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A geometric inequality in the Heisenberg group and its applications to stable solutions of semilinear problems

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Abstract In the Heisenberg group framework, we obtain a geometric inequality for stable solutions of $\Delta_{\mathbb{H}} u = f(u)$ in a domain $\Omega \subseteq \mathbb{H}$. More precisely, if we denote the horizontal intrinsic Hessian by Hu, the mean curvature of a level set by h, its imaginary curvature by p, the intrinsic normal by ν and the unit tangent by v, we have that

$$\int_{\Omega} |\nabla_{\mathbb{H}}\phi|^2 |\nabla_{\mathbb{H}} u|^2$$
$$\geq \int_{\Omega\cap\{\nabla_{\mathbb{H}}u\neq 0\}} \Big(|Hu|^2 - \langle (Hu)^2 \nu, \nu\rangle_{\mathbb{H}} - 2\,(TYuXu - TXuYu)\Big)\phi^2$$
$$= \int_{\Omega\cap\{\nabla_{\mathbb{H}}u\neq 0\}} |\nabla_{\mathbb{H}} u|^2 \left[h^2 + \left(p + \frac{\langle Huv, \nu\rangle_{\mathbb{H}}}{|\nabla_{\mathbb{H}} u|}\right)^2 + 2\langle T\nu, v\rangle_{\mathbb{H}}\right]\phi^2$$

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for any $\phi \in C_0^\infty(\Omega)$. Stable solutions in the entire $\mathbb{H}$ satisfying a suitably weighted energy growth and such that $\langle T\nu, v\rangle_{\mathbb{H}} \geq 0$ are then shown to have level sets with vanishing mean curvature.

0 Introduction

The purpose of this paper is to provide geometric estimates for stable solutions of PDEs in the Heisenberg group, by suitably developing some techniques for level set analysis provided in [16,17,20,25–27]. Such estimates will then be used for giving a criterion under which the level sets of entire stable solutions are minimal surfaces, i.e., they have vanishing mean curvature.

For this purpose, let us briefly recall the definition and the basic properties of the Heisenberg group.

Let $\mathbb{H}$ be the Heisenberg group, namely $\mathbb{R}^3$ endowed with the following noncommutative internal law: for every $(x_1, y_1, t_1), (x_2, y_2, t_2) \in \mathbb{R}^3$

$$(x_1, y_1, t_1) \circ (x_2, y_2, t_2) = (x_1 + x_2, y_1 + y_2, t_1, t_2 + 2(x_2 y_1 - x_1 y_2)).$$

We shall denote $X = (1, 0, 2y)$ and $Y = (0, 1, -2x)$. With the same notation we denote the two vectorfields $X = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}$ and $Y = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t}$ generating the algebra. Notice that

$$[X, Y] = T = -4\frac{\partial}{\partial t},$$

i.e. X and Y do not commute.

In particular, on each fiber $\mathcal{H}_P = \text{span}\{X, Y\}$ an internal product is given as follows: for every $U, V \in \mathcal{H}_P$, with $U = \alpha_1 X + \beta_1 Y$ and $V = \alpha_2 X + \beta_2 Y$, we have

$$\langle U, V\rangle_{\mathbb{H}} = \alpha_1\alpha_2 + \beta_1\beta_2.$$

This internal product makes the vectors X and Y orthonormal on $\mathcal{H}_P$. We shall define the norm on $\mathcal{H}_P$ for every $U \in \mathcal{H}$ as

$$| U |_{\mathbb{H}} = \sqrt{\langle U, X\rangle_{\mathbb{H}}^2 + \langle U, Y\rangle_{\mathbb{H}}^2}.$$

No confusion should arise between the Euclidean objects $\langle\cdot,\cdot\rangle$ and $|\cdot|$ and that ones on the fibers in the Heisenberg group respectively denoted by $\langle\cdot,\cdot\rangle_{\mathbb{H}}$ and $|\cdot|_{\mathbb{H}}$.

For a smooth function u, we denote $\nabla_{\mathbb{H}} u(P) = (Xu(P), Yu(P))$ where $Xu(P)$ and $Yu(P)$ are the coordinates of the vector $\nabla_{\mathbb{H}} u(P)$ with respect to the basis given by X and Y at P. In jargon, $\nabla_{\mathbb{H}} u$ is called the intrinsic gradient of u. We remind that a point $P \in \Sigma$ is characteristic for the C^1 level set Σ of u when the fiber in P coincides with the Euclidean tangent space to Σ at P, namely: $\mathcal{H}_P = T_P\Sigma$. In particular if $\nabla_{\mathbb{H}} u(P) \neq 0$, then P is not characteristic. The size of characteristic points on regular surfaces is small in the following sense. The intrinsic Hausdorff measure $\mathcal{H}_{CC}^{(3)}$ of the set of characteristic points $C(\Sigma)$, of a C^1 surface Σ, is zero.

Moreover the same result holds even when we consider the 2-Hausdorff measure with respect to the Euclidean distance, i.e. $\mathcal{H}_E^{(2)}(C(\Sigma))$, of $C^{1,1}$ surfaces Σ; nevertheless below this regularity counterexamples exist, see [5].

Whenever $P \in \{u = k\} \cap \{\nabla_{\mathbb{H}} u \neq 0\}$, one can consider the smooth surface $\{u = k\}$ and define

$$\nu = \frac{\nabla_{\mathbb{H}} u(P)}{\mid \nabla_{\mathbb{H}} u(P) \mid}. \tag{1}$$

Usually, such ν is called the intrinsic normal to $\{u = k\}$ at P. Moreover, associated with ν, to any noncharacteristic point $P \in \{u = k\}$, there exists the so called intrinsic unit tangent direction to the level set $\{u = k\}$ at P defined as

$$v = \frac{(Yu(P), -Xu(P))}{\mid \nabla_{\mathbb{H}} u \mid}, \tag{2}$$

where the above coordinates are given with respect to the (X, Y)-frame.

We observe that $\langle \nu, v \rangle_{\mathbb{H}} = 0$.

The Kohn-Laplace operator on $\mathbb{H}$ is defined by

$$\Delta_{\mathbb{H}} u = X^2 u + Y^2 u.$$

Since a divergent operator is defined on each fiber, we can write

$$\Delta_{\mathbb{H}} u = \mathrm{div}_{\mathbb{H}}(\nabla_{\mathbb{H}} u) = X(Xu) + Y(Yu).$$

We define the horizontal intrinsic Hessian matrix as

$$Hu = \begin{bmatrix} XXu, & YXu \\ XYu, & YYu \end{bmatrix}.$$

Notice that Hu is not symmetric. Its norm is given by

$$|Hu| = \sqrt{(XXu)^2 + (YXu)^2 + (XYu)^2 + (YYu)^2}. \tag{3}$$

As usual, we set

$$(Hu)^2 = (Hu)(Hu)^T. \tag{4}$$

Let u be a C^2 stable weak solution of $\Delta_{\mathbb{H}} u = f(u)$, with[1] $f \in C^1$, in a domain $\Omega \subseteq \mathbb{H}$, i.e., we suppose that

$$-\int_{\mathbb{H}} \langle \nabla_{\mathbb{H}} u, \nabla_{\mathbb{H}} \phi \rangle_{\mathbb{H}} = \int_{\mathbb{H}} f(u)\phi, \tag{5}$$

[1] In fact, a locally Lipschitz f could also be considered with these techniques, up to performing some further minor technicalities in the proofs, see [17,18].

and

$$0 \leq \int\limits_{\mathbb{H}} \langle \nabla_{\mathbb{H}}\phi, \nabla_{\mathbb{H}}\phi \rangle_{\mathbb{H}} + \int\limits_{\mathbb{H}} f'(u)\phi^2, \tag{6}$$

for every $\phi \in C_0^\infty(\Omega)$.

