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Strichartz estimates via the Schrödinger maximal operator

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Abstract We consider the Schrödinger operator $e^{it\Delta}$ acting on initial data f in $\dot{H}^s$. We show that an affirmative answer to a question of Carleson, concerning the sharp range of s for which $\lim_{t\to 0} e^{it\Delta} f(x) = f(x)$ a.e. $x \in \mathbb{R}^n$, would imply an affirmative answer to a question of Planchon, concerning the sharp range of q and r for which $e^{it\Delta}$ is bounded in $L_x^q(\mathbb{R}^n, L_t^r(\mathbb{R}))$. When $n = 2$, we unconditionally improve the range for which the mixed norm estimates hold.

1 Introduction

The Schrödinger equation, $i\partial_t u + \Delta u = 0$, in $\mathbb{R}^{n+1}$, with initial datum f in the Sobolev space $\dot{H}^s(\mathbb{R}^n)$, has solution $e^{it\Delta} f$ which can be formally written as

$$e^{it\Delta} f(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{2\pi i (x\cdot\xi - 2\pi t|\xi|^2)} d\xi. \tag{1}$$

We define the dimensional or scaling relation $s(q, r)$ by

$$s(q, r) = n\left(\frac{1}{2} - \frac{1}{q}\right) - \frac{2}{r}.$$

Stein [26], Tomas [31], Strichartz [27], Ginibre and Velo [9], and Keel and Tao [11] have all played a role in proving the following theorem.

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Theorem 1 [11] *Let* $q \in [2,\infty)$, $r \in [2,\infty]$ *and* $\frac{n}{q}+\frac{2}{r} \le \frac{n}{2}$. *Then*

$$\|e^{it\Delta}f\|_{L^r_t(\mathbb{R},L^q_x(\mathbb{R}^n))} \le C\|f\|_{\dot{H}^{s(q,r)}(\mathbb{R}^n)}.$$

The theorem is sharp in the sense that it is not true when $q<2$, $r<2$, or $\frac{n}{q}+\frac{2}{r}>\frac{n}{2}$. When $q=\infty$, the estimate holds only occasionally (see [17,23]).

Changing the order of the integrals, the problem is more difficult. We will ignore the subtle endpoint questions in this article. In connection with his work on the cubic semilinear Schrödinger equation, Planchon [20] asked whether the following is true:

Conjecture 1 Let $q \in \left(\frac{2(n+1)}{n},\infty\right]$, $r \in [2,\infty)$ and $\frac{n+1}{q}+\frac{1}{r}<\frac{n}{2}$. Then

$$\|e^{it\Delta}f\|_{L^q_x(\mathbb{R}^n,\,L^r_t(\mathbb{R}))} \le C\|f\|_{\dot{H}^{s(q,r)}(\mathbb{R}^n)}.$$

In one spatial dimension, this had already been proven in the affirmative, including the endpoints, by Kenig, Ponce and Vega [12,32].

In higher dimensions, arguments originally due to Tao and Vargas [30] which were then refined by Planchon [20] (see also [21]), can be combined with Tao's bilinear restriction estimate [28] to yield the conjecture in the range $q>\frac{2(n+3)}{n+1}$. When $q>r$, the endpoints can be included, and the key bound follows from the original Stein–Tomas theorem (see [10,20,32]). Note that $s(q,r)$ can be negative in this range.

We will prove that the conjecture would follow from a positive resolution of a question of Carleson concerning the sharp range of s for which

$$\lim_{t\to 0} e^{it\Delta}f(x)=f(x), \quad \text{a.e. } x\in\mathbb{R}^n, \qquad f\in H^s(\mathbb{R}^n).$$

By standard arguments, the convergence follows from the estimate

$$\|\sup_{0<t<1}|e^{it\Delta}f|\,\|_{L^2_x(\mathbb{B}^n)} \le C_s\|f\|_{H^s(\mathbb{R}^n)}, \tag{A}$$

where $\mathbb{B}^n$ is the unit ball in $\mathbb{R}^n$. If we restrict time to a sequence, then the convergence and a nonendpoint version of the maximal estimate are equivalent (see [25]).

Conjecture 2 (A) holds for all $s>1/4$.

In one spatial dimension, the convergence was originally proven by Carleson [5] via an L^1-estimate, and Kenig and Ruiz [13] showed that (A) holds for all $s\ge 1/4$. Dahlberg and Kenig [7] showed that this is sharp in the sense that (A) cannot hold when $s<1/4$.

In two spatial dimensions, significant contributions were made by Bourgain [1,2], Moyua et al. [18,19], and Tao and Vargas [28–30]. The best known result is due to Lee [15] who showed that (A) holds when $s>3/8$.

In higher dimensions, significant contributions were made by Carbery [4] and Cowling [6]. The best known result is independently due to Sjölin [22] and Vega [33] who showed that (A) holds when $s>1/2$.

For notational convenience, we rewrite estimate (A) as

$$\|\sup_{0<t<1} |e^{it\Delta} f|\, \|_{L^2_x(\mathbb{B}^n)} \leq C\|f\|_{H^{1/4+\kappa}(\mathbb{R}^n)}, \tag{A_κ}$$

where $\kappa \geq 0$, and define the dual exponents q_κ and q'_κ by

$$q_\kappa = \frac{n+1+8\kappa}{n+4\kappa} \quad \text{and} \quad q'_\kappa = \frac{n+1+8\kappa}{1+4\kappa}.$$

Theorem 2 *Let $q \in (2q_\kappa, \infty]$, $r \in (2q'_\kappa, \infty)$ and $\frac{n}{2q'_\kappa} + \frac{n}{q} + \frac{1}{r} < \frac{n}{2}$. If ($A_\kappa$) holds, then*

$$\|e^{it\Delta} f\|_{L^q_x(\mathbb{R}^n,\, L^r_t(\mathbb{R}))} \leq C\|f\|_{\dot{H}^{s(q,r)}(\mathbb{R}^n)}.$$

Note that $2q_\kappa$ and $\frac{2q'_\kappa}{n}$ both tend to $\frac{2(n+1)}{n}$ as κ tends to zero. Comparing with Conjecture 1, we see that (q, r) can approach the endpoint $(\frac{2(n+1)}{n}, \infty)$;

Corollary 1 *Conjecture 2 $\Rightarrow$ Conjecture 1.*

Combining the identity $D^s_t e^{it\Delta} f = e^{it\Delta} D^{2s}_x f$ with Sobolev embedding, Theorem 2 also yields estimates for the maximal operator. Indeed, applying Hölder to obtain local L^2-bounds, we see that

$$(A_\kappa) \;\Rightarrow\; (A_{\kappa'}), \qquad \kappa' > n\left(\frac{1}{2} - \frac{1}{2q_\kappa}\right) - \frac{1}{4}.$$

There is an improvement in regularity when $\kappa > (n-1)/8$. Taking $n = 2$ and iterating, we can suppress κ to be arbitrarily close to $1/8$, which recovers Lee's result.

Moreover, we see that a global version holds;

Corollary 2 *Let $q > 16/5$. Then for all $s > 1 - 2/q$,*

$$\|\sup_{t\in\mathbb{R}} |e^{it\Delta} f|\, \|_{L^q(\mathbb{R}^2)} \leq C_s\|f\|_{H^s(\mathbb{R}^2)}.$$

Taking more care with the range of r, we will also improve Planchon's estimate.

