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Compactness of the complex Green operator on CR-manifolds of hypersurface type

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Abstract The purpose of this article is to study compactness of the complex Green operator on CR manifolds of hypersurface type. We introduce (CR-P_q), a potential theoretic condition on $(0, q)$-forms that generalizes Catlin's property (P_q) to CR manifolds of arbitrary codimension. We prove that if an embedded CR-manifold of hypersurface type of real dimension at least five satisfies (CR-P_q) and (CR-P_{n-1-q}), then the complex Green operator is a compact operator on the Sobolev spaces $H^s_{0,q}(M)$ and $H^s_{0,n-1-q}(M)$, if $1 \leq q \leq n-2$ and $s \geq 0$. We use CR-plurisubharmonic functions to build a microlocal norm that controls the totally real direction of the tangent bundle.

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1 Introduction and results

In this article, we introduce property (CR-P_q), a potential theoretic condition on $(0, q)$-forms. We show that if an embedded CR-manifold of hypersurface type satisfies (CR-P_q) and (CR-P_{n-1-q}), then the complex Green operator is a compact operator on the Sobolev spaces $H^s_{0,q}(M)$ and $H^s_{0,n-1-q}(M)$ if $1 \leq q \leq n-2$. We use CR-plurisubharmonic functions to build a microlocal norm that controls the "bad" direction of the tangent bundle. We first prove the closed range and compactness results on $L^2_{0,q}(M)$ and use an elliptic regularization argument to pass to higher Sobolev spaces.

A CR-manifold of hypersurface type M is the generalization to higher codimension of the boundary of a pseudoconvex domain. Let $\Omega \subset \mathbb{C}^N$ be a pseudoconvex domain and H be a holomorphic function on the closure of Ω. If h is the boundary value of

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H, then h satisfies the tangential Cauchy–Riemann equations $\bar{\partial}_b h = 0$. As with the Cauchy–Riemann operator, $\bar{\partial}_b$ gives rise to a complex that is a useful tool for analyzing the behavior of forms on and near the boundary. A CR-manifold of hypersurface type is a $(2n-1)$-dimensional manifold that is locally equivalent to a hypersurface in $\mathbb{C}^n$. The tangential Cauchy–Riemann operator $\bar{\partial}_b$ can again be thought of as the restriction of $\bar{\partial}$ to M.

The L^2-theory of $\bar{\partial}_b$ has been studied when M is a CR-manifold of hypersurface type. When M is the boundary of a pseudoconvex domain, it is by now classical that $\bar{\partial}_b$ has closed range [2,18,25]. More recent work by Nicoara [22] shows the same result holds when M a CR-manifold of hypersurface type. The approach to analyze $\bar{\partial}_b$-problems proceeds down one of two paths. One is to follow Shaw's approach and use $\bar{\partial}$-techniques and jump formulas, and the other path is to use Kohn's ideas and develop a microlocal analysis to control the totally real or "bad" direction of the tangent bundle. When M is not a hypersurface, microlocal analysis seems to be a more natural approach, and we will use this approach.

The method that we use to solve the $\bar{\partial}_b$-equation is to introduce the Kohn Laplacian $\Box_b = \bar{\partial}_b^*\bar{\partial}_b + \bar{\partial}_b\bar{\partial}_b^*$ and invert it. The inverse (modulo its null space) is called the complex Green operator and denoted G_q when it acts on $L^2_{0,q}(M)$, and the canonical solution to $\bar{\partial}_b u = f$ is given by $u = \bar{\partial}_b^* G_q f$ (assuming f satisfies the appropriate compatibility condition, e.g., $\bar{\partial}_b f = 0$ when $1 \leq q \leq n-2$). Closed range of $\bar{\partial}_b$ implies that G_q exists and is bounded on L^2, though geometric and potential theoretic properties of M can give G_q much stronger regularity properties. These additional regularity properties, however, have only been explored when $M = \mathrm{b}\Omega$ is the boundary of a pseudoconvex domain. In this case, subellipticity of G_q holds if and only if M satisfies a curvature condition called finite type (at the symmetric level q and $n-1-q$) [4,6,8,14,19,21,23]. Optimal subelliptic estimates (so called maximal estimates) were obtained in [13] under the additional condition that all eigenvalues of the Levi form are comparable. This work unifies earlier results for strictly pseudoconvex domains and for domains of finite type in $\mathbb{C}^2$. For general domains, it is known that if Ω admits a defining function that is plurisubharmonic at points of the boundary, then G_q preserves the Sobolev spaces $H^s(\mathrm{b}\Omega)$, $s \geq 0$ [3]. A defining function is called plurisubharmonic at the boundary when its complex Hessian at points of the boundary is positive semidefinite in all directions. For example, all convex domains admit such defining functions.

On a pseudoconvex domain $\Omega \subset \mathbb{C}^N$, the $\bar{\partial}$-Neumann operator is the inverse to the $\bar{\partial}$-Neumann Laplacian $\Box = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ on $L^2_{0,q}(\Omega)$. When $q = 1$, a necessary and sufficient condition for subellipticity of the $\bar{\partial}$-Neumann operator on Ω is the existence of a plurisubharmonic function whose complex Hessian blows up proportional to a reciprocal power of the distance to the boundary [4,6,27]. In [5], Catlin introduces a weakened version of complex Hessian blowup condition and instead requires only that there exist plurisubharmonic functions with arbitrarily large complex Hessians. He calls this condition property (P) and its natural generalization to $(0,q)$-forms, called (P_q), is now a well known sufficient condition for compactness of the $\bar{\partial}$-Neumann operator (see [11,28] for a discussion of compactness in the $\bar{\partial}$-Neumann problem). In [23], Emil Straube and I show that if $M = \mathrm{b}\Omega$ is the boundary of a smooth, bounded, pseudoconvex domain and satisfies (P_q) and (P_{n-1-q}), then G_q is a compact operator on $L^2_{0,q}(M)$. We also show that compactness of G_q implies compactness of the $\bar\partial$-Neumann operator on $(0,q)$-forms on Ω and if $b\Omega$ is locally convexifiable then (P_q) and (P_{n-1-q}) is equivalent to compactness of G_q (see [10] as well). Our methods involve $\bar\partial$-techniques, a jump formula in the spirit of Shaw (and Boas) [2,25], and a detailed study of compactness of the $\bar\partial$-Neumann operator on the annulus between two pseudoconvex domains. Applying $\bar\partial$-techniques to investigate the complex Green operator in the higher codimension case investigated in this article seems to be difficult if $q > 1$ because it is unknown if (P_q) is invariant under CR-equivalences (or even biholomorphisms that are not conformal mappings) if $q > 1$.

The goal of this article is to generalize the compactness result of [23] to the case when M is a CR-manifold of hypersurface type. We introduce property (CR-P_q), a generalization of (P_q) for CR-manifolds of hypersurface type, and show that it is a sufficient condition for compactness of the complex Green operator.

Let

$$\mathcal{H}^q_{\tau p} = \{\varphi \in L^2_{0,q}(M) \cap \mathrm{Dom}(\bar\partial_b) \cap \mathrm{Dom}(\bar\partial_b^*) : \bar\partial_b\varphi = 0, \bar\partial_b^*\varphi = 0\}$$

be the space of harmonic forms and

$${}^\perp\mathcal{H}^q_{\tau p} = \{\varphi \in L^2_{0,q}(M) : (\varphi, \phi)_0 = 0, \text{ for all } \phi \in \mathcal{H}^q_{\tau p}\}.$$

Our main result is the following theorem.

Theorem 1.1 *Let $M \subset \mathbb{C}^N$ be a smooth, compact, orientable weakly pseudoconvex CR-manifold of hypersurface type of real dimension $(2n-1)$ that satisfies (CR-P_q) and (CR-P_{n-1-q}). If $1 \le q \le n-2$ and $s \ge 0$, then*

- (i) *$\bar\partial_b$ and $\bar\partial_b^*$ acting on $H^s_{0,q}(M)$ have closed range,*
- (ii) *the complex Green operator G_q exists and is a compact operator on $H^s_{0,q}(M)$,*
- (iii) *$\mathcal{H}^q_{\tau p}$ is finite dimensional.*

The assumption that $1 \le q \le n-2$ excludes the endpoints $q = 0$ and $q = n-1$. For the endpoint case, it is not clear what (CR-P_0) should be. However, one can check (in analogy to the $\bar\partial$-Neumann operator) that $G_0 = \bar\partial_b^* G_1^2 \bar\partial_b = \bar\partial_b^* G_1 (\bar\partial_b^* G_1)^*$, and thus it follows that (CR-P_1) is a sufficient condition for compactness of G_0 (and G_{n-1} as well). The requirement that the dimension of M is at least five is a seemingly technical assumption concerning the eigenvalues of a Hermitian matrix. In particular, if $H = (h_{jk})_{j,k=1}^{n-1}$ is a Hermitian, positive definite matrix, then $(\delta_{jk}\sum_{\ell=1}^{n-1} h_{\ell\ell} - h_{jk})_{j,k=1}^{n-1}$ is a Hermitian, positive definite matrix if $n \ge 3$. This fact is false when $n = 2$, and this causes the three dimensional case to remain open.

The symmetric requirements at levels q and $n-1-q$ are necessary [14,17,23]. To a $(0,q)$-form u on $b\Omega$, there is an associated $(0,n-1-q)$-form $\tilde u$ (obtained through a modified Hodge-$*$ construction) such that $\|u\| \approx \|\tilde u\|$, $\bar\partial_b \tilde u = (-1)^q \widetilde{(\bar\partial_b^* u)}$, and $\bar\partial_b^* \tilde u = (-1)^{q+1}\widetilde{(\bar\partial_b u)}$, modulo terms that are $O(\|u\|)$. Consequently, a compactness estimate holds for $(0,q)$-forms if and only if the corresponding estimate holds for $(0, n-1-q)$-forms. In view of the characterization of compactness on convex domains [10], such a symmetry between form levels is absent in the $\bar\partial$-Neumann problem. (The analogous construction performed for forms on Ω yields a form $\tilde u$ that in general is not in the domain of $\bar\partial^*$.)

A consequence of Theorem 1.1 and Corollary 3.3 is the following generalization of Theorem 1.4 in [23].

Corollary 1.2 *Let $M^{2n-1} \subset \mathbb{C}^N$ be a smooth, compact, orientable weakly pseudoconvex CR-manifold of hypersurface type that satisfies (P_q). Then M satisfies $(CR\text{-}P_q)$. In particular, if M satisfies (P_q) and (P_{n-1-q}) and is of real dimension at least five, then the conclusions of Theorem 1.1 hold.*

2 Definitions and notation

2.1 CR-manifolds and the tangential Cauchy–Riemann operator $\bar\partial_b$

Definition 2.1 Let $M \subset \mathbb{C}^N$ be a smooth manifold of real dimension $2n-1$. The *CR-structure on M* is given by a complex subbundle $T^{1,0}(M)$ of the complexified tangent bundle $T(M) \otimes \mathbb{C}$ that satisfies the following conditions:

(i) The complex dimension of each fiber of $T^{1,0}(M)$ is $n-1$ for all $p \in M$;
(ii) If we define $T^{0,1}(M) = \overline{T^{1,0}(M)}$, then $T^{1,0}(M) \cap T^{0,1}(M) = \{0\}$;
(iii) If $L, L' \in T^{1,0}(M)$ are two vector fields defined near M, then their commutator $[L, L'] = LL' - L'L$ is also an element of $T^{1,0}(M)$.

A manifold M endowed with a CR-structure is called a *CR-manifold*.

Since M is a submanifold of $\mathbb{C}^N$, we can take $T^{1,0}_z(M) = T^{1,0}_z(\mathbb{C}^N) \cap T_z(M) \otimes \mathbb{C}$ (under the natural inclusions). If the complex dimension of $T^{1,0}_z(M)$ is $n-1$ for all $z \in M$, we can then let $T^{1,0}(M) = \bigcup_{z\in M} T^{1,0}_z(M)$, and this defines the induced CR-structure on M. Observe that conditions (ii) and (iii) are automatically satisfied in this case.

For the remainder of this article, M is a smooth, orientable CR-manifold of real dimension $2n-1$ embedded $\mathbb{C}^N$ for some $N \geq n$. Let $B^q(M) = \bigwedge^q(T^{0,1}(M)^*)$ (the bundle of $(0, q)$ forms that consists of skew-symmetric multilinear maps of $T^{0,1}(M)^q$ into $\mathbb{C}$). We can choose our Riemannian metric to be the restriction on $T(M) \otimes \mathbb{C}$ of the usual Hermitian inner product on $\mathbb{C}^N$. We can define a Hermitian inner product on $B^q(M)$ by

$$(\varphi, \psi) = \int_M \langle \varphi, \psi \rangle_x \, dV,$$

where dV is the volume element on M and $\langle \varphi, \psi \rangle_x$ is the induced inner product on $B^q(M)$. This metric is compatible with the induced CR-structure, i.e., the vector spaces $T^{1,0}_z(M)$ and $T^{0,1}_z(M)$ are orthogonal under the inner product.

The involution condition (iii) of Definition 2.1 means that there is a restriction of the de Rham exterior derivative d to $B^q(M)$, which we denote by $\bar\partial_b$. The inner product gives rise to an L^2-norm $\|\cdot\|_0$, and we also denote the closure of $\bar\partial_b$ in this norm by $\bar\partial_b$ (by an abuse of notation). In this way, $\bar\partial_b : L^2_{0,q}(M) \to L^2_{0,q+1}(M)$ is a well-defined, closed, densely defined operator, and we define $\bar\partial_b^* : L^2_{0,q+1}(M) \to L^2_{0,q}(M)$ to be the L^2-adjoint of $\bar\partial_b$. The Kohn Laplacian $\Box_b : L^2_{0,q}(M) \to L^2_{0,q}(M)$ is defined as

$$\Box_b = \bar\partial_b^*\bar\partial_b + \bar\partial_b\bar\partial_b^*,$$

and its inverse on $(0, q)$-forms (up to $(\Box_b)$) is called the complex Green operator and denoted by G_q.

The induced CR-structure $T^{1,0}(M)$ has a local basis $L_1, \dots, L_{n-1}$ in a neighborhood U of each point $x \in M$. Let $\omega_1, \dots, \omega_{n-1}$ be the dual basis of $(1, 0)$-forms that satisfy $\langle \omega_j, L_k\rangle = \delta_{jk}$. Then $\bar L_1, \dots, \bar L_{n-1}$ is a local basis for the $(0, 1)$-vector fields with dual basis $\bar\omega_1, \dots, \bar\omega_{n-1}$ in U. Also, $T(U)$ is spanned by $L_1, \dots, L_{n-1}$, $\bar L_1, \dots, \bar L_{n-1}$ and one more vector T taken to be purely imaginary (so $\bar T = -T$). Let γ be the purely imaginary global 1-form on M that annihilates $T^{1,0}(M) \oplus T^{0,1}(M)$ and is normalized so that $\langle \gamma, T\rangle = -1$.

Definition 2.2 The *Levi form at a point* $x \in M$ is the Hermitian form given by $\langle d\gamma_x, L \wedge \bar L'\rangle$ where $L, L' \in T^{1,0}_x(U)$, U a neighborhood of $x \in M$. We call M *weakly pseudoconvex* if the Levi form is positive semi-definite for all $x \in M$ and *strictly pseudoconvex* if there is a form γ such that the Levi form is positive definite at all $x \in M$.

2.2 Property (CR-P_q) and CR-plurisubharmonic functions

Definition 2.3 A smooth function $\varphi : \Omega \to \mathbb{C}$ is called *plurisubharmonic on* $(0, q)$-*forms* if the sum of any q eigenvalues of the complex Hessian of φ at $z \in \Omega$ is at least $C \ge 0$. The constant C is the *constant of plurisubharmonicity* (of φ at z).

Definition 2.4 A surface $S \subset \mathbb{R}^k$ satisfies *property* (P_q) if for every $C > 0$, there exists a function φ and a neighborhood $U \supset S$ so that $0 \le \varphi \le 1$ and φ is plurisubharmonic on $(0, q)$-forms on U with plurisubharmonicity constant C.

As discussed above, property (P_q) has played a crucial role in the development of the compactness theory for the $\bar\partial$-Neumann operator and now we define its analog for the compactness theory of the complex Green operator on CR-manifolds of hypersurface type.

Definition 2.5 Let M be a CR-manifold. A real-valued $\mathcal{C}^\infty_c$ function λ defined in a neighborhood of M is called *strictly CR-plurisubharmonic on* $(0, q)$*-forms* if there exist constants $A_0, A_\lambda > 0$ so that for any orthonormal $Z_j \in T^{1,0}(M)$, $1 \le j \le q$,

$$\sum_{j=1}^{q} \left\langle \frac{1}{2}\left(\partial_b\bar\partial_b\lambda - \bar\partial_b\partial_b\lambda\right) + A_0 d\gamma, Z_j \wedge \bar Z_j \right\rangle \ge A_\lambda$$

where $d\gamma$ is the invariant expression of the Levi form. λ is called *weakly CR-plurisubharmonic on* $(0, q)$*-forms* if $A_\lambda \geq 0$. A_λ is called the *CR-plurisubharmonicity constant*.

CR-plurisubharmonic functions were used by Nicoara [22] to prove closed range of $\bar\partial_b$ on CR manifolds of hypersurface type.

Definition 2.6 A surface $S \subset \mathbb{R}^k$ satisfies *property* (*CR-P_q*) if for every $A > 0$, there exists a function λ and a neighborhood $U \supset S$ so that $0 \leq \lambda \leq 1$ and λ is CR-plurisubharmonic on $(0, q)$-forms on U with CR-plurisubharmonicity constant A.

Appendix A contains results from multilinear algebra that help to explain the relationship of (P_q) and (CR-P_q).

In this article, constants with no subscripts may depend on n, N, M but not the CR-plurisubharmonic functions λ^+, λ^-, or any quantities associated with λ^+ or λ^-. Those constants will be denoted with an λ^+, λ^-, or $\pm$ in the subscript. The constant A will be reserved the constant in the construction of pseudodifferential operators in Section 3 (though A with subscripts will not).

3 Computations in local coordinates

3.1 Local coordinates and CR-plurisubharmonicity

The microlocal analysis that we will use relies the existence of suitable local coordinates. The first such result is Lemma 3.2 from [22], recorded here as the following result.