Condition (5) is the usual definition of weak solution.

Condition (6) is also classical and it is natural in the calculus of variation framework—in particular, it says that the second variation of the associated functional has a sign, as it happens, for instance, for local minima.

The main result of this paper is to provide a geometric inequality for such u. Namely, in the subsequent Theorems 1.3 and 2.3, we shall prove that

$$\begin{aligned} &\int\limits_{\Omega} |\nabla_{\mathbb{H}}\phi|^2 |\nabla_{\mathbb{H}}u|^2 \\ &\geq \int\limits_{\Omega\cap\{\nabla_{\mathbb{H}}u\neq 0\}} \Big(|Hu|^2 - \langle (Hu)^2\nu, \nu\rangle_{\mathbb{H}} - 2\,(TYuXu - TXuYu)\Big)\phi^2 \\ &= \int\limits_{\Omega\cap\{\nabla_{\mathbb{H}}u\neq 0\}} |\nabla_{\mathbb{H}}u|^2 \left[h^2 + \left(p + \frac{\langle Hu\upsilon, \upsilon\rangle_{\mathbb{H}}}{|\nabla_{\mathbb{H}}u|}\right)^2 + 2\langle T\nu, \upsilon\rangle_{\mathbb{H}}\right]\phi^2, \end{aligned} \tag{7}$$

for any $\phi \in C_0^\infty(\Omega)$, where h and p are respectively the intrinsic mean curvature and the imaginary curvature of the noncharacteristic points belonging to the level sets Σ of u. Concerning the notion of intrinsic mean curvature we refer to [2,3,12,14,21,24], while for the notion of imaginary curvature and its geometric meaning we refer to [2,3].

Shortly, we just recall here that the mean curvature h, in a noncharacteristic point $P \in \Sigma$ of the level surface given by u, is defined as

$$h = \mathrm{div}_{\mathbb{H}}\nu(P), \tag{8}$$

while the imaginary curvature p at the point $P \in \Sigma$ of the level surface Σ, given by u, is defined as

$$p = -\frac{Tu(P)}{|\nabla_{\mathbb{H}}u(P)|}. \tag{9}$$

Another geometric interpretation of (7) will be provided in (40), where the imaginary curvature p will be related to the tangential gradient of $|\nabla_{\mathbb{H}}u|$ along the level set.

We observe that (7) may be interpreted in two ways. One way is to think that some interesting geometric objects which describe u, such as its intrinsic Hessian and the curvatures of its level sets, are bounded by an energy term. These quantities involved in the inequality are weighted by a test function ϕ which can be chosen as we wish.

Another point of view consists in thinking that (7) bounds a suitably weighted L^2-norm of any test function ϕ with a suitably weighted L^2-norm of its gradient. The weights here are given by the stable solution u.

The latter interpretation sees (7) as a Sobolev-Poincaré inequality for any test function ϕ and suitable choices of u may lead to interesting weighted inequalities, as shown in [20].

In this paper, we shall emphasize the first point of view. Indeed, by a suitable choice of ϕ, it will be possible to give a criterion for the level sets of u to have vanishing curvature (that is, to be minimal surfaces). Such criterion will be explicitly discussed in Corollary 3.2.

We recall that the study of geometric inequalities for semilinear equations goes back to [26,27], where uniformly elliptic PDEs in the Euclidean spaces were taken into account, and further important developments have been performed in [16].

The techniques of [26,27] have been then applied in [17] for singular and degenerate PDEs and in [25] for fractional operators. Related techniques have also been exploited, in a different framework, by [11].

Here, we perform an interplay between the techniques of [26,27] and a geometric analysis on the Heisenberg group, in order to obtain our results. For this, some formulas will be borrowed from [2,3].

The relation between entire stable solutions and minimal surfaces, as performed in Corollary 3.2 below, is inspired by a famous conjecture of De Giorgi (see [15]) in the Euclidean setting. Though several results have been obtained in the Heisenberg group analogue for such a conjecture (see [8,9]), and even in the general framework of Carnot groups (see [7]), many questions are still open (see [4,10,18]).

In the subsequent Sect. 1 we shall develop the analytical tools towards (7). In particular, one part of (7) will be given in Theorem 1.3.

Then, in Sect. 2, the geometry of the Heisenberg group will be investigated, in order to complete the proof of (7) (this will be accomplished in Lemma 2.2 and Theorem 2.3).

Finally, Sect. 3 contains the application to the stable solutions in the entire space, see in particular Theorem 3.7, which shows that suitably low energy stable solutions in the Heisenberg group cannot exist.

1 Analytical computations

We begin with an elementary observation:

Lemma 1.1 *We have that*

$$X\Delta_{\mathbb{H}}u = \Delta_{\mathbb{H}}Xu + 2TYu \tag{10}$$

and

$$Y\Delta_{\mathbb{H}}u = \Delta_{\mathbb{H}}Yu - 2TXu. \tag{11}$$

Proof

$$\begin{aligned} X\Delta_{\mathbb{H}}u &= X(XXu) + X(YYu) = XX(Xu) + XY(Yu) \\ &= \Delta_{\mathbb{H}}Xu - YY(Xu) + XY(Yu) \\ &= \Delta_{\mathbb{H}}Xu - YY(Xu) + XY(Yu) - YX(Yu) + YX(Yu) \end{aligned}$$

$$= \Delta_{\mathbb{H}} Xu - YY(Xu) + TYu + YX(Yu)$$
$$= \Delta_{\mathbb{H}} Xu + TYu + YTu = \Delta_{\mathbb{H}} Xu + 2TYu.$$

This proves (10). The proof of (11) is analogous. □

We observe that, if u satisfies $\Delta_{\mathbb{H}} u = f(u)$, then, by Lemma 1.1,

$$\begin{aligned} \Delta_{\mathbb{H}} Xu + 2TYu &= f'(u)Xu \\ \Delta_{\mathbb{H}} Yu - 2TXu &= f'(u)Yu. \end{aligned} \tag{12}$$

Next result is a version in the Heisenberg group of a classical result (we give details for the reader's facility):

Lemma 1.2 *Let $c \in \mathbb{R}$. Suppose that Ω is an open domain of $\mathbb{H}$ and that $w : \Omega \to \mathbb{R}$ is Lipschitz with respect to the metric structure of $\mathbb{H}$.*

Then, $\nabla_{\mathbb{H}} w = 0$ for almost any $x \in \{w = c\}$.

Proof By Coarea Formula (see (1.4) in [23]), given an integrable function $\gamma : \Omega \to \mathbb{R}$, we have that

$$\int_{\Omega} \gamma |\nabla_{\mathbb{H}} w| = \int_{-\infty}^{+\infty} \Psi(t)\, dt\ ,$$

where

$$\Psi(t) = \int_{\{w=t\}} \gamma\, d\mu_t$$

and μ_t is the Heisenberg group perimeter $\|\partial\{w > t\}\|$ (see [23] for details).

We take a bounded domain $U \subseteq \Omega$ and γ to be the characteristic function $\chi_{U \cap \{w=c\}}$. Then, $\Psi(t) = 0$ for any $t \neq c$ and so

$$\int_{U \cap \{w=c\}} |\nabla_{\mathbb{H}} w| = 0,$$

hence $|\nabla_{\mathbb{H}} w| = 0$ almost everywhere in $U \cap \{w = c\}$. □

Next result gives the first part of the inequality in (7).

