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Determination of holomorphic modular forms by primitive Fourier coefficients

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Abstract We prove that Siegel modular forms of degree greater than one, integral weight and level N, with respect to a Dirichlet character χ of conductor $\mathfrak{f}_\chi$ are uniquely determined by their Fourier coefficients indexed by matrices whose contents run over all divisors of $N/\mathfrak{f}_\chi$. The cases of other major types of holomorphic modular forms are included.

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1 Introduction

It is an interesting problem to give a reasonable sufficient condition on the Fourier coefficients under which modular forms must be zero.

To this problem, there are some results available for cuspidal Hecke eigenforms.

For example, Breulmann and Kohnen [1] have showed that cuspidal Hecke eigenforms on $\mathrm{Sp}_2(\mathbb{Z})$ are uniquely determined by their Fourier coefficients indexed by matrices of square-free content. We here define the content $\epsilon(h)$ of h via

$$\epsilon(h) = \max\{a \in \mathbb{N} \mid a^{-1}h \text{ is half-integral}\}$$

for each nonzero symmetric half-integral matrix h. We call matrices of content one primitive. This result was extended to higher degrees by Katsurada [4]. A similar kind of result was independently obtained in [6]. Namely, every cuspidal Siegel-Hecke

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eigenform of arbitrary degree is uniquely determined by all of its primitive Fourier coefficients and a certain subset of its Hecke eigenvalues. There are no holomorphic modular forms on the split exceptional group G_2. Nevertheless, an analogous result for certain modular forms on G_2 was obtained at the end of [2].

Whereas [1] uses analytic methods involving twisted Koecher-Maass series and spinor zeta functions, [2,4,6] use algebraic arguments based on the explicit description of the action of the Hecke operators on the Fourier coefficients.

These results look as if they might be analogous to the classical fact, i.e., if

$$f(\tau) = \sum_{t=1}^{\infty} a(t) q^t \in S_k(\mathrm{SL}_2(\mathbb{Z})), \quad q = \mathbf{e}(\tau) = e^{2\pi\sqrt{-1}\tau}$$

is a Hecke eigenform, then the knowledge of $a(1)$ and $\{a(p)\}_p$, where p runs over all prime numbers, is sufficient to determine f. However, the following stronger result, which had been implicit in the proof of Theorem 1 of [9] by Zagier (see [9, p. 387]), reveals that primitive Fourier coefficients less compellingly serve as a substitute for the lack of $a(1)$.

Theorem 1 (Zagier) *Let κ be a positive integer, and*

$$F(Z) = \sum_h A(h) \mathbf{e}(\mathrm{tr}(hZ))$$

an element of $M_\kappa(\mathrm{Sp}_2(\mathbb{Z}))$. Assume that $A(h) = 0$ for all primitive h. Then $F = 0$.

Remark 1 Heim [3] has recently obtained a somewhat elaborate result concerning $S_\kappa(\mathrm{Sp}_2(\mathbb{Z}))$ with even weight κ by essentially the same way.

Zagier's proof in [9] is brilliant and is based on the Taylor expansions of the Fourier–Jacobi coefficients. In this paper we extend Theorem 1 to holomorphic modular forms on classical tube domains, generalizing his method.

For simplicity, we here state our result only in the case of Siegel modular forms of integral weight. Let N be a positive integer and χ a character of $(\mathbb{Z}/N\mathbb{Z})^\times$ of conductor $\mathfrak{f}_\chi$. Put

$$\Gamma_0^n(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_n(\mathbb{Z}) \;\middle|\; c \equiv 0 \pmod{N} \right\}.$$

When $n \geq 2$, a modular form $F \in M_\kappa(\Gamma_0^n(N), \chi)$ is a holomorphic function on the upper half-space of degree n which satisfies

$$F(\gamma Z) = \chi(\det d) \det(cZ + d)^\kappa F(Z), \quad \gamma Z = (aZ + b)(cZ + d)^{-1}$$

for every $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0^n(N)$.

