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Weighted Poincaré inequality and heat kernel estimates for finite range jump processes

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Abstract It is well-known that there is a deep interplay between analysis and probability theory. For example, for a Markovian infinitesimal generator $\mathcal{L}$, the transition density function $p(t, x, y)$ of the Markov process associated with $\mathcal{L}$ (if it exists) is the fundamental solution (or heat kernel) of $\mathcal{L}$. A fundamental problem in analysis and in probability theory is to obtain sharp estimates of $p(t, x, y)$. In this paper, we consider a class of non-local (integro-differential) operators $\mathcal{L}$ on $\mathbb{R}^d$ of the form

$$\mathcal{L}u(x) = \lim_{\varepsilon \downarrow 0} \int_{\{y \in \mathbb{R}^d :\, |y-x| > \varepsilon\}} (u(y) - u(x)) J(x, y) dy,$$

where $J(x, y) = \dfrac{c(x, y)}{|x - y|^{d+\alpha}} \mathbf{1}_{\{|x-y| \le \kappa\}}$ for some constant $\kappa > 0$ and a measurable symmetric function $c(x, y)$ that is bounded between two positive constants. Associated

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with such a non-local operator $\mathcal{L}$ is an $\mathbb{R}^d$-valued symmetric jump process of finite range with jumping kernel $J(x, y)$. We establish sharp two-sided heat kernel estimate and derive parabolic Harnack principle for them. Along the way, some new heat kernel estimates are obtained for more general finite range jump processes that were studied in (Barlow et al. in Trans Am Math Soc, 2008). One of our key tools is a new form of weighted Poincaré inequality of fractional order, which corresponds to the one established by Jerison in (Duke Math J 53(2):503–523, 1986) for differential operators. Using Meyer's construction of adding new jumps, we also obtain various a priori estimates such as Hölder continuity estimates for parabolic functions of jump processes (not necessarily of finite range) where only a very mild integrability condition is assumed for large jumps. To establish these results, we employ methods from both probability theory and analysis extensively.

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1 Introduction and main results

The second order elliptic differential operators and diffusion processes take up, respectively, an central place in the theory of partial differential equations (PDE) and in probability theory, see [16,20] for example. There are close relationships between these two subjects. For a large class of second order elliptic differential operators $\mathcal{L}$ on $\mathbb{R}^d$, there is a diffusion process X on $\mathbb{R}^d$ associated with it so that $\mathcal{L}$ is the infinitesimal generator of X, and vice versa. The connection between $\mathcal{L}$ and X can also be seen as follows. The fundamental solution (also called heat kernel) for $\mathcal{L}$ is the transition density function of X.

Recently there are intense interests in studying discontinuous Markov processes, due to their importance both in theory and in application. See, for example, [5,21,26] and the references therein. The infinitesimal generator of an discontinuous Markov process in $\mathbb{R}^d$ is no longer a differential operator but rather a non-local (or, integro-differential) operator. For example, the infinitesimal generator of a isotropically symmetric α-stable process in $\mathbb{R}^d$ with $\alpha \in (0, 2)$ is a fractional Laplacian operator $c\,(-\Delta)^{\alpha/2}$. Recently there are also many interests from the theory of PDE (such as singular obstacle problems) to study non-local operators; see, for example, [7,31] and the references therein.

In this paper, we consider the following type of non-local (integro-differential) operators $\mathcal{L}$ on $\mathbb{R}^d$ with measurable symmetric kernel J:

$$\mathcal{L}u(x) = \lim_{\varepsilon\downarrow 0} \int\limits_{\{y\in\mathbb{R}^d:\, |y-x|>\varepsilon\}} (u(y) - u(x))J(x, y)dy,$$

where

$$J(x, y) = \frac{c(x, y)}{|x - y|^{d+\alpha}} \mathbf{1}_{\{|x-y|\le\kappa\}} \tag{1.1}$$

for some constant $\kappa > 0$ and a measurable symmetric function $c(x, y)$ that is bounded between two positive constants. Associated with such a non-local operator $\mathcal{L}$ is an $\mathbb{R}^d$-valued finite range symmetric jump process X with jumping kernel $J(x, y)$. We will be concerned with obtaining sharp two-sided heat kernel estimates for $\mathcal{L}$ (or, equivalently, for X), as well as establishing parabolic Harnack inequality and a priori joint Hölder continuity estimate for parabolic functions of $\mathcal{L}$. Our approach employs a combination of probabilistic and analytic techniques.

Two-sided heat kernel estimates for diffusions (or second order elliptic differential operators) have a long history and many beautiful results have been established. But two-sided heat kernel estimates for jump processes in $\mathbb{R}^d$ have only been studied recently. In [27], Kolokoltsov obtained two-sided heat kernel estimates for certain stable-like processes in $\mathbb{R}^d$, whose infinitesimal generators are a class of pseudo-differential operators having smooth symbols. Bass and Levin [4] used a completely different approach to obtain similar estimates for discrete time Markov chain on $\mathbb{Z}^d$ where the conductance between x and y is comparable to $|x-y|^{-d-\alpha}$ for $\alpha \in (0, 2)$. In [11], two-sided heat kernel estimates and a scale-invariant parabolic Harnack inequality for symmetric α-stable-like processes on d-sets are obtained. (See [18] for some extensions.) Very recently in [12], parabolic Harnack inequality and two-sided heat kernel estimates are even established for non-local operators of variable order. But so far the two-sided heat kernel estimates for non-local operators have been established only for the case that the jumping kernel has full support on the state space. See [1] for some results on parabolic Harnack inequality and heat kernel estimate for non-local operators of variable order on $\mathbb{R}^d$, whose jumping kernel is supported on jump size less than or equal to 1.

Throughout this paper, $d \geq 1$ and $\alpha \in (0, 2)$. Let the jump kernel J be defined by (1.1) and let

$$\mathcal{Q}(u, v) := \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(x) - u(y))(v(x) - v(y)) J(x, y) dx dy, \tag{1.2}$$

$$\mathcal{D} := \left\{ f \in L^2(\mathbb{R}^d, dx) : \mathcal{Q}(f, f) < \infty \right\}. \tag{1.3}$$

It is easy to check that $(\mathcal{Q}, \mathcal{D})$ is a regular Dirichlet form on $\mathbb{R}^d$ and so there is a Hunt process X associated with it. When the jumping kernel $J(x, y)$ is the unrestricted $\frac{c(x,y)}{|x-y|^{d+\alpha}}$, the associated process is the symmetric α-stable-like process Y on $\mathbb{R}^d$ studied in [11]. Among other things, it is shown in [11] that Y has Hölder continuous transition density function and so Y can be modified to start from every $x \in \mathbb{R}^d$. Since X can be constructed from Y by removing jumps of size larger than κ via Meyer's construction (see [1,3]), X is conservative and can be modified to start from every point in $\mathbb{R}^d$. For this reason, in the sequel, we will call such X a finite range (or truncated) α-stable-like process. It is proved in [1, Theorem 3.1] that there is a properly exceptional set $\mathcal{N} \subset \mathbb{R}^d$ and a positive symmetric kernel $p(t, x, y)$ defined on $(0, \infty) \times (\mathbb{R}^d \backslash \mathcal{N}) \times (\mathbb{R}^d \backslash \mathcal{N})$ such that $p(t, x, y)$ is the transition density function of X (starting from $x \in \mathbb{R}^d \backslash \mathcal{N}$) with respect to the Lebesgue measure on $\mathbb{R}^d$, and for each $y \in \mathbb{R}^d \backslash \mathcal{N}$ and $t > 0$, $x \mapsto p(t, x, y)$ is quasi-continuous. It is this version of the transition density function of X we will take throughout this paper. Here a set $\mathcal{N} \subset \mathbb{R}^d$ is called properly exceptional with respect to the process X if it has zero Lebesgue measure and

$$\mathbb{P}^x\left(\{X_t, X_{t-}\} \subset \mathbb{R}^d \backslash \mathcal{N} \text{ for every } t > 0\right) = 1 \qquad \text{for } x \in \mathbb{R}^d \backslash \mathcal{N}.$$

It is well-known (see [15]) that every exceptional set is $\mathcal{Q}$-polar and every $\mathcal{Q}$-polar set is contained in a properly exceptional set. Later we will show in Theorem 4.3, $p(t, x, y)$ in fact has a Hölder continuous version and so we can take $\mathcal{N} = \emptyset$. The purpose of this paper is to obtain sharp upper and lower estimates on $p(t, x, y)$. The jump size cutoff constant κ in (1.1) plays no special role, so for convenience we will simply take $\kappa = 1$ for the rest of this paper.

When $c(x, y)$ is a constant, X is a finite range (also called truncated) isotropically symmetric α-stable process in $\mathbb{R}^d$ with jumps of size larger than 1 removed. The potential theory of this Lévy process is studied in [24,25]. One interesting fact is that, even though scale-invariant elliptic Harnack principle is true for such a process, the boundary Harnack principle is only valid for the positive harmonic functions of this process in bounded convex domains (see the last section of [24] for a counterexample). Since the parabolic Harnack principle implies elliptic Harnack principle, our Theorem 4.1 extends the result on Harnack principle in [24] to the case that $c(x, y)$ is not necessarily constant.

Finite range stable processes, more generally finite range jump processes, are very important both in theory and in application. Finite range jump processes are very natural in applications where jumps only up to a certain size are allowed. Moreover, in some aspects, finite range jump processes have nicer behaviors and are more preferable than unrestricted jump processes. For instance, in [13], to show certain property of Schramm–Loewner evolution driven by symmetric stable processes, finite range (or truncated) stable process has been used as a tool. However, as we shall see below, in some other respects, finite range jump processes are much more delicate to study than unrestricted jump processes.

In the sequel, for two non-negative functions f and g, the notation $f \asymp g$ means that there are positive constants c_1, c_2, c_3 and c_4 so that $c_1 g(c_2 x) \le f(x) \le c_3 g(c_4 x)$ in the common domain of definition for f and g. The Euclidean distance between x and y will be denoted as $|x - y|$. For $a, b \in \mathbb{R}$, $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. We will use μ_d or dx to denote the Lebesgue measure in $\mathbb{R}^d$. A statement that is said to be hold quasi-everywhere (q.e. in abbreviation) on a set $A \subset \mathbb{R}^d$ if there is an $\mathcal{Q}$-polar set $\mathcal{N}$ such that the statement holds for every point in $A \backslash \mathcal{N}$.

Our theorems on the heat kernel estimate on $p(t, x, y)$ can be stated as follows (in the figure, R_* is a constant in $(0, 1)$):

(i) (Proposition 2.1 and Theorem 3.6) In the regions D_1 and D_2, we have

$$p(t, x, y) \asymp \left(t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}}\right).$$

(More precisely, $p(t, x, y) \asymp t^{-d/\alpha}$ in D_1 and $p(t, x, y) \asymp \frac{t}{|x-y|^{d+\alpha}}$ in D_2).

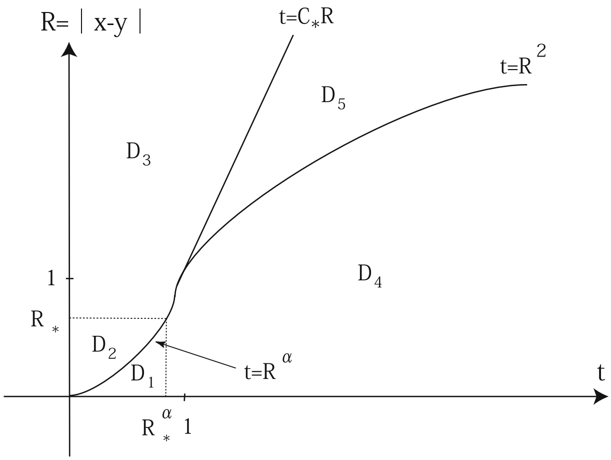
(ii) (Theorem 2.3 and Theorem 3.6) In the region D_3, we have

$$p(t,x,y) \asymp \left(\frac{t}{|x-y|}\right)^{c|x-y|} = \exp\left(-c|x-y|\log\frac{|x-y|}{t}\right).$$

(iii) (Theorems 2.3 and 3.6) In the regions D_4 and D_5, we have

$$p(t,x,y) \asymp t^{-d/2}\exp\left(-\frac{c|x-y|^2}{t}\right).$$

(More precisely, $p(t,x,y) \asymp t^{-d/2}$ in D_4 and $p(t,x,y) \asymp t^{-d/2}\exp\left(-\frac{c|x-y|^2}{t}\right)$ in D_5).



As we see, the heat kernel estimate is of α-stable type in (i), of Poisson type in (ii) and of Gaussian type in (iii). Such behavior of the heat kernel, in particular (i) and (iii), may be useful in applications. For example, in mathematical finance, it has been observed that even though discontinuous stable processes provide better representations of financial data than Gaussian processes (cf. [19]), financial data tend to become more Gaussian over a longer time-scale (see [28] and the references therein). Our heat kernel estimates show that finite range stable-like processes have this type of property: they behave like discontinuous stable processes in small scale and behave like Brownian motion in large scale. Furthermore, they avoid large sizes of jumps which can be considered as impossibly huge changes of financial data in short time.

In fact, some of our heat kernel estimates for $t \geq 1$ will be stated and proved for a more general class of finite range jump processes that is studied in [1] (see (2.16), Theorems 2.4 and 3.5 below). These heat kernel estimates improve the estimates given in [1, Theorems 1.2 and 1.3] significantly. They are also used in Sect. 4 to show the two-sided estimates for Green functions of these processes for $|x-y| \geq 1$.

To get the near diagonal lower bound of the heat kernel $p(t,x,y)$, we introduce and prove a general scaling version of weighted Poincaré inequality of fractional order (see

Theorem 5.1 below). This inequality may be of independent interest. (For the details on (weighted) Poincaré inequality and lower bound estimate of heat kernels for diffusions, we refer our readers to [14,22,29,30] and the references therein.) The proof of our weighted Poincaré inequality is quite long and involved. To keep the flow of the main ideas of our proof for the heat kernel estimates, we put the proof of the weighted Poincaré inequality in the last section. We hope that the establishing of such a scaling version of weighted Poincaré inequality and its usage in getting the heat kernel lower bound estimate will shed new light on our understanding of the heat kernel behavior of more general Markov processes.

Using the heat kernel estimates, we derive the parabolic Harnack inequality for the finite range jump processes. Our proof uses a combination of the techniques developed in [1,2,11,12]. As a direct consequence of the heat kernel estimates, we derive a two-sided sharp estimate for Green functions in $\mathbb{R}^d$ for $d \geq 3$. From the heat kernel estimates and the parabolic Harnack inequality, we also obtain the Hölder continuity of the parabolic functions of finite range stable-like processes. In particular, we note that the Hölder continuity for bounded parabolic functions is a consequence of the local heat kernel estimate, while the parabolic Harnack inequality at small size scale can be obtained from the local heat kernel estimate and some mild condition on the jumping kernel for large jumps. This allows us to establish the parabolic Harnack inequality and the joint Hölder continuity for parabolic functions for a larger class of symmetric processes that can be obtained from finite range stable-like process by adding larger jumps with uniformly bounded (total) jumping intensity for those jumps of size larger than 1 through Meyer's construction. See Theorem 4.5 for details.

The remainder of this paper is organized as follows. In Sect. 2, we prove the upper bound estimates of the heat kernel. Section 3 contains the results on the lower bound estimates of the heat kernel. In Sect. 4, we establish parabolic Harnack principle and the two-sided estimates for Green functions of the finite range jump processes as well as Hölder continuity of heat kernels. In the last section, we give the proof of weighted Poincaré inequality of fractional order.

2 Heat kernel upper bound estimate

In this section, we will state the results on the upper bound estimates of the heat kernel for the finite range symmetric α-stable-like process X more precisely and present proofs. Most of the heat kernel estimates in this section and next one are established for quasi-everywhere (q.e.) point in $\mathbb{R}^d$. However in Theorem 4.3 of Sect. 4, we will show that the heat kernels of finite range stable processes are Hölder continuous and therefore these estimates hold for every point in $\mathbb{R}^d$.

Proposition 2.1 (i) *For each $T^* > 0$, there exists $c_1 = c_1(T^*) > 0$ such that*

$$p(t, x, y) \leq c_1 \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \right)$$

for all $t \in (0, T^*]$ and q.e. $x, y \in \mathbb{R}^d$.

(ii) *There exist* $0 < R_* < 1$ *and* $c_2 > 0$ *such that*

$$c_2\left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right) \le p(t,x,y)$$

for all $t \in (0, T_*]$ *and q.e.* $x, y \in \mathbb{R}^d$ *with* $|x-y| \in (0, R_*]$ *where* $T_* := R_*^\alpha$.

Proof The estimates on these regions can be deduced from the existing results. Let $p_0(t,x,y)$ be the transition density function of stable-like process Y on $\mathbb{R}^d$ whose jumping kernel is $\frac{c(x,y)}{|x-y|^{d+\alpha}}$. Since X can be constructed from Y by removing jumps of size larger than 1 via Meyer's construction, by [1, Lemma 3.6] and [3, Lemma 3.1(c)] we have

$$p(t,x,y) \le e^{t\|\mathcal{J}\|_\infty} p_0(t,x,y) \quad \text{and} \quad p_0(t,x,y) \le p(t,x,y) + t\|J_1\|_\infty$$

where

$$J_1(x,y) := \frac{c(x,y)}{|x-y|^{d+\alpha}} \mathbf{1}_{\{|x-y|>1\}} \quad \text{and} \quad \mathcal{J}(x) := \int_{\mathbb{R}^d} J_1(x,y)dy.$$

Applying the estimates on $p_0(t,x,y)$ in [11] to the above two inequalities, we have

$$p(t,x,y) \le c_1 e^{t\|\mathcal{J}\|_\infty}\left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right) \tag{2.1}$$

and

$$\frac{1}{c_1}\left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right) - t\|J_1\|_\infty \le p(t,x,y). \tag{2.2}$$

Now (i) follows immediately from (2.1). Since

$$\frac{1}{2c_1}t^{-d/\alpha} \le \frac{1}{c_1}t^{-d/\alpha} - t\|J_1\|_\infty \qquad \text{if } t \le (2c_1\|J_1\|_\infty)^{-\frac{\alpha}{d+\alpha}}$$

and

$$\frac{1}{2c_1}\frac{t}{|x-y|^{d+\alpha}} \le \frac{1}{c_1}\frac{t}{|x-y|^{d+\alpha}} - t\|J_1\|_\infty \qquad \text{if } |x-y| \le (2c_1\|J_1\|_\infty)^{-\frac{1}{d+\alpha}},$$

we get (ii) from (2.2). □

The discrete Markov chain analogue of the following result is established in [4, Proposition 2.1]. See also [8, Sect. 2] where the following result is discussed when $c(x,y)$ is a constant and $\alpha = 1$.

Proposition 2.2 *There exist* $c_1, c_2 > 0$ *such that*

$$p(t,x,y) \le \begin{cases} c_1 t^{-d/\alpha} & \text{for } t \in (0,1], \\ c_2 t^{-d/2} & \text{for } t \in [1,\infty). \end{cases} \tag{2.3}$$

Proof By Proposition 2.1(i), we only need to show (2.3) for $t \in [1,\infty)$.