Theorem 3 *Let $n = 2$. Then Conjecture 1 is true for $q > 16/5$.*

To illustrate, this is a nonendpoint version of

$$\|e^{it\Delta} f\|_{L^{16/5}_x(\mathbb{R}^2, L^{16}_t(\mathbb{R}))} \leq C\|f\|_{\dot{H}^{1/4}(\mathbb{R}^2)}.$$

We follow the approach of Lee in that we adapt the proof of Tao's bilinear theorem [28], rather than applying the estimate directly.

Throughout, c and C will denote positive constants that may depend on the dimensions and exponents of the Lebesgue spaces. The constants C will sometimes depend on the small parameters ε, δ and β, but never on the functions f or g, and never on the large parameters R or N. It will occasionally be made explicit when they depend on other factors like the Sobolev index. Their values may change from line to line. The following are notations that will be used frequently:

$L^q_x(\mathbb{R}^n, L^r_t(I))$: the Lebesgue space with norms $\left(\int_{\mathbb{R}^n} \left|\int_I |f(x,t)|^r dt\right|^{q/r} dx\right)^{1/q}$
D^s: the derivative defined by $\widehat{D^s g}(\xi) = (2\pi|\xi|)^s \widehat{g}(\xi)$
$\dot{H}^s(\mathbb{R}^n)$: the homogeneous Sobolev space with s derivatives in $L^2(\mathbb{R}^n)$
$H^s(\mathbb{R}^n)$: the inhomogeneous Sobolev space with s derivatives in $L^2(\mathbb{R}^n)$
$\mathbb{B}^n := \{ x \in \mathbb{R}^n : |x| \le 1 \}$
$B_1(Ne_1) := \{ \xi \in \mathbb{R}^n : |\xi - Ne_1| \le 1 \}$
ξ_j: a member of the lattice $R^{-1/2}\mathbb{Z}^n$
x_k: a member of the lattice $R^{1/2}\mathbb{Z}^n$
$T_{jk} := \{(x,t) \in \mathbb{R}^n \times [0,R] : |x - (x_k + 4\pi t\xi_j)| \le R^{1/2}\}$.
$Q_R := [-R/4, R/4] \times \ldots \times [-R/4, R/4]$
$P_R(l) := \{(x,t) \in \mathbb{R}^n \times [R/2, R] : x - (lR/2 + 4\pi tN)e_1 \in Q_R\}$
$s(q,r) := n(1/2 - 1/q) - 2/r$
$q_\kappa := \frac{n+1+8\kappa}{n+4\kappa}$
$\widehat{\psi}$: a positive and smooth function, supported in $B_{\sqrt{n}}$.
$\widehat{\eta}$: a positive and smooth function, supported in $\mathbb{B}^n$, and equal to 1 at the origin.

2 Globalization lemmas

The following lemma provides convenient estimates with which we will interpolate.

Lemma 1 *For all $N \gg 1$, $r \ge 2$, and f frequency supported in $B_1(Ne_1)$,*

$$\|e^{it\Delta} f\|_{L^\infty_x(\mathbb{R}^n, L^r_t(\mathbb{R}))} \le CN^{-1/r} \|f\|_{L^2(\mathbb{R}^n)}.$$

Proof We suppose that $n \ge 2$; the 1-dimensional case was proven in [12]. By interpolation with the trivial L^∞-estimate, we may also take $r = 2$. By writing the square as a double integral,

$$\|e^{it\Delta} f(x)\|^2_{L^2_t(\mathbb{R})} = \int_{\mathbb{R}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \widehat{f}(\xi)\widehat{f}(y) e^{2\pi i (x\cdot(\xi - y) - 4\pi t(|\xi|^2 - |y|^2))} d\xi dy dt,$$

so that, by an application of Fubini, and integrating in t,

$$\|e^{it\Delta} f(x)\|^2_{L^2_t(\mathbb{R})} \le \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\widehat{f}(\xi)\widehat{f}(y)|}{||\xi|^2 - |y|^2|} d\xi dy.$$

Writing $|\xi|^2 - |y|^2 = (\xi + y) \cdot (\xi - y)$, and recalling that $y, \xi \in B_1(Ne_1)$, we see that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\widehat{f}(\xi)\widehat{f}(y)|}{|\,|\xi|^2 - |y|^2|} d\xi dy \leq \frac{C}{N} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\widehat{f}(\xi)\widehat{f}(y)|}{|\xi - y|} d\xi dy.$$

Thus, by the Hardy–Littlewood–Sobolev inequality,

$$\|e^{it\Delta} f(x)\|^2_{L^2_t(\mathbb{R})} \leq CN^{-1} \|\widehat{f}\|^2_{L^{\frac{2n}{2n-1}}(\mathbb{R}^n)},$$

and, as $\operatorname{supp} \widehat{f} \subset B_1(Ne_1)$, by Hölder and Plancherel we complete the proof.

As in the arguments of Fefferman [8], Bourgain [3], Wolff [34], Tao [28], and Lee [15], we decompose into wave-packets at scale $R \gg 1$.

Fix a positive and smooth function $\widehat{\psi}$, supported in $B_{\sqrt{n}}$, such that

$$\sum_j \widehat{\psi}(\xi - R^{1/2}\xi_j) = 1,$$

where $\xi_j \in R^{-1/2}\mathbb{Z}^n$. We also fix a positive and smooth function $\widehat{\eta}$, supported in $\mathbb{B}^n$ and equal to one at the origin, so that by the Poisson summation formula,

$$\sum_k \eta(x - \frac{x_k}{R^{1/2}}) = 1,$$

where $x_k \in R^{1/2}\mathbb{Z}^n$. Now, for any Schwartz function f we have the decompositions

$$\widehat{f}(\xi) = \sum_j \widehat{f}_j(\xi) = \sum_j \widehat{\psi}(R^{1/2}(\xi - \xi_j))\, \widehat{f}(\xi), \tag{2}$$

$$f(x) = \sum_{j,k} f_{jk}(x) = \sum_{j,k} \eta\left(\frac{x - x_k}{R^{1/2}}\right) f_j(x). \tag{3}$$

Note that $\widehat{f}_{jk}$ is supported in the ball of radius $(\sqrt{n} + 1)R^{-1/2}$ with centre ξ_j.

We recall the Hardy–Littlewood maximal operator $M : L^1_{loc}(\mathbb{R}^n) \to L^1_{loc}(\mathbb{R}^n)$ defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B_r|} \int_{B_r} |f(y - x)|\, dy.$$

For a proof of the following lemma see [28] or [14].

Lemma 2 *Let $t \in [-R, R]$. Then for all $K \in \mathbb{N}$ there exist constants C_K, such that*

$$|e^{it\Delta} f_{jk}(x)| \leq C_K Mf_j(x_k) \left(1 + \frac{|x - (x_k + 4\pi t\xi_j|}{R^{1/2}}\right)^{-K}.$$

We note that when $t \in [0, R]$, the wave-packets $e^{it\Delta} f_{jk}$ are essentially supported in the tubes T_{jk} with direction $(4\pi\xi_j, 1)$ defined by

$$T_{jk} = \{(x, t) \in \mathbb{R}^n \times [0, R] : |x - (x_k + 4\pi t\xi_j)| \le R^{1/2}\}.$$

We see that a translation of the frequency support of the data corresponds to an affine translation of the essential supports of the wave-packets.

Similarly, for $l \in \mathbb{Z}$, we define parallelepipeds $P_R(l)$ by

$$P_R(l) = \{(x, t) \in \mathbb{R}^n \times [R/2, R] : x - (lR/2 + 4\pi t N)e_1 \in Q_R\},$$

where Q_R is the n-dimensional cube of side $R/2$, centred at the origin. Note that when $\xi_j \in B_1(Ne_1)$, the tubes and parallelepipeds point approximately in the same direction.