Lemma 3.1 *Let M be a compact smooth, $(2n-1)$-dimensional weakly pseudoconvex CR-manifold of hypersurface type embedded in a complex space $\mathbb{C}^N$ such that $N \geq n$ and endowed with an induced CR-structure. For each point $P \in M$, there exists a neighborhood U so that $M \cap U$ is CR-equivalent to a hypersurface in $\mathbb{C}^n$. Additionally, on U there is a local orthonormal basis $L_1, \dots, L_n, \bar L_1, \dots, \bar L_n$ of the n-dimensional complex bundle containing TM so that*:

(i) $L_j\big|_P = \frac{\partial}{\partial w_j}$ *for* $1 \leq j \leq n$ *where* $(w_1, \dots, w_N)$ *are the coordinates of* $\mathbb{C}^N$*, and*

(ii) $[L_j, \bar L_k]\big|_P = c_{jk} T$ *where* $T = L_n - \bar L_n$ *and* c_{jk} *are the coefficients of the Levi form in* $L_1, \dots, L_{n-1}, \bar L_1, \dots, \bar L_{n-1}, T$*, a local orthonormal basis for* TM.

The local coordinates from Lemma 3.1 allow us to make a careful comparison of the Levi form with its $\bar\partial_b$-analog.

Proposition 3.2 *Let M be as in Lemma 3.1. If λ is a smooth function near M, $L \in T^{1,0}(M)$, and $\nu = L_n + \bar L_n$ is the "real normal" to M, then on M,*

$$\left\langle \frac{1}{2}\left(\partial\bar\partial\lambda - \bar\partial\partial\lambda\right), L \wedge \bar L \right\rangle - \left\langle \frac{1}{2}\left(\partial_b\bar\partial_b\lambda - \bar\partial_b\partial_b\lambda\right), L \wedge \bar L \right\rangle = \frac{1}{2}\nu\{\lambda\}\langle d\gamma, L \wedge \bar L\rangle.$$

Proof Using Lemma 3.1, there exists an orthonormal basis of $\mathbb{C}T(\mathbb{C}^N)$ given by $L_1,\dots,L_N, \bar{L}_1,\dots,\bar{L}_N$ so that $L_1,\dots,L_{n-1}$ and $\bar{L}_1,\dots,\bar{L}_{n-1}$ are orthonormal bases of $T^{1,0}(M)$ and $T^{0,1}(M)$, respectively, $T = L_n - \bar{L}_n \in TM$ is a purely imaginary tangent vector, and $\nu = L_n + \bar{L}_n$ is the "real normal" tangent vector to M. Let $\omega_1,\dots,\omega_N,\bar{\omega}_1,\dots,\bar{\omega}_N$ be the dual cotangent vectors to $L_1,\dots,L_N,\bar{L}_1,\dots,\bar{L}_N$, respectively. Assume that the coordinates are centered around $P \in M$ in sense of Lemma 3.1.

Recall that $\partial\bar\partial = -\bar\partial\partial$, so $\partial\bar\partial = \frac{1}{2}(\partial\bar\partial - \bar\partial\partial)$. We now compute

$$\partial\bar\partial\lambda = \partial\left(\sum_{k=1}^N \bar{L}_k\lambda\,\bar{\omega}_k\right) = \sum_{j,k=1}^N L_j\bar{L}_k\lambda\,\omega_j\wedge\bar{\omega}_k + \sum_{\ell=1}^N \bar{L}_\ell\lambda\,\partial\bar{\omega}_\ell. \tag{1}$$

Also,

$$\bar\partial\partial\lambda = \bar\partial\left(\sum_{j=1}^N L_j\lambda\,\omega_j\right) = -\sum_{j,k=1}^N \bar{L}_k L_j\lambda\,\omega_j\wedge\bar{\omega}_k + \sum_{\ell=1}^N L_\ell\lambda\,\bar\partial\omega_\ell. \tag{2}$$

Let $L = \sum_{j=1}^{n-1}\xi_j L_j$ be a complex tangent vector on M. Then

$$\begin{aligned}\langle\bar\partial\omega_\ell, L_j\wedge\bar{L}_k\rangle\big|_P &= L_j\{\langle\omega_\ell,\bar{L}_k\rangle\}\big|_P - \bar{L}_j\{\langle\omega_\ell, L_k\rangle\}\big|_P - \langle\omega_\ell,[L_j,\bar{L}_k]\rangle\big|_P\\ &= -\langle(\omega_\ell, c_{jk}T\rangle\big|_P = -\delta_{\ell n}c_{jk}(P).\end{aligned}$$

Similarly, since $T = L_n - \bar{L}_n$,

$$\begin{aligned}\langle\partial\bar\omega_\ell, L_j\wedge\bar{L}_k\rangle\big|_P &= L_j\{\langle\bar\omega_\ell,\bar{L}_k\rangle\}\big|_P - \bar{L}_j\{\langle\bar\omega_\ell, L_k\rangle\}\big|_P - \langle\bar\omega_\ell,[L_j,\bar{L}_k]\rangle\big|_P\\ &= -\langle\bar\omega_\ell, c_{jk}T\rangle\big|_P = \delta_{\ell n}c_{jk}(P).\end{aligned}$$

Consequently, for $1\le j,k\le n-1$,

$$\left\langle\sum_{\ell=1}^N\left(\bar{L}_\ell\lambda\,\partial\bar\omega_\ell - L_\ell\lambda\,\partial\bar\omega_\ell\right), L_j\wedge\bar{L}_k\right\rangle\Bigg|_P = (\bar{L}_n\{\lambda\}+L_n\{\lambda\})c_{jk}(P) = c_{jk}(P)\nu(\lambda)\big|_P.$$

If $K = \sum_{j=1}^N\xi_j L_j + \sum_{k=1}^N\zeta_k\bar{L}_k$, then

$$\langle\omega_j\wedge\bar\omega_k, K\wedge\bar{K}\rangle = \omega_j(K)\bar\omega_k(\bar{K}) - \omega_j(\bar{K})\bar\omega_k(K) = \xi_j\bar\xi_k - \bar\zeta_j\zeta_k.$$

We now perform the same calculations with $\bar\partial_b$. We have

$$\partial_b\bar\partial_b\lambda = \partial_b\left(\sum_{k=1}^n \bar L_k\lambda\,\bar\omega_k\right) = \sum_{j,k=1}^n L_j\bar L_k\lambda\,\omega_j\wedge\bar\omega_k + \sum_{\ell=1}^n \bar L_\ell\lambda\,\partial_b\bar\omega_\ell,$$
$$\bar\partial_b\partial_b\lambda = \bar\partial_b\left(\sum_{j=1}^n L_j\lambda\,\omega_j\right) = -\sum_{j,k=1}^n \bar L_k L_j\lambda\,\omega_j\wedge\bar\omega_k + \sum_{\ell=1}^n L_\ell\lambda\,\bar\partial_b\omega_\ell.$$

By Boggess [1, p. 132], if d_b is the exterior derivative on M and ϕ is a 1-form, then

$$\langle d_b\phi, L_j\wedge\bar L_k\rangle = L_j\{\langle\phi,\bar L_k\rangle\} - \bar L_k\{\langle\phi, L_j\rangle\} - \langle\phi,[L_j,\bar L_k]\rangle.$$

Since $\bar\partial_b$ on $(0,q)$-forms is the projection of d_b onto the space of $(0,q+1)$-forms, if ϕ is a $(1,0)$-form, then the fact that d_b maps (1,0)-form to the sum of a $(2,0)$-form and a $(1,1)$-form means that $\langle d_b\omega_\ell, L_j\wedge\bar L_k\rangle = \langle\bar\partial_b\omega_\ell, L_j\wedge\bar L_k\rangle$. Thus, for $1\le\ell\le n$,

$$\langle\bar\partial_b\omega_\ell, L_j\wedge\bar L_k\rangle\big|_P = -\langle\omega_\ell,[L_j,\bar L_k]\rangle\big|_P = 0.$$

Similarly,

$$\langle\partial_b\bar\omega_\ell, L_j\wedge\bar L_k\rangle\big|_P = -\langle\bar\omega_\ell,[L_j,\bar L_k]\rangle\big|_P = 0.$$

Putting our equations together, if $L=\sum_{j=1}^{n-1}\xi_j L_j$, then we compute

$$\left\langle\frac{1}{2}\left(\partial\bar\partial-\bar\partial\partial\right)\lambda, L\wedge\bar L'\right\rangle\Big|_P - \left\langle\frac{1}{2}\left(\partial_b\bar\partial_b-\bar\partial_b\partial_b\right)\lambda, L\wedge\bar L'\right\rangle\Big|_P$$
$$= \sum_{j,k=1}^n \frac{c_{jk}(P)}{2}\left(\bar L_n\lambda + L_n\lambda\right)\xi_j\bar\xi'_k\Big|_P.$$

□

We can already see from Proposition 3.2 the importance of CR-plurisubharmonic functions. On a compact (smooth) manifold, $\nu\{\lambda\}$ will be a bounded quantity, and multiples of Levi-form are controlled by CR-plurisubharmonicity.

As a consequence of Proposition 3.2 and Lemma A.1, we learn that functions that are plurisubharmonic on $(0,q)$-forms near M are CR-plurisubharmonic on $(0,q)$-forms.

Corollary 3.3 *Let M be as in Lemma 3.1. If λ is a smooth, real-valued function that is plurisubharmonic on $(0,q)$-forms near M and has CR-plurisubharmonicity constant A_λ, then λ is CR-plurisubharmonic on $(0,q)$-forms with CR-plurisubharmonicity constant A_λ.*

3.2 Pseudodifferential operators

From Lemma 3.1, there exists a finite cover $\{U_\nu\}_\nu$ so each U_ν has a special boundary system and can be parameterized by a hypersurface in $\mathbb{C}^n$ (U_ν may be shrunk as necessary). To set up the microlocal analysis, we need to define appropriate pseudodifferential operators on each U_ν. Let $\xi = (\xi_1, \ldots, \xi_{2n-2}, \xi_{2n-1}) = (\xi', \xi_{2n-1})$ be the coordinates in Fourier space so that ξ' is dual to the part of $T(M)$ in the maximal complex subspace (i.e., $T^{1,0}(M) \oplus T^{0,1}(M)$) and ξ_{2n-1} is dual to the totally real part of $T(M)$, i.e., the "bad" direction T. Define

$$\mathcal{C}^+ = \left\{\xi : \xi_{2n-1} \ge \frac{1}{2}|\xi'| \text{ and } |\xi| \ge 1\right\};$$
$$\mathcal{C}^- = \{\xi : -\xi \in \mathcal{C}^+\};$$
$$\mathcal{C}^0 = \left\{\xi : -\frac{3}{4}|\xi'| \le \xi_{2n-1} \le \frac{3}{4}|\xi'|\} \cup \{\xi : |\xi| \le 1\right\}.$$

Note that $\mathcal{C}^+$ and $\mathcal{C}^-$ are disjoint, but both intersect $\mathcal{C}^0$ nontrivially. Next, we define functions on $\{|\xi| : |\xi|^2 = 1\}$. Let

$$\psi^+(\xi) = 1 \text{ when } \xi_{2n-1} \ge \frac{3}{4}|\xi'| \quad \text{and} \quad \operatorname{supp}\psi^+ \subset \left\{\xi : \xi_{2n-1} \ge \frac{1}{2}|\xi'|\right\};$$
$$\psi^-(\xi) = \psi^+(-\xi);$$
$$\psi^0(\xi) \text{ satisfies } \psi^0(\xi)^2 = 1 - \psi^+(\xi)^2 - \psi^-(\xi)^2.$$

Extend ψ^+, ψ^-, and ψ^0 homogeneously outside of the unit ball, i.e., if $|\xi| \ge 1$, then

$$\psi^+(\xi) = \psi^+(\xi/|\xi|),\ \psi^-(\xi) = \psi^-(\xi/|\xi|), \quad \text{and} \quad \psi^0(\xi) = \psi^0(\xi/|\xi|).$$

Also, extend ψ^+, ψ^-, and ψ^0 smoothly inside the unit ball so that $(\psi^+)^2 + (\psi^-)^2 + (\psi^0)^2 = 1$. Finally, for A to be chosen later, define

$$\psi_A^+(\xi) = \psi(\xi/A),\ \psi_A^-(\xi) = \psi^-(\xi/A), \quad \text{and} \quad \psi_A^0(\xi) = \psi^0(\xi/A).$$

Next, let Ψ_A^+, Ψ_A^-, and Ψ^0 be the pseudodifferential operators of order zero with symbols ψ_A^+, ψ_A^-, and ψ_A^0, respectively. The equality $(\psi_A^+)^2 + (\psi_A^-)^2 + (\psi_A^0)^2 = 1$ implies that

$$(\Psi_A^+)^*\Psi_A^+ + (\Psi_A^0)^*\Psi_A^0 + (\Psi_A^-)^*\Psi_A^- = Id.$$

We will also have use for pseudodifferential operators that "dominate" a given pseudodifferential operator. Let ψ be cut-off function and $\tilde{\psi}$ be another cut-off function so that $\tilde{\psi}|_{\operatorname{supp}\psi} \equiv 1$. If Ψ and $\tilde{\Psi}$ are pseudodifferential operators with symbols ψ and $\tilde{\psi}$, respectively, then we say that $\tilde{\Psi}$ dominates Ψ.

For each U_ν, we have a local CR-equivalence to a hypersurface in $\mathbb{C}^n$, and we can define Ψ_A^+, Ψ_A^-, and Ψ_A^0 to act on functions or forms supported in U_ν, so let $\Psi_{\nu,A}^+$,

$\Psi^-_{\nu,A}$, and $\Psi^0_{\nu,A}$ be the pseudodifferential operators of order zero defined on U_ν and $\mathcal{C}^+_\nu$, and $\mathcal{C}^-_\nu$, and $\mathcal{C}^0_\nu$ be the regions of ξ-space dual to U_ν on which the symbol of each of those pseudodifferential operators is supported. Then it follows that:

$$(\Psi^+_{\nu,A})^*\Psi^+_{\nu,A} + (\Psi^0_{\nu,A})^*\Psi^0_{\nu,A} + (\Psi^-_{\nu,A})^*\Psi^-_{\nu,A} = Id.$$

Let $\tilde{\Psi}^+_{\mu,A}$ and $\tilde{\Psi}^-_{\mu,A}$ be pseudodifferential operators that dominate $\Psi^+_{\mu,A}$ and $\Psi^-_{\mu,A}$, respectively (where $\Psi^+_{\mu,A}$ and $\Psi^-_{\mu,A}$ are defined on some U_μ). If $\tilde{\mathcal{C}}^+_\mu$ and $\tilde{\mathcal{C}}^-_\mu$ are the supports of $\tilde{\Psi}^+_{\mu,A}$ and $\tilde{\Psi}^-_{\mu,A}$, respectively, then we can choose $\{U_\mu\}$, $\tilde{\psi}^+_{\mu,A}$, and $\tilde{\psi}^-_{\mu,A}$ so that the following result holds.

Lemma 3.4 *Let M be a compact, orientable, embedded CR-manifold. There is a finite open covering $\{U_\mu\}_\mu$ of M so that if $U_\mu, U_\nu \in \{U_\mu\}$ have nonempty intersection, then there exists a diffeomorphism ϑ between U_ν and U_μ with Jacobian $\mathcal{J}_\vartheta$ so that*:

(i) ${}^t\mathcal{J}_\vartheta(\tilde{\mathcal{C}}^+_\mu) \cap \mathcal{C}^-_\nu = \emptyset$ *and* $\mathcal{C}^+_\nu \cap {}^t\mathcal{J}_\vartheta(\tilde{\mathcal{C}}^-_\mu) = \emptyset$ *where* ${}^t\mathcal{J}_\vartheta$ *is the inverse of the transpose of* $\mathcal{J}_\vartheta$;

(ii) *Let* ${}^\vartheta\Psi^+_{\mu,A}$, ${}^\vartheta\Psi^-_{\mu,A}$, *and* ${}^\vartheta\Psi^0_{\mu,A}$ *be the transfers of* $\Psi^+_{\mu,A}$, $\Psi^-_{\mu,A}$, *and* $\Psi^0_{\mu,A}$, *respectively via* ϑ. *Then on* $\{\xi : \xi_{2n-1} \geq \frac{4}{5}|\xi'|$ *and* $|\xi| \geq (1+\epsilon)A\}$, *then principal symbol of* ${}^\vartheta\Psi^+_{\mu,A}$ *is identically* 1, *on* $\{\xi : \xi_{2n-1} \leq -\frac{4}{5}|\xi'|$ *and* $|\xi| \geq (1+\epsilon)A\}$, *then principal symbol of* ${}^\vartheta\Psi^-_{\mu,A}$ *is identically* 1, *and on* $\{\xi : -\frac{1}{3}\xi_{2n-1} \geq \frac{1}{3}|\xi'|$ *and* $|\xi| \geq (1+\epsilon)A\}$, *then principal symbol of* ${}^\vartheta\Psi^0_{\mu,A}$ *is identically* 1, *where* $\epsilon > 0$ *and can be very small*;

(iii) *Let* ${}^\vartheta\tilde{\Psi}^+_{\mu,A}$, ${}^\vartheta\tilde{\Psi}^-_{\mu,A}$ *be the transfers via* ϑ *of* $\tilde{\Psi}^+_{\mu,A}$ *and* $\tilde{\Psi}^-_{\mu,A}$, *respectively. Then the principal symbol of* ${}^\vartheta\tilde{\Psi}^+_{\mu,A}$ *is identically 1 on* $\mathcal{C}^+_\nu$ *and the principal symbol of* ${}^\vartheta\tilde{\Psi}^-_{\mu,A}$ *is identically 1 on* $\mathcal{C}^-_\nu$;

(iv) $\tilde{\mathcal{C}}^+_\mu \cap \tilde{\mathcal{C}}^-_\mu = \emptyset$.

We will suppress the left superscript ϑ as it should be clear from the context which pseudodifferential operator must be transferred. The proof of this lemma is contained in Lemma 4.3 and its subsequent discussion in [22].

3.3 Norms

Let $\mathcal{I}_q = \{J = (j_1, \ldots, j_q) \in \mathbb{N}^q : 1 \leq j_1 < \cdots < j_q \leq n-1\}$.

