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# On the CR analogue of Obata's theorem in a pseudohermitian 3-manifold

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**Abstract** In this paper, we first prove the CR analogue of Obata's theorem on a closed pseudohermitian 3-manifold with zero pseudohermitian torsion. Secondly, instead of zero torsion, we have the CR analogue of Li-Yau's eigenvalue estimate on the lower bound estimate of positive first eigenvalue of the sub-Laplacian in a closed pseudohermitian 3-manifold with nonnegative CR Paneitz operator $P_0$. Finally, we have a criterion for the positivity of first eigenvalue of the sub-Laplacian on a complete noncompact pseudohermitian 3-manifold with nonnegative CR Paneitz operator. The key step is a discovery of integral CR analogue of Bochner formula which involving the CR Paneitz operator.

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## 1 Introduction

Let $(M, J, \theta)$ be a closed pseudohermitian $(2n+1)$-manifold (see definition below). Greenleaf [10] proved the pseudohermitian analogue of Lichnerowicz's Theorem [13] for the lower bound of the first positive eigenvalue $\mu_1$ of the sublaplacian for a closed pseudohermitian manifold $M^{2n+1}$ with $n \geq 3$. More precisely, under a condition on

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the pseudohermitian Ricci curvature and the pseudohermitian torsion[1]

$$Ric_m(Z, Z) - (\frac{n+1}{2})Tor_m(Z, Z) \geq k_0 \langle Z, Z\rangle_{L_\theta} \tag{1.1}$$

for all $m \in M$, $Z \in T_{1,0}$, and for some positive constant $k_0$, one can have

$$\mu_1 \geq \frac{nk_0}{(n+1)}.$$

In [17], Li and Luk proved the same result for the cases $n = 1, n = 2$. However, in the case $n = 1$, they needed a condition depending not only on the Webster Ricci curvature and the pseudohermitian torsion, but also on a covariant derivative of the pseudohermitian torsion.

Recently, it was proved by the authors that the same result [3,4] holds on a pseudohermitian 3-manifold $(M^3, J, \theta)$ under more geometric condition which involving the positivity of the CR Paneitz operator $P_0$ (see definition below) with respect to $(J, \theta)$.

We observe that for a standard pseduhermitian $(2n + 1)$-sphere $(S^{2n+1}, \widehat{J}, \widehat{\theta})$ with the induced natural CR structure from $\mathbf{C}^{n+1}$ and the standard contact form $\widehat{\theta}$, one can show that [1,4]

$$\mu_1 = \frac{nk_0}{(n+1)}.$$

Here $Ric_m(Z, Z) = k_0 \langle Z, Z\rangle_{L_\theta}$ for all $m \in M$, $Z \in T_{1,0}$, and for some positive constant $k_0$. Thus one have the sharp estimate of $\mu_1$ on the standard sphere $(S^{2n+1}, \widehat{J}, \widehat{\theta})$.

Then it is natural to conjecture the CR analogue of Obata's Theorem [18] on a closed $(2n + 1)$-dimensional pseudohermitian manifold $(M, J, \theta)$.

*Conjecture 1.1* Let $(M, J, \theta)$ be a closed $(2n+1)$-dimensional pseudohermitian manifold with

$$Ric_m(Z, Z) - \left(\frac{n+1}{2}\right) Tor_m(Z, Z) \geq k_0 \langle Z, Z\rangle_{L_\theta}$$

for all $m \in M$, $Z \in T_{1,0}$, and for a positive constant $k_0$. Suppose that

$$\mu_1 = \frac{nk_0}{(n+1)}.$$

Then $(M, J, \theta)$ is the standard pseudohermitian $(2n + 1)$-sphere $(S^{2n+1}, \widehat{J}, \widehat{\theta})$ with $Ric_m(Z, Z) = k_0 \langle Z, Z\rangle_{L_\theta}$.

[1] The commutation relation (3.11) of Greenleaf has something wrong (see [15]). Then the coefficient of torsion term in the Bochner formula (4.2) of Greenleaf should be $-\frac{n}{2}$. Hence the Bochner formula in our paper is 2 times the one of Greenleaf. Therefore the correct coefficient of torsion term in condition (1.1) is $-\frac{(n+1)}{2}$.

In this paper, among the others, we confirm the Conjecture 1.1 on a closed 3-dimensional pseudohermitian manifold $(M, J, \theta)$ with zero torsion.

We first give a brief introduction to pseudohermitian geometry on a closed 3-manifold (see [15,16] for more details). Let $M$ be a closed 3-manifold with an oriented contact structure $\xi$. There always exists a global contact form $\theta$, obtained by patching together local ones with a partition of unity. The characteristic vector field of $\theta$ is the unique vector field $T$ such that $\theta(T) = 1$ and $\mathcal{L}_T\theta = 0$ or $d\theta(T, \cdot) = 0$. A $CR$ structure compatible with $\xi$ is a smooth endomorphism $J : \xi \to \xi$ such that $J^2 = -Id$. A pseudohermitian structure compatible with $\xi$ is a $CR$-structure $J$ compatible with $\xi$ together with a global contact form $\theta$. The CR structure $J$ can extend to $\mathbf{C} \otimes \xi$ and decomposes $\mathbf{C} \otimes \xi$ into the direct sum of $T_{1,0}$ and $T_{0,1}$ which are eigenspaces of $J$ with respect to $i$ and $-i$, respectively.

Let $\{T, Z_1, Z_{\bar{1}}\}$ be a frame of $TM \otimes \mathbf{C}$, where $Z_1$ is any local frame of $T_{1,0}$, $Z_{\bar{1}} = \overline{Z_1} \in T_{0,1}$ and $T$ is the characteristic vector field. Then $\left\{\theta, \theta^1, \theta^{\bar{1}}\right\}$, the coframe dual to $\{T, Z_1, Z_{\bar{1}}\}$, satisfies

$$d\theta = i h_{1\bar{1}} \theta^1 \wedge \theta^{\bar{1}}, \tag{1.2}$$

for some positive function $h_{1\bar{1}}$. Actually we can always choose $Z_1$ such that $h_{1\bar{1}} = 1$; hence, throughout this paper, we assume $h_{1\bar{1}} = 1$.

The Levi form $\langle\ ,\ \rangle_{L_\theta}$ is the Hermitian form on $T_{1,0}$ defined by

$$\langle Z, Y\rangle_{L_\theta} = -i\left\langle d\theta, Z \wedge \overline{Y}\right\rangle.$$

We can extend $\langle\ ,\ \rangle_{L_\theta}$ to $T_{0,1}$ by defining $\left\langle \overline{Z}, \overline{Y}\right\rangle_{L_\theta} = \overline{\langle Z, Y\rangle}_{L_\theta}$ for all $Z, Y \in T_{1,0}$. The Levi form induces naturally a Hermitian form on the dual bundle of $T_{1,0}$, denoted by $\langle\ ,\ \rangle_{L_\theta^*}$, and hence on all the induced tensor bundles. Integrating the Hermitian form (when acting on sections) over $M$ with respect to the volume form $d\mu = \theta \wedge d\theta$, we get an inner product on the space of sections of each tensor bundle. We denote the inner product by the notation $\langle\ ,\ \rangle$. For example

$$\langle \varphi, \psi\rangle = \int_M \varphi\bar{\psi}\, d\mu, \tag{1.3}$$

for functions $\varphi$ and $\psi$.