Definition 1 We say that E_1 and E_2 are *1-separated* if they are measurable sets that satisfy

$$\inf\{ |\xi_1 - \xi_2| : \xi_1 \in E_1, \quad \xi_2 \in E_2\} \ge 1/2.$$

The following lemma is a key ingredient. It allows us to deduce estimates on balls from estimates restricted to parallelepipeds. We will see later that parallelepipeds are the natural domain on which to attack the problem.

Lemma 3 *Let $r \ge q$ and $\alpha \ge \frac{1}{q} - \frac{1}{r}$. Suppose that*

$$\|e^{it\Delta} f\, e^{it\Delta} g\|_{L^q_x L^r_t(P_R(0))} \le C R^{\varepsilon} N^{\alpha} \|f\|_2 \|g\|_2$$

whenever $R, N \gg 1$, and $\widehat{f}, \widehat{g}$ are supported on 1-separated subsets of $B_1(Ne_1)$. Then

$$\|e^{it\Delta} f\, e^{it\Delta} g\|_{L^q_x(Q_R, L^r_t[R/2,R])} \le C R^{\varepsilon} N^{\alpha} \|f\|_2 \|g\|_2.$$

Proof We decompose the solution into wave-packets at scale R,

$$e^{it\Delta} f = \sum_{j,k} e^{it\Delta} f_{jk}.$$

Letting P_l denote the short, fat tubes defined by

$$P_l = \{(x, t) \in \mathbb{R}^n \times [R/2, R] : |x - (lR/2 + 4\pi t N)e_1| \le 50R\},$$

where $l \in \mathbb{Z}$, we write

$$f_l = \sum_{j,k\,:T_{jk} \cap P_l \neq \emptyset} f_{jk},$$

so that $e^{it\Delta} f_l$ consists of the wave-packets that pass near to $P_R(l)$. As the tubes and the parallelepipeds point in essentially the same direction, a tube T_{jk} can intersect P_l for at most a constant number of l, so we note for later that

$$\begin{aligned}\sum_l \|f_l\|^2_{L^2(\mathbb{R}^n)} &\leq C\sum_l \sum_{j,k\,:\,T_{jk}\cap P_l\neq\emptyset} \|f_{jk}\|^2_{L^2(\mathbb{R}^n)}\\ &\leq C\sum_{j,k}\|f_{jk}\|^2_{L^2(\mathbb{R}^n)}\\ &\leq C\|f\|^2_{L^2(\mathbb{R}^n)},\end{aligned}$$

and we will refer to this as almost orthogonality.

We consider the pointwise bound

$$|e^{it\Delta}f| \leq |e^{it\Delta}f_l| + \left|\sum_{j,k\,:T_{jk}\cap P_l=\emptyset} e^{it\Delta}f_{jk}\right|, \tag{4}$$

and use the rapid decay to show that the last term is of negligible size on $P_R(l)$.

Writing $\overline{x} = x - 4\pi t N e_1$, we have $|x-(x_k+4\pi t\xi_j)| \approx |\overline{x}-x_k|$ whenever $(x,t)\in P_R(l)$ and $T_{jk}\cap P_l=\emptyset$, so by Lemma 2,

$$\left|\sum_{j,k\,:T_{jk}\cap P_l=\emptyset} e^{it\Delta}f_{jk}(x)\right| \leq C_{K'}R^{K'/2}\sum_{j=1}^{cR^{n/2}}\sum_{k\,:\,|\overline{x}-x_k|\geq R}\frac{Mf_j(x_k)}{|\overline{x}-x_k|^{K'}}$$

for all $K'\in\mathbb{N}$. Choosing K' sufficiently large, we see that for all $K\in\mathbb{N}$,

$$\left|\sum_{j,k\,:T_{jk}\cap P_l=\emptyset} e^{it\Delta}f_{jk}(x)\right| \leq C_K R^{-K}\sum_{j=1}^{cR^{n/2}}\sum_{k\,:\,|\overline{x}-x_k|\geq R}\frac{Mf_j(x_k)}{|\overline{x}-x_k|^{2n}}. \tag{5}$$

Writing $\psi_R = R^{-n/2}\psi(R^{-1/2}\,\cdot)$, by (2) we have

$$|f_j| = |\psi_R * f|, \tag{6}$$

so that $Mf_j(x')\approx Mf_j(x_k)$ whenever $|x'-x_k|\leq\sqrt{n}R^{1/2}$. Now observe that

$$\begin{aligned}\sum_{k\,:\,|\overline{x}-x_k|\geq R}\frac{Mf_j(x_k)}{|\overline{x}-x_k|^{2n}} &\leq CR^{-n/2}\left(1+\frac{|\cdot|}{R^{1/2}}\right)^{-2n} *\ Mf_j(\overline{x})\\ &\leq CMMf_j(\overline{x}),\end{aligned} \tag{7}$$

so the error term is not only going to be small, but also square integrable. Substituting (6) and (7) into (5),

$$\left|\sum_{j,k:T_{jk}\cap P_l=\emptyset} e^{it\Delta} f_{jk}(x)\right| \leq C_K R^{-K} MM[\psi_R * f](\overline{x}),$$

and substituting this into (4), we see that for all $K \in \mathbb{N}$ there exist C_K such that

$$|e^{it\Delta} f(x)| \leq |e^{it\Delta} f_l(x)| + C_K R^{-K} MM[\psi_R * f](x - 4\pi t N e_1)$$

whenever $(x, t) \in P_R(l)$.

We use these pointwise bounds on parallelepipeds, to obtain an $L^q(Q_R, L_t^r[R/2, R])$ bound. Fix a large K and define $Lf(x,t) := R^{-K} MM[\psi_R * f](x - 4\pi t N e_1)$. We also write $\overline{P}_R(l) := Q_R \times [R/2, R] \cap P_R(l)$, so that by concavity

$$\begin{aligned}
&\|e^{it\Delta} f\, e^{it\Delta} g\|^q_{L^q(Q_R, L_t^r[R/2,R])} \\
&\quad\leq \sum_l \|e^{it\Delta} f\, e^{it\Delta} g\|^q_{L_x^q L_t^r(\overline{P}_R(l))} \\
&\quad\leq C_K^{2q} \sum_l \|(|e^{it\Delta} f_l| + Lf)(|e^{it\Delta} g_l| + Lg)\|^q_{L_x^q L_t^r(\overline{P}_R(l))} \\
&\quad\leq C_K^{2q} \sum_l \Big(\|e^{it\Delta} f_l\, e^{it\Delta} g_l\|^q_{L_x^q L_t^r(\overline{P}_R(l))} + \|Lf\, e^{it\Delta} g_l\|^q_{L_x^q L_t^r(\overline{P}_R(l))} \\
&\qquad + \|e^{it\Delta} f_l\, Lg\|^q_{L_x^q L_t^r(\overline{P}_R(l))} + \|Lf\, Lg\|^q_{L_x^q L_t^r(\overline{P}_R(l))}\Big).
\end{aligned} \tag{8}$$