We have a volume form dV on M, and we define the following inner products and norms on functions (with their natural generalizations to forms). Let λ^+ and λ^- be functions defined on M. Set

$$(\phi, \varphi)_0 = \int_M \phi\bar{\varphi}\, dV, \quad \text{and} \quad \|\varphi\|_0^2 = (\varphi, \varphi)_0;$$

$$(\phi, \varphi)_{\lambda^+} = \int_M \phi\bar{\varphi}\, e^{-\lambda^+}\, dV, \quad \text{and} \quad \|\varphi\|^2_{\lambda^+} = (\varphi, \varphi)_{\lambda^+};$$

$$(\phi,\varphi)_{\lambda^-} = \int_M \phi\bar{\varphi}\, e^{\lambda^-}\, dV, \quad \text{and} \quad \|\varphi\|^2_{\lambda^-} = (\varphi,\varphi)_{\lambda^-}.$$

If $\varphi = \sum_{j\in\mathcal{I}_q} \varphi_j\, \bar{\omega}_J$, then we use the common shorthand $\|\varphi\| = \sum_{j\in\mathcal{I}_q} \|\varphi_J\|$ where $\|\cdot\|$ represents a generic norm applied to φ.

We also need a norm that is well-suited for the microlocal arguments. Let $\{\zeta_\nu\}$ be a partition of unity subordinate to the covering $\{U_\nu\}$ satisfying $\sum_\nu \zeta_\nu^2 = 1$. Also, for each ν, let $\tilde{\zeta}_\nu$ be a cutoff function that dominates ζ_ν so that $\operatorname{supp}\tilde{\zeta}_\nu \subset U_\nu$. Then we define the global inner product and norm as follows:

$$\begin{aligned}\langle\phi,\varphi\rangle_{\lambda^+,\lambda^-} = \langle\phi,\varphi\rangle_{\pm} = \sum_\nu \Big(&(\tilde{\zeta}_\nu\Psi^+_{\nu,A}\zeta_\nu\phi^\nu, \tilde{\zeta}_\nu\Psi^+_{\nu,A}\zeta_\nu\varphi^\nu)_{\lambda^+}\\ &+(\tilde{\zeta}_\nu\Psi^0_{\nu,A}\zeta_\nu\phi^\nu, \tilde{\zeta}_\nu\Psi^0_{\nu,A}\zeta_\nu\varphi^\nu)_0\\ &+(\tilde{\zeta}_\nu\Psi^-_{\nu,A}\zeta_\nu\phi^\nu, \tilde{\zeta}_\nu\Psi^-_{\nu,A}\zeta_\nu\varphi^\nu)_{\lambda^-}\Big)\end{aligned}$$

and

$$|||\varphi|||^2_{\lambda^+,\lambda^-} = |||\varphi|||^2_{\pm} = \sum_\nu \left(\|\tilde{\zeta}_\nu\Psi^+_{\nu,A}\zeta_\nu\varphi^\nu\|^2_{\lambda^+} + \|\tilde{\zeta}_\nu\Psi^0_{\nu,A}\zeta_\nu\varphi^\nu\|^2_0 + \|\tilde{\zeta}_\nu\Psi^-_{\nu,A}\zeta_\nu\varphi^\nu\|^2_{\lambda^-}\right),$$

where φ^ν is the form φ expressed in the local coordinates on U_ν. The superscript ν will often be omitted.

For a form φ supported on M, the Sobolev norm of order s is given by the following:

$$\|\varphi\|^2_s = \sum_\nu \|\tilde{\zeta}_\nu\Lambda^s\zeta_\nu\varphi^\nu\|^2_0$$

where Λ is defined to be the pseudodifferential operator with symbol $(1+|\xi|^2)^{1/2}$.

It will be essential for us to pass from the unweighted L^2-norm on M and the microlocal norm defined above. The following lemma says that we can do this without any loss of information.

Lemma 3.5 *Let λ^+, λ^- be smooth functions on M with $0 \le \lambda^+, \lambda^- \le 1$. Then there exist constants $C_1, C_2 > 0$ so that*

$$C_1\|\varphi\|^2_0 \le |||\varphi|||^2_{\pm} \le C_2\|\varphi\|^2_0.$$

Proof It is enough to check this when φ is a function. Since $0 \le \lambda^+, \lambda^- \le 1$,

$$|||\varphi|||^2_{\pm} \le e\sum_\nu \left(\|\tilde{\zeta}_\nu\Psi^+_{\nu,A}\zeta_\nu\varphi^\nu\|^2_0 + \|\tilde{\zeta}_\nu\Psi^0_{\nu,A}\zeta_\nu\varphi^\nu\|^2_0 + \|\tilde{\zeta}_\nu\Psi^-_{\nu,A}\zeta_\nu\varphi^\nu\|^2_0\right).$$

We can express $\tilde{\zeta}_\nu\Psi^+_{\nu,A}\zeta_\nu\varphi^\nu = \Psi^+_{\nu,A}\zeta_\nu\varphi^\nu - (1-\tilde{\zeta}_\nu)\Psi^+_{\nu,A}\zeta_\nu\varphi^\nu$. $(1-\tilde{\zeta}_\nu)\Psi^+_{\nu,A}\zeta_\nu$ is infinitely smoothing, but using this bound would lead to a constant depending on A.

We wish to avoid constants depending on A. Observe that

$$\begin{aligned}
&(1-\tilde{\zeta}_\nu(x))\Psi^+_{\nu,A}\zeta_\nu(x)\varphi^\nu(x)\\
&= \frac{1}{(2\pi)^{2n-1}}(1-\tilde{\zeta}_\nu(x)) \int\limits_{\mathbb{R}^{2n-1}} e^{ix\cdot\xi}\psi^+_{\nu,A}(\xi)\widehat{\zeta_\nu\varphi^\nu}(\xi)\,d\xi\\
&= \frac{1}{(2\pi)^{2n-1}} \int\limits_{\mathbb{R}^{2n-1}} \varphi^\nu(y) \int\limits_{\mathbb{R}^{2n-1}} (1-\tilde{\zeta}_\nu(x))\zeta_\nu(y)e^{i(x-y)\xi}\psi^+_{\nu,A}(\xi)\,d\xi\,dy.
\end{aligned}$$

Define $K(x,y) = \frac{1}{(2\pi)^{2n-1}}\int_{\mathbb{R}^{2n-1}}(1-\tilde{\zeta}_\nu(x))\zeta_\nu(y)e^{i(x-y)\cdot\xi}\psi^+_{\nu,A}(\xi)\,d\xi$. By integration by parts, for any multiindex α,

$$K(x,y) = (1-\tilde{\zeta}_\nu(x))\zeta_\nu(y)\frac{(-i)^\alpha}{(2\pi(x-y)^\alpha)^{2n-1}} \int\limits_{\mathbb{R}^{2n-1}} e^{i(x-y)\cdot\xi}D^\alpha\psi^+_{\nu,A}(\xi)\,d\xi.$$

Recall that $\psi^+_{\nu,A}(\xi) = \psi^+(\xi/A)$, so requiring that $A \geq 1$ means that $|D^\alpha\psi^+_{\nu,A}(\xi)| \leq C_\alpha$ where C_α does not depend on A. However, $\operatorname{supp}(1-\tilde{\zeta}_\nu)\cap\operatorname{supp}\zeta_\nu = \emptyset$, so for any N, there exists C_N so that

$$|K(x,y)| \leq |1-\tilde{\zeta}_\nu(x)||\zeta_\nu(y)|\frac{C_N}{(1+|x-y|)^N},$$

where C_N does not depend on A. Consequently,

$$\|(1-\tilde{\zeta}_\nu)\Psi^+_{\nu,A}\zeta_\nu\varphi^\nu(x)\|_0^2 \leq \tilde{C}\|\zeta_\nu\varphi^\nu\|_0^2.$$

The range of $\Psi^+_{\nu,A}\zeta_\nu$ is not $L^2(U_\nu)$ but $L^2(\mathbb{R}^{2n-1})$, but this problem is mitigated by the fact that $\Psi^+_{\nu,A}\zeta_\nu$ is a smoothing operator outside of $\operatorname{Dom}(\zeta_\nu)$. Also, $\Psi^+_A\zeta_\nu$ is a contraction on $L^2(\mathbb{R}^{2n-1})$, so

$$\|\tilde{\zeta}_\nu\Psi^+_{\nu,A}\zeta_\nu\varphi^\nu\|_0^2 \leq 2\|\Psi^+_{\nu,A}\zeta_\nu\varphi^\nu\|_0^2 + 2\|(1-\tilde{\zeta}_\nu)\Psi^+_{\nu,A}\zeta_\nu\varphi^\nu\|_0^2 \leq C\|\zeta_\nu\varphi^\nu\|_0^2$$

for some C independent of A. By (possibly) increasing C, a similar bound will also hold for for $\Psi^0_{\nu,A}$ and $\Psi^-_{\nu,A}$. The upper bound of the lemma therefore follows (since the sum over ν is finite and $0 \leq \zeta_\nu \leq 1$).

We now show the lower bound. Note that $\sum_\nu \zeta_\nu^2 = 1 = \sum_\nu \tilde{\zeta}_\nu\zeta_\nu^2$. Consequently,

$$\begin{aligned}
\|\varphi\|_0^2 &= \left(\sum_\nu \zeta_\nu^2\varphi,\varphi\right)_0 = \sum_\nu \|\zeta_\nu\varphi^\nu\|_0^2\\
&= \sum_\nu \left(\left((\Psi^+_{\nu,A})^*\Psi^+_{\nu,A} + (\Psi^0_{\nu,A})^*\Psi^0_{\nu,A} + (\Psi^-_{\nu,A})^*\Psi^-_{\nu,A}\right)\zeta_\nu\varphi^\nu,\zeta_\nu\varphi^\nu\right)_0
\end{aligned}$$

$$= \sum_{\nu} \Big(\|(\tilde{\zeta}_\nu + (1-\tilde{\zeta}_\nu))\Psi^+_{\nu,A}\zeta_\nu\varphi^\nu\|_0^2 + \|(\tilde{\zeta}_\nu + (1-\tilde{\zeta}_\nu))\Psi^0_{\nu,A}\zeta_\nu\varphi^\nu\|_0^2$$

$$+ \|(\tilde{\zeta}_\nu + (1-\tilde{\zeta}_\nu))\Psi^-_{\nu,A}\zeta_\nu\varphi^\nu\|_0^2 \Big)$$

However, $\|(\tilde{\zeta}_\nu + (1-\tilde{\zeta}_\nu))\Psi^+_{\nu,A}\zeta_\nu\varphi^\nu\|_0^2 \le 2(\|\tilde{\zeta}_\nu\Psi^+_{\nu,A}\zeta_\nu\varphi^\nu\|_0^2 + \|(1-\tilde{\zeta}_\nu)\Psi^+_{\nu,A}\zeta_\nu\varphi^\nu\|_0^2)$, and $\Psi^+_{\nu,A}\zeta_\nu\varphi^\nu$ is pseudolocal (indeed, $(1-\tilde{\zeta}_\nu))\Psi^+_{\nu,A}\zeta_\nu\varphi^\nu$ is infinitely smoothing), so $\|\tilde{\zeta}_\nu\Psi^+_{\nu,A}\zeta_\nu\varphi^\nu\|_0^2$ controls $\|(1-\tilde{\zeta}_\nu)\Psi^+_{\nu,A}\zeta_\nu\varphi^\nu\|_0^2$ and similarly for $\Psi^-_{\nu,A}$ and $\Psi^0_{\nu,A}$. As a result,

$$\|\varphi\|_0^2 \le C \sum_{\nu} \Big(\|\tilde{\zeta}_\nu\Psi^+_{\nu,A}\zeta_\nu\varphi^\nu\|_0^2 + \|\tilde{\zeta}_\nu\Psi^0_{\nu,A}\zeta_\nu\varphi^\nu\|_0^2 + \|\tilde{\zeta}_\nu\Psi^-_{\nu,A}\zeta_\nu\varphi^\nu\|_0^2 \Big)$$

$$\le C \sum_{\nu} \Big(\|\tilde{\zeta}_\nu\Psi^+_{\nu,A}\zeta_\nu\varphi^\nu\|_{\lambda^+}^2 + \|\tilde{\zeta}_\nu\Psi^0_{\nu,A}\zeta_\nu\varphi^\nu\|_0^2 + \|\tilde{\zeta}_\nu\Psi^-_{\nu,A}\zeta_\nu\varphi^\nu\|_{\lambda^-}^2 \Big)$$

since λ^+ and λ^- are positive, bounded, and bounded away from zero. □

The meaning of Lemma 3.5 is that $|||\varphi|||_\pm \sim \|\varphi\|_0^2$ with constants independent of A, so the Riesz Representation Theorem implies the following corollary (see Corollary 4.6 in [22]).

Corollary 3.6 *There exists a self-adjoint operator $E_{\lambda^+,\lambda^-} = E_\pm$ so that*

$$(\varphi, \phi)_0 = \langle \varphi, E_\pm \phi \rangle_\pm$$

for any two forms φ and ϕ in $L^2(M)$. $E_\pm$ is the inverse of

$$F_\pm = \sum_{\nu} \Big(\zeta_\nu (\Psi^+_{\nu,A})^* \tilde{\zeta}_\nu e^{-\lambda^+} \tilde{\zeta}_\nu \Psi^+_{\nu,A}\zeta_\nu + \zeta_\nu (\Psi^0_{\nu,A})^* \tilde{\zeta}_\nu^2 \Psi^0_{\nu,A}\zeta_\nu$$

$$+ \zeta_\nu (\Psi^-_{\nu,A})^* \tilde{\zeta}_\nu e^{\lambda^-} \tilde{\zeta}_\nu \Psi^-_{\nu,A}\tilde{\zeta}_\nu \Big).$$

$E_\pm$ and $F_\pm$ are bounded in $L^2(M)$ independently of $A \ge 1$ since $0 \le \lambda^+, \lambda^- \le 1$.

3.4 $\bar{\partial}_b$ and its adjoints

If f is a function on M, in local coordinates,

$$\bar{\partial}_b f = \sum_{j=1}^{n-1} \bar{L}_j f \, \bar{\omega}_j,$$

while if φ is a $(0,q)$-form, there exist functions m_K^J so that

$$\bar{\partial}_b\varphi = \sum_{\substack{J\in\mathcal{I}_q\\ K\in\mathcal{I}'_{q+1}}}\sum_{j=1}^{n-1}\epsilon_K^{jJ}\bar{L}_j\varphi_J\,\bar{\omega}_K + \sum_{\substack{J\in\mathcal{I}_q\\ K\in\mathcal{I}'_{q+1}}}\varphi_J m_K^J\,\bar{\omega}_K.$$

Let $\bar{L}_j^*$ be the adjoint of $\bar{L}_j$ in $(\cdot,\cdot)_0$, $\bar{L}_j^{*,+}$ be the adjoint of $\bar{L}_j$ in $(\cdot,\cdot)_{\lambda^+}$, and $\bar{L}_j^{*,-}$ be the adjoint of $\bar{L}_j$ in $(\cdot,\cdot)_{\lambda^-}$. Then we define $\bar{\partial}_b^*$, $\bar{\partial}_b^{*,+}$, and $\bar{\partial}_b^{*,-}$ to be the adjoints of $\bar{\partial}_b$ in $L^2(M)$, $L^2(M,e^{-\lambda^+})$, and $L^2(M,e^{\lambda^-})$, respectively. On a $(0,q)$-form φ, we have (for some functions $f_j\in C^\infty(U)$)

$$\begin{aligned}
\bar{\partial}_b^*\varphi &= \sum_{\substack{I\in\mathcal{I}'_{q-1}\\ J\in\mathcal{I}_q}}\sum_{j=1}^{n-1}\epsilon_J^{jI}\bar{L}_j^*\varphi_J\,\bar{\omega}_I + \sum_{\substack{I\in\mathcal{I}'_{q-1}\\ J\in\mathcal{I}_q}}\overline{m_J^I}\varphi_J\,\bar{\omega}_I\\
&= -\sum_{\substack{I\in\mathcal{I}'_{q-1}\\ J\in\mathcal{I}_q}}\sum_{j=1}^{n-1}\epsilon_J^{jI}\left(L_j\varphi_J + f_j\varphi_J\right)\bar{\omega}_I + \sum_{\substack{I\in\mathcal{I}'_{q-1}\\ J\in\mathcal{I}_q}}\overline{m_J^I}\varphi_J\,\bar{\omega}_I;\\
\bar{\partial}_b^{*,+}\varphi &= \sum_{\substack{I\in\mathcal{I}'_{q-1}\\ J\in\mathcal{I}_q}}\sum_{j=1}^{n-1}\epsilon_J^{jI}\bar{L}_j^{*,+}\varphi_J\,\bar{\omega}_I + \sum_{\substack{I\in\mathcal{I}'_{q-1}\\ J\in\mathcal{I}_q}}\overline{m_J^I}\varphi_J\,\bar{\omega}_I \qquad (3)\\
&= -\sum_{\substack{I\in\mathcal{I}'_{q-1}\\ J\in\mathcal{I}_q}}\sum_{j=1}^{n-1}\epsilon_J^{jI}\left(L_j\varphi_J - L_j\lambda^+\varphi_J + f_j\varphi_J\right)\bar{\omega}_I + \sum_{\substack{I\in\mathcal{I}'_{q-1}\\ J\in\mathcal{I}_q}}\overline{m_J^I}\varphi_J\,\bar{\omega}_I;\\
\bar{\partial}_b^{*,-}\varphi &= \sum_{\substack{I\in\mathcal{I}'_{q-1}\\ J\in\mathcal{I}_q}}\sum_{j=1}^{n-1}\epsilon_J^{jI}\bar{L}_j^{*,-}\varphi_J\,\bar{\omega}_I + \sum_{\substack{I\in\mathcal{I}'_{q-1}\\ J\in\mathcal{I}_q}}\overline{m_J^I}\varphi_J\,\bar{\omega}_I\\
&= -\sum_{\substack{I\in\mathcal{I}'_{q-1}\\ J\in\mathcal{I}_q}}\sum_{j=1}^{n-1}\epsilon_J^{jI}\left(L_j\varphi_J + L_j\lambda^-\varphi_J + f_j\varphi_J\right)\bar{\omega}_I + \sum_{\substack{I\in\mathcal{I}'_{q-1}\\ J\in\mathcal{I}_q}}\overline{m_J^I}\varphi_J\,\bar{\omega}_I.
\end{aligned}$$

Consequently, we see that

$$\bar{\partial}_b^{*,+} = \bar{\partial}_b^* - [\bar{\partial}_b^*,\lambda^+] \quad\text{and}\quad \bar{\partial}_b^{*,-} = \bar{\partial}_b^* + [\bar{\partial}_b^*,\lambda^-],$$

and all three adjoints have the same domain. Finally, let $\bar{\partial}_{b,\pm}^*$ be the adjoint of $\bar{\partial}_b$ with respect to $\langle\cdot,\cdot\rangle_\pm$.

