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Hyperfunctions and (analytic) hypoellipticity

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Abstract In this work we discuss the problem of smooth and analytic regularity for hyperfunction solutions to linear partial differential equations with analytic coefficients. In particular we show that some well known "sum of squares" operators, which satisfy Hörmander's condition and consequently are hypoelliptic, admit hyperfunction solutions that are not smooth (in particular they are not distributions).

Mathematics Subject Classification (2000) Primary 35B60; Secondary 35A20

1 Introduction

At a conference in Pienza, Italy (November 2005), J. M. Bony suggested that when a "sum of squares operator" with analytic coefficients satisfies Hörmander's bracket condition, and is not analytic–hypoelliptic (for distribution solutions), then it should possess non-smooth hyperfunction solutions. He also suggested that when analytic-hypoellipticity holds, then all hyperfunction solutions should be real-analytic.

These questions motivated us to write this article with the study of a new concept of (analytic) hypoellipticity for general linear partial differential operators with

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real-analytic coefficients, where we now test smoothness (real-analyticity) for hyperfunction solutions. We shall call these properties $\mathfrak{h}$-(*analytic*) *hypoellipticity*.

We study the cases of constant coefficient operators, principal type operators and "sums of squares" operators satisfying Hörmander's condition. For this last class we were able to prove that if *such an operator is analytic–hypoelliptic then its transpose is* $\mathfrak{h}$*-hypoelliptic* (Corollary 2). We also determined a subclass of such operators that are not $\mathfrak{h}$-hypoelliptic (Theorem 4) although, of course, they are hypoelliptic thanks to Hörmander's theorem [7]. This subclass includes the well known examples of non-analytic–hypoelliptic operators introduced by Baouendi and Goulaouic [1] and Oleinik [13].

To conclude, we observe that the result in Corollary 2 provides a new method of proving non-analytic-hypoellipticity for "sum of squares" operators: it suffices to construct singular hyperfunction solutions for the transposed equation. We hope, in a future publication, to return to this question.

2 Definitions and general remarks

Let Ω be an open subset of $\mathbb{R}^N$ and let $P = P(x, D)$ be a linear partial differential operator with real-analytic coefficients in Ω. We shall say that $P(x, D)$ is $\mathfrak{h}$*-hypoelliptic* (resp. $\mathfrak{h}$*-analytic–hypoelliptic) in* Ω if for every open subset U of Ω and if for every hyperfunction u in U the following is true: if Pu is smooth (resp. real-analytic) in U then the same is true for u.

Denote by $\mathcal{B}(U)$ the space of hyperfunctions on U. Since $\mathcal{D}'(U) \subset \mathcal{B}(U)$ it follows that $\mathfrak{h}$-hypoellipticity (resp. $\mathfrak{h}$-analytic-hypoelliptiticy) implies hypoellipticity (resp. analytic-hypoelliptiticy). The converse is in general not true and the study of this converse will be the main subject of this work.

Notice that if $P(x, D)$ is locally smoothly solvable at every point of Ω then $\mathfrak{h}$-analytic-hypoellipticity implies $\mathfrak{h}$-hypoellipticity. In particular it follows that, thanks to a celebrated result due to Sato ([8], Theorem 9.5.1), if $P(x, D)$ is *elliptic* in Ω then $P(x, D)$ is $\mathfrak{h}$-hypoelliptic and $\mathfrak{h}$-analytic–hypoelliptic.

3 Infinite order operators

In order to embark in the analysis of these properties we first pause to introduce some necessary tools.

We recall that a *local operator in the sense of Sato* is an operator of the form

$$J(D) = \sum_{\alpha \in \mathbb{Z}_+^N} a_\alpha D^\alpha, \quad D = \left(\frac{1}{i} \frac{\partial}{\partial x_1}, \ldots, \frac{1}{i} \frac{\partial}{\partial x_N} \right)$$

where $\{a_\alpha\} \subset \mathbb{C}$ satisfies the following property: for every $\epsilon > 0$ there is $C_\epsilon > 0$ such that

$$|a_\alpha| \leq C_\epsilon \epsilon^{|\alpha|} / \alpha!, \quad \alpha \in \mathbb{Z}_+^{\mathbb{N}}. \tag{1}$$

If Ω is an open subset of $\mathbb{R}^N$ then $J(D)$ define endomorphisms

$$J(D) : \mathcal{C}^\omega(\Omega) \longrightarrow \mathcal{C}^\omega(\Omega) \tag{2}$$

and

$$J(D) : \mathcal{B}(\Omega) \longrightarrow \mathcal{B}(\Omega). \tag{3}$$

Indeed (2) follows directly from the definition, whereas (3) follows from duality when Ω is bounded and from a standard extension argument in general. In particular it follows that $J(D)u$ makes sense for $u \in \mathcal{D}'(\Omega)$ although, in general, $J(D)u$ is not a distribution.