Now, by two applications of Hölder,

$$\begin{aligned}
\sum_l \|Lf\|^{2q}_{L_x^{2q} L_t^{2r}(Q_R\times[R/2,R]\cap P_R(l))} &\leq C R^{nq(\frac{1}{q}-\frac{1}{r})} \sum_{l=-N}^{N} \|Lf\|^{2q}_{L_x^{2r} L_t^{2r}(P_R(l))} \\
&\leq C R^{nq(\frac{1}{q}-\frac{1}{r})} N^{q(\frac{1}{q}-\frac{1}{r})} \left(\sum_l \|Lf\|^{2r}_{L_x^{2r} L_t^{2r}(P_R(l))}\right)^{\frac{q}{r}}
\end{aligned}$$

By summing up, applying Fubini and making an affine change of variables,

$$\begin{aligned}
\sum_l \|Lf\|^{2r}_{L_x^{2r} L_t^{2r}(P_R(l))} &\leq C R^{-2rK+1} \|MM[\psi_R * f]\|^{2r}_{L_x^{2r}(\mathbb{R}^n)} \\
&\leq C R^{-2rK+1} \|f\|^{2r}_{L^{2r}(\mathbb{R}^n)},
\end{aligned}$$

where the second inequality is by the Hardy–Littlewood maximal theorem and Young's inequality. As $\widehat{f}$ is supported in $B_1(Ne_1)$, together with Bernstein's inequality, these estimates yield

$$\sum_l \|Lf\|^{2q}_{L^{2q}_x L^{2r}_t(\overline{P}_R(l))} \le CR^{-qK} N^{q(\frac{1}{q}-\frac{1}{r})} \|f\|^{2q}_{L^2(\mathbb{R}^n)}.$$

We have the same inequality for g, so that, by two applications of Cauchy–Schwarz,

$$\sum_l \|Lf\, Lg\|^{q}_{L^{q}_x L^{r}_t(\overline{P}_R(l))} \le CR^{-qK} N^{q(\frac{1}{q}-\frac{1}{r})} \|f\|^{q}_2 \, \|g\|^q_2.$$

On the other hand, by Hölder and Lemma 1,

$$\begin{aligned} \|e^{it\Delta} f_l\|_{L^{2q}_x L^{2r}_t(\overline{P}_R(l))} &\le CR^{\frac{n}{2q}} \|e^{it\Delta} f_l\|_{L^{\infty}_x L^{2r}_t(\mathbb{R}^{n+1})} \\ &\le CR^{\frac{n}{2q}} N^{-\frac{1}{2r}} \|f_l\|_2. \end{aligned}$$

Thus, by two applications of Cauchy–Schwarz,

$$\begin{aligned} \sum_l \|e^{it\Delta} f_l\, Lg\|^{q}_{L^{q}_x L^{r}_t(\overline{P}_R(l))} &\le CR^{-\frac{qK}{2}} N^{\frac{q}{2}(\frac{1}{q}-\frac{1}{r})} \left(\sum_l \|e^{it\Delta} f_l\|^{2q}_{L^{2q}_x L^{2r}_t(\overline{P}_R(l))}\right)^{1/2} \|g\|^q_2 \\ &\le CR^{-\frac{qK}{4}} N^{q(\frac{1}{q}-\frac{1}{r})} \left(\sum_l \|f_l\|^{2q}_2\right)^{1/2} \|g\|^q_2 \\ &\le CR^{-\frac{qK}{4}} N^{q(\frac{1}{q}-\frac{1}{r})} \|f\|^q_2 \, \|g\|^q_2, \end{aligned}$$

where in the third inequality we have used convexity and the almost orthogonality derived earlier. Similarly, we have

$$\sum_l \|Lf\, e^{it\Delta} g_l\|^{q}_{L^{q}_x L^{r}_t(\overline{P}_R(l))} \le CR^{-\frac{qK}{4}} N^{q(\frac{1}{q}-\frac{1}{r})} \|f\|^{q}_2 \, \|g\|^q_2.$$

Finally, by spatial translation invariance and the hypothesis,

$$\|e^{it\Delta} f_l\, e^{it\Delta} g_l\|_{L^{q}_x L^{r}_t(P_R(l))} \le CR^{\varepsilon} N^{\alpha} \|f_l\|_2 \|g_l\|_2,$$

so that, by Cauchy–Schwarz,

$$\begin{aligned} \sum_l \|e^{it\Delta} f_l\, e^{it\Delta} g_l\|^{q}_{L^{q}_x L^{r}_t(P_R(l))} &\le CR^{q\varepsilon} N^{q\alpha} \left(\sum_l \|f_l\|^{2q}_2\right)^{1/2} \left(\sum_l \|g_l\|^{2q}_2\right)^{1/2} \\ &\le CR^{q\varepsilon} N^{q\alpha} \|f\|^q_2 \, \|g\|^q_2, \end{aligned}$$

again using convexity and the almost orthogonality.

Comparing the terms in (8), we see that

$$\|e^{it\Delta}f\,e^{it\Delta}g\|_{L^q(Q_R,\,L^r_t[R/2,R])}\le CR^\varepsilon N^\alpha\|f\|_2\|g\|_2,$$

and we are done.

The following mixed norm 'epsilon removal' lemma is due to Lee and Vargas [16] (see also [2,29]). In their work, the spatial integral is evaluated before the temporal integral and as such the estimates are invariant under translation on the frequency side. A careful reading of the proof reveals that only small changes are required to reverse the order.

Lemma 4 *Suppose that for all $\varepsilon>0$ and $\alpha>\frac{1}{q_0}-\frac{1}{r_0}$,*

$$\|e^{it\Delta}f\,e^{it\Delta}g\|_{L^{q_0}_x(Q_R,L^{r_0}_t[R/2,R])}\le C_{\varepsilon,\alpha}R^\varepsilon N^\alpha\|f\|_2\|g\|_2$$

whenever $R,N\gg 1$, and $\widehat{f},\widehat{g}$ are supported on 1-separated subsets of $B_1(Ne_1)$. Then provided that $\frac{q}{r}>\frac{q_0}{r_0}$, $q(1-\frac{1}{r})>q_0(1-\frac{1}{r_0})$, and $\alpha>\frac{1}{q}-\frac{1}{r}$,

$$\|e^{it\Delta}f\,e^{it\Delta}g\|_{L^q_xL^r_t(\mathbb{R}^{n+1})}\le C_{q,r,\alpha}N^\alpha\|f\|_2\|g\|_2.$$

Proof The proof is the same as that of Lemma 4.4 and Remark 4.5 in [16], with the following changes:

The measures $d\sigma_i$ are replaced by the canonical pull-back measure on

$$\{\,(\xi,-2\pi|\xi|^2)\in\mathbb{R}^{n+1}\ :\ \xi\in B_1(Ne_1)\,\}$$

which we denote by $d\sigma_N$. By a well-known calculation,

$$\begin{aligned}|\widehat{d\sigma_N}(x,t)|=|e^{it\Delta}(\chi_{B_1(Ne_1)})^\vee(x)|&\le C(1+|x-4\pi tNe_1|+|t|)^{-n/2}\\&\le CN^{n/2}(1+|x|+|t|)^{-n/2}.\end{aligned}$$