Let $(\mathcal{E}^0, \mathcal{F}^0)$ be the Dirichlet form for the finite range isotropically symmetric α-stable process with jumps of size larger than 1 removed. That is,

$$\mathcal{E}^0(u,u) = \int_{\mathbb{R}^d\times\mathbb{R}^d} (u(x)-u(y))^2 \frac{c_0(d,\alpha)}{|x-y|^{d+\alpha}} \mathbf{1}_{\{|x-y|\le 1\}} dxdy,$$

$$\mathcal{F}^0 = \left\{ u \in L^2(\mathbb{R}^d, dx) : \mathcal{E}^0(u,u) < \infty \right\},$$

where $c_0(d,\alpha) > 0$ is a constant. Note that $\mathcal{D} \subset \mathcal{F}^0$ and there is a constant $\kappa := \kappa(d,\alpha) > 0$ such that

$$\mathcal{E}^0(u,u) \le \kappa\, \mathcal{Q}(u,u) \qquad \text{for } u \in \mathcal{F}. \tag{2.4}$$

By the Fourier transform, we have

$$\mathcal{E}^0(f,g) = c_0 \int_{\mathbb{R}^d} \hat{g}(\xi)\bar{\hat{f}}(\xi)\phi(\xi)d\xi,$$

where $\hat{f}(\xi) := (2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{i\xi\cdot y} f(y)dy$ is the Fourier transform of u and

$$\phi(\xi) := \int_{\{|y|<1\}} \frac{1-\cos(\xi\cdot y)}{|y|^{d+\alpha}} dy. \tag{2.5}$$

By the change of variable $y = x/|\xi|$, we have from (2.5)

$$\phi(\xi) = |\xi|^\alpha \int_{\{|x|<|\xi|\}} \frac{1-\cos\left(\frac{\xi}{|\xi|}\cdot x\right)}{|x|^{d+\alpha}} dx. \tag{2.6}$$

Note that $1-\cos\left(\frac{\xi}{|\xi|}\cdot x\right)$ behaves like $|x|^2$ for small $|x|$. Moreover, as $|\xi|$ goes to infinity, the integral in the above equation goes to a positive constant. Thus it is easy to see that there exist $M > 1$ and $c_1 > 0$ such that

$$\phi(\xi) \ge \begin{cases} c_1|\xi|^\alpha, & \text{for all } |\xi| > M, \\ c_1|\xi|^2, & \text{for all } |\xi| \le M. \end{cases}$$

Thus for every $r \le 1$, we have

$$\int\limits_{\{|\xi|>r\}} |\hat{f}(\xi)|^2 d\xi \le \int\limits_{\{M>|\xi|>r\}} \left(\frac{|\xi|}{r}\right)^2 |\hat{f}(\xi)|^2 d\xi + \int\limits_{\{|\xi|\ge M\}} \left(\frac{|\xi|}{r}\right)^\alpha |\hat{f}(\xi)|^2 d\xi$$
$$\le c_2 \left(r^{-2} \int\limits_{\{M>|\xi|>r\}} \phi(\xi)|\hat{f}(\xi)|^2 d\xi + r^{-\alpha} \int\limits_{\{|\xi|\ge M\}} \phi(\xi)|\hat{f}(\xi)|^2 d\xi \right)$$
$$\le c_2 r^{-2} \int\limits_{\mathbb{R}^d} \phi(\xi)|\hat{f}(\xi)|^2 d\xi = c_3 r^{-2} \mathcal{E}^0(f,f) \le c_3 \kappa r^{-2} \mathcal{Q}(f,f),$$

where the last inequality is due to (2.4). Using the above inequality, we get

$$\|f\|_2^2 = \int\limits_{\{|\xi|>r\}} |\hat{f}(\xi)|^2 d\xi + \int\limits_{\{|\xi|\le r\}} |\hat{f}(\xi)|^2 d\xi$$
$$\le c_4(\kappa) \left(r^{-2} \mathcal{Q}(f,f) + 2 r^d \|f\|_1^2 \right), \quad r \le 1. \tag{2.7}$$

Note that, if $a \le b$, the function $r \to h(r) := ar^{-2} + 2br^d$ has a local minimum at

$$r = \left(\frac{a}{db}\right)^{\frac{1}{d+2}} \le 1.$$

Thus by minimizing the right-hand side of (2.7) for $\mathcal{Q}(f,f) \le \|f\|_1^2$, we get

$$\|f\|_2^2 \le c_5 \, \mathcal{E}^0(f,f)^{\frac{d}{d+2}} \, \|f\|_1^{\frac{4}{d+2}} \le c_6 \, \mathcal{Q}(f,f)^{\frac{d}{d+2}} \, \|f\|_1^{\frac{4}{d+2}}.$$

Therefore by Theorem 2.9 in [8], we conclude that

$$p(t,x,y) \le c_7 \, t^{-d/2} \qquad \text{for all } t \in [1,\infty).$$

□

Theorem 2.3 *There exist $C_* < 1$ and $c_1, c_2, c_3, c_4 > 0$ such that*

$$p(t,x,y) \le c_1 \left(\frac{t}{|x-y|}\right)^{c_2|x-y|} = c_1 \exp\left(-c_2|x-y| \log \frac{|x-y|}{t}\right) \tag{2.8}$$

for q.e. $x, y \in \mathbb{R}^d$ with $(t, |x-y|) \in \{(t,R) : R \ge \max\{t/C_*, R_*\}\}$ and

$$p(t,x,y) \le c_3 t^{-d/2} \exp\left(-\frac{c_4|x-y|^2}{t}\right) \tag{2.9}$$

for q.e. $x, y \in \mathbb{R}^d$ *with* $(t, |x-y|) \in \{(t, R) : R_* \le R \le t/C_*\}$*, where* R_* *is given in Proposition 2.1.*

Proof Using Proposition 2.3 above, [8, Corollary 3.28] and [1, Theorem 3.1], we have

$$p(t, x, y) \le c(t^{-d/\alpha} \vee t^{-d/2}) \exp(-E(2t, x, y)) \qquad \text{for q.e. } x, y \in \mathbb{R}^d. \tag{2.10}$$

Here $E(2t, x, y)$ is given by the following:

$$\begin{aligned} \Gamma(\psi)(x) &= \int (e^{\psi(x)-\psi(y)} - 1)^2 J(x, y)dy, \\ \Lambda(\psi)^2 &= \|\Gamma(\psi)\|_\infty \vee \|\Gamma(-\psi)\|_\infty, \\ E(t, x, y) &= \sup\{|\psi(x) - \psi(y)| - t\Lambda(\psi)^2 : \ \psi \in Lip_0 \text{ with } \Lambda(\psi) < \infty\}, \end{aligned}$$

where Lip_0 is a space of compactly supported Lipschitz continuous functions on $\mathbb{R}^d$.
Fix $x_0, y_0 \in \mathbb{R}^d$ and let $R = |x_0 - y_0| \ge R_*$. Define

$$\psi(x) = \lambda(R - |x_0 - x|)^+.$$

So $|\psi(x) - \psi(y)| \le \lambda|x - y|$. Note that $|e^t - 1|^2 \le t^2 e^{2|t|}$. Hence

$$\Gamma(\psi)(x) = \int (e^{\psi(x)-\psi(y)} - 1)^2 J(x, y)dy \le e^{2\lambda}\lambda^2 \int |x-y|^2 J(x, y)dy \le c_1 \lambda^2 e^{2\lambda}.$$

So we have

$$-E(2t, x_0, y_0) \le -\lambda R + c_1 t\lambda^2 e^{2\lambda}. \tag{2.11}$$

For each t and R, take $\lambda_0 > 0$ such that

$$\lambda_0 e^{2\lambda_0} = \frac{R}{2c_1 t}. \tag{2.12}$$

Since xe^{2x} is strictly increasing, it is easy to check that such λ_0 exists uniquely. Then the right hand side of (2.11) is equal to $-\lambda_0 R/2$. Let $C_* = (2c_1 e)^{-1}$ which is less than 1 by taking c_1 large. When $R/(2c_1 t) \ge e$ (i.e., $t \le C_* R$), (2.12) holds with $\lambda_0 \asymp \log(R/t)$, and when $R/(2c_1 t) < e$ (i.e., $t \ge C_* R$), (2.12) holds with $\lambda_0 \asymp R/t$. Putting these into (2.10), we obtain the following; In the region $\{(t, R) : t \le C_* R, R \ge R_*\}$,

$$\begin{aligned} p(t, x, y) &\le c'(t^{-d/\alpha} \vee t^{-d/2}) \exp\left(-c_2 R \log \frac{R}{t}\right) \\ &= c'\left(t^{-d/\alpha} \vee t^{-d/2}\right)\left(\frac{t}{R}\right)^{c_2 R}, \end{aligned} \tag{2.13}$$

and in the region $\{(t, R) : t \ge C_* R, R \ge R_*\}$,

$$p(t, x, y) \le c' t^{-d/2} \exp(-c'' R^2/t),$$

which gives (2.9).

To complete the proof, we need to discuss the former case more. When $t \geq 1$, the right hand side of (2.13) is bounded from above by $c'(t/R)^{c_2 R}$, and when $t \leq 1$ and $R \geq R^*$ for some large $R^* > 1$, it is bounded from above by $c'(t/R)^{c_2 R - d/\alpha} \leq c'(t/R)^{c_3 R}$, both of which give (2.8). So all we need is to consider the case $t \leq 1$ and $R_* \leq R \leq R^*$. But in this case, the desired estimate is already established in Proposition 2.1(i). □

Now let's consider a more general non-local Dirichlet form $(\mathcal{E}, \mathcal{F})$. Set

$$\mathcal{E}(f, f) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (f(y) - f(x))^2 J(x, y)\, dx\, dy, \tag{2.14}$$

$$\mathcal{F} = \overline{C_c^1(\mathbb{R}^d)}^{\mathcal{E}_1}, \tag{2.15}$$

where the jump kernel $J(x, y)$ is a symmetric non-negative function of x and y such that $J(x, y) = 0$ for $|x - y| \geq 1$ and there exist $\alpha, \beta \in (0, 2)$, $\beta > \alpha$ and positive κ_1, κ_2 such that

$$\kappa_1 |y - x|^{-d-\alpha} \leq J(x, y) \leq \kappa_2 |y - x|^{-d-\beta} \qquad \text{for } |y - x| < 1. \tag{2.16}$$

Here $\mathcal{E}_1(f, f) := \mathcal{E}(f, f) + \|f\|_2^2$, $C_c^1(\mathbb{R}^d)$ denotes the space of C^1 functions on $\mathbb{R}^d$ with compact support, and $\mathcal{F}$ is the closure of $C_c^1(\mathbb{R}^d)$ with respect to the metric $\mathcal{E}_1(f, f)^{1/2}$. The Dirichlet form $(\mathcal{E}, \mathcal{F})$ is regular on $\mathbb{R}^d$ and so it associates a Hunt process Z, starting from quasi-everywhere in $\mathbb{R}^d$. It is proved in [1] that Z is conservative and has quasi-continuous transition density function $q(t, x, y)$ with respect to the Lebesgue measure on $\mathbb{R}^d$.

When $t \in [1, \infty)$, only the upper bound of the jumping kernel played a role in the proofs of Proposition 2.2 and Theorem 2.3. Thus, combining with Theorem 1.2 in [1], the following is true for Z.

Theorem 2.4 *There is a constant $c > 0$ such that*

$$q(t, x, y) \leq c\,(t^{-d/\alpha} \vee t^{-d/2}) \quad \textit{for q.e. } x, y \in \mathbb{R}^d.$$

Moreover, there exist $C_1 < 1$, $R_1 \leq \frac{1}{4}$ and $c_1, c_2, c_3, c_4 > 0$ such that

$$q(t, x, y) \leq c_1 \left(\frac{t}{|x - y|}\right)^{c_2 |x-y|} = c_1 \exp\left(-c_2 |x - y| \log \frac{|x - y|}{t}\right) \quad \textit{for q.e. } x, y \in \mathbb{R}^d \tag{2.17}$$

with $(t, |x - y|) \in \{(t, R) : t \geq 1, R \geq \max\{t/C_1,\ R_1\}\}$ and

$$q(t, x, y) \leq c_3 t^{-d/2} \exp\left(-\frac{c_4 |x - y|^2}{t}\right) \quad \textit{for q.e. } x, y \in \mathbb{R}^d \tag{2.18}$$

with $(t, |x - y|) \in \{(t, R) : t \geq 1, R_1 \leq R \leq t/C_1\}$.

The above theorem will be used in the next section to prove the near-diagonal lower bound for $q(t,x,y)$.

3 Heat kernel lower bound estimate

In this section, we give the proof of the lower bound estimate of the heat kernel. We first record a simple observation, which sheds lights on the different heat kernel behaviors at small (stable) and large (Gaussian) scale. Recall that a finite range isotropically symmetric α-stable process in $\mathbb{R}^d$ with jumps of size larger than 1 removed is the Lévy process with Lévy measure $c_0(d,\alpha)|h|^{-d-\alpha}\mathbf{1}_{\{|h|\le 1\}}dh$.

Lemma 3.1 *Let X be finite range isotropically symmetric α-stable process in $\mathbb{R}^d$ with jumps of size larger than 1 removed. For $\lambda > 0$, define*

$$Y_t^{(\lambda)} := Y_0^{(\lambda)} + \lambda^{-1/2}(X_{\lambda t} - X_0) \quad \textit{and} \quad Z_t^{(\lambda)} := Z_0^{(\lambda)} + \lambda^{-1/\alpha}(X_{\lambda t} - X_0).$$

Then the process $Y^{(\lambda)}$ converges in finite-dimensional distributions to a Brownian motion on $\mathbb{R}^d$ as $\lambda \to \infty$ and $Z^{(\lambda)}$ converges in finite-dimensional distributions to the isotropically symmetric α-stable process as $\lambda \to 0$.

Proof Recall that the Lévy exponent ϕ of X is given by (2.5). Clearly $Y^{(\lambda)}$ and $Z^{(\lambda)}$ are Lévy processes as well, with

$$\mathbb{E}\left[e^{i\xi\cdot(Y_t^{(\lambda)}-Y_0^{(\lambda)})}\right] = \mathbb{E}\left[e^{i\xi\cdot\lambda^{-1/2}(X_{\lambda t}-X_0)}\right] = e^{\lambda t\phi(\lambda^{-1/2}\xi)}, \qquad \xi \in \mathbb{R}^d$$

and

$$\mathbb{E}\left[e^{i\xi\cdot(Z_t^{(\lambda)}-Z_0^{(\lambda)})}\right] = \mathbb{E}\left[e^{i\xi\cdot\lambda^{-1/\alpha}(X_{\lambda t}-X_0)}\right] = e^{\lambda t\phi(\lambda^{-1/\alpha}\xi)}, \qquad \xi \in \mathbb{R}^d.$$

Let $\phi_\lambda(\xi)$ and $\psi_\lambda(\xi)$ denote the Lévy exponents of $Y^{(\lambda)}$ and $Z^{(\lambda)}$, respectively. Then we have by above and (2.6) that

$$\phi_\lambda(\xi) = \lambda\phi(\lambda^{-1/2}\xi) = \lambda^{1-\alpha/2}|\xi|^\alpha \int_{\{x\in\mathbb{R}^d:|x|\le\lambda^{-1/2}|\xi|\}} \frac{1-\cos x_1}{|x|^{d+\alpha}}\,dx \tag{3.1}$$

which converges to $c|\xi|^2$ as $\lambda \to \infty$. Moreover, there is $c_1 > 0$ so that

$$|\phi_\lambda(\xi)| \le c_1|\xi|^2 \qquad \text{for every } \xi \in \mathbb{R}^d \quad \text{and} \quad \lambda > 0. \tag{3.2}$$

Similarly,

$$\psi_\lambda(\xi) = \lambda\phi(\lambda^{-1/\alpha}\xi) = |\xi|^\alpha \int_{\{x\in\mathbb{R}^d:|x|\le\lambda^{-1/\alpha}|\xi|\}} \frac{1-\cos x_1}{|x|^{d+\alpha}}\,dx \tag{3.3}$$

which increases to $c_2|\xi|^\alpha$ as $\lambda \downarrow 0$, where $c_2 = \int_{\mathbb{R}^d} \frac{1-\cos(x_1)}{|x|^{d+\alpha}} dx$. This proves the lemma. □

Inequality (3.2) will be used later in the proof of Theorem 3.4.

Now let's consider the more general non-local Dirichlet form $(\mathcal{E}, \mathcal{F})$ in (2.14)–(2.15). Recall that the jump kernel $J(x, y)$ for $(\mathcal{E}, \mathcal{F})$ is zero for $|x - y| \geq 1$ and satisfies the condition (2.16), and $q(t, x, y)$ is the transition density function for the associated Hunt process Z with respect to the Lebesgue measure on $\mathbb{R}^d$.

Define

$$\phi(x) = c\left(1 - |x|^2\right)^{12/(2-\beta)} \mathbf{1}_{B(0,1)}(x),$$

where $c > 0$ is the normalizing constant so that $\int_{\mathbb{R}^d} \phi(x)dx = 1$.

The following proposition is an immediate consequence of the Assumption (2.16) and Theorem 5.1 in Sect. 5 below. As mentioned earlier, to keep the flow of our proof for heat kernel estimates, we will postpone its proof to Sect. 5.

Proposition 3.2 *There is a positive constant $c_1 = c_1(d, \alpha, \beta)$ independent of $r > 1$, such that for every $u \in L^1(B(0, 1), \phi dx)$,*

$$\int_{B(0,1)} (u(x) - u_\phi)^2 \phi(x)dx$$
$$\leq c_1 \int_{B(0,1)\times B(0,1)} (u(x) - u(y))^2\, r^{d+2} J(rx, ry)\sqrt{\phi(x)\phi(y)}\, dxdy.$$

Here $u_\phi := \int_{B(0,1)} u(x)\phi(x)dx$.

Remark 3.3 The above weighted Poincaré inequality in fact holds for more general weight function ϕ. See Sect. 5 for the details. □

For $\delta \in (0, 1)$, set

$$J_\delta(x, y) = \begin{cases} J(x, y) & \text{for } |x - y| \geq \delta; \\ \kappa_2 |y - x|^{-d-\beta} & \text{for } |x - y| < \delta, \end{cases} \tag{3.4}$$

and define $(\mathcal{E}^\delta, \mathcal{F}^\delta)$ in the same way as we defined $(\mathcal{E}, \mathcal{F})$ in (2.14)–(2.15).

For $\delta \in (0, 1)$, let Z^δ be the symmetric Markov process associated with $(\mathcal{E}^\delta, \mathcal{F}^\delta)$. By [1], the process Z^δ can be modified to start from every point in $\mathbb{R}^d$ and is conservative; moreover Z^δ has a quasi-continuous transition density function $q^\delta(t, x, y)$ defined on $[0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$, with respect to the Lebesgue measure on $\mathbb{R}^d$.

The idea of the proof of the following theorem is motivated by that of Proposition 4.9 in [1]. For ball $B(x_0, r) \subset \mathbb{R}^d$, let $q^{\delta, B(x_0,r)}(t, x, y)$ denote the transition density function of the subprocess $Z^{B(x_0,r)}$ of Z killed upon leaving the ball $B(x_0, r)$.