The computations proving Lemma 4.8 and Lemma 4.9 and equation (4.4) in [22] can be applied here with only a change of notation, so we have the following two results, recorded here as Lemma 3.7 and Lemma 3.8. The meaning of the results is that $\bar\partial^*_{b,\pm}$ acts like $\bar\partial_b^{*,+}$ for forms whose support is basically $\mathcal{C}^+$ and $\bar\partial_b^{*,-}$ on forms whose support is basically $\mathcal{C}^-$.

Lemma 3.7 *On smooth* $(0,q)$*-forms,*

$$\begin{aligned}\bar\partial^*_{b,\pm} &= \bar\partial^*_b - \sum_\mu \zeta_\mu^2 \tilde\Psi^+_{\mu,A}[\bar\partial^*_b,\lambda^+] + \sum_\mu \zeta_\mu^2\tilde\Psi^-_{\mu,A}[\bar\partial^*_b,\lambda^-]\\ &\quad+\sum_\mu\Big(\tilde\zeta_\mu[\tilde\zeta_\mu\Psi^+_{\mu,A}\zeta_\mu,\bar\partial_b]^*\tilde\zeta_\mu\Psi^+_{\mu,A}\zeta_\mu+\zeta_\mu(\Psi^+_{\mu,A})^*\tilde\zeta_\mu[\bar\partial_b^{*,+},\tilde\zeta_\mu\Psi^+_{\mu,A}\zeta_\mu]\tilde\zeta_\mu\\ &\quad+\tilde\zeta_\mu[\tilde\zeta_\mu\Psi^-_{\mu,A}\zeta_\mu,\bar\partial_b]^*\tilde\zeta_\mu\Psi^-_{\mu,A}\zeta_\mu+\zeta_\mu(\Psi^+_{\mu,A})^*\tilde\zeta_\mu[\bar\partial_b^{*,-},\tilde\zeta_\mu\Psi^-_{\mu,A}\zeta_\mu]\tilde\zeta_\mu+E_A\Big),\end{aligned}$$

where the error term E_A *is a sum of order zero terms and lower order terms. Also, the symbol of* E_A *is supported in* $\mathcal{C}^0_\mu$ *for each* μ*.*

We are now ready to define the energy forms that we use. Let

$$\begin{aligned}Q_{b,\pm}(\phi,\varphi)&=\langle\bar\partial_b\phi,\bar\partial_b\varphi\rangle_\pm+\langle\bar\partial^*_{b,\pm}\phi,\bar\partial^*_{b,\pm}\varphi\rangle_\pm;\\ Q_{b,+}(\phi,\varphi)&=(\bar\partial_b\phi,\bar\partial_b\varphi)_{\lambda^+}+(\bar\partial_b^{*,+}\phi,\bar\partial_b^{*,+}\varphi)_{\lambda^+};\\ Q_{b,0}(\phi,\varphi)&=(\bar\partial_b\phi,\bar\partial_b\varphi)_0+(\bar\partial^*_b\phi,\bar\partial^*_b\varphi)_0;\\ Q_{b,-}(\phi,\varphi)&=(\bar\partial_b\phi,\bar\partial_b\varphi)_{\lambda^-}+(\bar\partial_b^{*,-}\phi,\bar\partial_b^{*,-}\varphi)_{\lambda^-}.\end{aligned}$$

Lemma 3.8 *If* φ *is a smooth* $(0,q)$*-form on* M*, then there exist constants* K, $K_\pm$, K' *with* $K\ge 1$ *so that*

$$\begin{aligned}&K Q_{b,\pm}(\varphi,\varphi)+K_\pm\sum_\nu\|\tilde\zeta_\nu\tilde\Psi^0_{\nu,A}\zeta_\nu\varphi^\nu\|_0^2+K'\|\varphi\|_0^2+O_\pm(\|\varphi\|^2_{-1})\\ &\ge\sum_\nu\Big[Q_{b,+}(\tilde\zeta_\nu\Psi^+_{\nu,A}\zeta_\nu\varphi^\nu,\tilde\zeta_\nu\Psi^+_{\nu,A}\zeta_\nu\varphi^\nu)+Q_{b,0}(\tilde\zeta_\nu\Psi^0_{\nu,A}\zeta_\nu\varphi^\nu,\tilde\zeta_\nu\Psi^0_{\nu,A}\zeta_\nu\varphi^\nu)\\ &\quad+Q_{b,-}(\tilde\zeta_\nu\Psi^-_{\nu,A}\zeta_\nu\varphi^\nu,\tilde\zeta_\nu\Psi^-_{\nu,A}\zeta_\nu\varphi^\nu)\Big]\end{aligned}\tag{4}$$

where K *and* K' *do not depend on* A*.*

Many of the subsequent proofs make use of the“lc/sc” argument: $-\epsilon\|x\|^2-\epsilon^{-1}\|y\|^2\le 2\,\mathrm{Re}((x,y))\le\epsilon\|x\|^2+\epsilon^{-1}\|y\|^2$ where $(\cdot,\cdot)$ is any Hermitian inner product with associated norm $\|\cdot\|$. Also, since that $\bar\partial_b^{*,+}=\bar\partial_b+$“lower order”, commuting $\bar\partial_b^{*,+}$ by $\Psi^+_{\nu,A}$ creates error terms of order 0 that do not depend on λ^+ and lower order terms that may depend on λ^+.

4 The basic estimate

The goal of this section is to prove a basic estimate for smooth forms on M.

Proposition 4.1 *Let $M \subset \mathbb{C}^N$ be a compact, orientable, weakly pseudoconvex CR-manifold of dimension $n \geq 5$ and $1 \leq q \leq n-2$. Assume that M admits functions λ^+ and λ^- where λ^+ is strictly CR-plurisubharmonic on $(0, q)$-forms and λ^- is strictly CR-plurisubharmonic on $(0, n-1-q)$-forms. Let $\varphi \in \mathrm{Dom}(\bar\partial_b) \cap \mathrm{Dom}(\bar\partial_b^*)$. There exist constants K, $K_\pm$, and $K'_\pm$ where K does not depend on λ^+, λ^-, or A so that*

$$A_\pm |||\varphi|||_\pm^2 \leq K Q_{b,\pm}(\varphi, \varphi) + K |||\varphi|||_\pm^2 + K_\pm \sum_\nu \sum_{J \in \mathcal{I}_q} \|\tilde\zeta_\nu \tilde\Psi^0_{\nu,A} \zeta_\nu \varphi^\nu_J\|_0^2 + K'_\pm \|\varphi\|_{-1}^2.$$

The constant $A_\pm > 0$ is the minimum of the CR-plurisubharmonicity constants A_{λ^+} and A_{λ^-}.

The proof of Proposition 4.1 comes as the culmination of a series of calculations.

4.1 Local estimates

We work on a fixed $U = U_\nu$. On this neighborhood, as above, there exists an orthonormal basis of vector fields $L_1, \ldots, L_n, \bar L_1, \ldots, \bar L_n$ so that

$$[L_j, \bar L_k] = c_{jk} T + \sum_{\ell=1}^{n-1} (d^\ell_{jk} L_\ell - \bar d^\ell_{kj} \bar L_\ell) \tag{5}$$

if $1 \leq j, k \leq n-1$, and $T = L_n - \bar L_n$, and for some fixed point P,

$$[L_j, \bar L_k]\big|_P = c_{jk} T.$$

Note that c_{jk} are the coefficients of the Levi form. Recall that $\bar L^{*,+}$, $\bar L^*$, and $\bar L^{*,-}$ are the adjoints of $\bar L$ in $(\cdot, \cdot)_{\lambda^+}$, $(\cdot, \cdot)_0$, and $(\cdot, \cdot)_{\lambda^-}$, respectively. From (3), we see that

$$\bar L_j^{*,+} = -L_j + L_j(\lambda^+) - f_j \quad \text{and} \quad \bar L_j^{*,-} = -L_j - L_j(\lambda^-) - f_j,$$

and plugging this into (5), we have

$$[\bar L_j^{*,+}, \bar L_k] = -c_{jk} T - \sum_{\ell=1}^{n-1} \left(d^\ell_{jk} L_\ell - \bar d^\ell_{kj} \bar L_\ell \right) - \bar L_k L_j \lambda^+ + \bar L_k f_j$$

$$[\bar L_j^{*,-}, \bar L_k] = -c_{jk} T - \sum_{k=1}^{n-1} \left(d^\ell_{jk} L_\ell - \bar d^\ell_{kj} \bar L_k \right) + \bar L_k L_j \lambda^- + \bar L_k f_j$$

For the inner product $Q_{b,+}(\varphi, \varphi)$, we have the following estimate.

Lemma 4.2 *Let φ be a $(0, q)$-form supported in U, $\varphi \in \mathrm{Dom}(\bar\partial_b) \cap \mathrm{Dom}(\bar\partial_b^*)$. There exists $0 < \epsilon' \ll 1$ so that*

$$Q_{b,+}(\varphi,\varphi) \geq (1-\epsilon') \sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1} \|\bar{L}_j\varphi_J\|^2_{\lambda^+} + \sum_{J\in\mathcal{I}_q}\sum_{j\in J}\Bigg[\operatorname{Re}\left\{(c_{jj}T\varphi_J,\varphi_J)_{\lambda^+}\right\}$$
$$+\frac{1}{2}\left((\bar{L}_jL_j(\lambda^+)+L_j\bar{L}_j(\lambda^+))\varphi_J,\varphi_J)_{\lambda^+}\right)$$
$$+\frac{1}{2}\sum_{\ell=1}^{n-1}\left((d^\ell_{jj}L_\ell(\lambda^+)+\bar{d}^\ell_{jj}\bar{L}_\ell(\lambda^+))\varphi_J,\varphi_J\right)_{\lambda^+}\Bigg]$$
$$-\sum_{J,J'\in\mathcal{I}_q}\sum_{\substack{1\leq j,k\leq n-1\\ j\neq k}}\epsilon^{kJ}_{jJ'}\Big[\operatorname{Re}\left\{(c_{jk}T\varphi_J,\varphi_{J'})_{\lambda^+}\right\}$$
$$+\frac{1}{2}\left((\bar{L}_kL_j(\lambda^+)+L_j\bar{L}_k(\lambda^+))\varphi_J,\varphi_{J'})_{\lambda^+}\right)$$
$$+\frac{1}{2}\sum_{\ell=1}^{n-1}\left((d^\ell_{jk}L_\ell(\lambda^+)+\bar{d}^\ell_{kj}\bar{L}_\ell(\lambda^+))\varphi_J,\varphi_{J'}\right)_{\lambda^+}\Big]+O(\|\varphi\|_0^2).$$

Proof First, observe

$$(\bar{\partial}^{*,+}_b\varphi,\bar{\partial}^{*,+}_b\varphi)_{\lambda^+} = \sum_{\substack{I\in\mathcal{I}'_{q-1}\\ J,J'\in\mathcal{I}_q}}\sum_{\substack{1\leq j,k\leq n-1\\ j\neq k}}\epsilon^{jI}_J\epsilon^{kI}_{J'}(\bar{L}^{*,+}_j\varphi_J,\bar{L}^{*,+}_k\varphi_{J'})_{\lambda^+}$$
$$+O\left(\|\varphi\|^2_{\lambda^+}+\left(\sum_{j=1}^{n-1}\|\bar{L}_j\varphi\|^2_{\lambda^+}\right)^{1/2}\|\varphi\|_{\lambda^+}\right).$$

However, if $j\neq k$, then $\epsilon^{jI}_J\epsilon^{kI}_{J'}=\epsilon^{kjI}_{kJ}\epsilon^{jkI}_{jJ'}=-\epsilon^{jkI}_{kJ}\epsilon^{jkI}_{jJ'}=-\epsilon^{kJ}_{jJ'}$. Consequently,

$$\|\bar{\partial}^{*,+}_b\varphi\|^2_{\lambda^+} = \sum_{J\in\mathcal{I}_q}\sum_{j\in J}\|\bar{L}^{*,+}_j\varphi_J\|^2_{\lambda^+} - \sum_{J,J'\in\mathcal{I}_q}\sum_{\substack{1\leq j,k\leq n-1\\ j\neq k}}\epsilon^{kJ}_{jJ'}(\bar{L}^{*,+}_j\varphi_J,\bar{L}^{*,+}_k\varphi_{J'})_{\lambda^+}$$
$$+O\left(\|\varphi\|^2_{\lambda^+}+\left(\sum_{j=1}^{n-1}\|\bar{L}_j\varphi\|^2_{\lambda^+}\right)^{1/2}\|\varphi\|_{\lambda^+}\right)$$
$$=\sum_{J\in\mathcal{I}_q}\sum_{j\in J}\|\bar{L}_j\varphi_J\|^2_{\lambda^+}+\sum_{J\in\mathcal{I}_q}\sum_{j\in J}\left([\bar{L}_j,\bar{L}^{*,+}_j]\varphi_J,\varphi_J\right)_{\lambda^+}$$
$$-\sum_{J,J'\in\mathcal{I}_q}\sum_{\substack{1\leq j,k\leq n-1\\ j\neq k}}\epsilon^{kJ}_{jJ'}(\bar{L}^{*,+}_j\varphi_J,\bar{L}^{*,+}_k\varphi_{J'})_{\lambda^+}$$
$$+O\left(\|\varphi\|^2_{\lambda^+}+\left(\sum_{j=1}^{n-1}\|\bar{L}_j\varphi\|^2_{\lambda^+}\right)^{1/2}\|\varphi\|_{\lambda^+}\right)$$

Second, from the calculation of $\bar\partial_b$ above, we compute

$$\begin{aligned}
\|\bar\partial_b\varphi\|_{\lambda+}^2 &= \sum_{\substack{J,J'\in\mathcal{I}_q\\ K\in\mathcal{I}'_{q+1}}}\ \sum_{1\le j,k\le n-1} \epsilon^{kJ}_K\,\epsilon^{jJ'}_K\,(\bar L_k\varphi_J,\bar L_j\varphi_{J'})_{\lambda+} + O\left(\|\varphi\|_{\lambda+}^2 + \left(\sum_{j=1}^{n-1}\|\bar L_j\varphi\|_{\lambda+}^2\right)^{1/2}\|\varphi\|_{\lambda+}\right)\\
&= \sum_{J\in\mathcal{I}_q}\sum_{j\notin J}\|\bar L_j\varphi_J\|_{\lambda+}^2 + \sum_{J,J'\in\mathcal{I}_q}\ \sum_{\substack{1\le j,k\le n-1\\ j\ne k}} \epsilon^{kJ}_{jJ'}(\bar L_k\varphi_J,\bar L_j\varphi_{J'})_{\lambda+}\\
&\quad + O\left(\|\varphi\|_{\lambda+}^2 + \left(\sum_{j=1}^{n-1}\|\bar L_j\varphi\|_{\lambda+}^2\right)^{1/2}\|\varphi\|_{\lambda+}\right)\\
&= \sum_{J\in\mathcal{I}_q}\sum_{j\notin J}\|\bar L_j\varphi_J\|_{\lambda+}^2 + \sum_{J,J'\in\mathcal{I}_q}\ \sum_{\substack{1\le j,k\le n-1\\ j\ne k}} \epsilon^{kJ}_{jJ'}(\bar L_j^{*,+}\varphi_J,\bar L_k^{*,+}\varphi_{J'})_{\lambda+}\\
&\quad + \sum_{J,J'\in\mathcal{I}_q}\ \sum_{\substack{1\le j,k\le n-1\\ j\ne k}} \epsilon^{kJ}_{jJ'}\left([\bar L_j^{*,+},\bar L_k]\varphi_J,\varphi_{J'}\right)_{\lambda+}\\
&\quad + O\left(\|\varphi\|_{\lambda+}^2 + \left(\sum_{j=1}^{n-1}\|\bar L_j\varphi\|_{\lambda+}^2\right)^{1/2}\|\varphi\|_{\lambda+}\right).
\end{aligned}$$

By a lc/sc argument,

$$\left(\sum_{j=1}^{n-1}\|\bar L_j\varphi\|_{\lambda+}^2\right)^{1/2}\|\varphi\|_{\lambda+} \ge -\epsilon\sum_{j=1}^{n-1}\|\bar L_j\varphi\|_{\lambda+}^2 - \frac{1}{\epsilon}\|\varphi\|_{\lambda+}^2,$$

so adding together our computations yields

$$\begin{aligned}
Q_{b,+}(\varphi,\varphi) &\ge (1-\epsilon)\sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}\|\bar L_j\varphi_J\|_{\lambda+}^2 + \sum_{J\in\mathcal{I}_q}\sum_{j\in J}\left([\bar L_j,\bar L_j^{*,+}]\varphi_J,\varphi_J\right)_{\lambda+}\\
&\quad + \sum_{J,J'\in\mathcal{I}_q}\ \sum_{\substack{1\le j,k\le n-1\\ j\ne k}} \epsilon^{kJ}_{jJ'}\left([\bar L_j^{*,+},\bar L_k]\varphi_J,\varphi_{J'}\right)_{\lambda+} + O(\|\varphi\|_{\lambda+}^2). \qquad (6)
\end{aligned}$$

Recall that the commutator

$$[\bar L_j^{*,+},\bar L_k] = -c_{jk}T - \sum_{\ell=1}^{n-1}\left(d^\ell_{jk}L_\ell - \bar d^\ell_{kj}\bar L_\ell\right) - \bar L_k L_j\lambda^+ + \bar L_k f_j,$$

and note that

$$\left|\left(\bar d^\ell_{kj}\bar L_\ell\varphi_J,\varphi_{J'}\right)_{\lambda+}\right| \le \epsilon\|\bar L_\ell\varphi_J\|_{\lambda+}^2 + C_\epsilon\|\varphi\|_{\lambda+}^2.$$

Consequently,

$$
\begin{aligned}
Q_{b,+}(\varphi,\varphi) \geq (1-\epsilon) \sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1} \|\bar{L}_j\varphi_J\|^2_{\lambda^+} \\
+\mathrm{Re}\left\{\sum_{J\in\mathcal{I}_q}\sum_{j\in J}\left[\left(c_{jj}T\varphi_J,\varphi_J\right)_{\lambda^+}+\sum_{\ell=1}^{n-1}\left(d^\ell_{jj}L_\ell\varphi_J,\varphi_J\right)_{\lambda^+}+\left(\bar{L}_jL_j\lambda^+\varphi_J,\varphi_J\right)_{\lambda^+}\right]\right\} \\
-\mathrm{Re}\left\{\sum_{J,J'\in\mathcal{I}_q}\sum_{\substack{1\leq j,k\leq n-1\\ j\neq k}}\epsilon^{kJ}_{jJ'}\left[\left(c_{jk}T\varphi_J,\varphi_{J'}\right)_{\lambda^+}+\sum_{\ell=1}^{n-1}\left(d^\ell_{jk}L_\ell\varphi_J,\varphi_{J'}\right)_{\lambda^+}\right.\right. \\
\left.\left.+\left(\bar{L}_kL_j\lambda^+\varphi_J,\varphi_{J'}\right)_{\lambda^+}\right]\right\} + O(\|\varphi\|_0^2).
\end{aligned}
$$

