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Fibred multilinks and singularities $f\overline{g}$

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Abstract In this article we extend Milnor's fibration theorem to the case of functions of the form $f_{\bar{g}}$ with f, g holomorphic, defined on a complex analytic (possibly singular) germ $(X, 0)$. We further refine this fibration theorem by looking not only at the link of $\{f\bar{g} = 0\}$, but also at its multi-link structure, which is more subtle. We mostly focus on the case when X has complex dimension two. Our main result (Theorem 4.4) gives in this case the equivalence of the following three statements:

(i) The real analytic germ $f_{\bar{g}} : (X, p) \to (\mathbb{R}^2, 0)$ has 0 as an isolated critical value;
(ii) the multilink $L_f \cup -L_g$ is fibered; and
(iii) if $\pi : \tilde{X} \to X$ is a resolution of the holomorphic germ $fg : (X, p) \to (\mathbb{C}, 0)$, then for each rupture vertex (j) of the decorated dual graph of π one has that the corresponding multiplicities of f, g satisfy: $m_j^f \neq m_j^g$.

Moreover one has that if these conditions hold, then the Milnor-Lê fibration $\Psi_{f\bar{g}} : \mathcal{L}_X \backslash (L_f \cup L_g) \to \mathbb{S}^1_\eta$ of $f_{\bar{g}}$ is a fibration of the multilink $L_f \cup -L_g$. We also give a combinatorial criterium to decide whether or not the multilink $L_f \cup -L_g$ is fibered. If the meromorphic germ f/g is semitame, then we show that the Milnor-Lê fibration

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given by $\Psi_{f\bar{g}}$ is equivalent to the usual Milnor fibration given by $f\bar{g}/|f\bar{g}|$. We finish this article by discussing several realization problems.

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0 Introduction

Milnor's fibration theorem for complex singularities [26] states that if $(\mathbb{C}^n, 0) \xrightarrow{f} (\mathbb{C}, 0)$ is a holomorphic map, then for every sufficiently small sphere $\mathbb{S}_\varepsilon = \partial\mathbb{B}_\varepsilon$ around $0 \in \mathbb{C}^n$ one has a fibre bundle

$$\phi = \frac{f}{|f|} : \mathbb{S}_\varepsilon \backslash K \longrightarrow \mathbb{S}^1,$$

where $K = f^{-1}(0) \cap \mathbb{S}_\varepsilon$ is the link of f at 0. In this work, we give generalizations and refinements of this theorem for germs of maps of the form $(X, 0) \xrightarrow{f\bar{g}} (\mathbb{C}, 0)$, where X is a complex analytic variety in $\mathbb{C}^n$ with an isolated singularity at 0 and f and g are both holomorphic maps. Our main interest is when X is a complex surface, but in Sects. 1 and 5 we consider more general settings.

Our first result (Theorem 1.3) is for real analytic germs in general, inspired by the corresponding theorem of Milnor in [26, Chap. 10]. We consider an equidimensional real analytic space X in $\mathbb{R}^m$ with an isolated singularity at $0 \in \mathbb{R}^m$ and an analytic map $f : (X, 0) \to (\mathbb{R}^k, 0)$ with an isolated critical value at $0 \in \mathbb{R}^k$. Let $\mathbb{D}^k_\eta$ be a small ball around 0 in $\mathbb{R}^k$ and let $\mathbb{S}^{k-1}_\eta$ be its boundary. We prove that if f has the Thom property (see 1.2), then for every $\varepsilon > 0$ sufficiently small and $\eta > 0$ sufficiently small with respect to ε, the map

$$f : N(\varepsilon, \eta) \longrightarrow \mathbb{S}^{k-1}_\eta;$$

is the projection of a locally trivial fibre bundle, where $N(\varepsilon, \eta)$ is the Milnor tube $f^{-1}(\mathbb{S}^{k-1}_\eta) \cap \mathbb{B}_\varepsilon$. Moreover, if we let $\mathcal{L}_X = X \cap \mathbb{S}^{m-1}_\varepsilon$ be the link of X and $T_{\varepsilon,\eta}$ be the intersection $f^{-1}(\mathbb{D}_\eta) \cap \mathbb{S}^{m-1}_\varepsilon$, then, by inflating the tube $N(\varepsilon, \eta)$, one gets an induced fibre bundle:

$$\Psi_f : \mathcal{L}_X \backslash \mathrm{Int}(T_{\varepsilon,\eta}) \longrightarrow \mathbb{S}^{k-1}_\eta,$$

where Ψ_f coincides with f on the boundary $\partial T_{\varepsilon,\eta} = f^{-1}(\mathbb{S}^{k-1}_\eta) \cap \mathbb{S}_\varepsilon$.

This result is certainly known to various people and the first part of it is implicit in [20]; in particular, this proves that $N(\varepsilon, \eta) \backslash \partial T_{\varepsilon,\eta}$ is diffeomorphic to $\mathcal{L}_X \backslash L_f$, the complement of the link $L_f := f^{-1}(0) \cap \mathbb{S}_\varepsilon$ of f in the link of X. We also give several examples of real analytic functions with an isolated critical value that have the Thom property.

In Sects. 2–4 we focus on the case when X is a complex surface with an isolated singularity at $0 \in X$. In this context, we prove (1.4 and 1.7) that every map $X \to \mathbb{C}$ of the form $f\bar{g}$ with an isolated critical value and f, g holomorphic has Thom's property, and that $\Psi_{f\bar{g}}$ extends to the complement of the link $L_{f\bar{g}} = L_f \cup L_g$ in $\mathcal{L}_X$. We obtain what we call the Milnor–Lê fibration of $f\bar{g}$:

$$\Psi_{f\bar{g}} : \mathcal{L}_X \backslash L_{f\bar{g}} \longrightarrow \mathbb{S}^1_\eta.$$

We do not know whether or not an analogous result holds when X is a complex variety of dimension greater than 2.

We remark that examples of maps of the form $f\bar{g}$ with a Milnor fibration have already appeared in [1,15,33–36].

We define an orientation on the link $L_{f\bar{g}}$ by setting $L_{f\bar{g}} = L_f \cup -L_g$ where L_f and L_g are equipped with their natural orientations as the boundaries of the complex curves $f^{-1}(0) \cap \mathbb{B}_\varepsilon$ and $g^{-1}(0) \cap \mathbb{B}_\varepsilon$. Then the link $L_{f\bar{g}}$ is in fact a *plumbing multilink*. This means that it is union of $\mathbb{S}^1$-leaves (i.e., Seifert leaves) in a Waldhausen decomposition of $\mathcal{L}_X$ and each component of the link comes with an orientation and a multiplicity.

In Sect. 2 we envisage fibered plumbing multilinks in general, and we give an amazingly simple combinatorial criterium (2.11), which is necessary and sufficient, to decide whether or not a plumbing multilink L can be fibered (see 2.2 for the precise definition). This result was first proved in [7] and generalizes a result of [11] for multilinks in integral homology spheres. We use this to show (2.14) that the multilink $L_{f\bar{g}} \subset \mathcal{L}_X$ fibers (as a multilink), if the multiplicities of f and g, are distinct at each rupture vertex of the plumbing graph (decorated with arrows) of some resolution of the holomorphic germ fg, which is part of the main result of this work, Theorem 4.4.

In Sect. 3, we look at the geometry of the Milnor–Lê fibration $\Psi_{f\bar{g}} : \mathcal{L}_X \backslash L_{f\bar{g}} \longrightarrow \mathbb{S}^1_\eta$ near the multilink $L_{f\bar{g}}$. The main Lemma 3.1 in this section implies that $\Psi_{f\bar{g}}$ is in fact a fibration of the multilink $L_f \cup -L_g$, i.e., it equips $\mathcal{L}_X$ with an open-book decomposition with multiplicities having $L_f \cup -L_g$ as binding. This is essential for the proof of Theorem 4.4 in Sect. 4, which gives the equivalence of the following three statements:

(i) The real analytic germ $f\bar{g} : (X, p) \to (\mathbb{R}^2, 0)$ has 0 as an isolated critical value;
(ii) the multilink $L_f \cup -L_g$ is fibered; and
(iii) if $\pi : \tilde{X} \to X$ is a resolution of the holomorphic germ $fg : (X, p) \to (\mathbb{C}, 0)$, then for each rupture vertex (j) of the decorated dual graph of π one has $m_j^f \neq m_j^g$.
Moreover, one has that if these conditions hold, then the Milnor–Lê fibration $\Psi_{f\bar{g}} : \mathcal{L}_X \backslash (L_f \cup L_g) \to \mathbb{S}^1_\eta$ of $f\bar{g}$ is a fibration of the multilink $L_f \cup -L_g$.

Notice that condition (ii) is topological while the others are analytic.

In Sect. 5, we compare the geometry of the Milnor–Lê fibration $\Psi_{f\bar{g}} : \mathcal{L}_X \backslash \mathrm{Int}(T_{\varepsilon,\eta}) \to \mathbb{S}^1_\eta$ with the Milnor fibration $\phi_{f/g} : \mathcal{L}_X \backslash L_{fg} \to \mathbb{S}^1$ of the meromorphic function f/g defined in [4] by

$$\phi_{f/g}(z) = \frac{(f/g)(z)}{|(f/g)(z)|}$$

under the assumption that f/g is semitame at 0 (Definition 5.2). We show (Theorem 5.5) that the two fibrations are equivalent. The proof is based on ideas of [26] adapted in [4] to semitame meromorphic germs, and a canonical decomposition of the space X associated to the map $f\bar{g}$, which is introduced in [8] for holomorphic germs.

It is of course interesting to compare these fibrations with the local fibrations for meromorphic germs studied for instance in [13,14,38,39]; this is done in [5].

At the end Sect. 5 we compare the two assumptions "f/g is semitame" and "$f\bar{g}$ has an isolated critical value at 0"; we notice (Theorem 5.8) that when X is $\mathbb{C}^2$ these two conditions turn out to be equivalent. Our proof of this equivalence is somehow intricate and it is not clear to us how these two conditions are actually related, neither we know whether the equivalence given in Theorem 5.8 extends to a more general setting.

Finally, in Sect. 6 we look at a "realization question" regarding fibered multilinks. We prove the following: let M be either $\mathbb{S}^3$ or a 3-manifold homeomorphic to the link of a taut surface singularity; let L_1, L_2 be two plumbing fibered multilinks in a suitable plumbing decomposition of M, with positive multiplicities, such that $L_1 \cup -L_2$ is also fibered and any two components K_1 of L_1 and K_2 of L_2 belong to distinct plumbing pieces of M. Then there exist two holomorphic germs $f, g : (X, p) \to (\mathbb{C}, 0)$ without common branches, such that L_1 and L_2 are the multilinks of f and g, respectively, $L_1 - L_2$ is the multilink of $f\bar{g}$ and

$$\frac{f\bar{g}}{|f\bar{g}|} : \mathcal{L}_X \backslash (L_1 \cup L_2) \longrightarrow \mathbb{S}^1,$$

is a fibre bundle that realizes $L_1 - L_2$ as a fibered multilink.

The proof is based on [32] (5.4 and 5.5) and its generalisation to multilinks in [31], where it is shown that every fibered positive plumbing multilink in a normal surface singularity link can be realized by a holomorphic map.

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1 Milnor's fibration and the Thom $\mathbf{a_f}$ condition

Let U be an open neighbourhood of 0 in $\mathbb{R}^{n+k}$ and let $X \subset U$ be a real analytic variety of dimension $m > 0$ with an isolated singularity at 0. Let $\tilde{f} : (U, 0) \to (\mathbb{R}^k, 0)$ be a real analytic germ which is generically a submersion, i.e., its jacobian matrix $D\tilde{f}$ has rank k on a dense open subset of U. We denote by f the restriction of $\tilde{f}$ to X. Following Thom and others, we say that $x \in X\backslash\{0\}$ is *a regular point* of f if $Df_x : T_x X \to \mathbb{R}^k$ is a submersion, otherwise x is *a critical point*. A point $y \in \mathbb{R}^k$ is *a regular value* of f if there is no critical point in $f^{-1}(y)$; otherwise y is *a critical value*. We say that f has *an isolated critical value* at $0 \in \mathbb{R}^k$ if there is a neighbourhood $\mathbb{D}_\delta$ of 0 in $\mathbb{R}^k$ so that all points $y \in \mathbb{D}_\delta\backslash\{0\}$ are regular values of f.

