Math. Ann. (2008) 341:201–221
DOI 10.1007/s00208-007-0189-9

Mathematische Annalen

On the discriminant locus of a Lagrangian fibration

Justin Sawon

Received: 4 October 2006 / Published online: 6 December 2007

Abstract Let $X \to \mathbb{P}^n$ be an irreducible holomorphic symplectic manifold of dimension $2n$ fibred over $\mathbb{P}^n$. Matsushita proved that the generic fibre is a holomorphic Lagrangian abelian variety. In this article we study the discriminant locus $\Delta \subset \mathbb{P}^n$ parametrizing singular fibres. Our main result is a formula for the degree of Δ, leading to bounds on the degree when X is a fourfold.

Mathematics Subject Classification (2000) 53C26 · 14D06

1 Introduction

Due primarily to the work of Matsushita [14,15], much is now known about the structure of fibrations on irreducible holomorphic symplectic manifolds. In particular, the generic fibre must be a holomorphic Lagrangian complex torus and it is expected that the base must be projective space. In fact, 10 years earlier Mukai [17] already posed the question: when is a fibration $X \to \mathbb{P}^n$ by n-dimensional complex tori a holomorphic symplectic manifold? Our goal in this article is to find restrictions on the degree of the discriminant locus $\Delta \subset \mathbb{P}^n$ in the case that X is holomorphic symplectic.

To begin with, we assume the fibration is the relative Jacobian of a family of curves, where the curves degenerate in a controlled manner over a generic point of Δ (they acquire a single node). We prove that the degree of Δ is given by

$$\deg\Delta = 24\left(n!\sqrt{\hat{A}}[X]\right)^{\frac{1}{n}},$$

J. Sawon (✉)
Department of Mathematics, Colorado State University, Fort Collins, CO 80523-1874, USA
e-mail: sawon@math.colostate.edu
URL: http://www.math.colostate.edu/∼sawon

where $\sqrt{\hat{A}}[X]$ the characteristic number of X coming from the square root of the $\hat{A}$-polynomial. This same characteristic number arose in earlier work of Hitchin and Sawon [10], and it appears to play a fundamental rôle in holomorphic symplectic geometry. The important point in this case is that we have a model for the singular fibre X_t above a generic point $t \in \Delta$. Indeed the above formula for $\deg\Delta$ readily generalizes to projective X fibred by principally polarized abelian varieties, provided the singular fibre X_t for generic $t \in \Delta$ conforms to this same model (Theorem 1). We then generalize our formula to fibrations by non-principally polarized abelian varieties, whose singular fibres conform to slightly different models (Theorem 2). All of these models come from toroidal compactifications of moduli spaces of abelian varieties, due to Igusa [13] and Mumford [19] (see also [1]). In addition to constructing these compactifications, Mumford [18] described degenerations of abelian varieties which sit above the boundary. Our assumption is that for generic $t \in \Delta$, the singular fibre X_t is a (semi-stable) degeneration of an abelian variety which occurs over a generic point in a codimension one component of the boundary. We will of course give an explicit description of how these degenerate abelian varieties look.

It is worth noting that in four dimensions Matsushita [16] has classified all possible singular fibres that can occur over a generic point of Δ, and given local models. Some of these look like products of smooth and singular elliptic curves, up to étale cover, and occur in examples like the Hilbert scheme $\mathrm{Hilb}^2 S$ of two points on an elliptic K3 surface $S \to \mathbb{P}^1$ (Example 3.5 in [23]). Excluding such Lagrangian fibrations, where the generic fibre is a product of elliptic curves, the singular fibres considered in this article are the only ones known to occur in global examples. It would be good to extend our results to allow any of the singular fibres on Matsushita's list, though the existence of non-reduced components creates some difficulties.

In [9] Guan proved that the characteristic numbers of a holomorphic symplectic fourfold are bounded; so when X is a fourfold our formulae give bounds on the degree of Δ. We briefly indicate why such bounds might be useful. Suppose $X \to \mathbb{P}^2$ is a fibration by abelian surfaces with polarization of type $(1, d)$. This leads to a morphism

$$\phi : \mathbb{P}^2 \backslash \Delta \longrightarrow \mathcal{A}^\circ(1, d),$$

where $\mathcal{A}^\circ(1, d)$ is the moduli space of abelian surfaces with this polarization. If the singular fibres X_t for generic $t \in \Delta$ are well-behaved, then this map can be extended to a morphism between (partial) compactifications

$$\phi^* : \mathbb{P}^2 \backslash \Delta_{\text{sing}} \longrightarrow \mathcal{A}^{\circ *}(1, d).$$

The construction and properties of $\mathcal{A}^{\circ *}(1, d)$ when d is prime are well-described in the book by Hulek et al. [11]. The hope then is that the degree of Δ can be used to control the degree of the morphism ϕ^*, implying finiteness of the number of deformation classes of holomorphic symplectic fourfolds which admit Lagrangian fibrations (cf. the comments at the end of the introduction in [26]). Unfortunately it is not immediately clear how to achieve this: the degree of Δ only tells us about the intersection of $\phi^*(\mathbb{P}^2 \backslash \Delta_{\text{sing}})$ with the boundary divisor in $\mathcal{A}^{\circ *}(1, d)$, and the latter is not ample in general.