The pseudohermitian connection of $(J, \theta)$ is the connection $\nabla$ on $TM \otimes \mathbf{C}$ (and extended to tensors) given in terms of a local frame $Z_1 \in T_{1,0}$ by

$$\nabla Z_1 = \theta_1{}^1 \otimes Z_1, \quad \nabla Z_{\bar{1}} = \theta_{\bar{1}}{}^{\bar{1}} \otimes Z_{\bar{1}}, \quad \nabla T = 0,$$

where $\theta_1{}^1$ is the 1-form uniquely determined by the following equations:

$$\begin{aligned} d\theta^1 &= \theta^1 \wedge \theta_1{}^1 + \theta \wedge \tau^1, \\ \tau^1 &\equiv 0 \mod \theta^{\bar{1}}, \\ 0 &= \theta_1{}^1 + \theta_{\bar{1}}{}^{\bar{1}}, \end{aligned} \tag{1.4}$$

where $\tau^1$ is the pseudohermitian torsion. Put $\tau^1 = A^1{}_{\bar{1}}\theta^{\bar{1}}$. The structure equation for the pseudohermitian connection is

$$d\theta_1{}^1 = W\theta^1 \wedge \theta^{\bar{1}} + 2iIm(A^{\bar{1}}{}_{1,\bar{1}}\theta^1 \wedge \theta), \tag{1.5}$$

where $W$ is the Tanaka–Webster scalar curvature.

We will denote components of covariant derivatives with indices preceded by comma; thus write $A^{\bar{1}}{}_{1,\bar{1}}\theta^1 \wedge \theta$. The indices $\{0, 1, \bar{1}\}$ indicate derivatives with respect to $\{T, Z_1, Z_{\bar{1}}\}$. For derivatives of a scalar function, we will often omit the comma, for instance, $\varphi_1 = Z_1\varphi,\ \varphi_{1\bar{1}} = Z_{\bar{1}}Z_1\varphi - \theta_1^1(Z_{\bar{1}})Z_1\varphi,\ \varphi_0 = T\varphi$ for a (smooth) function.

For a real function $\varphi$, the subgradient $\nabla_b$ is defined by $\nabla_b\varphi \in \xi$ and $\langle Z, \nabla_b\varphi\rangle_{L_\theta} = d\varphi(Z)$ for all vector fields $Z$ tangent to contact plane. Locally $\nabla_b\varphi = \varphi_{\bar{1}}Z_1 + \varphi_1 Z_{\bar{1}}$. We can use the connection to define the subhessian as the complex linear map

$$(\nabla^H)^2\varphi : T_{1,0} \oplus T_{0,1} \to T_{1,0} \oplus T_{0,1},$$

by

$$(\nabla^H)^2\varphi(Z) = \nabla_Z\nabla_b\varphi.$$

Also

$$\Delta_b\varphi = Tr\left((\nabla^H)^2\varphi\right) = (\varphi_{1\bar{1}} + \varphi_{\bar{1}1}).$$

For all $Z = x^1 Z_1 \in T_{1,0}$, we define

$$Ric(Z, Z) = Wx^1x^{\bar{1}} = W|Z|^2_{L_\theta},$$
$$Tor(Z, Z) = 2Re\, iA_{\bar{1}\bar{1}}x^{\bar{1}}x^{\bar{1}}.$$

Next we recall some definitions.

**Definition 1.1** Let $(M, J, \theta)$ be a closed three-dimensional pseudohermitian manifold. A piecewise smooth curve $\gamma : [0, 1] \to M$ is said to be horizontal if $\gamma'(t) \in \xi$ whenever $\gamma'(t)$ exists. The length of $\gamma$ is then defined by

$$l(\gamma) = \int_0^1 h(\gamma'(t), \gamma'(t))^{\frac{1}{2}} dt.$$

The Carnot-Carathéodory distance $d_c$ between two points $p, q \in M$ is defined by

$$d_c(p, q) = \inf\left\{l(\gamma) |\ \gamma \in C_{p,q}\right\},$$

where $C_{p,q}$ is the set of all horizontal curves which join $p$ and $q$. By Chow connectivity theorem [5], there always exists a horizontal curve joining $p$ and $q$, so the distance is finite. We say $M$ is complete if it is complete as a metric space $(M, d_c)$.

**Definition 1.2** Let $(M, J, \theta)$ be a closed three-dimensional pseudohermitian manifold. We define [15]

$$P\varphi = (\varphi_{\bar{1}}{}^{\bar{1}}{}_{1} + iA_{11}\varphi^{1})\theta^{1} = P\varphi = (P_{1}\varphi)\theta^{1},$$

which is an operator that characterizes CR-pluriharmonic functions. Here $P_1\varphi = \varphi_{\bar{1}}{}^{\bar{1}}{}_{1} + iA_{11}\varphi^1$ and $\overline{P}\varphi = (\overline{P}_1)\theta^{\bar{1}}$, the conjugate of $P$. The CR Paneitz operator $P_0$ is defined by

$$P_0\varphi = 4\left(\delta_b(P\varphi) + \bar{\delta}_b(\overline{P}\varphi)\right), \tag{1.6}$$

where $\delta_b$ is the divergence operator that takes $(1,0)$-forms to functions by $\delta_b(\sigma_1\theta^1) = \sigma_{1,}{}^{1}$, and similarly, $\bar{\delta}_b(\sigma_{\bar{1}}\theta^{\bar{1}}) = \sigma_{\bar{1},}{}^{\bar{1}}$.

We observe that

$$\int_M \langle P\varphi + \overline{P}\varphi, d_b\varphi\rangle_{L_\theta^*}\, d\mu = -\frac{1}{4}\int_M P_0\varphi\cdot\varphi\, d\mu \tag{1.7}$$

with $d\mu = \theta\wedge d\theta$. One can check that $P_0$ is self-adjoint, that is, $\langle P_0\varphi, \psi\rangle = \langle\varphi, P_0\psi\rangle$ for all smooth functions $\varphi$ and $\psi$. For the details about these operators, the reader can make reference to [7–9,11,15].

**Definition 1.3** On a complete pseudohermitian 3-manifold $(M, J, \theta)$, we call the Paneitz operator $P_0$ with respect to $(J, \theta)$ essentially positive if there exists a constant $\Lambda > 0$ such that

$$\int_M P_0\varphi\cdot\varphi d\mu \geq \Lambda\int_M \varphi^2 d\mu. \tag{1.8}$$

for all real $C^\infty$ smooth functions $\varphi \in (\ker P_0)^\perp$ (i.e. perpendicular to the kernel of $P_0$ in the $L^2$ norm with respect to the volume form $d\mu = \theta\wedge d\theta$). We say that $P_0$ is nonnegative if

$$\int_M P_0\varphi\cdot\varphi d\mu \geq 0$$

for all real $C^\infty$ smooth functions.