In fact, $J(D)$ defines a sheaf homomorphism $J(D) : \mathcal{B} \to \mathcal{B}$, that is, if $U \subset \Omega$ is open and if $u \in \mathcal{B}(\Omega)$ then

$$[J(D)u]|_U = J(D)[u|_U].$$

This justifies the terminology *local* employed by Sato.

We shall make use of a particular class of such operators. If $s > 1$ we set

$$Q_s(D) = \sum_{\alpha \in \mathbb{Z}_+^N} b_\alpha D^\alpha \doteq \prod_{q=1}^{\infty} \left(1 - \frac{1}{q^{2s}} \Delta\right).$$

Here Δ is the usual Laplace operator in $\mathbb{R}^N$.

The next result lists the key properties of the operators Q_s that will be important for us. As usual, $G^s(\Omega)$ will denote the class of Gevrey functions of order s defined on the open subset Ω of $\mathbb{R}^N$. We shall also set $G^s_c(\Omega) \doteq G^s(\Omega) \cap \mathcal{C}^\infty_c(\Omega)$.

Proposition 1 *There are constants $A > 0$, $L > 0$ such that*

$$|b_\alpha| \le A L^{|\alpha|} / \alpha!^s. \tag{4}$$

In particular $Q_s(D)$ are local operators in the sense of Sato. Moreover, for every open subset Ω of $\mathbb{R}^N$, the following properties also hold:

(a) *The operator Q_s define continuous endomorphisms of $G^{s'}(\Omega)$ and of $G^{s'}_c(\Omega)$, for every $1 \le s' < s$.*
(b) *If $u \in \mathcal{B}(\Omega)$ and $Q_s(D)u \in \mathcal{C}^\omega(\Omega)$ then $u \in \mathcal{C}^\omega(\Omega)$.*
(c) *If $f \in \mathcal{C}^\infty(\Omega)$ and if $U \subset\subset \Omega$ is open there is $u \in G^s(U)$ solving $Q_s(D)u = f$ in U.*

Proof We note that (4) is due to Komatsu [9]. This property also implies that if $1 \le s' < s$ and if $\epsilon > 0$ are given then there is $C > 0$ such that

$$|b_\alpha| \le C \epsilon^{|\alpha|} / \alpha!^{s'}.$$

This shows that $Q_s(D)$ are indeed local in the sense of Sato and also that property (a) holds. Property (b) is also due to Sato and very well known. We shall proceed to the proof of property (c).

The symbol of $Q_s(D)$ is the entire function

$$Q_s(\zeta) = \prod_{q=1}^{\infty} \left(1 + \frac{1}{q^{2s}}(\zeta_1^2 + \cdots + \zeta_N^2)\right),$$

which satisfies the following key estimates:

$$|Q_s(\xi)| \geq |\xi|^p p!^{-s}, \quad p \in \mathbb{Z}_+, \quad \xi \in \mathbb{R}^{\mathbb{N}}. \tag{5}$$

Indeed, for all $p \in \mathbb{Z}_+$,

$$|Q_s(\xi)| \geq \prod_{q=1}^{p} \left(1 + \frac{|\xi|^2}{q^{2s}}\right) \geq \prod_{q=1}^{p} \frac{|\xi|}{q^s} = |\xi|^p p!^{-s}.$$

Let now $f \in \mathcal{C}^\infty(\Omega)$. If $\psi \in \mathcal{C}_c^\infty(\Omega)$, $\psi = 1$ in an open neighborhood of $\overline{U}$, and if $\epsilon > 0$ we form

$$u_\epsilon(x) = \frac{1}{(2\pi)^N} \iint \frac{1}{Q_s(\xi)} \mathrm{e}^{i(x-y)\cdot\xi - \epsilon|\xi|^2} \psi(y) f(y) \mathrm{d}y \, \mathrm{d}\xi.$$