Also,

$$
\begin{aligned}
\epsilon^{kJ}_{jJ'}\,\mathrm{Re}\left\{\left(d^\ell_{jk}L_\ell\varphi_J,\varphi_{J'}\right)_{\lambda^+}\right\} &= \epsilon^{kJ}_{jJ'}\,\mathrm{Re}\left\{\left(L_\ell(d^\ell_{jk}\varphi_J),\varphi_{J'}\right)_{\lambda^+} - \epsilon^{kJ}_{jJ'}\,\mathrm{Re}\{\left(L_\ell(d^\ell_{jk})\varphi_J,\varphi_{J'}\right)_{\lambda^+}\right\} \\
&= \epsilon^{kJ}_{jJ'}\,\mathrm{Re}\left\{\left(-\bar{L}^{*,+}_\ell(d^\ell_{jk}\varphi_J),\varphi_{J'}\right)_{\lambda^+}+\left(d^\ell_{jk}L_\ell(\lambda^+)\varphi_J,\varphi_{J'}\right)_{\lambda^+}\right\} + O(\|\varphi\|^2_{\lambda^+}) \\
&\geq -\epsilon\|\bar{L}_\ell\varphi_{J'}\|^2_{\lambda^+} + \epsilon^{kJ}_{jJ'}\,\mathrm{Re}\left\{\left(d^\ell_{jk}L_\ell(\lambda^+)\varphi_J,\varphi_{J'}\right)_{\lambda^+}\right\} + O(\|\varphi\|^2_{\lambda^+}).
\end{aligned}
$$

Recalling that $\mathrm{Re}\, z = \mathrm{Re}\, \bar{z}$ for any complex number z, we have

$$
\begin{aligned}
&\sum_{J,J'\in\mathcal{I}_q}\sum_{j,k,\ell=1}^{n-1}\epsilon^{kJ}_{jJ'}\,\mathrm{Re}\left\{\left(d^\ell_{jk}L_\ell(\lambda^+)\varphi_J,\varphi_{J'}\right)_{\lambda^+}\right\} \\
&= \frac{1}{2}\sum_{J,J'\in\mathcal{I}_q}\sum_{j,k,\ell=1}^{n-1}\epsilon^{kJ}_{jJ'}\,\mathrm{Re}\left\{\left(d^\ell_{jk}L_\ell(\lambda^+)\varphi_J,\varphi_{J'}\right)_{\lambda^+}+\left(d^\ell_{kj}L_\ell(\lambda^+)\varphi_{J'},\varphi_J\right)_{\lambda^+}\right\} \\
&= \frac{1}{2}\sum_{J,J'\in\mathcal{I}_q}\sum_{j,k,\ell=1}^{n-1}\epsilon^{kJ}_{jJ'}\,\mathrm{Re}\left\{\left(d^\ell_{jk}L_\ell(\lambda^+)\varphi_J,\varphi_{J'}\right)_{\lambda^+}+\left(\varphi_{J'},\bar{d}^\ell_{kj}\bar{L}_\ell(\lambda^+)\varphi_J\right)_{\lambda^+}\right\} \\
&= \frac{1}{2}\sum_{J,J'\in\mathcal{I}_q}\sum_{j,k,\ell=1}^{n-1}\epsilon^{kJ}_{jJ'}\,\mathrm{Re}\left\{\left(d^\ell_{jk}L_\ell(\lambda^+)\varphi_J,\varphi_{J'}\right)_{\lambda^+}+\overline{\left(\bar{d}^\ell_{kj}\bar{L}_\ell(\lambda^+)\varphi_J,\varphi_{J'}\right)_{\lambda^+}}\right\} \\
&= \frac{1}{2}\sum_{J,J'\in\mathcal{I}_q}\sum_{j,k,\ell=1}^{n-1}\epsilon^{kJ}_{jJ'}\left(\left(d^\ell_{jk}L_\ell(\lambda^+)\varphi_J,\varphi_{J'}\right)_{\lambda^+}+\left(\bar{d}^\ell_{kj}\bar{L}_\ell(\lambda^+)\varphi_J,\varphi_{J'}\right)_{\lambda^+}\right).
\end{aligned}
$$

Similarly,

$$\operatorname{Re}\left\{ \sum_{J,J'\in\mathcal{I}_q} \sum_{\substack{1\le j,k\le n-1\\ j\ne k}} \epsilon^{kJ}_{jJ'} \left(\bar L_k L_j \lambda^+ \varphi_J, \varphi_{J'}\right)_{\lambda^+} \right\}$$
$$= \frac12 \sum_{J,J'\in\mathcal{I}_q} \sum_{\substack{1\le j,k\le n-1\\ j\ne k}} \epsilon^{kJ}_{jJ'} \left(\left(\bar L_k L_j \lambda^+ \varphi_J, \varphi_{J'}\right)_{\lambda^+} + \left(L_j \bar L_k \lambda^+ \varphi_J, \varphi_{J'}\right)_{\lambda^+}\right).$$

□

Next, we concentrate on the $Q_{b,-}(\varphi,\varphi)$ term.

Lemma 4.3 *Let φ be a $(0,q)$-form supported in U, $\varphi \in \mathrm{Dom}(\bar\partial_b)\cap\mathrm{Dom}(\bar\partial_b^*)$. There exists $0<\epsilon'\ll 1$ so that*

$$Q_{b,-}(\varphi,\varphi) \ge (1-\epsilon') \sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1} \|\bar L_j^{*,-}\varphi_J\|^2_{\lambda^-} + \sum_{J\in\mathcal{I}_q}\sum_{j\notin J}\Bigg[\operatorname{Re}\left\{-(c_{jj}T\varphi_J,\varphi_J)_{\lambda^-}\right\}$$
$$+\frac12\left((\bar L_j L_j(\lambda^-) + L_j\bar L_j(\lambda^-))\varphi_J,\varphi_J)_{\lambda^-}\right)$$
$$+\frac12\sum_{\ell=1}^{n-1}\left((d^\ell_{jj}L_\ell(\lambda^-)+\bar d^\ell_{jj}\bar L_\ell(\lambda^-))\varphi_J,\varphi_J\right)_{\lambda^-}\Bigg]$$
$$+\sum_{J,J'\in\mathcal{I}_q}\sum_{\substack{1\le j,k\le n-1\\ j\ne k}}\epsilon^{kJ}_{jJ'}\Bigg[\operatorname{Re}\left\{-(c_{jk}T\varphi_J,\varphi_{J'})_{\lambda^-}\right\}$$
$$+\frac12\left((\bar L_k L_j(\lambda^-)+L_j\bar L_k(\lambda^-))\varphi_J,\varphi_{J'})_{\lambda^-}\right)$$
$$+\frac12\sum_{\ell=1}^{n-1}\left((d^\ell_{jk}L_\ell(\lambda^-)+\bar d^\ell_{kj}\bar L_\ell(\lambda^-))\varphi_J,\varphi_{J'}\right)_{\lambda^-}\Bigg] + O(\|\varphi\|_0^2).$$

Proof This lemma is proved with the same techniques as the previous lemma. By the argument leading up to (6), we have

$$Q_{b,-}(\varphi,\varphi) = \sum_{J\in\mathcal{I}_q}\sum_{j\notin J}\|\bar L_j\varphi_J\|^2_{\lambda^-} + \sum_{J,J'\in\mathcal{I}_q}\sum_{\substack{1\le j,k\le n-1\\ j\ne k}}\epsilon^{kJ}_{jJ'}(\bar L_j^{*,-}\varphi_J,\bar L_k^{*,-}\varphi_{J'})_{\lambda^-}$$
$$+\sum_{J,J'\in\mathcal{I}_q}\sum_{\substack{1\le j,k\le n-1\\ j\ne k}}\epsilon^{kJ}_{jJ'}\left([\bar L_j^{*,-},\bar L_k]\varphi_J,\varphi_{J'}\right)_{\lambda^-}$$

$$+\sum_{J\in\mathcal{I}_q}\sum_{j\in J}\|\bar{L}_j^{*,-}\varphi_J\|_{\lambda^-}^2-\sum_{J,J'\in\mathcal{I}_q}\sum_{\substack{1\le j,k\le n-1\\ j\neq k}}\epsilon^{kJ}_{jJ'}(\bar{L}_j^{*,-}\varphi_J,\bar{L}_k^{*,-}\varphi_{J'})_{\lambda^-}$$

$$+O\left(\|\varphi\|_{\lambda^-}^2+\left(\sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}\|\bar{L}_j^{*,-}\varphi_J\|_{\lambda^-}^2\right)^{1/2}\|\varphi\|_{\lambda^-}\right).$$

By integration by parts,

$$\|\bar{L}_j\varphi_J\|_{\lambda^-}^2=\|\bar{L}_j^{*,-}\varphi_J\|_{\lambda^-}^2+\left([\bar{L}_j^{*,-},\bar{L}_j]\varphi_J,\varphi_J\right)_{\lambda^-}.$$

Thus,

$$Q_{b,-}(\varphi,\varphi)=\sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}\|\bar{L}_j^{*,-}\varphi_J\|_{\lambda^-}^2+\sum_{J\in\mathcal{I}_q}\sum_{j\notin J}\left([\bar{L}_j^{*,-},\bar{L}_j]\varphi_J,\varphi_J\right)_{\lambda^-}$$

$$+\sum_{J,J'\in\mathcal{I}_q}\sum_{\substack{1\le j,k\le n-1\\ j\neq k}}\epsilon^{kJ}_{jJ'}\left([\bar{L}_j^{*,-},\bar{L}_k]\varphi_J,\varphi_{J'}\right)_{\lambda^-}$$

$$+O\left(\|\varphi\|_{\lambda^-}^2+\left(\sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}\|\bar{L}_j^{*,-}\varphi_J\|_{\lambda^-}^2\right)^{1/2}\|\varphi\|_{\lambda^-}\right).$$

Following the argument of Lemma 4.2, we proceed as above. □

The significance of the estimates in Lemma 4.2 and Lemma 4.3 is demonstrated by the multilinear algebra in Appendix, and it highlights the need for (CR-P_q) as well as (CR-P_{n-1-q}).

We need the following versions of the sharp Gårding inequality. This is Theorem 7.1 in [22] written for forms. It can be proved by following proofs of Theorem 3.1 and Theorem 3.2 in [20] line by line (making the obvious modifications).

Theorem 4.4 *Suppose that $P=(p_{jk}(z,D))$ is a matrix of first order pseudodifferential operators. If $p(z,\xi)$ is Hermitian and the sum of any collection of q eigenvalues is nonnegative, then there exists a constant $C>0$ so that for any $(0,q)$-form u,*

$$\operatorname{Re}\left\{\sum_{J\in\mathcal{I}_q}\sum_{j\in J}\left(p_{jj}(\cdot,D)u_J,u_J\right)-\sum_{J,J'\in\mathcal{I}_q}\sum_{\substack{1\le j,k\le m\\ j\neq k}}\epsilon^{kJ}_{jJ'}\left(p_{jk}(\cdot,D)u_J,u_{J'}\right)\right\}\ge -C\|u\|^2.$$

If $p(z,\xi)$ is Hermitian and the sum of any collection of $(n-1-q)$ eigenvalues is nonnegative, then

$$\operatorname{Re}\left\{\sum_{J\in\mathcal{I}_q}\sum_{j\notin J}\big(p_{jj}(\cdot,D)u_J,u_J\big)+\sum_{J,J'\in\mathcal{I}_q}\sum_{\substack{1\le j,k\le m\\ j\neq k}}\epsilon^{kJ}_{jJ'}\big(p_{jk}(\cdot,D)u_J,u_{J'}\big)\right\}\geq -C\|u\|^2.$$

Corollary 4.5 *Let R be a first order pseudodifferential operator such that $\sigma(R)\geq\kappa$ where κ is some positive constant and (h_{jk}) a hermitian matrix (that does not depend on ξ). Then there exists a constant C such that if the sum of any q eigenvalue of (h_{jk}) is nonnegative, then*

$$\operatorname{Re}\left\{\sum_{J\in\mathcal{I}_q}\sum_{j\in J}\big(h_{jj}Ru_J,u_J\big)-\sum_{J,J'\in\mathcal{I}_q}\sum_{\substack{1\le j,k\le m\\ j\neq k}}\epsilon^{kJ}_{jJ'}\big(h_{jk}Ru_J,u_{J'}\big)\right\}$$
$$\geq\kappa\operatorname{Re}\left\{\sum_{J\in\mathcal{I}_q}\sum_{j\in J}\big((h_{jj}u_J,u_J\big)-\sum_{J,J'\in\mathcal{I}_q}\sum_{\substack{1\le j,k\le m\\ j\neq k}}\epsilon^{kJ}_{jJ'}\big(h_{jk}u_J,u_{J'}\big)\right\}-C\|u\|^2.$$

and if the the sum of any collection of $(n-1-q)$ eigenvalues of (h_{jk}) is nonnegative, then

$$\operatorname{Re}\left\{\sum_{J\in\mathcal{I}_q}\sum_{j\notin J}\big((h_{jj}Ru_J,u_J\big)+\sum_{J,J'\in\mathcal{I}_q}\sum_{\substack{1\le j,k\le m\\ j\neq k}}\epsilon^{kJ}_{jJ'}\big(h_{jk}Ru_J,u_{J'}\big)\right\}$$
$$\geq\kappa\operatorname{Re}\left\{\sum_{J\in\mathcal{I}_q}\sum_{j\notin J}\big((h_{jj}u_J,u_J\big)+\sum_{J,J'\in\mathcal{I}_q}\sum_{\substack{1\le j,k\le m\\ j\neq k}}\epsilon^{kJ}_{jJ'}\big(h_{jk}u_J,u_{J'}\big)\right\}-C\|u\|^2.$$

Note that (h_{jk}) may be a matrix-valued function in z but may not depend on ξ.

Proof Apply the previous theorem with P where $p_{jk}=h_{jk}(R-\kappa)$. □

We need Gårding's inequality to prove the following analog to Lemma 4.12 in [22].

Lemma 4.6 *Let M be a weakly pseudoconvex CR-manifold and φ a $(0,q)$-form supported on U' so that up to a smooth term $\hat{\varphi}$ is supported in $\mathcal{C}^+$. Then*

$$\operatorname{Re}\left\{\sum_{J\in\mathcal{I}_q}\sum_{j\in J}\big(c_{jj}T\varphi_J,\varphi_J\big)_{\lambda^+}-\sum_{J,J'\in\mathcal{I}_q}\sum_{\substack{1\le j,k\le m\\ j\neq k}}\epsilon^{kJ}_{jJ'}\big(c_{jk}T\varphi_J,\varphi_{J'}\big)_{\lambda^+}\right\}$$
$$\geq A\operatorname{Re}\left\{\sum_{J\in\mathcal{I}_q}\sum_{j\in J}\big((c_{jj}\varphi_J,\varphi_J\big)_{\lambda^+}-\sum_{J,J'\in\mathcal{I}_q}\sum_{\substack{1\le j,k\le m\\ j\neq k}}\epsilon^{kJ}_{jJ'}\big(c_{jk}\varphi_J,\varphi_{J'}\big)_{\lambda^+}\right\}$$
$$+O(\|\varphi\|^2_{\lambda^+})+O_A(\|\tilde{\zeta}_\nu\tilde{\Psi}^0_A\varphi\|^2_0).$$

where the constant in $O(\|\varphi\|^2_{\lambda^+})$ does not depend on A.