*Remark 1.1* 1. Let $(M, J, \theta)$ be a closed three-dimensional pseudohermitian manifold. The essentially positivity of $P_0$ is a CR invariant in the sense that it is independent of the choice of the contact form $\theta$.
2. Let $(M, J, \theta)$ be a closed pseudohermitian 3-manifold with zero torsion. Then the corresponding CR Paneitz operator is essentially positive [1,2].

We first recall the second author's previous result.

**Proposition 1.2** ([4,17]) *Let $(M, J, \theta)$ be a closed three-dimensional pseudohermitian manifold with nonnegative Paneitz operator $P_0$. Suppose that*

$$Ric_m(Z, Z) - Tor_m(Z, Z) \geq k_0 \langle Z, Z\rangle_{L_\theta}, \tag{1.9}$$

*for all $m \in M$, $Z \in T_{1,0}$, and for some positive constant $k_0$. Let $\mu_1$ be the first positive eigenvalue with respect to $\Delta_b$ . Then*

$$\mu_1 \geq \frac{k_0}{2} > 0.$$

*As a consequence, the same result holds for a closed three-dimensional pseudo-hermitian manifold with*

$$A_{11} = 0 \quad \text{and} \quad W \geq k_0.$$

*Remark 1.2* The result of last part of Proposition 1.2 is done also in [17].

Define the Levi metric $h$ on $\ker\theta$ by

$$h(X, Y) = d\theta(X, JY).$$

A family of Webster (adapted) metrics $h_\lambda$ of $(M, J, \theta)$ are the Riemannian metrics

$$h_\lambda = h + \lambda^{-2}\theta^2,\ \lambda > 0.$$

Let $\mu_1^\lambda$ be the first positive eigenvalue of the Laplacian $\Delta_\lambda$ with respect to the metric $h_\lambda$. We denote that $\max|A_{11}| = \tau_0$. Here is the main Theorem in the present paper.

**Theorem 1.3** *(i) Let $(M, J, \theta)$ be a closed pseudohermitian 3-manifold with the nonnegative Paneitz operator $P_0$. Then*

$$\mu_1^\lambda \leq (2 + \lambda^2\mu_1 + 2\lambda^2\tau_0)\mu_1.$$

*(ii) Let $(M, J, \theta)$ be a complete noncompact pseudohermitian 3 -manifold with the nonnegative CR Paneitz operator $P_0$. Then*

$$\mu_1^\lambda \leq (2 + \lambda^2\mu_1 + 2\lambda^2\tau_0)\mu_1.$$

As our first consequence, by choosing $k_0 = 2\lambda^{-2}$ as in Proposition 1.2, one obtains the following CR analogue of Obata's Theorem.

**Corollary 1.4** *Let $(M, J, \theta)$ be a closed three-dimensional pseudohermitian manifold with*

$$A_{11} = 0 \quad \text{and} \quad W \geq 2\lambda^{-2}.$$

*If* $\mu_1 = \lambda^{-2}$, *then* $(M, J, \theta)$ *is the standard pseudohermitian* 3*-sphere* $(S^3, \widehat{J}, \widehat{\theta})$ *with constant Tanaka–Webster curvature* $2\lambda^{-2}$.

For the second consequence of Theorem 1.3, by applying the CR analogue of Li-Yau eigenvalue estimate [14], one can derive an eigenvalue estimate on a closed pseudohermitian 3-manifold with nonvanishing torsion and nonnegative CR Paneitz operator $P_0$ which is served as a generalization of our previous results [3].

**Corollary 1.5** *Let* $(M, J, \theta)$ *be a closed pseudohermitian* 3*-manifold with nonnegative CR Paneitz operator* $P_0$. *Assume that there exist positive constants* $k_1, k_2$ *with*

$$W \geq -k_1$$

*and*

$$k_2 = \max\{|A_{11}|,\ |A_{11,0}|^{\frac{1}{2}}, |A_{11,\bar{1}}|^{\frac{2}{3}}\}.$$

*Then for a positive constant* $\lambda < 1$ *with* $\lambda^2 k \leq 1$ *and* $k = \max\{k_1, k_2\}$,*either (i)*

$$\lambda^{-2} \leq \mu_1$$

*or (ii)*

$$\mu_1 \geq \frac{\exp\{-[1 + (1 + 7c^2 D^2 \lambda^{-2})^{\frac{1}{2}}]\}}{5cD^2}$$

*where D is the diameter of* $(M, J, \theta)$.

*Remark 1.3* In Corollary 1.5, the pseudohermitian torsion is nonzero which is the main different from our previous works [3,4].

Third, we have a criterion for the positivity of first eigenvalue of the sub-Laplacian on a complete noncompact pseudohermitian 3-manifold with nonnegative CR Paneitz operator.

**Corollary 1.6** *Let* $(M, J, \theta)$ *be a complete noncompact pseudohermitian* 3*-manifold with nonnegative CR Paneitz operator* $P_0$. *Suppose that* $\mu_1^{\lambda} > 0$ *on* $(M, h_\lambda)$ *for some* $\lambda > 0$, *then*

$$\mu_1 > 0$$

*on* $(M, J, \theta)$.

We briefly describe the methods used in our proofs. In Sect. 2, we derive our crucial integral CR analogue of Bochner formula which involving CR Paneitz operator (Lemma 2.2). In Sect. 3, we obtain the relations between the Riemannian Ricci tensors with respect to the Webster metric and pseudohermitian Ricci tensors. Finally, by comparing the Laplacian w.r.t. Webster metric and sub-Laplacian, the proofs of main theorems on CR analogue of Obata's Theorem (Corollary 1.4) and Li-Yau's eigenvalue estimate (Corollary 1.5) are complete as in Sect. 4.

## 2 CR analogue of Bochner formula and Paneitz operator

In this section, we derive the following CR analogue of pointwise and integral version of Bochner formula in a complete pseudohermitian 3 -manifold.

**Lemma 2.1** *Let $(M, J, \theta)$ be a complete pseudohermitian 3 -manifold. For a real function $\varphi$,*

$$\begin{aligned}\frac{1}{2}\Delta_b|\nabla_b\varphi|^2 = {} & |(\nabla^H)^2\varphi|^2 + 3\langle\nabla_b\varphi, \nabla_b\Delta_b\varphi\rangle_{L_\theta}\\ & +(2Ric - 3Tor)((\nabla_b\varphi)_{\mathbf{C}}, (\nabla_b\varphi)_{\mathbf{C}})\\ & -4\langle P\varphi + \overline{P}\varphi, d_b\varphi\rangle_{L_\theta^*}.\end{aligned}$$

*Here $(\nabla_b\varphi)_{\mathbf{C}} = \varphi_{\bar{1}}Z_1$ is the corresponding complex $(1, 0)$-vector field of $\nabla_b\varphi$ and $d_b\varphi = \varphi_1\theta^1 + \varphi_{\bar{1}}\theta^{\bar{1}}$.*

*Proof* First from [10], we have for a real function $\varphi$

$$\begin{aligned}\Delta_b|\nabla_b\varphi|^2 = {} & 2|(\nabla^H)^2\varphi|^2 + 2\langle\nabla_b\varphi, \nabla_b\Delta_b\varphi\rangle_{L_\theta}\\ & +(4Ric + 2Tor)((\nabla_b\varphi)_{\mathbf{C}}, (\nabla_b\varphi)_{\mathbf{C}})\\ & +4\langle J\nabla_b\varphi, \nabla_b\varphi_0\rangle_{L_\theta}.\end{aligned} \tag{2.1}$$

