1.1 for R_∞ is:

Theorem 1.2 *Let $(R, \mathfrak{m})$ be a Noetherian local domain of prime characteristic.*

(i) *Any prime ideal of R_∞ has a bounded free resolution of countably generated free R_∞-modules.*
(ii) *If* $\dim R < 3$, *then* $\sup\{\mathrm{pd}_{R_\infty}(R_\infty/\mathfrak{p}) : \mathfrak{p} \in \mathrm{Spec}(R_\infty)\} < \infty$.
(iii) *If R_∞ is coherent (this holds if R is regular), then* $\mathrm{gldim}(R_\infty) \leq \dim R + 1$.
(iv) *If R is regular and of dimension one, then* $\mathrm{gldim}(R_\infty) = 2$.

Throughout this paper all rings are commutative, with identity, and all modules are unital. The organization of this paper is as follows:

In Sect. 2, we summarize some results concerning projective and flat dimensions of modules over non-Noetherian rings. Also, for the convenience of reader, we collect some known results about R^+ and R_∞ which will be used throughout this work. The *Local Global Principle Theorem* [5, Theorem 2.2.7] indicates that a Noetherian local ring has finite global dimension if and only if its residue field has finite projective dimension. The main result of Sect. 3 gives an analogue of this theorem for a certain class of non-Noetherian rings. Sections 4 and 5 are devoted to the proof of Theorem 1.1. In Sect. 6 by applying Theorem 1.1, we examine the cohomological dimension and the cohomological depth of almost zero functors that are naturally associated to the different classes of almost zero modules. We close Sect. 6 by an application of almost zero modules, see Proposition 6.8 below. Section 7 is concerned with the proof of Theorem 1.2. In Sect. 8, as another application of Theorem 1.1, we show that some known results for Noetherian rings do not hold for general commutative rings. We include some questions in a final section.

2 Preliminaries

In this section we set notation and discuss some facts which will be used throughout the paper. In the first subsection we collect some results concerning projective and flat dimensions of modules over non-Noetherian rings. Also, we recall some known results concerning the perfect and absolute integral closures of Noetherian domains. We do this task in Sect. 2.2.

2.1 Homological dimensions of modules over certain non-Noetherian rings

By $\aleph_{-1}$ we denote the cardinality of finite sets. By $\aleph_0$ we mean the cardinality of the set of all natural numbers. Consider the set $\Omega := \{\alpha : \alpha \text{ is a countable ordinal}$[1]$\text{number}\}$. By definition $\aleph_1$, is the cardinality of Ω. Inductively, $\aleph_n$ can be defined for all $n \in \mathbb{N}$. For more details on this notion we recommend the reader to [22, Appendix]. A ring is called $\aleph_n$-Noetherian if each of its ideals can be generated by a set of cardinality bounded by $\aleph_n$. So, Noetherian rings are exactly $\aleph_{-1}$-Noetherian rings. In the next result we give some basic properties of $\aleph_n$-Noetherian rings.

Lemma 2.1 *The following assertions hold.*

(i) *Let A be a $\aleph_n$-Noetherian ring and let S be a multiplicative closed subset of A. Then $S^{-1}A$ is $\aleph_n$-Noetherian.*
(ii) *Let A be a $\aleph_n$-Noetherian ring. Then any homomorphic image of A is $\aleph_n$-Noetherian.*
(iii) *Let $\{A_n : n \in \mathbb{N}\}$ be a chain of commutative Noetherian rings and let $A := \bigcup A_n$. Then A is $\aleph_0$-Noetherian. In particular, if A is Noetherian, then the ring $A[X_1, X_2, \ldots] := \bigcup_{n=1}^{\infty} A[X_1, \ldots, X_n]$ is $\aleph_0$-Noetherian.*

Proof The proof is straightforward and we leave it to the reader. □

In what follows we will use the following remarkable results of Osofsky several times. It is worth to recall that if $n = -1$, Lemma 2.2(iii) below, is just a rephrasing of the fact that over a Noetherian ring projective and flat dimensions of a cycle module coincide.

Lemma 2.2 *The following assertions hold.*

[1] Recall that a set X is an ordinal if X is totally ordered with respect to inclusion and every element of X is also a subset of X. Also, one can see that Ω is itself an ordinal number larger than all countable ones, so it is an uncountable set.

(i) ([23, p. 14]) *Let V be a valuation domain and let $\mathfrak{r}$ be an ideal of V. Then* $\mathrm{pd}(\mathfrak{r}) = n + 1$ *if and only if $\mathfrak{r}$ is generated by $\aleph_n$ but no fewer elements.*
(ii) ([22, Proposition 2.62]) *Let n be any nonnegative integer or ∞. Then there exists a valuation domain V with global dimension n.*
(iii) (*See the proof of* [22, Corollary 2.47]) *Let $\mathfrak{a}$ be an ideal of a $\aleph_n$-Noetherian ring A. Then* $\mathrm{pd}_A(A/\mathfrak{a}) \le \mathrm{fd}_A(A/\mathfrak{a}) + n + 1$.

Recall that a ring is coherent if each of its finitely generated ideals are finitely presented. A typical example is a valuation domain. In the sequel we will need the following result.

Lemma 2.3 *The following assertions hold.*

(i) ([10, Theorem 1.3.9]) *Let A be a ring. Then*

$$\begin{aligned}\mathrm{wdim}\, A &:= \sup\{\mathrm{fd}(M) : M \text{ is an } A\text{-module}\}\\ &= \sup\{\mathrm{fd}(A/\mathfrak{a}) : \mathfrak{a} \text{ is a finitely generated ideal of } A\}.\end{aligned}$$

(ii) ([10, Corollary 2.5.10]) *Let A be a coherent ring and let M be a finitely presented A-module. Then* $\mathrm{pd}_A(M) \le n$ *if and only if* $\mathrm{Tor}^A_{n+1}(M, A/\mathfrak{m}) = 0$ *for all maximal ideals $\mathfrak{m}$ of A.*
(iii) (*Auslander's global dimension Theorem;* [22, Theorem 2.17]) *Let A be a ring. Then*

$$\mathrm{gldim}\, A = \sup\{\mathrm{pd}_A(A/\mathfrak{a}) : \mathfrak{a} \trianglelefteq A\}.$$

2.2 The perfect and absolute integral closure of Noetherian domains

Recall that the absolute integral closure R^+ is defined as the integral closure of a domain R inside an algebraic closure of the field of fractions of R. Throughout this paper p is a prime number. In this subsection we summarize the basic results concerning R^+ and the perfect closure of R (when $\mathrm{char}\, R = p$):

$$R_\infty := \{x \in R^+ : x^{p^n} \in R \text{ for some } n \in \mathbb{N} \cup \{0\}\}.$$

Assume that $\mathrm{char}\, R = p$ and let A be either R^+ or R_∞ and let $x \in A$. By $(x^\infty)A$ we mean that $(x^{1/p^n} : n \in \mathbb{N} \cup \{0\})A$. Now, assume that $\mathrm{char}\, R = 0$ and let $x \in R^+$. By $(x^\infty)R^+$ we mean $(x^{1/n} : n \in \mathbb{N})R^+$. Note that a local domain $(R, \mathfrak{m}, k)$ has mixed characteristic p, if $\mathrm{char}\, R = 0$ and $\mathrm{char}\, k = p$. We say that R has residue prime characteristic if $\mathrm{char}\, k = p$. We now list some properties of R^+ and R_∞.

Lemma 2.4 *Let $(R, \mathfrak{m})$ be a Noetherian local domain.*

(i) *There is a $\mathbb{Q}$-valued valuation map on R^+ which is nonnegative on R^+ and positive on $\mathfrak{m}R^+$.*
(ii) *Let ϵ be a real number and let $\mathfrak{a}_\epsilon := \{x \in R^+ | v(x) > \epsilon\}$. Then $\mathfrak{a}_\epsilon$ is an ideal of R^+.*

(iii) *Assume that R has prime characteristic p. Let A be either R^+ or R_∞ and let $x_1, \dots, x_\ell$ be a finite sequence of elements of A. Then $\sum_{i=1}^{\ell}(x_i^\infty)A$ is a radical ideal of A.*

(iv) *Assume that R has mixed characteristic p. Let $p, x_2, \dots, x_\ell$ be a finite sequence of elements of R^+. Then $(p^\infty)R^+ + \sum_{i=2}^{\ell}(x_i^\infty)R^+$ is a radical ideal of R^+.*

(v) *If $(R, \mathfrak{m})$ is complete, then R^+ is a directed union of module-finite extensions of R which are complete, local and normal.*

(vi) *Let S be any multiplicative closed subset of R. Then $S^{-1}R^+ \cong (S^{-1}R)^+$.*

(vii) *Assume that R has residue prime characteristic. Let A be either R^+ or R_∞ and let $x_1, \dots, x_\ell$ be a finite sequence of elements of A. Then $\mathrm{fd}_A(A/\sum_{i=1}^{\ell}(x_i^\infty)A) \leq \ell$.*

Proof (i) This is in [14, p. 28]. Note that in that argument R does not need to be complete.

(ii) This is easy to check and we leave it to the reader.

For (iii) and (iv), see parts (1) and (2) of [13, Propostion 2.11]. Note that (iv) may be checked modulo $(p^\infty)R^+$, to translates the prime characteristic case.

(v) By the proof of [18, Lemma 4.8.1], R^+ is a directed union of module-finite extensions of R which are complete and local. Let $\overline{R'}$ be the integral closure of R' in its field of fractions. Recall from [18, Theorem 4.3.4] that the integral closure of a complete local domain in its field of fractions is Noetherian and local. Thus $\overline{R'}$ is a Noetherian complete local normal domain. To conclude, it remains to recall that $\overline{R'} \subseteq R^+$.

(vi) See the second paragraph of [1, Sect. 2].

(vii) See [13, Proposition 2.15]. □

We close this section by the following corollary of Lemma 2.4.

Corollary 2.5 *Let $(R, \mathfrak{m})$ be a Noetherian local domain of dimension d.*

(i) *Assume that R has prime characteristic p. Let A be either R^+ or R_∞ and let $x_1, \dots, x_d$ be a system of parameters for R. Then $\sum_{i=1}^{d}(x_i^\infty)A$ is a maximal ideal of A.*

(ii) *Assume that R has mixed characteristic p. Let $p, x_2, \dots, x_d$ be a system of parameters for R. Then $(p^\infty)R^+ + \sum_{i=2}^{d}(x_i^\infty)R^+$ is a maximal ideal of R^+.*

Proof It follows by the fact that a radical ideal $\mathfrak{a}$ of a ring A is maximal if $\mathrm{ht}(\mathfrak{a}) = \dim A < \infty$. □

3 A local global principle theorem

By Auslander's global dimension Theorem, $\mathrm{gldim}\, A = \sup\{\mathrm{pd}_A(A/\mathfrak{a}) : \mathfrak{a} \trianglelefteq A\}$. So, in order to study the global dimension of A, it is enough for us to commute $\mathrm{pd}_A(A/\mathfrak{a})$ for all ideals $\mathfrak{a}$ of A. It would be interesting to know whether the same equality remains true for some special types of ideals. For instance, we know that if A is Noetherian, then

$$\operatorname{gldim} A = \sup\{\operatorname{pd}_A(A/\mathfrak{m}) : \mathfrak{m} \in \operatorname{Max} A\} = \sup\{\operatorname{pd}_A(A/\mathfrak{p}) : \mathfrak{p} \in \operatorname{Spec} A\}.$$

Thus, the maximal ideals, prime ideals and finitely generated ideals might be appropriate candidate for our proposed ideals.

Definition 3.1 Let Σ be a subset of the set of all ideals of A. We say that A has finite global dimension on Σ, if $\sup\{\operatorname{pd}_A(A/\mathfrak{a}) : \mathfrak{a} \in \Sigma\} < \infty$.