Theorem 3.4 *Suppose the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is given by (2.14)–(2.15) with the jumping kernel J satisfying the condition (2.16) and J_δ is given by (3.4). For each*

$\delta_0 > 0$, *there exists* $c = c(\delta_0) > 0$, *independent of* $\delta \in (0, 1)$ *such that for every* $x_0 \in \mathbb{R}^d$ *and* $t \geq \delta_0$,

$$q^{\delta, B(x_0, t^{1/2})}(t, x, y) \geq c\, t^{-d/2} \quad \textit{for every } t \geq \delta_0 \quad \textit{and} \quad \textit{q.e.} x, y \in B(x_0, \sqrt{t}/2) \tag{3.5}$$

and

$$q^{\delta}(t, x, y) \geq c\, t^{-d/2} \quad \textit{for every } t \geq \delta_0 \quad \textit{and} \quad \textit{q.e.} x, y \textit{ with } |x - y|^2 \leq t. \tag{3.6}$$

Proof In view of [1, Theorem 4.10], it suffices to prove that there are $t_0 < 1/2$ and $c > 0$, independent of $\delta \in (0, 1)$, such that (3.5)–(3.6) hold for $t \geq t_0^{-1}$. In fact, if $\delta_0 < t_0^{-1}$ and $\delta_0 \leq t \leq t_0^{-1}$, we let $n_0 = 1 + [2/\sqrt{t_0 \delta_0}]$, where $[a]$ is the largest integer which is no larger than a. By [1, Theorem 4.10], we have

$$q^{\delta, B(x_0, \delta_0^{1/2})}(s, x, y) \geq c_0, \quad \text{for every } \frac{\delta_0}{n_0} \leq s \leq t_0^{-1} \quad \text{and} \quad x, y \in B(x_0, 3\delta_0^{1/2}/4) \tag{3.7}$$

where the constant c_0 is independent of δ and $x_0 \in \mathbb{R}^d$. Given $x, y \in B(x_0, \sqrt{t}/2)$, let $z_1 \cdots z_{n_0 - 1}$ be equally spaced points on the line segment joining x and y such that $x \in B(z_1, 3\delta_0^{1/2}/4) \subset B(z_1, \delta_0^{1/2}) \subset B(x_0, t^{1/2})$ and $y \in B(z_{n_0-1}, 3\delta_0^{1/2}/4) \subset B(z_{n_0-1}, \delta_0^{1/2}) \subset B(x_0, t^{1/2})$. Using (3.7) and the semigroup property, we have

$$\begin{aligned}
& q^{\delta, B(x_0, t^{1/2})}(t, x, y) \\
& \quad = \int_{B(x_0, t^{1/2})} \cdots \int_{B(x_0, t^{1/2})} q^{\delta, B(x_0, t^{1/2})}(t/n_0, x, w_1) \ldots \\
& \qquad q^{\delta, B(x_0, t^{1/2})}(t/n_0, w_{n_0-1}, y) dw_1 \ldots dw_{n_0-1} \\
& \quad \geq \int_{B(z_1, 3\delta_0^{1/2}/4)} \cdots \int_{B(z_{n_0-1}, 3\delta_0^{1/2}/4)} q^{\delta, B(z_1, \delta_0^{1/2})}(t/n_0, x, w_1) \ldots \\
& \qquad q^{\delta, B(z_{n_0}, \delta_0^{1/2})}(t/n_0, w_{n_0-1}, y) dw_1 \ldots dw_{n_0-1} \geq \widetilde{c_0} \geq \widetilde{c_0} \delta_0^{d/2} t^{-d/2}.
\end{aligned}$$

Similar argument gives (3.6) when $\delta_0 < t_0^{-1}$ and $t \in [\delta_0, t_0^{-1}]$.

Fix $\delta \in (0, 1)$ and, for simplicity, in this proof we sometimes drop the superscript "δ" from Z^δ and $q^\delta(t, x, y)$. For ball $B_r := B(0, r) \subset \mathbb{R}^d$, let $q^{B_r}(t, x, y)$ denote the transition density function of the subprocess Z^{B_r} of Z killed upon leaving the ball B_r. Then by the proof of Proposition 4.3 in [1], there is a constant $c_1 = c_1(\delta, r) > 0$ such that

$$q^{B_r}(t, x, y) \geq c_1 (r - |x|)^\beta (r - |y|)^\beta \quad \text{for every } t \in [r^2/8,\ r^2/4] \quad \text{and} \quad x, y \in B_r.$$

Define

$$\varphi_r(x) = \left(r^2 - |x|^2\right)^{12/(2-\beta)} \mathbf{1}_{B_r}(x).$$

It follows from Lemmas 4.5 and 4.6 of [1] that for every $t > 0$ and $y_0 \in B_r$, $q^{B_r}(t, x, y_0) \in \mathcal{F}^{B_r}$ and $\varphi_r(\cdot)/q^{B_r}(t, x, y_0) \in \mathcal{F}^{B_r}$, where $(\mathcal{E}, \mathcal{F}^{B_r})$ is the Dirichlet form for the killed process Z^{B_r}.

Note that the Dirichlet form of $\left\{r^{-1}Z_{r^2 t}, t \geq 0\right\}$ is $(\mathcal{E}^{(r)}, \mathcal{F}^{(r)})$, where

$$\mathcal{E}^{(r)}(u, u) = \int\limits_{\mathbb{R}^d \times \mathbb{R}^d} (u(x) - u(y))^2 r^{d+2} J_\delta(rx, ry)dxdy \tag{3.8}$$
$$\mathcal{F}^{(r)} = \left\{u \in L^2(u, u) : \mathcal{E}^{(r)}(u, u) < \infty\right\} = W^{\beta/2,2}(\mathbb{R}^d).$$

By (2.16) and (3.2), there are constants $c_2, c_3 > 0$ independent of $r \geq 1$ and $\delta \in (0, 1)$ such that for every $u \in W^{1,2}(\mathbb{R}^n) \subset W^{\beta/2,2}(\mathbb{R}^d)$,

$$\mathcal{E}^{(r)}(u, u) \leq c_2 \int\limits_{\mathbb{R}^d} \phi_r(\xi)|\widehat{u}(\xi)|^2 d\xi \leq c_3 \int\limits_{\mathbb{R}^d} |\nabla u(x)|^2 dx. \tag{3.9}$$

Here $\widehat{u}$ denotes the Fourier transform of u.

Define

$$q_r^B(t, x, y) := r^d q^{B_r}(r^2 t, rx, ry). \tag{3.10}$$

It is easy to see $q_r^B(t, x, y)$ is the transition density function for process $r^{-1}Z^{B_r}_{r^2 t}$. The latter is the subprocess of $\{r^{-1}Z_{r^2 t}, t \geq 0\}$ killed upon leaving the unit ball $B(0, 1)$, whose Dirichlet form will be denoted as $(\mathcal{E}^{(r)}, \mathcal{F}^{(r),B})$. It follows from above there is a constant $c_4 = c_4(\delta, r) > 0$ such that

$$q_r^B(t, x, y) \geq c_4(1-|x|)^\beta(1 - |y|)^\beta \qquad \text{for every } t \in [1/8,\ 1/4] \text{ and } x, y \in B(0, 1).$$

Recall that

$$\phi(x) = c_5 \left(1 - |x|^2\right)^{12/(2-\beta)} \mathbf{1}_{B(0,1)}(x),$$

where c_5 is a normalizing constant so that $\int_{\mathbb{R}^d} \phi(x)dx = 1$. Let $x_0 \in B(0, 1)$ and Define

$$\begin{aligned}
u(t,x) &:= q_r^B(t,x,x_0), \quad v(t,x) \ := \ q_r^B(t,x,x_0)/\phi(x)^{1/2},\\
H(t) &:= \int_{B(0,1)} \phi(y)\log u(t,y)dy,\\
G(t) &:= \int_{B(0,1)} \phi(y)\log v(t,y)dy\\
&= \int_{B(0,1)} \phi(y)\log u(t,y)dy - \frac{1}{2}\int_{B(0,1)} \phi(x)\log\phi(x)dx\\
&= H(t) - c_6.
\end{aligned}$$

By Lemma 4.7 of [1],

$$G'(t) = -\mathcal{E}^{(r)}\left(u(t,\cdot), \frac{\phi}{u(t,\cdot)}\right). \tag{3.11}$$

(The reason we work with J_δ rather than J is so that we can use [1, Lemma 4.7] to obtain above (3.11). The remainder of the argument does not use the condition on J_δ, and in particular the constants can be taken to be independent of $\delta \in (0,1)$).

Write $J^{(r)}(x,y) := r^{d+2}J_\delta(rx,ry)$ and $\kappa_B^{(r)}(x) := 2\int_{\mathbb{R}^d\setminus B(0,1)} J^{(r)}(x,y)dy$ for $x \in B := B(0,1)$. Then we have from (3.8) and (3.11),

$$\begin{aligned}
G'(t) = &-\int_B\int_B \frac{[u(t,y)-u(t,x)]}{u(t,x)u(t,y)}[u(t,x)\phi(y)-\phi(x)u(t,y)]J^{(r)}(x,y)\,dy\,dx\\
&-\int_B \phi(x)\kappa_B^{(r)}(x)dx.
\end{aligned}$$

The main step is to show that for all t in (0,1] one has

$$G'(t) \ge -c_7 + c_8\int_B (\log u(t,y) - H(t))^2\phi(y)\,dy\,. \tag{3.12}$$

for positive constants c_7, c_8.

Setting $a = u(t,y)/u(t,x)$ and $b = \phi(y)/\phi(x)$, we see that

$$\begin{aligned}
&\frac{[u(t,y)-u(t,x)]}{u(t,x)u(t,y)}\ [u(t,x)\phi(y)-\phi(x)u(t,y)]\\
&\qquad = \phi(x)\left(b - \frac{b}{a} - a + 1\right)\\
&\qquad = \phi(x)\left[\left((1-b^{1/2}\right)^2 - b^{1/2}\left(\frac{a}{b^{1/2}} + \frac{b^{1/2}}{a} - 2\right)\right].
\end{aligned} \tag{3.13}$$

Using the inequality

$$A + \frac{1}{A} - 2 \geq (\log A)^2, \qquad A > 0,$$

with $A = a/\sqrt{b}$, the right hand side of (3.13) is bounded above by

$$(\phi(x)^{1/2} - \phi(y)^{1/2})^2 - \sqrt{\phi(x)\phi(y)}\,(\log v(t,y) - \log v(t,x))^2.$$

Substituting in the formula for $G'(t)$ and using Proposition 3.2,

$$\begin{aligned}
H'(t) = G'(t) &\geq -c_9 + \int_B \int_B (\log v(t,y) - \log v(t,x))^2 \sqrt{\phi(x)\phi(y)}\, J^{(r)}(x,y)\,dx\,dy \\
&\geq -c_9 + c_{10} \int_B (\log v(t,y) - G(t))^2 \phi(y)\,dy \\
&\geq -c_{11} + c_{12} \int_B (\log u(t,y) - H(t))^2 \phi(y)\,dy\ ,
\end{aligned}$$

which gives (3.12). Note that in the first inequality we used the fact that

$$\begin{aligned}
&\int_B \int_B (\phi(x)^{1/2} - \phi(y)^{1/2})^2 J^{(r)}(x,y)\,dx\,dy + \int_B \phi(x)\kappa_B^{(r)}(x)dx \\
&\quad = \mathcal{E}^{(r)}(\phi^{1/2}, \phi^{1/2}) < \infty,
\end{aligned}$$

which follows from (3.9) and in the last inequality we used the fact that

$$\begin{aligned}
&\int_B (\log u(t,y) - H(t))^2 \phi(y)\,dy \\
&\quad = \int_B \left(\log v(t,y) - G(t) + \tfrac{1}{2}\log \phi(y) - c_6\right)^2 \phi(y)\,dy \\
&\quad \leq 2 \int_B (\log v(t,y) - G(t))^2\, \phi(y)\,dy + 2 \int_B \left(\tfrac{1}{2}\log \phi(y) - c_6\right)^2 \phi(y)\,dy \\
&\quad = 2 \int_B (\log v(t,y) - G(t))^2\, \phi(y)\,dy + c_{13}.
\end{aligned}$$

Let $q_r(t,x,y) := r^d q(r^2 t, rx, ry)$, which is the transition density function with respect to the Lebesgue measure on $\mathbb{R}^d$ for the process $Z_t^{(r)} := r^{-1} Z_{r^2 t}$, whose non-local Dirichlet form is given by the jumping intensity measure $r^{d+2} J(rx, ry)$. Using

Theorem 2.4 and the fact that $R_1 \le \frac{1}{4} \le \frac{r}{4}$ where R_1 is given in Theorem 2.4, for $r^2 t \ge 1$,

$$\begin{aligned}
&\mathbb{P}_x\left(Z_t^{(r)} \notin B(x, 1/4)\right) \\
&\quad = \int_{B(x,1/4)^c} r^d q(r^2 t, rx, ry) dy \\
&\quad = \int_{B(rx,r/4)^c} q(r^2 t, rx, z) dz \\
&\quad \le c_{14} \int_{\{z\in\mathbb{R}^d : C_1|z-rx| \ge \max\{C_1 r/4,\, r^2 t\}\}} e^{-c_{15}|z-rx|}\, dz \\
&\qquad + c_{16} \int_{\{z\in\mathbb{R}^d : r^2 t \ge C_1|z-rz| \ge C_1 r/4\}} r^{-d} t^{-d/2} \exp\left(-\frac{c_{17}|z-rz|^2}{r^2 t}\right) dz \\
&\quad \le c_{18} \int_{\{w\in\mathbb{R}^d : |w| \ge r/4\}} e^{-c_{15}|w|}\, dw + c_{19} \int_{r/4}^{r^2 t/C_1} r^{-d} t^{-d/2} \exp\left(-\frac{c_{17} s^2}{r^2 t}\right) s^{d-1} ds \\
&\quad \le c_{18} \int_{\{w\in\mathbb{R}^d : |w| \ge r/4\}} e^{-c_{15}|w|}\, dw + c_{19} \int_{1/(4\sqrt{t})}^{\infty} \exp\left(-c_{17} u^2\right) u^{d-1} du.
\end{aligned}$$

Let $t_0 \in (0, 1/2)$ be small so that

$$c_{19} \int_{1/(4\sqrt{t_0})}^{\infty} \exp\left(-c_{17} u^2\right) u^{d-1} du < 1/16$$

and

$$c_{18} \int_{\{w\in\mathbb{R}^d : |w| \ge 1/(4\sqrt{t_0})\}} e^{-c_{15}|w|}\, dw < 1/16.$$

We then have

$$\mathbb{P}_x\left(Z_t^{(r)} \notin B(x, 1/4)\right) < 1/16 + 1/16 = 1/8 \qquad \text{for every } r \ge t_0^{-1/2} \text{ and } 0 < t \le t_0.$$

By Lemma 3.8 of [1], we have for every $r \ge t_0^{-1/2}$,

$$\mathbb{P}_x\left(\sup_{s\in[0,t_0]} |Z_s^{(r)} - Z_0^{(r)}| > 1/4\right) \le 1/4.$$

Therefore, with $r \geq t_0^{-1/2}$, for every $t \leq t_0$,

$$\int_{B(0,1/4)} u(t,x)\,dx \geq \mathbb{P}_0\left(\sup_{s\in[0,t_0]} |Z_s^{(r)} - Z_0^{(r)}| < 1/4\right)$$
$$= 1 - \mathbb{P}_0\left(\sup_{s\in[0,t_0]} |Z_s^{(r)} - Z_0^{(r)}| \geq 1/4\right) \geq \tfrac{3}{4}.$$

Here the conservativeness of $Z_t^{(r)}$ is used in the first equality.

Choose K such that $\mu_d(B(0,1/4))e^{-K} = \frac{1}{4}$ and define

$$D_t := \{x \in B(0,1/4) : u(t,x) \geq e^{-K}\}.$$

By Proposition 2.2, if $t \leq t_0$

$$\frac{3}{4} \leq \int_{B(0,1/4)} u(t,x)dx = \int_{D_t} u(t,x)\,dx + \int_{B(0,1/4)\setminus D_t} u(t,x)\,dx$$
$$\leq c_{20}t^{-d/\alpha}\mu_d(D_t) + \mu_d(B(0,1/4))e^{-K}$$
$$= c_{20}t^{-d/\alpha}\mu_d(D_t) + \frac{1}{4}.$$

Therefore

$$\mu_d(D_t) \geq \frac{t^{d/\alpha}}{c_{21}} \geq c_{22} > 0 \qquad \text{if } t \in [\varepsilon/4, t_0].$$

Note that the positive constant $c_{22} = c_{22}(\varepsilon)$ can be chosen to be independent of $r \geq t_0^{-1/2}$ and $x_0 \in B(0,1/2)$.

Jensen's inequality tells us that if $t \leq t_0$

$$H(t) = \int_B \big(\log u(t,x)\big)\phi(x)\,dx \leq \log \int_B u(t,x)\phi(x)\,dx \leq \log \|\phi\|_\infty := \overline{H}.$$

On D_t, $\log u(t,x) \geq -K$ so there are only four possible cases:

(a) If $\log u(t,x) > 0$ and $H(t) \leq 0$, then $(\log u(t,x) - H(t))^2 \geq H(t)^2$.

(b) If $\log u(t,x) > 0$ and $0 < H(t) \leq \overline{H}$, then

$$(\log u(t,x) - H(t))^2 \geq 0 \geq H(t)^2 - \overline{H}^2.$$

(c) If $-K \leq \log u(t,x) \leq 0$ and $|H(t)| \geq 2K$, then $(\log u(t,x) - H(t))^2 \geq \frac{1}{4}H(t)^2$.

(d) If $-K \le \log u(t,x) \le 0$ and $|H(t)| < 2K$, then

$$(\log u(t,x) - H(t))^2 \ge 0 \ge \frac{1}{4}H(t)^2 - K^2.$$

Thus we conclude

$$(\log u(t,x) - H(t))^2 \ge \frac{1}{4}H(t)^2 - (\overline{H} \vee K)^2 \qquad \text{on } D_t.$$

Since ϕ is bounded below by $c_{23} > 0$ on $B(0, 1/4)$, then

$$\begin{aligned} c_{12}\int_B (\log u(t,x) - H(t))^2 \phi(x)dx - c_{11} &\ge c_{12}\int_{D_t} (\log u(t,x) - H(t))^2 \phi(x)dx - c_{11} \\ &\ge c_{24}\mu_d(D_t)\left(\tfrac{1}{4}H(t)^2 - (\overline{H} \vee K)^2\right) - c_{11}. \end{aligned}$$

We therefore have

$$H'(t) \ge FH(t)^2 - E, \qquad t \in [\varepsilon/4, t_0]$$

for some positive constants E and F that are independent of $r \ge t_0^{-1/2}$.

Now we do some calculus. Let $t_2 \in [\varepsilon/2, t_0 \wedge 2]$ and let $Q := \max(16E, (16E/F)^{1/2})$. Suppose $H(t_2) \le -Q$. Since $H'(t) \ge -E$ and $t_2 - t < t_0 \wedge 2 \le 2$,

$$H(t_2) - H(t) \ge -2E \qquad \text{for } t \in [\varepsilon/4, t_2]. \tag{3.14}$$

This implies $H(t) \le -Q/2$. Since $FQ^2/4 \ge 4E$, $E < \frac{F}{2}H(t)^2$ and hence

$$H'(t) \ge \frac{F}{2}H(t)^2.$$

Integrating $H'/H^2 \ge F/2$ over $[\frac{\varepsilon}{4}, t_2]$ yields

$$\frac{1}{H(t_2)} - \frac{1}{H(\varepsilon/4)} \le -\frac{F}{2}(t_2 - \varepsilon/4) \le -\frac{F\varepsilon}{8}.$$

Since $H(\varepsilon/4) \le -Q/2 < 0$, we have $1/H(t_2) \le -F\varepsilon/16$, that is,

$$H(t_2) \ge -\frac{16}{F\varepsilon}.$$

This proves that either $H(t_2) \ge -Q$ or $H(t_2) \ge -16/(F\varepsilon)$. Thus in either case, $H(t_2) \ge -U$, where $U = U(\varepsilon) := \max\{Q,\ 16/(F\varepsilon)\} > 0$, and so $G(t_2) = H(t_2) - c_6 \ge -U - c_6$.