Then Lemma 2.1 follows from (2.1) and the following (2.2).

$$\begin{aligned}\langle J\nabla_b\varphi, \nabla_b\varphi_0\rangle_{L_\theta} = {} & \langle\nabla_b\varphi, \nabla_b\Delta_b\varphi\rangle_{L_\theta}\\ & -2Tor((\nabla_b\varphi)_{\mathbf{C}}, (\nabla_b\varphi)_{\mathbf{C}})\\ & -2\langle P\varphi + \overline{P}\varphi, d_b\varphi\rangle_{L_\theta^*}.\end{aligned} \tag{2.2}$$

We give a proof of (2.2) below. By using commutation relation $i\varphi_0 = \varphi_{1\bar{1}} - \varphi_{\bar{1}1}$ [15], we have $\varphi_{1\bar{1}1} - \varphi_{\bar{1}11} = i\varphi_{01}$. Thus

$$\begin{aligned}\langle J\nabla_b\varphi, \nabla_b\varphi_0\rangle_{L_\theta} &= i(\varphi_{\bar{1}}\varphi_{01} - \varphi_1\varphi_{0\bar{1}})\\ &= \varphi_{\bar{1}}(\varphi_{1\bar{1}1} - \varphi_{\bar{1}11}) + \varphi_1(\varphi_{\bar{1}1\bar{1}} - \varphi_{1\bar{1}\bar{1}}).\end{aligned} \tag{2.3}$$

On the other hand

$$\begin{aligned}\langle\nabla_b\varphi, \nabla_b\Delta_b\varphi\rangle_{L_\theta} &= \varphi_{\bar{1}}(\Delta_b\varphi)_1 + \varphi_1(\Delta_b\varphi)_{\bar{1}}\\ &= \varphi_{\bar{1}}(\varphi_{1\bar{1}1} + \varphi_{\bar{1}11}) + \varphi_1(\varphi_{\bar{1}1\bar{1}} + \varphi_{1\bar{1}\bar{1}}).\end{aligned} \tag{2.4}$$

It follows from (2.3) and (2.4) that

$$\begin{aligned}&\langle J\nabla_b\varphi, \nabla_b\varphi_0\rangle_{L_\theta} - \langle\nabla_b\varphi, \nabla_b\Delta_b\varphi\rangle_{L_\theta}\\ &\quad= -2\varphi_{\bar{1}}\varphi_{\bar{1}11} - 2\varphi_1\varphi_{1\bar{1}\bar{1}}\\ &\quad= -2\varphi_{\bar{1}}\left(P_1\varphi - iA_{11}\varphi_{\bar{1}}\right) - 2\varphi_1\left(\overline{P}_1\varphi + iA_{\bar{1}\bar{1}}\varphi_1\right)\\ &\quad= -2\,Tor((\nabla_b\varphi), (\nabla_b\varphi)) - 2\langle P\varphi + \overline{P}\varphi, d_b\varphi\rangle_{L_\theta^*}.\end{aligned}$$

This completes the proof of the Lemma. □

On the other hand, one can have the integral version of CR Bochner formula.

**Lemma 2.2** *(i) Let $(M, J, \theta)$ be a closed pseudohermitian 3 -manifold. Then*

$$\int_M \varphi_0^2 \, d\mu = \int_M (\Delta_b\varphi)^2 \, d\mu + 2\int_M Tor((\nabla_b\varphi)_{\mathbf{C}}, (\nabla_b\varphi)_{\mathbf{C}}) d\mu - \frac{1}{2}\int_M \varphi P_0\varphi \, d\mu$$

*for any $\varphi \in C^\infty(M)$. In additional, if the CR Paneitz operator $P_0$ is nonnegative, then*

$$\int_M \varphi_0^2 \, d\mu \le \int_M (\Delta_b\varphi)^2 \, d\mu + 2\tau_0 \int_M |\nabla_b\varphi|^2 \, d\mu.$$

*(ii) Let $(M, J, \theta)$ be a complete pseudohermitian 3-manifold. Then*

$$\int_M \varphi_0^2 \, d\mu = \int_M (\Delta_b\varphi)^2 \, d\mu + 2\int_M Tor((\nabla_b\varphi)_{\mathbf{C}}, (\nabla_b\varphi)_{\mathbf{C}}) d\mu - \frac{1}{2}\int_M P_0\varphi \cdot \varphi \, d\mu$$

*for any $\varphi \in C_0^\infty(\Omega)$ with $\Omega \subset\subset M$. In additional, if the CR Paneitz operator $P_0$ is nonnegative, then*

$$\int \varphi_0^2 \, d\mu \le \int (\Delta_b\varphi)^2 \, d\mu + 2\tau_0 \int |\nabla_b\varphi|^2 \, d\mu.$$

*Proof* (i) Let $(M, J, \theta)$ be a closed pseudohermitian 3-manifold. Then by integrating (2.2) and using (1.7), we have

$$\int \varphi_0^2 dV = \int (\Delta_b\varphi)^2 dV + 2\int Tor((\nabla_b\varphi), (\nabla_b\varphi)) dV - \frac{1}{2}\int P_0\varphi \cdot \varphi dV. \quad (2.5)$$

(ii) The same method holds for a complete pseudohermitian 3-manifold with a compact support smooth $\varphi \in C_0^\infty(\Omega)$, $\Omega \subset\subset M$.

This completes the proof of the Lemma. □

## 3 Curvature tensors for the Webster metric

In this section, we derive the relations between the Ricci tensors $R_{ij}^\lambda$ with respect to the Webster metric $h_\lambda$ and the pseudohermitian Ricci tensors.

We first write $\theta_1^1 = i\omega$ for some real 1-form $\omega$ by

$$d\theta^1 = \theta^1 \wedge \theta_1{}^1 + \theta \wedge \tau^1,$$
$$0 = \theta_1{}^1 + \theta_{\bar{1}}{}^{\bar{1}},$$

and $Z_1 = \frac{1}{2}(e_1 - ie_2)$ for real vectors $e_1, e_2$. It follows $e_2 = Je_1$. Let $e^1 = \mathrm{Re}(\theta^1)$, $e^2 = \mathrm{Im}(\theta^1)$. Then $\{e^1, e^2, \theta = e^3\}$ is dual to $\{e_1, e_2, e_3 = T\}$.

We recall the family of Webster metrics $h_\lambda$ of $(M, J, \theta)$ as following:

$$h_\lambda = h + \lambda^{-2}\theta^2,\ \lambda > 0$$

with

$$h(X, Y) = d\theta(X, JY).$$

Now put

$$\omega^1 = e^1,\ \omega^2 = e^2, \omega^3 = \lambda^{-1}e^3.$$

Then we have the Riemannian structure equations with respect to $h_\lambda$:

$$\begin{aligned} d\omega^i &= \omega^j \wedge \omega_j^i, \quad 1 \le i, j \le 3, \\ 0 &= \omega_i^j + \omega_j^i, \\ d\omega_i^j &= \omega_i^k \wedge \omega_k^j + \frac{1}{2}R^\lambda_{ijkl}\omega^k \wedge \omega^l, \quad 1 \le i, j, k, l \le 3. \end{aligned} \tag{3.1}$$

and $\theta^1 = \omega^1 + i\omega^2$ which satisfies the structure equations as in (1.4) and (1.5). Here $R^\lambda_{ijkl}$ is the Riemannian curvature tensor.