It follows from (5) and from the fact that $Q_s(0) = 1$ that u_ϵ are well defined (indeed they extend as entire functions to $\mathbb{C}^N$). Also

$$Q_s(D)u_\epsilon \to \psi f \text{ uniformly in } \mathbb{R}^N \text{ when } \epsilon \to 0^+.$$

On the other hand we can also write, after an integration by parts,

$$u_\epsilon(x) = \frac{1}{(2\pi)^N} \iint \frac{1}{Q_s(\xi)} \mathrm{e}^{i(x-y)\cdot\xi - \epsilon|\xi|^2} \frac{(1-\Delta)^r[\psi(y) f(y)]}{(1+|\xi|^2)^r} \mathrm{d}y \, \mathrm{d}\xi.$$

Here r is any integer $> N/2$. Now, for any $\alpha \in \mathbb{Z}_+^{\mathbb{N}}$ we have

$$D_x^\alpha u_\epsilon(x) = \frac{1}{(2\pi)^N} \iint \frac{\xi^\alpha}{Q_s(\xi)} \mathrm{e}^{i(x-y)\cdot\xi - \epsilon|\xi|^2} \frac{(1-\Delta)^r[\psi(y) f(y)]}{(1+|\xi|^2)^r} \mathrm{d}y \, \mathrm{d}\xi$$

and thus, from (5) we obtain

$$|D_x^\alpha u_\epsilon(x)| \leq \frac{1}{(2\pi)^N} \iint \frac{|\xi|^{|\alpha|}}{|Q_s(\xi)|} \frac{(1-\Delta)^r[\psi(y) f(y)]}{(1+|\xi|^2)^r} \mathrm{d}y \, \mathrm{d}\xi \leq C|\alpha|!^s.$$

Since we can assume, without loss of generality, that U has a regular boundary, we can apply Ascoli's theorem: there is a sequence $\epsilon_j \to 0^+$ such that u_{ϵ_j} converges in $G^s(\overline{U})$ to some $u \in G^s(\overline{U})$. In particular the convergence occurs in $\mathcal{D}'(U)$ and consequently $Q_s(D)u_{\epsilon_j} \to Q_s(D)u$ in $G_c^{s'}(U)^*$, where $1 < s' < s$. Since we also have $Q_s(D)u_{\epsilon_j} \to f$ in $G_c^{s'}(U)^*$, it follows that $Q_s(D)u = f$ in U. □

4 Constant coefficient operators

We now study the constant coefficient case, for which we give a complete answer. The following theorem follows from known results. For us, the most interesting point is the fact that (d) implies (a). This can be found in [17], p. 103, Corollary 2. The proof we give of this fact is new and will be of foremost importance for Sect. 8.[1]

Theorem 1 *Let $P(D)$ be a constant coefficient partial differential operator in $\mathbb{R}^N$. The following properties are equivalent*:

(a) *$P(D)$ is elliptic*;

(b) *$P(D)$ is analytic–hypoelliptic in $\mathbb{R}^N$*;

(c) *$P(D)$ is $\mathfrak{h}$-analytic–hypoelliptic in $\mathbb{R}^N$*;

(d) *$P(D)$ is $\mathfrak{h}$-hypoelliptic in $\mathbb{R}^N$*.

Proof The equivalence between (a) and (b) is a classical result due to Petrowsky [14] and, as we have already pointed out in Sect. 2, (a) implies (c) and (c) implies (d). It then suffices to show that (d) implies (a). For this it suffices to show that if $P(D)$ is hypoelliptic but not elliptic then $P(D)$ cannot be $\mathfrak{h}$-hypoelliptic.