Example 1.1 (a) Define a map $\mathbb{C}^2 \xrightarrow{f} \mathbb{C}$ by $f(z_1, z_2) = \bar{z}_1^3(z_1^2 + z_2^3)$. Its critical set is the axis $\{z_1 = 0\}$ and $0 \in \mathbb{C}$ is its only critical value.

(b) Define $h : \mathbb{C}^n \to \mathbb{C}$, $n > 1$, by $(z_1, \ldots, z_n) \mapsto z_1^{a_1}\bar{z}_{\sigma_1} + \cdots + z_n^{a_n}\bar{z}_{\sigma_n}$, with all a_i integers greater than 1 and $\{z_{\sigma_1}, \ldots, z_{\sigma_n}\}$ a permutation of the coordinates $\{z_1, \ldots, z_n\}$. Then, by [35] or [37, Chap. VII], h has an isolated critical point at $0 \in \mathbb{C}^n$.

(c) Now define $h : \mathbb{C}^n \to \mathbb{C}$, $n > 1$, by $h = f\bar{g}$ where $g(z_1, \ldots, z_n) = z_1 \ldots z_n$ and f is the Pham–Brieskorn polynomial

$$f(z_1, \ldots, z_n) = z_1^{a_1} + \cdots z_n^{a_n} \quad a_i \geq 2,$$

A straightforward computation shows that h has an isolated critical value at $0 \in \mathbb{R}^2$ whenever the a_i satisfy $\sum_{i=1}^{n} \frac{1}{a_i} \neq 1$. Notice that for $n = 2$ this means that one a_i is more than 2. For $n = 3$ the condition is satisfied whenever the unordered triple (p, q, r) is not (2, 3, 6), (2, 4, 4) or (3, 3, 3).

Our aim now is to extend Milnor's fibration theorem [26, Chap. 11] to real singularities with an isolated critical value at the origin and satisfying an additional condition: the Thom property. Let us explain what this means (cf. [21]).

Let $\tilde{f} : (U, 0) \to (\mathbb{R}^k, 0)$ be again a real analytic map, and assume $f = \tilde{f}|_X$ has an isolated critical value at $0 \in \mathbb{R}^k$. We set $V = f^{-1}(0) = \tilde{f}^{-1}(0) \cap X$. According to [16,21], there exists a Whitney stratification $(V_\alpha)_{\alpha \in A}$ of U adapted to X and V.

Definition 1.2 The Whitney stratification $(V_\alpha)_{\alpha \in A}$ satisfies *the Thom a_f condition* with respect to f if for every sequence of points $\{x_n\} \in X\backslash V$ converging to a point x_0 in a stratum $V_\alpha \subset V$ such that the sequence of tangent spaces $T_{x_n}(f^{-1}(f(x_n))$ has a limit T, one has that T contains the tangent space of V_α at x_0. We say that f has *the Thom property* if such a stratification exists.

Since the Whitney stratification $(V_\alpha)_{\alpha \in A}$ has finitely many strata containing 0 in their closures, by [26,40] one has that each sufficiently small sphere $\mathbb{S}_\varepsilon$ intersects each stratum V_α transversally and the homeomorphism type of the intersection $\mathcal{L}_X = X \cap \mathbb{S}_\varepsilon$ does not depend on ε (cf. [6,21]). By Thom's transversality, if the stratification satisfies the a_f-condition, then given such ε one has that for every sufficiently small disc $\mathbb{D}_\eta^k = \{x \in \mathbb{R}^k / ||x|| \leq \eta\}$ and for every $t \in \mathbb{D}_\eta^k$, the level surface $f^{-1}(t)$ also intersects transversally the sphere $\mathbb{S}_\varepsilon$. Let us set $\mathbb{S}_\eta^{k-1} = \partial\mathbb{D}_\eta^k$ and $N(\varepsilon, \eta) = f^{-1}(\mathbb{S}_\eta^{k-1}) \cap \mathbb{B}_\varepsilon$, where $\mathbb{B}_\varepsilon$ is the ball bounded by $\mathbb{S}_\varepsilon$. Let $T_{\varepsilon,\eta} = f^{-1}(\mathbb{D}_\eta^k) \cap \mathcal{L}_X$, so it is an algebraic neighbourhood of the link $L_f = f^{-1}(0) \cap \mathcal{L}_X$ in the sense of [10]. We refer to $N(\varepsilon, \eta)$ as a *Milnor tube* for f.

The following result is an extension of the Milnor–Lê fibration Theorem [26, 11.2], [20].

Theorem 1.3 *Let the analytic map $f : X \to \mathbb{R}^k$ have $0 \in \mathbb{R}^k$ as an isolated critical value and assume it has the Thom property. Let $\mathcal{L}_X = X \cap \mathbb{S}_\varepsilon$ be the link of X. Then for every $\varepsilon > 0$ sufficiently small and $\eta > 0$ sufficiently small with respect to ε, the map*

$$f : N(\varepsilon, \eta) \to \mathbb{S}_\eta^{k-1},$$

is the projection of a locally trivial fibre bundle. Moreover, the manifold $N(\varepsilon, \eta)$ *is diffeomorphic to* $\mathcal{L}_X \backslash Int(T_{\varepsilon,\eta})$ *and the previous fibration induces a fibre bundle*:

$$\Psi_f : \mathcal{L}_X \backslash \mathrm{Int}(\mathrm{T}_{\varepsilon,\eta}) \longrightarrow \mathbb{S}_\eta^{\mathrm{k}-1},$$

with $\Psi_f = f$ *restricted to the boundary* $\partial T_{\varepsilon,\eta} = f^{-1}(\mathbb{S}_\eta^{k-1}) \cap \mathbb{S}_\varepsilon$. *In particular, one has that* $N(\varepsilon, \eta) \backslash \partial T_{\varepsilon,\eta}$ *is diffeomorphic to* $\mathcal{L}_X \backslash L_f$.

Proof The first statement is essentially an application of Ehresmann's Fibration Theorem. Choose a Whitney stratification (V_α) for which f satisfies the Thom condition. As before, choose a small enough sphere $\mathbb{S}_\varepsilon$ around 0 that intersects all strata transversally, and so does every other smaller sphere around 0. Choose $\eta > 0$ sufficiently small so that for each $t \in \mathbb{S}_\eta^{k-1}$ the fibre $f^{-1}(t)$ intersects $\mathbb{S}_\varepsilon$ transversally. One has a submersion: $N(\varepsilon, \eta) \xrightarrow{f} \mathbb{S}_\eta^{k-1}$ with compact fibers $f^{-1}(t) \cap \mathbb{B}_\varepsilon$, which is a fibre bundle, as in Ehresmann's Theorem.

The aim now, to prove the second statement, is to construct a tangent vector field v on $(X \backslash V) \cap U$, with U a neighbourhood of 0, which is transversal to all the spheres around 0 and also transversal to the "tube" $f^{-1}(\mathbb{S}_\eta^{k-1})$ near 0. Since the construction is local, we can assume X is $\mathbb{R}^m$, for otherwise we construct the vector field locally and then glue the pieces together by a partition of unity. This is done as in Milnor's work. For instance, let $f_1, \ldots, f_k$ be the components of f and define a function

$$r(x) = f_1^2(x) + \cdots + f_k^2(x).$$

Let ∇r be its gradient. Following [25], define a vector field on $(X \cap \mathbb{B}_\varepsilon) \backslash V$ by:

$$v(x) = \|x\| \cdot \nabla r + \|\nabla r\| \cdot x.$$

This is a smooth vector field and the Curve Selection Lemma implies it has the properties we want. The rest of the proof is as in [26, Chap. 5] and is left to the reader. □

Our main interest in this article is when X is a complex analytic surface and the map is of the form $f\bar{g}$ with f, g holomorphic. One has:

Proposition 1.4 *Let* X *be a complex analytic surface in* $\mathbb{C}^N$ *with an isolated singularity at* 0 *and let* $f, g : (X, 0) \to (\mathbb{C}, 0)$ *be the restriction to* X *of two germs of holomorphic functions* $(\mathbb{C}^N, 0) \to (\mathbb{C}, 0)$. *Assume further that* f *and* g *have no common branch and* $f\bar{g}$ *has an isolated critical value at* $0 \in \mathbb{R}^2 \cong \mathbb{C}$. *Then* $f\bar{g}$ *has the Thom property.*

Our proof actually shows that if the local ring $\mathcal{O}_{X,0}$ is with unique factorization, then the corresponding Whitney stratification is canonical.

Proof Suppose first that X is $\mathbb{C}^2$. Let $f_1, \ldots, f_m$ be the irreducible factors of f and let $g_1, \ldots, g_n$ be those of g. Let us equip a small open neighbourhood $U \subset \mathbb{C}^2$ of 0 with the Whitney stratification whose strata are :

$$U\backslash(fg)^{-1}(0); \quad V_i = f_i^{-1}(0)\backslash\{0\}, \quad i = 1, \ldots, m;$$
$$V'_j = g_j^{-1}(0)\backslash\{0\}, \quad j = 1, \ldots, n; \quad \text{and} \quad \{0\}.$$

We claim that this stratification satisfies the (a_f)-condition with respect to $f\bar{g}$. It suffices to check the condition on the strata $V_i, i = 1, \ldots, m$ and $V'_j, j = 1, \ldots, n$. Let us check this on V_1; the arguments are the same for the other strata.

Let us decompose f as $f = f_1^p\, h$, $p \geq 1$, in such a way that f_1 is not a factor of h. Then the jacobian matrix of $f\bar{g}$ with respect to the coordinates $(z_1, \bar{z}_1, z_2, \bar{z}_2)$ in $\mathbb{R}^4$ is given by

$$D(f\bar{g})(z_1, \bar{z_1}, z_2, \bar{z_2}) = \begin{pmatrix} \frac{\partial(\Re(f\bar{g}))}{\partial z_1} & \frac{\partial(\Re(f\bar{g}))}{\partial \bar{z}_1} & \frac{\partial(\Re(f\bar{g}))}{\partial z_2} & \frac{\partial(\Re(f\bar{g}))}{\partial \bar{z}_2} \\ \frac{\partial(\Im(f\bar{g}))}{\partial z_1} & \frac{\partial(\Im(f\bar{g}))}{\partial \bar{z}_1} & \frac{\partial(\Im(f\bar{g}))}{\partial z_2} & \frac{\partial(\Im(f\bar{g}))}{\partial \bar{z}_2} \end{pmatrix}. \quad (1.5)$$

Set:

$$a_i = p f_1^{p-1} \frac{\partial f_1}{\partial z_i} h\bar{g} + f_1^p \frac{\partial h}{\partial z_i} \bar{g} \quad \text{and} \quad b_i = \bar{f_1}^p \bar{h} \frac{\partial g}{\partial z_i}$$

Then

$$\begin{aligned} \frac{\partial(\Re(f\bar{g}))}{\partial z_i} &= \frac{1}{2}(a_i + b_i) \\ \frac{\partial(\Re(f\bar{g}))}{\partial \bar{z}_i} &= \frac{1}{2}\,\overline{(a_i + b_i)}, \\ \frac{\partial(\Im(f\bar{g}))}{\partial z_i} &= \frac{1}{2i}\,(a_i - b_i), \\ \frac{\partial(\Im(f\bar{g}))}{\partial \bar{z}_i} &= -\frac{1}{2i}\,\overline{(a_i - b_i)}, \end{aligned}$$