We replace the estimate

$$\|e^{it\Delta}f\,e^{it\Delta}g\|_{L^{q_0}_tL^{r_0}_x(Q)}\le C_\varepsilon R^\varepsilon\|f\|_2\|g\|_2$$

for all $n+1$ dimensional cubes Q of side length $R/2$, by

$$\|e^{it\Delta}f\,e^{it\Delta}g\|_{L^{q_0}_xL^{r_0}_t(Q)}\le C_{\varepsilon,\alpha}R^\varepsilon N^\alpha\|f\|_2\|g\|_2 \tag{9}$$

for all $\alpha>\frac{1}{q_0}-\frac{1}{r_0}$, which follows from the hypothesis and translation invariance. The estimate

$$\|e^{it\Delta}f\,e^{it\Delta}g\|_{L^\infty_tL^1_x(\mathbb{R}^{n+1})}\le\|f\|_2\|g\|_2,$$

is replaced with

$$\begin{aligned}\|e^{it\Delta}f\,e^{it\Delta}g\|_{L^\infty_x L^1_t(\mathbb{R}^{n+1})} &\le CN^{-1}\|f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)}\\ &= CN^{\frac{1}{\infty}-\frac{1}{1}}\|f\|_2\|g\|_2,\end{aligned} \tag{10}$$

which follows by Cauchy–Schwarz from Lemma 1. The third interpolation point is unchanged;

$$\begin{aligned}\|e^{it\Delta}f\,e^{it\Delta}g\|_{L^\infty_x L^\infty_t(\mathbb{R}^{n+1})} &\le C\|f\|_2\|g\|_2\\ &= CN^{\frac{1}{\infty}-\frac{1}{\infty}}\|f\|_2\|g\|_2.\end{aligned} \tag{11}$$

Interpolating between (9), (10), and (11), we note that

$$\begin{aligned}\alpha_\theta &:= \theta\alpha_0 + (1-\theta)\alpha_1\\ &\ge \theta\left(\frac{1}{q_0}-\frac{1}{r_0}\right) + (1-\theta)\left(\frac{1}{q_1}-\frac{1}{r_1}\right)\\ &= \left(\frac{\theta}{q_0}+\frac{1-\theta}{q_1}\right) - \left(\frac{\theta}{r_0}+\frac{1-\theta}{r_1}\right)\\ &=: \frac{1}{q_\theta}-\frac{1}{r_\theta},\end{aligned}$$

so that the powers of N behave as desired.

We will require a version of the previous lemma for dealing with nonsharp powers of N. Note that the interpolation points with $q=\infty$ of the previous proof are α-improving so that the following lemma follows in the same way.

Lemma 5 *Suppose that for some $\alpha_0>0$ and for all $\varepsilon>0$,*

$$\|e^{it\Delta}f\,e^{it\Delta}g\|_{L^{q_0}_x(Q_R, L^{r_0}_t[R/2,R])} \le C_\varepsilon R^\varepsilon N^{\alpha_0}\|f\|_2\|g\|_2$$

whenever $R, N \gg 1$, and $\widehat{f}, \widehat{g}$ are supported on 1-separated subsets of $B_2(Ne_1)$. Then provided that $\frac{q}{r} > \frac{q_0}{r_0}$, $q(1-\frac{1}{r}) > q_0(1-\frac{1}{r_0})$, and $\alpha > \alpha_0$,

$$\|e^{it\Delta}f\,e^{it\Delta}g\|_{L^q_x L^r_t(\mathbb{R}^{n+1})} \le C_{q,r,\alpha}N^\alpha\|f\|_2\|g\|_2.$$

3 Bilinear estimates

By the globalizing lemmas, it will suffice to prove local estimates. We use the following notation:

Definition 2 Let $R^*(2\times 2 \to q, r, \alpha, \beta)$ denote the estimate

$$\|e^{it\Delta}f\,e^{it\Delta}g\|_{L^q_x L^r_t(P_R)} \le CR^\beta N^\alpha\|f\|_2\|g\|_2$$

whenever $R, N \gg 1$, $\widehat{f}, \widehat{g}$ are supported on 1-separated subsets of $B_1(Ne_1)$, and P_R is a parallelepiped of side $R/2$ and direction $(4\pi Ne_1, 1)$.

Recall the notional estimate

$$\| \sup_{0<t<1} |e^{it\Delta} f| \|_{L^2_x(\mathbb{B}^n)} \leq C \|f\|_{H^{1/4+\kappa}}, \qquad (A_\kappa)$$

and the dual exponents q_κ and q'_κ defined by

$$q_\kappa = \frac{n+1+8\kappa}{n+4\kappa} \quad \text{and} \quad q'_\kappa = \frac{n+1+8\kappa}{1+4\kappa}.$$

Theorem 4 *Suppose that (A_κ) holds. Then for all $q > q_\kappa$, $r > q'_\kappa$ and $\alpha > \frac{n}{q'_\kappa} - \frac{1}{r}$,*

$$\|e^{it\Delta} f\, e^{it\Delta} g\|_{L^q_x(\mathbb{R}^n,\, L^r_t(\mathbb{R}))} \leq C_\alpha N^\alpha \|f\|_2 \|g\|_2$$

whenever $N \gg 1$, and $\widehat{f}, \widehat{g}$ are supported on 1-separated subsets of $B_1(Ne_1)$.

Proof As f is frequency supported in $B_1(Ne_1)$, it is easy to calculate that the temporal Fourier transform of $e^{it\Delta} f$ is supported in an interval of length CN. Similarly this is true for $e^{it\Delta} f\, e^{it\Delta} g$, so that by Bernstein's inequality,

$$\|e^{it\Delta} f\, e^{it\Delta} g\|_{L^r_t(\mathbb{R})} \leq CN^{\frac{1}{p}-\frac{1}{r}} \|e^{it\Delta} f\, e^{it\Delta} g\|_{L^p_t(\mathbb{R})}.$$

Thus, by Lemmas 3 and 5, it will be enough to show that

$$\|e^{it\Delta} f\, e^{it\Delta} g\|_{L^{q_\kappa}_x L^{q'_\kappa}_t(P_R)} \leq C_\beta R^\beta N^{\frac{n}{q'_\kappa} - \frac{1}{q'_\kappa}} \|f\|_2 \|g\|_2 \qquad (12)$$

whenever $R \gg 1$, $\beta > 0$, and P_R is of side $R/2$ and direction $(4\pi Ne_1, 1)$.

We proceed by induction on scales. As P_R is contained in a cuboid, with long side $4\pi RN$, and short side R, by Hölder,

$$\begin{aligned}\|e^{it\Delta} f\, e^{it\Delta} g\|_{L^{q_\kappa}_x L^{q'_\kappa}_t(P_R)} &\leq C(R^n N)^{\frac{1}{q_\kappa} - \frac{1}{q'_\kappa}} \|e^{it\Delta} f\, e^{it\Delta} g\|_{L^{q'_\kappa}_x L^{q'_\kappa}_t(\mathbb{R}^{n+1})} \\ &\leq C(R^n N)^{\frac{1}{q_\kappa} - \frac{1}{q'_\kappa}} \|f\|_2 \|g\|_2,\end{aligned}$$

where the second inequality is by Cauchy–Schwarz, Fubini, and the linear Strichartz estimates of Theorem 1. Thus we have $R^*(2 \times 2 \to q_\kappa, q'_\kappa, (n-1)/q'_\kappa, \beta)$ for some large β. In fact we have a better power of α here than the $(n-1)/q'_\kappa$ that we get in the induction step. From now on we denote $(n-1)/q'_\kappa$ by α_κ. It will suffice to prove

$$R^*(2 \times 2 \to q_\kappa, q'_\kappa, \alpha_\kappa, \beta) \;\Rightarrow\; R^*(2 \times 2 \to q_\kappa, q'_\kappa, \alpha_\kappa, \max\{(1-\delta)\beta, c\delta\} + \varepsilon)$$

for all δ and $\varepsilon > 0$, where c is independent of δ and ε, as (12) would follow by iteration.