In order to do so we apply standard results on Gevrey hypoellipticity (see for instance [16], pp. 68–69): given such $P(D)$ there is $s > 1$ such that, for every $\Omega \subset \mathbb{R}^N$ open there is $v \in \mathcal{C}^\infty(\Omega) \backslash G^s(\Omega)$ solving $P(D)v = 0$. Since $P(D)$ and $Q_s(D)$ commute we have $P(D)u = 0$ if $u \doteq Q_s(D)v$. If u were a distribution in Ω it would belong to $\mathcal{C}^\infty(\Omega)$. Thus, for $U \subset\subset \Omega$ arbitrary, Proposition 1(c) would imply the existence of $g \in G^s(U)$ solving $Q_s(D)g = u$ in U. In particular $Q_s(D)[v - g] = 0$ and then $v - g \in \mathcal{C}^\omega(U)$ (Proposition 1(b)). Thus $v|_U \in G^s(U)$, and since U is arbitrary we conclude that $v \in G^s(\Omega)$, which is a contradiction. □

5 An abstract result

Let Ω be an open subset of $\mathbb{R}^n$. We shall denote by $\mathfrak{S}(\Omega)$ the class of all linear partial differential operators $P = P(x, D)$, with analytic coefficients in Ω, which satisfy the following property: for every $U \subset\subset \Omega$ open there is a constant $C > 0$ such that

$$\|\phi\|_{L^2} \leq C\| \, {}^t P\phi\|_{L^2}, \quad \phi \in \mathcal{C}_c^\infty(U). \tag{6}$$

[1] In [10] it is proved that the heat operator is not $\mathfrak{h}$-hypoelliptic.

Our main result in this section is the following:

Theorem 2 *Let* $P \in \mathfrak{S}(\Omega)$ *and assume that* P *is also analytic–hypoelliptic in* Ω. *Then the following property holds*:

- $\forall U \subset\subset \Omega$, $\forall u \in \mathcal{B}(U)$, ${}^tPu \in L^2_{\text{loc}}(U)$ *implies* $u \in L^2_{\text{loc}}(U)$.

Proof It is well known that (6) implies that for every $U \subset \Omega$ open there is a bounded linear operator $K_U : L^2(U) \to L^2(U)$ such that

$$PK_U = I \quad \text{in} \quad L^2(U). \tag{7}$$

We now fix $U \subset\subset \Omega$ open and take $V \subset\subset U$ also open. Since P is analytic–hypoelliptic we have a well defined, continuous linear map

$$T_{U,V} : \mathcal{O}(\overline{U}) \longrightarrow \mathcal{O}(\overline{V})$$

defined in the following way: if $h \in \mathcal{O}(\overline{U})$ then $T_{U,V}(h)$ is equal to the class of the extension of $K_U(h|_U)$ to some complex neighborhood of $\overline{V}$. Notice that the continuity of $T_{U,V}$ follows from the closed graph theorem (cf. [5], p. 136). Notice also that (7) implies

$$PT_{U,V}(h) = r_{U,V}(h), \quad h \in \mathcal{O}(\overline{U}), \tag{8}$$

where $r_{U,V} : \mathcal{O}(\overline{U}) \to \mathcal{O}(\overline{V})$ is induced by the natural restriction map. By transposition we obtain a continuous linear map

$${}^tT_{U,V} : \mathcal{O}'(\overline{V}) \longrightarrow \mathcal{O}'(\overline{U})$$

such that

$${}^tT_{U,V}\,{}^tP = \iota_{U,V} \ \text{ on } \mathcal{O}'\ (\overline{U}), \tag{9}$$

where $\iota_{U,V} : \mathcal{O}'(\overline{V}) \to \mathcal{O}'(\overline{U})$ is the natural inclusion map. □

Lemma 1 *Let* $\beta \in \mathcal{O}'(\partial V)$ *and let* $v \in \mathcal{B}(U)$ *be defined by* ${}^tT_{U,V}\beta \in \mathcal{O}'(\overline{U})$. *Then* $v|_V \in L^2_{loc}(V)$.

Let us assume this result for a moment and conclude the proof of the theorem. Let $u \in \mathcal{B}(U)$ be such that $f \doteq {}^tPu \in L^2_{\text{loc}}(U)$ and let $\mu \in \mathcal{O}'(\overline{V})$ represent $u|_V$. We have $\beta \doteq {}^tP\mu - f|_V \in \mathcal{O}'(\partial V)$ and (9) gives

$${}^tT_{U,V}\beta = \mu - {}^tT_{U,V}(f|_V) \tag{10}$$

as elements of $\mathcal{O}'(\overline{U})$. If $v \in \mathcal{B}(U)$ is defined by ${}^tT_{U,V}\beta$ then $u - v$ is defined, modulo an element in $\mathcal{O}'(\overline{U}\setminus V)$, by the analytic functional

$$\mathcal{O}(\mathbb{C}^n) \ni h \mapsto \lambda(h) = \int_V f(x)\, K_U(h|_U)\, \mathrm{d}x.$$

We have $|\lambda(h)| \le C\|h|_U\|_{L^2(U)}$ and thus, by Lemma 1, it follows that $u|_V \in L^2_{\text{loc}}(V)$.