Theorem 2 (cf. Theorem 3) *Suppose $\kappa \in \mathbb{N}$, $n \geq 2$, and*

$$F(Z) = \sum_h A(h) \mathbf{e}(\mathrm{tr}(hZ))$$

is an element of $M_\kappa(\Gamma_0^n(N), \chi)$*. Assume that* $A(h) = 0$ *for all nonzero symmetric half-integral matrices* h *such that* $\epsilon(h)$ *divides* $N/\mathfrak{f}_\chi$*. Then we have* $F = 0$.

After reviewing several definitions and notations in Sect. 2, we state our main result in Sect. 3. Section 4 contains an elementary lemma on elliptic modular forms, which helps us to complete the proof in Sect. 5.

2 Domains and forms

There are four major types of tube domains, which we refer to as Cases I–IV. The first three of them are defined with respect to the division algebra. Let D denote the rational number field, an imaginary quadratic field or a definite quaternion algebra over $\mathbb{Q}$ in Cases I–III, respectively. For $x \in \mathrm{M}_{lm}(D)$, we put $x^* = {}^t\overline{x}$, where $\overline{\cdot}$ is to be interpreted as the main involution in Case III and the complex conjugate otherwise. Fix a positive integer n. In Case IV we fix an $(n-1)$-dimensional vector space X over $\mathbb{Q}$ and a positive definite quadratic form $q : X \to \mathbb{Q}$.

Remark 2 The systematic use of Jordan algebras would allow us to include exceptional domains. However, we think that our formulation is better suited to clarify the picture.

We define a vector space V_n over $\mathbb{Q}$ by

$$V_n = \{x \in \mathrm{M}_n(D) \mid x^* = x\}, \qquad \text{(I–III)}$$
$$V_n = \mathbb{Q}e \oplus X \oplus \mathbb{Q}f. \qquad \text{(IV)}$$

In Case IV we define the quadratic form $Q : V_n \to \mathbb{Q}$ by

$$Q[ae + x + bf] = 2ab - q[x] \quad \text{for } a, b \in \mathbb{Q},\ x \in X. \qquad \text{(IV)}$$

Whenever we speak of quadratic forms q and Q, we identify q and Q with symmetric matrices and understand that

$$Q[x] = {}^t x Q x, \quad Q(x, y) = {}^t x Q y \quad \text{for } x, y \in V_n.$$

Let $\tau : \mathrm{M}_n(D) \to \mathbb{Q}$ be the reduced trace. Let ν be the reduced norm on $\mathrm{M}_n(D)$ in Case III and the determinant otherwise. We define also in Cases I–III the bilinear form Q on V_n by

$$Q(x, y) = \tau(xy), \qquad \text{(I)}$$
$$Q(x, y) = \tau(xy)/2. \qquad \text{(II, III)}$$

For any $\mathbb{Q}$-algebra R, we extend ν to the polynomial map over R on $\mathrm{M}_n(D \otimes_{\mathbb{Q}} R)$ and extend Q to the R-bilinear form on $V_n \otimes_{\mathbb{Q}} R$.

Putting $V_{\mathbb{R}} = V_n \otimes_{\mathbb{Q}} \mathbb{R}$ and letting $\mathbb{H}$ be the Hamilton quaternion division algebra, we define the complexification $V_{\mathbb{C}}$ of $V_{\mathbb{R}}$ by

$$V_{\mathbb{C}} = \{x \in \mathrm{M}_n(\mathbb{C}) \mid {}^t x = x\}, \tag{I}$$
$$V_{\mathbb{C}} = \mathrm{M}_n(\mathbb{C}), \tag{II}$$
$$V_{\mathbb{C}} = \{x \in \mathrm{M}_n(\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}) \mid x^* = x\}, \tag{III}$$
$$V_{\mathbb{C}} = V_n \otimes_{\mathbb{Q}} \mathbb{C}. \tag{IV}$$