First we consider the problem when the frequency supports are close to the origin. We define $\widetilde{f}$ and $\widetilde{g}$ by

$$\widehat{\widetilde{f}} = \widehat{f}(\xi - Ne_1) \quad \text{and} \quad \widehat{\widetilde{g}} = \widehat{g}(\xi - Ne_1),$$

and we break up the solutions into wave-packets at scale R, so that

$$e^{it\Delta}\widetilde{f} = \sum_{j,k} e^{it\Delta}\widetilde{f}_{jk} \quad \text{and} \quad e^{it\Delta}\widetilde{g} = \sum_{j,k} e^{it\Delta}\widetilde{g}_{jk}.$$

Recall that the wave-packets $e^{it\Delta}\widetilde{f}_{jk}$ are essentially supported on tubes $\widetilde{T}_{jk}$, and we denote the tubes associated to $e^{it\Delta}\widetilde{g}_{jk}$ by $\widetilde{T}'_{jk}$. We also cover the cube $Q_R \times [R/2, R]$ by cubes $\widetilde{P} \in \widetilde{\mathcal{P}}$ of side $R^{1-\delta}$. The following orthogonality lemma is the key ingredient of Tao's bilinear restriction theorem.

Lemma 6 [28] *There exists a relationship $\sim$ between tubes $\widetilde{T}_{jk}$ and cubes $\widetilde{P}$ such that, for all tubes $\widetilde{T}_{jk}$,*

$$\#\{\widetilde{P} \in \widetilde{\mathcal{P}} \, : \, \widetilde{T}_{jk} \sim \widetilde{P}\} \leq CR^{\varepsilon}, \tag{13}$$

and for a constant c independent of δ and ε,

$$\left\| \Bigg(\sum_{\widetilde{T}_{jk} \sim \widetilde{P}} e^{it\Delta}\widetilde{f}_{jk} \Bigg) \Bigg(\sum_{\widetilde{T}'_{jk} \nsim \widetilde{P}} e^{it\Delta}\widetilde{g}_{jk} \Bigg) \right\|_{L^2(\widetilde{P})} \leq CR^{\varepsilon + c\delta - \frac{n-1}{4}} \|f\|_2 \|g\|_2,$$

and

$$\left\| \Bigg(\sum_{\widetilde{T}_{jk} \nsim \widetilde{P}} e^{it\Delta}\widetilde{f}_{jk} \Bigg) \Bigg(\sum_{\widetilde{T}'_{jk} \nsim \widetilde{P}} e^{it\Delta}\widetilde{g}_{jk} \Bigg) \right\|_{L^2(\widetilde{P})} \leq CR^{\varepsilon + c\delta - \frac{n-1}{4}} \|f\|_2 \|g\|_2.$$

We refer the interested reader to [28] for the precise definition of the relation $\sim$. It can be thought of as saying that the wave-packets are concentrated on the cubes.

As a translation of the frequency supports corresponds to an affine translation of the spatial support, returning to the original problem, we can suppose that P_R is the affine translation of $Q_R \times [R/2, R]$ under the mapping $x_1 \to x_1 + 4\pi t N e_1$. We cover this by parallelepipeds $P \in \mathcal{P}$ that correspond to the cubes $\widetilde{P}$ under the same affine translation. Similarly we break up the solutions into wave-packets with associated tubes T_{jk} and T'_{jk}, that correspond to $\widetilde{T}_{jk}$ and $\widetilde{T}'_{jk}$ under the affine translation. Thus, we have the induced relation $T_{jk} \sim P$ if $\widetilde{T}_{jk} \sim \widetilde{P}$.

As we have covered P_R by smaller parallelepipeds P, by the triangle inequality, it will suffice to show

$$\sum_{P\in\mathcal{P}}\left\|\left(\sum_{T_{jk}}e^{it\Delta}f_{jk}\right)\left(\sum_{T'_{jk}}e^{it\Delta}g_{jk}\right)\right\|_{L_x^{q_\kappa}L_t^{q'_\kappa}(P)}\le C_\beta R^{\max\{(1-\delta)\beta,c\delta\}+\varepsilon}N^{\alpha_\kappa}\|f\|_{L^2}\|g\|_{L^2}.$$

By the triangle inequality again, it will suffice to bound the 'local' part,

$$\sum_{P\in\mathcal{P}}\left\|\left(\sum_{T_{jk}\sim P}e^{it\Delta}f_{jk}\right)\left(\sum_{T'_{jk}\sim P}e^{it\Delta}g_{jk}\right)\right\|_{L_x^{q_\kappa}L_t^{q'_\kappa}(P)}$$

and the 'global' parts,

$$\sum_{P\in\mathcal{P}}\left\|\left(\sum_{T_{jk}\sim P}e^{it\Delta}f_{jk}\right)\left(\sum_{T'_{jk}\nsim P}e^{it\Delta}g_{jk}\right)\right\|_{L_x^{q_\kappa}L_t^{q'_\kappa}(P)},$$

$$\sum_{P\in\mathcal{P}}\left\|\left(\sum_{T_{jk}\nsim P}e^{it\Delta}f_{jk}\right)\left(\sum_{T'_{jk}\sim P}e^{it\Delta}g_{jk}\right)\right\|_{L_x^{q_\kappa}L_t^{q'_\kappa}(P)},$$

$$\sum_{P\in\mathcal{P}}\left\|\left(\sum_{T_{jk}\nsim P}e^{it\Delta}f_{jk}\right)\left(\sum_{T'_{jk}\nsim P}e^{it\Delta}g_{jk}\right)\right\|_{L_x^{q_\kappa}L_t^{q'_\kappa}(P)}.$$

To bound the local part, we simply invoke the induction hypothesis;

$$\begin{aligned}
&\sum_{P\in\mathcal{P}}\left\|\left(\sum_{T_{jk}\sim P}e^{it\Delta}f_{jk}\right)\left(\sum_{T'_{jk}\sim P}e^{it\Delta}g_{jk}\right)\right\|_{L_x^{q_\kappa}L_t^{q'_\kappa}(P)}\\
&\quad\le\sum_{P\in\mathcal{P}}CR^{(1-\delta)\beta}N^{\alpha_\kappa}\left\|\sum_{T_{jk}\sim P}f_{jk}\right\|_2\left\|\sum_{T'_{jk}\sim P}g_{jk}\right\|_2\\
&\quad\le CR^{(1-\delta)\beta}N^{\alpha_\kappa}\left(\sum_{P\in\mathcal{P}}\left\|\sum_{T_{jk}\sim P}f_{jk}\right\|_2^2\right)^{1/2}\left(\sum_{P\in\mathcal{P}}\left\|\sum_{T'_{jk}\sim P}g_{jk}\right\|_2^2\right)^{1/2}\\
&\quad\le CR^{(1-\delta)\beta+\varepsilon}N^{\alpha_\kappa}\|f\|_2\|g\|_2,
\end{aligned}$$

where the second inequality is by Cauchy–Schwarz, and the third by (13) and almost orthogonality. This bound is acceptable.