Lemma 3.2 *Let A be a $\aleph_n$-Noetherian ring. Then, A has finite global dimension on finitely generated ideals if and only if A has finite global dimension.*

Proof By Auslander's global dimension Theorem, $\operatorname{gldim} A = \sup\{\operatorname{pd}_A(A/\mathfrak{a}) : \mathfrak{a} \trianglelefteq A\}$. By applying Lemma 2.2(iii), we get that

$$\operatorname{pd}_A(A/\mathfrak{a}) \leq \operatorname{fd}_A(A/\mathfrak{a}) + n + 1.$$

Thus $\operatorname{gldim} A \leq \operatorname{wdim} A + n + 1$. In view of Lemma 2.3(i), we see that

$$\operatorname{wdim} A = \sup\{\operatorname{fd}(A/\mathfrak{a}) : \mathfrak{a} \textit{ is a finitely generated ideal of } A\}.$$

We incorporate these observations in to $\operatorname{fd}_A(A/\mathfrak{a}) \leq \operatorname{pd}_A(A/\mathfrak{a})$ for all ideals $\mathfrak{a}$ of A. It turns out that

$$\operatorname{wdim} A \leq \sup\{\operatorname{pd}(A/\mathfrak{a}) : \mathfrak{a} \textit{ is a finitely generated ideal of } A\} < \infty.$$

This completes the proof. □

Lemma 3.3 *Let A be a coherent ring of finite global dimension on maximal ideals. Then, A has finite global dimension on finitely generated ideals.*

Proof Let $\mathfrak{a}$ be a finitely generated ideal of A. Then $A/\mathfrak{a}$ is finitely presented. By Lemma 2.3(ii), $\operatorname{pd}_A(A/\mathfrak{a}) \leq n$ if and only if $\operatorname{Tor}^A_{n+1}(A/\mathfrak{a}, A/\mathfrak{m}) = 0$ for all maximal ideals $\mathfrak{m}$ of A. This completes the proof. □

The following is our main result in this section.

Theorem 3.4 (Local Global Principle) *Let A be a coherent ring which is $\aleph_n$-Noetherian for some integer $n \geq -1$. Then the following are equivalent*:

(i) *A has finite global dimension,*
(ii) *A has finite global dimension on radical ideals,*
(iii) *A has finite global dimension on prime spectrum,*
(iv) *A has finite global dimension on maximal ideals, and*
(v) *A has finite global dimension on finitely generated ideals.*

Proof The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ and $(iii) \Rightarrow (iv)$ are trivial.

$(iv) \Rightarrow (v)$ This follows by Lemma 3.3.

$(v) \Rightarrow (i)$ This is in Lemma 3.2. □

Remark 3.5 Consider the family of coherent quasi-local $\aleph_n$-Noetherian rings of finite global dimensions. This family contains strictly the class of Noetherian regular local rings. Indeed, let $n \geq 0$ be an integer. In view of Lemma 2.2(ii), there exists a valuation domain A of global dimension $n+2$. Clearly, A is coherent. By applying Lemma 2.2(i), we see that A is $\aleph_n$-Noetherian but not $\aleph_{n-1}$-Noetherian. In particular, A is not Noetherian.

It is noteworthy to remark that the assumptions of the previous results are really needed.

Example 3.6 (i) There exists a ring A of finite global dimension on finitely generated ideals but not of finite global dimension. To see this, let A be a valuation domain of infinite global dimension. Note that such a ring exists, see Lemma 2.2(ii). Since, any finitely generated ideal of A is principal, it turns out that A is coherent and it has finite global dimension on finitely generated ideals. Clearly, by Lemma 3.2(i), A is not $\aleph_n$-Noetherian for all integer $n \geq -1$.

(ii) There exists a ring A of finite global dimension on radical ideals but not of finite global dimension on finitely generated ideals. Let A_0 be the ring of polynomials with nonnegative rational exponents in an indeterminant x over a field. Let T be the localization of A_0 at $(x^\alpha : \alpha > 0)$ and set $A := T/(x^\alpha u : \ u \text{ is unit, } \alpha > 1)$. Then by [22, p. 53], A has finite global dimension on maximal ideals and $\mathrm{pd}(x^{1/2}A) = \infty$. Note that $\dim A = 0$, and so $\operatorname{Spec} A = \operatorname{Max} A$. Thus, A has finite global dimension on radical ideals but not of finite global dimension on finitely generated ideals. By Lemma 2.1, A is $\aleph_0$-Noetherian. Note that Lemma 3.3 asserts that A is not coherent.

Let $\mathfrak{p}$ be a prime ideal of a ring A. In [21, Theorem 3], Northcott proved that $\operatorname{rank}_{A/\mathfrak{p}} \mathfrak{p}/\mathfrak{p}^2 \leq \operatorname{wdim} A$. Here, we give an application of this Theorem.

Corollary 3.7 *Let $(V, \mathfrak{m})$ be a valuation domain. Then the following are equivalent*:

(i) $\bigcap \mathfrak{m}^n = 0$,
(ii) *V is Noetherian,*
(iii) *V is a principal ideal domain, and*
(iv) *V is an unique factorization domain.*

Proof Without loss of generality, we can assume that V is not a field.

$(i) \Rightarrow (ii)$ First, note that any finitely generated ideal of V is principal. Then by Lemma 2.3(i),

$$\operatorname{wdim} V = \sup\{\operatorname{fd}(V/\mathfrak{r}) : \mathfrak{r} \text{ is a finitely generated ideal of } V\} = 1.$$

By [21, Theorem 3], $\operatorname{rank}_{V/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) \leq \operatorname{wdim} V = 1$. Thus, $\mathfrak{m} = aV + \mathfrak{m}^2$ for some $a \in V$. As the ideals of a valuation domain are linearly ordered by means of inclusion, one has either $aV \subseteq \mathfrak{m}^2$ or $\mathfrak{m}^2 \subseteq aV$. Due to the Hausdorff assumption on $\mathfrak{m}$ we can assume that $\mathfrak{m} \neq \mathfrak{m}^2$, and so $\mathfrak{m} = aV$. In view of [20, Exercise 3.3], V is a discrete valuation domain.

$(ii) \Rightarrow (iii)$ and $(iii) \Rightarrow (iv)$ are well-known.

$(vi) \Rightarrow (i)$ By [3, Remark 3.13], ht $\mathfrak{m} = 1$. Thus $\mathfrak{m} = xV$ for some x, because V is an unique factorization domain. Therefore V is Noetherian, since each of its prime ideals are finitely generated. The argument can now by completed by applying Krull's Intersection Theorem. □

4 Projective dimension of certain modules over R^+

Throughout this section R is a domain. The aim of this section is to establish the projective dimensions of residue ring of certain ideals of R^+. First we give a couple of lemmas.

Lemma 4.1 *Let $(R, \mathfrak{m})$ be a one-dimensional quasi-local domain and has residue prime characteristic p. Then R^+ has an unique maximal ideal.*

Proof Let x be any nonzero element of $\mathfrak{m}$. In mixed characteristic case we assume in addition that $x = p$. Let $\mathfrak{M} \in \operatorname{Max} R^+$. Since the ring extension R^+/R is integral, $\mathfrak{m}R^+ \subseteq \mathfrak{M}$. In particular, $x \in \mathfrak{M}$. In view of Lemma 2.4(iii) and (iv), we know that $(x^\infty)R^+$ is a radical ideal of R^+ of height one, which is contained in $\mathfrak{M}$. So $\mathfrak{M} = (x^\infty)R^+$. □

Lemma 4.2 *Let R be a domain and let $\mathfrak{a}$ be a radical ideal of R^+. Then $\mathfrak{a} = \mathfrak{a}^n$ for all $n \in \mathbb{N}$.*

Proof It is enough to show that $\mathfrak{a} = \mathfrak{a}^2$. Let $x \in \mathfrak{a}$. Then the polynomial $f(X) = X^2 - x \in R^+[X]$ has a root $\zeta \in R^+$. As $\mathfrak{a}$ is radical, from $\zeta^2 = x \in \mathfrak{a}$, we deduce that $\zeta \in \mathfrak{a}$. So $x = \zeta.\zeta \in \mathfrak{a}^2$. □

Now, we recall the following result of Auslander.

Lemma 4.3 ([22, Lemma 2.18]) *Let A be a ring and let Γ be a well-ordered set. Suppose that $\{N_\gamma : \gamma \in \Gamma\}$ is a collection of submodules of an A-module M such that $\gamma' \leq \gamma$ implies $N_{\gamma'} \subseteq N_\gamma$ and $M = \bigcup_{\gamma\in\Gamma} N_\gamma$. Suppose that* $\operatorname{pd}_A(N_\gamma/\bigcup_{\gamma'<\gamma} N_{\gamma'}) \leq n$ *for all $\gamma \in \Gamma$. Then $pd_A(M) \leq n$.*

Lemma 4.4 *Let R be a domain and let x be a nonzero and nonunit element of R^+.*

(i) *If R is of characteristic zero, then $(x^\infty)R^+$ has a bounded free resolution of countably generated free R^+-modules of length one. In particular,* $\operatorname{pd}_{R^+}((x^\infty)R^+) \leq 1$. *The equality holds, if R is Noetherian Henselian and local.*
(ii) *If R is of prime characteristic p, then $(x^\infty)R^+$ has a bounded free resolution of countably generated free R^+-modules of length one. In fact,* $\operatorname{pd}_{R^+}((x^\infty)R^+) = 1$.

Proof (i) Clearly, $\frac{1}{n} - \frac{1}{n+1} > 0$, and so $x^{\frac{1}{n}-\frac{1}{n+1}} \in R^+$. Note that $x^{1/n} = x^{1/n+1}x^{\frac{1}{n}-\frac{1}{n+1}}$. In particular, $x^{1/n}R^+ \subseteq x^{1/n+1}R^+$. For each n, the exact sequence

$$0 \longrightarrow x^{1/n}R^+ \longrightarrow x^{1/n+1}R^+ \longrightarrow x^{1/n+1}R^+/x^{1/n}R^+ \longrightarrow 0$$

is a projective resolution of $x^{1/n+1}R^+/x^{1/n}R^+$. So $\mathrm{pd}_{R^+}(x^{1/n+1}R^+/x^{1/n}R^+) \leq 1$. By Lemma 4.3, $\mathrm{pd}_{R^+}((x^\infty)R^+) \leq 1$. Now, we construct the concrete free resolution of $(x^\infty)R^+$. Let F_0 be a free R^+-module with base $\{e_n : n \in \mathbb{N}\}$. The assignment $e_n \mapsto x^{1/n}$ provides a natural epimorphism $\varphi : F_0 \longrightarrow (x^\infty)R^+$. We will show that $\ker\varphi$ is free over R^+. For each integer n set $\eta_n := e_n - x^{1/n(n+1)}e_{n+1}$. Let F_1 be a submodule of $\ker\varphi$ generated by $\{\eta_n : n \in \mathbb{N}\}$. It is easy to see that $F_1 \subseteq \ker\varphi$. In fact, the equality holds. Let $\Sigma_{i=1}^n \alpha_i e_i$ be in $\ker\varphi$, where $\alpha_i \in R^+$ for all $1 \leq i \leq n$. It is straightforward to check that

$$\alpha_n x^{1/n} = -\Sigma_{i=1}^{n-1}\alpha_i x^{1/i} = \beta_{n-1}x^{1/n-1} = \beta_{n-1}x^{1/n}x^{1/n^2-n},$$

where $\beta_{n-1} := -(\alpha_1 x^{\frac{n-2}{n-1}} + \alpha_2 x^{\frac{n-3}{2(n-1)}} + \alpha_3 x^{\frac{n-4}{3(n-1)}} + \cdots + \alpha_{n-1})$. Thus $\alpha_n = \beta_{n-1}x^{1/n^2-n}$, because R^+ is an integral domain. Hence

$$\Sigma_{i=1}^n \alpha_i e_i + \beta_{n-1}\eta_{n-1} = \Sigma_{i=1}^{n-2}\alpha_i e_i + (\alpha_{n-1} + \beta_{n-1})e_{n-1} \in \ker\varphi.$$

By using induction on n, one deduces that $\Sigma_{i=1}^{n-2}\alpha_i e_i + (\alpha_{n-1}+\beta_{n-1})e_{n-1} \in F_1$. This yields that $\Sigma_{i=1}^n \alpha_i e_i \in F_1$, because $\beta_{n-1}\eta_{n-1} \in \ker\varphi$. Thus $\ker\varphi$ is generated by the set $\{\eta_n | n \in \mathbb{N}\}$. In order to establish $\mathrm{pd}_{R^+}((x^\infty)R^+) \leq 1$, it is therefore enough for us to prove that F_1 is a free R^+-module with base $\{\eta_n : n \in \mathbb{N}\}$. Assume that $\Sigma_{i=1}^n \alpha_i \eta_i = 0$, where $\alpha_i \in R^+$. View this equality in F_0. The coefficient of e_{n+1} in the left hand side of the equality is $-x^{1/n^2+n}\alpha_n$. So $\alpha_n = 0$. Continuing inductively, we get that $\alpha_{n-1} = \cdots = \alpha_1 = 0$. Hence F_1 is a free R^+-module. This yields the desired claim.[2]
Now, we assume that R is Noetherian local and Henselian. It turns out that R^+ is quasi-local. If $\mathrm{pd}_{R^+}((x^\infty)R^+) = 1$ was not be the case, then $(x^\infty)R^+$ should be projective and consequently free, see [20, Theorem 2.5]. Over a domain A an ideal $\mathfrak{a}$ is free if and only if it is principal. Indeed, suppose on the contrary that $\mathfrak{a}$ is free with the base set $\Lambda := \{\lambda_\gamma : \gamma \in \Gamma\}$ and $|\Gamma| \geq 2$. Let $\lambda_1, \lambda_2 \in \Lambda$. The equation $r\lambda_1 + s\lambda_2 = 0$ has a nonzero solution as $r := \lambda_2$ and $s := -\lambda_1$, a contradiction. So, $(x^{1/n} : n \in \mathbb{N})R^+ = cR^+$ for some $c \in R^+$. We adopt the notation of Lemma 2.4(i). We see that $v(r) \geq v(c) > 0$ for all $r \in (x^\infty)R^+$. Now, let $n \in \mathbb{N}$ be such that $v(c) > v(x)/n$. Then $v(x^{1/n}) = v(x)/n < v(c)$, a contradiction.