We put

$$\mathcal{P}_n = \{x \in V_{\mathbb{R}} \mid x \text{ is positive}\}, \tag{I–III}$$
$$\mathcal{P}_n = \{x \in V_{\mathbb{R}} \mid Q[x] > 0,\ \ Q(x, e + f) > 0\}, \tag{IV}$$

where we call x positive if $y^* x y > 0$ for all $y \in \mathbb{D}^n \backslash \{0\}$ ($\mathbb{D} = D \otimes_{\mathbb{Q}} \mathbb{R}$). The upper half-space $\mathfrak{H}_n$ is defined by

$$\mathfrak{H}_n = \{x + \sqrt{-1}y \in V_{\mathbb{C}} \mid x \in V_{\mathbb{R}},\ \ y \in \mathcal{P}_n\}.$$

We consider the algebraic group G_n whose group of R-valued points is given by

$$G_n(R) = \{g \in \mathrm{SL}_{2n}(D \otimes_{\mathbb{Q}} R) \mid g^* \eta_n g = \eta_n\}, \tag{I–III}$$
$$G_n(R) = \{g \in \mathrm{SL}_{n+3}(R) \mid {}^t g Q' g = Q'\} \tag{IV}$$

for any $\mathbb{Q}$-algebra R, where

$$\eta_n = \begin{pmatrix} 0 & -\mathbf{1}_n \\ \mathbf{1}_n & 0 \end{pmatrix}, \quad Q' = \begin{pmatrix} & & 1 \\ & Q & \\ 1 & & \end{pmatrix}.$$

The connected component of the identity in the topological group $G_n(\mathbb{R})$ is denoted by $G_n(\mathbb{R})^\circ$. Obviously, $G_n(\mathbb{R})^\circ = G_n(\mathbb{R})$ in Cases I–III. The action of $G_n(\mathbb{R})^\circ$ on $\mathfrak{H}_n$ and the automorphy factor $j(g, Z)$ on $G_n(\mathbb{R})^\circ \times \mathfrak{H}_n$ are defined by

$$gZ = (aZ + b)(cZ + d)^{-1}, \quad j(g, Z) = \nu(cZ + d), \tag{I–III}$$
$$gZ^{\sim} = (gZ)^{\sim} j(g, Z), \quad Z^{\sim} = \begin{pmatrix} -Q[Z]/2 \\ Z \\ 1 \end{pmatrix}. \tag{IV}$$

Here, we write g in the form $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with matrices a, b, c, d of size n in Cases I–III. Assuming that κ is even in Case III, we put

$$j_\kappa(g, Z) = j(g, Z)^\kappa, \tag{I, II, IV}$$
$$j_\kappa(g, Z) = j(g, Z)^{\kappa/2}. \tag{III}$$

We fix once and for all a maximal order $\mathcal{O}$ of D and a positive integer N. Of course, $\mathcal{O}$ is the ring of integers of D in Cases I, II. Writing a typical element $\gamma \in G_n(\mathbb{Q})$ in the form $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with a matrix d of size n in Cases I–III (resp. size 1 in Case IV), we denote by $\Gamma_0^n(N)$ the subgroup of $G_n(\mathbb{Q})$ consisting of element γ for which all the entries of a, b, $N^{-1}c$, d belong to $\mathcal{O}$ (resp. $\mathbb{Z}$).