Proof Let $\tilde{\Psi}^+_A$ be a pseudodifferential operator of order zero whose symbol dominates $\hat{\phi}$ (up to a smooth error) and is supported in $\tilde{\mathcal{C}}^+$. By the support conditions of φ and $\hat{\varphi}$,

$$\begin{aligned}
&\sum_{J\in\mathcal{I}_q}\sum_{j\in J}\left(c_{jj}T\varphi_J,\varphi_J\right)_{\lambda^+} - \sum_{J,J'\in\mathcal{I}_q}\sum_{\substack{1\le j,k\le m\\ j\ne k}}\epsilon^{kJ}_{jJ'}\left(c_{jk}T\varphi_J,\varphi_{J'}\right)_{\lambda^+}\\
&=\sum_{J\in\mathcal{I}_q}\sum_{j\in J}\Big(c_{jj}T\varphi_J,((\tilde{\Psi}^+_A)^*\tilde{\Psi}^+_A+(Id-(\tilde{\Psi}^+_A)^*\tilde{\Psi}^+_A))\varphi_J\Big)_{\lambda^+}\\
&\quad-\sum_{J,J'\in\mathcal{I}_q}\sum_{\substack{1\le j,k\le m\\ j\ne k}}\epsilon^{kJ}_{jJ'}\Big(c_{jk}T\varphi_J,((\tilde{\Psi}^+_A)^*\tilde{\Psi}^+_A+(Id-(\tilde{\Psi}^+_A)^*\tilde{\Psi}^+_A))\varphi_{J'}\Big)_{\lambda^+}\\
&=\sum_{J\in\mathcal{I}_q}\sum_{j\in J}\Big(c_{jj}T\varphi_J,(\tilde{\Psi}^+_A)^*\tilde{\Psi}^+_A\varphi_J\Big)_{\lambda^+}\\
&\quad-\sum_{J,J'\in\mathcal{I}_q}\sum_{\substack{1\le j,k\le m\\ j\ne k}}\epsilon^{kJ}_{jJ'}\Big(c_{jk}T\varphi_J,(\tilde{\Psi}^+_A)^*\tilde{\Psi}^+_A\varphi_{J'}\Big)_{\lambda^+}+\textit{smoother terms}\\
&=\sum_{J\in\mathcal{I}_q}\sum_{j\in J}\Big(\tilde{\zeta}e^{-\lambda^+}c_{jj}\tilde{\Psi}^+_A T\varphi_J,\tilde{\zeta}\tilde{\Psi}^+_A\varphi_J\Big)_0\\
&\quad-\sum_{J,J'\in\mathcal{I}_q}\sum_{\substack{1\le j,k\le m\\ j\ne k}}\epsilon^{kJ}_{jJ'}\Big(\tilde{\zeta}e^{-\lambda^+}c_{jk}\tilde{\Psi}^+_A T\varphi_J,\tilde{\zeta}\tilde{\Psi}^+_A\varphi_{J'}\Big)_0+\textit{smoother terms}\\
&=\sum_{J\in\mathcal{I}_q}\sum_{j\in J}\Big(\tilde{\zeta}(\tilde{\Psi}^+_A)^*\tilde{\zeta}^2e^{-\lambda^+}c_{jj}\tilde{\Psi}^+_A T\varphi_J,\varphi_J\Big)_0\\
&\quad-\sum_{J,J'\in\mathcal{I}_q}\sum_{\substack{1\le j,k\le m\\ j\ne k}}\epsilon^{kJ}_{jJ'}\Big(\tilde{\zeta}(\tilde{\Psi}^+_A)^*\tilde{\zeta}^2e^{-\lambda^+}c_{jk}\tilde{\Psi}^+_A T\varphi_J,\varphi_{J'}\Big)_0+\textit{smoother terms}
\end{aligned}$$

where smoother terms are $O(\|\varphi\|^2_{-1})$ or better (and the constant may depend on A). One fact quickly computed and used implicitly above is that $\sigma((\tilde{\Psi}^+_A)^*T)=\sigma(T(\tilde{\Psi}^+_A)^*)=\xi_{2n-1}\tilde{\psi}^+_A(\xi)$ (up to smooth terms) when applied to φ. Next, we will compute $\sigma((\tilde{\Psi}^+_A)^*\tilde{\zeta}^2_\nu e^{-\lambda^+}c_{jk})$. $\sigma(\tilde{\Psi}^+_A)\equiv 1$ on $\mathcal{C}^+$, so $\sigma((\tilde{\Psi}^+_A)^*)\equiv 1$ on $\mathcal{C}^+$ as well, and it follows that $\sigma((\tilde{\Psi}^+_A)^*)=\tilde{\psi}^+_A(\xi)$ up to terms supported in $\mathcal{C}^0\backslash\mathcal{C}^+$. Thus, up to errors on $\mathcal{C}^0\backslash\mathcal{C}^+$,

$$\sigma((\tilde{\Psi}^+_A)^*\tilde{\zeta}^2_\nu e^{-\lambda^+}c_{jk})=\sum_{\beta\ge0}\frac{1}{\beta!}\partial^\beta_\xi\tilde{\psi}^+_A(\xi)D^\beta_x(\tilde{\zeta}^2_\nu e^{-\lambda^+}c_{jk})=\tilde{\psi}^+_A(\xi)\tilde{\zeta}^2_\nu e^{-\lambda^+}c_{jk},$$

and on $\mathcal{C}^+$,

$$\begin{aligned}
\sigma((\tilde{\Psi}^+_A)^*\tilde{\zeta}^2_\nu e^{-\lambda^+}c_{jk}T\Psi^+_A)&=\sum_\alpha\frac{1}{\alpha!}\partial^\alpha_\xi\sigma((\tilde{\Psi}^+_A)^*\tilde{\zeta}^2_\nu e^{-\lambda^+}c_{jk})D^\alpha_x\sigma(T\tilde{\Psi}^+_A)\\
&=\sum_\alpha\frac{1}{\alpha!}\partial^\alpha_\xi(\tilde{\psi}^+_A(\xi)\tilde{\zeta}^2_\nu e^{-\lambda^+}c_{jk})D^\alpha_x\sigma(\xi_{2n-1}\tilde{\psi}^+_A(\xi))=\tilde{\zeta}^2_\nu e^{-\lambda^+}c_{jk}\xi_{2n-1}.
\end{aligned}$$

By construction, $\xi_{2n-1} \geq A$ on $\mathcal{C}^+$ and $(\tilde{\zeta}_\nu e^{-\lambda^+} c_{jk})$ is positive semi-definite (and hence the sum of any q eigenvalues is nonnegative), so we can apply Corollary 4.5 with T as R and $(e^{-\lambda^+} c_{jk})$ as (h_{jk}) to conclude that there exists a constant C independent of A so that

$$\sum_{J\in\mathcal{I}_q}\sum_{j\in J}\left(c_{jj}T\varphi_J,\varphi_J\right)_{\lambda^+} - \sum_{J,J'\in\mathcal{I}_q}\sum_{\substack{1\leq j,k\leq m\\ j\neq k}} \epsilon^{kJ}_{jJ'}\left(c_{jk}T\varphi_J,\varphi_{J'}\right)_{\lambda^+}$$
$$\geq A\left(\sum_{J\in\mathcal{I}_q}\sum_{j\in J}\left(\tilde{\zeta}^2 e^{-\lambda^+}c_{jj}\varphi_J,\varphi_J\right)_0 - \sum_{J,J'\in\mathcal{I}_q}\sum_{\substack{1\leq j,k\leq m\\ j\neq k}} \epsilon^{kJ}_{jJ'}\left(\tilde{\zeta}^2 e^{-\lambda^+}c_{jk}\varphi_J,\varphi_{J'}\right)_0\right)$$
$$-C\|\varphi\|^2_{\lambda^+} + O(\|\varphi\|^2_{-1}) + O_A(\|\tilde{\zeta}_\nu\tilde{\Psi}^0_A\varphi\|^2_0)$$
$$= A\sum_{J\in\mathcal{I}_q}\sum_{j\in J}\left(c_{jj}\varphi_J,\varphi_J\right)_{\lambda^+} - \sum_{J,J'\in\mathcal{I}_q}\sum_{\substack{1\leq j,k\leq m\\ j\neq k}} \epsilon^{kJ}_{jJ'}\left(c_{jk}\varphi_J,\varphi_{J'}\right)_{\lambda^+}$$
$$+O(\|\varphi\|^2_0) + O_A(\|\tilde{\zeta}\tilde{\Psi}^0_A\varphi\|^2_0).$$

□

By the same argument, we have the following:

Lemma 4.7 *Let φ be a $(0,q)$-form supported on U so that up to a smooth term, $\hat{\varphi}$ is supported in $\mathcal{C}^-$, then*

$$\sum_{J\in\mathcal{I}_q}\sum_{j\notin J}\left(c_{jj}(-T)\varphi_J,\varphi_J\right)_{\lambda^-} + \sum_{J,J'\in\mathcal{I}_q}\sum_{\substack{1\leq j,k\leq m\\ j\neq k}} \epsilon^{kJ}_{jJ'}\left(c_{jk}(-T)\varphi_J,\varphi_{J'}\right)_{\lambda^-}$$
$$\geq A\left(\sum_{J\in\mathcal{I}_q}\sum_{j\notin J}\left(c_{jj}\varphi_J,\varphi_J\right)_{\lambda^-} + \sum_{J,J'\in\mathcal{I}_q}\sum_{\substack{1\leq j,k\leq m\\ j\neq k}} \epsilon^{kJ}_{jJ'}\left(c_{jk}\varphi_J,\varphi_{J'}\right)_{\lambda^-}\right)$$
$$+O(\|\varphi\|^2_{\lambda^-}) + O_A(\|\tilde{\zeta}_\nu\tilde{\Psi}^0_A\varphi\|^2_0).$$

We now review the two local results from [22] that are crucial in proving the basic estimate Proposition 4.1. Let $(s^+_{jk})^{n-1}_{j,k=1}$ be the matrix defined by

$$s^+_{jk} = \frac{1}{2}\left(\bar{L}_k L_j(\lambda^+) + L_j\bar{L}_k(\lambda^+) + \sum_{\ell=1}^{n-1}(d^\ell_{jk}L_\ell(\lambda^+) + \bar{d}^\ell_{kj}\bar{L}_\ell(\lambda^+))\right) + A_0 c_{jk}.$$

Proposition 4.8 *Let $\varphi \in \mathrm{Dom}(\bar{\partial}_b)\cap\mathrm{Dom}(\bar{\partial}^*_b)$ be a $(0,q)$-form supported in U. Assume that λ^+ is a strictly CR-plurisubharmonic function on $(0,q)$-forms with CR-plurisubharmonicity constant A_{λ^+}. Then there exists a constant C that is independent of A_{λ^+} so that*

$$Q_{b,+}(\tilde{\zeta}\Psi^+_A\varphi,\tilde{\zeta}\Psi^+_A\varphi) + C\|\tilde{\zeta}\Psi^+_A\varphi\|^2_{\lambda^+} + O_{\lambda^+}(\|\tilde{\zeta}\tilde{\Psi}^0_A\varphi\|^2_0) \geq A_{\lambda^+}\|\tilde{\zeta}\Psi^+_A\varphi\|^2_{\lambda^+}.$$

Proof Since $\varphi \in \mathrm{Dom}(\bar\partial_b)\cap\mathrm{Dom}(\bar\partial_b^*)$, it follows that $\tilde\zeta\Psi_A^+\varphi \in \mathrm{Dom}(\bar\partial_b)\cap\mathrm{Dom}(\bar\partial_b^*)$. Moreover, $\mathrm{supp}(\tilde\zeta\Psi_A^+\varphi)\subset U'$. By Lemma 4.2,

$$
\begin{aligned}
&Q_{b,+}(\tilde\zeta\Psi_A^+\varphi,\tilde\zeta\Psi_A^+\varphi) \ge (1-\epsilon')\sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}\|\bar L_j\tilde\zeta\Psi_A^+\varphi_J\|_{\lambda^+}^2\\
&+\mathrm{Re}\left\{\sum_{J\in\mathcal{I}_q}\sum_{j\in J}(c_{jj}T\tilde\zeta\Psi_A^+\varphi_J,\tilde\zeta\Psi_A^+\varphi_J)_{\lambda^+}\right.\\
&\left.-\sum_{J,J'\in\mathcal{I}_q}\sum_{\substack{1\le j,k\le n-1\\ j\ne k}}\epsilon^{kJ}_{jJ'}(c_{jk}T\tilde\zeta\Psi_A^+\varphi_J,\tilde\zeta\Psi_A^+\varphi_{J'})_{\lambda^+}\right\}\\
&+\frac12\sum_{J\in\mathcal{I}_q}\sum_{j\in J}\Bigg[\Big((\bar L_jL_j(\lambda^+)+L_j\bar L_j(\lambda^+))\tilde\zeta\Psi_A^+\varphi_J,\tilde\zeta\Psi_A^+\varphi_J)_{\lambda^+}\Big)\\
&+\sum_{\ell=1}^{n-1}\Big((d^\ell_{jj}L_\ell(\lambda^+)+\bar d^\ell_{jj}\bar L_\ell(\lambda^+))\tilde\zeta\Psi_A^+\varphi_J,\tilde\zeta\Psi_A^+\varphi_J\Big)_{\lambda^+}\Bigg]\\
&-\frac12\sum_{J,J'\in\mathcal{I}_q}\sum_{\substack{1\le j,k\le n-1\\ j\ne k}}\epsilon^{kJ}_{jJ'}\Bigg[\Big((\bar L_kL_j(\lambda^+)+L_j\bar L_k(\lambda^+))\tilde\zeta\Psi_A^+\varphi_J,\tilde\zeta\Psi_A^+\varphi_{J'})_{\lambda^+}\Big)\\
&+\sum_{\ell=1}^{n-1}\Big((d^\ell_{jk}L_\ell(\lambda^+)+\bar d^\ell_{kj}\bar L_\ell(\lambda^+))\tilde\zeta\Psi_A^+\varphi_J,\tilde\zeta\Psi_A^+\varphi_{J'}\Big)_{\lambda^+}\Bigg]+O(\|\tilde\zeta\Psi_A^+\varphi\|_0^2).
\end{aligned}
$$

To control the T terms, we use Lemma 4.6 since $\mathrm{supp}\,\tilde\zeta\subset U'$, and the Fourier transform of $\tilde\zeta\Psi_A^+\varphi$ is supported in $\mathcal{C}^+$ up to a smooth term. Indeed, with $A=A_0$ (and A_0 from the definition of (CR-P_q)), we have

$$
\begin{aligned}
&\mathrm{Re}\left\{\sum_{J\in\mathcal{I}_q}\sum_{j\in J}(c_{jj}T\tilde\zeta\Psi_A^+\varphi_J,\tilde\zeta\Psi_A^+\varphi_J)_{\lambda^+}\right.\\
&\left.-\sum_{J,J'\in\mathcal{I}_q}\sum_{\substack{1\le j,k\le n-1\\ j\ne k}}\epsilon^{kJ}_{jJ'}(c_{jk}T\tilde\zeta\Psi_A^+\varphi_J,\tilde\zeta\Psi_A^+\varphi_{J'})_{\lambda^+}\right\}\\
&\ge A_0\Bigg[\sum_{J\in\mathcal{I}_q}\sum_{j\in J}(c_{jj}\tilde\zeta\Psi_A^+\varphi_J,\tilde\zeta\Psi_A^+\varphi_J)_{\lambda^+}
\end{aligned}
$$

$$- \sum_{J,J' \in \mathcal{I}_q} \sum_{\substack{1 \le j,k \le n-1 \\ j \ne k}} \epsilon^{kJ}_{jJ'} (c_{jk} \tilde{\zeta} \Psi^+_A \varphi_J, \tilde{\zeta} \Psi^+_A \varphi_{J'})_{\lambda^+} \Bigg]$$
$$+ O(\|\tilde{\zeta} \Psi^+_A \varphi\|^2_{\lambda^+}) + O_{\lambda^+}(\|\tilde{\zeta} \tilde{\Psi}^0_A \varphi\|^2_0).$$

Putting these estimates together, we have

$$Q_{b,+}(\tilde{\zeta} \Psi^+_A \varphi, \tilde{\zeta} \Psi^+_A \varphi)$$
$$\ge \sum_{J \in \mathcal{I}_q} \sum_{j \in J} (s^+_{jj} \tilde{\zeta} \Psi^+_A \varphi_J, \tilde{\zeta} \Psi^+_A \varphi_J)_{\lambda^+} - \sum_{J,J' \in \mathcal{I}_q} \sum_{\substack{1 \le j,k \le n-1 \\ j \ne k}} \epsilon^{kJ}_{jJ'} (s^+_{jk} \tilde{\zeta} \Psi^+_A \varphi_J, \tilde{\zeta} \Psi^+_A \varphi_{J'})_{\lambda^+}$$
$$+ O(\|\tilde{\zeta} \Psi^+_A \varphi\|^2_{\lambda^+}) + O_{\lambda^+}(\|\tilde{\zeta} \tilde{\Psi}^0_A \varphi\|^2_0)).$$

Recall that λ^+ is strictly CR-plurisubharmonic on $(0, q)$-forms with CR-plurisubharmonicity constant A_{λ^+}. In local coordinates, if $L = \sum_{j=1}^{n-1} \xi_j L_j$, then

$$\left\langle \frac{1}{2} \left(\partial_b \bar{\partial}_b \lambda^+ - \bar{\partial}_b \partial_b \lambda^+ \right) + A_0 d\gamma, L \wedge \bar{L} \right\rangle = \sum_{j,k=1}^{n-1} s^+_{jk} \xi_j \bar{\xi}_k,$$

and (s^+_{jk}) is a Hermitian matrix. Therefore, by the multilinear algebra lemmas, Lemma A.1 and Lemma A.2,

$$Q_{b,+}(\tilde{\zeta} \Psi^+_A \varphi, \tilde{\zeta} \Psi^+_A \varphi) + C \|\tilde{\zeta} \Psi^+_A \varphi\|^2_{\lambda^+} + O_{\lambda^+}(\|\tilde{\zeta} \tilde{\Psi}^0_A \varphi\|^2_0) \ge A_{\lambda^+} \|\tilde{\zeta} \Psi^+_A \varphi\|^2_{\lambda^+}.$$

where the constant C is independent of A_{λ^+}. □

Let

$$s^-_{jk} = \frac{1}{2} \left(\bar{L}_k L_j(\lambda^-) + L_j \bar{L}_k(\lambda^-) + \sum_{\ell=1}^{n-1} (d^\ell_{jk} L_\ell(\lambda^-) + \bar{d}^\ell_{kj} \bar{L}_\ell(\lambda^-)) \right) + A_0 c_{jk}.$$

Proposition 4.9 *Let* $\varphi \in \mathrm{Dom}(\bar{\partial}_b) \cap \mathrm{Dom}(\bar{\partial}^*_b)$ *be a* $(0, q)$*-form supported in* U. *Assume that* λ^- *is a strictly CR-plurisubharmonic function on* $(0, n-1-q)$*-forms with CR-plurisubharmonicity constant* A_{λ^-} *Then there exists a constant* C *that is independent of* A_{λ^-} *so that*

$$Q_{b,-}(\tilde{\zeta} \Psi^-_A \varphi, \tilde{\zeta} \Psi^-_A \varphi) + C \|\tilde{\zeta} \Psi^-_A \varphi\|^2_{\lambda^+} + O_{\lambda^-}(\|\tilde{\zeta} \tilde{\Psi}^0_A \varphi\|^2_0) \ge A_{\lambda^-} \|\tilde{\zeta} \Psi^-_A \varphi\|^2_{\lambda^-}.$$

Proof Similarly to the proof of Lemma 4.8, we can apply Lemma 4.3 to $\tilde{\zeta} \Psi^-_A \varphi$ which gives (for some $1 \gg \epsilon > 0$)

$$Q_{b,-}(\tilde{\zeta}\Psi_A^-\varphi,\tilde{\zeta}\Psi_A^-\varphi) \geq (1-\epsilon')\sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}\|\bar{L}_j^{*,-}\tilde{\zeta}\Psi_A^-\varphi_J\|^2_{\lambda^-}$$

$$+\operatorname{Re}\left\{\sum_{J\in\mathcal{I}_q}\sum_{j\notin J}(c_{jj}(-T)\tilde{\zeta}\Psi_A^-\varphi_J,\tilde{\zeta}\Psi_A^-\varphi_J)_{\lambda^-}\right.$$

$$\left.+\sum_{J,J'\in\mathcal{I}_q}\sum_{\substack{1\leq j,k\leq n-1\\ j\neq k}}\epsilon^{kJ}_{jJ'}(c_{jk}(-T)\tilde{\zeta}\Psi_A^-\varphi_J,\tilde{\zeta}\Psi_A^-\varphi_{J'})_{\lambda^-}\right\}$$

$$+\frac{1}{2}\sum_{J\in\mathcal{I}_q}\sum_{j\notin J}\left[\Big((\bar{L}_jL_j(\lambda^-)+L_j\bar{L}_j(\lambda^-))\tilde{\zeta}\Psi_A^-\varphi_J,\tilde{\zeta}\Psi_A^-\varphi_J)_{\lambda^-}\Big)\right.$$

$$\left.+\sum_{\ell=1}^{n-1}\Big((d^\ell_{jj}L_\ell(\lambda^-)+\bar{d}^\ell_{jj}\bar{L}_\ell(\lambda^-))\tilde{\zeta}\Psi_A^-\varphi_J,\tilde{\zeta}\Psi_A^-\varphi_J\Big)_{\lambda^-}\right]$$

$$+\frac{1}{2}\sum_{J,J'\in\mathcal{I}_q}\sum_{\substack{1\leq j,k\leq n-1\\ j\neq k}}\epsilon^{kJ}_{jJ'}\left[\Big((\bar{L}_kL_j(\lambda^-)+L_j\bar{L}_k(\lambda^-))\tilde{\zeta}\Psi_A^-\varphi_J,\tilde{\zeta}\Psi_A^-\varphi_{J'})_{\lambda^-}\Big)\right.$$

$$\left.+\sum_{\ell=1}^{n-1}\Big((d^\ell_{jk}L_\ell(\lambda^-)+\bar{d}^\ell_{kj}\bar{L}_\ell(\lambda^-))\tilde{\zeta}\Psi_A^-\varphi_J,\tilde{\zeta}\Psi_A^-\varphi_{J'}\Big)_{\lambda^-}\right]+O(\|\tilde{\zeta}\Psi_A^-\varphi\|_0^2).$$