(ii) Let F_0 be a free R^+-module with base $\{e_n : n \in \mathbb{N} \cup \{0\}\}$. The assignment $e_n \mapsto x^{1/p^n}$ provides a natural epimorphism $\varphi : F \longrightarrow (x^\infty)R^+$. Set $\eta'_n := e_n - x^{\frac{p-1}{p^{n+1}}}e_{n+1}$. In the proof of (i) replace η_n by η'_n. By making straightforward modification of the proof (i), one can check easily that $\mathrm{pd}_{R^+}(\mathfrak{a}) \leq 1$. Let $\mathfrak{p}$ be a height one prime ideal of R such that $x \in \mathfrak{p}$ and let $S = R\backslash\mathfrak{p}$. By Lemma 2.4 vi), $S^{-1}R^+ \cong (S^{-1}R)^+$. If $\mathrm{pd}_{R^+}((x^\infty)R^+) = 1$ was not be the case, then

[2] Note that we proved $\mathrm{pd}_{R^+}((x^\infty)R^+) \leq 1$ by two different methods. The first one is an easy application of a result Auslander, see Lemma 4.3. The second actually proved a more stronger result: $(x^\infty)R^+$ has a free resolution of countably generated free R^+-modules of length bounded by one.

the ideal $(x^\infty)S^{-1}R^+$ of $S^{-1}R^+$ should be projective. Since by Lemma 4.1, $S^{-1}R^+$ is quasi-local, it turns out that $(x^\infty)S^{-1}R^+$ is free over $S^{-1}R^+$. As, we have seen above, over a domain an ideal is free if and only if it is principal. This observation along with Lemma 2.4(iii) shows that $(x^\infty)S^{-1}R^+$ is principal and radical. Now, by using Nakayama's Lemma and Lemma 4.2, we achieved at a contradiction. □

Now, we establish another preliminary lemma.

Lemma 4.5 *Let $(R, \mathfrak{m})$ be a quasi-local domain of residue prime characteristic p. Let $x_1, \dots, x_\ell$ be a finite sequence of nonzero and nonunit elements of R^+. In mixed characteristic case assume in addition that $x_1 = p$. Then $R^+/\sum_{i=1}^{\ell}(x_i^\infty)R^+$ has a bounded free resolution of countably generated free R^+-modules of length 2ℓ.*

Proof In view of Lemma 4.4, $R^+/(x_i^\infty)R^+$ has a bounded free resolution $\mathbf{Q}^i$ of countably generated free R^+-modules of length 2. By using induction on ℓ, we will show that $\bigotimes_{i=1}^{\ell}\mathbf{Q}^i$ is a bounded free resolution of $R^+/\sum_{i=1}^{\ell}(x_i^\infty)R^+$ consisting of countably generated free R^+-modules of length at most 2ℓ. By the induction hypothesis, $\mathbf{P} := \bigotimes_{i=2}^{\ell}\mathbf{Q}^i$ is a bounded free resolution of $R^+/\sum_{i=2}^{\ell}(x_i^\infty)R^+$ consisting of countably generated free R^+-modules of length at most $2\ell - 2$. In light of [25, Theorem 11.21], we see that

$$H_n(\mathbf{Q}^1 \otimes_{R^+} \mathbf{P}) \cong \operatorname{Tor}_n^{R^+}(R^+/\mathfrak{a}_1, R^+/\sum_{i=2}^{\ell}(x_i^\infty)R^+).$$

The ideal $(x_1^\infty)R^+$ is a directed union of free R^+-modules, and so it is a flat R^+-module. Hence, for each $n > 1$ we have $\operatorname{Tor}_n^{R^+}(R^+/(x_1^\infty)R^+, R^+/\sum_{i=2}^{\ell}(x_i^\infty)R^+) = 0$. Also,

$$\begin{aligned}&\operatorname{Tor}_1^{R^+}(R^+/(x_1^\infty)R^+, R^+/\sum_{i=2}^{\ell}(x_i^\infty)R^+)\\ &\quad\cong (x_1^\infty)R^+ \cap \sum_{i=2}^{\ell}(x_i^\infty)R^+/(x_1^\infty)R^+\sum_{i=2}^{\ell}(x_i^\infty)R^+,\end{aligned}$$

which is zero by [13, Propostion 2.11(3)]. On the other hand,

$$H_0(\mathbf{Q}^1 \otimes_{R^+} \mathbf{P}) \cong R^+/(x_1^\infty)R^+ \otimes_{R^+} R^+/\sum_{i=2}^{\ell}(x_i^\infty)R^+ \cong R^+/\sum_{i=1}^{\ell}(x_i^\infty)R^+.$$

This implies that $\bigotimes_{i=1}^{\ell}\mathbf{Q}^i = \mathbf{Q}^1 \otimes_{R^+} \mathbf{P}$ is a bounded free resolution of $R^+/\sum_{i=1}^{\ell}(x_i^\infty)R^+$ consisting of countably generated free R^+-modules of length at most 2ℓ, which is precisely what we wish to prove. □

We immediately exploit Lemma 4.5 to prove some parts of Theorem 1.1.

Theorem 4.6 *Let $(R, \mathfrak{m})$ be a Noetherian Henselian local domain of residue prime characteristic p.*

(i) *The R^+-module $R^+/\mathfrak{m}_{R^+}$ has a free resolution of countably generated free R^+-modules of length bounded by* $2\dim R$.
(ii) *If* $\dim R = 1$, *then* $\mathrm{pd}_{R^+}(R^+/\mathfrak{m}_{R^+}) = 2$.

Proof (i) Let $d := \dim R$ and let $\{x_1, \ldots, x_d\}$ be a system of parameters for R. In the mixed characteristic case, we may and do assume in addition that $x_1 = p$. By Lemma 4.5, $R^+/\sum_{i=1}^{d}(x_i^{\infty})R^+$ has a free resolution of countably generated free R^+-modules of length bounded by $2d$. It remains to recall from Corollary 2.5 that $\sum_{i=1}^{d}(x_i^{\infty})R^+ = \mathfrak{m}_{R^+}$.
(ii) This follows by part (i) and Corollary 2.5. □

Here, we give some more examples of ideals of R^+ of finite projective dimension.

Example 4.7 (i) Let $(R, \mathfrak{m})$ be a Noetherian complete local domain of prime characteristic. Let $\mathfrak{a}$ be a finitely generated ideal of R^+ with the property that $\mathrm{ht}\,\mathfrak{a} \geq \mu(\mathfrak{a})$, where $\mu(\mathfrak{a})$ is the minimal number of elements that needs to generate $\mathfrak{a}$. We show that $\mathrm{pd}_{R^+}(R^+/\mathfrak{a}) \leq \mu(\mathfrak{a}) \leq \dim R$. Indeed, let $\underline{a} := a_1, \ldots, a_n$ be a minimal generating set for $\mathfrak{a}$. Keep in mind that R^+ is a directed union of its subrings R' which are module-finite extensions of R. In view of Lemma 2.4(v), R' is a complete local domain. Let R' be one of them, which contains a_i for all $1 \leq i \leq n$. Set $\mathfrak{b} := (a_1, \ldots, a_n)R'$. Then $\mathfrak{b}R^+ = \mathfrak{a}$. Also, we have $n \leq \mathrm{ht}\,\mathfrak{a} \leq \mathrm{ht}\,\mathfrak{b} \leq \mu(\mathfrak{a}) \leq n$, because R^+ is an integral extension of a Noetherian ring R'. So $n := \mu(\mathfrak{b}) = \mathrm{ht}\,\mathfrak{b}$. In view of the equality $(R')^+ = R^+$, we can and do assume that $a_i \in R$ for all $1 \leq i \leq n$. This yields that $\underline{a}$ is a part of a system of parameter for R. By [15, Theorem 5.15], R^+ is a balanced big Cohen–Macaulay R-algebra. So $\underline{a}$ is a regular sequence on R^+. Therefore, the Koszul complex of R^+ with respect to $\underline{a}$ provides a projective resolution of $R^+/\mathfrak{a}$ of length $\mu(\mathfrak{a})$.
(ii) Let $\mathbb{F}$ be a field of prime characteristic. Consider the local ring $R := \mathbb{F}[[X^2, Y^2, XY]]$. Its maximal ideal is $\mathfrak{m} := (X^2, Y^2, XY)R$. We show that $\mathrm{pd}_{R^+}(R^+/\mathfrak{m}R^+) = 2$. Indeed, consider the complete regular local ring $A := \mathbb{F}[[X, Y]]$. One has $A^+ = R^+$. By [15, 6.7, Flatness], A^+ is flat over A. Therefore,

$$\mathrm{pd}_{R^+}(R^+/\mathfrak{m}R^+) = \mathrm{pd}_{A^+}(A^+/\mathfrak{m}A^+) \leq \mathrm{pd}_A(A/\mathfrak{m}A) \leq 2.$$

(iii) Let k be the algebraic closure of $\mathbb{Z}/p\mathbb{Z}$ and let $(R, \mathfrak{m}, k)$ be a 2-dimensional complete normal domain. Then [1, Remark 4.10] says that every two generated ideal of R^+ has finite projective dimension.

Here, we give a simple application of Lemma 2.4(vii).

Remark 4.8 Wodzicki has constructed an example of a ring A such that its Jacobson radical $\mathfrak{r}$ is nonzero and satisfies $\mathrm{Tor}_i^A(A/\mathfrak{r}, A/\mathfrak{r}) = 0$ for all $i \geq 1$. This class of rings are of interest due to Telescope Conjecture, see [19, p. 1234]. Now, assume that $(R, \mathfrak{m})$ is a one-dimensional local domain of prime characteristic. We show that R^+ belongs to such class of rings. Indeed, by Lemma 2.4(vii), we know that $\mathrm{Tor}_i^{R^+}(R^+/\mathfrak{m}_{R^+}, R^+/\mathfrak{m}_{R^+}) = 0$ for all $i \geq 2$ and

$$\mathrm{Tor}_1^{R^+}(R^+/\mathfrak{m}_{R^+}, R^+/\mathfrak{m}_{R^+}) = \mathfrak{m}_{R^+}/\mathfrak{m}_{R^+}^2.$$

To conclude, it remains recall from Lemma 4.2 that $\mathfrak{m}_{R^+} = \mathfrak{m}_{R^+}^2$.

5 Global dimension of R^+ in one special case

The aim of this section is to complete the proof of Theorem 1.1. We state our results in more general setting and include some of their converse parts.

Lemma 5.1 *Let $(R, \mathfrak{m})$ be a one-dimensional quasi-local domain and has residue prime characteristic. Then R^+ is $\aleph_0$-Noetherian.*

Proof By Lemma 4.1, R^+ is quasi-local and its maximal ideal $\mathfrak{m}_{R^+}$ is countably generated. By adopting the proof of Cohen's Theorem [20, Theorem 3.4], one can prove that if every prime ideal of a ring A can be generated by a countable set, then A is $\aleph_0$-Noetherian. This completes the proof. □

Lemma 5.2 *Let R be a Noetherian domain and of dimension greater than one. Then $\bigcap_{0\neq\mathfrak{p}\in\mathrm{Spec}\, R^+}\mathfrak{p} = 0$.*

Proof Let $\mathfrak{a}$ be the intersection of all height one prime ideals of R. Assume that $\mathfrak{a} \neq 0$. Then having [20, Theorem 31.2] in mind, we find that $\mathfrak{a}$ has infinitely many minimal prime ideals, which is impossible as R is Noetherian. Thus $\mathfrak{a} = 0$, and so $\bigcap_{0\neq\mathfrak{p}\in\mathrm{Spec}\, R}\mathfrak{p} = 0$. We now suppose that $\bigcap_{0\neq\mathfrak{p}\in\mathrm{Spec}\, R^+}\mathfrak{p} \neq 0$ and look for a contradiction. Let x be a nonzero element of $\bigcap_{0\neq\mathfrak{p}\in\mathrm{Spec}\, R^+}\mathfrak{p}$. Keep in mind that R^+ is a directed union of its subrings R' which are module-finite over R. Let R' be one of them which contains x. From the equality $(R')^+ = R^+$, we may and do assume that $x \in R$. Thus one can find a nonzero prime ideal $\mathfrak{q} \in \mathrm{Spec}\, R$ such that x does not belong to $\mathfrak{q}$. Since the ring extension $R \longrightarrow R^+$ is integral, by [20, Theorem 9.3], there exists a prime ideal $\mathfrak{p}$ of R^+ lying over $\mathfrak{q}$. By the choice of x, we have $x \in \mathfrak{p}$. In particular $x \in \mathfrak{q}$, a contradiction. □

Lemma 5.3 *Let $(R, \mathfrak{m})$ be a Noetherian complete local domain.*

(i) *If* $\dim R^+ = 1$*, then R^+ is a valuation domain with value group $\mathbb{Q}$.*
(ii) *If R^+ is a valuation domain, then* $\dim R^+ \leq 1$.