We include modular forms of half-integral weight in Case I. By a half-integer, we mean an element $\kappa \in 2^{-1}\mathbb{Z}$ such that 2κ is odd. We refer to this case as I$'$ when it requires a separate treatment. Let us require N to be a multiple of 4 in Case I$'$. Given such a κ and $\gamma \in \Gamma_0^n(4)$, we put

$$j_\kappa(\gamma, Z) = (\Theta(\gamma Z)/\Theta(Z))^{2\kappa}, \tag{I$'$}$$

where

$$\Theta(Z) = \sum_{r \in \mathbb{Z}^n} \mathbf{e}({}^t r Z r).$$

As is well-known,

$$(\Theta(\gamma Z)/\Theta(Z))^2 = \chi_{-1}(\det d)\det(cZ + d) \tag{2.1}$$

for any $\gamma = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \Gamma_0^n(4)$, where $\chi_{-1}(t) = (-1)^{(t-1)/2}$ (cf. [10]).

Fix a Dirichlet character χ modulo N. For $\gamma = \left(\begin{smallmatrix} * & * \\ * & d \end{smallmatrix}\right) \in \Gamma_0^n(N)$, we put

$$\chi(\gamma) = \chi(\nu(d)), \tag{I–III}$$

$$\chi(\gamma) = \chi(d). \tag{IV}$$

Remark 3 (1) Note that $\nu(d)$ is an element of $\mathbb{Q}$ in Case II by Lemma 1.1 of [8].
(2) In Case I$'$, we can easily see that $M_\kappa(\Gamma_0^n(N), \chi) = \{0\}$ if $\chi(-1)^n = -1$.

Provided that $n \geq 2$, we call a holomorphic function F on $\mathfrak{H}_n$ a modular form of weight κ with respect to $\Gamma_0^n(N)$ and χ if

$$F(\gamma Z) = \chi(\gamma) j_\kappa(\gamma, Z) F(Z) \tag{2.2}$$

for every $\gamma \in \Gamma_0^n(N)$. When $n = 1$, we further require F to be holomorphic at all cusps. The space of modular forms of weight κ with respect to $\Gamma_0^n(N)$ and χ is denoted by $M_\kappa(\Gamma_0^n(N), \chi)$.

3 Main theorem

Fix an even-integral lattice L of X and define $\Gamma_0^n(N)$ with respect to a basis of L in Case IV. Put

$$\widetilde{\mathcal{O}} = \{\alpha \in D \mid \tau(\alpha\beta) \in \mathbb{Z} \text{ for every } \beta \in \mathcal{O}\}, \tag{I–III}$$

$$\widetilde{L} = \{\alpha \in X \mid q(\alpha, \beta) \in \mathbb{Z} \text{ for every } \beta \in L\}. \tag{IV}$$

The set of semi-integral hermitian matrices T_n is defined by

$$T_n = \{(\alpha_{ij}) \in V_n \mid \alpha_{ii} \in \mathbb{Z},\ \alpha_{ij} \in \widetilde{\mathcal{O}}\ (i \neq j)\}, \qquad \text{(I–III)}$$
$$T_n = \mathbb{Z}e \oplus \widetilde{L} \oplus \mathbb{Z}f. \qquad \text{(IV)}$$

It is important to note that $F \in M_\kappa(\Gamma_0^n(N), \chi)$ has a Fourier expansion of the form

$$F(Z) = \sum_{h \in T_n} A(h)\mathbf{e}(Q(h, Z)),$$

where $A(h) = 0$ unless $Q[h] \geq 0$ and $Q(e + f, h) \geq 0$ in Case IV (resp. h is positive semi-definite in Cases I–III) by the Koecher principle.

For each nonzero element $h \in T_n$, we put

$$\epsilon(h) = \max\{a \in \mathbb{N} \mid a^{-1}h \in T_n\}.$$

We call a Dirichlet character χ quadratic if $\chi^2 = 1$, and call χ even or odd according as $\chi(-1) = 1$ or -1. If χ is even (resp. odd), then we write $\mathfrak{F}_\chi$ for the greatest common divisor of $\{\mathfrak{f}_{\chi\psi}\}_\psi$, where ψ runs through all even (resp. odd) quadratic characters modulo N.