Now for every $x_0, x_1 \in B(0, 1/2)$, applying the above first with x_0 and then with x_0 replaced by x_1, we have

$$\begin{aligned}
\log q_r^B(2t_2, x_0, x_1) &= \log \int_B q_r^B(t_2, x_0, z) q_r^B(t_2, x_1, z)\, dz \\
&\geq \log \int_B q_r^B(t_2, x_0, z) q_r^B(t_2, x_1, z) \phi(z)\, dz - \log \|\phi\|_\infty \\
&\geq \int_B \log \left(q_r^B(t_2, x_0, z) q_r^B(t_2, x_1, z) \right) \phi(z)\, dz - \log \|\phi\|_\infty \\
&= \int_B \log q_r^B(t_2, x_0, z)\phi(z) dz + \int_B \log q_r^B(t_2, x_1, z)\phi(z) dz \\
&\quad - \log \|\phi\|_\infty \\
&\geq -2(U + c_{26}),
\end{aligned}$$

that is, $q_r^B(2t_2, x_0, x_1) \geq e^{-2(U+c_{26})}$. A repeated use of the semigroup property (but at most $2/t_2$ more times) then shows $q_r^B(t, x_0, x_1) \geq c_{27}(\varepsilon)$ for every $t \in [\varepsilon/2, 2]$ and $x_0, x_1 \in B(0, 1/2)$. Taking $\varepsilon = 1/4$, we have for every $r \geq t_0^{-1/2}$, $x, y \in B(0, 1/2)$ and $t \in [1/4, 2]$,

$$r^d q^{B_r}(r^2 t, rx, ry) = q_r^B(t, x, y) \geq c_{28},$$

in particular,

$$q^{B_r}(r^2, rx, ry) \geq c_{28} r^{-d}.$$

Thus we have

$$q^{B(0,\sqrt{t})}(t, x, y) \geq c_{28} t^{-d/2} \qquad \text{for } t \geq t_0^{-1} \quad \text{and} \quad x, y \in B(0, \sqrt{t}/2).$$

Clearly the above inequality holds with $B(0, \sqrt{t})$ and $B(0, \sqrt{t}/2)$ being replaced by any other ball $B(x_0, \sqrt{t})$ and $B(x_0, \sqrt{t}/2)$ of the same radius, respectively. Consequently,

$$q(t, x, y) \geq q^{B(x_0,\sqrt{t})}(t, x, y) \geq c_{28} t^{-d/2} \qquad \text{for } t \geq t_0^{-1} \quad \text{and} \quad |x - y|^2 \leq t.$$

This proves the theorem. □

For any ball $B \subset \mathbb{R}^d$, let $(\mathcal{E}^{\delta,B}, \mathcal{F}^{\delta,B})$ denote the Dirichlet form of the subprocess $Z^{\delta,B}$ of Z^δ killed upon leaving the ball B. It is shown in [1, Theorems 1.5 and 2.6] that $(\mathcal{E}^\delta, \mathcal{F}^\delta)$ and $(\mathcal{E}^{\delta,B}, \mathcal{F}^{\delta,B})$ converge as $\delta \to 0$ to $(\mathcal{E}, \mathcal{F})$ and $(\mathcal{E}^B, \mathcal{F}^B)$, respectively in the sense of Mosco, where B is a ball in $\mathbb{R}^d$. Therefore the semigroup of Z^δ and $Z^{\delta,B}$ converge in L^2 to that of Z and Z^B, respectively. By the same proof as that for

[1, Theorem 1.3], we deduce from Theorem 3.4 the following lower bound estimate for the heat kernel of Z, which extends Theorem 1.3 in [1].

Theorem 3.5 *Suppose the Dirichlet form* $(\mathcal{E}, \mathcal{F})$ *is given by* (2.14)–(2.15) *with the jumping kernel* J *satisfying the condition* (2.16). *For each* $t_0 > 0$, *there exists* $c_1 = c_1(t_0) > 0$, *such that for every* $x_0 \in \mathbb{R}^d$, $t \geq t_0$,

$$q^{B(x_0,t^{1/2})}(t,x,y) \geq c_1 t^{-d/2} \qquad \text{for q.e. } x, y \in B(x_0, \sqrt{t}/2)$$

and

$$q(t,x,y) \geq c_1 t^{-d/2} \qquad \text{for q.e. } x, y \text{ with } |x-y|^2 \leq t.$$

Now we return to the case for the Dirichlet form $(\mathcal{Q}, \mathcal{D})$ given by (1.2)–(1.3).

Theorem 3.6 *There exist* $c_0, c_1, c_2, c_3, c_4 > 0$ *such that*

$$p(t,x,y) \geq \begin{cases} c_0\, t^{-d/2} & \text{when } t \geq R_*^\alpha,\ |x-y|^2 \leq t, \\ c_1 \left(\frac{t}{|x-y|}\right)^{c_2|x-y|} & \text{when } |x-y| \geq \max\{t/C_*,\ R_*\}, \\ c_3 t^{-d/2} \exp\left(-\frac{c_4|x-y|^2}{t}\right) & \text{when } C_*|x-y| \leq t \leq |x-y|^2, \end{cases} \tag{3.15}$$

where R_* *and* C_* *are the constants given in Proposition* 2.1 *and in Theorem* 2.3, *respectively.*

Proof By Theorem 3.5, we only need to show the second and third inequalities in (3.15). We first prove the second inequality in (3.15). Let $R := |x-y|$ and $c_+ := (4/R_*) \vee (C_*/T_*) \geq 1$. Let $l \geq 2$ be a positive integer such that $c_+R < l \leq c_+R + 1$ and let $x = x_0, x_1, \ldots, x_l = y$ be such that $|x_i - x_{i+1}| \leq 2R/l \leq 2/c_+$ for $i = 1, \ldots, l-1$. (Here we used the fact that $\mathbb{R}^d$ is a geodesic space.) Since $t/l \leq C_*R/l \leq C_*/c_+ \leq T_*$ and $2R/l \leq 2/c_* \leq R_*/2$, by Proposition 2.1(ii), we have for all $(y_i, y_{i+1}) \in B(x_i, R_*/4) \times B(x_{i+1}, R_*/4)$

$$p(t/l, y_i, y_{i+1}) \geq c_0 \left((t/l)^{-d/\alpha} \wedge \frac{(t/l)}{(R/l)^{d+\alpha}}\right) \geq c_1 \left((t/l)^{-d/\alpha} \wedge (t/l)\right) = c_1 t/l, \tag{3.16}$$

since $t/l \leq T_* \leq 1$. Let $B_i = B(x_i, R_*/4)$. Using (3.16), we have

$$\begin{aligned} p(t,x,y) &\geq \int\limits_{B_1} \cdots \int\limits_{B_{l-1}} p(t/l, x, y_1) \ldots p(t/l, y_{l-1}, y) dy_1 \ldots dy_{l-1} \\ &\geq c_1(t/l)\Pi_{i=1}^{l-1} c_2(t/l) = (c_3 t/l)^l \asymp (c_4 t/R)^{c_+R+1} \geq c_5(t/R)^{c_6 R}, \end{aligned}$$

and the proof is completed.

We next prove the third inequality in (3.15). Take maximum $l \in \mathbb{N}$ such that $t/l \le (R/l)^2$; then $R^2/t - 1 < l \le R^2/t$. Since $t \ge C_* R$, we can take $t/l \ge C_*^2$. Let $x = x_0, x_1, \dots, x_l = y$ be such that $|x_i - x_{i+1}| \asymp R/l$ for $i = 1, \dots, l-1$. Since $(R/l)^2 \asymp t/l \ge C_*^2$, by Theorem 3.5, we have

$$p(t/l, x_i, x_{i+1}) \ge c_1 (t/l)^{-d/2}. \tag{3.17}$$

Using (3.17), we have

$$\begin{aligned} p(t,x,y) &\ge \int_{B_1} \dots \int_{B_{l-1}} p(t/l, x, y_1) \dots p(t/l, y_{l-1}, y) dy_1 \dots dy_{l-1} \\ &\ge c_1 (t/l)^{-d/2} \Pi_{i=1}^{l-1} \left(c_2 (t/l)^{-d/2} (R/l)^d \right) \\ &\ge c_1 (t/l)^{-d/2} c_2^{l-1} \\ &\ge c_1 (t/l)^{-d/2} \exp(-c_3 l) \\ &\ge c_4 t^{-d/2} \exp\left(-\frac{c_5 |x-y|^2}{t} \right). \end{aligned}$$

This completes the proof. □

Remark 3.7 In [12], the following two-sided transition density function estimate was obtained for the relativistic α-stable-like process where $J(x,y) \asymp |x-y|^{-d-\alpha} e^{-|x-y|}$: for $t \le 1$:

$$\begin{aligned} c_1 \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \right) e^{-c_2|x-y|} &\le p(t,x,y) \\ &\le c_3 \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \right) e^{-c_4|x-y|}. \end{aligned}$$

Theorems 2.3 and 3.6 show that when $|x-y| \to \infty$, the behavior of the heat kernel for finite range α-stable-like process is different from that of relativistic α-stable-like process.

We will use the following near diagonal lower bound for the killed process in the next section. Recall that $R_* \in (0,1)$ is the constant given in Proposition 2.1(ii).

Proposition 3.8 *For every $c_1 \in (0,1)$, $c_2, c_3 > 0$, there is a constant $c_4 > 0$ such that for every $x_0 \in \mathbb{R}^d$ and $r \le R_*$,*

$$p^{B(x_0,r)}(t,x,y) \ge c_4 t^{-d/\alpha} \quad \textit{for q.e. } x, y \in B(x_0, c_1 r) \quad \textit{and} \quad t \in [c_2 r^\alpha,\ c_3 r^\alpha]. \tag{3.18}$$

Proof Let $\kappa := c_2/(2c_3)$ and $B_r := B(x_0, r)$. We first show that there is a constant $c_5 \in (0,1)$ so that (3.18) holds for every $r \le R_*$, quasi-every $x, y \in B(x_0, c_1 r)$ and $t \in [\kappa\, c_5 r^\alpha,\ c_5 r^\alpha]$. We will use the following Dynkin–Hunt formula, which is easy to establish using the strong Markov property, since we know the existence of the heat kernels:

$$p^{B_r}(t,x,y) = p(t,x,y) - \mathbb{E}^x[1_{\{\tau_{B_r}\le t\}} p(t-\tau_{B_r}, X_{\tau_{B_r}}, y)]. \tag{3.19}$$

For $r \le R_*$ and $t \in [\kappa c_5 r^\alpha, c_5 r^\alpha]$, and $x, y \in B(x_0, c_1 r)$, by (3.19) and Proposition 2.1(i) and (ii) ($|x-y| \le 2c_1 r \le 2c_1(\kappa c_5)^{-1/\alpha} t^{-1/\alpha}$), we have

$$p^{B_r}(t,x,y) \ge c_6 c_5^{1+d/\alpha} t^{-d/\alpha} - c_7 \mathbb{E}^x\left[1_{\{\tau_{B_r}\le t\}}\left((t-\tau_{B_r})^{-d/\alpha} \wedge \frac{t-\tau_{B_r}}{|X_{\tau_{B_r}}-y|^{d+\alpha}}\right)\right], \tag{3.20}$$

where constants c_6, c_7 are independent of $c_5 \in (0,1]$. Observe that

$$|X_{\tau_{B_r}} - y| \ge (1-c_1)r, \qquad t-\tau_{B_r} \le t \le c_5 r^\alpha$$

and so

$$\frac{t-\tau_{B_r}}{|X_{\tau_{B_r}}-y|^{d+\alpha}} \le \frac{t-\tau_{B_r}}{((1-c_1)r))^{d+\alpha}} \le \frac{c_5^{1+d/\alpha}}{(1-c_1)^{d+\alpha}} t^{-d/\alpha}. \tag{3.21}$$

Note that if $c_5 < ((1-c_1)/2)^\alpha$, by Proposition 2.1 (i), for $t \le c_5 r^\alpha$

$$\mathbb{P}_x\left(X_t \notin B(x,(1-c_1)r/2)\right) = \int_{B(x,(1-c_1)r/2)^c} p(t,x,y)dy$$
$$\le c_7 \int_{B(x,(1-c_1)r/2)^c} \frac{t}{|x-y|^{d+\alpha}} dz \le c_8 \frac{t}{r^\alpha} \le c_8 c_5$$

where c_8 is independent of c_5. Now applying [1, Lemma 3.8], we have

$$\mathbb{P}_x\left(\tau_{B(x,(1-c_1)r)} \le t\right) \le 2c_8 c_5. \tag{3.22}$$

Consequently, we have from (3.20), (3.21) and (3.22)

$$\begin{aligned} p^{B_r}(t,x,y) &\ge \left(c_6 c_5^{1+d/\alpha} - c_7 \frac{c_5^{1+d/\alpha}}{(1-c_1)^{d+\alpha}} \mathbb{P}_x\left(\tau_{B_r} \le t\right)\right) t^{-d/\alpha} \\ &\ge \left(c_6 c_5^{1+d/\alpha} - c_7 \frac{c_5^{1+d/\alpha}}{(1-c_1)^{d+\alpha}} \mathbb{P}_x\left(\tau_{B(x,(1-c_1)r)} \le t\right)\right) t^{-d/\alpha} \\ &\ge c_5^{1+d/\alpha}\left(c_6 - 2c_8 c_7 \frac{c_5}{(1-c_1)^{d+\alpha}}\right) t^{-d/\alpha}. \end{aligned}$$

Clearly we can choose $c_5 < ((1-c_1)/2)^\alpha$ small so that $p^{B_r}(t,x,y) \geq c_9 t^{-d/\alpha}$. This establishes (3.18) for any $x_0 \in \mathbb{R}^d$, $r \leq R_*$ and $t \in [\kappa\, c_5 r^\alpha,\ c_5 r^\alpha]$.

Now for $r \leq R_*$ and $t \in [c_2 r^\alpha,\ c_3 r^\alpha]$, define $k_0 = [2c_3/c_5]+1$. Here for $a \geq 1$, $[a]$ denotes the largest integer that does not exceed a. Then $t/k_0 \in [\kappa\, c_5 r^\alpha,\ c_5 r^\alpha]$. Using semigroup k_0 times, we conclude that for q.e. $x, y \in B(x_0, c_1 r)$ and $t \in [c_2 r^\alpha,\ c_3 r^\alpha]$,

$$
\begin{aligned}
& p^{B(x_0,r)}(t,x,y) \\
&= \int_{B(x_0,r)} \cdots \int_{B(x_0,r)} p^{B(x_0,r)}(t/k_0, x, w_1) \ldots \\
&\quad p^{B(x_0,r)}(t/k_0, w_{n-1}, y) dw_1 \ldots dw_{n-1} \\
&\geq \int_{B(x_0,(t/k_0)^{1/\alpha}/2)} \cdots \int_{B(x_0,(t/k_0)^{1/\alpha}/2)} p^{B(x_0,r)}(t/k_0, x, w_1) \ldots \\
&\quad p^{B(x_0,r)}(t/k_0, w_{n-1}, y) dw_1 \ldots dw_{n-1} \\
&\geq c_9 (t/k_0)^{-d/\alpha} \left(c_9 (t/k_0)^{-d/\alpha}\, c_1 r^d \right)^{k_0 - 1} \\
&\geq c_{10} t^{-d/\alpha},
\end{aligned}
$$

where $c_{10} := c_9^{k_0} k_0^{d/\alpha} \left(c_1 c_9 c_5^{-d/\alpha} \right)^{k_0-1}$. The proof of (3.18) is now completed. □

4 Applications of heat kernel estimates

4.1 Parabolic Harnack inequality

We first introduce a space-time process $Z_s := (V_s, X_s)$, where $V_s = V_0 - s$. The filtration generated by Z satisfying the usual condition will be denoted as $\{\widetilde{\mathcal{F}}_s;\ s \geq 0\}$. The law of the space-time process $s \mapsto Z_s$ starting from (t,x) will be denoted as $\mathbb{P}^{(t,x)}$.

We say that a non-negative Borel measurable function $h(t,x)$ on $[0,\infty) \times \mathbb{R}^d$ is *parabolic* (or *caloric*) on $D = (a,b) \times B(x_0, r)$ if for every relatively compact open subset D_1 of D, $h(t,x) = \mathbb{E}^{(t,x)}[h(Z_{\tau_{D_1}})]$ for every $(t,x) \in D_1 \cap ([0,\infty) \times \mathbb{R}^d)$, where $\tau_{D_1} = \inf\{s > 0 :\ Z_s \notin D_1\}$.

For each $r > 0$, we define

$$
\psi(r) := r^\alpha \vee r^2.
$$

Theorem 4.1 *For every $\delta \in (0,1)$, there exists $c = c(\alpha, \delta) > 0$ such that for every $x_0 \in \mathbb{R}^d$, $t_0 \geq 0$, $R > 0$ and every non-negative function u on $[0,\infty) \times \mathbb{R}^d$ that is* *parabolic on* $(t_0, t_0 + 6\delta\psi(R)) \times B(x_0, 4R)$*,*

$$\sup_{(t_1,y_1)\in Q_-} u(t_1, y_1) \le c \inf_{(t_2,y_2)\in Q_+} u(t_2, y_2), \tag{4.1}$$

where $Q_- = (t_0 + \delta\psi(R), t_0 + 2\delta\psi(R)) \times B(x_0, R)$ *and* $Q_+ = (t_0 + 3\delta\psi(R), t_0 + 4\delta\psi(R)) \times B(x_0, R)$*.*

To prove the theorem, we need one notion and one lemma. According to [2], we say (UJS) holds if

$$J(x, y) \le \frac{c}{r^d} \int_{B(x,r)} J(x', y)dx' \quad \text{whenever } r \le \frac{1}{2}|x - y|,\ x, y \in \mathbb{R}^d. \tag{UJS}$$

For $R > 0$, we say (UJS)$_{\le R}$ holds if the above holds for all $x, y \in \mathbb{R}^d$ and $r \le \frac{|x-y|}{2} \wedge R$.

It is easy to check that finite range jump process satisfies (UJS)$_{\le 1}$.

The following lemma corresponds to [11, Lemma 4.9] (also [12, Lemmas 6.1]). The statement is changed (in the sense that the size of two space-time balls are different and the initial points are also different) and the proof requires major changes from the original ones.