Theorem 1.3 *For every $\phi \in C_0^\infty(\Omega)$*

$$\int_{\Omega \cap \{\nabla_{\mathbb{H}} u \neq 0\}} \Big(|\ Hu\ |^2 - \langle (Hu)^2 \nu, \nu \rangle_{\mathbb{H}} - 2\,(TYuXu - TXuYu) \Big) \phi^2 \leq \int_{\Omega} |\ \nabla_{\mathbb{H}} \phi\ |^2 |\ \nabla_{\mathbb{H}} u\ |^2\ . \tag{13}$$

Proof The proof is inspired by some computations in [16,17,26,27], suitably modified here in order to understand the complicated geometry of Heisenberg group.

Let us consider $\eta = Xu\phi^2$ and $\xi = Yu\phi^2$ as test functions in (12). We get, integrating by parts (12), that

$$-\int_{\mathbb{H}} \langle \nabla_{\mathbb{H}} Xu, \nabla_{\mathbb{H}}(Xu\phi^2)\rangle_{\mathbb{H}} + 2\left(\int_{\mathbb{H}} TYuXu\phi^2\right) = \int_{\mathbb{H}} f'(u)(Xu)^2\phi^2$$

and

$$-\int_{\mathbb{H}} \langle \nabla_{\mathbb{H}} Yu, \nabla_{\mathbb{H}}(Yu\phi^2)\rangle_{\mathbb{H}} - 2\left(\int_{\mathbb{H}} TXuYu\phi^2\right) = \int_{\mathbb{H}} f'(u)(Yu)^2\phi^2.$$

Then, by summing term by term, we get

$$\begin{aligned} &-\int_{\mathbb{H}} \mid \nabla_{\mathbb{H}} Xu \mid^2 \phi^2 - \int_{\mathbb{H}} \mid \nabla_{\mathbb{H}} Yu \mid^2 \phi^2 \\ &-\int_{\mathbb{H}} \langle \nabla_{\mathbb{H}} Xu, \nabla_{\mathbb{H}}(\phi^2)\rangle_{\mathbb{H}} Xu - \int_{\mathbb{H}} \langle \nabla_{\mathbb{H}} Yu, \nabla_{\mathbb{H}}(\phi^2)\rangle_{\mathbb{H}} Yu \\ &+2\int_{\mathbb{H}} (TYuXu - TXuYu)\,\phi^2 \\ &= \int_{\mathbb{H}} f'(u) \mid \nabla_{\mathbb{H}} u \mid^2 \phi^2. \end{aligned} \tag{14}$$

On the other hand, putting $\mid \nabla_{\mathbb{H}} u \mid \phi$ as test function in (6), we get:

$$0 \leq \int_{\mathbb{H}} \mid \nabla_{\mathbb{H}}(\mid \nabla_{\mathbb{H}} u \mid \phi) \mid^2 + \int_{\mathbb{H}} f'(u) \mid \nabla_{\mathbb{H}} u \mid^2 \phi^2. \tag{15}$$

We denote by $\mathbb{H}_o$ the set $\Omega \cap \{\nabla_{\mathbb{H}} u \neq 0\}$. Let us stress that, in $\mathbb{H}_o$,

$$X \mid \nabla_{\mathbb{H}} u \mid = \frac{1}{\mid \nabla_{\mathbb{H}} u \mid}(XuXXu + YuXYu) \tag{16}$$

and

$$Y \mid \nabla_{\mathbb{H}} u \mid = \frac{1}{\mid \nabla_{\mathbb{H}} u \mid}(XuYXu + YuYYu). \tag{17}$$

In particular,

$$\nabla_{\mathbb{H}} \mid \nabla_{\mathbb{H}} u \mid = \frac{1}{\mid \nabla_{\mathbb{H}} u \mid}(Hu)^T \nabla_{\mathbb{H}} u, \tag{18}$$

thus, in $\mathbb{H}_o$,

$$
\begin{aligned}
| \nabla_{\mathbb{H}}(| \nabla_{\mathbb{H}} u |) |^2 &= \frac{1}{| \nabla_{\mathbb{H}} u |^2} \langle (Hu)^T \nabla_{\mathbb{H}} u, (Hu)^T \nabla_{\mathbb{H}} u \rangle_{\mathbb{H}} \\
&= \frac{1}{| \nabla_{\mathbb{H}} u |^2} \langle (Hu)^2 \nabla_{\mathbb{H}} u, \nabla_{\mathbb{H}} u \rangle_{\mathbb{H}}, \qquad (19)
\end{aligned}
$$

due to (4).

Moreover, in $\mathbb{H}_o$,

$$
\nabla_{\mathbb{H}}(| \nabla_{\mathbb{H}} u | \phi) = \frac{\phi}{| \nabla_{\mathbb{H}} u |} (Hu)^T \nabla_{\mathbb{H}} u + \nabla_{\mathbb{H}} \phi \, | \nabla_{\mathbb{H}} u | .
$$

So, in $\mathbb{H}_o$,

$$
\begin{aligned}
&\left| \nabla_{\mathbb{H}}(| \nabla_{\mathbb{H}} u | \phi) \right|^2 \\
&\quad = \langle \nabla_{\mathbb{H}}(| \nabla_{\mathbb{H}} u | \phi), \nabla_{\mathbb{H}}(| \nabla_{\mathbb{H}} u | \phi) \rangle_{\mathbb{H}} \\
&\quad = \frac{\phi^2}{| \nabla_{\mathbb{H}} u |^2} \langle (Hu)^T \nabla_{\mathbb{H}} u, (Hu)^T \nabla_{\mathbb{H}} u \rangle_{\mathbb{H}} + 2 \langle \nabla_{\mathbb{H}} \phi, (Hu)^T \nabla_{\mathbb{H}} u \rangle_{\mathbb{H}} \phi + | \nabla_{\mathbb{H}} \phi |^2 | \nabla_{\mathbb{H}} u |^2 \\
&\quad = \phi^2 \langle (Hu)^T \nu, (Hu)^T \nu \rangle_{\mathbb{H}} + 2 \langle \nabla_{\mathbb{H}} \phi, (Hu)^T \nabla_{\mathbb{H}} u \rangle_{\mathbb{H}} \phi + | \nabla_{\mathbb{H}} \phi |^2 | \nabla_{\mathbb{H}} u |^2 \\
&\quad = \phi^2 \langle (Hu)^2 \nu, \nu \rangle_{\mathbb{H}} + 2 \langle Hu \nabla_{\mathbb{H}} \phi, \nabla_{\mathbb{H}} u \rangle_{\mathbb{H}} \phi + | \nabla_{\mathbb{H}} \phi |^2 | \nabla_{\mathbb{H}} u |^2 . \qquad (20)
\end{aligned}
$$

By exploiting Lemma 1.2 with $v = |\nabla_{\mathbb{H}} u|$, we obtain that $\nabla_{\mathbb{H}}(| \nabla_{\mathbb{H}} u | \phi) = 0$ almost everywhere outside $\mathbb{H}_o$. Analogously, using Lemma 1.2 with $v = Xu$ or $v = Yu$, we conclude that $\nabla_{\mathbb{H}} Xu = \nabla_{\mathbb{H}} Yu = 0$ almost everywhere outside $\mathbb{H}_o$.