**Theorem 3.1** *Let $(M, J, \theta)$ be a closed pseudohermitian 3-manifold and $R^\lambda_{ij}$ be the Ricci curvature tensors with respect to the Webster metric $h_\lambda$.*

*(i) If the pseudohermitian torsion $A_{11}$ is vanishing, then*

$$\left(R^\lambda_{ij}\right) = \begin{pmatrix} 2W - 2\lambda^{-2} & 0 & 0 \\ 0 & 2W - 2\lambda^{-2} & 0 \\ 0 & 0 & 2\lambda^{-2} \end{pmatrix},$$

*where $W$ is the Tanaka–Webster scalar curvature and $\lambda$ is any positive constant.*

*(ii) If the torsion is nonvanishing, then*

$$\begin{aligned} R^\lambda_{11} &= 2W - 2\lambda^{-2} - 2i\lambda^2\,\mathrm{Im}\,A_{\bar{1}\bar{1}}{\theta_1}^1(T) + 2\,\mathrm{Im}\,A_{\bar{1}\bar{1}} - \lambda^2 T\left(\mathrm{Re}\,A_{\bar{1}\bar{1}}\right), \\ R^\lambda_{22} &= 2W - 2\lambda^{-2} + 2i\lambda^2\,\mathrm{Im}\,A_{\bar{1}\bar{1}}{\theta_1}^1(T) - 2\,\mathrm{Im}\,A_{\bar{1}\bar{1}} + \lambda^2 T(\mathrm{Re}\,A_{\bar{1}\bar{1}}), \\ R^\lambda_{33} &= -2\lambda^2\left|A_{\bar{1}\bar{1}}\right|^2 + 2\lambda^{-2}, \\ R^\lambda_{12} &= 2i\lambda^2\,\mathrm{Re}\,A_{\bar{1}\bar{1}}{\theta_1}^1(T) - 2\,\mathrm{Re}\,A_{\bar{1}\bar{1}} - \lambda^2 T\left(\mathrm{Im}\,A_{\bar{1}\bar{1}}\right), \\ R^\lambda_{13} &= 2\lambda\,\mathrm{Re}\,A_{11,\bar{1}}, \\ R^\lambda_{23} &= -2\lambda\,\mathrm{Im}\,A_{11,\bar{1}}. \end{aligned}$$

*Proof* We first obtain the relation of ${\theta_1}^1,\ \tau^1$ with $\omega_i^j$. In fact

$$\begin{aligned} d\theta^1 &= d\omega^1 + i d\omega^2 \\ &= (\omega^1 + i\omega^2) \wedge i\omega_1^2 + \omega^3 \wedge (\omega_3^1 + i\omega_3^2). \end{aligned}$$

On the other hand

$$\begin{aligned} d\theta^1 &= \theta^1 \wedge \theta_1^1 + \theta \wedge \tau^1 \\ &= (\omega^1 + i\omega^2) \wedge \theta_1^1 + \lambda\omega^3 \wedge \tau^1. \end{aligned}$$

Hence by Cartan lemma

$$\begin{aligned} {\theta_1}^1 &= i\omega_1^2 + a(\omega^1 + i\omega^2) + b\omega^3, \\ \lambda\tau^1 &= \omega_3^1 + i\omega_3^2 + b(\omega^1 + i\omega^2) + c\omega^3, \end{aligned} \tag{3.2}$$

for some complex functions $a, b, c$.

Similarly

$$d\omega^3 = \omega^1 \wedge \omega_1^3 + \omega^2 \wedge \omega_2^3.$$

But

$$d\omega^3 = \lambda^{-1} d\theta = 2\lambda^{-1}\omega^1 \wedge \omega^2.$$

Then

$$\begin{aligned} \omega_1^3 &= A\omega^1 + (B + \lambda^{-1})\omega^2, \\ \omega_2^3 &= (B - \lambda^{-1})\omega^1 + D\omega^2, \end{aligned} \tag{3.3}$$

for some real functions $A, B, D$.

Substitute (3.3) into (3.2), we obtain

$$\begin{aligned} \lambda\tau^1 &= [b - A - i(B - \lambda^{-1})]\omega^1 + [ib - (B + \lambda^{-1}) - iD]\omega^2 + c\omega^3 \\ &\equiv 0 \bmod \theta^{\bar{1}}, \end{aligned}$$

and then

$$c = 0 \quad \text{and} \quad 2b = A + D - 2i\varepsilon^{-1}. \tag{3.4}$$

On the other hand from

$$\begin{aligned} 0 &= {\theta_1}^1 + {\theta_{\bar{1}}}^{\bar{1}} \\ &= (a + \bar{a})\omega^1 + (ia - i\bar{a})\omega^2 + (b + \bar{b})\omega^3, \end{aligned}$$

we have

$$a = 0 \quad \text{and} \quad \operatorname{Re} b = 0. \tag{3.5}$$

From (3.4) together with (3.5), we get

$$A + D = 0, \quad a = c = 0 \quad \text{and} \quad b = -i\lambda^{-1}.$$

These together with (3.2) show that

$$\begin{aligned} \theta_1{}^1 &= i(\omega_1^2 - \lambda^{-2}\theta), \\ \tau^1 &= \lambda^{-1}(-A - iB)\theta^{\bar{1}}. \end{aligned} \tag{3.6}$$

However, $\tau^1 = A^1{}_{\bar{1}}\theta^{\bar{1}} = A_{\bar{1}\bar{1}}\theta^{\bar{1}}$. Thus $A = -\lambda \operatorname{Re} A_{\bar{1}\bar{1}}$ and $B = -\lambda \operatorname{Im} A_{\bar{1}\bar{1}}$. Substituting into (3.3) we get

$$\begin{aligned} \omega_1^3 &= \left(-\lambda \operatorname{Re} A_{\bar{1}\bar{1}}\right)\omega^1 + \left(-\lambda \operatorname{Im} A_{\bar{1}\bar{1}} + \lambda^{-1}\right)\omega^2, \\ \omega_2^3 &= \left(-\lambda \ \operatorname{Im} \ A_{\bar{1}\bar{1}} - \lambda^{-1}\right)\omega^1 + \left(\lambda \operatorname{Re} A_{\bar{1}\bar{1}}\right)\omega^2. \end{aligned} \tag{3.7}$$

Note that

$$\begin{aligned} \Omega_1^2 &= -i d\theta_1{}^1 \\ &= -i[W\theta^1 \wedge \theta^{\bar{1}} + 2i \operatorname{Im}(A_{11,\bar{1}}\theta^1 \wedge \theta)] \\ &= -2We^1 \wedge e^2 + 2\operatorname{Im}(A_{11,\bar{1}}\theta^1 \wedge \theta) \\ &= -2We^1 \wedge e^2 + 2\operatorname{Im} A_{11,\bar{1}}e^1 \wedge e^3 + 2\operatorname{Re} A_{11,\bar{1}}e^2 \wedge e^3. \end{aligned} \tag{3.8}$$