For each $t \in \mathbb{R}^2\backslash\{0\}$ and for each $x \in (f\bar{g})^{-1}(t)$, the tangent space $T_x(f\bar{g})^{-1}(t)$ is defined by the equation $D(f\bar{g})(x)^t(v_1, \overline{v_1}, v_2, \overline{v_2}) = 0$, i.e., by the two equations,

$$\Re\,((a_1 + b_1)v_1 + (a_2 + b_2)v_2) = 0\,,$$

and

$$\Im\,((a_1 - b_1)v_1 + (a_2 - b_2)v_2) = 0\,;$$

or equivalently:

$$a_1 v_1 + \overline{b_1 v_1} + a_2 v_2 + \overline{b_2 v_2} = 0\,;$$

i.e.,

$$\left(p\frac{\partial f_1}{\partial z_1}h\bar{g}+f_1\frac{\partial h}{\partial z_1}\bar{g}\right)v_1+f_1\overline{\frac{\partial g}{\partial z_1}}h\bar{v}_1+\left(p\frac{\partial f_1}{\partial z_2}h\bar{g}+f_1\frac{\partial h}{\partial z_2}\bar{g}\right)v_2+f_1\overline{\frac{\partial g}{\partial z_2}}h\bar{v}_2 = 0.$$

Then, if $\{x_n\}$ denotes a sequence of points in $\mathbb{R}^4\setminus\{(fg)^{-1}(0)\}$ converging to a point $x \in f_1^{-1}(0)$, the limit T of the tangent planes $T_{x_n}(f\bar{g})^{-1}(t_n)$, where $t_n = (f\bar{g})(x_n)$, has equation:

$$\left(p\frac{\partial f_1}{\partial z_1}(x)h(x)\bar{g}(x)\right)v_1 + \left(p\frac{\partial f_1}{\partial z_2}(x)h(x)\bar{g}(x)\right)v_2 = 0.$$

Since $h(x)\bar{g}(x) \neq 0$, then T has equation:

$$\frac{\partial f_1}{\partial z_1}(x)v_1 + \frac{\partial f_1}{\partial z_2}(x)v_2 = 0.$$

Then T equals the plane tangent at x to the curve $f_1^{-1}(0)$; so one has that these singularities satisfy the Thom a_f-condition for the above Whitney stratification.

Now let $(X, 0)$ be a germ of a normal complex surface with an isolated singularity and let f, g be as in the statement of the theorem. Let us equip a coordinate chart $U \subset X$ around p with the Whitney stratification defined as above. The previous discussion shows that this stratification of U satisfies the a_f-condition with respect to $f\bar{g}$. □

Remark 1.6 (i) The previous arguments can be generalized to higher dimensions in many cases, as for instance to the singularities in Example 1.1.c; in fact one may conjecture that Proposition 1.4 also holds in higher dimensions. Notice that every polynomial map $\mathbb{R}^{2m} \to \mathbb{R}^2$ can be expressed as a sum of functions of type $f\bar{g}$; it would be interesting to determine the class of such singularities that have the Thom property, as for instance the examples in 1.1.b.

(ii) Notice that if $X = \mathbb{C}^2$ and f, g are as in Proposition 1.4, then for each holomorphic germ $h : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with an isolated critical value at $0 \in \mathbb{C}$, the real analytic germ $H : (\mathbb{C}^{n+2}, 0) \to (\mathbb{R}^2, 0)$ defined by

$$H(z_1, z_2, z_3, \dots, z_{n+2}) = f(z_1, z_2)\overline{g(z_1, z_2)} + h(z_3, \dots, z_{n+2})$$

obviously has 0 as an isolated critical value too. Since holomorphic functions have the Thom property, by [16], it follows that H also has the Thom property.

As a consequence of Theorem 1.3 and Proposition 1.4 one obtains:

Corollary 1.7 *Let X be a complex analytic surface in $\mathbb{C}^N$ with an isolated singularity at 0 and let $f, g : (X, 0) \to (\mathbb{C}, 0)$ be the restriction to X of two germs of holomorphic functions $(\mathbb{C}^N, 0) \to (\mathbb{C}, 0)$. Assume further that f and g have no common branch* *and $f\bar{g}$ has an isolated critical value at $0 \in \mathbb{R}^2 \cong \mathbb{C}$. Then one has a locally trivial fibration*

$$\Psi_{f\bar{g}} : \mathcal{L}_X \backslash L_{f\bar{g}} \longrightarrow \mathbb{S}^1_\eta$$

which restricted to $T_{(\varepsilon,\eta)} \backslash L_{f\bar{g}}$ is the map $x \mapsto \eta \frac{f\bar{g}}{|f\bar{g}|}$.

Notice that $\Psi_{f\bar{g}}$ depends on the choice of a diffeomorphism from $N(\varepsilon, \eta)$ to $\mathcal{L}_X \backslash Int(T_{\varepsilon,\eta})$ (see the proof of Theorem 1.3), but all such choices yield isomorphic fibrations.

Definition 1.8 We call $\Psi_{f\bar{g}} : \mathcal{L}_X \backslash L_{f\bar{g}} \longrightarrow \mathbb{S}^1_\eta$ *the Milnor–Lê fibration* of $f\bar{g}$.

2 Fibrations of plumbing multilinks

From now on the standard circle $\mathbb{S}^1$ is oriented as the boundary of the complex disk $\mathbb{D}^2 = \{z \in \mathbb{C}/|z| \leq 1\}$. A *circle* means a one-dimensional manifold diffeomorphic to $\mathbb{S}^1$.

Let M be a compact connected oriented 3-manifold. An (unoriented) *knot* in M is a circle embedded in M. A *link* L in M is a finite disjoint union of circles embedded in M. We let $L = K_1 \cup \ldots \cup K_l$ be the connected components of L. We say that L is an *oriented link* if an orientation is fixed on each K_i. Notice that the normal bundle of each K_i in M is trivial. Hence, if L is oriented then one has orientation preserving diffeomorphisms $N(K_i) \cong \mathbb{D}^2 \times \mathbb{S}^1$ defined on a neighbourhood of each K_i, that carry the oriented circles K_i into $\{0\} \times \mathbb{S}^1$ with its usual orientation.

A *multilink* is the data of an oriented link $L = K_1 \cup \ldots \cup K_l$ together with a *multiplicity* $n_i \in \mathbb{Z}$ associated with each component K_i. We denote such a multilink by

$$L = n_1 K_1 \cup \ldots \cup n_l K_l\,,$$

and we fix the convention that $n_i K_i = (-n_i)(-K_i)$, where $-K_i$ means K_i with the opposite orientation. We say that two multilinks $L \subset M$ and $L' \subset M'$ are *equivalent* if there exists an orientation-preserving homeomorphism $H : (M, L) \to (M', L')$ such that the multiplicities of the corresponding components coincide.

Example 2.1 Let X be a complex surface in $\mathbb{C}^N$ with a normal singularity at 0 and let $\mathcal{L}_X = X \cap \mathbb{S}^{2N-1}_\varepsilon$ be its link. We equip $\mathcal{L}_X$ with its natural orientation as boundary of $X \cap \mathbb{B}^{2N}_\varepsilon$. Let $f : (X, 0) \longrightarrow (\mathbb{C}, 0)$ be a holomorphic germ and $L_f = \mathcal{L}_X \cap f^{-1}(0)$ its link in $\mathcal{L}_X$, naturally oriented as the boundary of the complex curve $f^{-1}(0) \cap \mathbb{B}^{2N}_\epsilon$, $\epsilon > 0$ sufficiently small.

Let $\pi : \widetilde{X} \to X$ be a good resolution of the germ f, i.e., a resolution of X such that the irreducible components of the divisor $(f \circ \pi)^{-1}(0)$ are smooth and with normal crossings. Let $S_1, \ldots, S_l$ be the branches of the strict transform $\overline{\pi^{-1}(f^{-1}(0) \backslash \{0\})}$ of f. We denote by n_i the multiplicity of $f \circ \pi$ along the curve S_i. For each $i = 1, \ldots, l$, let K_i be the component of L_f associated with S_i, i.e., the knot $\pi(S_i) \cap \mathcal{L}_X$. By definition *the multilink associated with* f is:

$$L_f = \bigcup_{i=1}^{l} n_i K_i.$$

In particular, when the local ring $\mathcal{O}_{X,0}$ is a unique factorization ring (e.g. $X = \mathbb{C}^2$), let $f = u \prod_{i=1}^{l} f_i^{n_i}$ be its decomposition into irreducible factors, $u(0) \neq 0$; then the multilink associated with f is $L_f = \bigcup_{i=1}^{l} n_i L_{f_i}$.

Definition 2.2 Let M be a compact connected oriented 3-manifold. A multilink $L = n_1 K_1 \cup \ldots \cup n_l K_l$ in M is *fiberable* (or simply fibered) if there exists a map $\Phi : M \backslash L \longrightarrow \mathbb{S}^1$ which satisfies:

(i) the map Φ is a C^∞ locally trivial fibration; and
(ii) for each $i = 1, \ldots, l$, there exists a regular neighbourhood $N(K_i)$ of K_i in $M \backslash (L \backslash K_i)$, an orientation-preserving diffeomorphism $\tau : \mathbb{S}^1 \times \mathbb{D}^2 \to N(K_i)$ such that $\tau(\mathbb{S}^1 \times \{0\}) = K_i$ and an integer $k_i \in \mathbb{Z}$ such that for all $(t, z) \in \mathbb{S}^1 \times (\mathbb{D}^2 \backslash \{0\})$ one has:

$$(\Phi \circ \tau)(t, z) = \left(\frac{z}{|z|} \right)^{n_i} t^{k_i}$$

We say that such a Φ is a *fibration of the multilink* L.

Being precise, to say that L is fibered means that we have already chosen a fibration for it, but by abuse of notation we also say L is fibered meaning that it can be fibered.

Example 2.3 If $(X, 0)$ is a normal complex surface singularity and $f : (X, 0) \longrightarrow (\mathbb{C}, 0)$ is holomorphic, then the Milnor fibration $f(x) \big/ |f(x)| : \mathcal{L}_X \backslash L_f \to \mathbb{S}^1$ is a fibration of the multilink L_f.

Remark 2.4 (1) The integer k_i is non unique; it depends on the choice of τ, but its class modulo n_i is well defined.
(2) For each $t \in \mathbb{S}^1$ the intersection $\Phi^{-1}(t) \cap (N(K_i))$ has $\gcd(n_i, k_i)$ connected components, each of them being diffeomorphic to a half-open annulus $\mathbb{S}^1 \times [0, 1[$.
(3) The n_i are local data whereas the classes $k_i \pmod{n_i}$ are global. Indeed, let D be a meridian disk of $N(K_i)$ oriented as $\{1\} \times \mathbb{D}^2$ via τ and equip its boundary $m = \partial D$ with the induced orientation. Then n_i is the degree of the restriction of Φ to m. But k_i depends on the equivalence class of the multilink L in M, and in particular on all the multiplicities $n_i, i = 1, \ldots, n_l$. See [11, p. 30] where the inverse l_i of k_i modulo n_i is computed explicitly when the link is an integral homology sphere.
(4) If $k_i = 0$ for each $i = 1, \ldots, n$, and $n_i \in \{-1, +1\}$ then one has:

$$(\Phi \circ \tau)(t, z) = \frac{z}{|z|} \quad \text{if } n_i = +1; \quad \text{or} \quad (\Phi \circ \tau)(t, z) = \frac{\bar{z}}{|z|} \quad \text{if } n_i = -1.$$

In this case L is just an oriented link and Φ is a so-called open-book fibration of M with binding L; this case is studied in [33] when $M = \mathbb{S}^3$.

Definition 2.5 If $\Phi : M\backslash L \to \mathbb{S}^1$ and $\Phi' : M'\backslash L' \to \mathbb{S}^1$ are fibrations of the multilinks (M, L) and (M', L'), we say that Φ and Φ' are *topologically equivalent* if there exist orientation-preserving diffeomorphisms $H : (M, L) \to (M', L')$ and $\rho : \mathbb{S}^1 \to \mathbb{S}^1$ such that $\rho \circ \Phi = \Phi' \circ H|_{M\backslash L}$.