To control the T terms, we use Lemma 4.7 since $\operatorname{supp}\tilde{\zeta}\subset U'$, and the Fourier transform of $\tilde{\zeta}\Psi_A^-\varphi$ is supported in $\mathcal{C}^-$ up to a smooth term. Indeed, with $A=A_0$ where A_0 is from the definition of CR-plurisubharmonicity on $(0,q)$-forms,

$$\operatorname{Re}\left\{\sum_{J\in\mathcal{I}_q}\sum_{j\notin J}(c_{jj}(-T)\tilde{\zeta}\Psi_A^-\varphi_J,\tilde{\zeta}\Psi_A^-\varphi_J)_{\lambda^-}\right.$$

$$\left.+\sum_{J,J'\in\mathcal{I}_q}\sum_{\substack{1\leq j,k\leq n-1\\ j\neq k}}\epsilon^{kJ}_{jJ'}(c_{jk}(-T)\tilde{\zeta}\Psi_A^-\varphi_J,\tilde{\zeta}\Psi_A^-\varphi_{J'})_{\lambda^-}\right\}$$

$$\geq A_0\left[\sum_{J\in\mathcal{I}_q}\sum_{j\notin J}(c_{jj}\tilde{\zeta}\Psi_A^-\varphi_J,\tilde{\zeta}\Psi_A^-\varphi_J)_{\lambda^-}\right.$$

$$\left.+\sum_{J,J'\in\mathcal{I}_q}\sum_{\substack{1\leq j,k\leq n-1\\ j\neq k}}\epsilon^{kJ}_{jJ'}(c_{jk}\tilde{\zeta}\Psi_A^-\varphi_J,\tilde{\zeta}\Psi_A^+\varphi_{J'})_{\lambda^-}\right]$$

$$+O(\|\tilde{\zeta}\Psi_A^-\varphi\|^2_{\lambda^-})+O_{\lambda^-}(\|\tilde{\zeta}\tilde{\Psi}_A^0\varphi\|_0^2)$$

Putting these estimates together, we have

$$
\begin{aligned}
&Q_{b,-}(\tilde\zeta\Psi^-_A\varphi, \tilde\zeta\Psi^-_A\varphi)\\
&\geq \sum_{J\in\mathcal{I}_q}\sum_{j\notin J}(s^-_{jj}\tilde\zeta\Psi^-_A\varphi_J, \tilde\zeta\Psi^-_A\varphi_J)_{\lambda^-} + \sum_{J,J'\in\mathcal{I}_q}\sum_{\substack{1\leq j,k\leq n-1\\ j\neq k}}\epsilon^{kJ}_{jJ'}(s^-_{jk}\tilde\zeta\Psi^-_A\varphi_J, \tilde\zeta\Psi^-_A\varphi_{J'})_{\lambda^-}\\
&\quad + O(\|\tilde\zeta\Psi^-_A\varphi\|^2_{\lambda^-}) + O_{\lambda^-}(\|\tilde\zeta\tilde\Psi^0_A\varphi\|^2_0)).
\end{aligned}
$$

Recall that λ^- is strictly CR-plurisubharmonic on $(0, n-1-q)$-forms with CR-plurisubharmonicity constant A_{λ^-}. In local coordinates, if $L = \sum_{j=1}^{n-1}\xi_j L_j$, then

$$
\left\langle \frac{1}{2}\left(\partial_b\bar\partial_b\lambda^- - \bar\partial_b\partial_b\lambda^-\right) + A_0 d\gamma, L\wedge\bar L\right\rangle = \sum_{j,k=1}^{n-1} s^-_{jk}\xi_j\bar\xi_k,
$$

and (s^-_{jk}) is a Hermitian matrix. Therefore, by the multilinear algebra lemmas, Lemma A.1 and Lemma A.3,

$$
Q_{b,-}(\tilde\zeta\Psi^-_A\varphi, \tilde\zeta\Psi^-_A\varphi) + C\|\tilde\zeta\Psi^-_A\varphi\|^2_{\lambda^-} + O_{\lambda^-}(\|\tilde\zeta\tilde\Psi^0_A\varphi\|^2_0) \geq A_{\lambda^-}\|\tilde\zeta\Psi^-_A\varphi\|^2_{\lambda^-}.
$$

where the constant C is independent of A_{λ^-}. □

We are finally ready to prove the basic estimate.

Proof (Basic Estimate—Proposition 4.1) From (4), there exist constants K, $K_\pm$ so that

$$
\begin{aligned}
&K Q_{b,\pm}(\varphi,\varphi) + K_\pm\sum_\nu\|\tilde\zeta_\nu\tilde\Psi^0_{\nu,A}\zeta_\nu\varphi^\nu\|^2_0 + K'\|\varphi\|^2_0 + O_\pm(\|\varphi\|^2_{-1})\\
&\quad\geq \sum_\nu\left[Q_{b,+}(\tilde\zeta_\nu\Psi^+_{\nu,A}\zeta_\nu\varphi^\nu, \tilde\zeta_\nu\Psi^+_{\nu,A}\zeta_\nu\varphi^\nu) + Q_{b,-}(\tilde\zeta_\nu\Psi^-_{\nu,A}\zeta_\nu\varphi^\nu, \tilde\zeta_\nu\Psi^-_{\nu,A}\zeta_\nu\varphi^\nu)\right].
\end{aligned}
$$

From Proposition 4.8 and Proposition 4.9 it follows that by increasing the size of K, $K_\pm$, and K' (where K' does NOT depend on A) that

$$
K Q_{b,\pm}(\varphi,\varphi) + K_\pm\sum_\nu\|\tilde\zeta_\nu\tilde\Psi^0_{\nu,A}\zeta_\nu\varphi^\nu\|^2_0 + K'\|\varphi\|^2_0 + O_\pm(\|\varphi\|^2_{-1}) \geq A_\pm\|\varphi\|^2_0
$$

where $A_\pm = \min\{A_{\lambda^-}, A_{\lambda^+}\}$. □

4.2 A Sobolev estimate in the "elliptic directions"

For forms whose Fourier transforms are supported up to a smooth term in $\mathcal{C}^0$, we have better estimates. The following result is the $(0, q)$-form version of Lemma 4.18 in [22].

Lemma 4.10 *Let φ be a $(0,1)$-form supported in U_ν for some ν such that up to a smooth term, $\hat{\varphi}$ is supported in $\tilde{\mathcal{C}}^0_\nu$. There exist positive constants $C > 1$ and $C_1 > 0$ independent of A so that*

$$C Q_{b,\pm}(\varphi, E_\pm \varphi) + C_1 \|\varphi\|_0^2 \geq \|\varphi\|_1^2.$$

The proof in [22] also holds at level $(0, q)$.

We can use Lemma 4.10 to control terms of the form $\|\tilde{\zeta}_\nu \Psi^0_{\nu,A} \zeta_\nu \varphi^\nu\|_0^2$.

Proposition 4.11 *For any $\epsilon > 0$, there exists $C_{\epsilon,\pm} > 0$ so that*

$$\|\tilde{\zeta}_\nu \Psi^0_{\nu,A} \zeta_\nu \varphi^\nu\|_0^2 \leq \epsilon Q_{b,\pm}(\varphi^\nu, \varphi^\nu) + C_\pm \|\varphi^\nu\|_{-1}^2.$$

Proof Observe that $\|\tilde{\zeta}_\nu \Psi^0_{\nu,A} \zeta_\nu \varphi^\nu\|_0^2 = \|\Lambda^{-1} \tilde{\zeta}_\nu \Psi^0_{\nu,A} \zeta_\nu \varphi^\nu\|_1^2$. The $(0, q)$-form $\tilde{\zeta}_\nu \Psi^0_{\nu,A} \zeta_\nu \varphi^\nu$ is supported in $\mathcal{C}^0$, so Lemma 4.10 applies. Although the range of Λ^{-1} is outside U_ν, we can write $\Lambda^{-1}\tilde{\zeta}_\nu = \zeta'_\nu \Lambda^{-1} \tilde{\zeta}_\nu + (1 - \zeta'_\nu)\Lambda^{-1}\tilde{\zeta}_\nu$ where ζ'_ν is a smooth bump function that is identically one on the support of $\tilde{\zeta}_\nu$. Then $(1 - \zeta'_\nu)\Lambda^{-1}\tilde{\zeta}_\nu$ is infinitely smoothing and hence can be absorbed in the $\|\varphi\|_{-1}^2$ term. Let $P = \zeta'_\nu \Lambda^{-1}$ and $\psi = \tilde{\zeta}_\nu \Psi^0_{\nu,A} \zeta_\nu \varphi^\nu$. By Lemma 4.10 and the fact that P is a pseudodifferential operator of order -1,

$$\|\Lambda^{-1} \tilde{\zeta}_\nu \Psi^0_{\nu,A} \zeta_\nu \varphi^\nu\|_1^2 \leq \|P\psi\|_1^2 + C\|\varphi^\nu\|_{-1}^2 \leq C_1 Q_{b,\pm}(P\psi, P\psi) + C\|\varphi^\nu\|_{-1}^2.$$

The adjoint of P is $P^{*,\pm} = \zeta'_\nu \Lambda^{-1}$. Consequently $P - P^{*,\pm}$ is a pseudodifferential operator of order -2, and we can apply Lemma 2.4.2 in [9] to prove

$$\begin{aligned} Q_{b,\pm}(P\psi, P\psi) &= \operatorname{Re} Q_{b,\pm}(\psi, P^{*,\pm} P\psi) + C_\pm \|\varphi^\nu\|_{-1}^2 \\ &\leq \epsilon Q_{b,\pm}(\varphi^\nu, \varphi^\nu) + C_{\epsilon,\pm} \|\varphi^\nu\|_{-1}^2. \end{aligned}$$

□

The term $\epsilon Q_{b,\pm}(\varphi, \varphi)$ could be replaced by $\epsilon \|\Box_{b,\pm}\varphi\|_{-1}^2$ if we had a need for it.

5 Existence and compactness theorems for the complex Green operator

In this section, we use the basic estimate to prove existence and compactness theorems for the complex Green operator. As always, M is a compact, orientable, weakly pseudoconvex CR-manifold of dimension at least 5, endowed with strongly CR-plurisubharmonic functions λ^+ and λ^-.

5.1 Closed range for $\Box_{b,\pm}$

For $1 \le q \le n-2$, let

$$\begin{aligned}\mathcal{H}^q_\pm &= \{\varphi \in \mathrm{Dom}(\bar\partial_b) \cap \mathrm{Dom}(\bar\partial_b^*) : \bar\partial_b \varphi = 0, \bar\partial^*_{b,\pm}\varphi = 0\}\\ &= \{\varphi \in \mathrm{Dom}(\bar\partial) \cap \mathrm{Dom}(\bar\partial_b^*) : Q_{b,\pm}(\varphi,\varphi) = 0\}\end{aligned}$$

be the space of $\pm$-harmonic $(0, q)$-forms.

Lemma 5.1 *For $A_\pm$ suitably large and $1 \le q \le n-2$, $\mathcal{H}^q_\pm$ is finite dimensional and there exists $C > 0$ that does not depend on λ^+ or λ^- so that for all $(0,q)$-forms $\varphi \in \mathrm{Dom}(\bar\partial_b) \cap \mathrm{Dom}(\bar\partial_b^*)$ so that $\varphi \perp \mathcal{H}^q_\pm$ (with respect to $\langle\cdot,\cdot\rangle_\pm$).*

$$|||\varphi|||^2_\pm \le C Q_{b,\pm}(\varphi,\varphi). \tag{7}$$

Proof For $\varphi \in \mathcal{H}^q_\pm$, we can use Proposition 4.1 with $A_\pm$ suitably large (to absorb terms) so that

$$A_\pm |||\varphi|||^2_\pm \le C_\pm \left(\sum_\nu \|\tilde\zeta_\nu \Psi^0_{\nu,A}\zeta_\mu \varphi^\nu\|^2_0 + \|\varphi\|^2_{-1} \right).$$

Also, by Proposition 4.11,

$$\sum_\nu \|\tilde\zeta_\nu \Psi^0_{\nu,A}\zeta_\mu \varphi^\nu\|^2_0 \le C_\pm \|\varphi\|^2_{-1}.$$

since $Q_{b,\pm}(\varphi,\varphi) = 0$. The unit ball in $\mathcal{H}^q_\pm \cap L^2(M)$ is compact, and hence finite dimensional.

Assume that (7) fails. Then there exists $\varphi_k \perp \mathcal{H}^q_\pm$ with $|||\varphi_k|||_\pm = 1$ so that

$$|||\varphi_k|||^2_\pm \ge k Q_{b,\pm}(\varphi_k,\varphi_k). \tag{8}$$

For k suitably large, we can use Proposition 4.1 and the above argument to absorb $Q_{b,\pm}(\varphi_k,\varphi_k)$ by $A_\pm |||\varphi_k|||_\pm$ to get:

$$|||\varphi_k|||^2_\pm \le C_\pm \|\varphi_k\|^2_{-1}. \tag{9}$$

Since $L^2(M)$ is compact in $H^{-1}(M)$, there exists a subsequence φ_{k_j} that converges in $H^{-1}(M)$. However, (9) forces φ_{k_j} to converge in $L^2(M)$ as well. Although the norm $(Q_{b,\pm}(\cdot,\cdot) + |||\cdot|||^2_\pm)^{1/2}$ dominates the $L^2(M)$-norm, (8) applied to φ_{j_k} shows that φ_{j_k} converges in the $(Q_{b,\pm}(\cdot,\cdot) + |||\cdot|||^2_\pm)^{1/2}$ norm as well. The limit φ satisfies $|||\varphi|||_\pm = 1$ and $\varphi \perp \mathcal{H}^q_\pm$. However, a consequence of (8) is that $\varphi \in \mathcal{H}^q_\pm$. This is a contradiction and (8) holds. □

Let

$${}^\perp\mathcal{H}^q_\pm = \{\varphi \in L^2_{0,q}(M) : \langle\varphi,\phi\rangle_\pm = 0, \text{ for all } \phi \in \mathcal{H}^q_\pm\}.$$

On ${}^\perp\mathcal{H}^q_\pm$, define

$$\Box_{b,\pm} = \bar\partial_b\bar\partial^*_{b,\pm} + \bar\partial^*_{b,\pm}\bar\partial_b.$$

Since $\bar\partial^*_{b,\pm} = E_\pm\bar\partial^*_b + [\bar\partial^*_b, E_\pm]$, $\mathrm{Dom}(\bar\partial^*_{b,\pm}) = \mathrm{Dom}(\bar\partial^*_b)$. This causes

$$\begin{aligned}\mathrm{Dom}(\Box_{b,\pm}) = \{\varphi \in L^2_{0,q}(M) : \varphi \in \mathrm{Dom}(\bar\partial_b) \cap \mathrm{Dom}(\bar\partial^*_b),\\ \bar\partial_b\varphi \in \mathrm{Dom}(\bar\partial^*_b), \quad \text{and} \quad \bar\partial^*_b\varphi \in \mathrm{Dom}(\bar\partial_b)\}.\end{aligned}$$

5.2 Proof of Theorem 1.1 when $s = 0$

This subsection is devoted the proof of Theorem 1.1 when $s = 0$, i.e., the L^2-case.

As a consequence of Lemma 5.1, we may apply Theorem 1.1.2 in [12] to conclude that $\bar\partial_b : L^2_{(0,q)}(M) \to L^2_{(0,q+1)}(M)$ and $\bar\partial^*_{b,\pm} : L^2_{(0,q)}(M) \to L^2_{(0,q-1)}(M)$ have closed range. However, by Theorem 1.1.1 in [12], this also means that $\bar\partial_b : L^2_{(0,q-1)}(M) \to L^2_{(0,q)}(M)$ and $\bar\partial^*_{b,\pm} : L^2_{(0,q+1)}(M) \to L^2_{(0,q)}(M)$ have closed range (and satisfy the appropriate L^2 inequality with a constant that does not depend on λ^+ or λ^-). Again by Lemma 5.1, Theorem 1.1.1 in [12], and Lemma 3.5, $\bar\partial^*_b$ has closed range when acting on $L^2_{(0,q)}(M)$ or $L^2_{(0,q+1)}(M)$. Therefore, for a $(0, q)$-form $u \in \mathrm{Dom}(\bar\partial_b) \cap \mathrm{Dom}(\bar\partial^*_b)$, we have the estimates

$$\|u\|_0^2 \le C(\|\bar\partial_b u\|_0^2 + \|\bar\partial^*_b u\|_0^2 + \|H_q u\|_0^2) \tag{10}$$

and

$$\|u\|_0^2 \le C(Q_{b,\pm}(u, u) + \|H_{\pm,q}u\|_0^2) \tag{11}$$

where H_q is the projection of u onto $\mathcal{H}^q_{\tau p}$ and $H_{\pm,q}$ is the projection of u onto $\mathcal{H}^q_\pm$. This implies the existence of G_q and $G_{q,\pm}$ as bounded operators on $L^2_{(0,q)}(M)$ that invert $\Box_b$ on ${}^\perp\mathcal{H}^q_{\tau p}$ and $\Box_{b,\pm}$ on ${}^\perp\mathcal{H}^q_\pm$, respectively (see for example [24], Lemma 3.2 and its proof). Moreover, the solvability of $\bar\partial_b$ in $L^2_{(0,q)}(M)$ and weighted $L^2_{(0,q)}(M)$ forces

$$\ker(\bar\partial_b) = \underbrace{\mathrm{Range}(\bar\partial_b) \oplus \mathcal{H}^q_\pm}_{\oplus \text{ with respect to } \langle\cdot,\cdot\rangle_\pm} = \underbrace{\mathrm{Range}(\bar\partial_b) \oplus \mathcal{H}^q_{\tau p}}_{\oplus \text{ with respect to } (\cdot,\cdot)_0}.$$

Consequently, $\mathcal{H}^q_{\tau p}$ is finite dimensional.