Thus, plugging (20) in (15) we get

$$
\begin{aligned}
0 \le &\int_{\mathbb{H}_o} \left(\phi^2 \langle (Hu)^2 \nu, \nu \rangle_{\mathbb{H}} + 2 \langle Hu \nabla_{\mathbb{H}} \phi, \nabla_{\mathbb{H}} u \rangle_{\mathbb{H}} \phi + | \nabla_{\mathbb{H}} \phi |^2 | \nabla_{\mathbb{H}} u |^2 \right) \\
&+ \int_{\mathbb{H}} f'(u) \, | \nabla_{\mathbb{H}} u |^2 \, \phi^2. \qquad (21)
\end{aligned}
$$

Now recalling (14), it follows from (21) that

$$
\begin{aligned}
0 \le &\int_{\mathbb{H}_o} \left(\langle (Hu)^2 \nu, \nu \rangle_{\mathbb{H}} \phi^2 + 2 \langle Hu \nabla_{\mathbb{H}} \phi, \nabla_{\mathbb{H}} u \rangle_{\mathbb{H}} \phi + | \nabla_{\mathbb{H}} \phi |^2 | \nabla_{\mathbb{H}} u |^2 \right) \\
&- \int_{\mathbb{H}_o} | \nabla_{\mathbb{H}} Xu |^2 \, \phi^2 - \int_{\mathbb{H}_o} | \nabla_{\mathbb{H}} Yu |^2 \, \phi^2 \\
&- \int_{\mathbb{H}_o} \langle \nabla_{\mathbb{H}} Xu, \nabla_{\mathbb{H}} (\phi^2) \rangle_{\mathbb{H}} Xu - \int_{\mathbb{H}_o} \langle \nabla_{\mathbb{H}} Yu, \nabla_{\mathbb{H}} (\phi^2) \rangle_{\mathbb{H}} Yu + 2 \int_{\mathbb{H}_o} (TYuXu - TXuYu) \, \phi^2.
\end{aligned}
$$

Rearranging the terms, we get

$$0 \le \int\limits_{\mathbb{H}_o} \Big(\langle (Hu)^2\nu, \nu\rangle_{\mathbb{H}}\phi^2 + \mid \nabla_{\mathbb{H}}\phi \mid^2 \mid \nabla_{\mathbb{H}} u \mid^2\Big) - \int\limits_{\mathbb{H}_o} \mid \nabla_{\mathbb{H}} Xu \mid^2 \phi^2 - \int\limits_{\mathbb{H}_o} \mid \nabla_{\mathbb{H}} Yu \mid^2 \phi^2 + 2\int\limits_{\mathbb{H}_o} (TYuXu - TXuYu)\,\phi^2.$$

This is equivalent to

$$\int\limits_{\mathbb{H}_o} \Big(\mid \nabla_{\mathbb{H}} Xu \mid^2 + \mid \nabla_{\mathbb{H}} Yu \mid^2 - \langle (Hu)^2\nu, \nu\rangle_{\mathbb{H}} - 2\,(TYuXu - TXuYu)\Big)\,\phi^2 \le \int\limits_{\mathbb{H}} \mid \nabla_{\mathbb{H}}\phi \mid^2 \mid \nabla_{\mathbb{H}} u \mid^2 .$$

This, recalling (3), gives (13). □

2 Geometric computations

Having obtained the first part of (7) in the previous Theorem 1.3, our aim is now to obtain the second part. For this, we perform a geometric analysis of the level sets of u at nondegenerate points P where $\{\nabla_{\mathbb{H}} u \neq 0\}$. We denote by $\nabla^{\mathbb{H}}_{v}\nu_{\{u=k\}}(P)$ the intrinsic Weingarten map at the point P associated with the noncharacteristic smooth surface $\{u = k\}$. Let v be the unit tangent vector at $P \in \{u = k\}$, as defined in (2).

Moreover, let h be the mean curvature and p the imaginary curvature on the level surface $\{u = k\}$ (recall (8) and (9)).

We also define

$$H\nu = \begin{bmatrix} X\nu_1, & Y\nu_1 \\ X\nu_2, & Y\nu_2 \end{bmatrix}. \tag{22}$$

Then (see [2,3]), at any $P \in \{u = k\} \cap \{\nabla_{\mathbb{H}} u \neq 0\}$, we have that

$$\nabla^{\mathbb{H}}_{v}\nu_{\{u=k\}}(P) = H\nu v = -hv. \tag{23}$$

Lemma 2.1 *On* $\{u = k\} \cap \{\nabla_{\mathbb{H}} u \neq 0\}$,

$$\mid Hu \mid^2 - \langle (Hu)^2\nu, \nu\rangle_{\mathbb{H}} = \mid \nabla_{\mathbb{H}} u \mid^2 \left[h^2 + \left(p + \frac{\langle Huv, \nu\rangle_{\mathbb{H}}}{\mid \nabla_{\mathbb{H}} u \mid}\right)^2\right]. \tag{24}$$

Proof From (1) and (2),

$$\nu_1 = -v_2 \ \text{ and } \ \nu_2 = v_1. \tag{25}$$

Thus, $v \in \mathcal{H}_P$ is the unit tangent vector to $\{u = k\}$, namely the orthonormal vector to ν on $\mathcal{H}_P$. Then,

$$\nabla_{\mathbb{H}} Xu = \langle \nabla_{\mathbb{H}} Xu, \nu \rangle_{\mathbb{H}} \nu + \langle \nabla_{\mathbb{H}} Xu, v \rangle_{\mathbb{H}} v$$

and

$$\nabla_{\mathbb{H}} Yu = \langle \nabla_{\mathbb{H}} Yu, \nu \rangle_{\mathbb{H}} \nu + \langle \nabla_{\mathbb{H}} Yu, v \rangle_{\mathbb{H}} v.$$

Hence

$$|\nabla_{\mathbb{H}} Xu|^2 = |\langle \nabla_{\mathbb{H}} Xu, \nu \rangle_{\mathbb{H}} \nu + \langle \nabla_{\mathbb{H}} Xu, v \rangle_{\mathbb{H}} v|^2 = \langle \nabla_{\mathbb{H}} Xu, \nu \rangle_{\mathbb{H}}^2 + \langle \nabla_{\mathbb{H}} Xu, v \rangle_{\mathbb{H}}^2$$

and

$$|\nabla_{\mathbb{H}} Yu|^2 = |\langle \nabla_{\mathbb{H}} Yu, \nu \rangle_{\mathbb{H}} \nu + \langle \nabla_{\mathbb{H}} Yu, v \rangle_{\mathbb{H}} v|^2 = \langle \nabla_{\mathbb{H}} Yu, \nu \rangle_{\mathbb{H}}^2 + \langle \nabla_{\mathbb{H}} Yu, v \rangle_{\mathbb{H}}^2.$$

By developing the calculation, we come up with

$$\begin{aligned}
|Hu|^2 &= |\nabla_{\mathbb{H}} Xu|^2 + |\nabla_{\mathbb{H}} Yu|^2 \\
&= (XXu)^2 \nu_1^2 + (YXu)^2 \nu_2^2 + 2XXuYXu\nu_1\nu_2 \\
&\quad + (XXu)^2 v_1^2 + (YXu)^2 v_2^2 + 2XXuYXuv_1v_2 \\
&\quad + (XYu)^2 \nu_1^2 + (YYu)^2 \nu_2^2 + 2XYuYYu\nu_1\nu_2 \\
&\quad + (XYu)^2 v_1^2 + (YYu)^2 v_2^2 + 2XYuYYuv_1v_2 \\
&= (XXu)^2 \nu_1^2 + (YXu)^2 \nu_2^2 + (XXu)^2 v_1^2 + (YXu)^2 v_2^2 \\
&\quad + (XYu)^2 \nu_1^2 + (YYu)^2 \nu_2^2 + (XYu)^2 v_1^2 + (YYu)^2 v_2^2, \qquad (26)
\end{aligned}$$

due to (3) and (25).