Remark 2.6 (1) Such an H is an equivalence between the links (M, L) and (M', L').
(2) Notice that in general there can be several non-equivalent fibrations for a given multilink. When M is an integral homology sphere, a multilink $L \subset M$ determines a unique integral class $\alpha \in H^1(M\backslash N(L))$, where $N(L)$ is a regular neighbourhood of L. Identifying $H^1(M\backslash N(L)) \cong [M\backslash N(L), \mathbb{S}^1]$, the homotopy classes of maps, one has that L is fibered if and only if α contains a locally trivial fibration. Moreover, if there exists such a fibration then it is unique up to isotopy, as a consequence of a result of Blank and Laudenbach (see [11, p. 34] for details). Hence each fibered multilink $L \subset M$ in an integral homology sphere has a unique fibration $\Phi : M\backslash L \to \mathbb{S}^1$ up to topological equivalence (cf. Remark 2.13 (3)).

Now let M be a *plumbing manifold*, or equivalently a graph manifold in the sense of Waldhausen. So M is homeomorphic to the boundary of a 4-manifold obtained by plumbing together a finite number of $\mathbb{D}^2$-bundles $N_1, \ldots, N_r$ over oriented real surfaces $E_1, \ldots, E_r$, see [17] or [30]. If $T \subset M$ denotes the union of the plumbing tori, then $M\backslash T$ is a union of 3-manifolds equipped with an $\mathbb{S}^1$-foliation, the $\mathbb{S}^1$-leaves being boundaries of the $\mathbb{D}^2$-fibres of the N_i's.

Convention 2.7 In the sequel we make the **convention** that the surfaces E_i have genus ≥ 0 and are oriented. We also assume that each plumbing operation has positive ϵ in the sense of [30], and that no bundle N_i is plumbed with itself (so each edge in the associated plumbing graph joins two different vertices).

The homeomorphism class of a plumbing manifold M is completely encoded in the *plumbing graph* Γ of any plumbing decomposition of M, defined in the classical way: the vertices $(1), \ldots, (r)$ are in bijection with the bundles $N_1, \ldots, N_r$ and the edges joining two vertices (i) and (j) are in bijection with the plumbing operations between N_i and N_j. Each vertex (i) is weighted by the genus of the surface E_i (usually omitted when the genus is 0) and by the Euler class e_i of the bundle N_i.

We denote by M_Γ the *intersection matrix* associated with Γ, i.e., the $r \times r$ matrix $(a_{ij})_{1 \leq i,j \leq r}$ defined by $a_{ii} = e_i$ for all i and for $i \neq j$ the integer a_{ij} equals the number of edges of Γ joining the vertices (i) and (j).

Example 2.8 (cf. [30]) The link $\mathcal{L}_X$ of a normal complex surface (X, p) is a plumbing manifold and a plumbing graph of $\mathcal{L}_X$ can be obtained as the dual graph Γ of any resolution $\pi : \tilde{X} \to X$ of X such that the irreducible components $E_1, \ldots, E_r$ of the exceptional divisor $E = \pi^{-1}(p)$ are smooth curves with normal crossings. The intersection matrix M_Γ is then the intersection matrix of the E_i's in $\tilde{X}$ and it is negative definite [28].

Definition 2.9 Let M be a plumbing manifold. A link L in M is a *plumbing link* if it is a finite union of $\mathbb{S}^1$-leaves in a plumbing decomposition of M.

Notice that every plumbing link L in M has a natural orientation as the union of the boundaries of some fibres of the D^2-bundles N_i. In the sequel we consider always this natural orientation. Then a *plumbing multilink* is a multilink $L = n_1K_1 \cup \ldots \cup n_lK_l$ such that each K_i is an $\mathbb{S}^1$-leaf in a plumbing decomposition of M.

The equivalence class of a plumbing multilink L is fully described by its *decorated plumbing graph*, obtained from a plumbing graph for M by adding arrows corresponding to the components of L, each arrow endowed with its multiliplicity n_i.

For example if (X, p) is a normal complex surface singularity and $f : (X, p) \to (\mathbb{C}, 0)$ is a holomorphic germ, then L_f is a plumbing multilink and a plumbing graph of L_f can be obtained as the dual graph of any good resolution $\pi : \tilde{X} \to X$ of f.

Definition 2.10 Let Γ be a plumbing graph of a plumbing multilink. A vertex of Γ is a *rupture vertex* if it either carries genus > 0 or if it has at least three incident edges, the arrows being considered as edges.

The following theorem is a generalization of a result of Eisenbud and Neumann ([11], Theorem 11.2, see also p. 136), reformulated in terms of plumbing links. In [11] this result is proved for multilinks in $\mathbb{Z}$-homology spheres and formulated in terms of splicing diagrams. This result is also proved in the thesis of Chaves in terms of graph decompositions ([7], theorem 2.2.10). The sketch of proof given here is similar to that of [7] in the sense $\Rightarrow$, but for the other implication we use the results of [32] extended to the multilink case.

Theorem 2.11 *Let $L = n_1K_1 \cup \ldots \cup n_lK_l$ be a plumbing multilink with plumbing graph Γ and intersection matrix M_Γ, and let $(1), \ldots, (r)$ be the vertices of Γ. Let $b(L) = (b_1, \ldots, b_r) \in \mathbb{Z}^r$, where b_i is the sum of the multiplicities n_j carried by the arrows attached to the vertex (i). Then L is fibered if and only if there exist $(m_1, \ldots, m_r) \in \mathbb{Z}^r$ such that the following two conditions hold*:

(1) $M_\Gamma{}^t(m_1, \ldots, m_r) + {}^tb(L) = 0$, *where ${}^t(.)$ means the transposition.*
(2) *for each rupture vertex (j) of Γ, the integer m_j is* $\neq 0$.

The system of equations (1) is called the *monodromical system* of L (see [32]).

Proof Assume L is fibered and let $\Phi : M\backslash L \to \mathbb{S}^1$ be a fibration; let $F = \Phi^{-1}(t),\ t \in \mathbb{S}^1$ be a fibre. Notice that F is naturally oriented by lifting to $M\backslash L$ the canonical vector field on $\mathbb{S}^1$. For each $i = 1, \ldots, r$, let l_i be an $\mathbb{S}^1$-leaf of N_i, and let m_i be the intersection number $F \cdot l_i$ in M. Then $(m_1, \ldots m_r)$ satisfies condition (1) in the theorem (see [32], Thm 4.3). Moreover, when Γ has at least one rupture vertex then, according to arguments of Thurston-Roussarie et al. (see [11, 4.2]), F is a horizontal surface in $M\backslash N(L)$, where $N(L)$ is a regular neighbourhood of L in M. This means that for each rupture vertex (j) of Γ, F is transversal to L_j up to isotopy, thus $m_j \neq 0$.

Assume now that $N = (m_1, \ldots, m_r)$ verifies both conditions (1) and (2). Using the formulas of [32], one constructs a graph G^N from Γ and N. The conditions (1) and (2) imply that G^N is the Nielsen graph of the monodromy of a fibration of the multilink L ([32], Lemma 4.7). Therefore L is fibered. □

Example 2.12 (1) Let M be the plumbing 3-manifold whose graph is $-D_4$ (see Fig. 1), and let L_1, L_2, L_3 and L_4 be the four multilinks whose graphs are represented on Fig. 1. The numbers without parenthesis are the Euler classes and the numbers with parenthesis are the multiplicities m_i and n_i. Then L_1 is not fibered since the rupture vertex carries multiplicity 0. The multilink $L_2 = L_3 - L_4$ is fibered since it verifies the conditions 1 and 2 of 2.11, whereas L_3 and L_4 are not fibered because they do not verify the condition 1.

(2) Let X be the hypersurface in $\mathbb{C}^3$ with equation $x^3 + y^3 + z^3 = 0$. Then a resolution graph of the germ $(X, 0)$ consists of a single vertex with Euler class -3 and genus 1. Let L_5 be the plumbing multilink in $\mathcal{L}_X$ whose graph is represented on Fig. 1. Then L_5 is fibered if and only if 3 divides n.

Remark 2.13 (1) An easy computation (using for instance [30]) shows that the conditions in 2.11 do not depend on the choice of the plumbing graph.

(2) If the intersection matrix M_Γ is invertible (e.g. if M is the link of a normal surface singularity), then there is a unique solution $(m_1, \ldots, m_r) \in \mathbb{Q}^r$ to condition 1) in Theorem 2.11; this is given by ${}^t(m_1, \ldots, m_r) = -(M_\Gamma)^{-1}\, {}^t b(L)$. One obtains $m_i \in \mathbb{Z}$ for all i by considering the multilink $kL = (kn_1)K_1 \cup \ldots \cup (kn_l)K_l$ with $k = \det M_\Gamma$. Thus, if condition 2 holds for the multilink L, then the multilink kL is fiberable.

(3) An easy generalization of the arguments in [32] shows that each $(m_1, \ldots, m_r) \in \mathbb{Z}^r$ satifying the conditions (1) and (2) in Theorem 2.11 defines a finite number of topological equivalence classes of fibrations of L such that $F \cdot l_i = m_i$ for each $i = 1, \ldots, r$. When M_Γ is invertible, then any fibered multilink admits a finite number of fibrations up topological equivalence. Moreover, if M is a rational homology sphere then there is only one such class, by [32, 4.7 and 4.8].

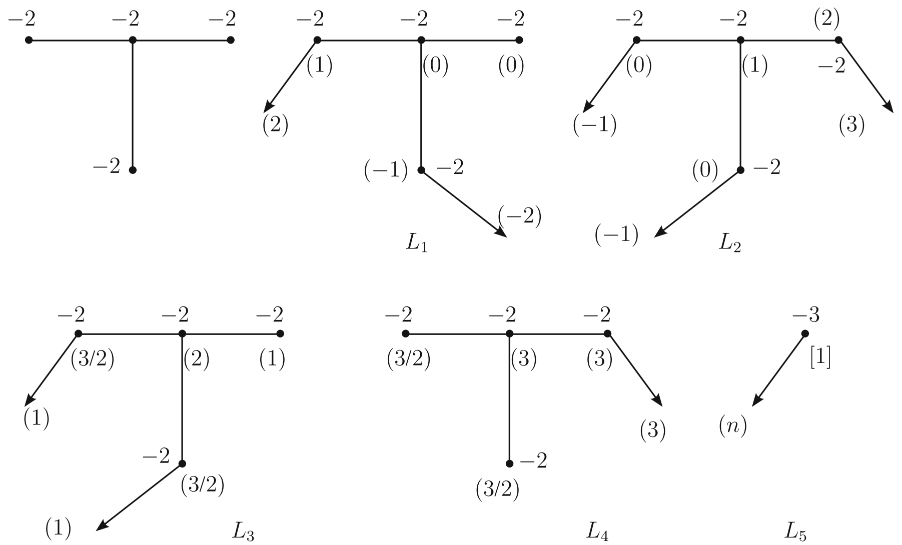

Fig. 1 Multilinks

Corollary 2.14 *Let (X, p) be a normal surface singularity and let $f, g : (X, p) \longrightarrow (\mathbb{C}, 0)$ be holomorphic germs. Let $\pi : \tilde{X} \to X$ be a good resolution of the holomorphic germ fg such that the total transform $(fg \circ \pi)^{-1}(0)$ has normal crossings, and let Γ be its dual graph; $(1), \ldots, (r)$ are the vertices of Γ. For each $i = 1, \ldots, r$, let m_i^f (respectively, m_i^g) be the multiplicity of $f \circ \pi$ (respectively, $g \circ \pi$) along the corresponding irreducible component of the exceptional divisor. Then the multilink $L_{f\bar{g}} = L_f \cup -L_g$ is fibered if and only if for each rupture vertex (j) of Γ one has $m_j^f \neq m_j^g$.*

Proof The links L_f and L_g are fibered by Milnor's theorem, and according to [18, 2.6] the solutions of their monodromical systems are $(m_1^f, \ldots, m_r^f)$ and $(m_1^g, \ldots, m_r^g)$, respectively. With the notations of the theorem one has:

$$(M_\Gamma)^{-1}\ \ {}^t b(L_f \cup -L_g) = (M_\Gamma)^{-1}\ \ {}^t(b(L_f) - b(L_g)).$$

Therefore

$$(M_\Gamma)^{-1}\ \ {}^t b(L_f \cup -L_g) = -{}^t(m_1^f - m_1^g, \ldots, m_r^f - m_r^g).$$

Hence $(m_1^f - m_1^g, \ldots, m_r^f - m_r^g)$ is the solution of the monodromical system of $L_f \cup -L_g$. One concludes 2.14 using 3.11. $\square$

3 The geometry near the multilink $L_{f\bar{g}}$

From now on (X, p) denotes a complex analytic surface with a normal singularity at p, and $f, g : (X, p) \to (\mathbb{C}, 0)$ are two holomorphic germs without common branches. The lemma below describes the behaviour of the real analytic germ $f\bar{g} : (X, p) \to (\mathbb{C}, 0)$ in a neighbourhood of its link $L_{f\bar{g}} \subset \mathcal{L}_X$.