Lemma 4.2 *Let* $R \le R_*$ *and* $\delta < 1$*.* $Q_1 = [t_0 + 2\delta R^\alpha/3,\ t_0 + 5\delta R^\alpha] \times B(x_0, 3R/2)$*,* $Q_2 = [t_0 + \delta R^\alpha/3,\ t_0 + 6\delta R^\alpha] \times B(x_0, 2R)$ *and define* Q_- *and* Q_+ *as in Theorem 4.1. Let* $h : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}_+$ *be bounded and supported in* $[0, \infty) \times B(x_0, 3R)^c$*. Then there exists* $C_1 = C_1(\delta) > 0$ *such that the following holds*:

$$\mathbb{E}^{(t_1,y_1)}[h(Z_{\tau_{Q_1}})] \le C_1 \mathbb{E}^{(t_2,y_2)}[h(Z_{\tau_{Q_2}})] \quad \text{for } (t_1, y_1) \in Q_- \text{ and } (t_2, y_2) \in Q_+.$$

Proof Without loss of generality, assume that $t_0 = 0$. Denote $B_{cR} = B(x_0, cR)$. Using the Lévy system formula,

$$\begin{aligned}
\mathbb{E}^{(t_2,y_2)}[h(Z_{\tau_{Q_2}})] &= \mathbb{E}^{(t_2,y_2)}[h(t_2 - (\tau_{B_{2R}} \wedge (t_2 - \delta R^\alpha/3)), X_{\tau_{B_{2R}} \wedge (t_2 - \delta R^\alpha/3)})] \\
&= \mathbb{E}^{(t_2,y_2)}\left[\int_0^{t_2-\delta R^\alpha/3} \mathbf{1}_{\{t \le \tau_{B_{2R}}\}} dt \int_{B_{3R}^c} h(t_2 - t, v) J(X_t, v) dv\right] \\
&= \int_0^{t_2-\delta R^\alpha/3} h(t_2 - t, v) dt \int_{B_{3R}^c} \mathbb{E}^{(t_2,y_2)}[\mathbf{1}_{\{t \le \tau_{B_{2R}}\}} J(X_t, v)] dv \\
&= \int_{\delta R^\alpha/3}^{t_2} h(s, v) ds \int_{B_{3R}^c} \mathbb{E}^{(0,y_2)}\left[\mathbf{1}_{\{t_2 - s \le \tau_{B_{2R}}\}} J(X_{t_2-s}, v)\right] dv
\end{aligned}$$

$$= \int_{\delta R^\alpha/3}^{t_2} ds \int_{B_{3R}^c} h(s,v)dv \int_{B_{2R}} p^{B_{2R}}(t_2-s, y_2, z) J(z,v)dz \tag{4.2}$$

$$\geq \int_{\delta R^\alpha/3}^{t_1} ds \int_{B_{3R}^c} h(s,v)dv \int_{B_{2R}} p^{B_{2R}}(t_2-s, y_2, z) J(z,v)dz$$

$$\geq \int_{\delta R^\alpha/3}^{t_1} ds \int_{B_{3R}^c} h(s,v)dv \int_{B_{3R/2}} p^{B_{2R}}(t_2-s, y_2, z) J(z,v)dz. \tag{4.3}$$

Since $6\delta R^\alpha \geq t_2 - s \geq t_2 - t_1 \geq \delta R^\alpha$ for $s \in [\delta R^\alpha/3,\ t_1]$, by Proposition 3.8, we have that the right hand side of (4.3) is greater than or equal to

$$\frac{c_1}{R^d} \int_{\delta R^\alpha/3}^{t_1} ds \int_{B_{3R}^c} h(s,v)dv \int_{B_{3R/2}} J(z,v)dz.$$

So, the proof is complete once we obtain

$$\mathbb{E}^{(t_1,y_1)}[h(Z_{\tau_{Q_1}})] \leq \frac{c_2}{R^d} \int_{\delta R^\alpha/3}^{t_1} ds \int_{B_{3R}^c} h(s,v)dv \int_{B_{3R/2}} J(z,v)dz. \tag{4.4}$$

Analogous to (4.2), we have by using the Lévy system,

$$\mathbb{E}^{(t_1,y_1)}[h(Z_{\tau_{Q_1}})] = \int_{2\delta R^\alpha/3}^{t_1} ds \int_{B_{3R}^c} h(s,v)dv \int_{B_{3R/2}} p^{B_{3R/2}}(t_1-s, y_1, z) J(z,v)dz.$$

Since

$$\int_{B_{3R/2}} p^{B_{3R/2}}(t_1-s, y_1, z) \int_{B_{3R}^c} J(z,v)h(s,v)dvdz$$
$$= \int_{B_{5R/4}} p^{B_{3R/2}}(t_1-s, y_1, z) \int_{B_{3R}^c} J(z,v)h(s,v)dvdz$$
$$+ \int_{B_{3R/2}\setminus B_{5R/4}} p^{B_{3R/2}}(t_1-s, y_1, z) \int_{B_{3R}^c} J(z,v)h(s,v)dvdz = I_1 + I_2.$$

When $z \in B_{3R/2}\setminus B_{5R/4}$, we have $|y_1 - z| \geq R/4$, so by Proposition 2.1(i), $p_s^{B_{3R/2}}(y_1, z) \leq c_3 R^{-d}$ for some constant $c_3 > 0$ and $\int_0^{t_1} I_2\, ds$ is less than or equal to the right hand side of (4.4). For $z \in B_{5R/4}$ by $\mathbf{UJS}_{\le 1}$,

$$\int_{B_{3R}^c} J(z,v)h(s,v)dv \le \frac{c_4}{R^d} \int_{B(z,R/6)} \int_{B_{3R}^c} J(z',v)h(s,v)dvdz' \le \frac{c_4}{R^d} \int_{B_{3R/2}} \int_{B_{3R}^c} J(z',v)h(s,v)dvdz'$$

since $B(z, R/6) \subset B_{3R/2}$. Since the right hand side of the above inequality does not depend on z anymore, multiplying both sides by $p^{B_{3R/2}}(t_1 - s, y_1, z)$ and integrating over $z \in B_{5R/4}$ (and further integrating over $\int_{2\delta R^\alpha/3}^{t_1} ds$), we obtain $\int_0^{t_1} I_1 ds$ is less than or equal to the right hand side of (4.4). This proves the lemma. □

Proof of Theorem 4.1 Let R_* denote the constant given in Proposition 2.1. We first consider the case that u is non-negative and bounded on $[0,\infty) \times \mathbb{R}^d$.

(1) Suppose $R \le R_*/2$. When $t \in (0, R_*^\alpha]$ and $|x - y| \le R_*$, one can prove [11, Lemmas 4.11] (also see [12, Lemma 6.2]) from our heat kernel estimates in Proposition 2.1. Given our Lemma 4.2 and the lemmas corresponding to [11, Lemmas 4.11 and 4.13], the proof of the parabolic Harnack inequality is similar to those in [11,12] with some modification. We skip the details here. Interested reader can find its full proof in [10].

(2) Suppose $R \in (R_*/2, 1]$ and let $(t_1, x_1) \in Q_-$ and $(t_2, x_2) \in Q_+$. Without loss of generality, we may assume $x_0 = 0$ and $t_0 = 0$. We further assume that $|x_1 - x_2| \le R_*/8$. If not, we just repeat the argument below at most $16[R/R^*]$ times.

For notational convenience, denote $R_*/2$ by r_* and let $B^1 = B(x_1, r_*)$, $B^2 = B(x_1, r_*/2)$. Define

$$Q_1 = \left(t_1 + \tfrac{\delta}{2}\psi(r_*),\ t_1 + \tfrac{3\delta}{4}\psi(r_*)\right) \times \left(B^1 \backslash B^2\right) \quad \text{and} \quad Q_2 = [0, t_2] \times B^2.$$

Since u is parabolic, by the case (1) but with

$$\left(t_1 - \tfrac{\delta}{4}\psi(r_*),\ t_1 + \tfrac{\delta}{4}\psi(r_*)\right) \times B^1 \quad \text{and} \quad \left(t_1 + \tfrac{\delta}{2}\psi(r_*),\ t_1 + \tfrac{3\delta}{4}\psi(r_*)\right) \times B^1$$

in place of Q_- and Q_+ respectively, we have

$$u(t_2, x_2) = \mathbb{E}^{(t_2,x_2)}\left[u(Z_{\tau_{Q_2}})\right] \ge \mathbb{E}^{(t_2,x_2)}\left[u(Z_{\tau_{Q_2}}) : Z_{\tau_{Q_2}} \in Q_1\right] \ge c_1\, u(t_1, x_1)\, \mathbb{P}^{(t_2,x_2)}\left(Z\tau_{Q_2} \in Q_1\right).$$

Since $|y-z| < 2r_* = R_* < 1$ for every $(y,z) \in B^2 \times B^1$, we have by the Lévy system formula for X that

$$\begin{aligned}\mathbb{P}^{(t_2,x_2)}\left(Z_{\tau_{Q_2}} \in Q_1\right) &= \mathbb{P}^{x_2}\left(X_{\tau_{B^2}} \in B^1, t_2-t_1-\tfrac{3\delta}{4}\psi(r_*) < \tau_{B^2} < t_2-t_1-\tfrac{\delta}{2}\psi(r_*)\right)\\ &\geq c_2 \int_{t_2-t_1-\frac{3\delta}{4}\psi(r_*)}^{t_2-t_1-\frac{\delta}{2}\psi(r_*)} \int_{B^2} \left(\int_{B^1\setminus\overline{B^2}} \frac{p^{B^2}(s,x_2,y)}{|y-z|^{d+\alpha}} dz \right) dy ds\\ &\geq c_3 \int_{t_2-t_1-\frac{3\delta}{4}\psi(r_*)}^{t_2-t_1-\frac{\delta}{2}\psi(r_*)} \int_{B^2} p^{B^2}(s,x_2,y) dy ds\end{aligned}$$

for some positive constants $c_2 = c_2(\alpha,d)$ and $c_3 = c_3(\alpha,d,R_*)$. Note that

$$\tfrac{\delta}{4}\psi(r_*) \leq t_2 - t_1 - \tfrac{3\delta}{4}\psi(r_*) \leq t_2 - t_1 - \tfrac{\delta}{2}\psi(r_*) \leq 3\delta\psi(2r_*).$$

Applying Proposition 3.8 to $p^{B^2}(s,x_2,y)$, we have

$$\begin{aligned}\mathbb{P}^{(t_2,x_2)}\left(Z_{\tau_{Q_2}} \in Q_1\right) &\geq c_4 \int_{t_2-t_1-\frac{3\delta}{4}\psi(r_*)}^{t_2-t_1-\frac{\delta}{2}\psi(r_*)} \int_{B(x_1,\psi(r_*/8))} s^{-d/\alpha}\mu_d(dy)ds\\ &\geq c_5 \tfrac{\delta}{4}\psi(r_*) > 0.\end{aligned}$$

This proves that $u(t_2,x_2) \geq c_6 u(t_1,x_1)$ for some positive constant $c_6{=}c_6(d,\alpha,R_*,\delta)$.

(3) Now let's consider the case $R \geq 1$. We will use balayage; see [6, Chap. VI] for details, and see [1, Theorem 1.7] and [2, Proposition 3.3] for similar arguments. Without loss of generality, we may assume $x_0 = 0$ and $t_0 = 0$. Let $B = B(0,4R)$, $B' = B(0,3R)$, $E = (0,6\delta\psi(R)) \times \overline{B'}$, $Q = (0,6\delta\psi(R)) \times B$. As in the proof of [1, Theorem 1.7], we define u_E, the réduite of u with respect to E by

$$u_E(s,x) = \mathbb{E}^{(s,x)}[u(V_{T_E}, X_{T_E}) : T_E < \tau_Q],$$

where $T_E = \inf\{s \geq 0 : Z_s \in E\}$; then $u = u_E$ on E. By the balayage formula, there exists a measure ν_E supported on $\bar{E}$ such that

$$u_E(t,x) = \int_E p^B(t-r,x,z)\nu_E(dr,dz) \qquad \text{for all } (t,x) \in Q, \tag{4.5}$$

where $p^B(s,x,y) = 0$ if $s < 0$.

Let $(t_1, x_1) \in Q_-$ and $(t_2, x_2) \in Q_+$ and observe that

$$3\delta\psi(R) \geq t_2 - r \geq t_2 - t_1 \geq \delta\psi(R) \qquad \text{for every } r \in [0, t_1].$$

It follows from Theorem 3.5 and semigroup property that

$$p^B(t_2 - r, y, z) \geq c_1 R^{-d}, \qquad \text{for all } y, z \in \overline{B'}, r \in [0, t_1].$$

The above gives us that

$$u_E(t_2, x_2) \geq \int\limits_{[0,t_1]\times\overline{B'}} p^B(t - r, x, z)\nu_E(dr, dz) \geq \frac{c_1}{R^d}\nu_E([0, t_1] \times \overline{B'}).$$

Thus in order to prove the parabolic Harnack inequality, it suffices to show the following for each $(t_1, x_1) \in Q_-$;

$$u_E(t_1, x_1) = \int\limits_{[0,t_1]\times\overline{B'}} p^B(t_1 - r, x_1, z)\nu_E(dr, dz) \leq \frac{c_2}{R^d}\nu_E([0, t_1] \times \overline{B'}). \quad (4.6)$$

Since the jumps of the process X are bounded by 1 and $R \geq 1$, u_E is parabolic (caloric) on $(0, 6\delta\psi(R)) \times B(0, 2R)$. It follows that the support of ν_E is contained in $\bar{E}\backslash(0, 6\delta\psi(R)) \times B(0, 2R)$. Thus, we can write

$$u_E(t_1, x_1) = \int\limits_{F_1(t_1)\cup F_2} p^B(t_1 - r, x_1, z)\nu_E(dr, dz),$$

where $F_1(t) := [0, t] \times (\overline{B'}\backslash B(0, R))$, $F_2 = \{0\} \times \overline{B'}$. If $(r, z) \in F_1(t_1)$, then $|x_1 - z| \geq R$, so by (2.3) when $t_1 - r \geq \delta\psi(R)$ and by Proposition 2.1 (i) and Theorem 2.3 otherwise, we have $p^B(t_1 - r, x_1, z) \leq c_3 R^{-d}$. If $(r, z) \in F_2$, then $t_1 - r \geq \delta\psi(R)$ and by (2.3) again, we have $p^B(t_1 - r, x_1, z) \leq c_3 R^{-d}$. Thus,

$$u_E(t_1, x_1) \leq \frac{c_3}{R^d}\nu_E(F_1(t_1) \cup F_2) \leq \frac{c_2}{R^d}\nu_E([0, t_1] \times \overline{B'})$$

and (4.6) is established.

Finally, we will prove (4.1) when u is not necessarily bounded on $[0, \infty) \times \mathbb{R}^d$. Let U be a bounded domain such that $\overline{Q_- \cup Q_+} \subset U \subset \overline{U} \subset (t_0, t_0 + 6\delta\psi(R)) \times B(x_0, 4R)$. For any $n \in \mathbb{N}$, define $u_n(t, x) = \mathbb{E}^{(t,x)}[(u \wedge n)(Z_{\tau_U})]$. Then u_n is non-negative and bounded on $[0, \infty) \times \mathbb{R}^d$, parabolic on U and $\lim_{n\to\infty} u_n(t, x) = u(t, x)$ for $x \in [0, \infty) \times \mathbb{R}^d$. From the above arguments, we see that (4.1) holds for u_n with the constant c independent of n. Letting $n \to \infty$, we obtain (4.1) for u. □

By the same proof as that for [11, Theorem 4.14] or [12, Proposition 4.14], we have the following Hölder continuity for parabolic functions.

Theorem 4.3 *For every $R_0 \in (0,1]$, there are constants $c = c(R_0) > 0$ and $\kappa > 0$ such that for every $0 < R \le R_0$ and every bounded parabolic function h in $Q(0, x_0, 2R) := (0, (2R)^\alpha) \times B(x_0, 2R)$,*

$$|h(s,x) - h(t,y)| \le c\, \|h\|_{\infty,R}\, R^{-\kappa} \left(|t-s|^{1/\alpha} + |x-y|\right)^{\kappa} \tag{4.7}$$

holds for $(s,x), (t,y) \in Q(0, x_0, R)$, where $\|h\|_{\infty,R} := \sup_{(t,y)\in[0,(2R)^\alpha]\times\mathbb{R}^d} |h(t,y)|$. In particular, for the transition density function $p(t,x,y)$ of X, for any $t_0 \in (0,1)$, there are constants $c = c(t_0) > 0$ and $\kappa > 0$ such that for any $t, s \in [t_0, 1]$ and $(x_i, y_i) \in \mathbb{R}^d \times \mathbb{R}^d$ with $i = 1, 2$,

$$|p(s,x_1,y_1) - p(t,x_2,y_2)| \le c\, t_0^{(-d+\kappa)/\alpha} \left(|t-s|^{1/\alpha} + |x_1-x_2| + |y_1-y_2|\right)^{\kappa}. \tag{4.8}$$

Remark 4.4 (i) Since the heat kernel $p(t,x,y)$ is Hölder continuous, the estimates derived in previous sections for $p(t,x,y)$ hold for every $x, y \in \mathbb{R}^d$.

(ii) Note that the proof of Theorem 4.3 needs only the short time heat kernel estimates in Proposition 2.1 on $p(t,x,y)$ for $t \in (0, T^*]$ and for q.e. $x, y \in \mathbb{R}^d$ having $|x-y| \le R_*$ for some $T_* \in (0,1)$ and $R_* \in (0,1]$. Therefore as long as a pure jump symmetric strong Markov process Y has a transition density function $p(t,x,y)$ that has the two sided short time finite range estimates as that in Theorem 4.3 for $t \in (0, t_0)$ and $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$ with $|x-y| \le r_0$ for some t_0 and $r_0 > 0$, it can be established directly from these heat kernel estimate that every bounded parabolic functions of Y is Hölder continuous. If in addition we have $(\mathrm{UJS})_{\le 1}$ and the lower bound on the heat kernel $p^B(t,x,y)$ as in Proposition 3.8, then the parabolic Harnack inequality (Theorem 4.1) can be proved for $R \le R_*/2$.

(iii) In fact, $(\mathrm{UJS})_{\le 1}$ is necessary for the parabolic Harnack inequality for $R \le 1$. This is proved in [2, Proposition 4.7] for the discrete space setting, and the proof for the continuous space case can be found in [10].

(iv) There is a minor gap in the proof of [12, Lemma 6.1]. Condition $(\mathrm{UJS})_{\le 1}$ should be imposed on the jumping kernel J for this lemma and consequently for the main results (such as Theorems 1.2 and 4.12) of [12]. Note that $(\mathrm{UJS})_{\le 1}$ is automatically satisfied if $\psi \equiv 1$ in (1.12) of [12] (corresponds to the case $\gamma_1 = \gamma_2 = 0$). A sufficient condition for J to satisfy condition $(\mathrm{UJS})_{\le 1}$ is that the function ψ in (1.12) of [12] has the property that $\psi(r+1) \le c_0\, \psi(r)$ for every $r \ge 1$.

Suppose that Y is the Hunt process associated with Dirichlet form $(\mathcal{Q}, \mathcal{D})$ given by (1.2)–(1.3) whose jumping intensity kernel $J(x,y)$ has the property that $J(x,y)\, \mathbf{1}_{\{(x,y):\, d(x,y)>\kappa\}}$ is bounded and

$$J(x,y) = \frac{c(x,y)}{|x-y|^{d+\alpha}} \quad \text{for } |x-y| \le 1 \qquad \text{and}$$
$$\sup_{x\in\mathbb{R}^d} \int_{\{y\in\mathbb{R}^d:\, |y-x|>1\}} J(x,y)dy < \infty, \tag{4.9}$$

where $c(x, y)$ is a function that is bounded between two positive constants and is symmetric in x and y. Then by the Meyer's construction method (see [1, Lemma 3.6] and [3, Lemmas 3.1(c) and (3.18)]), the process Y can be constructed from the finite range α-stable-like process X having jump intensity kernel $\frac{c(x,y)}{|x-y|^{d+\alpha}}\mathbf{1}_{\{|x-y|\le 1\}}$ and so Y has a transition density function $q(t, x, y)$ with respect to μ_d. Moreover, for any ball $B \subset \mathbb{R}^d$,

$$q(t, x, y) \ge e^{-t\|\mathcal{J}\|_\infty} p(t, x, y) \quad \text{and} \quad q^B(t, x, y) \ge e^{-t\|\mathcal{J}\|_\infty} p^B(t, x, y), \tag{4.10}$$

and

$$q(t, x, y) \ \le\ p(t, x, y) + t\|J_1\|_\infty \tag{4.11}$$

where $p(t, x, y)$ is the transition density function of X,

$$J_1(x, y) := J(x, y)\mathbf{1}_{\{d(x,y)>1\}} \quad \text{and} \quad \mathcal{J}(x) := \int_{\mathbb{R}^d} J_1(x, y)\mu_d(dy).$$

Thus using the heat kernel estimate for $p(t, x, y)$ in Proposition 2.1 and Proposition 3.8, by the same line of argument as that in the Remark 4.4(ii) we have the following.

Theorem 4.5 *The Hölder continuity estimate* (4.7) *holds for bounded parabolic functions of Y. In particular, all these applies to the transition density function $q(t, x, y)$ of Y. Moreover, if in addition, we assume (UJS)$_{\le R_1}$ then the parabolic Harnack inequality* (4.1) *holds for non-negative parabolic functions of Y with $R \le R_1/2$.*

The full detail of the above theorem will be given in more general context in [10].

Remark 4.6 Very recently, Kassmann [23, Theorem 1.1] proved by a quite different analytic method the Hölder continuity for bounded harmonic functions of symmetric pure jump processes whose jumping intensity kernel J satisfies the condition

$$\begin{aligned} &J(x, y) = c(x, y)|x - y|^{-d-\alpha} \quad \text{for } |x - y| \le 1 \quad \text{and} \\ &J(x, y) \le c|x - y|^{-d-\eta} \quad \text{for } |x - y| > 1, \end{aligned}$$

where $\eta > 0$ and $c > 0$ are two positive constants. Clearly, such type of jumping kernel is a special case of those given by (4.9). Since every harmonic function is parabolic, our Theorem 4.5 recovers and extends the main result of [23]. See [31] for some related work on the Hölder continuity of bounded harmonic functions for a class of non-local operators.