On the other hand, recalling (16), (17) and (19), we obtain that

$$\begin{aligned}
\langle (Hu)^2 \nu, \nu \rangle_{\mathbb{H}} &= |\ \nabla_{\mathbb{H}}\ |\ \nabla_{\mathbb{H}} u\ ||^2 \\
&= (XXu)^2 \nu_1^2 + (XYu)^2 \nu_2^2 + 2XXuXYu\nu_1\nu_2 \\
&\quad + (YYu)^2 \nu_2^2 + (YXu)^2 \nu_1^2 + 2YYuYXu\nu_1\nu_2. \qquad (27)
\end{aligned}$$

Hence, making use of (25), (26) and (27), we conclude that the left hand side of (24), which we denote by LHS, equals

$$\begin{aligned}
\text{LHS} &= (XXu)^2 v_1^2 - 2XXuXYuv_1v_2 + (XYu)^2 v_1^2 \\
&\quad + (YYu)^2 v_2^2 + (YXu)^2 v_2^2 - 2YYuYXuv_1v_2 \\
&= ((XXu)v_1 + (XYu)v_2)^2 + ((YXu)v_1 + (YYu)v_2)^2 \\
&= |\ (Hu)^T v\ |^2\ . \qquad (28)
\end{aligned}$$

On the other hand, from (22),

$$\begin{aligned} &| \nabla_{\mathbb{H}} u \mid H\nu + \frac{1}{| \nabla_{\mathbb{H}} u |} \begin{bmatrix} Xu\langle X\nabla_{\mathbb{H}} u, \nu\rangle_{\mathbb{H}}, & Xu\langle Y\nabla_{\mathbb{H}} u, \nu\rangle_{\mathbb{H}} \\ Yu\langle X\nabla_{\mathbb{H}} u, \nu\rangle_{\mathbb{H}}, & Yu\langle Y\nabla_{\mathbb{H}} u, \nu\rangle_{\mathbb{H}} \end{bmatrix} \\ &\quad = Hu \\ &\quad = \begin{bmatrix} 0, & -Tu \\ Tu, & 0 \end{bmatrix} + (Hu)^T. \end{aligned}$$

This and (23) imply that

$$\begin{aligned} (Hu)^T v = &- \begin{bmatrix} 0, & -Tu \\ Tu, & 0 \end{bmatrix} v - | \nabla_{\mathbb{H}} u \mid hv \\ &+ \frac{1}{| \nabla_{\mathbb{H}} u |} \begin{bmatrix} Xu\langle X\nabla_{\mathbb{H}} u, \nu\rangle_{\mathbb{H}}, & Xu\langle Y\nabla_{\mathbb{H}} u, \nu\rangle_{\mathbb{H}} \\ Yu\langle X\nabla_{\mathbb{H}} u, \nu\rangle_{\mathbb{H}}, & Yu\langle Y\nabla_{\mathbb{H}} u, \nu\rangle_{\mathbb{H}} \end{bmatrix} v. \end{aligned}$$

Hence, from (28), we obtain

$$\begin{aligned} \text{LHS} = &\langle (Hu)^T v, (Hu)^T v\rangle_{\mathbb{H}} = (Tu)^2 + h^2 \mid \nabla_{\mathbb{H}} u \mid^2 \\ &- 2h\langle \begin{bmatrix} Xu\langle X\nabla_{\mathbb{H}} u, \nu\rangle_{\mathbb{H}}, & Xu\langle Y\nabla_{\mathbb{H}} u, \nu\rangle_{\mathbb{H}} \\ Yu\langle X\nabla_{\mathbb{H}} u, \nu\rangle_{\mathbb{H}}, & Yu\langle Y\nabla_{\mathbb{H}} u, \nu\rangle_{\mathbb{H}} \end{bmatrix} v, v\rangle_{\mathbb{H}} \\ &+ \frac{1}{| \nabla_{\mathbb{H}} u |^2} \langle \begin{bmatrix} Xu\langle X\nabla_{\mathbb{H}} u, \nu\rangle_{\mathbb{H}}, & Xu\langle Y\nabla_{\mathbb{H}} u, \nu\rangle_{\mathbb{H}} \\ Yu\langle X\nabla_{\mathbb{H}} u, \nu\rangle_{\mathbb{H}}, & Yu\langle Y\nabla_{\mathbb{H}} u, \nu\rangle_{\mathbb{H}} \end{bmatrix} v, \\ &\begin{bmatrix} Xu\langle X\nabla_{\mathbb{H}} u, \nu\rangle_{\mathbb{H}}, & Xu\langle Y\nabla_{\mathbb{H}} u, \nu\rangle_{\mathbb{H}} \\ Yu\langle X\nabla_{\mathbb{H}} u, \nu\rangle_{\mathbb{H}}, & Yu\langle Y\nabla_{\mathbb{H}} u, \nu\rangle_{\mathbb{H}} \end{bmatrix} v\rangle_{\mathbb{H}} \\ &- \frac{2}{| \nabla_{\mathbb{H}} u |} \langle \begin{bmatrix} 0, & -Tu \\ Tu, & 0 \end{bmatrix} v, \begin{bmatrix} Xu\langle X\nabla_{\mathbb{H}} u, \nu\rangle_{\mathbb{H}}, & Xu\langle Y\nabla_{\mathbb{H}} u, \nu\rangle_{\mathbb{H}} \\ Yu\langle X\nabla_{\mathbb{H}} u, \nu\rangle_{\mathbb{H}}, & Yu\langle Y\nabla_{\mathbb{H}} u, \nu\rangle_{\mathbb{H}} \end{bmatrix} v\rangle_{\mathbb{H}}. \end{aligned} \tag{29}$$

Now let us notice that

$$\begin{aligned} &\langle \begin{bmatrix} Xu\langle X\nabla_{\mathbb{H}} u, \nu\rangle_{\mathbb{H}}, & Xu\langle Y\nabla_{\mathbb{H}} u, \nu\rangle_{\mathbb{H}} \\ Yu\langle X\nabla_{\mathbb{H}} u, \nu\rangle_{\mathbb{H}}, & Yu\langle Y\nabla_{\mathbb{H}} u, \nu\rangle_{\mathbb{H}} \end{bmatrix} v, v\rangle_{\mathbb{H}} \\ &\quad = \langle (v_1 \langle X\nabla_{\mathbb{H}} u, \nu\rangle_{\mathbb{H}} + v_2 \langle Y\nabla_{\mathbb{H}} u, \nu\rangle_{\mathbb{H}})\, \nabla_{\mathbb{H}} u, v\rangle_{\mathbb{H}} = 0. \end{aligned} \tag{30}$$

Furthermore, from (25), we have

$$\begin{aligned} &\langle \begin{bmatrix} 0, & -Tu \\ Tu, & 0 \end{bmatrix} v, \begin{bmatrix} Xu\langle X\nabla_{\mathbb{H}} u, \nu\rangle_{\mathbb{H}}, & Xu\langle Y\nabla_{\mathbb{H}} u, \nu\rangle_{\mathbb{H}} \\ Yu\langle X\nabla_{\mathbb{H}} u, \nu\rangle_{\mathbb{H}}, & Yu\langle Y\nabla_{\mathbb{H}} u, \nu\rangle_{\mathbb{H}} \end{bmatrix} v\rangle_{\mathbb{H}} \\ &\quad = Tu\, \langle \nu, (v_1 \langle X\nabla_{\mathbb{H}} u, \nu\rangle_{\mathbb{H}} + v_2 \langle Y\nabla_{\mathbb{H}} u, \nu\rangle_{\mathbb{H}})\, \nabla_{\mathbb{H}} u\rangle_{\mathbb{H}} = Tu \mid \nabla_{\mathbb{H}} u \mid \langle (Hu)^T \nu, v\rangle_{\mathbb{H}} \end{aligned}$$