An integer N_χ is defined via

$$N_\chi = N/\mathfrak{f}_\chi, \qquad \text{(I, II, IV)}$$
$$N_\chi = N/\mathfrak{f}_{\chi^2}, \qquad \text{(III)}$$
$$N_\chi = (N/4, N/\mathfrak{F}_\chi). \qquad \text{(I}'\text{)}$$

Theorem 3 *Suppose that $\kappa \geq \frac{1}{2}$, $n \geq 2$ and*

$$F(Z) = \sum_{h \in T_n} A(h)\mathbf{e}(Q(h, Z))$$

is an element of $M_\kappa(\Gamma_0^n(N), \chi)$. Assume that $A(h) = 0$ for all nonzero $h \in T_n$ such that N_χ is divisible by $\epsilon(h)$. Then we have $F = 0$.

Remark 4 In Case IV, we put

$$G'_n(\mathbb{R}) = \{\alpha \in \mathrm{SL}_{n+1}(\mathbb{R}) \mid {}^t\alpha Q\alpha = Q\}, \qquad \Delta_n = \mathrm{SL}_{n+1}(\mathbb{Z}) \cap G'_n(\mathbb{R})^\circ.$$

For given $h \in T_n$, we define the class $\mathrm{cls}(h)$ of h by

$$\mathrm{cls}(h) = \{\alpha^* h\alpha \mid \alpha \in \mathrm{GL}_n(\mathcal{O})\}, \qquad \text{(I–III)}$$
$$\mathrm{cls}(h) = \{\alpha h \mid \alpha \in \Delta_n\}. \qquad \text{(IV)}$$

Note that ϵ is class invariant and the vanishing of $A(h)$ depends only on $\mathrm{cls}(h)$.

4 Lemmas on elliptic modular forms

If l is an integer and the discriminant of $\mathbb{Q}(\sqrt{l})/\mathbb{Q}$ is $\mathfrak{d}$, then χ_l denotes the quadratic character of conductor $|\mathfrak{d}|$, i.e.,

$$\chi_l(d) = \left(\frac{\mathfrak{d}}{d}\right),$$

where the right hand side is the Kronecker symbol. The product $\chi\chi'$ of two characters χ and χ' is the primitive Dirichlet character associated with $d \mapsto \chi(d)\chi'(d)$.

For convenience, we first review two classical results. These are Lemma 4.6.5 and Theorem 4.6.8 of [5] respectively when k is an integer, and Lemma 4 and Theorem 1 of [7] respectively when k is a half-integer.

Lemma 1 *Let k be an element of $2^{-1}\mathbb{Z}$ and $f(\tau) = \sum_{t=0}^{\infty} a(t)q^t$ that of $M_k(\Gamma_0(N), \chi)$. For a positive integer l, we put*

$$g(\tau) = \sum_{(t,l)=1} a(t)q^t.$$

Then $g \in M_k(\Gamma_0(Nl^2), \chi)$.

In Case I$'$, for a positive integer l, we write $S_{N,\chi,l}$ for the set of prime factors p of $(l, N/4)$ such that N is divisible by $p\mathfrak{f}_{\chi\chi_p}$.

Lemma 2 *Let k, l and f be the ones in Lemma 1. Assume that $a(t) = 0$ for all t prime to l.*

1. *When k is a positive integer, the following assertions hold.*
   (a) *If $(l, N/\mathfrak{f}_\chi) = 1$, then $f = 0$.*
   (b) *If $(l, N/\mathfrak{f}_\chi) \neq 1$, then there exist $f_p \in M_k(\Gamma_0(N/p), \chi)$ for all prime factors p of $(l, N/\mathfrak{f}_\chi)$ such that*

$$f(\tau) = \sum_{p|(l,N/\mathfrak{f}_\chi)} f_p(p\tau).$$

2. *When k is a half-integer, the following assertions hold.*
   (c) *If $S_{N,\chi,l} = \emptyset$, then $f = 0$.*
   (d) *If $S_{N,\chi,l} \neq \emptyset$, then there exist $f_p \in M_k(\Gamma_0(N/p), \chi\chi_p)$ for all $p \in S_{N,\chi,l}$ such that*

$$f(\tau) = \sum_{p\in S_{N,\chi,l}} f_p(p\tau).$$

The following lemma is a generalization of Lemma 2 (a), (c).