4.2 Two-sided Green function estimates

When $d = 1, 2$, the finite range α-stable-like processes are all recurrent. So in this subsection, we assume $d \ge 3$ and give two-sided sharp estimates the Green function for $G(x,y)$ of finite range stable-like process X in $\mathbb{R}^d$ where

$$G(x,y) := \int_0^\infty p(t,x,y)dt, \qquad x,y \in \mathbb{R}^d.$$

Theorem 4.7 *There exists $c = c(\alpha, d) > 1$ such that*

$$c^{-1}\left(\frac{1}{|x-y|^{d-\alpha}} + \frac{1}{|x-y|^{d-2}}\right) \le G(x,y) \le c\left(\frac{1}{|x-y|^{d-\alpha}} + \frac{1}{|x-y|^{d-2}}\right), \qquad x,y \in \mathbb{R}^d.$$

Proof We first note that for every $T, M \in [0,\infty)$

$$\int_T^\infty t^{-\frac{d}{2}} e^{-\frac{M|x-y|^2}{2t}} dt = \frac{1}{|x-y|^{d-2}} \int_0^{\frac{|x-y|^2}{T}} u^{\frac{d-4}{2}} e^{-\frac{1}{2}Mu} du. \tag{4.12}$$

Recall that $R_* < 1$ and $T_* = R_*^\alpha$ are the constants from Proposition 2.1(ii). Using (4.12), it is easy to see that, if $|x-y| \le R_*$, by Proposition 2.1(i) and Theorem 2.3

$$G(x,y) \le c_1 \int_0^{|x-y|^\alpha} \frac{t}{|x-y|^{d+\alpha}} dt + c_2 \int_{|x-y|^\alpha}^{T_*} t^{-d/\alpha} dt + c_3 \int_{T_*}^\infty t^{-\frac{d}{2}} e^{-\frac{c_4|x-y|^2}{2t}} dt$$

$$\le \frac{c_5}{|x-y|^{d-\alpha}} + \frac{c_3}{|x-y|^{d-2}} \int_0^{\frac{|x-y|^2}{T_*}} u^{\frac{d-4}{2}} e^{-\frac{1}{2}c_4 u} du \le \frac{c_6}{|x-y|^{d-\alpha}}.$$

On the other hand if $|x-y| > R_*$, by Theorem 2.3 and (4.12)

$$G(x,y) \le c_7 \int_0^{C_*|x-y|} \exp\left(-c_8|x-y| \log\frac{|x-y|}{t}\right) dt + c_9 \int_{C_*|x-y|}^\infty t^{-\frac{d}{2}} e^{-\frac{c_4|x-y|^2}{2t}} dt$$

$$\le c_7 \int_0^{C_*|x-y|} \exp(-c_{10}|x-y|)\, dt + \frac{c_9}{|x-y|^{d-2}} \int_0^{\frac{|x-y|}{C_*}} u^{\frac{d-4}{2}} e^{-\frac{1}{2}c_4 u} du$$

$$\le c_7 C_*|x-y| \exp(-c_{10}|x-y|) + \frac{c_{11}}{|x-y|^{d-2}} \le \frac{c_{12}}{|x-y|^{d-2}}$$

where $C_* < 1$ is given in Theorem 2.3.

The lower bounded is easier. If $|x-y| \le R_*$, by Proposition 2.1(ii)

$$G(x,y) \ \ge \ c_{13} \int_0^{|x-y|^\alpha} \frac{t}{|x-y|^{d+\alpha}} dt = \frac{c_{13}}{2|x-y|^{d-\alpha}}.$$

If $|x-y| \ge R_*$, by Theorem 3.6 and (4.12)

$$G(x,y) \ \ge \ c_{14} \int_{|x-y|^2}^{\infty} t^{-\frac{d}{2}} dt = \frac{c_{14}}{|x-y|^{d-2}} \int_0^1 u^{\frac{d-4}{2}} du.$$

□

Remark 4.8 Under some mild assumptions on bounded open set D, when $c(x,y)$ is a constant, Green function $G_D(x,y)$ for X in D is comparable to the one for isotropically symmetric stable process in D (see [17,25]). Theorem 4.7 shows that, unlike bounded open sets, the behavior of the Green function for X in $\mathbb{R}^d$ is different from the behavior of the Green function for isotropically symmetric stable process in $\mathbb{R}^d$.

Now let's consider the more general non-local Dirichlet form $(\mathcal{E},\mathcal{F})$ in (2.14)–(2.15) with the jumping kernel J satisfying the condition (2.16). Recall that $q(t,x,y)$ is the transition density function for the associated Hunt process Z with respect to the Lebesgue measure on $\mathbb{R}^d$. For $d \ge 3$, let

$$V(x,y) := \int_0^\infty q(t,x,y) dt, \qquad x, y \in \mathbb{R}^d.$$

Using Theorems 2.4 and 3.5 instead of Theorems 2.3 and 3.6, respectively, in the proof of Theorem 4.7, we get the Green function estimate for the process Z for $|x-y| \ge 1$.

Theorem 4.9 *There exists $c = c(\alpha, d) > 1$ such that*

$$c^{-1} \frac{1}{|x-y|^{d-2}} \ \le \ V(x,y) \ \le \ c \frac{1}{|x-y|^{d-2}} \qquad \textit{for } |x-y| \ge 1.$$

4.3 Differentiability of spectral functions

In [32–34], the differentiability of spectral functions for symmetric stable processes are studied.

Recall that X is a finite range stable-like process considered in this paper whose Dirichlet form $(\mathcal{Q},\mathcal{D})$ is given by (1.2)–(1.3) whose jumping intensity kernel $J(x,y) = \frac{c(x,y)}{|x-y|^{d+\alpha}} \mathbf{1}_{\{|x-y| \le 1\}}$. Let μ be a signed measure in Kato class $\mathbf{K}_\infty(X)$ as introduced in [9]. The associated spectral function $C(\lambda)$ is defined to be

$$C(\lambda) = -\inf\left\{ \mathcal{Q}(u,u) + \lambda \int_{\mathbb{R}^d} u(x)^2 \mu(dx) : u \in \mathcal{D} \text{ with } \int_{\mathbb{R}^d} u(x)^2 dx = 1 \right\}.$$

Using the heat kernel estimates established in this paper, by an almost same argument as that in [32–34], it can be shown that if $d \le 4$ and if the extended Dirichlet space $(\mathcal{Q}, \mathcal{D}_e)$ is compactly embedded into $L^2(\mathbb{R}^d, |\mu|)$, then $\lambda \mapsto C(\lambda)$ is differentiable on $\mathbb{R}$. But we will not go into details about it.

5 Weighted Poincaré inequality of fractional order

Throughout this section, $r \ge 1$, $\sigma \in (0, \infty)$ and $\alpha \in (0, 2)$. Recall that μ_d denotes the Lebesgue measure in $\mathbb{R}^d$. In this section, the exact values of the constants c's are always independent of r and they might change from one appearance to another. Let $\mathcal{M}(\sigma)$ be the set of all non-increasing function Ψ from $[0, 1]$ to $[0, 1]$ such that $\Psi(s) > \Psi(1) = 0$ for every $s \in [0, 1)$ and

$$\Psi(s + \tfrac{1}{2}((1-s) \wedge \tfrac{1}{2})) \ge \sigma\, \Psi(s), \qquad s \in (0, 1). \tag{5.1}$$

We will use $\mathcal{N}(\sigma)$ to denote all the functions Φ of the form $c\Psi(|x|)$ for some $\Psi \in \mathcal{M}(\sigma)$ having $\int_{\mathbb{R}^d} \Phi(x)dx=1$. Note that, when $\beta\in(0, 2)$, $c(1-|x|^2)^{12/(2-\beta)}\mathbf{1}_{B(0,1)}(x)$ is in $\mathcal{N}((1/8)^{12/(2-\beta)})$. Condition (5.1) says that for each $\Phi \in \mathcal{N}(\sigma)$, values of Φ at points with comparable distance from the unit sphere $\partial B(0, 1)$ are comparable. This implies that values of Φ in balls in Whitney-type covering, which will be discussed below, are universally comparable to each other. This property will be used in many places below.

For $\Phi \in \mathcal{N}(\sigma)$, define

$$u_\Phi := \int_{B(0,1)} u(x)\Phi(x)dx.$$

This section is devoted to prove the following form of weighted Poincaré inequality.

Theorem 5.1 *For every $d \ge 1$, $0 < \alpha < 2$ and $\sigma \in (0, \infty)$, there exists a positive constant $c_1 = c_1(d, \alpha, \sigma)$ independent of $r \ge 1$, such that for every $\Phi \in \mathcal{N}(\sigma)$ and $u \in L^1(B(0, 1), \Phi(x)dx)$,*

$$\int_{B(0,1)} (u(x) - u_\Phi)^2 \Phi(x)dx$$

$$\le c_1 \int_{B(0,1)\times B(0,1)} (u(x) - u(y))^2 \frac{r^{2-\alpha}}{|x-y|^{d+\alpha}} \mathbf{1}_{\{|x-y|\le 1/r\}} (\Phi(x) \wedge \Phi(y))\, dxdy.$$

Moreover, the constant c_1 stays bounded for $\alpha \in (0, 2)$.

The exponent $2-\alpha$ of r in the integral above is quite delicate to get. We will prove the above theorem through several lemmas. For the remainder of this section, we fix $\sigma \in (0,\infty)$ and $\Phi \in \mathcal{N}(\sigma)$.

We first prove the following simple lemma. Let

$$u_{B(x,s)} := \frac{1}{\mu_d(B(x,s))} \int_{B(x,s)} u(y)dy.$$

Lemma 5.2 *For every $B(z,s) \subset B(0,1)$ and every $u \in L^1(B(z,s),dx)$,*

$$\int_{B(z,s)} (u(x) - u_{B(z,s)})^2 dx \le \frac{1}{\mu_d(B(z,s))} \int_{B(z,s)} \int_{B(z,s)} (u(x)-u(y))^2 dxdy.$$

Proof By Cauchy–Schwartz inequality,

$$\int_{B(z,s)} (u(x) - u_{B(z,s)})^2(x) dx = \int_{B(z,s)} \left(\frac{1}{\mu_d(B(z,s))} \int_{B(z,s)} (u(x)-u(y))dy \right)^2 dx$$

$$\le \frac{1}{\mu_d(B(z,s))} \int_{B(z,s)} \int_{B(z,s)} (u(x)-u(y))^2 dxdy.$$

□

Recall Whitney-type coverings (see [29, Sect. 5.3.3] for details): We first let

$$\overline{\mathcal{W}} := \left\{ B : \text{ the center of the ball } B \text{ is in } B(0,1) \text{ and } r(B) = \frac{1}{10^3}\rho(B) \right\}$$

where $r(B)$ is the radius of the ball B and $\rho(B)$ denotes the Euclidean distance between the ball B and $B(0,1)^c$. In the sequel, for $\lambda > 0$ and a ball $B = B(x,r)$ centered at x with radius r, we denote λB the concentric ball $B(x, \lambda r)$ with radius λr.

Start $\mathcal{W}$ by picking a ball $B^0 \in \overline{\mathcal{W}}$ with the largest possible radius. Pick the next ball B^1 to be a ball in $\overline{\mathcal{W}}$ which does not intersect B^0 and has maximal radius. Assuming that k balls $B^0, \ldots, B^{k-1}$ have already been picked, pick the next ball B^k to be a ball in $\overline{\mathcal{W}}$ which does not intersect $\cup_{j=0}^{k-1} B^j$ and has maximal radius. Though this procedure, we get a sequence of disjoint balls $\mathcal{W} := \{B^0, \ldots, B^{k-1}, B^k, \ldots\}$ from $\overline{\mathcal{W}}$. Moreover, the Whitney-type decomposition of the unit ball $B(0,1)$ has the following properties (see, for example, [29, p. 135]).

(1)

$$B(0,1) = \bigcup_{B \in \mathcal{W}} 2B.$$

(2) There exists a positive constant K such that

$$\sup_{y\in B(0,1)} \#\{B \in \mathcal{W} : y \in 10^2 B\} \le K \tag{5.2}$$

where $\#S$ is the number of elements in the set S.

There exists a ball $B(0) \in \mathcal{W}$ such that $0 \in 2B(0)$. We pick an fix such a ball $B(0)$ and call it the central ball of $\mathcal{W}$. For any $B \in \mathcal{W}$, let γ_B be the straight line segment between the center of B and the origin. Let

$$\overline{\mathcal{W}}(B) := \{A \in \mathcal{W} : 2A \cap \gamma_B \neq \emptyset\}.$$

Now we define the chain $\mathcal{W}(B) := (B_0, B_1, \cdots, B_{l(B)-1})$ with $B_0 = B(0)$ and $B_{l(B)-1} = B$ as follows; Starting from the origin, let y_0 be the first point along γ_B which does not belong to $2B_0$. Define B_1 to be (any) one of balls in $\overline{\mathcal{W}}(B)$ such that $y_0 \in 2B_1$. Inductively, having $B_0, B_1, \ldots, B_k$ constructed, let y_k be the first point along γ_B which does not belong to $\cup_{j=0}^k 2B_j$. Define B_{k+1} to be (any) one of balls in $\overline{\mathcal{W}}(B)$ such that $y_k \in 2B_{k+1}$. When the last chosen is not B, we simply add B as the last ball in $\mathcal{W}(B)$.

Using Lemma 5.2, the next lemma can be proved easily.

Lemma 5.3 *There exists a positive constant $c = c(d)$ such that for every $B \in \mathcal{W}$, $B_i, B_{i+1} \in \mathcal{W}(B)$ and for every $u \in L^1(B(0,1), \Phi dx)$,*

$$|u_{4B_i} - u_{4B_{i+1}}| \le \sum_{j=0}^{1} \frac{c}{\mu_d(B_{i+j})} \left(\int_{4B_{i+j}} \int_{4B_{i+j}} (u(x)-u(y))^2 dxdy \right)^{1/2}.$$

Proof Note that

$$\begin{aligned}
&(\mu_d(4B_i \cap 4B_{i+1}))^{1/2} |u_{4B_i} - u_{4B_{i+1}}| \\
&= \left(\int_{4B_i \cap 4B_{i+1}} |u_{4B_i} - u_{4B_{i+1}}|^2 \mu_d(dx) \right)^{1/2} \\
&\le \left(\int_{4B_i} |u(x) - u_{4B_i}|^2 \mu_d(dx) \right)^{1/2} + \left(\int_{4B_{i+1}} |u(x) - u_{4B_{i+1}}|^2 \mu_d(dx) \right)^{1/2}.
\end{aligned}$$

Now the lemma follows from our Lemma 5.2 and the fact that

$$\mu_d(4B_i \cap 4B_{i+1}) \ge c \max\{\mu_d(B_i), \mu_d(B_{i+1})\}$$

(see Lemma 5.3.7 in [29]). □

Lemma 5.4 *There exists a positive constant $c = c(d, \sigma)$ such that for every $B \in \mathcal{W}$, B_i, $B_{i+1} \in \mathcal{W}(B)$ and for every $u \in L^1(B(0, 1), \Phi dx)$,*

$$\sqrt{\Phi_B}\,|u_{4B_i} - u_{4B_{i+1}}| \leq \sum_{j=0}^{1} \frac{c}{\mu_d(B_{i+j})} \left(\int_{4B_{i+j}} \int_{4B_{i+j}} (u(x) - u(y))^2 (\Phi(x) \wedge \Phi(y)) dx dy \right)^{1/2}.$$

Proof Since the values of Φ are universally comparable to each other on $4B$ for every $B \in \mathcal{W}$, we have from Lemma 5.3

$$|u_{4B_i} - u_{4B_{i+1}}| \leq \sum_{j=0}^{1} \frac{c}{(\mu_d(B_{i+j}))^{1/2} (\int_{B_{i+j}} \Phi(y) dy)^{1/2}} \times \left(\int_{4B_{i+j}} \int_{4B_{i+j}} (u(x) - u(y))^2 (\Phi(x) \wedge \Phi(y)) dx dy \right)^{1/2}. \quad (5.3)$$

Note that

$$\rho(A) = 10^3 r(A) \geq \frac{10^3}{4} r(B) = \frac{1}{4} \rho(B) \qquad \text{for every } A \in \mathcal{W}(B). \quad (5.4)$$

(See Lemma 5.3.6 in [29].) Using (5.1), (5.4) and the fact that Ψ is non-increasing, there exists a positive constant c independent of B such that

$$\max_{y \in B} \Phi(y) \leq c \min_{y \in A} \Phi(y) \qquad \text{for every } A \in \mathcal{W}(B).$$

Thus we have

$$\Phi_B = \frac{1}{\mu_d(B)} \int_B \Phi(y) dy \leq c \frac{1}{\mu_d(B_i)} \int_{B_i} \Phi(y) dy \qquad \text{for every } B_i \in \mathcal{W}(B). \quad (5.5)$$

The lemma follows from (5.3) and (5.5). □

The proof of the next lemma is similar to that of Theorem 5.3.4 on pp. 141–143 of [29]. For reader's convenience, we nevertheless spell out the details of the proof here.

Lemma 5.5 *There exists a positive constant $c = c(d, \sigma)$ such that for every $u \in L^1(B(0, 1), \Phi dx)$,*

$$\int_{B(0,1)} (u(x) - u_\Phi)^2 \Phi(x) dx \leq c \sum_{A \in \mathcal{W}} \frac{1}{\mu_d(A)} \int_{4A \times 4A} (u(x) - u(y))^2 (\Phi(x) \wedge \Phi(y)) dx dy.$$

Proof Note that

$$
\begin{aligned}
&\int_{B(0,1)} (u(x)-u_\Phi)^2\Phi(x)dx\\
&\le 2\int_{B(0,1)} (u(x)-u_{4B(0)})^2\Phi(x)dx + 2\left(\int_{B(0,1)} \Phi(x)dx\right)(u_\Phi - u_{4B(0)})^2\\
&\le 2\int_{B(0,1)} (u(x)-u_{4B(0)})^2\Phi(x)dx + 2\int_{B(0,1)} (u(x)-u_{4B(0)})^2\Phi(x)dx\\
&\le 4\sum_{B\in\mathcal{W}}\int_{4B}(u(x)-u_{4B(0)})^2\Phi(x)dx\\
&\le 8\sum_{B\in\mathcal{W}}\int_{4B}(u(x)-u_{4B})^2\Phi(x)dx + 8\sum_{B\in\mathcal{W}}(u_{4B}-u_{4B(0)})^2\int_{4B}\Phi(x)dx\\
&\le c\sum_{B\in\mathcal{W}}\frac{1}{\mu_d(B)}\int_{4B\times 4B}(u(x)-u(y))^2(\Phi(x)\wedge\Phi(y))dxdy\\
&\quad + c\sum_{B\in\mathcal{W}}\int \mathbf{1}_B(z)\left(|u_{4B}-u_{4B(0)}|(\Phi_B)^{1/2}\right)^2 dz,
\end{aligned}
$$

where in the last inequality, we used the fact that the values of Φ are universally comparable to each other on $4B$ for every $B\in\mathcal{W}$. To establish the lemma, it suffices to deal with the second summation above.