and

$$\langle\begin{bmatrix} Xu\langle X\nabla_{\mathbb{H}}u, \nu\rangle_{\mathbb{H}}, & Xu\langle Y\nabla_{\mathbb{H}}u, \nu\rangle_{\mathbb{H}} \\ Yu\langle X\nabla_{\mathbb{H}}u, \nu\rangle_{\mathbb{H}}, & Yu\langle Y\nabla_{\mathbb{H}}u, \nu\rangle_{\mathbb{H}} \end{bmatrix}\upsilon, \begin{bmatrix} Xu\langle X\nabla_{\mathbb{H}}u, \nu\rangle_{\mathbb{H}}, & Xu\langle Y\nabla_{\mathbb{H}}u, \nu\rangle_{\mathbb{H}} \\ Yu\langle X\nabla_{\mathbb{H}}u, \nu\rangle_{\mathbb{H}}, & Yu\langle Y\nabla_{\mathbb{H}}u, \nu\rangle_{\mathbb{H}} \end{bmatrix}\upsilon\rangle_{\mathbb{H}}$$
$$=| \nabla_{\mathbb{H}}u |^2 (\upsilon_1\langle X\nabla_{\mathbb{H}}u, \nu\rangle_{\mathbb{H}} + \upsilon_2\langle Y\nabla_{\mathbb{H}}u, \nu\rangle_{\mathbb{H}})^2 =| \nabla_{\mathbb{H}}u |^2 \langle (Hu)^T\nu, \upsilon\rangle_{\mathbb{H}}^2.$$

Consequently, making use of (9),

$$\frac{(Tu)^2}{| \nabla_{\mathbb{H}}u |^2} - \frac{2}{| \nabla_{\mathbb{H}}u |^3}\langle\begin{bmatrix} 0, & -Tu \\ Tu, & 0 \end{bmatrix}\upsilon, \begin{bmatrix} Xu\langle X\nabla_{\mathbb{H}}u, \nu\rangle_{\mathbb{H}}, & Xu\langle Y\nabla_{\mathbb{H}}u, \nu\rangle_{\mathbb{H}} \\ Yu\langle X\nabla_{\mathbb{H}}u, \nu\rangle_{\mathbb{H}}, & Yu\langle Y\nabla_{\mathbb{H}}u, \nu\rangle_{\mathbb{H}} \end{bmatrix}\upsilon\rangle_{\mathbb{H}}$$
$$+\frac{1}{| \nabla_{\mathbb{H}}u |^4}\langle\begin{bmatrix} Xu\langle X\nabla_{\mathbb{H}}u, \nu\rangle_{\mathbb{H}}, & Xu\langle Y\nabla_{\mathbb{H}}u, \nu\rangle_{\mathbb{H}} \\ Yu\langle X\nabla_{\mathbb{H}}u, \nu\rangle_{\mathbb{H}}, & Yu\langle Y\nabla_{\mathbb{H}}u, \nu\rangle_{\mathbb{H}} \end{bmatrix}\upsilon,$$
$$\begin{bmatrix} Xu\langle X\nabla_{\mathbb{H}}u, \nu\rangle_{\mathbb{H}}, & Xu\langle Y\nabla_{\mathbb{H}}u, \nu\rangle_{\mathbb{H}} \\ Yu\langle X\nabla_{\mathbb{H}}u, \nu\rangle_{\mathbb{H}}, & Yu\langle Y\nabla_{\mathbb{H}}u, \nu\rangle_{\mathbb{H}} \end{bmatrix}\upsilon\rangle_{\mathbb{H}}$$
$$= p^2 + \frac{2p\langle (Hu)^T\nu, \upsilon\rangle_{\mathbb{H}}}{| \nabla_{\mathbb{H}}u |} + \frac{\langle (Hu)^T\nu, \upsilon\rangle_{\mathbb{H}}^2}{| \nabla_{\mathbb{H}}u |^2}$$
$$= \left(p + \frac{\langle (Hu)^T\nu, \upsilon\rangle_{\mathbb{H}}}{| \nabla_{\mathbb{H}}u |}\right)^2$$
$$= \left(p + \frac{\langle Hu\upsilon, \nu\rangle_{\mathbb{H}}}{| \nabla_{\mathbb{H}}u |}\right)^2.$$

From this, (29) and (30), we obtain (24). □

We are now in the position of relating interesting analytical and geometrical objects, in the following way:

Lemma 2.2 *On* $\{u = k\} \cap \{\nabla_{\mathbb{H}}u \neq 0\}$,

$$TYuXu - TXuYu = -| \nabla_{\mathbb{H}}u |^2 \langle T\nu, \upsilon\rangle_{\mathbb{H}}. \tag{31}$$

Proof By a straightforward calculation (see also [19]), it follows that

$$\langle T\nu, \upsilon\rangle_{\mathbb{H}} = \frac{YuTXu - XuTYu}{| \nabla_{\mathbb{H}}u |^2}.$$

This implies (31). □

Theorem 2.3 *For any* $\phi \in C_0^\infty(\Omega)$

$$\int_{\Omega\cap\{\nabla_{\mathbb{H}}u\neq 0\}} | \nabla_{\mathbb{H}}u |^2\left[h^2 + \left(p+\frac{\langle Hu\upsilon, \nu\rangle_{\mathbb{H}}}{| \nabla_{\mathbb{H}}u |}\right)^2 +2\langle T\nu, \upsilon\rangle_{\mathbb{H}}\right]\phi^2 \leq \int_{\Omega} | \nabla_{\mathbb{H}}\phi |^2 | \nabla_{\mathbb{H}}u |^2.$$

Proof The desired claim follows from Theorem 1.3 and Lemma 2.2. □

We remark that the general form of our geometric inequality given in (7) is a consequence of Theorem 1.3, Lemma 2.1, Lemma 2.2 and Theorem 2.3.

3 Application to entire stable solutions

We now apply Theorem 2.3 in order to deduce suitable geometric properties of global stable solutions.

For this, given $\xi = (x, y, t) \in \mathbb{H}$ we define its gauge norm as

$$|\xi| = \left((x^2 + y^2)^2 + t^2\right)^{1/4}.$$

Analogously, the gauge ball centered at 0 of radius R will be denoted by $B(0, R)$, that is we set

$$B(0, R) = \{\xi \in \mathbb{H} \text{ s.t. } |\xi| < R\}.$$

Lemma 3.1 *Let* $g \in L^\infty_{\text{loc}}(\mathbb{R}^n, [0, +\infty))$ *and let* $q > 0$. *Let also, for any* $\tau > 0$,

$$\eta(\tau) = \int\limits_{B(0,\tau)} g(\xi) d\xi. \tag{32}$$

Then, for every $0 < r < R$,

$$\int\limits_{B(0,R)\setminus B(0,r)} \frac{g(\xi)}{|\xi|^q} d\xi \leq q \int_r^R \frac{\eta(\tau)}{\tau^{q+1}} d\tau + \frac{1}{R^q}\eta(R).$$

Proof By changing order of integration,

$$\begin{aligned}
&\int\limits_{B(0,R)\setminus B(0,r)} \frac{g(\xi)}{|\xi|^q} d\xi \\
&= q \int\limits_{B(0,R)\setminus B(0,r)} \left(\int_{|\xi|}^R \frac{g(\xi)}{\tau^{q+1}} d\tau \right) d\xi + \frac{1}{R^q} \int\limits_{B(0,R)\setminus B(0,r)} g(\xi) d\xi \\
&\leq q \int_r^R \left(\int\limits_{B(0,\tau)} \frac{g(\xi)}{\tau^{q+1}} d\xi \right) d\tau + \frac{1}{R^q}\eta(R) \\
&\leq q \int_r^R \frac{\eta(\tau)}{\tau^{q+1}} d\tau + \frac{1}{R^q}\eta(R).
\end{aligned}$$