Next we compute the relations between the Tanaka–Webster curvature $W$, the pseudohermitian torsion $\tau^1$ and the curvatures with respect to $h_\lambda$.

$$\begin{aligned} \Omega_2^1 &= d\omega_2^1 + \lambda^{-2}d\theta \\ &= \omega_2^3 \wedge \omega_3^1 + R^\lambda_{2112}\omega^1 \wedge \omega^2 + R^\lambda_{2113}\omega^1 \wedge \omega^3 + R^\lambda_{2123}\omega^2 \wedge \omega^3 + 2\lambda^{-2}\omega^1 \wedge \omega^2 \\ &= \left(-\lambda^2 \left|A_{\bar{1}\bar{1}}\right|^2 + 3\lambda^{-2} + R^\lambda_{2112}\right)\omega^1 \wedge \omega^2 + R^\lambda_{2113}\omega^1 \wedge \omega^3 + R^\lambda_{2123}\omega^2 \wedge \omega^3. \end{aligned} \tag{3.9}$$

Using (3.7), (3.9) and comparing this with (3.8), we obtain

$$\begin{aligned} -R^\lambda_{1212} - \lambda^2 \left|A_{\bar{1}\bar{1}}\right|^2 + 3\lambda^{-2} &= 2W, \\ R^\lambda_{1213} &= 2\lambda\left(\operatorname{Im} A_{11,\bar{1}}\right), \\ R^\lambda_{1223} &= 2\lambda\left(\operatorname{Re} A_{11,\bar{1}}\right). \end{aligned} \tag{3.10}$$

Next from (3.1) and (3.7), we have

$$\begin{aligned} d\omega_1^3 &= d\left((-\lambda \operatorname{Re} A_{\bar{1}\bar{1}})\omega^1 + (-\lambda \operatorname{Im} A_{\bar{1}\bar{1}} + \lambda^{-1})\omega^2\right) \\ &= \left(-\lambda \operatorname{Re} A_{\bar{1}\bar{1}}\right) d\omega^1 + (-\lambda \operatorname{Im} A_{\bar{1}\bar{1}} + \lambda^{-1}) d\omega^2 \\ &\quad + d\left(-\lambda \operatorname{Re} A_{\bar{1}\bar{1}}\right) \wedge \omega^1 + d\left(-\lambda \operatorname{Im} A_{\bar{1}\bar{1}}\right) \wedge \omega^2. \end{aligned}$$

But

$$\begin{aligned} d\omega^1 &= \omega^2 \wedge \omega_2^1 + \omega^3 \wedge \omega_3^1 \\ &= \omega^2 \wedge \omega_2^1 + \left(-\lambda \operatorname{Re} A_{\bar{1}\bar{1}}\right) \omega^1 \wedge \omega^3 + (-\lambda \operatorname{Im} A_{\bar{1}\bar{1}} + \lambda^{-1})\omega^2 \wedge \omega^3 \end{aligned}$$

and

$$\begin{aligned} d\omega^2 &= \omega^1 \wedge \omega_1^2 + \omega^3 \wedge \omega_3^2 \\ &= \omega^1 \wedge \omega_1^2 + \left(\lambda \operatorname{Re} A_{\bar{1}\bar{1}}\right) \omega^2 \wedge \omega^3 + (-\lambda \operatorname{Im} A_{\bar{1}\bar{1}} - \lambda^{-1})\omega^1 \wedge \omega^3. \end{aligned}$$

Then

$$\begin{aligned} d\omega_1^3 &= \left(-\lambda \operatorname{Re} A_{\bar{1}\bar{1}}\right) \omega^2 \wedge \omega_2^1 + (-\lambda \operatorname{Im} A_{\bar{1}\bar{1}} + \lambda^{-1})\omega^1 \wedge \omega_1^2 \\ &\quad + \left(\lambda^2 \left|A_{\bar{1}\bar{1}}\right|^2 - \lambda^{-2}\right) \omega^1 \wedge \omega^3 + d\left(-\lambda \operatorname{Re} A_{\bar{1}\bar{1}}\right) \wedge \omega^1 + d\left(-\lambda \operatorname{Im} A_{\bar{1}\bar{1}}\right) \wedge \omega^2. \end{aligned}$$

On the other hand,

$$\begin{aligned} d\omega_1^3 &= \omega_1^2 \wedge \omega_2^3 + \sum_{1 \le i < j \le 3} R^\lambda_{13ij} \omega^i \wedge \omega^j \\ &= (\lambda \operatorname{Im} A_{\bar{1}\bar{1}} + \lambda^{-1})\omega^1 \wedge \omega_1^2 - \left(\lambda \operatorname{Re} A_{\bar{1}\bar{1}}\right) \omega^2 \wedge \omega_1^2 + \sum_{i<j} R^\lambda_{13ij} \omega^i \wedge \omega^j. \end{aligned}$$

Hence

$$\begin{aligned} \sum_{i<j} R^\lambda_{13ij} \omega^i \wedge \omega^j &= -2\lambda \operatorname{Re} A_{\bar{1}\bar{1}} \omega^2 \wedge \omega_2^1 - 2\lambda \operatorname{Im} A_{\bar{1}\bar{1}} \omega^1 \wedge \omega_1^2 \\ &\quad + \left(\lambda^2 \left|A_{\bar{1}\bar{1}}\right|^2 - \lambda^{-2}\right) \omega^1 \wedge \omega^3 \\ &\quad + d\left(-\lambda \operatorname{Re} A_{\bar{1}\bar{1}}\right) \wedge \omega^1 + d\left(-\lambda \operatorname{Im} A_{\bar{1}\bar{1}}\right) \wedge \omega^2. \end{aligned}$$

Therefore we get

$$\begin{aligned} R^\lambda_{1313} &= 2i\lambda^2 \operatorname{Im} A_{\bar{1}\bar{1}} {\theta_1}^1(T) - 2ImA_{\bar{1}\bar{1}} + \lambda^2 T\left(\operatorname{Re} A_{\bar{1}\bar{1}}\right) + \lambda^2 \left|A_{\bar{1}\bar{1}}\right|^2 - \lambda^{-2}, \\ R^\lambda_{1323} &= -2i\lambda^2 \operatorname{Re} A_{\bar{1}\bar{1}} {\theta_1}^1(T) + 2 \operatorname{Re} A_{\bar{1}\bar{1}} + \lambda^2 T \operatorname{Im} A_{\bar{1}\bar{1}}. \end{aligned} \tag{3.11}$$