Considering the first global part, by Fubini and the affine change of variables $x_1 \to x_1 + 4\pi t N e_1$, followed by Lemma 6, we have

$$\left\| \left(\sum_{T_{jk} \sim P} e^{it\Delta} f_{jk} \right) \left(\sum_{T'_{jk} \nsim P} e^{it\Delta} g_{jk} \right) \right\|_{L_x^2 L_t^2(P)} \leq C R^{\varepsilon + c\delta - \frac{n-1}{4}} \|f\|_2 \|g\|_2. \tag{14}$$

On the other hand, by scaling and the hypothesis,

$$\begin{aligned} \left\| \sum_{T_{jk} \sim P} e^{it\Delta} f_{jk} \right\|_{L_x^2 L_t^\infty(B_{NR})} &\leq C(RN^2)^{1/4+\kappa} \left\| \sum_{T_{jk} \sim P} f_{jk} \right\|_2 \\ &\leq C(RN^2)^{1/4+\kappa} \|f\|_2. \end{aligned}$$

Similarly

$$\left\| \sum_{T_{jk} \nsim P} e^{it\Delta} g_{jk} \right\|_{L_x^2 L_t^\infty(B_{NR})} \leq C(RN^2)^{1/4+\kappa} \|g\|_2,$$

so that by Cauchy–Schwarz,

$$\left\| \left(\sum_{T_{jk} \sim P} e^{it\Delta} f_{jk} \right) \left(\sum_{T'_{jk} \nsim P} e^{it\Delta} g_{jk} \right) \right\|_{L_x^1 L_t^\infty(P)} \leq C(RN^2)^{1/2+2\kappa} \|f\|_2 \|g\|_2. \tag{15}$$

Interpolating between (14) and (15), using Hölder, gives

$$\left\| \left(\sum_{T_{jk} \sim P} e^{it\Delta} f_{jk} \right) \left(\sum_{T'_{jk} \nsim P} e^{it\Delta} g_{jk} \right) \right\|_{L_x^{q_\kappa} L_t^{q'_\kappa}(P)} \leq C R^{\varepsilon + c\delta} N^{\alpha_\kappa} \|f\|_2 \|g\|_2,$$

so that, by summing,

$$\sum_P \left\| \left(\sum_{T_{jk} \sim P} e^{it\Delta} f_{jk} \right) \left(\sum_{T'_{jk} \nsim P} e^{it\Delta} g_{jk} \right) \right\|_{L_x^{q_\kappa} L_t^{q'_\kappa}(P)} \leq C R^{(c+n+1)\delta + \varepsilon} N^{\alpha_\kappa} \|f\|_2 \|g\|_2,$$

which is acceptable. The other two global parts are bounded in the same way, which completes the proof.

We now pass to the unconditional result in which the powers of N are improved. In the final section we will see that this improvement allows us to obtain the almost optimal range of r in Theorem 3. A refinement of Lemma 4, which preserved the precise powers of N, would allow α to equal $1/q - 1/r$ in the following.

Theorem 5 *Suppose that* $q \in (\frac{8}{5}, \frac{5}{3})$ *and* $\frac{4}{q} + \frac{1}{r} < 3$. *Then for all* $\alpha > \frac{1}{q} - \frac{1}{r}$,

$$\|e^{it\Delta} f\, e^{it\Delta} g\|_{L^q_x(\mathbb{R}^2,\, L^r_t(\mathbb{R}))} \leq C_\alpha N^\alpha \|f\|_2 \|g\|_2$$

whenever $N \gg 1$, *and* $\widehat{f}, \widehat{g}$ *are supported on* 1*-separated subsets of* $B_1(Ne_1)$.

Proof Combining the bilinear theorem of Tao [28] with Bernstein's inequality as before, we see that

$$\|e^{it\Delta} f\, e^{it\Delta} g\|_{L^q_x(\mathbb{R}^2,\, L^r_t(\mathbb{R}))} \leq C N^{\frac{1}{q} - \frac{1}{r}} \|f\|_2 \|g\|_2 \tag{16}$$

for all $r \geq q > 5/3$. Now, by interpolation combined with Lemmas 3 and 4, it will suffice to show that

$$\|e^{it\Delta} f\, e^{it\Delta} g\|_{L^{8/5}_x L^2_t(P_R)} \leq C R^\beta N^{1/8} \|f\|_2 \|g\|_2$$

whenever $R \gg 1$, $\beta > 0$, and P_R has side $R/2$ and direction $(4\pi Ne_1, 1)$.

Again, we proceed by induction on scales. As P_R is contained in a cuboid, with long side $4\pi RN$, and short side R, by Hölder,

$$\|e^{it\Delta} f\, e^{it\Delta} g\|_{L^{8/5}_x L^2_t(P_R)} \leq C(R^2 N)^{1/8} \|e^{it\Delta} f\, e^{it\Delta} g\|_{L^2_x L^2_t(\mathbb{R}^{2+1})},$$

so that by (16), we have

$$\|e^{it\Delta} f\, e^{it\Delta} g\|_{L^{8/5}_x L^2_t(P_R)} \leq C(R^2 N)^{1/8} \|f\|_2 \|g\|_2.$$

We see that $R^*(2 \times 2 \to 8/5, 2, 1/8, \beta)$ holds for a large β. Therefore, by iterating, it will suffice to prove that

$$R^*(2 \times 2 \to 8/5, 2, 1/8, \beta) \quad \Rightarrow \quad R^*(2 \times 2 \to 8/5, 2, 1/8, \max\{(1-\delta)\beta, c\delta\} + \varepsilon)$$

for all δ and $\varepsilon > 0$, where the constant c is independent of δ and ε.

As before, we cover P_R by smaller parallelepipeds P, so that it will suffice to bound the local part,

$$\sum_{P \in \mathcal{P}} \left\| \left(\sum_{T_{jk} \sim P} e^{it\Delta} f_{jk} \right) \left(\sum_{T'_{jk} \sim P} e^{it\Delta} g_{jk} \right) \right\|_{L^{8/5}_x L^2_t(P)},$$

which is dealt with via the induction hypothesis, and the global parts of type

$$\sum_{P\in\mathcal{P}}\left\|\left(\sum_{T_{jk}\sim P}e^{it\Delta}f_{jk}\right)\left(\sum_{T'_{jk}\nsim P}e^{it\Delta}g_{jk}\right)\right\|_{L_x^{8/5}L_t^2(P)}.$$

By Hölder, followed by Fubini and the affine change of variables $x_1\to x_1+4\pi tNe_1$,

$$\begin{aligned}\left\|\left(\sum_{T_{jk}\sim P}e^{it\Delta}f_{jk}\right)\left(\sum_{T'_{jk}\nsim P}e^{it\Delta}g_{jk}\right)\right\|_{L_x^{8/5}L_t^2(P)}\\ \le (R^2N)^{1/8}\left\|\left(\sum_{\widetilde{T}_{jk}\sim\widetilde{P}}e^{it\Delta}\widetilde{f}_{jk}\right)\left(\sum_{\widetilde{T}'_{jk}\nsim\widetilde{P}}e^{it\Delta}\widetilde{g}_{jk}\right)\right\|_{L^2_{x,t}(\widetilde{P})},\end{aligned}$$

so that by Lemma 6,

$$\left\|\left(\sum_{T_{jk}\sim P}e^{it\Delta}f_{jk}\right)\left(\sum_{T'_{jk}\nsim P}e^{it\Delta}g_{jk}\right)\right\|_{L_x^{8/5}L_t^2(P)}\le CR^{\varepsilon+c\delta}N^{1/8}\|f\|_2\|g\|_2,$$

where the constant c is independent of δ and ε. Summing, this yields

$$\sum_{P\in\mathcal{P}}\left\|\left(\sum_{T_{jk}\sim P}e^{it\Delta}f_{jk}\right)\left(\sum_{T'_{jk}\nsim P}e^{it\Delta}g_{jk}\right)\right\|_{L_x^{8/5}L_t^2(P)}\le CR^{(c+3)\delta+\varepsilon}N^{1/8}\|f\|_2\|g\|_2,$$

which is acceptable. The other two global parts are bounded in the same way, which completes the proof.