Lemma 3.1 *Let K be a component of the multilink $L_f \cup -L_g$ and let n be its multiplicity in $L_f \cup -L_g$. Then there exists a regular neighbourhood $N(K)$ of K in $\mathcal{L}_X$, an orientation-preserving diffeomorphism $\tau : \mathbb{S}^1 \times \mathbb{D}^2 \to N(K)$ such that $\tau(\mathbb{S}^1 \times \{0\}) = K$, and an integer $m \in \mathbb{Z}$ such that for all $(t, z) \in \mathbb{S}^1 \times (\mathbb{D}^2 \setminus \{0\})$ we have*:

$$\left(\frac{f\bar{g}}{|f\bar{g}|} \circ \tau\right)(t, z) = \left(\frac{z}{|z|}\right)^n t^m.$$

Proof The proof is adapted from that of [33, 3.1] for the case $X = \mathbb{C}^2$. Let $\pi : \tilde{X} \longrightarrow X$ be a good resolution of the holomorphic germ fg. Let S be a branch of the strict transform of f by π and let E be the irreducible component of the exceptional divisor $\pi^{-1}(p)$ which intersects S. We denote by n the multiplicity of f along the curve S and by m_f (respectively, m_g) the multiplicity of $f \circ \pi$ (respectively, $g \circ \pi$) along E. We choose local coordinates (z_1, z_2) in $\tilde{X}$ such that $S \cap E$ is $(0, 0)$, $z_2 = 0$ is a local equation for E. Let $\gamma \in \mathbb{C}\{\{z_1, z_2\}\}$ be such that $\gamma(z_1, z_2) = 0$ is a local equation for S, then in a neighbourhood of $(0, 0)$ we have

$$(f \circ \pi)(z_1, z_2) = z_2^{m_f} \gamma(z_1, z_2)^n u(z_1, z_2)\,,$$

and

$$(g \circ \pi)(z_1, z_2) = z_2^{m_g} v(z_1, z_2),$$

with u and v being units in $\mathbb{C}\{\{z_1, z_2\}\}$. Then, locally,

$$(f\bar{g} \circ \pi)(z_1, z_2) = z_2^{m_f}\, \overline{z_2}^{m_g}\, \gamma(z_1, z_2)^n\, u(z_1, z_2)\, \overline{v(z_1, z_2)}.$$

A suitable change of local coordinates $(z_1', z_2) = \alpha(z_1, z_2)$ leads to

$$(f\bar{g} \circ \pi)(z_1', z_2) = z_2^{m_f} \overline{z_2}^{m_g} {z_1'}^{n},$$

where $z_1' = 0$ is a local equation of S.

Now let us identify $\mathcal{L}_X$ with the boundary of a regular neighbourhood W of the exceptional divisor $\pi^{-1}(p)$ in $\tilde{X}$, defined locally by $W = \{(z_1', z_2)//|z_2| \leq \eta\}$, with $\eta \ll 1$; let us study the restriction of $\frac{f\bar{g}}{|f\bar{g}|} \circ \pi$ to a small tubular neighbourhood $N(K)$ of the component $K = S \cap \partial W$ of L_f, say

$$N(K) = \{(z_1', z_2)/|z_1'| \leq \eta',\ |z_2| = \eta\},$$

where $\eta' \ll \eta$. Let $\tau : \mathbb{S}^1 \times \mathbb{D}^2 \to N(K)$ be the orientation preserving diffeomorphism defined by $\tau(t, z) = (\eta' z, \eta t)$. Then for all $(t, z) \in \mathbb{S}^1 \times (\mathbb{D}^2 \setminus \{0\})$ one has:

$$\frac{f\bar{g}}{|f\bar{g}|} \circ \pi \circ \alpha \circ \tau\ (t, z) = \left(\frac{z}{|z|}\right)^n t^{m_f} \cdot \bar{t}^{m_g} = \left(\frac{z}{|z|}\right)^n t^{(m_f - m_g)}.$$

Similar computations near a branch of the strict transform of g complete the proof. □

The following immediate consequence of Lemma 3.1 refines Corollary 1.7 and is part of our main Theorem 4.4.

Corollary 3.2 *If $f\bar{g} : (X, 0) \to (\mathbb{R}^2, 0)$ has 0 as an isolated critical value, then its Milnor–Lê fibration $\Phi_{f\bar{g}} : \mathcal{L}_X \setminus L_{f\bar{g}} \to \mathbb{S}^1$ is a fibration of the multilink $L_f \cup -L_g$.*

Notice that $L_{f\bar{g}}$ is $L_f \cup L_g$ as unoriented links, but 3.2 states that $L_{f\bar{g}}$ actually fibres with its natural multilink structure. Thus 3.2 is much stronger than Corollary 1.7 since it describes this Milnor–Lê fibration as the fibration of the multilink, taking into account the "multiple open-book" structure near the binding, the orientations and the multiplicities. In particular, 3.2 enables us to describe (using [32]) the genus of the fibres and most of the data concerning the topology of the fibration (see for instance [34]).

4 $f\bar{g}$ and the discriminantal ratios of (g, f)

Let (X, p), f and g be as in Sects. 2 and 3. We now give a sufficient condition (Lemma 4.2) in terms of the discriminantal ratios of the germ (g, f), for the germ $f\bar{g}$ to have 0 as an isolated critical value. Then we use a result of [22] relating the discriminantal ratios to some topological invariants of the meromorphic germ f/g to prove our main Theorem 4.4.

The discriminantal ratios are defined in [22] as follows. Set $\pi = (g, f) : X \to \mathbb{C}^2$ and let $C \subset X$ be its critical locus. Let $\Delta = \pi(C)$ be its discriminant curve, and let $(\Delta_\alpha)_{\alpha \in A}$ be the set of branches of Δ which are not the coordinates axes.

Definition 4.1 The *discriminantial ratio* of Δ_α with respect to the canonical complex coordinates $u = g(x)$ and $v = f(x)$ of $(g, f)(X)$ is the rational number

$$d_\alpha = \frac{I(u = 0, \Delta_\alpha)}{I(v = 0, \Delta_\alpha)},$$

where $I(-, -)$ denotes the intersection number at 0 of complex analytic curves in $\mathbb{C}^2$.

Lemma 4.2 *If the discriminantal ratios of* $\pi = (g, f)$ *with respect to the complex coordinates* $u = g(x)$ *and* $v = f(x)$ *are all* $\neq 1$*, then* $f\bar{g} : (X, p) \to (\mathbb{C}, 0)$ *has* 0 *as an isolated critical value.*

Proof Let U be an open neighbourhood of p in X such that f and g have no critical points in $U\backslash(fg)^{-1}(0)$. Let $x \in U\backslash(fg)^{-1}(0)$, let $\phi : V \to W$ be a parametrisation of $X\backslash\{p\}$ in a neighbourhood W of x in X, with V an open set in $\mathbb{C}^2$, and let $y = \phi^{-1}(x)$. Since $D\phi(y) : T_y\mathbb{C}^2 \to T_x X$ is an isomorphism, one has that x is a critical point of $f\bar{g}$ if and only if y is a critical point of $(f\bar{g}) \circ \phi = (f \circ \phi) \cdot \overline{(g \circ \phi)} : V \to \mathbb{R}^2$.

Consider the holomorphic functions $F = f \circ \phi$, $G = g \circ \phi$ and the map $F\overline{G} = (f\bar{g}) \circ \phi$. We decompose $F\overline{G}$ in its real and imaginary parts:

$$\left(\Re(F\overline{G}), \Im(F\overline{G})\right) = \left(\frac{1}{2}(F\overline{G} + \overline{F}G), \frac{1}{2i}(F\overline{G} - \overline{F}G)\right).$$

Its jacobian matrix $D(F\overline{G})$ is given by the matrix in (1.5) replacing f by F and g by G. Therefore the rank of $D(F\overline{G})$ at $(z_1, z_2) \in V$ is not maximal if and only if all 2×2 minors of the jacobian matrix have zero determinant. This is equivalent to saying that for each pair of variables (z_i, z_j) one has the following equalities (cf. [33]):

$$FG\left(\frac{\partial F}{\partial z_1}\frac{\partial G}{\partial z_2} - \frac{\partial F}{\partial z_2}\frac{\partial G}{\partial z_1}\right) = 0 \qquad (1) \qquad (4.3)$$

$$\left| G\frac{\partial F}{\partial z_1} \right| = \left| F\frac{\partial G}{\partial z_1} \right| \qquad (2)$$

$$\left| G\frac{\partial F}{\partial z_2} \right| = \left| F\frac{\partial G}{\partial z_2} \right| \qquad (3)$$

$$| G |^2 \frac{\partial F}{\partial z_1}\overline{\frac{\partial F}{\partial z_2}} = | F |^2 \frac{\partial G}{\partial z_1}\overline{\frac{\partial G}{\partial z_2}} \tag{4}$$

Assume that $(z_1, z_2) \in V\backslash(FG)^{-1}(0)$ is such a point. Then the first equation in (4.3) implies

$$\frac{\partial F}{\partial z_1}(z_1, z_2)\frac{\partial G}{\partial z_2}(z_1, z_2) - \frac{\partial F}{\partial z_2}(z_1, z_2)\frac{\partial G}{\partial z_1}(z_1, z_2) = 0,$$

so that (z_1, z_2) belongs to the critical locus of the germ (G, F) and $\phi(z_1, z_2)$ is in the critical locus C of (g, f).

Now assume that $f\bar{g}$ does not have 0 as an isolated critical value. Then there exists a branch γ of C in U which is not a branch of $fg^{-1}(0)$, and a sequence $\{x_n\}_{n\in\mathbb{N}}$ of points of $\gamma\backslash\{p\}$ converging to p which satisfy that for each $n \in \mathbb{N}$ and for every parametrisation $\phi : V \to W \subset X\backslash\{p\}$ with $x_n \in W$, the rank of $D(f\bar{g} \circ \phi)(\phi^{-1}(x_n))$ is not maximal. Let us choose one such parametrisation $\phi : V \to W \subset X\backslash\{p\}$ with W contractible in $X\backslash\{p\}$ and $\gamma\backslash\{p\} \subset W$. We set $\beta = \phi^{-1}(\gamma\backslash\{p\})$, $F = f \circ \phi$, $G = g \circ \phi$, and for each $n \in \mathbb{N}$ set $(z_{n,1}, z_{n,2}) = \phi^{-1}(x_n)$ and $(u_n, v_n) = (g(x_n), f(x_n))$.