We now prove that G_q is compact. First observe, we have the following identity:

$$\begin{aligned}G_{q+1}\bar\partial_b u = G_{q+1}\bar\partial_b(\bar\partial_b\bar\partial^*_b + \bar\partial^*_b\bar\partial_b)G_q u &= G_{q+1}\bar\partial_b\bar\partial^*_b\bar\partial_b G_q u\\ &= G_{q+1}(\bar\partial_b\bar\partial^*_b + \bar\partial^*_b\bar\partial_b)\bar\partial_b G_q u = \bar\partial_b G_q u.\end{aligned}$$

Thus,

$$\bar{\partial}_b G_q = (\bar{\partial}_b^* G_{q+1})^*.$$

To prove that G_q is a compact operator, it suffices to show compactness on ${}^\perp\mathcal{H}^q_{\tau p}$ (since G_q is zero on $\mathcal{H}^q_{\tau p}$). When $u \in {}^\perp\mathcal{H}^q_{\tau p}$, Eq. (10) implies (since $G_q u \in {}^\perp\mathcal{H}^q_{\tau p}$)

$$\|G_q u\|_0^2 \lesssim \|\bar{\partial}_b G_q u\|_0^2 + \|\bar{\partial}_b^* G_q u\|_0^2 = \|(\bar{\partial}_b^* G_{q+1})^* u\|_0^2 + \|\bar{\partial}_b^* G_q u\|_0^2. \tag{12}$$

Therefore, we only need to show that both $\bar{\partial}_b^* G_q$ and $\bar{\partial}_b^* G_{q+1}$ are compact. Our main tool will be a strengthening of (11). We claim that

$$\|u\|_0^2 \le \frac{C}{A_\pm}\left(|||\bar{\partial}_b u|||_\pm^2 + |||\bar{\partial}_{b,\pm}^* u|||_\pm^2\right) + C_\pm \|u\|_{-1}^2. \tag{13}$$

To prove (13), we already know the estimate if $u \in \mathcal{H}^q_\pm$, so we can assume that $u \in {}^\perp\mathcal{H}^q_\pm$. we use Proposition 4.1 to see that

$$A_\pm |||u|||_\pm \le K Q_{b,\pm}(u,u) + K_\pm\left(\sum_\nu \|\tilde{\zeta}_\nu \Psi^0_{\nu,A}\zeta_\nu u^\nu\|_0^2 + \|u\|_{-1}^2\right).$$

Thus, to prove (13), we have to show that $K_\pm \sum_\nu \|\tilde{\zeta}_\nu \Psi^0_{\nu,A}\zeta_\nu u^\nu\|_0^2$ is well-controlled. Using Proposition 4.11, we have (with $\epsilon = 1/K_\pm$),

$$K_\pm \sum_\nu \|\tilde{\zeta}_\nu \Psi^0_{\nu,A}\zeta_\nu u^\nu\|_0^2 \le Q_{b,\pm}(u,u) + K'_\pm \|u\|_{-1}^2.$$

and (13) is proved.

When $\alpha \in \text{Range}(\bar{\partial}_b) \subset L^2_{0,q+1}(M)$, $\bar{\partial}_b^* G_{q+1}\alpha$ gives the L^2-norm minimizing solution to $\bar{\partial}_b v = \alpha$, and $\bar{\partial}_{b,\pm}^* G_{\pm,q+1}\alpha$ gives a the solution that minimizes the $|||\cdot|||_\pm$-norm. For such α, (13) therefore implies

$$\begin{aligned}\|\bar{\partial}_b^* G_{q+1}\alpha\|_0^2 &\le \|\bar{\partial}_{b,\pm}^* G_{\pm,q+1}\alpha\|_0^2 \le C|||\bar{\partial}_{b,\pm}^* G_{\pm,q+1}\alpha|||_\pm^2 \\ &\le \frac{C}{A_\pm}|||\alpha|||_\pm + C_\pm \|\bar{\partial}_{b,\pm}^* G_{\pm,q+1}\alpha\|_{-1}^2 \\ &\le \frac{C}{A_\pm}|||\alpha|||_\pm + C_\pm \|\bar{\partial}_{b,\pm}^* G_{\pm,q+1}\alpha\|_{-1}^2 \end{aligned} \tag{14}$$

Applying Lemma 5.1 to $\bar{\partial}_{b,\pm}^* G_{\pm,q+1}$ shows that $\bar{\partial}_{b,\pm}^* G_{\pm,q+1} : L^2_{0,q+1}(M) \to L^2_{0,q+1}(M)$ is a bounded operator with C is independent of $A_\pm$. Therefore, $L^2_{0,q+1}(M)$ embeds compactly in $W^{-1}_{0,q+1}(M)$. Moreover, $A_\pm$ can be made arbitrarily large since M satisfies (P_q) and (P_{n-1-q}). Equation (14) proves that $\bar{\partial}_{b,\pm}^* G_{\pm,q+1} : L^2_{0,q+1}(M) \to L^2_{0,q}(M)$ continuously, so the map $\bar{\partial}_{b,\pm}^* G_{\pm,q+1} : L^2_{0,q+1}(M) \to W^{-1}_{0,q}(M)$ is compact, and it follows that $\bar\partial_b^* G_{q+1}$ is compact on Range$(\bar\partial_b)$ by [7], Proposition V.2.3. On the orthogonal complement of Range$(\bar\partial_b)$, $\bar\partial_b^* G_{q+1} = 0$, so $\bar\partial_b^* G_{q+1} : L^2_{0,q+1}(M) \to L^2_{0,q+1}(M)$ is compact. To estimate $\bar\partial_b^* G_q\alpha$, we cannot invoke (13) directly because $\bar\partial_b^* G_q\alpha$ is a $(q-1)$-form. Instead, for $\alpha \in$ Range$(\bar\partial_b) \subset L^2_{0,q}(M)$,

$$
\begin{aligned}
|||\bar\partial_{b,\pm}^* G_{\pm,q}\alpha|||_\pm^2 &= \langle \bar\partial_b\bar\partial_{b,\pm}^* G_{\pm,q}\alpha, G_{\pm,q}\alpha\rangle_\pm = \langle \alpha, G_{\pm,q}\alpha\rangle_\pm \\
&\le \frac{2C}{A_\pm}|||\alpha|||_\pm^2 + \frac{A_\pm}{2C}|||G_{\pm,q}\alpha|||_\pm^2 \le \frac{2C}{A_\pm}|||\alpha|||_\pm^2 + \frac{1}{2}|||\bar\partial_{b,\pm}^* G_{\pm,q}\alpha|||_\pm^2 + C_\pm\|G_{\pm,q}\alpha\|_{-1}^2.
\end{aligned}
\tag{15}
$$

Here we have used that $\bar\partial_b\alpha = 0$ and that $\alpha \in {}^\perp\mathcal{H}^q_\pm$ (since $\alpha \in$ Range $\bar\partial_b$) in the second inequality. Also, the first inequality shows that the $|||\bar\partial_{b,\pm}^* G_{\pm,q}\alpha|||_\pm^2 < \infty$ and thus the term in the final inequality can be absorbed. Thus we can can prove $\bar\partial_b^* G_q L^2_{0,q}(M) \to L^2_{0,q-1}(M)$ is a compact operator by repeating the argument that follows (14) with $G_{\pm,q}$ replacing $\bar\partial_{b,\pm}^* G_{\pm,q+1}$.

5.3 End proof of Theorem 1.1—the $s > 0$ case

Fix $s > 0$. Recall that compactness G_q in $L^2_{0,q}(M)$ is equivalent to the following compactness estimate: for every $\epsilon > 0$, there exists $C_\epsilon > 0$ so that for every $u \in$ Dom$(\bar\partial_b) \cap$ Dom$(\bar\partial_b^*)$,

$$
\|u\|_0^2 \le \epsilon(\|\bar\partial_b u\|_0^2 + \|\bar\partial_b^* u\|_0^2) + C_\epsilon\|u\|_{-1}^2.
$$

We claim that this estimate also holds a priori in H^s, $s > 0$. Indeed, using the fact that the commutators $[\bar\partial_b, \Lambda^s]$ and $[\bar\partial_b^*, \Lambda^s]$ are pseudodifferential operators of order s (independent of ϵ), we have

$$
\begin{aligned}
\|u\|_s^2 = \|\Lambda^s u\|_0^2 &\le \epsilon(\|\bar\partial_b\Lambda^s u\|_0^2 + \|\bar\partial_b^*\Lambda^s u\|_0^2) + C_\epsilon\|\Lambda^s u\|_{-1}^2 \\
&\le \epsilon(\|\Lambda^s\bar\partial_b u\|_0^2 + \|\Lambda^s\bar\partial_b^* u\|_0^2) + \epsilon(\|[\bar\partial_b, \Lambda^s]u\|_0^2 + \|[\bar\partial_b^*, \Lambda^s]u\|_0^2) + C_\epsilon\|u\|_{s-1}^2 \\
&\le \epsilon(\|\bar\partial_b u\|_s^2 + \|\bar\partial_b^* u\|_s^2) + C\epsilon\|u\|_s^2 + C_\epsilon\|u\|_{s-1}^2.
\end{aligned}
$$

When $\epsilon < 1/2C$, the $C\epsilon\|u\|_s^2$ can be absorbed into the left-hand side of the equation. Thus, we have the estimate that for every $\epsilon > 0$, there exists $C_\epsilon > 0$ so that for every $u \in H^s_{0,q}(M)$ with $\bar\partial_b u \in H^s_{0,q+1}(M)$ and $\bar\partial_b^* u \in H^s_{0,q-1}(M)$,

$$
\|u\|_s^2 \le \epsilon(\|\bar\partial_b u\|_s^2 + \|\bar\partial_b^* u\|_s^2) + C_\epsilon\|u\|_{s-1}^2.
\tag{16}
$$

Unlike in L^2-case, this estimate does not imply that G_q is compact in H^s. The difficulty rests in the fact that while u may be in $H^s_{0,q}(M)$, we can only say that $G_q u \in L^2_{0,q}(M)$. We need to work with the family of regularized operators $G_{\delta,q}$,

$0 < \delta \le 1$, arising from the following regularization. Let $Q^{\delta}_{b,0}(\cdot,\cdot)$ be the quadratic form on $H^1_{0,q}(M)$ defined by

$$Q^{\delta}_{b,0}(u,v) = Q_{b,0}(u,v) + \delta Q_L(u,v)$$

where Q_L is the hermitian inner product associated to the de Rham exterior derivative d, i.e., $Q_L(u,v) = (du,dv)_0 + (d^*u, d^*v)_0$. The inner product Q_L has form domain $H^1_{0,q}(M)$. Consequently, $Q^{\delta}_{b,0}$ gives rise a unique, self-adjoint, elliptic operator $\Box_{b,\delta}$ with inverse $G_{q,\delta}$. Equivalently, for $u \in L^2_{0,q}(M)$ and $v \in H^1_{0,q}(M)$, $(u,v)_0 = Q^{\delta}_{b,0}(G_{q,\delta}u, v)$. By elliptic regularity, we know that if $u \in H^s_{0,q}(M)$, then $G_{q,\delta}u \in H^{s+2}_{0,q}(M)$. We claim that for any $\epsilon > 0$, there exists C_ϵ so that for any $u \in H^s_{0,q}(M)$,

$$\|G_{q,\delta}u\|_s^2 \le \epsilon\|u\|_s^2 + C_\epsilon\|u\|_{s-1}^2, \tag{17}$$

where the inequalities are uniform in $0 < \delta \le 1$. Estimates of the form (17) are well known to be equivalent to the compactness of $G_{q,\delta}$ on $H^s_{0,q}(M)$, (see, for example, [7, Proposition V.2.3]).

By the a priori estimate (16),

$$\|G_{q,\delta}u\|_s^2 \le \epsilon(\|\bar\partial_b G_{q,\delta}u\|_s^2 + \|\bar\partial_b^* G_{q,\delta}u\|_s^2) + C_\epsilon\|u\|_{s-1}^2.$$

The $\bar\partial_b$ and $\bar\partial_b^*$ terms can be estimated as follows:

$$\begin{aligned}\|\bar\partial_b G_{q,\delta}u\|_s^2 + \|\bar\partial_b^* G_{q,\delta}u\|_s^2 &\le Q_{b,0}(\Lambda^s G_{q,\delta}u, \Lambda^s G_{q,\delta}u) + C\|G_{q,\delta}u\|_s^2\\ &\le Q^{\delta}_{b,0}(\Lambda^s G_{q,\delta}u, \Lambda^s G_{q,\delta}u) + C\|G_{q,\delta}u\|_s^2\\ &\le |(\Lambda^s u, \Lambda^s G_{q,\delta}u)_0| + C\|u\|_s^2,\end{aligned}$$

where we have used the estimate $Q^{\delta}_{b,0}(\Lambda^s G_{q,\delta}u, \Lambda^s G_{q,\delta}u) \le |(\Lambda^s u, \Lambda^s G_{q,\delta}u)_0| + C\|G_{q,\delta}u\|_s^2$, which follows from [15, Lemma 3.1]. Thus, we have

$$\|G_{q,\delta}u\|_s^2 \le \epsilon(\|G_{q,\delta}u\|_s^2 + \|u\|_s^2) + C_\epsilon\|u\|_{s-1}^2,$$

By absorbing terms (and choosing $\epsilon < 1/2$), we have proven (17) with the constant C_ϵ independent of δ, $0 < \delta \le 1$.

We want to let $\delta \to 0$. If $u \in H^s_{0,q}(M)$, then $\{G_{q,\delta}u : 0 < \delta \le 1\}$ is bounded in $H^s_{0,q}(M)$. Thus, there exists a sequence $\delta_k \to 0$ and $\tilde u \in H^s_{0,q}(M)$ so that $G_{q,\delta_k}u \to \tilde u$ weakly in $H^s_{0,q}(M)$. Consequently, if $v \in H^2_{0,q}(M)$, then

$$\lim_{n\to\infty} Q^{\delta_k}_{b,0}(G_{q,\delta_k}u, v) = Q_{b,0}(\tilde u, v).$$

However,

$$Q^{\delta_k}_{b,0}(G_{q,\delta_k}u, v) = (u,v) = Q_{b,0}(G_q u, v),$$

so $G_q u = \tilde{u}$ and (17) is satisfied with $\delta = 0$. Thus, G_q is a compact operator on $H^s_{0,q}(\Omega)$. and Theorem 1.1 is proved.

Appendix A: Multilinear algebra

Some crucial multilinear algebra is contained in the following lemma from Straube [26].

Lemma A.1 *Let* $(\lambda_{jk})^m_{j,k=1}(z)$ *be an* $m \times m$ *matrix-valued function and* $1 \leq q \leq m$. *The following are equivalent*:

(1) $\sum_{K \in \mathcal{I}_{q-1}} \sum^m_{j,k=1} \lambda_{jk}(z) u_{jK} \overline{u_{kK}} \geq A|u|^2 \quad \forall u \in \Lambda^{(0,q)}_z$.
(2) *The sum of any* q *eigenvalues of* $(\lambda_{jk}(z))_{j,k}$ *is at least* A.
(3) *For any orthonormal* $\underline{t}^\ell \in \mathbb{C}^m$, $1 \leq \ell \leq q$,

$$\sum_{\ell=1}^{q} \lambda_{jk}(z)(\underline{t}^\ell)_j \overline{(\underline{t}^\ell)_k} \geq A$$

These are Lemma 6.3 and Lemma 6.4 in [22].

Lemma A.2 *Let* (b_{jk}) *be a Hermitian matrix and let* $1 \leq q \leq n-2$. *Then then* $\binom{n-1}{q}$ *by* $\binom{n-1}{q}$ *matrix* $(B^q_{JJ'})$ *given by*

$$B^q_{JJ} = \sum_{j \in J} b_{jj}$$

$$B^q_{JJ'} = -\sum_{\substack{1 \leq j,k \leq n-1 \\ j \neq k}} \epsilon^{kJ}_{jJ'} b_{jk} \qquad \text{if } J \neq J',$$

where J *and* J' *are multiindices,* $|J| = |J'| = q$ *is also Hermitian. Moreover, the eigenvalues of* $(B^q_{JJ'})$ *are sums of the eigenvalues of* (b_{jk}) *taken* q *at a time.*

Lemma A.3 *Let* (d_{jk}) *be a Hermitian matrix and let* $1 \leq q \leq n-2$. *Then then* $\binom{n-1}{q}$ *by* $\binom{n-1}{q}$ *matrix* $(D^q_{JJ'})$ *given by*

$$D^q_{JJ} = \sum_{j \notin J} d_{jj}$$

$$D^q_{JJ'} = \sum_{\substack{1 \leq j,k \leq n-1 \\ j \neq k}} \epsilon^{kJ}_{jJ'} d_{jk} \qquad \text{if } J \neq J',$$

where J *and* J' *are multiindices,* $|J| = |J'| = q$ *is also Hermitian. Moreover, the eigenvalues of* $(D^q_{JJ'})$ *are sums of the eigenvalues of* (d_{jk}) *taken* $n-1-q$ *at a time, so* $(D^q_{JJ'})$ *is positive definite if* (d_{jk}) *is positive definite and* $n-1-q > 0$*;* $(D^q_{JJ'})$ *is positive semi-definite if* (d_{jk}) *is positive semi-definite for any* n.

If $q = 1$, then Lemma A.3 says that if $n \geq 3$ and $H = (h_{jk})$ is a Hermitian, positive definite matrix, $1 \leq i, k \leq n-1$, then $(\delta_{jk} \sum_{\ell=1}^{n-1} h_{\ell\ell} - h_{jk})$ is a Hermitian, positive definite matrix. The requirement that $n \geq 3$ is the seemingly technical reason that Theorem 1.1 is stated for $2n - 1 \geq 5$, as well as the results in [22] and the fact that the work by Kohn and Nicoara in [16] assumes closed range of $\bar{\partial}_b$.

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