Proof of Lemma 1 Let $W \subset\subset V$ be open and let χ_W denote the characteristic function of W. If $f \in L^2(U)$ then $K_U(\chi_W f)$ extends as an element in $\mathcal{O}(\partial V)$ and the closed graph theorem gives a continuous linear map $L^2(U) \to \mathcal{O}(\partial V)$ which assigns to $f \in Ł^2(U)$ the class of the extension of $K_U(\chi_W f)$ to a complex neighborhood of ∂V. By the same argument we have a continuous linear map $\mathcal{O}(\overline{U} \setminus W) \to \mathcal{O}(\partial V)$ which assigns to h the class of the extension of $K_U((1-\chi_W)h|_U)$ to a complex neighborhood of ∂V. It is easily seen that $h \mapsto \langle \beta, K_U((1-\chi_W)h|_U)\rangle$ defines an element belonging to $\mathcal{O}'(\overline{U}\setminus W)$ and consequently ${}^tT_{U,V}\beta$ and the analytic functional $h \mapsto \langle \beta, K_U(\chi_W h|_U)\rangle$ define the same hyperfunction on W. Since the latter is defined by an element in $L^2(U)$, the result is proved. □

Corollary 1 *Let $P \in \mathfrak{S}(\Omega)$ be analytic–hypoelliptic in Ω. If ${}^tP$ is hypoelliptic (resp. analytic–hypoelliptic) in Ω then ${}^tP$ is $\mathfrak{h}$-hypoelliptic (resp. $\mathfrak{h}$-analytic–hypoelliptic) in Ω.*

6 Okaji's example

This example shows that hypothesis (6) is crucial for the conclusion of Corollary 1. According to a result due to Okaji [12], given any positive even integer k and any non-zero complex number c the operator in $\mathbb{R}^2$

$$P = \left(\frac{\partial}{\partial t} + it^k \frac{\partial}{\partial x}\right)^2 + c\,\frac{\partial}{\partial x} \tag{11}$$

is analytic–hypoelliptic in an open neighborhood Ω of the origin but its transpose ${}^tP$ is not solvable at the origin. On the other hand ${}^tP : \mathcal{B}(\Omega) \to \mathcal{B}(\Omega)$ is surjective [4] and consequently there is $u \in \mathcal{B}(\Omega)$ such that ${}^tPu \doteq f \in \mathcal{C}^\infty(\Omega)$, but such that u is not a distribution in any neighborhood of the origin.

7 Principal type operators

For principal type operators we can give a complete answer to the question we are studying. We recall that $P(x, D)$ (of order m) is said to be *subelliptic in* Ω if for every $U \subset\subset \Omega$ open there are constants $C > 0$, $0 < \delta \le 1$ such that

$$\|\phi\|_{H^{m-1+\delta}} \le C\|P\phi\|_{L^2}, \quad \phi \in \mathcal{C}_c^\infty(U). \tag{12}$$

We have

Theorem 3 *Let $P(x, D)$ be a principal type linear partial differential operator with analytic coefficients in an open subset Ω of $\mathbb{R}^N$. The following properties are equivalent:*

(a) *$P(x, D)$ is subelliptic in Ω;*

(b) *$P(x, D)$ is hypoelliptic in* Ω;
(c) *$P(x, D)$ is analytic–hypoelliptic in* Ω;
(d) *$P(x, D)$ is* $\mathfrak{h}$*-hypoelliptic in* Ω;
(e) *$P(x, D)$ is* $\mathfrak{h}$*-analytic–hypoelliptic in* Ω.