□

Corollary 3.2 *Let us assume that u is a stable solution of* $\Delta_{\mathbb{H}} u = f(u)$ *in the whole* $\mathbb{H}$ *such that*

$$\langle T\nu, \nu\rangle_{\mathbb{H}} \geq 0. \tag{33}$$

For any $\tau > 0$*, set*

$$\eta(\tau) = 4 \int\limits_{B(0,\tau)} |\nabla_{\mathbb{H}} u(x, y, t)|^2 (x^2 + y^2)\, d(x, y, t). \tag{34}$$

Suppose that

$$\liminf_{R\to\infty} \frac{\int_{\sqrt{R}}^{R} \frac{\eta(\tau)}{\tau^5}\, d\tau + \frac{\eta(R)}{R^4}}{(\log R)^2} = 0. \tag{35}$$

Then, the level sets of u in the vicinity of noncharacteristic points are minimal surfaces in the Heisenberg group (i.e., the curvature h vanishes identically) and on such surfaces the following equation holds

$$p = -\frac{1}{|\nabla_{\mathbb{H}} u|}\langle Hu\nu, \nu\rangle_{\mathbb{H}}. \tag{36}$$

Proof Let

$$g(\xi) = 4\, |\nabla_{\mathbb{H}} u(\xi)|^2 (x^2 + y^2).$$

Then, the function η defined in (34) is consistent with the notation in (32).

Therefore, Lemma 3.1 and (35) imply that

$$\liminf_{R\to\infty} \frac{1}{(\log R)^2} \int\limits_{B(0,R)\setminus B(0,\sqrt{R})} \frac{g(\xi)}{|\xi|^4} d\xi = 0. \tag{37}$$

Now, let us define the following test function for all positive R:

$$\phi(\xi) = \begin{cases} 1, & \text{if } \xi \in B(0, \sqrt{R}) \\ 0, & \text{if } \xi \in \mathbb{H}\setminus B(0, R) \\ \frac{2\log\left(\frac{R}{|\xi|}\right)}{\log R}, & \text{if } \xi \in B(0, R)\setminus B(0, \sqrt{R}). \end{cases}$$

Then

$$|X\phi| = \tfrac{2}{\log R}|\xi|^{-4}|(x^2 + y^2)x + yt|$$

and

$$|Y\phi| = \frac{2}{\log R}|\xi|^{-4}|(x^2 + y^2)y - xt|.$$

Therefore, for $\xi \in B(0, R)\backslash B(0, \sqrt{R})$,

$$
\begin{aligned}
|\nabla_{\mathbb{H}}\phi(\xi)|^2 &= \frac{4}{(\log R)^2} |\xi|^{-8} \left(((x^2+y^2)x+yt)^2 + ((x^2+y^2)y - xt)^2\right) \\
&= \frac{4}{(\log R)^2} \frac{(x^2+y^2)}{|\xi|^4}.
\end{aligned}
$$

So,

$$
\int_{\mathbb{H}} |\nabla_{\mathbb{H}}\phi|^2 |\nabla_{\mathbb{H}} u|^2 = \frac{1}{(\log R)^2} \int_{B(0,R)\backslash B(0,\sqrt{R})} \frac{g(\xi)}{|\xi|^4} d\xi.
$$

Hence, the claim follows from Theorem 2.3, (33) and (37). □

Remark 3.3 We observe that (35) may be seen as a condition on the growth of a suitably weighted energy η.

Notice also that if, for any R large enough,

$$
\eta(R) \leq C R^4,
$$

for some constant $C > 0$, then (35) is satisfied.

Remark 3.4 Bounded stable solutions that do not depend on t do not satisfy (35) unless they are constant.

Indeed such solutions would satisfy $\Delta u = f(u)$ in $\mathbb{R}^2$ and so, by well known results (see, e.g., [1,6,17,22]) they depend on only one variable, up to rotation.

That is, $u(x, y) = u_o(\omega_1 x + \omega_2 y)$, for any $(x, y) \in \mathbb{R}^2$, where $\omega = (\omega_1, \omega_2) \in S^1$.

As a consequence, if u were not constant, there would exist $\epsilon > 0$ and an open interval $I \subset \mathbb{R}$ for which $|u_o'(\theta)| \geq \epsilon$ for any $\theta \in I$.

Accordingly, by changing variables $\tilde{x} = \omega_1 x + \omega_2 y$, $\tilde{y} = -\omega_2 x + \omega_1 y$, we obtain

$$
\begin{aligned}
&\int_{B(0,\tau)} |\nabla_{\mathbb{H}} u(x, y)|^2 (x^2+y^2) d(x, y, t) \\
&\geq \int_{\{x^2+y^2 \leq \sqrt{\tau^4/2}\}} \int_{\{|t| \leq \sqrt{\tau^4/2}\}} |\nabla_{\mathbb{H}} u(x, y)|^2 (x^2+y^2)\, dt\, d(x, y) \\
&= C_0 \tau^2 \int_{\{x^2+y^2 \leq \sqrt{\tau^4/2}\}} |u_o'(\omega_1 x + \omega_2 y)|^2 (x^2+y^2)\, d(x, y) \\
&= C_0 \tau^2 \int_{\{\tilde{x}^2+\tilde{y}^2 \leq \sqrt{\tau^4/2}\}} |u_o'(\tilde{x})|^2 (\tilde{x}^2+\tilde{y}^2)\, d(\tilde{x}, \tilde{y}) \\
&\geq C_0 \tau^2 \int_{\{\tilde{x}^2+\tilde{y}^2 \leq \sqrt{\tau^4/2}\}} |u_o'(\tilde{x})|^2 \tilde{y}^2\, d(\tilde{x}, \tilde{y}),
\end{aligned}
$$

for some $C_0 > 0$, and so, when τ is large

$$\begin{aligned}
&\int_{B(0,\tau)} |\nabla_{\mathbb{H}} u(x,y)|^2 (x^2+y^2)\, d(x,y,t) \\
&\geq C_0 \tau^2 \int_{\{|\tilde{y}|\leq \sqrt[4]{\tau^4/4}\}} \int_{\tilde{x}\in I} |u_o'(\tilde{x})|^2 \tilde{y}^2\, d\tilde{x}\, d\tilde{y} \\
&\geq C_0 \epsilon^2\, |I|\, \tau^2 \int_{\{|\tilde{y}|\leq \sqrt[4]{\tau^4/4}\}} \tilde{y}^2\, d\tilde{y} \\
&\geq C_1 \epsilon^2\, |I|\, \tau^5,
\end{aligned}$$

which cannot be compatible with (35).

Remark 3.5 The minimal surface condition $h = 0$ implies that the level set of any noncharacteristic point contains a horizonal straight segment (see Lemma 4.1 and Theorem 4.1 in [3]). In particular, when $\{\nabla_{\mathbb{H}} u = 0\} = \emptyset$ (or, at least, when the level set does not contain characteristic points), it follows that level sets satifying $h = 0$ are ruled surfaces, that is, each point has a horizonal straight line passing through it.

Remark 3.6 Inspired by rigidity properties developed in the Euclidean setting after a conjecture of De Giorgi (see [1,6,15,22]), one may wonder whether the minimal surface condition $h = 0$, possibly together with (36), implies that stable solutions boil down to one-dimensional Euclidean ones (this and Remark 3.4 would imply that such solutions are, in fact, constant).

Analogously, one may wonder whether the minimal surface condition $h = 0$ together with (36) implies, locally, that the level surface is a flat Euclidean plane.