By Lemma 5.4, we get

$$
\begin{aligned}
&|u_{4B}-u_{4B(0)}|(\Phi_B)^{1/2}\mathbf{1}_B(z)\\
&\le \sum_{i=0}^{l(B)-2}|u_{4B_i}-u_{4B_{i+1}}|(\Phi_B)^{1/2}\mathbf{1}_B(z)\\
&\le c\sum_{i=0}^{l(B)-1}\frac{1}{\mu_d(B_i)}\left(\int_{4B_i}\int_{4B_i}(u(x)-u(y))^2(\Phi(x)\wedge\Phi(y))dxdy\right)^{1/2}\mathbf{1}_B(z)\\
&= c\sum_{i=0}^{l(B)-1}\frac{1}{\mu_d(B_i)}\left(\int_{4B_i}\int_{4B_i}(u(x)-u(y))^2(\Phi(x)\wedge\Phi(y))dxdy\right)^{1/2}\mathbf{1}_{10^4B_i}(z)\mathbf{1}_B(z)\\
&\le c\sum_{A\in\mathcal{W}}\frac{1}{\mu_d(A)}\left(\int_{4A}\int_{4A}(u(x)-u(y))^2(\Phi(x)\wedge\Phi(y))dxdy\right)^{1/2}\mathbf{1}_{10^4A}(z)\mathbf{1}_B(z).
\end{aligned}
$$

In the first equality above, we have used the fact that $B \subset 10^4 B_i$ (Lemma 5.3.8 in [29]). Since the balls in $\mathcal{W}$ are disjoint, summing both sides over $B \in \mathcal{W}$ and taking the square, we get

$$\sum_{B\in\mathcal{W}} \mathbf{1}_B(z) \left(|u_{4B} - u_{4B(0)}|(\Phi_B)^{1/2}\right)^2$$
$$\leq c \left(\sum_{A\in\mathcal{W}} \frac{1}{\mu_d(A)} \left(\int_{4A}\int_{4A} (u(x) - u(y))^2 (\Phi(x) \wedge \Phi(y)) dx dy \right)^{1/2} \mathbf{1}_{10^4 A}(z) \right)^2 .$$

Integrating over $z \in B(0, 1)$, and using Lemma 5.3.12 in [29] and the fact the balls in $\mathcal{W}$ are disjoint, we have

$$\sum_{B\in\mathcal{W}} \int \mathbf{1}_B(z) \left(|u_{4B} - u_{4B(0)}|\Phi_B^{1/2}\right)^2 dz$$
$$\leq c \int \left(\sum_{A\in\mathcal{W}} \frac{1}{\mu_d(A)} \left(\int_{4A}\int_{4A} (u(x) - u(y))^2 (\Phi(x) \wedge \Phi(y)) dx dy \right)^{1/2} \mathbf{1}_{10^4 A}(z) \right)^2 dz$$
$$\leq c \int \left(\sum_{A\in\mathcal{W}} \frac{1}{\mu_d(A)} \left(\int_{4A}\int_{4A} (u(x) - u(y))^2 (\Phi(x) \wedge \Phi(y)) dx dy \right)^{1/2} \mathbf{1}_{A}(z) \right)^2 dz$$
$$\leq c \int \sum_{A\in\mathcal{W}} \frac{1}{(\mu_d(A))^2} \left(\int_{4A}\int_{4A} (u(x) - u(y))^2 (\Phi(x) \wedge \Phi(y)) dx dy \right) \mathbf{1}_{A}(z) dz$$
$$\leq c \sum_{A\in\mathcal{W}} \frac{1}{\mu_d(A)} \left(\int_{4A}\int_{4A} (u(x) - u(y))^2 (\Phi(x) \wedge \Phi(y)) dx dy \right) .$$

This completes the proof for the lemma. □

Lemma 5.6 *There exists a positive constant $c = c(d, \sigma)$ such that for every $u \in L^1(B(0, 1), \Phi dx)$,*

$$\int_{B(0,1)} (u(x) - u_\Phi)^2 \Phi(x) dx$$
$$\leq \frac{c}{10^{2\alpha}} \int_{B(0,1)\times B(0,1)} \frac{(u(x) - u(y))^2}{|x - y|^{d+\alpha}} \mathbf{1}_{\{|x-y| \leq \frac{1}{10^2}\}} (\Phi(x) \wedge \Phi(y)) dx dy.$$

Proof Since $|x-y| \le 8r(A) \le \frac{1}{10^2}$ if $x, y \in 4A$, we have for every $A \in \mathcal{W}$

$$
\begin{aligned}
&\frac{1}{\mu_d(A)} \int\limits_{4A\times 4A} (u(x)-u(y))^2(\Phi(x)\wedge\Phi(y))dxdy \\
&\le \frac{c}{(r(A))^d} \int\limits_{4A\times 4A} \frac{(u(x)-u(y))^2|x-y|^{d+\alpha}}{|x-y|^{d+\alpha}} \mathbf{1}_{\{|x-y|\le \frac{1}{10^2}\}}(\Phi(x)\wedge\Phi(y))dxdy \\
&\le \frac{c}{10^{2\alpha}} \int\limits_{4A\times 4A} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}} \mathbf{1}_{\{|x-y|\le \frac{1}{10^2}\}}(\Phi(x)\wedge\Phi(y))dxdy.
\end{aligned}
$$

It then follows from Lemma 5.5 and (5.2) that

$$
\begin{aligned}
&\int\limits_{B(0,1)} (u(x)-u_\Phi)^2\Phi(x)dx \\
&\quad\le \frac{c}{10^{2\alpha}} \sum_{A\in\mathcal{W}} \int\limits_{4A\times 4A} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}} \mathbf{1}_{\{|x-y|\le \frac{1}{10^2}\}}(\Phi(x)\wedge\Phi(y))dxdy \\
&\quad\le \frac{c}{10^{2\alpha}} \int\limits_{B(0,1)\times B(0,1)} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}} \mathbf{1}_{\{|x-y|\le \frac{1}{10^2}\}}(\Phi(x)\wedge\Phi(y))dxdy.
\end{aligned}
$$

□

Due to Lemma 5.6, we have Theorem 5.1 for $1 \le r \le 10^2$. So, from now we may assume $r > 10^2$.

Lemma 5.7 *There exists a positive constant $c = c(d,\sigma)$ such that for every $r > 10^2$ for every $u \in L^1(B(0,1), \Phi dx)$,*

$$
\begin{aligned}
&\int\limits_{B(0,1)} (u(x)-u_\Phi)^2\Phi(x)dx \\
&\le c \int\limits_{B(0,1)\times B(0,1)} (u(x)-u(y))^2 \frac{r^{-\alpha}}{|x-y|^{d+\alpha}} \mathbf{1}_{\{|x-y|<1/r\}}(\Phi(x)\wedge\Phi(y))dxdy \\
&\quad + c \int\limits_{B(0,1-\frac{10}{r})\times B(0,1-\frac{10}{r})} \frac{(u(x)-u(y))^2}{|x-y|^{d}} \mathbf{1}_{\{|x-y|<\frac{1}{10^2}\}}(\Phi(x)\wedge\Phi(y))dxdy.
\end{aligned}
$$

Proof By Lemma 5.5, we have

$$\begin{aligned}
&\int_{B(0,1)} (u(x)-u_\Phi)^2\Phi(x)dx \\
&\le c\sum_{A\in\mathcal{W}}\int_{4A\times 4A}\frac{(u(x)-u(y))^2}{|x-y|^d}\left(\frac{|x-y|}{r(A)}\right)^d(\Phi(x)\wedge\Phi(y))dxdy \\
&\le c\left(\sum_{A\in\mathcal{W}:\, r(A)\le\frac{1}{10r}}+\sum_{A\in\mathcal{W}:\, r(A)>\frac{1}{10r}}\right)\int_{4A\times 4A}\frac{(u(x)-u(y))^2}{|x-y|^d}(\Phi(x)\wedge\Phi(y))dxdy \\
&=: I+II.
\end{aligned}$$

If $A\in\mathcal{W}$ and $r(A)\le\frac{1}{10r}$, then $|x-y|\le 8r(A)<\frac{1}{r}$ for every $x, y\in 4A$. So using (5.2), we have

$$\begin{aligned}
I &\le c\sum_{A\in\mathcal{W}:\, r(A)\le\frac{1}{10r}}\int_{4A\times 4A}(u(x)-u(y))^2\frac{r^{-\alpha}}{|x-y|^{d+\alpha}}\mathbf{1}_{\{|x-y|\le 1/r\}}(\Phi(x)\wedge\Phi(y))dxdy \\
&\le c\int_{B(0,1)\times B(0,1)}(u(x)-u(y))^2\frac{r^{-\alpha}}{|x-y|^{d+\alpha}}\mathbf{1}_{\{|x-y|<1/r\}}(\Phi(x)\wedge\Phi(y))dxdy.
\end{aligned}$$

On the other hands, if $A\in\mathcal{W}$ and $r(A)>\frac{1}{10r}$, then for every pair of points x, y in $4A$, we have $|x-y|\le 8r(A)<\frac{1}{10^2}$ and

$$\mathrm{dist}(x,\partial B(0,1))\ge\rho(A)-4r(A)>10^2 r(A)\ge\frac{10}{r}.$$

Therefore, using (5.2) we have

$$\begin{aligned}
II &\le c\sum_{A\in\mathcal{W}:\, r(A)>\frac{1}{10r}}\int_{4A\times 4A}\frac{(u(x)-u(y))^2}{|x-y|^d}\mathbf{1}_{\{|x-y|\le\frac{1}{10^2}\}}(\Phi(x)\wedge\Phi(y))dxdy \\
&\le c\int_{B(0,1-\frac{10}{r})\times B(0,1-\frac{10}{r})}\frac{(u(x)-u(y))^2}{|x-y|^d}\mathbf{1}_{\{|x-y|<\frac{1}{10^2}\}}(\Phi(x)\wedge\Phi(y))dxdy.
\end{aligned}$$

□

For our purpose, we need to construct another covering; For each $r>10^2$, we let $\mathcal{V}=\mathcal{V}_r:=\{B^1,\dots,B^{k(r)}\}$ be a maximum sequence of disjoint balls with radius $\frac{1}{400r}$ that we can put inside $B(0,1-\frac{10}{r})$. Note that

$$B\left(0,1-\frac{10}{r}\right)\subset\bigcup_{B\in\mathcal{V}}2B\subset\bigcup_{B\in\mathcal{V}}10^2B\subset B\left(0,1-\frac{9}{r}\right).$$

For every $y \in B(0,1)$, since $\bigcup_{B\in\mathcal{V}: y\in 2B} B \subset B(y, \frac{3}{400r})$,

$$\#\{B \in \mathcal{V} : y \in 2B\} \cdot \mu_d(B(0, \tfrac{1}{400r})) \le \mu_d(B(y, \tfrac{3}{400r})).$$

Therefore we have

$$\sup_{y\in B(0,1)} \#\{B \in \mathcal{V} : y \in 2B\} \le 3^d. \tag{5.6}$$

Recall that $\rho(B)$ denotes the Euclidean distance between the ball B and $B(0,1)^c$. For balls A and B in $\mathcal{V}$ with $\mathrm{dist}(A, B) > \frac{1}{40r}$ and $\rho(B) \ge \rho(A)$, we construct the path $\gamma_{A,B}$ starting from A in the following way. Let x_A be the center of A and x_B be the center of B. If $|x_B| \ge 1/(400r)$, then let $y_B := \frac{|x_A|}{|x_B|} x_B$ so that x_B is in the straight line segment from y_B to 0. Let $\gamma^2_{A,B}$ be the straight line segment from y_B to x_B. We also let $\gamma^1_{A,B}$ be the shortest path from x_A to y_B with $\gamma^1_{A,B} \subset \partial B(0, |x_A|)$. In this case, $\gamma_{A,B}$ is the union of $\gamma^1_{A,B}$ and $\gamma^2_{A,B}$ starting from x_A and ending at x_B via y_B. If $|x_B| < 1/(400r)$, let $\gamma_{A,B}$ be simply a straight line segment between 0 and x_A.

For $A, B \in \mathcal{V}$ with $\rho(B) \ge \rho(A)$, let

$$\overline{\mathcal{V}}(A, B) := \{C \in \mathcal{V} : 2C \cap \gamma_{A,B} \neq \emptyset\}$$

and define the chain $\mathcal{V}(A, B) := (C_0, C_1, \ldots, C_{l(A,B)-1})$ with $C_0 = A$ and $C_{l(A,B)-1} = B$ similar to the chain in the Whitney-type coverings; Starting from the center of A, let y_0 be the first point along $\gamma_{A,B}$ which does not belong to $2C_0$. Define C_1 to be one of balls in $\overline{\mathcal{V}}(A, B)$ such that $y_0 \in 2C_1$. Inductively, having $C_0, C_1, \ldots, C_k$ constructed, let y_k be the first point along $\gamma_{A,B}$ which does not belong to $\cup_{j=0}^k 2C_j$. Define C_{k+1} to be one of balls in $\overline{\mathcal{V}}(A, B)$ such that $y_k \in 2C_{k+1}$. When the last chosen is not B, we add B as the last ball in $\mathcal{V}(A, B)$.

In the sequel, for every path γ in $\mathbb{R}^d$ we denote by $|\gamma|$ the length of γ.

Lemma 5.8 *There exists a positive constant $c = c(d)$ such that for every $r > 10^2$ and every $A, B \in \mathcal{V}$ with $\rho(B) \ge \rho(A)$, $|\gamma_{A,B}| > \frac{1}{4r}$ and* $\mathrm{dist}(A, B) \le \frac{1}{50}$,

$$|x - y| \ge \frac{c}{r} \#\overline{\mathcal{V}}(A, B) \ge \frac{c}{r} \#\mathcal{V}(A, B) \ge |\gamma_{A,B}|, \qquad \textit{for every } (x, y) \in 2A \times 2B. \tag{5.7}$$

In particular,

$$\#\mathcal{V}(A, B) \le \#\overline{\mathcal{V}}(A, B) \le cr. \tag{5.8}$$

Proof It is easy to see that the length of $\gamma_{A,B}$ is less than or equal to $4|x - y|$ for every $(x, y) \in A \times B$. Thus by using the fact that balls C's in $\overline{\mathcal{V}}(A, B)$ are disjoint and that

$\cup_{C\in\overline{\mathcal{V}}(A,B)} C$ is within the $\frac{1}{100r}$-neighborhood of γ_{AB}, we have

$$\#\overline{\mathcal{V}}(A,B)\cdot\left(\frac{1}{400r}\right)^d = c\sum_{C\in\overline{\mathcal{V}}(A,B)}\mu_d(C) \le c|x-y|\,r^{1-d}$$

and so $\#\overline{\mathcal{V}}(A,B) \le cr\,|x-y|$.

On the other hand, since $2C$'s in $\mathcal{V}(A,B)$ covers $\gamma_{A,B}$, it is easy to see that

$$E := \left\{x\in B(0,1):\ \mathrm{dist}(x,\gamma_{A,B}) < \frac{1}{400r}\right\} \subset \bigcup_{C\in\mathcal{V}(A,B)} 3C$$

and that

$$\mu_d(E) \ge c\,|\gamma_{A,B}|\left(\frac{1}{r}\right)^{d-1}.$$

Thus

$$c|\gamma_{A,B}|r^{1-d} \le \mu_d(E) \le \sum_{C\in\overline{\mathcal{V}}(A,B)}\mu_d(3C) = \#\mathcal{V}(A,B)\cdot\left(\frac{3}{400r}\right)^d$$

and so $|\gamma_{A,B}| \le \frac{c}{r}\#\mathcal{V}(A,B)$. The lemma is proved. □

The proof of the next lemma is similar to the one of Lemma 5.3. So we skip its proof.

Lemma 5.9 *Let $A, B \in \mathcal{V}$ with $\rho(B) \ge \rho(A)$. There exists a positive constant $c = c(d)$ such that for every $C_i, C_{i+1} \in \mathcal{V}(A,B)$ and for every $u \in L^1(B(0,1),\Phi dx)$,*

$$|u_{2C_i} - u_{2C_{i+1}}|^2 \le \sum_{j=0}^{1}\frac{c}{(\mu_d(2C_{i+j}))^2}\int_{2C_{i+j}}\int_{2C_{i+j}}(u(x)-u(y))^2dxdy.$$

Lemma 5.10 *There exist positive constants $c = c(d,\sigma)$ and $c_1 = c_1(d)$ such that for every $r > 10^2$ and every $A, B \in \mathcal{V}$ with $\rho(B) \ge \rho(A)$ and $|\gamma_{A,B}| \ge \frac{1}{4r}$,*

$$\int_{2A}\int_{2B}\frac{(u(x)-u(y))^2}{|x-y|^d}\mathbf{1}_{\{|x-y|<\frac{1}{100}\}}(\Phi(x)\wedge\Phi(y))dxdy$$
$$\le c\,c_1^{\alpha}\,(\#\mathcal{V}(A,B))^{1-d-\alpha}\sum_{C\in\mathcal{V}(A,B)}\int_{2C}\int_{2C}\frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}(\Phi(x)\wedge\Phi(y))\,dxdy.$$

Proof Let $l := \#\mathcal{V}(A,B) \geq 2$. For every $y \in A$ and $x \in B$,

$$\begin{aligned}
&(u(x)-u(y))^2(\Phi(x)\wedge\Phi(y))\\
&\quad\leq (l+2)(\Phi(x)\wedge\Phi(y))\left(|u(x)-u_{2A}|^2+|u(x)-u_{2B}|^2+\sum_{i=0}^{l-1}|u_{2C_i}-u_{2C_{i+1}}|^2\right)\\
&\quad\leq 2l\Bigg((\Phi(x)\wedge\Phi(y))|u(y)-u_{2A}|^2 + (\Phi(x)\wedge\Phi(y))|u(x)-u_{2B}|^2\\
&\qquad+\sum_{i=0}^{l-1}(\Phi(x)\wedge\Phi(y))|u_{2C_i}-u_{2C_{i+1}}|^2\Bigg).
\end{aligned}$$

Note that from the construction of the chain $\mathcal{V}(A,B)$, it is easy to see that there exists a constant c independent of r such that for every $A, B \in \mathcal{V}$ and $C \in \mathcal{V}(A,B)$,

$$\int_{2A}\int_{2B}(\Phi(x)\wedge\Phi(y))dxdy \leq c\int_{2C_i}\int_{2C_{i+1}}(\Phi(x)\wedge\Phi(y))\,dxdy.$$

Obviously

$$\int_{2A}\int_{2B}|u(x)-u_{2B}|^2(\Phi(x)\wedge\Phi(y))dxdy \leq \mu_d(2B)\int_{2B}|u(x)-u_{2B}|^2\Phi(x)dx$$

and

$$\int_{2A}\int_{2B}|u(y)-u_{2A}|^2(\Phi(x)\wedge\Phi(y))dxdy \leq \mu_d(2A)\int_{2A}|u(y)-u_{2A}|^2\Phi(y)dy.$$

Thus we have, for every $y \in A$ and $x \in B$,

$$\begin{aligned}
&\int_{2A}\int_{2B}(u(x)-u(y))^2(\Phi(x)\wedge\Phi(y))dxdy\\
&\quad\leq 2l\Bigg(\int_{2A}\int_{2B}(\Phi(x)\wedge\Phi(y))|u(y)-u_{2A}|^2dxdy\\
&\qquad+\int_{2A}\int_{2B}(\Phi(x)\wedge\Phi(y))|u(x)-u_{2B}|^2dxdy\\
&\qquad+\sum_{i=0}^{l-1}\int_{2A}\int_{2B}(\Phi(x)\wedge\Phi(y))|u_{2C_i}-u_{2C_{i+1}}|^2dxdy\Bigg)
\end{aligned}$$

$$\leq cl \left(\mu_d(2A)| \int_{2A} |u(y) - u_{2A}|^2 \Phi(y)dy + \mu_d(2B) \int_{2B} |u(x) - u_{2B}|^2 \Phi(x)dx \right.$$
$$\left. + \sum_{i=0}^{l-1} |u_{2C_i} - u_{2C_{i+1}}|^2 \int_{2C_i} \int_{2C_{i+1}} (\Phi(x) \wedge \Phi(y))\, dxdy \right).$$

We apply Lemma 5.2 to the first two integrals in the above and apply Lemma 5.9 to the integrals in the summation above. Then using the fact that the values of Φ are universally comparable on each A, B, C_i, we get that

$$\int_{2A} \int_{2B} (u(x) - u(y))^2 (\Phi(x) \wedge \Phi(y)) dxdy$$
$$\leq c\,l \sum_{C \in \mathcal{V}(A,B)} \int_{2C} \int_{2C} (u(x) - u(y))^2 (\Phi(x) \wedge \Phi(y))\, dxdy. \tag{5.9}$$

Note that, using (5.7), we have that for $x \in B$ and $y \in A$ with $|x - y| < \frac{1}{100}$

$$\frac{1}{100} \geq |x - y| \geq c\,\frac{l}{r} \geq c\,l\,|z - w|, \qquad \forall z, w \in C \in \mathcal{V}(A, B). \tag{5.10}$$

Therefore, from (5.9)–(5.10), we conclude that

$$\int_{2A} \int_{2B} \frac{(u(x) - u(y))^2}{|x - y|^d} (\Phi(x) \wedge \Phi(y)) \mathbf{1}_{\{|x-y|<\frac{1}{100}\}} dxdy$$
$$\leq \frac{1}{10^{2\alpha}} \int_{2A} \int_{2B} \frac{(u(x) - u(y))^2}{|x - y|^{d+\alpha}} (\Phi(x) \wedge \Phi(y)) \mathbf{1}_{\{|x-y|<\frac{1}{100}\}} dxdy$$
$$= \frac{1}{10^{2\alpha}} \left(c\frac{r}{l}\right)^{d+\alpha} \int_{2A} \int_{2B} (u(x) - u(y))^2 (\Phi(x) \wedge \Phi(y)) \mathbf{1}_{\{|x-y|<\frac{1}{100}\}} dxdy$$
$$\leq c c_1^{\alpha}\, l^{1-d-\alpha} \sum_{C \in \mathcal{V}(A,B)} \int_{2C} \int_{2C} \frac{(u(z) - u(w))^2}{|z - w|^{d+\alpha}} (\Phi(z) \wedge \Phi(w))\, dzdw.$$

□

Recall that $[a]$ denote the largest integer which is no larger than a and define for $C \in \mathcal{V}$

$$C(\mathcal{V}) := \{(A, B) : A, B \in \mathcal{V} \text{ with } \rho(B) \geq \rho(A) \quad \text{and} \quad C \in \mathcal{V}(A, B)\}.$$

The following is a key lemma to count the number of chains containing each $C \in \mathcal{V}$.