Proof We refer to [18], where it is proved the equivalence of (a)–(c). For the other implications we firstly notice that the characterization of subellipticity for principal type operators is invariant under transposition and, secondly, that (a) implies that $P(x, D)$ belongs to $\mathfrak{S}(\Omega)$. Consequently, by Corollary 1, we conclude that (c) implies (d) and (e). □

8 "Sum of squares" operators

If Ω is an open subset $\mathbb{R}^N$, we shall denote by $\mathfrak{X}(\Omega)$ the set of all partial differential operators of the form

$$Q = \sum_{j=1}^{\nu} X_j^2 + X_0 + f, \tag{13}$$

where $X_0, X_1, \ldots, X_\nu$ are real-analytic, real vector fields on Ω satisfying the Hörmander condition:

$$\mathcal{L}_x(X_0, X_1, \ldots, X_\nu) = T_x\Omega, \quad \forall x \in \Omega,$$

and f is a real-analytic, complex-valued function on Ω. We recall that $\mathcal{L}_x(X_0, X_1, \ldots, X_\nu)$ is the Lie algebra generated by the vector fields $X_0, X_1, \ldots, X_\nu$ at the point x.

It is a result due to Hörmander [7] that every $Q \in \mathfrak{X}(\Omega)$ is $\mathcal{C}^\infty$-hypoelliptic in Ω and also that $\mathfrak{X}(\Omega) \subset \mathfrak{S}(\Omega)$. Moreover, it is easy to see that $\mathfrak{X}(\Omega)$ is closed by transposition. From Corollary 1 we obtain:

Corollary 2 *Let $Q \in \mathfrak{X}(\Omega)$. If Q is analytic–hypoelliptic in Ω then ${}^tQ$ is* $\mathfrak{h}$*-hypoelliptic in* Ω.

Example For any $p \in \mathbb{Z}_+$ the Grushin operator

$$Q = \frac{\partial^2}{\partial t^2} + t^{2p}\frac{\partial^2}{\partial x^2},$$

is $\mathfrak{h}$-hypoelliptic in $\mathbb{R}^2$. This result was already obtained by Matsuzawa [11] for a more general class of Grushin operators.

Using a similar argument as in the proof of Theorem 1 we can find a class of operators in the form of "sum of squares" which are not $\mathfrak{h}$-hypoelliptic.

Theorem 4 *Write the coordinates in* $\mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^n$ *as* $x_1, \ldots, x_m, t_1, \ldots, t_n$ *and consider the real-analytic vector fields*

$$Y_j = \sum_{\ell=1}^{n} a_{j\ell}(t)\frac{\partial}{\partial t_\ell} + \sum_{k=1}^{m} b_{jk}(t)\frac{\partial}{\partial x_k}, \quad j = 0, \ldots, \nu,$$

where $a_{j\ell}$, b_{jk} *are real-analytic in an open neighborhood* V *of the origin in* $\mathbb{R}^n$. *Let also*

$$\mathcal{P} = \sum_{j=1}^{\nu} Y_j^2 + Y_0 + f(t),$$

where f *is also real-analytic on* V. *Assume that*

(a) $\mathcal{P} \in \mathfrak{X}(\mathbb{R}^m \times V)$;
(b) *The vector fields* $\sum_{\ell=1}^{n} a_{j\ell}(t)\partial/\partial t_\ell$, $j = 1, \ldots, \nu$ *span* TV *at every point.*
(c) *There is* $u \in \mathcal{C}^\infty(\Omega)$ *satisfying* $\mathcal{P}u = 0$, $\Omega \subset \mathbb{R}^m \times V$ *being an open neighborhood of the origin, but such that, for some* $s > 1$, u *is not of Gevrey class* s *in any neighborhood of the origin.*

Then there is $f \in \mathcal{B}(\Omega)$ *satisfying* $\mathcal{P}f = 0$ *which is not a distribution on any neighborhood of the origin. In particular* $\mathcal{P}$ *is not* $\mathfrak{h}$*-hypoelliptic in* $\mathbb{R}^m \times V$.

Proof We start by taking an open neighborhood of the origin of the form $U \times \Theta \subset\subset \Omega$, where U (resp. Θ) is an open ball centered at the origin in $\mathbb{R}^m$ (resp. $\mathbb{R}^n$).