Let Δ_α be a branch of the discriminant curve Δ such that $\Delta_\alpha = (g, f)(\gamma)$. According to the definition of the discriminantal ratio d_α, the curve Δ_α admits a Puiseux expansion of the form:

$$u = \xi(v) = v^{d_\alpha}\left(a + \sum_{k\in\mathbb{N}^*} b_k v^{\frac{k}{m}}\right).$$

Then $G_{|\beta} = (\xi \circ F)_{|\beta}$ and for each point (z_1, z_2) of the curve β, the linear maps $DG(z_1, z_2)$ and $D(\xi \circ F)(z_1, z_2)$ coincide on the complex tangent line $T_{(z_1,z_2)}\beta$. Set $v = F(z_1, z_2)$ and let $h \in T_{(z_1,z_2)}\beta\backslash\{0\}$. Then $DG(z_1, z_2) \cdot h = \xi'(v)DF(z_1, z_2) \cdot h$. As (z_1, z_2) belongs to the jacobian locus of (G, F), for all $(z_1, z_2) \in \beta$ we obtain:

$$\left(\frac{\partial G}{\partial z_1}(z_1, z_2), \frac{\partial G}{\partial z_2}(z_1, z_2)\right) = \xi'(v)\left(\frac{\partial F}{\partial z_1}(z_1, z_2), \frac{\partial F}{\partial z_2}(z_1, z_2)\right).$$

This holds in particular for each $(z_{n,1}, z_{n,2})$, $n \in \mathbb{N}$.

Since f and g have no singular point in $U\backslash(fg)^{-1}(0)$, then for each $n \in \mathbb{N}$, either $\left(\frac{\partial F}{\partial z_1}(z_{n,1}, z_{n,2}), \frac{\partial G}{\partial z_1}(z_{n,1}, z_{n,2})\right) \neq (0, 0)$ or $\left(\frac{\partial F}{\partial z_2}(z_{n,1}, z_{n,2}), \frac{\partial G}{\partial z_2}(z_{n,1}, z_{n,2})\right) \neq (0, 0)$. Then, perhaps after replacing $\{x_n\}_{n\in\mathbb{N}}$ by a subsequence, we can assume that either

(a) for all $n \in \mathbb{N}$ one has $\left(\frac{\partial F}{\partial z_1}(z_{n,1}, z_{n,2}), \frac{\partial G}{\partial z_1}(z_{n,1}, z_{n,2})\right) \neq (0, 0)$; or

(b) for all $n \in \mathbb{N}$ one has $\left(\frac{\partial F}{\partial z_2}(z_{n,1}, z_{n,2}), \frac{\partial G}{\partial z_2}(z_{n,1}, z_{n,2})\right) \neq (0, 0)$.

If (a) holds then the second equation in (4.3) implies that for all $n \in \mathbb{N}$ one has $|\xi(v_n)| = |v_n \xi'(v_n)|$, where $v_n = f(x_n)$. That is,

$$|ad_\alpha + \sum_{k\in\mathbb{N}^*} b_k \left(d_\alpha + \frac{k}{m}\right) v_n{}^{\frac{k}{m}}| = |a + \sum_{k\in\mathbb{N}^*} b_k v_n{}^{\frac{k}{m}}|,$$

for all $n \in \mathbb{N}$. Taking the limit when $n \to \infty$ leads to $|ad_\alpha| = |a|$. Hence $d_\alpha = 1$.
Condition (b) also leads to $d_\alpha = 1$, so we arrive to Lemma 4.2. □

The following result is a generalization of ([33], theorem 5.1), which studied the case where $X = \mathbb{C}^2$ and f and g have an isolated critical point at $0 \in \mathbb{C}^2$.

Theorem 4.4 *Let (X, p) be a normal surface singularity and let $f, g : (X, p) \to (\mathbb{C}, 0)$ be two holomorphic germs with no common branches. Then the following conditions are equivalent*

(i) *The real analytic germ $f\bar{g} : (X, p) \to (\mathbb{R}^2, 0)$ has 0 as an isolated critical value.*
(ii) *The multilink $L_f \cup -L_g$ is fibered.*
(iii) *If $\pi : \tilde{X} \to X$ is a resolution of the holomorphic germ $fg : (X, p) \to (\mathbb{C}, 0)$, then for each rupture vertex (j) of the dual graph of π one has $m_j^f - m_j^g \neq 0$.*

Moreover, if these conditions hold, then the Milnor–Lê fibration $\Psi_{f\bar{g}} : \mathcal{L}_X \setminus L_{f\bar{g}} \to \mathbb{S}^1_\eta$ of $f\bar{g}$ is a fibration of the multilink $L_f \cup -L_g$.

The key point in the proof of ([33], 5.1) is the theorem 1.1 of [24], which relates the determinantal ratios of the germ $(g, f) : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ with some topological invariants of the meromorphic function f/g. The key point in the proof of theorem 4.4 is the following theorem of [22] that generalizes [24, theorem 1.1]. For this we need:

Definition 4.5 Let $\pi : \tilde{X} \to X$ be a good resolution of the holomorphic germ fg, let Γ be the associated dual graph and let $(1), \ldots, (r)$ be the vertices of Γ. For each $i = 1, \ldots, r$ let m_i^f (respectively, m_i^g) be the multiplicity of $f \circ \pi$ (respectively, $g \circ \pi$) along the irreducible component of the exceptional divisor $\pi^{-1}(p)$ represented by (i). The *contact quotient* associated with (i) is the rational number m_i^f / m_i^g.

Theorem 4.6 ([22], Theorem 0.3) *The set of the discriminantal ratios of the germ $(g, f) : (X, p) \to (\mathbb{C}^2, 0)$ equals the set T of contact quotients of (g, f) associated with the rupture vertices of the dual graph of a good resolution of the germ fg.*

Notice that the set T does not depend on the choice of the good resolution.

Proof of 4.4. By 3.2, if $f\bar{g}$ has 0 as an isolated critical value, then its Milnor–Lê fibration (1.8) is a fibration of the multilink $L_f \cup -L_g$ and (i) ⇒ (ii).

Assume now that the plumbing multilink $L_f \cup -L_g$ is fibered. Let $\pi : \tilde{X} \to X$ be a good resolution of the germ fg and let Γ be its dual graph. Let $(1), \ldots, (r)$ be the vertices of Γ. Let us denote by $\mathcal{R}$ the set of rupture vertices of Γ. Then, according to

2.14, for each rupture vertex (j) one has $m_j^f \neq m_j^g$, so (ii) $\Rightarrow$ (iii). Now, by definition 4.5 m_j^f/m_j^g is the contact quotient of (g, f) associated with the vertex (j), and 4.6 says that the set of contact quotients of (g, f) associated to the rupture vertices correspond to the determinantal ratios of the germ (g, f). If we now assume (iii), then Lemma 4.2 implies that $f\bar{g}$ has 0 as an isolated critical value, completing the proof of 4.4. □

5 The Milnor fibration for functions f/g

In this section, we compare the geometry of the Milnor–Lê fibration $\Psi_{f\bar{g}}$ with that of the Milnor fibration of the meromorphic function f/g introduced in [4], which we now recall. Let us consider an equidimensional, reduced complex analytic isolated singularity $(X, 0)$ of dimension n in $\mathbb{C}^N$ and two germs of holomorphic functions $f, g : (X, 0) \to (\mathbb{C}, 0)$ such that $f^{-1}(0)$ and $g^{-1}(0)$ have no common irreducible components.

Let us denote by $\mathcal{L}_X$ the link of $(X, 0)$ and by L_{fg} the link of fg in $\mathcal{L}_X$, i.e., the intersection of $(fg)^{-1}(0)$ with $\mathcal{L}_X$.

The meromorphic function f/g is well defined on $\mathcal{L}_X \backslash L_{fg}$ and takes values in $\mathbb{P}^1\backslash\{0, \infty\}$. Notice that, as observed in [15], away from $V_{fg} = \{fg = 0\}$ one has: $f\bar{g} = \frac{f}{g} \cdot |g|^2$ and therefore:

$$\frac{f\bar{g}}{|f\bar{g}|} = \frac{f/g}{|f/g|}.$$

The following definitions were introduced in [4] when the germ of X at 0 is smooth, following ideas of [29].

Let $M(f/g)$ be the set of points in $X\backslash V_{fg}$ where the fibers of $h := f/g$ are tangent to the spheres in $\mathbb{C}^N$ centered at 0. That is:

$$M(f/g) = \{x \in X\backslash V_{fg} | T_x \left((h^{-1}(h(x))\right) \subset T_x\mathbb{S}^{2N-1}_{||x||}\}.$$

Definition 5.1 The *bifurcation set* $B \subset \mathbb{P}^1$ of the meromorphic function f/g is the union of $\{0, \infty\}$ and the set of $c \in \mathbb{P}^1$ such that there exists a sequence of points $(x_k)_{k\in\mathbb{N}}$ in $M(f/g)$ such that

$$\lim_{k\to\infty} x_k = 0 \quad \text{and} \quad \lim_{k\to\infty} (f/g)(x_k) = c.$$

Definition 5.2 The meromorphic function f/g is *semitame* at 0 if $B = \{0, \infty\}$.

Adapting the proof of Theorem 2.6 of [4] which concerned the case when X is smooth, we obtain the first part of the following theorem. The second part follows by the same arguments as in Lemma 3.1.

Theorem 5.3 *Consider an equidimensional, reduced complex analytic isolated singularity* $(X, 0)$ *of dimension* n *in* $\mathbb{C}^N$ *and two germs of holomorphic functions* f, g :

$(X,0) \to (\mathbb{C},0)$ such that $f^{-1}(0)$ and $g^{-1}(0)$ have no common irreducible components. If f/g is semitame at 0, then the map

$$\phi_{f/g} = \frac{f/g}{|f/g|} : \mathcal{L}_X \backslash L_{fg} \longrightarrow \mathbb{S}^1,$$

is a locally trivial C^∞ fibre bundle. Moreover, $\phi_{f/g}$ is a fibration of the multilink $L_f \cup -L_g$.

Definition 5.4 We call $\phi_{f/g} : \mathcal{L}_X \backslash L_{fg} \longrightarrow \mathbb{S}^1$ the *Milnor fibration* of the meromorphic germ f/g.

When the link is a rational homology sphere, then the arguments of ([32], 4.7, 4.8) show that the Milnor fibration and the Milnor–Lê fibration are topologically equivalent (see Remark 2.13, 3). The following theorem states that these two fibrations are in fact equivalent whatever the link $\mathcal{L}_X$ may be.

Theorem 5.5 *Let $f, g : (X,0) \to (\mathbb{C},0)$ be holomorphic germs such that $f^{-1}(0)$ and $g^{-1}(0)$ have no common irreducible components, $f\bar{g}$ has an isolated critical value at 0, $f\bar{g}$ has the Thom property and f/g is semitame at 0. Let $T_{\varepsilon,\eta} = (f/g)^{-1}(\mathbb{D}^2_\eta) \cap \mathcal{L}_X$ be an algebraic neighbourhood of L_f in $\mathcal{L}_X$ as in Theorem 1.3. Then the Milnor–Lê fibration of $f\bar{g}$ and the Milnor fibration of f/g are equivalent in the sense that there exists a diffeomorphism $H : (\mathcal{L}_X, L_{fg}) \to (\mathcal{L}_X, L_{fg})$ such that $\frac{1}{\eta}\Psi_{f\bar{g}} = \phi_{f/g} \circ H$ on $\mathcal{L}_X \backslash \mathrm{Int}(\mathrm{T}_{\varepsilon,\eta})$.*

We recall that when X has dimension 2 the Milnor–Lê fibration $\Psi_{f\bar{g}}$ is defined on all $\mathcal{L}_X \backslash L_{fg}$. Then one has:

Corollary 5.6 *The two fibrations $\Psi_{f\bar{g}}$ and $\phi_{f/g}$ of the multilink $L_f - L_g$ are topologically equivalent in the sense of Definition 2.2.*

Proof of Theorem 5.5. We follow the constructions used in [8] to refine the classical Milnor fibration theorem. For simplicity, assume the representative of the germ X is small enough so that $f\bar{g}$ has no critical points away from the hypersurface $V = (fg)^{-1}(0)$ and X has a Whitney stratification for which V is a union of strata, $\{0\}$ is a stratum and every stratum has $\{0\}$ in its closure. We further assume that each stratum meets transversally every sphere in $\mathbb{C}^N$ centered at 0. Such a stratification exists by the Bertini–Sard theorem of Verdier in [40].