Similarly we have

$$\begin{aligned} d\omega_2^3 &= \left(\lambda \operatorname{Re} A_{\bar{1}\bar{1}}\right) d\omega^2 + (-\lambda \operatorname{Im} A_{\bar{1}\bar{1}} - \lambda^{-1}) d\omega^1 \\ &\quad + d\left(\lambda \operatorname{Re} A_{\bar{1}\bar{1}}\right) \wedge \omega^2 + d\left(-\lambda \operatorname{Im} A_{\bar{1}\bar{1}}\right) \wedge \omega^1 \\ &= \left(\lambda \operatorname{Re} A_{\bar{1}\bar{1}}\right) \omega^1 \wedge \omega_1^2 + (-\lambda \operatorname{Im} A_{\bar{1}\bar{1}} - \lambda^{-1}) \omega^2 \wedge \omega_2^1 \\ &\quad + \left(\lambda^2 \left|A_{\bar{1}\bar{1}}\right|^2 - \lambda^{-2}\right) \omega^2 \wedge \omega^3 \\ &\quad + d\left(\lambda \operatorname{Re} A_{\bar{1}\bar{1}}\right) \wedge \omega^2 + d\left(-\lambda \operatorname{Im} A_{\bar{1}\bar{1}}\right) \wedge \omega^1. \end{aligned}$$

On the other hand,

$$\begin{aligned} d\omega_2^3 &= \omega_2^1 \wedge \omega_1^3 + \sum_{1 \le i < j \le 3} R_{23ij}^{\lambda} \omega^i \wedge \omega^j \\ &= (\lambda \operatorname{Im} A_{\bar{1}\bar{1}} - \lambda^{-1}) \omega^2 \wedge \omega_2^1 + \left(\lambda \operatorname{Re} A_{\bar{1}\bar{1}}\right) \omega^1 \wedge \omega_2^1 + \sum_{i<j} R_{23ij}^{\lambda} \omega^i \wedge \omega^j. \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{i<j} R_{23ij}^{\lambda} \omega^i \wedge \omega^j &= 2\lambda \operatorname{Re} A_{\bar{1}\bar{1}} \omega^1 \wedge \omega_1^2 - 2\lambda \operatorname{Im} A_{\bar{1}\bar{1}} \omega^2 \wedge \omega_2^1 \\ &\quad + \left(\lambda^2 \left|A_{\bar{1}\bar{1}}\right|^2 - \lambda^{-2}\right) \omega^2 \wedge \omega^3 \\ &\quad + d\left(-\lambda \operatorname{Im} A_{\bar{1}\bar{1}}\right) \wedge \omega^1 + d\left(\lambda \operatorname{Re} A_{\bar{1}\bar{1}}\right) \wedge \omega^2. \end{aligned}$$

Thus

$$R_{2323}^{\lambda} = -2i\lambda^2 \operatorname{Im} A_{\bar{1}\bar{1}} {\theta_1}^1(T) + 2 \operatorname{Im} A_{\bar{1}\bar{1}} - \lambda^2 T(\operatorname{Re} A_{\bar{1}\bar{1}}) + \lambda^2 \left|A_{\bar{1}\bar{1}}\right|^2 - \lambda^{-2} \tag{3.12}$$

and

$$R_{1313}^{\lambda} + R_{2323}^{\lambda} = 2\lambda^2 \left|A_{\bar{1}\bar{1}}\right|^2 - 2\lambda^{-2}.$$

All together imply

$$4W = -2R_{1212}^{\lambda} - R_{1313}^{\lambda} - R_{2323}^{\lambda} + 4\lambda^{-2}.$$

Together with (3.10), (3.11) and (3.12), we get that the Ricci curvature $R^{\lambda}_{ij}$ of $M$ with respect to the Webster metric $h_{\lambda}$ is

$$\begin{aligned}
R^{\lambda}_{11} &= 2W - 2\lambda^{-2} - 2i\lambda^2 \operatorname{Im} A_{\bar{1}\bar{1}}\theta_1{}^1(T) + 2\operatorname{Im} A_{\bar{1}\bar{1}} - \lambda^2 T\left(\operatorname{Re} A_{\bar{1}\bar{1}}\right),\\
R^{\lambda}_{22} &= 2W - 2\lambda^{-2} + 2i\lambda^2 \operatorname{Im} A_{\bar{1}\bar{1}}\theta_1{}^1(T) - 2\operatorname{Im} A_{\bar{1}\bar{1}} + \lambda^2 T(\operatorname{Re} A_{\bar{1}\bar{1}}),\\
R^{\lambda}_{33} &= -2\lambda^2\left|A_{\bar{1}\bar{1}}\right|^2 + 2\lambda^{-2},\\
R^{\lambda}_{12} &= 2i\lambda^2 \operatorname{Re} A_{\bar{1}\bar{1}}\theta_1{}^1(T) - 2\operatorname{Re} A_{\bar{1}\bar{1}} - \lambda^2 T\left(\operatorname{Im} A_{\bar{1}\bar{1}}\right),\\
R^{\lambda}_{13} &= 2\lambda \operatorname{Re} A_{11,\bar{1}},\\
R^{\lambda}_{23} &= -2\lambda \operatorname{Im} A_{11,\bar{1}}.
\end{aligned} \tag{3.13}$$

In particular, for $A_{11} = 0$

$$\left(R^{\lambda}_{ij}\right) = \begin{pmatrix} 2W - 2\lambda^{-2} & 0 & 0\\ 0 & 2W - 2\lambda^{-2} & 0\\ 0 & 0 & 2\lambda^{-2} \end{pmatrix}. \tag{3.14}$$

This completes the proof. □

*Remark 3.1* If the torsion is nonvanishing, one can choose a suitable coordinate with $\theta_1{}^1(T) = 0$ at a point. It follows that

$$\begin{aligned}
R^{\lambda}_{11} &= 2W - 2\lambda^{-2} + 2\operatorname{Im} A_{\bar{1}\bar{1}} - \lambda^2 T\left(\operatorname{Re} A_{\bar{1}\bar{1}}\right),\\
R^{\lambda}_{22} &= 2W - 2\lambda^{-2} - 2\operatorname{Im} A_{\bar{1}\bar{1}} + \lambda^2 T(\operatorname{Re} A_{\bar{1}\bar{1}}),\\
R^{\lambda}_{33} &= -2\lambda^2\left|A_{\bar{1}\bar{1}}\right|^2 + 2\lambda^{-2},\\
R^{\lambda}_{12} &= -2\operatorname{Re} A_{\bar{1}\bar{1}} - \lambda^2 T\left(\operatorname{Im} A_{\bar{1}\bar{1}}\right),\\
R^{\lambda}_{13} &= 2\lambda \operatorname{Re} A_{11,\bar{1}},\\
R^{\lambda}_{23} &= -2\lambda \operatorname{Im} A_{11,\bar{1}}.
\end{aligned} \tag{3.15}$$

## 4 The proofs

Let $d_{\lambda}$ the distance with respect to the Webster metric $h_{\lambda}$. Fukaya [6] observed that the metric space $(M, d_{\lambda})$ converges to a Carnot-Carathéodory metric space $(M, d_c)$ and $\Delta_{\lambda}$ converges to a sub-Laplacian $\Delta_b$ as $\lambda \to 0$.