Proof (i) Keep in mind that R^+ is a directed union of its subrings R' which are module-finite over R. Let R' be one of them. In view of Lemma 2.4(v), R' is a complete local normal domain. Due to [20, Theorem 11.2], we know that R' is a discrete valuation domain. Note that $R^+ = R'^+$. Thus, R^+ is a directed union of discrete valuation domains. Since the directed union of valuation domains is again valuation domain, R^+ is a valuation domain. Now, we compute the value group of R^+. By Cohen's Structure Theorem, we can assume that $(R, \mathfrak{m}) =$

(V, tV) is a discrete valuation domain. By Lemma 2.4(i), there exists a value map as $v : R^+ \longrightarrow \mathbb{Q} \cup \{\infty\}$ such that $v(t) = 1$. Let $Q(R^+)$ be the field of fractions of R^+ and consider $R_v^+ := \{x \in Q(R^+) | v(x) \geq 0\}$. Clearly, $R^+ \subseteq R_v^+ \subsetneq Q(R^+)$. In light of [20, Exercise 10.5], that there does not exist any ring properly between R^+ and $Q(R^+)$. Thus $R^+ = R_v^+$. In particular, the value group of R^+ is $\mathbb{Q}$ (note that $v((t^{1/n})^m) = m/n$ for all $m/n \in \mathbb{Q}_{\geq 0}$).

(ii) Without loss of generality we can assume that $\dim R \neq 0$. As the ideals of valuation rings are linearly ordered, we have $\bigcap_{0 \neq \mathfrak{p} \in \operatorname{Spec} R^+} \mathfrak{p} \neq 0$. Then by applying Lemma 5.2, we get the claim. □

The preparation of Theorem 1.1 is finished. Now, we proceed to the proof of it. We repeat Theorem 1.1 to give its proof.

Theorem 5.4 *Let $(R, \mathfrak{m})$ be a Noetherian local Henselian domain has residue prime characteristic.*

(i) *The R^+-module $R^+/\mathfrak{m}_{R^+}$ has a free resolution of countably generated free R^+-modules of length bounded by* $2 \dim R$. *In particular,* $\operatorname{pd}_{R^+}(R^+/\mathfrak{m}_{R^+}) \leq 2 \dim R$.

(ii) *If R is one-dimensional and complete, then* $\operatorname{gldim} R^+ = \operatorname{pd}_{R^+}(R^+/\mathfrak{m}_{R^+}) = 2$.

Proof (i) This is in Theorem 4.6.

(ii) By Lemma 5.3(i), we know that R^+ is a valuation domain, and so any finitely generated ideal of R^+ is principal. This along with Lemma 2.3(i) shows that

$$\operatorname{wdim} R^+ = \sup\{\operatorname{fd}(R^+/\mathfrak{a}) : \mathfrak{a} \textit{ is a finitely generated ideal of } R^+\} = 1.$$

Also, Lemma 5.1 indicates that R^+ is $\aleph_0$-Noetherian. By applying this along with Lemma 2.2(iii), we find that $\operatorname{gldim} R^+ \leq 2$. The proof can now be completed by an appeal to Theorem 4.6(ii). □

Let $i : S \hookrightarrow R$ be a ring extension. Recall that i is called a cyclically pure extension if $\mathfrak{a}R \cap S = \mathfrak{a}$ for every ideal $\mathfrak{a}$ of S. We close this section by the following application of Lemma 5.3.

Proposition 5.5 *Let $(R, \mathfrak{m})$ be a one-dimensional Noetherian complete local domain which contains a field. Then, there exists a subring R' of R such that $R' \hookrightarrow R$ is not a cyclically pure extension.*

Proof By Cohen's Structure Theorem, there exists a complete regular local subring $(A, \mathfrak{m}_A)$ of R such that R is finitely generated over A and A contains a field $\mathbb{F}$. Any finitely generated torsion-free module over the principal ideal domain A is free. Thus R is free and R^+ is flat as modules over A. Assume that the claim holds for regular rings. Then there is a subring A' of A such that $A' \hookrightarrow A$ is not cyclically pure. It yields that $A' \hookrightarrow A \hookrightarrow R$ is not a cyclically pure extension. Then, without loss of generality we can assume that $A = R$ is regular. By $w : R \longrightarrow \mathbb{Z} \bigcup \{\infty\}$ we mean the value map of the discrete valuation ring R. Since R^+ is an integral extension of R, w can be extended to a valuation map on R^+. We denote it by $w : R^+ \longrightarrow \mathbb{Q} \bigcup \{\infty\}$.

Note that w is positive on $\mathfrak{m}_{R^+}$. Assume that t is an uniformising element of R, i.e. $\mathfrak{m} = tR$. Let $\sum$ be the class of all subrings R' of R such that $ut \notin R'$ for all $u \in R\backslash\mathfrak{m}$. It is clear that $w(u) = 0$ for all $u \in R\backslash\mathfrak{m}$. It turns out that $\mathbb{F} \in \sum$. So $\sum \neq \emptyset$. We can partially order $\sum$ by means of inclusion. Thus, $\sum$ is an inductive system. Let $\{R_\lambda\}_{\lambda\in\Lambda}$ be a nonempty totally ordered subset of $\sum$ and let R' be their union. In view of definition, $R' \in \sum$. By Zorn's Lemma, $\sum$ contains a maximal member R'. Next, we show that R' is our proposed ring. To this end, we show that $t^2, t^3 \in R'$. For this, we show that $R'[t^2], R'[t^3] \in \sum$. Suppose on the contrary that $R'[t^2] \notin \sum$. So $ut \in R'[t^2]$ for some $u \in R\backslash\mathfrak{m}$. Hence there exists a nonnegative integer n and $r_0, \ldots, r_n \in R'$ such that $ut = r_0 + r_1t^2 + \cdots + r_nt^{2n}$. This implies $r_0 = u't$, where either $u' := u - r_1t - \cdots - r_nt^{2n-1}$ in the case $n \neq 0$ or $u' := u$ in the case $n = 0$. A contradiction we search is that $u' \in R\backslash\mathfrak{m}$. Similarly, $t^3 \in R'$. Note that R^+ is a flat extension of R, and so cyclically pure. Now, consider the ideal $\mathfrak{a} := \{\alpha \in R^+ | w(\alpha) \geq 3\}$. In view of Lemma 5.3(i), either $\mathfrak{a} \subseteq t^2R^+$ or $t^2R^+ \subseteq \mathfrak{a}$ are necessarily true. The second possibility not be the case, since $t^2 \notin \mathfrak{a}$. Thus, $\mathfrak{a} \subseteq t^2R^+$. If the extension $R' \hookrightarrow R$, was not the case, then we should have $R' \hookrightarrow R^+$ is cyclically. Thus $t^2R^+ \cap R' = t^2R'$, and so $\mathfrak{a} \cap R' \subseteq t^2R'$. Hence $t^3 = rt^2$ for some $r \in R'$. Clearly, $r = t$. This is a contradiction by $R' \in \sum$. □

6 Application: almost zero modules

Let $(R, \mathfrak{m})$ be a Noetherian complete local domain. In view of Lemma 2.4(i), there is a valuation map $v : R^+ \longrightarrow \mathbb{Q} \bigcup \{\infty\}$ which is nonnegative on R^+ and positive on $\mathfrak{m}_{R^+}$. Fix such a valuation map and consider the following definition.

Definition 6.1 Let M be an R^+-module and let x be a nonzero element of $\mathfrak{m}_{R^+}$.

- (i) ([24, Definition 1.1]) M is called almost zero with respect to v, if for all $m \in M$ and for all $\epsilon > 0$ there is an element $a \in R^+$ with $v(a) < \epsilon$ such that $am = 0$. The notation $\mathfrak{T}_v$ stands for the class of almost zero R^+-modules with respect to v.
- (ii) M is called almost zero with respect to $\mathfrak{m}_{R^+}$, if $\mathfrak{m}_{R^+}M = 0$. We use the notation $\mathcal{GR}$ for the class of such modules, see [9, Definition 2.2].
- (iii) M is called almost zero with respect to x, if $x^{1/n}$ kills M for arbitrarily large n. $\mathfrak{T}_x$ stands for the class of almost zero modules with respect to x, see [8, Definition 1.1].

Classes of almost zero modules are (hereditary) torsion theories:

Definition 6.2 Let A be a ring and let $\mathfrak{T}$ be a subclass of A-modules.

- (i) The class $\mathfrak{T}$ is called a Serre class, if it is closed under taking submodules, quotients and extensions, i.e. for any exact sequence of A-modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$

the module $M \in \mathfrak{T}$ if and only if $M' \in \mathfrak{T}$ and $M'' \in \mathfrak{T}$.

(ii) A Serre class which is closed under taking the directed limit of any directed system of its objects is called a torsion theory.

Let $\mathfrak{T}$ be a torsion theory of A-modules. For an A-module M, let Σ be the family of all submodules of M, that belongs to $\mathfrak{T}$. We can partially order Σ by means of inclusion. This inductive system has an unique maximal element, call it $t(M)$. The assignment M to $t(M)$ provides a left exact functor. We denote it by $\mathfrak{T}$. We shall denote by $\mathbf{R}^i\mathfrak{T}$, the ith right derived functor of $\mathfrak{T}$. To make things easier, we first recall some notions. Consider the Gabriel filtration $\mathfrak{F} := \{\mathfrak{a} \trianglelefteq A : A/\mathfrak{a} \in \mathfrak{T}\}$. We define a partial order on $\mathfrak{F}$ by letting $\mathfrak{a} \leq \mathfrak{b}$, if $\mathfrak{a} \supseteq \mathfrak{b}$ for $\mathfrak{a}, \mathfrak{b} \in \mathfrak{F}$. Assume that E is an injective A-module. It is clear that $\mathbf{R}^i\mathfrak{T}(E)$ and $\varinjlim_{\mathfrak{a}\in\mathfrak{F}} \operatorname{Ext}^i_A(A/\mathfrak{a}, E)$ are zero for all $i > 0$. Thus, in view of the following natural isomorphism

$$\mathbf{R}^0\mathfrak{T}(M) = \{m \in M : \mathfrak{a}m = 0, \exists \mathfrak{a} \in \mathfrak{F}\} \cong \varinjlim_{\mathfrak{a}\in\mathfrak{F}} \operatorname{Hom}_A(A/\mathfrak{a}, M),$$

one can find that $\mathbf{R}^i\mathfrak{T}(-) \cong \varinjlim_{\mathfrak{a}\in\mathfrak{F}} \operatorname{Ext}^i_A(A/\mathfrak{a}, -)$. Let $\mathfrak{a}$ be an ideal of A. By $\mathfrak{T}_\mathfrak{a}$, we mean that $\{M : M_\mathfrak{p} = 0, \forall \mathfrak{p} \in \operatorname{Spec} A\setminus \operatorname{V}(\mathfrak{a})\}$. From the above computations, one can see that for Grothendieck's local cohomology modules with respect to $\mathfrak{a}$, there are isomorphisms

$$H^i_\mathfrak{a}(M) = \mathbf{R}^i\mathfrak{T}_\mathfrak{a}(M) \cong \varinjlim_n \operatorname{Ext}^i_A(A/\mathfrak{a}^n, M).$$

An important numerical invariant associated to $\mathfrak{T}$ is $cd(\mathfrak{T})$, the cohomological dimension of $\mathfrak{T}$, which is the largest integer i such that $\mathbf{R}^i\mathfrak{T} \neq 0$.