Define $f = Q_s(D_x)u$ (here the action is only in the variables x_j). Since $\mathcal{P}$ and $Q_s(D_x)$ commute we have $\mathcal{P}f = 0$. If f were a distribution then it would belong to $\mathcal{C}^\infty(\Omega)$. By an elementary extension of Proposition 1(c), where we allow f depending on parameters there is $v \in \mathcal{C}^\infty(\Theta, G^s(U))$ solving $Q_s(D_x)v = f$ in $U \times V$.

Since $Q_s(D_x)[v - u] = 0$ we have $v - u \in C^\infty(\Theta, G^1(U))$ (this also follows from the arguments in the proof of Proposition 3.2 in [2]) and consequently $u \in \mathcal{C}^\infty(\Theta, G^s(U))$. Thanks to property (b) we conclude that $u|_{U\times\Theta} \in G^s(U \times \Theta)$, which is a contradiction. □

Example The Baouendi–Goulaouic operator in $\mathbb{R}^3$

$$B = \frac{\partial^2}{\partial t^2} + t^2\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \tag{14}$$

satisfies the hypotheses of Theorem 4, with $1 < s < 2$ (cf. [1], and [8], p. 310). In this case we can present explicitly a hyperfuntion singular solution to the equation $Bu = 0$.

Indeed let $(w, z_1, z_2) \in \mathbb{C}^3$ be the complexification of (t, x_1, x_2). We define an open cone $V \subset \mathbb{R}^3$ as follows:

$$V = \{(\Im w, \Im z_1, \Im z_2) \in \mathbb{R}^3 : \Im z_1 > |\Im w| \text{ and } |\Im w| < 1\}.$$

Note that in V we have

$$\frac{1}{2}(\Im w)^2 - \Im z_1 < 0. \tag{15}$$

For $(w, z_1, z_2) \in \mathbb{R}^3 + iV$, we define u as follows:

$$H(w, z_1, z_2) = \int_0^\infty e^{iz_1\xi + z_2\sqrt{\xi} - \frac{1}{2}w^2\xi} d\xi.$$

A similar formula appears in [8].

Note that H is holomorphic on $\mathbb{R}^3 + iV$. Indeed, if $K \subset \mathbb{R}^3 + iV$ is compact then there are $\epsilon > 0$ and $\rho > 0$ such that if $(w, z_1, z_2) \in K$ and $\xi \geq \rho$ then

$$\begin{aligned}\Re\left(iz_1\xi + z_2\sqrt{\xi} - \frac{1}{2}w^2\xi\right) &= -(\Im z_1)\,\xi + x_2\sqrt{\xi} - \frac{1}{2}\left(t^2 - (\Im w)^2\right)\xi \\ &\leq \left(\frac{1}{2}(\Im w)^2 - \Im z_1\right)\xi + x_2\sqrt{\xi} \\ &\leq -\epsilon\,\xi.\end{aligned}$$

Such an estimate allows us to apply the Cauchy–Riemann operator under the integral sign. We conclude that H is holomorphic on $\mathbb{R}^3 + iV$ and hence it determines a hyperfunction $u = b(H) \in \mathcal{B}(\mathbb{R}^3)$.

By direct computation we have $Bu = 0$ on $\mathbb{R}^3$.

Now we claim that u is not a distribution. If it were it would be a smooth function, which means that H would be a smooth function up to the edge $\mathbb{R}^3 \times \{0\}$. But

$$H(0, i\Im z_1, 0) = \int_0^\infty e^{-\Im z_1 \xi} d\xi = \frac{1}{\Im z_1},$$

which proves our claim. □

Similar analysis can be made for the Oleinik operators [13]

$$P = \frac{\partial^2}{\partial t^2} + t^{2p}\frac{\partial^2}{\partial x_1^2} + t^{2q}\frac{\partial^2}{\partial x_2^2},$$

where $1 \leq p < q$, as well for the operators

$$P = \frac{\partial^2}{\partial t^2} + \left(\frac{\partial}{\partial x_1} + t^k\frac{\partial}{\partial x_2}\right)^2$$

for $k \geq 2$, which where studied in [3,6,15].

All these operators satisfy the hypotheses of Theorem 4 and consequently are not $\mathfrak{h}$-hypoelliptic. Since, moreover, the proofs of non-analytic-hypoellipticity presented in these works provide the existence of explicit non-analytic solutions of $Pu = 0$, a similar analysis as above produce also explicit hyperfunction solutions which are not distributions.

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