4 Linear estimates

The following lemma is a simple consequence of the Littlewood–Paley inequality (see [24]). Let $\vartheta\in C_0^\infty(\mathbb{R})$ and $\phi=\vartheta(2\pi|\cdot|^2)$ satisfy

$$\sum_{k=-\infty}^{\infty}\vartheta(4^{-k}|\cdot|)=1\quad\text{and}\quad\sum_{k=-\infty}^{\infty}\phi(2^{-k}|\cdot|)=1.$$

Defining f_k by $\widehat{f_k}=\phi(2^{-k}|\cdot|)\widehat{f}$, it can be calculated that

$$\left(\vartheta(4^{-k}|\tau|)\left(e^{it\Delta}f\right)^{\wedge_t}(\tau)\right)^{\vee_t}(t)=e^{it\Delta}f_k.$$

Lemma 7 *Let $q \in [2, \infty]$ and $r \in [2, \infty)$. Then*

$$\|e^{it\Delta} f\|^2_{L^q_x(\mathbb{R}^n,\, L^r_t(\mathbb{R}))} \leq C \sum_{k=-\infty}^{\infty} \|e^{it\Delta} f_k\|^2_{L^q_x(\mathbb{R}^n,\, L^r_t(\mathbb{R}))}.$$

We are now in a position to prove the linear estimates as stated in the introduction. There are two types of restriction on r; those which come from the restriction on r in the bilinear theorem are generally less restrictive than those related to the power of N.

Theorem 2 *Let $q \in (2q_\kappa, \infty]$, $r \in (2q'_\kappa, \infty)$ and $\frac{n}{2q'_\kappa} + \frac{n}{q} + \frac{1}{r} < \frac{n}{2}$. If ($A_\kappa$) holds, then*

$$\|e^{it\Delta} f\|_{L^q_x(\mathbb{R}^n,\, L^r_t(\mathbb{R}))} \leq C\|f\|_{\dot{H}^{s(q,r)}(\mathbb{R}^n)}.$$

Proof By scaling and Lemma 7, it will suffice to prove that

$$\|e^{it\Delta} f\|_{L^q_x L^r_t(\mathbb{R}^{n+1})} \leq C\|f\|_{L^2(\mathbb{R}^n)}$$

whenever $\widehat{f}$ is supported in $\{\, 1/2 \leq |\xi| \leq 1 \,\}$. In order to apply our bilinear theorem, we square the integral, so that

$$\|e^{it\Delta} f\|^2_{L^q_x L^r_t(\mathbb{R}^{n+1})} = \|e^{it\Delta} f\, e^{it\Delta} f\|_{L^{q/2}_x L^{r/2}_t(\mathbb{R}^{n+1})}.$$

Now, for each $j \in \mathbb{N}$ we can break up the support of $\widehat{f}$ into dyadic cubes τ^j_k of side 2^{-j}. We write $\tau^j_k \sim \tau^j_{k'}$ if τ^j_k and $\tau^j_{k'}$ have adjacent parents, but are not adjacent. Writing $\widehat{f} = \sum_k \widehat{f^j_k}$, where $\widehat{f^j_k} = \widehat{f}\chi_{\tau^j_k}$, we have

$$\begin{aligned} e^{it\Delta} f(x)\, e^{it\Delta} f(x) &= \int\int \widehat{f}(\xi)\widehat{f}(y) e^{2\pi i(x\cdot(\xi+y)-2\pi t(|\xi|^2+|y|^2))} d\xi dy \\ &= \sum_{j,k,k':\tau^j_k \sim \tau^j_{k'}} \int\int \widehat{f^j_k}(\xi)\widehat{f^j_{k'}}(y) e^{2\pi i(x\cdot(\xi+y)-2\pi t(|\xi|^2+|y|^2))} d\xi dy \\ &= \sum_{j,k,k':\tau^j_k \sim \tau^j_{k'}} e^{it\Delta} f^j_k(x)\, e^{it\Delta} f^j_{k'}(x). \end{aligned}$$

By the triangle inequality, we see that

$$\|e^{it\Delta} f\|^2_{L^q_x L^r_t(\mathbb{R}^{n+1})} \leq \sum_{j,k,k':\tau^j_k \sim \tau^j_{k'}} \|e^{it\Delta} f^j_k\, e^{it\Delta} f^j_{k'}\|_{L^{q/2}_x L^{r/2}_t(\mathbb{R}^{n+1})}.$$

Now, scaling out, applying Theorem 4 taking into account the rotational symmetry, then scaling in again, we see that

$$\|e^{it\Delta}f\|^2_{L^q_xL^r_t(\mathbb{R}^{n+1})} \leq C_\alpha \sum_{j,k,k':\tau^j_k\sim\tau^j_{k'}} 2^{-j(n-\frac{2n}{q}-\frac{4}{r})}2^{j\alpha}\|f^j_k\|_{L^2(\mathbb{R}^n)}\|f^j_{k'}\|_{L^2(\mathbb{R}^n)}$$

for all $\alpha > \frac{n}{q'_\kappa} - \frac{2}{r}$, where $q > 2q_\kappa$ and $r > 2q'_\kappa$.

Finally, as $\operatorname{supp}\widehat{f}^j_k$, $\operatorname{supp}\widehat{f}^j_{k'} \subset \operatorname{supp}\widehat{f}^{j-2}_{k''}$ for some k'', we have

$$\sum_{k,k':\tau^j_k\sim\tau^j_{k'}} \|f^j_k\|_{L^2(\mathbb{R}^n)}\|f^j_{k'}\|_{L^2(\mathbb{R}^n)} \leq C\|f\|^2_{L^2(\mathbb{R}^n)},$$

and the sum in j converges by hypothesis, which completes the proof.

Observe that if the power of N in the bilinear estimate was improved to $\alpha > 1/q - 1/r$, then we would obtain the almost sharp restriction, $\frac{n+1}{q} + \frac{1}{r} < \frac{n}{2}$, in the linear estimates. We state this formally.

Definition 3 Let $R^*(2\times 2 \to q, r)$ denote the estimate

$$\|e^{it\Delta}f\, e^{it\Delta}g\|_{L^q_x(\mathbb{R}^n,\, L^r_t(\mathbb{R}))} \leq C_\alpha N^\alpha \|f\|_2\|g\|_2$$

whenever $N \gg 1$, $\alpha > \frac{1}{q} - \frac{1}{r}$, and $\widehat{f}, \widehat{g}$ are supported on 1-separated subsets of $B_1(Ne_1)$.

Definition 4 Let $R^*(2 \to q, r)$ denote the estimate

$$\|e^{it\Delta}f\|_{L^q_x(\mathbb{R}^n,\, L^r_t(\mathbb{R}))} \leq C\|f\|_{L^2(\mathbb{R}^n)}$$

whenever $\widehat{f}$ is supported in $\{1/2 \leq |\xi| \leq 1\}$.

Lemma 8 *Let* $\frac{n+1}{q} + \frac{1}{r} < \frac{n}{2}$. *Then* $R^*(2\times 2 \to \frac{q}{2}, \frac{r}{2}) \Rightarrow R^*(2 \to q, r)$.

It remains to prove Theorem 3. By scaling and Lemma 7, it suffices to consider functions with frequency support in the unit annulus. Combining Theorem 5 with Lemma 8, we note that the condition $8/q + 2/r < 3$ that comes from the former is less restrictive than $3/q + 1/r < 1$ which comes from the latter, and we are done.

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