Proposition 6.3 *Let $(R, \mathfrak{m})$ be a Henselian local domain.*

(i) *If $(R, \mathfrak{m})$ has residue prime characteristic, then* $\operatorname{cd}(\mathcal{GR}) \leq 2\dim R$.
(ii) *If x is a nonzero element of $\mathfrak{m}_{R^+}$, then* $\operatorname{cd}(\mathfrak{T}_x) = 2$.
(iii) *Let $\mathfrak{F}_v := \{\mathfrak{a} \trianglelefteq R^+ : R^+/\mathfrak{a} \in \mathfrak{T}_v\}$. Then $\boldsymbol{R}^i\mathfrak{T}_v(-) \cong \varinjlim_{\mathfrak{a}\in\mathfrak{F}} H^i_\mathfrak{a}(-)$. In particular,*

$$\operatorname{cd}(\mathfrak{T}_v) \leq \sup\{\sup\{i \in \mathbb{N} \cup \{0\} : H^i_\mathfrak{a}(-) \neq 0\} | \mathfrak{a} \in \mathfrak{F}_v\}.$$

Proof (i) Consider the $\mathfrak{F}_{\mathcal{GR}} := \{\mathfrak{a} \trianglelefteq R^+ : R^+/\mathfrak{a} \in \mathcal{GR}\}$. Clearly, $\mathfrak{F}_{\mathcal{GR}} = \{\mathfrak{m}_{R^+}, R^+\}$. It turns out that

$$\mathbf{R}^i(\mathcal{GR})(-) \cong \varinjlim_{\mathfrak{a}\in\mathfrak{F}_{\mathcal{GR}}} \operatorname{Ext}^i_{R^+}(R^+/\mathfrak{a}, -) \cong \operatorname{Ext}^i_{R^+}(R^+/\mathfrak{m}_{R^+}, -).$$

So $\operatorname{cd}(\mathcal{GR}) = \operatorname{pd}_{R^+}(R^+/\mathfrak{m}_{R^+})$. The proof can now be completed by an appeal to Theorem 1.1.

(ii) Let M be an almost zero R^+-module with respect to x. In view of Definition 6.1(iii), there are sufficiently large integers ℓ such that $x^{1/\ell}M = 0$. Assume that $\ell \neq 1$. Then $x^{\frac{1}{\ell-1}}M = x^{\frac{1}{\ell(\ell-1)}}x^{\frac{1}{\ell}}M = 0$. Continuing inductively, $x^{1/n}M = 0$

for all $n \in \mathbb{N}$. Consider the Gabriel filtration $\mathfrak{F}_x := \{\mathfrak{a} \trianglelefteq R^+ : R^+/\mathfrak{a} \in \mathfrak{T}_x\}$. Observe that $\mathfrak{F}_x = \{\mathfrak{a} \trianglelefteq R^+ : (x^\infty) \subseteq \mathfrak{a}\}$. Recall that the partial order on $\mathfrak{F}_x$ is defined by letting $\mathfrak{a} \leq \mathfrak{b}$, if $\mathfrak{a} \supseteq \mathfrak{b}$ for $\mathfrak{a}, \mathfrak{b} \in \mathfrak{F}_x$. Thus $\{(x^\infty)\}$ is cofinal with $\mathfrak{F}_x$. Therefore,

$$\mathbf{R}^i\mathfrak{T}_x(-) \cong \varinjlim_{\mathfrak{a}\in\mathfrak{F}_x} \operatorname{Ext}^i_{R^+}(R^+/\mathfrak{a}, -) \cong \operatorname{Ext}^i_{R^+}(R^+/(x^\infty), -),$$

and so $\operatorname{cd}(\mathfrak{T}_x) = \operatorname{pd}_{R^+}(R^+/(x^\infty))$. This along with Lemma 4.4 completes the proof of (ii).

(iii) Let $\mathfrak{a} \in \mathfrak{F}_v$ and $n \in \mathbb{N}$. Assume that $\epsilon > 0$ is a rational number. There is an element a in $\mathfrak{a}$ with $v(a) < \epsilon/n$. Consequently, $v(a^n) < \epsilon$. Thus, $\mathfrak{a}^n$ has elements of small order, i.e. $\mathfrak{a}^n \in \mathfrak{F}_v$. Therefore,

$$\begin{aligned}\mathbf{R}^i\mathfrak{T}_v(M) &\cong \varinjlim_{\mathfrak{a}\in\mathfrak{F}_v} \operatorname{Ext}^i_{R^+}(R^+/\mathfrak{a}, M) \cong \varinjlim_{\mathfrak{a}\in\mathfrak{F}_v} (\varinjlim_{n\in\mathbb{N}} \operatorname{Ext}^i_{R^+}(R^+/\mathfrak{a}^n, M)) \\ &\cong \varinjlim_{\mathfrak{a}\in\mathfrak{F}_v} H^i_{\mathfrak{a}}(M),\end{aligned}$$

as claimed. □

Each valuation map may gives a different class of almost zero modules over R^+. However, in the case $\dim R = 1$, one has the following.

Proposition 6.4 *Let $(R, \mathfrak{m})$ be a one-dimensional Noetherian complete local domain. If $x \in \mathfrak{m}_{R^+}$ is nonzero, then $\mathfrak{T}_v = \mathcal{GR} = \mathfrak{T}_x$. In particular,* $\operatorname{cd}(\mathfrak{T}_v) = 2$.

Proof Clearly, $\mathcal{GR} \subseteq \mathfrak{T}_x \subseteq \mathfrak{T}_v$. Let $M \in \mathfrak{T}_v$ be nonzero and let $0 \neq m \in M$. We show that $\mathfrak{m}_{R^+} = (0 :_{R^+} m)$. This is trivial that $(0 :_{R^+} m) \subseteq \mathfrak{m}_{R^+}$. Assume that $x \in \mathfrak{m}_{R^+}$and observe that $R^+/(0 :_{R^+} m) \cong R^+m \in \mathfrak{T}_v$. By Lemma 5.3(i), either $(0 :_{R^+} m) \subseteq xR^+$ or $xR^+ \subseteq (0 :_{R^+} m)$. If $(0 :_{R^+} m) \subseteq xR^+$ was the case, then by the natural epimorphism $R^+/(0 :_{R^+} m) \longrightarrow R^+/xR^+$, we should have $R^+/xR^+ \in \mathfrak{T}_v$. This implies that xR^+ contains elements of sufficiently small order, contradicting the fact that $v(rx) = v(r) + v(x) \geq v(x)$ for all $r \in R^+$. So, $xR^+ \subseteq (0 :_{R^+} m)$. Thus, $\mathfrak{m}_{R^+} \subseteq (0 :_{R^+} m)$. Hence $\mathfrak{m}_{R^+}M = 0$, and so $M \in \mathcal{GR}$. □

Let A be a ring and $\mathfrak{T}$ a torsion theory of A-modules. Recall that any A-module M has a largest submodule $t(M)$ which belongs to $\mathfrak{T}$. The A-module M is called torsion-free with respect to $\mathfrak{T}$, if $t(M) = 0$. The collection of all torsion-free modules with respect to $\mathfrak{T}$ is denoted by $\mathcal{F}$. By $\mathfrak{T} - \operatorname{depth}_A(M) \geq n$ we mean that there is an injective resolution of M such that whose first n-terms are torsion-free with respect to $\mathfrak{T}$, see [6, Definition 1.1]. In light of [6, Proposition 1.5], we see that

$$\mathfrak{T} - \operatorname{depth}_A(M) = \inf\{i \in \mathbb{N} \cup \{0\} : \mathbf{R}^i\mathfrak{T}(M) \neq 0\}.$$

Here inf is formed in $\mathbb{Z} \cup \{\infty\}$ with the convention that $\inf \emptyset = \infty$.

Proposition 6.5 *Let $(R, \mathfrak{m})$ be a Noetherian Henselian local domain and let M be an R^+-module such that $\operatorname{Ext}^i_{R^+}(R^+/\mathfrak{m}_{R^+}, M) \neq 0$ for some i.*

(i) $\mathfrak{T}_v - \operatorname{depth}_{R^+}(M) \in \{0, 1, 2\}$.

(ii) *Let $\mathfrak{T}$ be any nonzero torsion theory of R^+-modules. If either R has prime characteristic or has mixed characteristic, then $\mathfrak{T} - \operatorname{depth}_{R^+}(M) \leq 2 \dim R$.*

Proof (i) Recall that $\mathcal{GR} \subseteq \mathfrak{T}_x \subseteq \mathfrak{T}_v$. This yields that

$$\mathfrak{T}_v - \operatorname{depth}_{R^+}(M) \leq \mathfrak{T}_x - \operatorname{depth}_{R^+}(M) \leq \mathcal{GR} - \operatorname{depth}_{R^+}(M).$$

By Proposition 6.3(ii), it is enough for us to prove that $\mathcal{GR}-\operatorname{depth}_{R^+}(M) < \infty$. But this holds, because $\operatorname{Ext}^i_{R^+}(R^+/\mathfrak{m}_{R^+}, M) \neq 0$ for some i.

(ii) Let $L \in \mathfrak{T}$ be a nonzero R^+-module and let ℓ be a nonzero element of L. So $R^+\ell \in \mathfrak{T}$. From the natural epimorphism $R^+\ell \cong R^+/(0 :_{R^+} \ell) \longrightarrow R^+/\mathfrak{m}_{R^+}$, we get that $R^+/\mathfrak{m}_{R^+} \in \mathfrak{T}$. It turns out that $\mathcal{GR}$ is minimal with respect to inclusion, among all nonzero torsion theories. At this point, (ii) becomes clear from the proof of (i). □

Lemma 6.6 *Let $(R, \mathfrak{m})$ be a Noetherian local domain. If $0 \neq \mathfrak{p} \in \operatorname{Spec} R^+$, then $R^+/\mathfrak{p} \in \mathfrak{T}_v$.*

Proof This is proved in [4]. But for the sake of completion, we present its short proof here. Let $x \in \mathfrak{p}$. For any positive integer n set $f_n(X) := X^n - x \in R^+[X]$. Let ζ_n be a root of f_n in R^+. It follows that $\zeta_n \in \mathfrak{p}$, since $(\zeta_n)^n = x \in \mathfrak{p}$. Keep in mind that v is positive on $\mathfrak{p}$. The equality $v(\zeta_n) = v(x)/n$ indicates that $\mathfrak{p}$ has elements of small order. Therefore, $R^+/\mathfrak{p} \in \mathfrak{T}_v$. □

The notation $\mathcal{F}_v$ stands for the class of torsion-free R^+-modules with respect to $\mathfrak{T}_v$. Let ϵ be a real number. Recall from Lemma 2.4(ii) that $\mathfrak{a}_\epsilon := \{x \in R^+ | v(x) > \epsilon\}$ is an ideal of R^+.

Lemma 6.7 *Let $(R, \mathfrak{m})$ be a Noetherian local domain.*

(i) *Let ϵ be a positive nonrational real number. Then $R^+/\mathfrak{a}_\epsilon \in \mathcal{F}_v$.*

(ii) *Assume that R is complete and has prime characteristic. If x is a nonzero element of R^+, then $R^+/xR^+ \in \mathcal{F}_v$.*

Proof (i) If $R^+/\mathfrak{a}_\epsilon \notin \mathcal{F}_v$, then there exists a nonzero element $x + \mathfrak{a}_\epsilon$ 'say in the torsion part of $R^+/\mathfrak{a}_\epsilon$ with respect to $\mathfrak{T}_v$. It follows that $v(x) \leq \epsilon$. As ϵ is nonrational, one has $v(x) < \epsilon$. By Definition 6.1(i), there exists an element a in R^+ such that $ax \in \mathfrak{a}_\epsilon$ and $v(a) < \epsilon - v(x)$. Thus, $\epsilon < v(ax) = v(a) + v(x) < \epsilon$. This is a contradiction.

(ii) The ring R^+ is a directed union of module-finite extensions R' of R. So $x \in R'$ for some of them. We can assume that R' is complete local and normal, see Lemma 2.4(v). Since $(R')^+ = R^+$, without loss of generality we can replace R by R'. Thus, we may and do assume that $x \in R$. Let $y + xR^+$ be a torsion element of R^+/xR^+ with respect to $\mathfrak{T}_v$. Without loss of generality, we can assume that $y \in R$. There are elements $a_n \in R^+$ of arbitrarily small order such that $a_n y \in xR^+$. [16, Theorem 3.1] asserts that $y \in (xR)^*$, the tight closure of xR. By [5, Corollary 10.2.7], $(xR)^* = \overline{xR}$, the integral closure of xR. Recall from [5, Proposition 10.2.3] that $xR = \overline{xR}$, since R is normal. So $y + xR^+ = 0$. Therefore, R^+/xR^+ is torsion-free with respect to $\mathfrak{T}_v$. □

Let A be a ring and let M be an A-module. Recall that a prime ideal $\mathfrak{p}$ is said to be associated to M if $\mathfrak{p} = (0 :_A m)$ for some $m \in M$. We denote the set of all associated prime ideals of M by $\operatorname{Ass}_A(M)$. The following result is an application of almost zero modules.