To see that additional assumptions are needed to obtain such a result, one may consider the simple example given by $u(x, y, t) = t - 2xy$. Indeed, such u satisfies $\Delta_{\mathbb{H}} u = 0$, and so it is a stable solution of a semilinear equation, the mean curvature h of any level set $\{u = k\}$ vanishes and (36) is satisfied on $\{x \neq 0\}$, though u does depend on all the variables $(x, y, t) \in \mathbb{H}$ and its level sets are not flat Euclidean planes.

In our framework, it is natural to define the horizontal derivative of $| \nabla_{\mathbb{H}} u |$ along the unit tangent direction v at P as

$$D_v^{\mathbb{H}} |\nabla_{\mathbb{H}} u| = \langle \nabla_{\mathbb{H}} \mid \nabla_{\mathbb{H}} u \mid, v\rangle_{\mathbb{H}}. \tag{38}$$

So, recalling (18), we get

$$D_v^{\mathbb{H}} \mid \nabla_{\mathbb{H}} u \mid = \frac{1}{\mid \nabla_{\mathbb{H}} u \mid} \langle (Hu)^T \nabla_{\mathbb{H}} u, v\rangle_{\mathbb{H}} = \langle (Hu)^T \nu, v\rangle_{\mathbb{H}} = \langle Hu v, \nu\rangle_{\mathbb{H}}. \tag{39}$$

This equation allows us to rewrite (7) as

$$\int_{\Omega} |\nabla_{\mathbb{H}}\phi|^2 |\nabla_{\mathbb{H}}u|^2 \geq \int_{\Omega\cap\{\nabla_{\mathbb{H}}u\neq 0\}} |\nabla_{\mathbb{H}}u|^2 \left[h^2 + \left(p + \frac{D_v^{\mathbb{H}} |\nabla_{\mathbb{H}}u|}{|\nabla_{\mathbb{H}}u|} \right)^2 + 2\langle T\nu, v\rangle_{\mathbb{H}} \right] \phi^2, \quad (40)$$

for any $\phi \in C_0^\infty(\Omega)$.

Notice that, if u does not depend on t, inequality (40) reduces to the Euclidean one in [26,27].

Analogously, one may rewrite (36) as

$$p = -\frac{D_v^{\mathbb{H}} |\nabla_{\mathbb{H}}u|}{|\nabla_{\mathbb{H}}u|}. \quad (41)$$

Keeping in mind the discussion in Remark 3.6, we may then state a nonexistence result:

Theorem 3.7 *There exists no u which is a $C^2(\mathbb{H})$ stable solution of $\Delta_{\mathbb{H}}u = f(u)$ satisfying*

$$\{\nabla_{\mathbb{H}}u = 0\} = \emptyset, \quad (42)$$

$$u \in L^\infty(\mathbb{H}), \quad (43)$$

$$\langle T\nu, v\rangle_{\mathbb{H}} \geq 0 \quad (44)$$

and

$$\liminf_{R\to\infty} \frac{\int_{\sqrt{R}}^{R} \frac{\eta(\tau)}{\tau^5}\, d\tau + \frac{\eta(R)}{R^4}}{(\log R)^2} = 0, \quad (45)$$

where η is as in (32).

Proof We argue by contradition, considering u satisfying the above properties.

So, by (43) and Theorem 2.10 in [13],

$$|\nabla_{\mathbb{H}}u| \in L^\infty(\mathbb{H}). \quad (46)$$

We claim that

$$u(x, y, t) \text{ does not depend on } t. \quad (47)$$

For this, we argue again by contradiction, supposing that there exists $P_o \in \mathbb{H}$ for which

$$u_t(P_o) \neq 0. \quad (48)$$

We thus consider the solution $\phi : \mathbb{R} \to \mathbb{H}$ of the Cauchy problem

$$\begin{cases} \phi'(s) = v(\phi(s)) \\ \phi(0) = P_o \end{cases}$$

Note that such solution is global, that is, it is defined for any $s \in \mathbb{R}$, because v is always well defined, due to (42), and it is bounded, having norm 1 in $\mathcal{H}_P$.

Since ϕ' is horizontal, that is $\phi' \in \mathcal{H}_P$, we thus have

$$\frac{d}{ds}u(\phi(s)) = \langle \nabla_{\mathbb{H}} u(\phi(s)),\ \phi'(s)\rangle_{\mathbb{H}} = |\nabla_{\mathbb{H}} u(\phi(s))|\, \langle \nu(\phi(s)), v(\phi(s))\rangle_{\mathbb{H}} = 0,$$

thus ϕ lies on the level set $\{u = u(P_o)\}$.

Moreover,

$$\frac{d}{ds} \mid \nabla_{\mathbb{H}} u(\phi(s)) \mid = \langle \nabla_{\mathbb{H}} |\nabla_{\mathbb{H}} u|(\phi(s)),\ \phi'(s)\rangle_{\mathbb{H}} = \left(D_v^{\mathbb{H}} \mid \nabla_{\mathbb{H}} u \mid\right)(\phi(s)), \tag{49}$$

due to (38).

Also, using (44), (45), Corollary 3.2 and (41), we deduce that

$$Tu(\phi(s)) = \left(D_v^{\mathbb{H}} \mid \nabla_{\mathbb{H}} u \mid\right)(\phi(s)). \tag{50}$$

Therefore, by (49) and (50),

$$\frac{d}{ds} \mid \nabla_{\mathbb{H}} u(\phi(s)) \mid = -4u_t(\phi(s)), \tag{51}$$

which, via (48), gives that

$$\frac{d}{ds} \mid \nabla_{\mathbb{H}} u(\phi(s)) \mid \Big|_{s=0} \neq 0. \tag{52}$$

Furthermore, from (51),

$$\begin{aligned} \frac{d^2 \mid \nabla_{\mathbb{H}} u(\phi(s)) \mid}{ds^2} &= \frac{d}{ds} Tu(\phi(s)) \\ &= \langle \nabla_{\mathbb{H}} Tu(\phi(s)), \phi'(s)\rangle_{\mathbb{H}} = \langle \nabla_{\mathbb{H}} Tu(\phi(s)), v(\phi(s))\rangle_{\mathbb{H}}. \end{aligned} \tag{53}$$

Since

$$\langle \nabla_{\mathbb{H}} Tu, v\rangle_{\mathbb{H}} = \langle T(|\nabla_{\mathbb{H}} u|\nu), v\rangle_{\mathbb{H}} = \langle |\nabla_{\mathbb{H}} u| T\nu, v\rangle_{\mathbb{H}},$$

because ν and v are orthogonal with respect to $\langle ., .\rangle_{\mathbb{H}}$, we deduce from (53) and (44) that

$$\frac{d^2 \mid \nabla_{\mathbb{H}} u(\phi(s)) \mid}{ds^2} = |\nabla_{\mathbb{H}} u| \langle T\nu(\phi(s)), v(\phi(s))\rangle_{\mathbb{H}} \geq 0. \tag{54}$$

Therefore, if we set

$$\Psi(s) = |\nabla_{\mathbb{H}} u(\phi(s))| - |\nabla_{\mathbb{H}} u(P_o)|,$$

we have that $\Psi \in C^2(\mathbb{R})$, $\Psi(0) = 0$, $\Psi'(0) \neq 0$ and $\Psi'' \geq 0$, thanks to (52) and (54).

Consequently,

$$\sup_{\mathbb{R}} \Psi = +\infty.$$

This is a contradiction with (46) and it thus proves (47).

Then, by (43), (45), (47) and Remark 3.4, we conclude that u is constant.

Since this cannot be, because of (42), we have obtained the contradiction which proves Theorem 3.7. □

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