Lemma 3 *Let k, l and f be the ones in Lemma 1.*

1. *Suppose that k is a positive integer. If $a(t) = 0$ for all t such that $N/\mathfrak{f}_\chi$ is divisible by (l, t), then $f = 0$.*
2. *Suppose that k is a half-integer. If $a(t) = 0$ for all t such that $(N/4, N/\mathfrak{F}_\chi)$ is divisible by (l, t), then $f = 0$.*

Proof We will prove only (2). The proof of (1) is almost the same.

The proof is by induction on $\delta = (l, N/4, N/\mathfrak{F}_\chi)$. If $\delta = 1$, then $f = 0$ by Lemma 2(c). We suppose that $\delta > 1$, assuming the result up to $\delta - 1$ for all N and χ.

Fix a prime factor p of δ. Put $L = lp^{-\operatorname{ord}_p l}$ and

$$g(\tau) = \sum_{(t,p)=1} a(t)q^t.$$

Then $g \in M_k(\Gamma_0(Np^2), \chi)$ by Lemma 1. Observe that $a(t) = 0$ if $(t, p) = 1$ and if $(Np^2/4, Np^2/\mathfrak{F}_\chi)$ is divisible by (t, L). Noting that $(L, Np^2/4, Np^2/\mathfrak{F}_\chi) < \delta$, we have $g = 0$ by induction. Put

$$\begin{aligned} f'(\tau) &= \sum_{t=0}^{\infty} b(t)q^t, \quad b(t) = a(pt), \\ N' &= N/p, \quad \chi' = \chi\chi_p, \quad l' = l/p. \end{aligned}$$

Employing Lemma 2 (2), we have $f' \in M_k(\Gamma_0(N'), \chi')$ if N' is divisible by $\mathfrak{f}_{\chi'}$, and $f' = 0$ otherwise. Observe that $b(t) = 0$ if $(N'/4, N'/\mathfrak{F}_{\chi'})$ is divisible by (l', t). Since $(l', N'/4, N'/\mathfrak{F}_{\chi'}) = \delta/p$, we have $f' = 0$ by induction. Hence $f = 0$. □

5 Proof of Theorem 3

Seeking a contradiction, we assume that $F \neq 0$. In view of Remark 4 we can take a nonzero hermitian matrix S of size $n-1$ in Cases I–III (resp. size 1 in Case IV) such that the Sth Fourier–Jacobi coefficient F_S is not identically zero. To be uniform, we write $A(S, r, t) = A\left(\begin{smallmatrix} S & r \\ r^* & t \end{smallmatrix}\right)$ for $r \in \widetilde{\mathcal{O}}^{n-1}$ and $t \in \mathbb{Z}$ in Cases I–III. We then have

$$F_S(\tau, w) = \sum_{r,\,t} A(S, r, t)q^t \mathbf{e}(B(r, w)),$$

where

$$B(r, w) = 2\,{}^t r w \quad \text{for} \quad w \in \mathbb{C}^{n-1}, \tag{I}$$

$$B(r, w) = r^* w_1 + {}^t r w_2 \quad \text{for} \quad w \in \mathbb{C}^{n-1} \times \mathbb{C}^{n-1}, \tag{II}$$

$$B(r, w) = \tau(r^* w) \quad \text{for} \quad w \in \mathbb{H}^{n-1} \otimes_{\mathbb{R}} \mathbb{C}, \tag{III}$$

$$B(r, w) = q(r, w) \quad \text{for} \quad w \in X \otimes_{\mathbb{Q}} \mathbb{C}. \tag{IV}$$