Proposition 6.8 *Let $(R, \mathfrak{m})$ be a Noetherian local domain which is not a field and let $\mathfrak{p}$ be a prime ideal of R^+.*

(i) *Let ϵ be a positive nonrational real number. Then $\mathfrak{a}_\epsilon$ is not finitely generated.*
(ii) *Adopt the above notation and assumptions. Then $\operatorname{Hom}_{R^+}(R^+/\mathfrak{p}, R^+/\mathfrak{a}_\epsilon) \neq 0$ if and only if $\mathfrak{p} = 0$. In particular, $\operatorname{Ass}_{R^+}(R^+/\mathfrak{a}_\epsilon) = \emptyset$.*
(iii) *Assume that R is complete and has prime characteristic and let $x \in R^+$ be a nonzero. Then $\operatorname{Hom}_{R^+}(R^+/\mathfrak{p}, R^+/xR^+) \neq 0$ if and only if $\mathfrak{p} = 0$. In particular, $\operatorname{Ass}_{R^+}(R^+/xR^+) = \emptyset$.*

Proof (i) Suppose on the contrary that $\mathfrak{a}_\epsilon$ has a finite generating set $\{x_1, \ldots, x_n\}$. Set $\alpha := \min\{v(x_i) | 1 \leq i \leq n\}$. Then $\epsilon < \alpha$. Let δ be a rational number, strictly between ϵ and α. As we saw in the proof of Lemma 5.3(i), there is an element a in R^+ such that $v(a) = \delta$. So $a \in \mathfrak{a}_\epsilon$. Thus, $a = \Sigma_{i=1}^n r_i x_i$ for some $r_i \in R^+$. It turns out that

$$\begin{aligned} v(a) &\geq \min\{v(r_i x_i) : 1 \leq i \leq n\} \\ &= \min\{v(r_i) + v(x_i) : 1 \leq i \leq n\} \\ &\geq \min\{v(x_i) : 1 \leq i \leq n\} \\ &> \delta. \end{aligned}$$

This contradiction shows that $\mathfrak{a}_\epsilon$ is not finitely generated.

(ii) Recall that the pair $(\mathfrak{T}_v, \mathcal{F}_v)$ is a maximal pair with having the property that $\operatorname{Hom}_{R^+}(T, F) = 0$ for all $T \in \mathfrak{T}_v$ and $F \in \mathcal{F}_v$. Incorporated this observation and Lemma 6.7 along with Lemma 6.7 to prove the first claim.
Now, we show that $\operatorname{Ass}_{R^+}(R^+/\mathfrak{a}_\epsilon) = \emptyset$. Suppose on the contrary that $\operatorname{Ass}_{R^+}(R^+/\mathfrak{a}_\epsilon)$ is nonempty and look for a contradiction. Let $\mathfrak{p} \in \operatorname{Ass}_{R^+}(R^+/\mathfrak{a}_\epsilon)$. Then $R^+/\mathfrak{p} \hookrightarrow R^+/\mathfrak{a}_\epsilon$. Due to the first claim we know that $\mathfrak{p} = 0$. On the other hand $\dim_{R^+}(R^+/\mathfrak{a}_\epsilon) < \dim_{R^+}(R^+)$. This provides a contradiction.

(iii) By Lemma 6.7(ii), the proof of the claim is a repetition of the proof of (ii).

□

We close this section by the following corollary.

Corollary 6.9 *Let $(R, \mathfrak{m})$ be a 1-dimensional complete local domain of prime characteristic. Then $\operatorname{Ass}_{R^+}(R^+/\mathfrak{a}) = \emptyset$ for any nonzero finitely generated ideal $\mathfrak{a}$ of R^+.*

Proof By Lemma 5.3 R^+ is a valuation domain. So its finitely generated ideals are principal. Therefore, the claim follows by Proposition 6.8. □

7 Perfect subrings of R^+

In this section we deal with a subring of R^+ with a lot of elements of small order. Let A be an integral domain with a valuation map $v : A \longrightarrow \mathbb{R} \bigcup \{\infty\}$ which is nonnegative on A. We say that an A-module M is almost zero with respect to v, if for all $m \in M$ and all $\epsilon > 0$, there is an element $a \in A$ with $v(a) < \epsilon$ such that $am = 0$. The notation $\mathfrak{T}_v^A$ stands for the class of almost zero A-modules with respect to v.

Lemma 7.1 *Adapt the above notation and assumptions. If $\mathfrak{T}_v^A \neq \emptyset$, then A is a non-Noetherian ring.*

Proof Let M be an almost zero A-module with respect to v. The existence of a such module provides a sequence $(a_n : n \in \mathbb{N})$ of elements of A such that

$$\cdots < v(a_{n+1}) < v(a_n) < \cdots < v(a_1) < 1.$$

Consider the following chain of ideals of A:

$$(a_1) \subsetneqq (a_1, a_2) \subsetneqq \cdots \subsetneqq (a_1, \ldots, a_n) \subsetneqq \cdots .$$

Note that, because v is nonnegative on A, one has $a_{n+1} \notin (a_1, \ldots, a_n)$ for all $n \in \mathbb{N}$. $\square$

In the following we recall the definition of R_∞.

Definition 7.2 Let $(R, \mathfrak{m}, k)$ be a Noetherian local domain.

(i) Assume that char $R = p$. By R_∞ we denote $\{x \in R^+ | x^{p^n} \in R$ for some $n \in \mathbb{N} \cup \{0\}\}$.

(ii) Assume that R is complete regular and has mixed characteristic p. First, consider the case $p \notin \mathfrak{m}^2$. Due to Cohen's Structure Theorem, we know that R is of the form $V[[x_2, \ldots, x_d]]$ for a discrete valuation ring V. By Faltings algebra, we mean that

$$R_\infty := \varinjlim_n V[p^{1/p^n}][[x_2^{1/p^n}, \ldots, x_d^{1/p^n}]].$$

Now, consider the case $p \in \mathfrak{m}^2$. Then Cohen's Structure Theorem gives a discrete valuation ring V and an element u of $\mathfrak{m} \backslash \mathfrak{m}^2$ such that R is of the form $V[[x_2, \ldots, x_d]]/(u)$. By R_∞ we mean that

$$R_\infty := \varinjlim_n V[p^{1/p^n}][[x_2^{1/p^n}, \ldots, x_d^{1/p^n}]]/(u).$$

(iii) Assume that R is complete regular and has equicharacteristic zero, i.e. char $R =$ char $k = 0$. Then by Cohen's Structure Theorem, R is of the form $k[[x_1, \ldots, x_d]]$. Take p be any prime number. By R_∞ we mean that $\varinjlim_n k[[x_1^{1/p^n}, \ldots, x_d^{1/p^n}]]$.

Theorem 7.10 is our main result in this section. To prove it, we need a couple of lemmas.

Lemma 7.3 *Let* $(R, \mathfrak{m})$ *be as Definition 7.2. Then* R_∞ *is* $\aleph_0$*-Noetherian.*

Proof First assume that char $R = 0$. Then by Definition 7.2, R_∞ is the directed union of Noetherian rings. In view of Lemma 2.1(iii), R_∞ is $\aleph_0$-Noetherian. Now assume that char $R = p$ For each positive integer n, set $R_n := \{x \in R_\infty | x^{p^n} \in R\}$. Thus $R_\infty = \bigcup R_n$. Let $\mathfrak{p}_n \in \operatorname{Spec} R_n$ and let $x_1, \ldots, x_\ell$ be a generating set for $\mathfrak{p}_0 := \mathfrak{p}_n \cap R$. Assume that $x \in \mathfrak{p}_n$. Hence, $x^{p^n} \in \mathfrak{p}_n \cap R = \mathfrak{p}_0$. So, there is $r_1, \ldots, r_\ell \in R$ such that $x^{p^n} = \sum_{1 \le i \le \ell} r_i x_i$. Let $\zeta_i \in R_n$ be a p^nth root of 1, i.e. $\zeta_i^{p^n} = 1$. By taking p^nth root, it yields that $x = \sum_{1 \le i \le \ell} \zeta_i r_i^{1/p^n} x_i^{1/p^n}$, where $r_i^{1/p^n} \in R_n$ and $\zeta_i \in R_n$ is a p^nth root of 1. It turns out that

$$\mathfrak{p}_n \subseteq (\zeta_1 x_1^{1/p^n}, \ldots, \zeta_\ell x_\ell^{1/p^n})R_n \subseteq (x_1^{1/p^n}, \ldots, x_\ell^{1/p^n})R_n \subseteq \mathfrak{p}_n,$$

and so

$$\mathfrak{p}_n = (x_1^{1/p^n}, \ldots, x_\ell^{1/p^n})R_n.$$

In particular R_n is Noetherian, because all of its prime ideals are finitely generated. In view of Lemma 2.1(iii), R_∞ is $\aleph_0$-Noetherian. □

Lemma 7.4 ([10, Theorem 2.3.3]) *Let* $\{R_\gamma : \gamma \in \Gamma\}$ *be a directed system of rings and let* $A = \varinjlim_{\gamma \in \Gamma} R_\gamma$. *Suppose that for* $\gamma \le \gamma'$, $R_{\gamma'}$ *is flat over* R_γ *and that* R_γ *is a coherent ring for all* $\gamma \in \Gamma$, *then* A *is a coherent ring.*

Remark 7.5 ([7, VI, Exercise 17]) Let $\{A_n\}$ be a directed system of rings with directed limit A. For each n let M_n and N_n be A_n-modules. Assume that $\{M_n\}$ and $\{N_n\}$ are directed systems. Then the respective directed limits M and N are A-modules, and for all i we may identify $\operatorname{Tor}_i^A(M, N)$ with the directed limit of the modules $\operatorname{Tor}_i^{A_n}(M_n, N_n)$.

Lemma 7.6 *The following assertions hold.*

(i) *Let* $\{(R_n, \mathfrak{m}_n) : n \in \mathbb{N}\}$ *be a directed chain of Noetherian local rings such that for each* $n < m$, R_m *is flat over* R_n *and let* $A = \bigcup R_n$. *Then* $\operatorname{gldim}(A) \le \sup\{\operatorname{gldim}(R_n) : n \in \mathbb{N}\} + 1$.
(ii) *Let* $(R, \mathfrak{m})$ *be a Noetherian regular local ring. Then* $\operatorname{gldim}(R_\infty) \le \dim R + 1$.

Proof (i) Without loss of generality we can assume that $d := \sup\{\operatorname{gldim}(R_n) : n \in \mathbb{N}\} < \infty$. First, we show that $\operatorname{pd}_A(A/\mathfrak{a}) \le d$ for all finitely generated ideals $\mathfrak{a}$ of A. Note that A is quasi-local with unique maximal ideal $\mathfrak{m}_A := \bigcup \mathfrak{m}_n$. In view of Lemma 7.4, A is coherent. Since $A/\mathfrak{a}$ is a finitely presented A-module, by Lemma 2.3(ii), $\operatorname{pd}_A(A/\mathfrak{a}) \le d$ if and only if $\operatorname{Tor}_{d+1}^A(A/\mathfrak{a}, A/\mathfrak{m}_A) = 0$. Set $\mathfrak{a}_i := \mathfrak{a} \cap R_i$, and recall that $\mathfrak{m}_i = \mathfrak{m}_A \cap R_i$. For any $j > d$ and for any i we observe that $\operatorname{Tor}_j^{R_i}(R_i/\mathfrak{a}_i, R_i/\mathfrak{m}_i) = 0$, because $j > d \ge \operatorname{gldim}(R_i)$.

A typical element of $\varinjlim_i R_i/\mathfrak{m}_i$ can be expressed by $[a_i + \mathfrak{m}_i]$. The assignment $[a_i + \mathfrak{m}_i] \mapsto [a_i] + \mathfrak{m} \in A/\mathfrak{m}$ provides a well-define map which is in fact a natural isomorphism between the rings $\varinjlim_i R_i/\mathfrak{m}_i$ and $A/\mathfrak{m}$. Similarly, $\varinjlim_i R_i/\mathfrak{a}_i \cong A/\mathfrak{a}$. In view of Remark 7.5, we see that

$$\operatorname{Tor}_j^A(A/\mathfrak{a}, A/\mathfrak{m}_A) \cong \varinjlim_i \operatorname{Tor}_j^{R_i}(R_i/\mathfrak{a}_i, R_i/\mathfrak{m}_i) = 0,$$

which shows that $\operatorname{pd}_A(A/\mathfrak{a}) \leq d$ for all finitely generated ideals $\mathfrak{a}$ of A.