Notice one has the following canonical decomposition of X associated to $f\bar{g}$: consider the set $\{l_\theta\}$ of all the real lines through the origin in $\mathbb{C}$, where θ is the angle of the corresponding line with respect to the positive real axis; this set is of course parametrized by $P^1_{\mathbb{R}}$. Set $X_\theta = \{z \in X | f\bar{g}(z) \in l_\theta\}$. Then one obviously has:

$$X = \bigcup_{\theta \in [0,\pi[} X_\theta \quad \text{and} \quad V = \bigcap_{\theta \in [0,\pi[} X_\theta.$$

We claim each X_θ is a real analytic hypersurface, non-singular away from $\mathrm{Sing}(V)$, the singular set of V. To prove this, let $l_\theta^\perp = l_{\theta+\frac{\pi}{2}}$ be the line orthogonal to l_θ, and let $\pi_\theta : \mathbb{C} \to l_\theta^\perp$ be the orthogonal projection. Then

$$X_\theta = (\pi_\theta \circ f\bar{g})^{-1}(0)\,,$$

so X_θ is analytic and its singular points are the critical points of $\pi_\theta \circ f\bar{g}$ contained in X_θ. By hypothesis the critical values of $f\bar{g}$ in X are exactly the singularities of V. If z is a regular point of $f\bar{g}$, then the differential map $D(f\bar{g})(z)$ has rank 2. Since the projection π_θ is a submersion one has that $D(\pi_\theta \circ f\bar{g})(z)$ has rank 1, hence z is a regular value of $\pi_\theta \circ f\bar{g}$ and the claim follows.

Therefore X decomposes as the union of the real analytic hypersurfaces X_θ which spin around V forming a "pencil" with axis V and they are non-singular away from Sing(V). Notice that each manifold $X_\theta^* = X_\theta \setminus V$ is a union of $(f\bar{g})$-fibers, and it is also a union of (f/g)-fibers; i.e.,

$$X_\theta^* = \bigcup_{t\in(l_\theta\setminus\{0\})} (f\bar{g})^{-1}(t) \;=\; \bigcup_{t\in(l_\theta\setminus\{0\})} (f/g)^{-1}(t).$$

Since by hypothesis the germ f/g is semitame at 0, for $t \neq 0$ the fibers $(f/g)^{-1}(t)$ are transversal to the spheres in $\mathbb{C}^N$ centered at 0, and therefore every sufficiently small sphere in $\mathbb{C}^N$ centered at 0 meets each manifold X_θ^* transversally.

We need the following:

Lemma 5.7 *For $\varepsilon_o > 0$ sufficiently small, we have that each manifold X_θ^* is transversal to every sphere centered at 0 of radius $\leq \varepsilon_o$.*

Proof of the Lemma Just as in Milnor's proof of [26, Lemma 4.6], using [4, Lemma 2.7] and the Curve Selection Lemma of Milnor, we have that given f, g as above, such that f/g is semitame, there is a number $\varepsilon_o > 0$ such that for all $z \in \mathbb{C}^n \setminus V$ with $\|z\| \leq \varepsilon_o$, the two vectors z and $i\, grad\, log\, \frac{f}{g}(z)$ are either linearly independent over $\mathbb{C}$ or else the argument of the complex number

$$\lambda = \left(i\, grad\, log\, \frac{f}{g}((z))\right)/z$$

has absolute value more than, say, $\pi/4$.

We claim this implies that there exists a C^∞ vector field v on $\mathbb{D}_\varepsilon \setminus V$ so that for all $z \in \mathbb{D}_\varepsilon \setminus V$ one has:

(i) the hermitian product $\langle v(z),\ grad\, log\, \frac{f}{g}(z)\rangle$ is real and positive; and
(ii) the hermitian product $\langle v(z), z\rangle$ has positive real part.

Indeed it suffices to construct this vector field locally, in the neighbourhood of some given point $z_o \in \mathbb{D}_\varepsilon \setminus V$, for we can afterwards glue all these vector fields by a partition of unity and get a vector field defined globally and satisfying the conditions of the lemma. We work on a coordinate chart for X at z_o, so we think of it as being identified with an open ball U in $\mathbb{C}^n$. We recall that the real part of the hermitian product is the usual inner product in $\mathbb{R}^{2n}$.

For simplicity set $\nabla(z_o) = grad\, log\, \frac{f}{g}(z_o)$, and let E denote the real orthogonal complement of $i\nabla(z_o)$ in $\mathbb{C}^n$; every w in E satisfies that $\langle v(z_o), \nabla(z_o)\rangle$ is real; the vector $\nabla(z_o)$ defines an oriented real line ℓ in E and every vector in the half-space E^+ of vectors in E whose projection to ℓ is positive satisfies condition (i). We need to show that there are vectors in E^+ satisfying also condition (ii). If z_o and $\nabla(z_o)$ are linearly dependent over $\mathbb{C}$, so that $\lambda z_o = \nabla(z_o)$ for some λ in $\mathbb{C}$, then one has

$$|argument\, \lambda| < \pi/4 ,$$

and we can just take $v = z_o$. If the vectors z_o and $\nabla(z_o)$ are linearly independent over $\mathbb{C}$, then $\nabla(z_o)$ and z_o are linearly independent over $\mathbb{R}$ and every vector in E^+ for which the real inner product with z_o is positive works for us.

Conditions (i) and (ii) imply one has a vector field on $\mathbb{D}_\varepsilon \setminus V$ which is everywhere tangent to the X_θ^* and everywhere transversal to the spheres. □

We notice that condition (i) in the above proof actually gives more than we need: the vector field one gets is transversal to the "pinched Milnor tubes" $(f/g)^{-1}(\mathbb{S}^1_\eta)$ of f/g. What we actually need for Theorem 5.5 is a vector field which is transversal to the Milnor tubes $N(\varepsilon, \eta) = (f\bar{g})^{-1}(\mathbb{D}_\eta) \cap \mathcal{L}_X$ of $f\bar{g}$, being also transversal to the spheres around 0 and tangent to the X_θ^*. For this we use the transversality of the X_θ^* with the spheres to construct the vector field we need, following [8].

We claim Lemma 5.7 implies that the map

$$z \mapsto \ \rho(z) := \|z\| \cdot \frac{f\bar{g}(z)}{|f\bar{g}(z)|} ,$$

is a submersion for all $z \in X \cap \mathbb{B}_{\varepsilon_0} \setminus V$. Notice the fibers of ρ are the intersections of the X_θ with the spheres around 0.

To prove that ρ is a submersion for each $z \in X \cap \mathbb{B}_{\varepsilon_0} \setminus V$, let X_θ^* be the corresponding manifold that contains the point z. Then we know that X_θ^* is transversal to $\mathbb{S}_{\|z\|}$, the sphere in $\mathbb{C}^N$ with center at 0 and radius $\|z\|$. Thus we know that the tangent space $T_z X_\theta^*$ contains a vector v_1 which is not tangent to $\mathbb{S}_{\|z\|}$. Since the fibers of ρ are contained in the spheres, this implies that the derivative $D_z(\rho)$ carries v_1 into a non-zero vector in $T_{\rho(z)}\mathbb{C}$ transversal to the circle $\mathbb{S}^1_{\|x\|}$ of radius $\|x\|$, which contains the point $\rho(z)$.

Similarly, by transversality, the intersection $N_z := X_\theta^* \cap \mathbb{S}_{\|z\|}$ is a smooth submanifold of the sphere of codimension 1. Let v_2 be a vector normal to N_z. Then $D_z(\rho)(v_1)$ is a non-zero vector in $T_{\rho(z)}\mathbb{C}$ transverse to the line l_θ, so it is tangent to the circle $\mathbb{S}^1_{\|x\|}$ and transversal to the line spanned by the vector $D_z(\rho)(v_1)$. Hence ρ is a submersion for each $z \in X \cap \mathbb{B}_{\varepsilon_0} \setminus V$.

We can now finish the proof of the theorem: let v_{rad} be the vector field in $\mathbb{C}$ defined by $v_{\text{rad}}(x) = x$, so its solutions are the real half-lines that emanate from the origin. Since $f\bar{g}$ has an isolated critical value at 0, we can assume it is a submersion at each point in some open ball $\mathbb{B}_{\varepsilon_1}$ centered at 0. Hence we can lift v_{rad} to a differentiable vector field $\tilde{v}_{\text{rad}}$ in $\mathbb{B}_{\varepsilon_1}$ which, by construction, is transversal to all Milnor tubes $(f\bar{g})^{-1}(\mathbb{S}^1_\eta)$ for all circles $\mathbb{S}^1_\eta$ in $\mathbb{C}$ centered at 0 and sufficiently small with respect to ε_1. By construction $D(f\bar{g})$ maps $\tilde{v}_{\text{rad}}$ into v_{rad}, so that the integral curves of $\tilde{v}_{\text{rad}}$ are tangent to the manifolds X_θ^*.

Similarly, if ε_0 is chosen so that the second claim above is also satisfied, then we can lift v_{rad} to a differentiable vector field $\widehat{v}_{\text{rad}}$ in $\mathbb{B}_{\varepsilon_1}$ which, by construction, is transversal to the spheres around 0 and the integral curves of $\widehat{v}_{\text{rad}}$ are tangent to the manifolds X_θ^*.

We now observe that by the Curve Selection Lemma of Milnor [26], there are no points in X sufficiently near 0 where these two vector fields point in exactly opposite directions. Hence the vector field $\zeta(z) = \tilde{v}_{\text{rad}}(z) + \widehat{v}_{\text{rad}}(z)$ is non-singular on $(X\backslash V) \cap \mathbb{B}_{\varepsilon_1}$, it is integrable, it is tangent to the X_θ^*, transversal to all Milnor tubes $(f\bar{g})^{-1}(\mathbb{S}_\eta^1)$ and transversal to the intersection of X with all sufficiently small spheres around 0. The rest of the proof is now as in Milnor's book: we use the flow defined by this vector field to inflate the tube $N(\varepsilon_0, \eta)$ that defines the Milnor–Lê fibration in such a way that the corresponding fibers in the sphere are the fibers of the Milnor fibration $\frac{f/g}{|f/g|}$. □

A natural question now is to compare the two hypothesis which lead, respectively, to the Milnor and the Milnor–Lê fibration:

1. f/g is semitame at 0
2. $f\bar{g}$ has an isolated critical value at 0 and the Thom a_f property

We recall that when X has dimension 2 the Thom a_f property is automatically satisfied, by Proposition 1.4. If one further has $X = \mathbb{C}^2$ then one can prove that the two conditions above are actually equivalent. Moreover, one has the following theorem, which follows immediately from previous results in this and other works:

Theorem 5.8 *Let $f, g : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be holomorphic germs such that $f^{-1}(0)$ and $g^{-1}(0)$ have no common factors. The following are equivalent*:

(i) *$f\bar{g}$ has an isolated critical value at* 0*,*
(ii) *The map $\Psi_{f\bar{g}} : \mathcal{L}_X\backslash L_{fg} \to \mathbb{S}_\eta^1$ is a fibration of the multilink $L_f \cup -L_g$,*
(iii) *The multilink $L_f \cup -L_g$ is fibered,*
(iv) *Each $c \neq 0, \infty$ is a generic value of the local pencil generated by f and g,*
(v) *The map $\phi_{f/g} : \mathcal{L}_X\backslash L_{fg} \to \mathbb{S}^1$ is a fibration of the multilink $L_f \cup -L_g$,*
(vi) *f/g is semitame at* 0

Moreover, if these condition hold, then the Milnor–Lê fibration $\Psi_{f\bar{g}}$ and the Milnor fibration $\phi_{f/g}$ are equivalent fibrations of the multilink $L_f \cup -L_g$.