For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, we define $\iota(\gamma) \in \Gamma_0^n(N)$ by

$$\iota(\gamma) = \begin{pmatrix} \mathbf{1}_{n-1} & & & \\ & a & & b \\ & & \mathbf{1}_{n-1} & \\ & c & & d \end{pmatrix}, \tag{I–III}$$

$$\iota(\gamma) = \begin{pmatrix} a & -b & & & \\ -c & d & & & \\ & & \mathbf{1}_{n-1} & & \\ & & & a & b \\ & & & c & d \end{pmatrix}. \tag{IV}$$

Let us define S' and χ' by

$$S' = S, \tag{I–III}$$
$$S' = (S/2) \cdot q, \tag{IV}$$
$$\chi' = \chi, \tag{I, II, IV}$$
$$\chi' = \chi^2. \tag{III}$$

Substituting

$$\iota(\gamma) \begin{pmatrix} z & w \\ w^* & \tau \end{pmatrix} = \begin{pmatrix} z - \dfrac{cww^*}{c\tau + d} & \dfrac{w}{c\tau + d} \\ \dfrac{w^*}{c\tau + d} & \gamma\tau \end{pmatrix}, \tag{I–III}$$

$$\iota(\gamma) \begin{pmatrix} z \\ w \\ \tau \end{pmatrix} = \begin{pmatrix} z - \dfrac{cq[w]}{2(c\tau + d)} \\ \dfrac{w}{c\tau + d} \\ \gamma\tau \end{pmatrix}, \tag{IV}$$

$$j_\kappa(\iota(\gamma), Z) = j_\kappa(\gamma, \tau)$$

into (2.2), we obtain the following transformation equation

$$F_S\left(\gamma\tau, \frac{w}{c\tau + d}\right) = \chi'(d) j_\kappa(\gamma, \tau) \mathbf{e}\left(\frac{cS'[w]}{c\tau + d}\right) F_S(\tau, w) \tag{5.1}$$

for every $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$.

Developing F_S in a Taylor expansion around $w = 0$, we have

$$F_S(\tau, w) = \sum_{\nu=0}^{\infty} \lambda_\nu(\tau, w),$$

where λ_ν is the homogeneous part of F_S of degree ν, i.e.,

$$\lambda_\nu(\tau, w) = \frac{1}{\nu!} \sum_{r,\, t} (2\pi\sqrt{-1}B(r, w))^\nu A(S, r, t) q^t.$$

Put $\nu_0 = \min\{\nu \mid \lambda_\nu \neq 0\}$. Applying (5.1), we have

$$\sum_{\nu=\nu_0}^{\infty} \frac{\lambda_\nu(\gamma\tau, w)}{j_\kappa(\gamma, \tau)(c\tau + d)^\nu} = \chi'(d) \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{2\pi\sqrt{-1}cS'[w]}{c\tau + d} \right)^j \sum_{\nu=\nu_0}^{\infty} \lambda_\nu(\tau, w).$$

It follows that

$$\lambda_{\nu_0}(\gamma\tau, w) = \chi''(d) j_{\kappa+\nu_0}(\gamma, \tau) \lambda_{\nu_0}(\tau, w),$$

where

$$\chi'' = \chi', \qquad \text{(I–IV)}$$
$$\chi'' = \chi\chi_{(-1)^{\nu_0}} \qquad \text{(I}'\text{)}$$

[see (2.1)]. Note that χ'' must be even in Case I$'$ by Remark 3 (2).

Thus the leading part λ_{ν_0} is a polynomial of w, whose coefficients are elements of $M_{\kappa+\nu_0}(\Gamma_0(N), \chi'')$. By assumption, we have $A(S, r, t) = 0$ if N_χ is divisible by $(\epsilon(S), t)$. It follows from Lemma 3 that $\lambda_{\nu_0} = 0$. This is a contradiction. □

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