Now, we show that $\operatorname{gldim}(A) \leq d + 1$. In view of Lemma 2.3(iii), we need to prove that $\operatorname{pd}_A(A/\mathfrak{b}) \leq d + 1$ for all ideals $\mathfrak{b}$ of A. Lemma 2.1(iii) asserts that A is $\aleph_0$-Noetherian. Therefore, Lemma 2.2(iii) along with Lemma 2.3(i) yield that

$$\begin{aligned}\operatorname{pd}_A(A/\mathfrak{b}) &\leq \operatorname{fd}_A(A/\mathfrak{b}) + 1\\ &\leq \sup\{\operatorname{fd}(A/\mathfrak{a}) : \mathfrak{a} \textit{ is a finitely generated ideal of } A\} + 1\\ &\leq \sup\{\operatorname{pd}(A/\mathfrak{a}) : \mathfrak{a} \textit{ is a finitely generated ideal of } A\} + 1\\ &\leq d + 1,\end{aligned}$$

which completes the proof.

(ii) First assume that $\operatorname{char} R = 0$. Then by Definition 7.2, R_∞ is the directed union of regular local rings each of them are free over the preceding. So the claim follows by (i). Now, assume that R is regular and $\operatorname{char} R = p$. Let $d := \dim R$. There exists a system of parameters $\{x_1, \ldots, x_d\}$ of R such that $\mathfrak{m} = (x_1, \ldots, x_d)$. For each positive integer n, set $R_n := \{x \in R_\infty | x^{p^n} \in R\}$. As we saw in the proof of Lemma 7.3 the ring R_n is Noetherian and local with the maximal ideal $(x_1^{1/p^n}, \ldots, x_d^{1/p^n})R_n$ and $\dim R_n = d$, because R_n is integral over R. Hence R_n is regular, since its maximal ideal can be generated by $\dim R_n$ elements. By using a result of Kunz [5, Corollary 8.2.8], one can find that R_m is flat over R_n for all $n < m$. Now, (i) completes the proof. □

From here on it will be assumed that $\operatorname{char} R = p$ and we shall seek to give results analogue to the results of previous sections over R_∞. Note that R_∞ is quasi-local. We denote its unique maximal ideal by $\mathfrak{m}_{R_\infty}$.

Lemma 7.7 *Let $(R, \mathfrak{m})$ be a Noetherian local domain of prime characteristic p. Let $x_1, \ldots, x_\ell$ be a finite sequence of nonzero and nonunit elements of R_∞.*

(i) *$R_\infty/\sum_{i=1}^{\ell}(x_i^\infty)R_\infty$ has a free resolution of countably generated free R_∞-modules of length bounded by 2ℓ.*
(ii) *$\operatorname{pd}_{R_\infty}(R_\infty/\sum_{i=1}^{\ell}(x_i^\infty)R_\infty) \leq \ell + 1$. In particular, $\operatorname{pd}_{R_\infty}(R_\infty/\mathfrak{m}_{R_\infty}) \leq \dim R + 1$.*

Proof (i) The proof of the first claim is a repetition of the proof of Theorem 4.6(i).
(ii) Recall from Lemma 2.4(vii) that $\operatorname{fd}_{R_\infty}(R_\infty/\sum_{i=1}^{\ell}(x_i^\infty)R_\infty) \leq \ell$. By Lemma 7.3, R_∞ is $\aleph_0$-Noetherian. Thus, Lemma 2.2(iii) shows that $\operatorname{pd}_{R_\infty}(R_\infty/$

$\sum_{i=1}^{\ell}(x_i^{\infty})R_{\infty}) \leq \ell + 1$. Let $d := \dim R$ and assume that $\{x_1, \dots, x_d\}$ is a system of parameters for R. To conclude the Lemma, it remains recall from Corollary 2.5 that $\mathfrak{m}_{R_{\infty}} = \sum_{i=1}^{d}(x_i^{\infty})R_{\infty}$. □

Now, we recall the following result of Sally and Vasconcelos.

Lemma 7.8 ([26, Corollary 1.5]) *Prime ideals in a 2-dimensional Noetherian local ring admit a bounded number of generators.*

Lemma 7.9 *Let* $(R, \mathfrak{m})$ *be a Noetherian local domain of prime characteristic* p. *Then any prime ideal* $\mathfrak{p}$ *of* R_{∞} *is of the form* $\sum_{i=1}^{\ell}(x_i^{\infty})$ *for some* $x_1, \dots, x_{\ell} \in R_{\infty}$. *Furthermore, if* $\dim R < 3$, ℓ *can be chosen such that it does not depend on the choice of* $\mathfrak{p}$.

Proof If $\mathfrak{p} = 0$, there is nothing to prove. So let $\mathfrak{p}$ be nonzero. Let $\underline{x} := x_1, \dots, x_{\ell}$ be a generating set for $\mathfrak{p}_0 := \mathfrak{p} \cap R$. When $\dim R < 3$ by Lemma 7.8, ℓ can be chosen such that it does not depend on the choice of $\mathfrak{p}$. Clearly, we have

$$\operatorname{rad}((x_1^{\infty}) + \cdots + (x_{\ell}^{\infty})) \subseteq \mathfrak{p}. \quad (*)$$

By Lemma 2.4(iii), it is enough to show that $\operatorname{rad}((x_1^{\infty}) + \cdots + (x_{\ell}^{\infty})) = \mathfrak{p}$. For each positive integer n, set $R_n := \{x \in R_{\infty} | x^{p^n} \in R\}$ and $\mathfrak{a}_n := \operatorname{rad}((x_1^{\infty}) + \cdots + (x_{\ell}^{\infty})) \cap R_n$. Recall that $R_{\infty} = \bigcup R_n$. It remains to show that $\mathfrak{p}_n := \mathfrak{p} \cap R_n \subseteq \mathfrak{a}_n$. In the proof of Lemma 7.3 we have seen that $\mathfrak{p}_n = (x_1^{1/p^n}, \dots, x_{\ell}^{1/p^n})R_n$. The ideal $\mathfrak{a}_n$ is radical, because it is a contraction of a radical ideal. It turns out that there exists a finite subset $\{\mathfrak{q}_1, \dots, \mathfrak{q}_{i_n}\}$ of prime spectrum of R_n such that $\mathfrak{a}_n = \bigcap_{j=1}^{i_n} \mathfrak{q}_j$. Without loss of generality we can assume that $\mathfrak{q}_1 \subseteq \mathfrak{p}_n$, because $\mathfrak{a}_n \subseteq \mathfrak{p}_n$, see (∗). One has

$$\mathfrak{p}_0 R_n \subseteq \mathfrak{q}_1 \subseteq \mathfrak{p}_n.$$

By Incomparability Theorem for the integral extension R_n/R, we get that $\mathfrak{q}_1 = \mathfrak{p}_n$. Let $1 \leq j \leq \ell$. Note that $x_j \in \mathfrak{q}_i$ for all i. Having $x_j^{1/p^n} \in R_n$ and $(x_j^{1/p^n})^{p^n} = x_j \in \mathfrak{q}_i$ in mind, we see that $x_j^{1/p^n} \in \mathfrak{q}_i$. Hence, $(x_1^{1/p^n}, \dots, x_{\ell}^{1/p^n})R_n \subseteq \mathfrak{q}_i$, and so $\mathfrak{p}_n \subseteq \mathfrak{q}_i$ for all i. Thus, the decomposition $\mathfrak{a}_n = \bigcap_{j=1}^{i_n} \mathfrak{q}_j$, yields that $\mathfrak{a}_n = \mathfrak{q}_1 = \mathfrak{p}_n$. This completes the proof. □

Now, we are ready to prove Theorem 1.2.

Theorem 7.10 *Let* $(R, \mathfrak{m})$ *be a Noetherian local domain of prime characteristic.*

- (i) *Any prime ideal of* R_{∞} *has a bounded free resolution of countably generated free* R_{∞}*-modules.*
- (ii) *If* $\dim R < 3$, *then* $\sup\{\operatorname{pd}_{R_{\infty}}(R_{\infty}/\mathfrak{p}) : \mathfrak{p} \in \operatorname{Spec}(R_{\infty})\} < \infty$.
- (iii) *If* R_{∞} *is coherent (this holds if R is regular), then* $\operatorname{gldim}(R_{\infty}) \leq \dim R + 1$.
- (iv) *If R is regular and of dimension one, then* $\operatorname{gldim}(R_{\infty}) = 2$.

Proof We prove (i) and (ii) at the same time. Let $\mathfrak{p}$ be a nonzero prime ideal of R_∞. By Lemma 7.9, there exists a finite sequence $x_1, \ldots, x_\ell$ of elements of R_∞ such that $\mathfrak{p} = \sum_{i=1}^{\ell}(x_i^{\infty})$. Also, when $\dim R < 3$, the integer ℓ can be chosen such that it does not depend on the choice of $\mathfrak{p}$. Putting this along with Lemma 7.7(i), we get that $\operatorname{pd}_{R_\infty}(R_\infty/\mathfrak{p}) \leq 2\ell$.

(iii) By Lemma 7.3 R_∞ is $\aleph_0$-Noetherian. Also, R_∞ is coherent. Incorporate these observations along with the proof of Lemma 7.6(i) to see that $\operatorname{gldim}(R_\infty) \leq \dim R+1$.

(iv) Due to the proof of Lemma 7.6(ii), we know that R_∞ is a directed union of discrete valuation domains, and so R_∞ is a $\aleph_0$-Noetherian valuation ring. By Lemma 7.1, R_∞ is not Noetherian. The only sticky point is to recall from Lemma 2.2(i) that $\operatorname{pd} \mathfrak{a} = n+1$ exactly if $\mathfrak{a}$ can be generated by $\aleph_n$ elements, where $\mathfrak{a}$ is an ideal of R_∞. □

Example 7.11 (i) Let $\mathbb{F}$ be a perfect field of characteristic 2. Consider the ring $R := \mathbb{F}[[x^2, x^3]]$. This ring is not regular. Note that $x \in R^+$ and $x^2 \in R$. Thus $x \in R_1 := \{r \in R^+ : r^2 \in R\}$. In particular, $R \subseteq \mathbb{F}[[x]] \subseteq R_1$, and so $R_\infty \subseteq (\mathbb{F}[[x]])_\infty \subseteq R_\infty$. Thus, in view of Theorem 7.10(iv), R_∞ is coherent and $\operatorname{gldim}(R_\infty) = 2 < \infty$.

(ii) Let A be a domain of finite global dimension on prime spectrum. If any quadratic polynomial with coefficient in A has a root in A, then A is of finite global dimension on radical ideals. Indeed, let $\mathfrak{a} := \bigcap_{i\in I} \mathfrak{p}$ be a radical ideal of A, where $\{\mathfrak{p}_i : i \in I\}$ is a family of prime ideals of A. By [15, Theorem 9.2], we know that $P := \sum_{i\in I} \mathfrak{p}_i$ is either a prime ideal or it is equal to A. Therefore, the desired claim follows from the following short exact sequence of A-modules

$$0 \longrightarrow A/\mathfrak{a} \longrightarrow \bigoplus_{i\in I} A/\mathfrak{p}_i \longrightarrow A/P \longrightarrow 0.$$

Definition 7.12 Let $(R, \mathfrak{m})$ be a Noetherian local domain of prime characteristic p and let M be an R_∞-module.

(i) Let $v_\infty : R_\infty \longrightarrow \mathbb{Q}\bigcup\{\infty\}$ be the restriction of $v : R^+ \longrightarrow \mathbb{Q}\bigcup\{\infty\}$ to R_∞. This map provides a torsion theory for R_∞. We denote it by $\mathfrak{T}_v^\infty$.

(ii) Let x be an element of R_∞. The R_∞-module M is called almost zero with respect to x, if $x^{1/p^n} M = 0$ for all $n \in \mathbb{N}$. The notion $\mathfrak{T}_x^\infty$ stands for the torsion theory of almost zero R_∞-modules with respect to x.

(iii) Let $\mathfrak{m}_{R_\infty}$ be the unique maximal ideal of R_∞. One has $\mathfrak{m}_{R_\infty}^2 = \mathfrak{m}_{R_\infty}$. The R_∞-module M is called almost zero with respect to $\mathfrak{m}_{R_\infty}$, if $\mathfrak{m}_{R_\infty} M = 0$. The notion $\mathcal{GR}^\infty$ stands for the torsion theory of almost zero R_∞-modules with respect to $\mathfrak{m}_{R^\infty}$.