The equivalence between (i), (ii) and (iii) is proved in [33] when $f\bar{g}$ has an isolated critical point. The general case is given by Theorem 4.4 above. The equivalence between (iii), (iv), (v) and (vi) is proved in [4, Theorem 5.6].

The previous result states that the semitameness of f/g is equivalent for $f\bar{g}$ to have an isolated critical value at 0. In fact, this equivalence is obtained as a consequence of several implications. In particular, (i) ⇒ (vi) is a consequence of Lemma 4.2 ((i) ⇒ (iii)), Proposition 2.13 of [4] ((iii) ⇒ (iv)) and an adaptation of some results of A. Durfee ([9]) to the local case which lead to (iv) ⇔ (vi).

We do not know how to produce a direct proof of the equivalence (when $X = \mathbb{C}^2$) between the statements "$f\bar{g}$ has an isolated critical value" and "f/g is semitame", i.e., the equivalence (i) ⇔ (iv). In fact, it is not obvious for us how these two conditions are related even in simple examples. For instance, consider the two germs

$f, g : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ given by $f(x, y) = x(x + y^2) + y^3$ and $g(x, y) = y^3$. As noticed in [4], the bifurcation set of f/g is $B = \{0, 1, \infty\}$, so f/g is not semitame at 0. On the other hand, a straight-forward computation shows that the map $f\bar{g}$ has a non-isolated critical value at 0. However the non semitameness of f/g leading to the bifurcation value $1 \in B$ is along the two-dimensional real set $C_1 \subset \mathbb{C}^2 \setminus \{fg = 0\}$ with equation:

$$(2x + y^2)|y|^2 + (3x + y)|x|^2 = 0,$$

whereas the real curve of critical points of $f\bar{g}$ has equations:

$$2x + y^2 = 0 \text{ and } 2|y - 3| = 3|y - 2|.$$

These two sets are disjoint and it is not clear to us how their existences are related.

When X is not $\mathbb{C}^2$, the comparison between the properties (1) and (2) remains open. Perhaps the arguments of [9] can be adapted to the case when X is a surface, but it seems that new ideas must be introduced for higher dimensions.

6 Realization of fibered multilinks by singularities : results and open questions

"Realization questions" arise naturally in the study of the topology of singularities. Roughly speaking these ask whether a given topological object is realized by polynomial or analytic equations.

For example, it is proved in [2] that all knots and links can be realised by algebraic equations in the following sense. Given a compact smooth submanifold with boundary U of $\mathbb{S}^{k-1}$ of codimension ≥ 1 and trivial normal bundle, there exists a real algebraic set $Z \subset \mathbb{R}^k$ with an isolated singularity at the origin such that the pair $(\mathbb{S}^{k-1}, \mathbb{S}^{k-1} \cap Z)$ is diffeomorphic to $(\mathbb{S}^{k-1}, \partial U)$. If U is a Seifert surface of a knot K in $\mathbb{S}^{k-1}$ the theorem yields the result stated as title of that article: "All knots are algebraic" (cf. [27]).

Regarding the results of the present paper, a natural question is to ask which fibered plumbing multilinks (M, L) are realized, up to diffeomorphism, as the binding of the Milnor fibration of a holomorphic or real analytic germ. In this section, we restrict to the case when M is homeomorphic to a surface singularity link.

Recall that in this paper, we only consider connected plumbing graphs whose vertices carry non-negative genus and strictly negative Euler numbers ; moreover, all plumbing operations have positive ϵ in the sense of [30].

In [31] we give a complete answer when $L = n_1 K_1 \cup \ldots \cup n_l K_l$ is positive, *i.e.* $n_i > 0$ for all $i = 1, \ldots, l$: all such multilinks are realized by holomorphic germs. More precisely,

Theorem 6.1 ([31], 2.1) *Let $L = n_1 K_1 \cup \ldots \cup n_l K_l$ be a multilink in a 3-manifold M with positive multiplicities n_i. The following are equivalent*:

(i) *There exists a normal complex surface singularity (X, p) and a holomorphic germ $f : (X, p) \to (\mathbb{C}, 0)$ such that the pair (M, L) is diffeomorphic to the pair $(\mathcal{L}_X, L_f)$.*

(ii) (M, L) *is a plumbing multilink admitting a plumbing graph* Γ *whose monodromical system admits a solution* $(m_1, \ldots, m_r) \in (\mathbb{N}_{>0})^r$.

Remark 6.2 (1) In fact, it is proved in [31] that (X, p) and $f : (X, p) \to (\mathbb{C}, 0)$ can be chosen in such a way that Γ is a plumbing graph of $(\mathcal{L}_X, L_f)$. This is stronger than just having a diffeomorphism between (M, L) and $(\mathcal{L}_X, L_f)$.

The following formulation of (ii) is very useful:

Proposition 6.3 *The condition (ii) is equivalent to*:

(ii') (M, L) is a fibered plumbing multilink admitting a plumbing graph Γ whose intersection matrix is negative definite.

Proof According to 2.11, the condition (ii) implies that L_f is fibred. Morevover, Zariski's lemma ([3]) shows that (ii) implies Grauert's condition : the intersection matrix of Γ is negative definite.

Conversely, assume that (ii') holds. Let $(m_1, \ldots, m_r)$ be the solution of the monodromical system of L. According to 2.11, $m_i \in \mathbb{Z}^*$. Then one has to prove that $m_i > 0 \ \forall i = 1, \ldots, r$. Let us consider the extremities $j = 1, \ldots, l$ of the arrows decorating Γ as vertices of Γ, and let us set $m_j = n_j$.

Let (i) and (j) be two vertices of Γ joined by an edge, weighted by the Euler classes e_i and e_j. We call formal blow-up the operation consisting of

1. replacing $(i)o - o(j)$ by a string of 3 vertices $(i)o - o - o(j)$ by adding one vertex (k) between (i) and (j), and
2. replacing the Euler classes e_i and e_j by $e_i - 1$ and $e_j - 1$, respectively, and of weighting (k) by the Euler class $e_k = -1$.

One assigns to (k) the multiplicity $m_k = m_i + m_j$. Then the intersection matrix of the new graph is again negative definite, and the multiplicities m_j are solution of its monodromical system.

Assume that there exists i such that $m_i < 0$. One can perform a finite number of formal blow-ups in such a way that, in the new graph Γ, there is no pair of neighbour vertices (i) and (j) such that $m_i < 0$ and $m_j > 0$. Let Γ' be the maximal subgraph of Γ whose vertices (i) verify $m_i < 0$. Then any vertex (k) of $\Gamma \setminus \Gamma'$ which is neighbour of a vertex of Γ' carries $m_k = 0$. Let us denote by $(\sigma_1), \ldots (\sigma_h)$ the vertices of Γ', and by $M_{\Gamma'}$ the intersection matrix associated with Γ'. Then $(m_{\sigma_1}, \ldots m_{\sigma_h})$ is solution of the monodromical system of Γ':

$$M_{\Gamma'} \, {}^t(m_{\sigma_1}, \ldots m_{\sigma_h}) = 0$$

But $M_{\Gamma'}$ is definite negative, as the intersection matrix associated with a subgraph of Γ. Contradiction. □

Now, let us consider a fibered plumbing multilink (M, L) such that the weights n_i of $L = n_1 K_1 \cup \ldots \cup n_l K_l$ are not all positive. Considering Theorem 5.3, a natural tentative to realize (M, L) by a real analytic germ is to try to construct two holomorphic germs $f : (X, p) \to (\mathbb{C}, 0)$ and $g : (X, p) \to (\mathbb{C}, 0)$ such that (M, L) is diffeomorphic to the pair $(\mathcal{L}_X, L_f \cup -L_g)$.

The results of ([31], Sect. 2) together with Theorem 5.3 enable one to perform this construction in a particular case, which is described in the following result. We recall [19] that a surface singularity is *taut* if its topology determines the analytic type. Of course, this includes the germ of $\mathbb{C}^2$ at the origin.

Theorem 6.4 *Let M be orientation preserving diffeomorphic to the link $\mathcal{L}_X$ of a taut surface singularity (X, p). Let L_1 and L_2 be two plumbing multilinks with positive multiplicities in a plumbing decomposition of M with plumbing graph Γ whose intersection matrix is negative definite, and such that*

1. *The multilinks L_1, L_2 and $L_1 \cup (-L_2)$ are fibered*
2. *For any components K_1 of L_1 and K_2 of L_2, the arrows representing K_1 and K_2 are carried by distincts vertices of Γ.*

Then there exist two holomorphic germs $f, g : (X, p) \to (\mathbb{C}, 0)$ without common branches such that (M, L_1), (M, L_2) and $(M, L_1 \cup -L_2)$ are, up to diffeomorphism, the multilinks $(\mathcal{L}_X, L_f)$, $(\mathcal{L}_X, L_g)$ and $(\mathcal{L}_X, L_f \cup -L_g)$ associated with the germs f, g and $f\bar{g} : (X, p) \to (\mathbb{C}, 0)$; and

$$\frac{f\bar{g}}{|f\bar{g}|} : \mathcal{L}_X \backslash (L_1 \cup -L_2) \longrightarrow \mathbb{S}^1,$$

is a fibre bundle that realises $L_1 \cup -L_2$ as a fibered multilink.

Proof Let Γ_1 (resp. Γ_2) be the graph Γ decorated with arrows corresponding to L_1 (resp. L_2). The graphs Γ_1 and Γ_2 hold the condition (ii'). There then exist two normal surface singularities (X_1, p_1) and (X_2, p_2) and two analytic germs $f : (X_1, p_1) \to (\mathbb{C}, 0)$ and $g : (X_2, p_2) \to (\mathbb{C}, 0)$ such that the multilink $(\mathcal{L}_{X_1}, L_f)$ (resp. $(\mathcal{L}_{X_2}, L_g)$ has Γ_1 (resp. Γ_2) as plumbing graph. The tautness hypothesis implies that one can take $X_1 = X_2 = X$. Moreover, the condition 2. guarantees that f and g does not have common branches. One concludes using Theorem 5.3. □

The realization problem of a fibered plumbing multilink (M, L) by a real analytic germ remains open in the general case even when M is a surface singularity link. The following two situations show that there exists some (M, L) such that M is a surface singularity link but which cannot be realized by real analytic germs of the form $f\bar{g} : (X, p) \to (\mathbb{C}, 0)$.

1. Assume that M is homeomorphic to the link $\mathcal{L}_X$ of a normal surface singularity which is not taut, and take two plumbing multilinks L_1 and L_2 in M with positive multiplicities and such that the three multilinks $L_1 \cup -L_2$, L_1 and L_2 are fibered. In this case, Theorem 6.1 implies that the multilinks L_1 and L_2 are realized as the multilinks of two holomorphic germs $f : (X_1, p_1) \to (\mathbb{C}, 0)$ and $g : (X_2, p_2) \to (\mathbb{C}, 0)$. But the analytic types of (X_1, p_1) and (X_2, p_2) may not coincide. Therefore the germ $f\bar{g}$ may not be defined and, a priori, the multilink link $L_1 \cup -L_2$ may not be realizable as the multilink of a real analytic germ $f\bar{g}$. In [NP], the authors give some examples of such links such that the analytic type of (X_1, p_1) and (X_2, p_2) cannot coincide.

2. It could happen that $L_1 \cup -L_2$ in 6.4 is fibered, but the multilinks L_1 and L_2 are not fibered (e.g. the multilink $L_3 \cup -L_4$ of example 2.12).
Since L_i has positive multiplicities, then, by the same arguments as in the proof of 6.3, $m_j^{(i)} > 0$ for all $j \in \{1, \dots, r\}$, where

$$^t(m_1^{(i)}, \dots, m_r^{(i)},) = -(M_\Gamma)^{-1} \quad {}^t b(L_i).$$

Therefore, this situation occurs when there exists $k \in \{1, \dots, r\}$ such that $m_k^{(i)} \notin \mathbb{Z}$ whereas for each $j = 1, \dots, r$, $m_j^{(1)} - m_j^{(2)} \in \mathbb{Z}^*$.

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