For a real function $\varphi$ and $e^{\lambda}_1 = e_1, e^{\lambda}_2 = e_2, e^{\lambda}_3 = \lambda T$, we have

$$\Delta_b = \frac{1}{2}(\varphi_{e_1e_1} + \varphi_{e_2e_2}), \quad \omega^{\alpha}_3(T) = 0.$$

**Lemma 4.1** *Let $(M, J, \theta)$ be a closed pseudohermitian 3-manifold. For a real function $\varphi$, we have*

$$\Delta_\lambda \varphi = 2\Delta_b \varphi + \lambda^2 T^2 \varphi.$$

*Proof* We compute that

$$\begin{aligned}\Delta_\lambda \varphi &= \sum_{j=1}^{3} e_j^\lambda(e_j^\lambda \varphi) - \omega_j^k(e_j^\lambda)e_k^\lambda \varphi \\ &= 2\Delta_b \varphi + e_3^\lambda(e_3^\lambda \varphi) - \omega_3^\alpha(e_3^\lambda)e_\alpha^\lambda \varphi - \omega_\beta^3(e_\beta^\lambda)e_3^\lambda \varphi \\ &= 2\Delta_b \varphi + \lambda^2 T^2 \varphi - \lambda \omega_1^3(e_1)T\varphi - \lambda \omega_2^3(e_2)T\varphi \\ &= 2\Delta_b \varphi + \lambda^2 T^2 \varphi - \lambda A T\varphi + \lambda A T\varphi \\ &= 2\Delta_b \varphi + \lambda^2 T^2 \varphi, \;\; A = -\operatorname{Re} A_{\bar{1}\bar{1}}.\end{aligned}$$

□

*Proof of Theorem 1.3* First note that $\omega^1 = e^1,\ \omega^2 = e^2, \omega^3 = \lambda^{-1}e^3$. Then

$$d\mu^\lambda = \omega^1 \wedge \omega^2 \wedge \omega^3 = \lambda^{-1}e^1 \wedge e^2 \wedge e^3$$

and

$$d\mu^\lambda = \frac{1}{2}\lambda^{-1} d\mu.$$

Therefore from Lemma 4.1

$$2\lambda \int \varphi \Delta_\lambda \varphi d\mu^\lambda = 2\int \varphi \Delta_b \varphi \, d\mu + \lambda^2 \int \varphi T^2 \varphi \, d\mu$$

and then

$$2\lambda \int |\nabla^\lambda \varphi|^2 d\mu^\lambda = 2\int |\nabla_b \varphi|^2 \, d\mu + \lambda^2 \int \varphi_0^2 \, d\mu. \tag{4.1}$$

Now from (4.1) and Lemma 2.2, it follows that

$$2\lambda \int |\nabla^\lambda \varphi|^2 d\mu^\lambda \leq 2\int |\nabla_b \varphi|^2 \, d\mu + \lambda^2 \int (\Delta_b \varphi)^2 \, d\mu + \lambda^2 \tau_0 \int |\nabla_b \varphi|^2 \, d\mu.$$

Suppose that

$$\Delta_b \varphi = -\mu_1 \varphi.$$

Then

$$\mu_1 = \frac{\int |\nabla_b \varphi|^2 \, d\mu}{\int \varphi^2 \, d\mu}$$

and

$$\frac{\int |\nabla^\lambda \varphi|^2 d\mu^\lambda}{\int \varphi^2 d\mu^\lambda} \le (2 + \lambda^2 \mu_1 + 2\lambda^2 \tau_0)\mu_1.$$

Now by definition for the eigenvalue, we have

$$\mu_1^\lambda \le \frac{\int |\nabla^\lambda \varphi|^2 d\mu^\lambda}{\int \varphi^2 d\mu^\lambda} \le (2 + \lambda^2 \mu_1 + 2\lambda^2 \tau_0)\mu_1.$$

This completes the proof. □

**Definition 4.1** We call a CR structure $J$ spherical if Cartan curvature tensor $Q_{11}$ vanishes identically. Here

$$Q_{11} = \frac{1}{6} W_{11} + \frac{i}{2} W A_{11} - A_{11,0} - \frac{2i}{3} A_{11,\bar{1}1}.$$

Note that $(M, J, \theta)$ is called a spherical pseudohermitian 3-manifold if $J$ is a spherical structure. We observe that the spherical structure is CR invariant and a closed spherical pseudohermitian 3-manifold $(M, J, \theta)$ is locally CR equivalent to the standard pseudohermitian 3 -sphere $(\mathbf{S}^3, \widehat{J}, \widehat{\theta})$. In additional, if $M$ is simply connected, then $(M, J, \theta)$ is the standard pseudohermitian 3-sphere. We refer to [12] for some details.

*Proof of Corollary 1.4* It follows from (3.14) that

$$\left(R_{ij}^\lambda\right) = \begin{pmatrix} 2W - 2\lambda^{-2} & 0 & 0 \\ 0 & 2W - 2\lambda^{-2} & 0 \\ 0 & 0 & 2\lambda^{-2} \end{pmatrix}$$

and then

$$Ric(h_\lambda) \ge (3-1)\lambda^{-2} = 2\lambda^{-2}.$$

On the other hand, Lichnerowicz Theorem implies

$$\mu_1^\lambda \ge 3\lambda^{-2}.$$

It follows from Theorem 1.3 that

$$3\lambda^{-2} \le \mu_1^\lambda \le (2 + \lambda^2 \mu_1)\mu_1.$$

Again from Proposition 1.2

$$\mu_1 \geq \lambda^{-2}.$$

Hence if $\mu_1 = \lambda^{-2}$, it follows that

$$\mu_1^\lambda = 3\lambda^{-2}.$$

Thus, due to Obata Theorem, $M$ is simply connected and $W = 2\lambda^{-2}$. On the other hand, $A_{11} = 0$. Then $M$ is spherical as well. All these imply $(M, J, \theta)$ is the standard pseudohermitian 3-sphere $(S^3, \widehat{J}, \widehat{\theta})$. □

*Proof of Corollary 1.5* By our assumptions and (3.15), we have

$$Ric^\lambda \geq -7\lambda^{-2} = -2 \cdot \frac{7}{2}\lambda^{-2}.$$

Therefore by Li-Yau eigenvalue estimate [14], we have

$$\mu_1^\lambda \geq \frac{\exp\{-[1 + (1 + 7c^2 D_\lambda^2 \lambda^{-2})^{\frac{1}{2}}]\}}{c D_\lambda^2}$$

for some $c$ depending on the dimension of $M$ and $D_\lambda^2 = \mathrm{diam}(M, h_\lambda)$. But we first note that $d_\lambda(x, x_0)$ is an increasing function of $\lambda^{-1}$ and then

$$d_\lambda(x, x_0) \leq d_c(x, x_0)$$

for the distance $d_\lambda(x, x_0)$ with respect to the metric $h_\lambda$. Hence

$$D_\lambda \leq D.$$

It follows that

$$(2 + \lambda^2 \mu_1 + 2\lambda^2 \tau_0)\mu_1 \geq \frac{\exp\{-[1 + (1 + 7c^2 D^2 \lambda^{-2})^{\frac{1}{2}}]\}}{c D^2}.$$

Now if $\mu_1 < \lambda^{-2}$, then

$$5\mu_1 \geq (2 + \lambda^2 \mu_1 + 2\lambda^2 \tau_0)\mu_1 \geq \frac{\exp\{-[1 + (1 + 7c^2 D^2 \lambda^{-2})^{\frac{1}{2}}]\}}{c D^2}.$$

That is

$$\mu_1 \geq \frac{\exp\{-[1 + (1 + 7c^2 D^2 \lambda^{-2})^{\frac{1}{2}}]\}}{5c D^2}.$$

□

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