As we mentioned before, any torsion theory determines a functor. In the next result we examine the cohomological dimension and cohomological depth of functors induced by torsion theories of Definition 7.12.

Corollary 7.13 *Let $(R, \mathfrak{m})$ be a Noetherian local domain of prime characteristic p and let x be a nonzero element of $\mathfrak{m}_{R_\infty}$.*

(i) $\mathrm{cd}(\mathfrak{T}_x^\infty) = 2$.
(ii) $\mathrm{cd}(\mathcal{GR}_\infty) \leq \dim R + 1$.
(iii) *If R is regular, then* $\mathrm{cd}(\mathfrak{T}_v^\infty) \leq \dim R + 1$.
(iv) *Let M be an R_∞-module with the property that* $\mathrm{Ext}^i_{R_\infty}(R_\infty/\mathfrak{m}_{R^\infty}, M) \neq 0$ *for some i. Then*

$$\inf\{i \in \mathbb{N}_0 : \boldsymbol{R}^i \mathfrak{T}_v^\infty(M) \neq 0\} \in \{0, 1, 2\}.$$

(v) *Let ϵ be a positive nonrational real number and let* $\mathfrak{a}_\epsilon^\infty := \{x \in R_\infty | v(x) > \epsilon\}$. *Then* $\mathrm{Ass}_{R_\infty}(R_\infty/\mathfrak{a}_\epsilon^\infty) = \emptyset$.

Proof (i) and (ii) are immediately follows by Lemma 7.7, because

$$\mathrm{cd}(\mathfrak{T}_x^\infty) = \mathrm{pd}_{R_\infty}(R_\infty/(x^\infty)R_\infty) = 2$$

and

$$\mathrm{cd}(\mathcal{GR}_\infty) = \mathrm{pd}_{R_\infty}(R_\infty/\mathfrak{m}_{R_\infty}).$$

(iii) is trivial by Theorem 1.2(iv).
(iv) follows by repeating the proof of Proposition 6.5.
(v) follows by repeating the proof of Proposition 6.8(i). □

8 Application: homological impossibilities over non-Noetherian rings

In this section we list some homological properties of Noetherian local rings which are not hold over general commutative rings. Our main tool for doing this is Theorem 1.1. Recall that a ring is called regular, if each of its finitely generated ideal has finite projective dimension. For example, valuation domains are regular. Over a Noetherian local ring $(R, \mathfrak{m})$, one can check that $\mathrm{id}_R(R/\mathfrak{m}) = \mathrm{pd}_R(R/\mathfrak{m})$. By the following, this not be the case in the context of non-Noetherian rings, even if the ring is coherent and regular.

Example 8.1 Let $(R, \mathfrak{m})$ be a Noetherian complete local domain of prime characteristic. Let E be the injective envelope of $R^+/\mathfrak{m}_{R^+}$.

(i) Let F be an R^+-module. Then F is flat if and only if the functor $- \otimes_{R^+} F$ is exact. This is the case if and only if the functor

$$\mathrm{Hom}_{R^+}(- \otimes_{R^+} F, E) \cong \mathrm{Hom}_{R^+}(-, \mathrm{Hom}_{R^+}(F, E))$$

is exact. Or equivalently, $\mathrm{Hom}_{R^+}(F, E)$ is an injective R^+-module.

(ii) Recall from Lemma 2.4(vii) that $R^+/\mathfrak{m}_{R^+}$ has a flat resolution $\mathbf{F}_\bullet$ of length bounded by $\dim R$. In view of (i) and by an application of the functor $\mathrm{Hom}_{R^+}(-, E)$ to $\mathbf{F}_\bullet$, we get to an injective resolution of $\mathrm{Hom}_{R^+}(R^+/\mathfrak{m}_{R^+}, E)$. Note that $R^+/\mathfrak{m}_{R^+}$ is a direct summand of $\mathrm{Hom}_{R^+}(R^+/\mathfrak{m}_{R^+}, E)$. So, $\mathrm{id}_{R^+}(R^+/\mathfrak{m}_{R^+}) \leq \mathrm{id}_{R^+}(\mathrm{Hom}_{R^+}(R^+/\mathfrak{m}_{R^+}, E)) \leq \dim R$.

(iii) Assume that $\dim R = 1$. We show that $\mathrm{id}_{R^+}(R^+/\mathfrak{m}_{R^+}) = 1$ and $\mathrm{pd}_{R^+}(R^+/\mathfrak{m}_{R^+}) = 2$. To see this, in light of Theorem 1.1 and parts (i) and (ii), it is enough to prove that $R^+/\mathfrak{m}_{R^+}$ is not an injective R^+-module. But, as $\mathrm{Hom}_{R^+}(R^+/\mathfrak{m}_{R^+}, E) \cong R^+/\mathfrak{m}_{R^+}$ is not a flat R^+-module, it turns out that $R^+/\mathfrak{m}_{R^+}$ is not an injective R^+-module.

The following gives some results on local cohomology modules in the context of non-Noetherian rings.

Example 8.2 (i) Recall that for a Noetherian ring A, Grothendieck's Vanishing Theorem [5, Theorem 3.5.7(a)] asserts that $H^i_{\mathfrak{a}}(-) := \lim_{\overrightarrow{n}} \mathrm{Ext}^i_A(A/\mathfrak{a}^n, -) = 0$ for all $i > \dim A$ and all ideals $\mathfrak{a}$ of A. Now, let $(R, \mathfrak{m})$ be a one-dimensional Noetherian complete local domain of prime characteristic. By Lemma 4.2, we observe that $\mathfrak{m}^n_{R^+} = \mathfrak{m}_{R^+}$ for all $n \in \mathbb{N}$. We incorporate this observation with Theorem 1.1, to see that

$$H^2_{\mathfrak{m}_{R^+}}(-) = \lim_{\overrightarrow{n}} \mathrm{Ext}^2_{R^+}(R^+/\mathfrak{m}^n_{R^+}, -) = \mathrm{Ext}^2_{R^+}(R^+/\mathfrak{m}_{R^+}, -) \neq 0.$$

Therefore, Grothendieck's Vanishing Theorem does not true for non-Noetherian rings, even if the ring is coherent and regular.

(ii) Let $(A, \mathfrak{m})$ be a Noetherian local ring and let M be a finitely generated A-module. Grothendieck's non Vanishing Theorem [5, Theorem 3.5.7(b)] asserts that $H^{\dim M}_{\mathfrak{m}}(M) \neq 0$. Let $(V, \mathfrak{m}_V)$ be a valuation domain with the value group $\mathbb{Z} \bigoplus \mathbb{Z}$ such that its maximal ideal $\mathfrak{m}_V = vV$ is principal. Such a ring exists, see [20, p. 79]. Recall from [27, Definition 2.3] that a sequence $\underline{x}$ is called weak proregular if $H^i_{(\underline{x})}(-) \cong H^i_{\underline{x}}(-)$ for all i, where $H^i_{\underline{x}}(M)$ denotes the ith cohomology module of *Čech* complex of M with respect to $\underline{x}$. Keep in mind that over integral domains any nonzero element x is weak proregular. Note that V is non-Noetherian, since its value group does not isomorphic with $\mathbb{Z}$. Also, recall that a ring is Noetherian if and only if each of its prime ideals are finitely generated. Thus $d := \dim V \geq 2$, and so $H^d_{\mathfrak{m}_V}(V) \cong H^d_{\underline{v}}(V) = 0$. Therefore, Grothendieck's non Vanishing Theorem does not true for non-Noetherian rings, even if the ring is coherent and regular.

(iii) Let $(R, \mathfrak{m})$ be a 1-dimensional Noetherian complete local domain of prime characteristic and let v be a non-zero element of $\mathfrak{m}_{R^+}$. Clearly, $H^2_{\underline{v}}(-) = 0$. Recall that $H^i_{(v)}(-) \cong H^i_{\underline{v}}(-)$ for all i. Thus $H^2_{(v)}(-) = 0$. Clearly, $\mathrm{rad}((v)R^+) = \mathfrak{m}_{R^+}$. By part (ii), $H^2_{\mathfrak{m}_{R^+}}(-) \neq 0$. Therefore, $H^i_{\mathfrak{a}}(-)$ and $H^i_{\mathrm{rad}(\mathfrak{a})}(-)$ are not necessarily isomorphic, where $\mathfrak{a}$ is an ideal of a general commutative ring.

Next, we exploit Example 8.2(ii) to show that the Intersection Theorem is not true for a general commutative ring, even if the ring is coherent and regular.

Example 8.3 Let $(R, \mathfrak{m})$ be a Noetherian local ring and $M, N \neq 0$ are finitely generated R-modules such that $M \otimes_R N$ has finite length. Recall from [5, Sect. 9.4] that the Intersection Theorem asserts that $\dim N \leq \mathrm{pd}\, M$. Now, let $(V, \mathfrak{m}_V)$ be the valuation domain of Example 8.2(ii). Then $\dim V \geq 2$. Now, consider the V-modules

$M := V/\mathfrak{m}_V$ and $N := V$. Clearly, $\ell(M \otimes_V N) = 1$. But $\dim N \geq 2$ and $\operatorname{pd} M = 1$. Thus Intersection Theorem is not true for coherent and regular rings.

9 Some questions

Theorem 1.1 gives no information when $\mathbb{Q} \subseteq R$. So, we ask the following:

Question 9.1 Let $(R, \mathfrak{m})$ be a Noetherian complete local domain of equicharacteristic zero. Is $\operatorname{pd}_{R^+}(R^+/\mathfrak{m}_{R^+}) < \infty$?

Theorem 1.2 gives some partial answers to:

Question 9.2 Let $(R, \mathfrak{m})$ be a Noetherian local domain of prime characteristic. Let $\mathfrak{a}$ be an ideal of R_∞. Under what conditions $\operatorname{pd}_{R_\infty}(R_\infty/\mathfrak{a}) < \infty$?

Proposition 6.4 gives no information when $\dim R \geq 2$. So, we ask the following:

Question 9.3 Let $(R, \mathfrak{m})$ be a Noetherian complete local domain of dimension greater than one. Is $\operatorname{cd}(\mathfrak{T}_v)$ bounded?

Let A be a ring and $\mathfrak{T}$ a torsion theory of A-modules. We say that an A-module M has finite almost (flat) projective dimension with respect to $\mathfrak{T}$ if there exists the following complex of (flat) projective R^+-modules

$$\mathbf{P}_\bullet : 0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0$$

such that $H_0(\mathbf{P}_\bullet) = M$ and $H_i(\mathbf{P}_\bullet) \in \mathfrak{T}$ for all $i > 0$. We say that a ring A is almost regular with respect to $\mathfrak{T}$ if any module has finite almost projective dimension with respect to $\mathfrak{T}$.

Example 9.4 Let $(R, \mathfrak{m})$ be a three-dimensional complete local domain of mixed characteristic. Recall that $H^i_{\underline{m}}(R^+)$ is the ith cohomology module of *Čech* complex of R^+ with respect to a generating set of $\mathfrak{m}$. By Lemma 2.4 v), $R^+ = \bigcup R'$, where the rings R' are Noetherian and normal. Any normal ring satisfies the Serre's condition $\mathbb{S}_2$. Hence, any length two system of parameters for R is a regular sequence for R', and so for R^+. Thus, the classical grade of $\underline{\mathfrak{m}}R^+$ on R^+ is greater than 1. Consequently, $H^1_{\underline{m}}(R^+) = H^0_{\underline{m}}(R^+) = 0$, because *Čech* grade is an upper bound for the classical grade, see e.g. [3, Proposition 2.3(i)]. Now, let x be a nonzero element of $\mathfrak{m}$. In light of [12, Theorem 0.2], we see that $x^{1/n}$ kills $H^2_{\underline{m}}(R^+)$ for arbitrarily large n. By inspection of *Čech* complex of R^+ with respect to $\underline{\mathfrak{m}}R^+$, we see that $H^3_{\underline{m}}(R^+)$ has finite almost flat dimension with respect to $\mathfrak{T}_x$.

Question 9.5 Let $(R, \mathfrak{m})$ be a Noetherian local domain. Let $\mathfrak{T}_v$ be as Definition 6.1(i) and $\mathfrak{T}^\infty_v$ be as Definition 7.12(i).

(i) Assume that R is complete. Is R^+ almost regular with respect to $\mathfrak{T}_v$?
(ii) Assume that R has prime characteristic. Is R_∞ almost regular with respect to $\mathfrak{T}^\infty_v$?

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