Lemma 5.11 *There exists a positive constant $c = c(d)$ such that for every $r > 10^2$, $30 \le l \le [16r]$ and $C \in \mathcal{V}$,*

$$\#\left\{(A, B) \in C(\mathcal{V}) : \frac{100 + l}{400r} < |\gamma_{A,B}| \le \frac{101 + l}{400r}\right\} \le c\, l^d. \tag{5.11}$$

Proof Without loss of generality, we assume $d \ge 2$. (The case of $d = 1$ is easier.) Fix $r > 10^2$, $30 \le l \le [16r]$ and $C \in \mathcal{V}$. We will order $(A, B) \in C(\mathcal{V})$ so that $\rho(B) \ge \rho(A)$. Let x_C be the center of the ball C. If $|x_C| \le 4/(400r)$, then $|x_B| \le 6/(400r)$, so the number of possible choice for B is less than $c2^d$. Since $(100 + l)/(400r) \le |\gamma_{A,B}| \le (101 + l)/(400r)$, the number of possible choice for A is cl^{d-1}, so (5.11) holds in this case. We thus assume $|x_C| > 4/(400r)$. Define $H_{x_C} := B(0, |x_C|+2/(400r))\setminus B(0, |x_C|-2/(400r))$. Since $2C\cap\gamma_{A,B} \ne \emptyset$, $H_{x_C}\cap\gamma_{A,B} \ne \emptyset$. Let y'_B be the first point along $\gamma_{A,B}$ (starting from x_B) which belongs to $H_{x_C} \cap \gamma_{A,B}$. Also, let $z_{A,B}$ be the first point along $\gamma_{A,B}$ (starting from x_B) which belongs to $2C$, and let γ_B be the sub-path of $\gamma_{A,B}$ starting from $z_{A,B}$ ending at x_B.

Let $m/(400r) \le |\gamma_B| < (m + 1)/(400r)$ where $0 \le m \le l + 100$ and consider the following two cases:

Case 1 $|y'_B - z_{A,B}| \le \frac{5}{400r}$.

Case 2 $|y'_B - z_{A,B}| > \frac{5}{400r}$.

For Case 1, the number of possible choices for y'_B and B is less than $c2^d$ when C is given and m is fixed. Once y'_B is fixed, the number of possible choice for A is $c(l - m + 106)^{d-1}$, since the arclength between $z_{A,B}$ and x_A along the curve $\gamma_{A,B}$ is at most $\frac{101+l-m}{400r}$ and $|y'_B - z_{A,B}| \le 5/(400r)$. Summing over m, the number of possible choices for A and B is less than

$$c' \sum_{m=0}^{l+100} (l - m + 106)^{d-1} \le c'' l^d.$$

For Case 2, let $i \le m$ be such that $i/(400r) \le |z_{A,B} - y_B| < (i + 1)/(400r)$ where $y_B := \frac{|x_A|}{|x_B|} x_B$. In this case, $|y_B - y'_B| \le 4/(400r)$ and $i \ge 1$. Since $y_B \in \partial B(0, |x_A|) \subset H_{x_C}$, given C, the number of possible choices for y_B and B is less than ci^{d-2} when m and i are fixed. Observe that given C and B, y'_B and x_B are determined. Since $x_A \in \partial B(0, |x_A|) \subset H_{x_C}$, given C and B, the number of possible choice for x_A is less than $c((l - m + i + 101)/i)^{d-2}$ when m and i are fixed. Summing over m and i, the number of possible choices for A and B is less than

$$c' \sum_{m=1}^{l+100} \sum_{i=1}^{m} i^{d-2} \left(\frac{l-m + i + 101}{i}\right)^{d-2} = c' \sum_{m=1}^{l+100} \sum_{i=1}^{m} (l - m + i + 101)^{d-2} \le c'' l^d.$$

We thus obtain (5.11). □

Lemma 5.12 *There exist positive constants $c = c(d, \sigma)$ and $c_1 = c_1(d)$ such that for every $r \geq 10^2$*

$$\sum_{\substack{A,B\in\mathcal{V}\\ dist(A,B)>\frac{1}{3r}}} \int_{2A}\int_{2B} \frac{(u(x)-u(y))^2}{|x-y|^d}\mathbf{1}_{\{|x-y|\leq\frac{1}{100}\}}(\Phi(x)\wedge\Phi(y))dxdy$$
$$\leq c\, c_1^\alpha \int_{B(0,1)\times B(0,1)} (u(x)-u(y))^2 \frac{r^{2-\alpha}}{|x-y|^{d+\alpha}}\mathbf{1}_{\{|x-y|\leq\frac{1}{r}\}}(\Phi(x)\wedge\Phi(y))\, dxdy.$$

Proof For $(x, y) \in 2A \times 2B$ with $|x-y| \leq \frac{1}{100}$, it is elementary to check that $|\gamma_{A,B}| < \frac{1}{25}$. Thus, by Lemma 5.10, we have

$$\sum_{\substack{A,B\in\mathcal{V}\\ \mathrm{dist}(A,B)>\frac{1}{3r}}} \int_{2A}\int_{2B} \frac{(u(x)-u(y))^2}{|x-y|^d}\mathbf{1}_{\{|x-y|\leq\frac{1}{100}\}}(\Phi(x)\wedge\Phi(y))dxdy$$
$$\leq c\, c_1^\alpha \sum_{\substack{A,B\in\mathcal{V}:\rho(B)\geq\rho(A)\\ \frac{130}{400r}<|\gamma_{A,B}|<\frac{1}{25}}} (\#\mathcal{V}(A,B))^{1-d-\alpha}$$
$$\times \sum_{C\in\mathcal{V}(A,B)} \int_{2C}\int_{2C} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}(\Phi(x)\wedge\Phi(y))\, dxdy$$
$$\leq c\, c_1^\alpha \sum_{C\in\mathcal{V}} \left(\sum_{l=30}^{[16r]} \sum_{\substack{(A,B)\in C(\mathcal{V})\\ \frac{100+l}{400r}<|\gamma_{A,B}|\leq\frac{100+l+1}{400r}}} (\#\mathcal{V}(A,B))^{1-d-\alpha} \right)$$
$$\times \int_{2C}\int_{2C} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}\mathbf{1}_{\{|x-y|\leq\frac{1}{r}\}}(\Phi(x)\wedge\Phi(y))\, dxdy.$$

Applying (5.7), we see that

$$\sum_{\substack{A,B\in\mathcal{V}\\ \mathrm{dist}(A,B)>\frac{1}{3r}}} \int_{2A}\int_{2B} \frac{(u(x)-u(y))^2}{|x-y|^d}\mathbf{1}_{\{|x-y|\leq\frac{1}{100}\}}(\Phi(x)\wedge\Phi(y))dxdy$$
$$\leq c\, c_1^\alpha \sum_{C\in\mathcal{V}} \left(\sum_{l=30}^{[16r]} l^{1-d-\alpha} \cdot \# \left\{ (A,B)\in C(\mathcal{V}) : \frac{100+l}{400r} < |\gamma_{A,B}| \leq \frac{101+l}{400r} \right\} \right)$$
$$\times \int_{2C}\int_{2C} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}\mathbf{1}_{\{|x-y|\leq\frac{1}{r}\}}(\Phi(x)\wedge\Phi(y))\, dxdy.$$

By Lemma 5.11,

$$\sum_{l=30}^{[16r]} l^{1-d-\alpha} \cdot \# \left\{ (A,B) \in C(\mathcal{V}) : \frac{100+l}{400r} < |\gamma_{A,B}| \leq \frac{101+l}{400r} \right\} \leq c \sum_{l=30}^{[16r]} l^{1-\alpha} \leq c r^{2-\alpha}.$$

Thus we conclude that

$$\begin{aligned} &\sum_{\substack{A,B\in\mathcal{V} \\ \mathrm{dist}(A,B)>\frac{1}{3r}}} \int_{2A}\int_{2B} \frac{(u(x)-u(y))^2}{|x-y|^d} \mathbf{1}_{\{|x-y|\leq\frac{1}{100}\}}(\Phi(x)\wedge\Phi(y))dxdy \\ &\leq c\, c_1^{\alpha} r^{2-\alpha} \sum_{C\in\mathcal{V}} \int_{2C}\int_{2C} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}} \mathbf{1}_{\{|x-y|\leq\frac{1}{r}\}}(\Phi(x)\wedge\Phi(y))\,dxdy \\ &\leq c\, c_1^{\alpha} \int_{B(0,1)\times B(0,1)} (u(x)-u(y))^2 \frac{r^{2-\alpha}}{|x-y|^{d+\alpha}} \mathbf{1}_{\{|x-y|\leq\frac{1}{r}\}}(\Phi(x)\wedge\Phi(y))\,dxdy. \end{aligned}$$

In the last inequality above, we have used (5.6). □

Proof of Theorem 5.1 By Lemma 5.7, it is enough to show the following claim; there exist constants $c = c(d,\sigma) > 0$ and $c_1(d) > 0$ such that for every $r > 10^2$ and $u \in L^1(B(0,1), \Phi dx)$

$$\begin{aligned} &\int_{B(0,1-\frac{10}{r})\times B(0,1-\frac{10}{r})} \frac{(u(x)-u(y))^2}{|x-y|^d} \mathbf{1}_{\{|x-y|\leq\frac{1}{100}\}}(\Phi(x)\wedge\Phi(y))dxdy \\ &\leq c\, c_1^{\alpha} \int_{B(0,1)\times B(0,1)} (u(x)-u(y))^2 \frac{r^{2-\alpha}}{|x-y|^{d+\alpha}} \mathbf{1}_{\{|x-y|\leq\frac{1}{r}\}}(\Phi(x)\wedge\Phi(y))dxdy. \end{aligned} \tag{5.12}$$

Note that

$$\begin{aligned} &\int_{B(0,1-\frac{10}{r})\times B(0,1-\frac{10}{r})} \frac{(u(x)-u(y))^2}{|x-y|^d} \mathbf{1}_{\{|x-y|\leq\frac{1}{100}\}}(\Phi(x)\wedge\Phi(y))dxdy \\ &\leq \sum_{A,B\in\mathcal{V}} \int_{2A}\int_{2B} \frac{(u(x)-u(y))^2}{|x-y|^d} \mathbf{1}_{\{|x-y|\leq\frac{1}{100}\}}(\Phi(x)\wedge\Phi(y))dxdy \\ &\leq \sum_{\substack{A,\,B\,\in\,\mathcal{V} \\ \mathrm{dist}(A,\,B)\,\leq\,\frac{1}{3r}}} \int_{2A}\int_{2B} \frac{(u(x)-u(y))^2}{|x-y|^d} \mathbf{1}_{\{|x-y|\leq\frac{1}{r}\}}(\Phi(x)\wedge\Phi(y))dxdy \end{aligned}$$

$$+ \sum_{\substack{A,\,B\in\mathcal{V}\\ \mathrm{dist}(A,\,B)>\frac{1}{3r}}} \int_{2A}\int_{2B} \frac{(u(x)-u(y))^2}{|x-y|^d}\mathbf{1}_{\{|x-y|\le\frac{1}{100}\}}(\Phi(x)\wedge\Phi(y))dxdy$$

$$\le c\,r^{2-\alpha} \int_{B(0,1)\times B(0,1)} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}\mathbf{1}_{\{|x-y|\le\frac{1}{r}\}}(\Phi(x)\wedge\Phi(y))dxdy$$

$$+ \sum_{\substack{A,\,B\in\mathcal{V}\\ \mathrm{dist}(A,\,B)>\frac{1}{3r}}} \int_{2A}\int_{2B} \frac{(u(x)-u(y))^2}{|x-y|^d}\mathbf{1}_{\{|x-y|\le\frac{1}{100}\}}(\Phi(x)\wedge\Phi(y))dxdy.$$

In the last inequality above, we have used (5.6) and the fact $r^{2-\alpha}\ge 1$. Thus (5.12) follows from Lemma 5.12. □

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References

1. Barlow, M.T., Bass, R.F., Chen, Z.-Q., Kassmann, M.: Non-local Dirichlet forms and symmetric jump processes. Trans. Am. Math. Soc. (to appear)
2. Barlow, M.T., Bass, R.F., Kumagai, T.: Parabolic Harnack inequality and heat kernel estimates for random walks with long range jumps. Math. Z. (to appear)
3. Barlow, M.T., Grigor'yan, A., Kumagai, T.: Heat kernel upper bounds for jump processes. J. Reine Angew. Math. (to appear)
4. Bass, R.F., Levin, D.A.: Transition probabilities for symmetric jump processes. Trans. Am. Math. Soc. **354**, 2933–2953 (2002)
5. Bertoin, J.: Lévy Processes. Cambridge University Press, Cambridge (1996)
6. Blumenthal, R.M., Getoor, R.K.: Markov Processes and Potential Theory. Academic Press, Reading (1968)
7. Caffarelli, L.A., Salsa, S., Silvestre, L.: Regularity estimates for the solution and the free boundary to the obstacle problem for the fractional Laplacian. Invent. Math. **171**(1), 425–461 (2008)
8. Carlen, E.A., Kusuoka, S., Stroock, D.W.: Upper bounds for symmetric Markov transition functions. Ann. Inst. H. Poincaré Prob. Stat. **23**, 245–287 (1987)
9. Chen, Z.-Q.: Gaugeability and conditional gaugeability. Trans. Am. Math. Soc. **354**, 4639–4679 (2002)
10. Chen, Z.-Q., Kim, P., Kumagai, T.: Notes on heat kernel estimates and parabolic Harnack inequality for jump processes, in preparation
11. Chen, Z.-Q., Kumagai, T.: Heat kernel estimates for stable-like processes on d-sets. Stoch. Process Appl. **108**, 27–62 (2003)
12. Chen, Z.-Q., Kumagai, T.: Heat kernel estimates for jump processes of mixed types on metric measure spaces. Prob. Theory Relat. Fields **140**, 277–317 (2008)
13. Chen, Z.-Q., Rohde, S.: SLE driven by symmetric stable processes. Preprint (2007)
14. Fabes, E.B., Stroock, D.W.: A new proof of Moser's parabolic Harnack inequality using the old ideas of Nash. Arch. Rational Mech. Anal. **96**(4), 327–338 (1986)
15. Fukushima, M., Oshima, Y., Takeda, M.: Dirichlet forms and symmetric Markov processes. de Gruyter, Berlin (1994)
16. Gilbarg, D., Trudinger, N.S.: Elliptic Partial Differential Equations of Second Order, 2nd edn. Springer, Heidelberg (1983)
17. Grzywny, T., Ryznar, M.: Estimates of Green function for some perturbations of fractional Laplacian. Illinois J. Math. (to appear)

18. Hu, J., Kumagai, T.: Nash-type inequalities and heat kernels for non-local Dirichlet forms. Kyushu J. Math. **60**, 245–265 (2006)
19. Hurst, S.R., Platen, E., Rachev, S.T.: Option pricing for a logstable asset price model. Math. Comput. Model. **29**, 105–119 (1999)
20. Ikeda, N., Watanabe, S.: Stochastic Differential Equations and Diffusion Processes, 2nd edn. North-Holland Publishing Co., Amsterdam (1989)
21. Janicki, A., Weron, A.: Simulation and Chaotic Behavior of α-Stable Processes. Dekker, New York (1994)
22. Jerison, D.: The Poincaré inequality for vector fields satisfying Hörmander's condition. Duke Math. J. **53**(2), 503–523 (1986)
23. Kassmann, M.: A priori estimates for integro-differential operators with measurable kernels. Calc. Var. Partial Diff. Equat. (to appear)
24. Kim, P., Song, R.: Potential theory of truncated stable processes. Math. Z. **256**(1), 139–173 (2007)
25. Kim, P., Song, R.: Boundary behavior of harmonic functions for truncated stable processes. J. Theor. Prob. **21**, 287–321 (2008)
26. Klafter, J., Shlesinger, M.F., Zumofen, G.: Beyond Brownian motion. Phys. Today **49**, 33–39 (1996)
27. Kolokoltsov, V.: Symmetric stable laws and stable-like jump-diffusions. Proc. Lond. Math. Soc. **80**, 725–768 (2000)
28. Matacz, A.: Financial modeling and option theory with the truncated Lévy process. Int. J. Theor. Appl. Finance **3**(1), 143–160 (2000)
29. Saloff-Coste, L.: Aspects of Sobolev-type Inequalities. Cambridge University Press, Cambridge (2002)
30. Saloff-Coste, L., Stroock, D.W.: Opérateurs uniformément sous-elliptiques sur les groupes de Lie. J. Funct. Anal. **98**(1), 97–121 (1991)
31. Silvestre, L.: Hölder estimates for solutions of integro differential equations like the fractional Laplace. Indiana Univ. Math. J. **55**, 1155–1174 (2006)
32. Takeda, M., Tsuchida, K.: Criticality of generalized Schrödinger operators and differentiability of spectral functions. Adv. Stud. Pure Math. **41**, 333–350 (2004)
33. Takeda, M., Tsuchida, K.: Differentiability of spectral functions for symmetric α-stable processes. Trans. Am. Math. Soc. **359**, 4031–4054 (2007)
34. Tsuchida, K.: Differentiability of spectral functions for relativistic α-stable processes with application to large deviations. Potential Anal. **28**(1), 17–33 (2008)