Instead we end the article by pursuing a slightly different direction. We use Guan's Theorem to show that both d and the degree of Δ are bounded (Theorem 4). Under an additional hypothesis, concerning the polarization of X, we are able to show that the pair consisting of d and the degree of Δ can take only thirteen possible values (Theorem 5). We expect that further work will eliminate many of these possibilities.

2 Good singular fibres

In this article a *Lagrangian fibration* shall mean an irreducible holomorphic symplectic manifold X of dimension $2n$ which is fibred over projective space

$$f : X \to \mathbb{P}^n.$$

Let $\Delta \subset \mathbb{P}^n$ be the discriminant locus over which the Jacobian of f drops rank; it is a divisor parametrizing singular fibres of f. The singular locus Δ_{sing} of Δ will be codimension at least two in $\mathbb{P}^n$, which will mean that it can effectively be ignored in most of our calculations. We write $\Delta_{\text{sm}} := \Delta \backslash \Delta_{\text{sing}}$ for the smooth locus of Δ.

Matsushita [14,15] proved that the generic fibre of f must be a (holomorphic Lagrangian) complex torus. We begin by describing an example where the fibres are Jacobians of genus n curves.

Example 1 (The Beauville–Mukai integrable system [3]) Let S be a K3 surface which contains a smooth genus n curve C, and assume for simplicity that the Picard group of S is generated (over $\mathbb{Z}$) by this curve. Then C moves in an n-dimensional linear system $|C| \cong \mathbb{P}^n$ and every curve in this family

$$\mathcal{C} \to \mathbb{P}^n$$

is integral (reduced and irreducible). The relative compactified Jacobian $X = \bar{J}_0(\mathcal{C}/\mathbb{P}^n)$ is then a (smooth) Lagrangian fibration over $\mathbb{P}^n$. Here the compactified Jacobian $\bar{J}_0 C_t$ of an integral curve C_t is defined to be the moduli space of rank-one torsion-free sheaves of Euler characteristic zero, i.e., degree $n-1$ (see [6]).

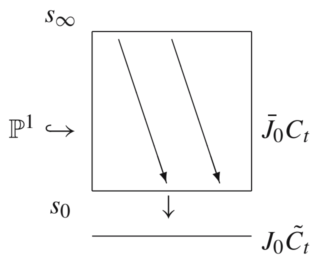
There are two features of this fibration to which we wish to draw attention. Firstly, each smooth fibre contains a canonical theta divisor Θ (the image of

$$\text{Sym}^{n-1} C_t \to \text{Pic}^{n-1} C_t = J_0 C_t,$$

which can be defined without reference to a basepoint). When $t \in \Delta_{\text{sm}}$, the curve C_t acquires a single node. In this case too there is a (generalized) theta divisor Θ on $\bar{J}_0 C_t$ (for example, see [7]). So we have a relative theta divisor over $\mathbb{P}^n \backslash \Delta_{\text{sing}}$, whose closure gives a divisor Y in X.

Secondly, consider the structure of a singular fibre $\bar{J}_0 C_t$ for $t \in \Delta_{\text{sm}}$. The following description of the compactified Jacobian of a curve C_t with a single node is well known (see [12]; Example (1) on page 83 of [20] describes the genus two case, which can easily be generalized). Let $\tilde{C}_t$ be the normalization of C_t. The normalization of $\bar{J}_0 C_t$

is then a certain $\mathbb{P}^1$-bundle over $J_0\tilde{C}_t$. The zero and infinity sections s_0 and s_∞ of the $\mathbb{P}^1$-bundle are canonically isomorphic to $J_0\tilde{C}_t$, but we instead identify them using a certain translation in $J_0\tilde{C}_t$. Then $\bar{J}_0C_t$ is given by taking the $\mathbb{P}^1$-bundle and gluing s_0 and s_∞ using the above identification.



Definition 1 Let $X \to \mathbb{P}^n$ be a Lagrangian fibration by principally polarized abelian varieties such that the generic singular fibre X_t for $t \in \Delta_{\mathrm{sm}}$ is obtained by gluing together the zero and infinity sections of a $\mathbb{P}^1$-bundle over a principally polarized abelian variety of dimension $n-1$, just as in the example above. Then we say $X \to \mathbb{P}^n$ has *good* singular fibres.

Remark 1 As mentioned in the introduction, Igusa [13] and Mumford [19] constructed compactifications of the moduli space of abelian varieties. Although this involves some choices, in the principally polarized case there is just one boundary component of codimension one. Moreover, Mumford [18] also gave a construction of degenerating abelian varieties; a generic point of the boundary then corresponds to a degenerate abelian variety as described above, i.e., a good singular fibre, which can therefore be regarded as the generic semi-stable degeneration of a principally polarized abelian variety.

For a Lagrangian fibration with good singular fibres we arrive at the following picture of the local structure of the fibration $f : X \to \mathbb{P}^n$ over Δ_{sm}. In a neighbourhood of the singular locus of a fibre over Δ_{sm} there exist local coordinates $(z_1, \dots, z_n, w_1, \dots, w_n)$ on X such that f is given by

$$f : (z_1, \dots, z_n, w_1, \dots, w_n) \mapsto (z_1 w_1, z_2, \dots, z_n).$$

Here Δ_{sm} is given by the vanishing of the first component, locally on $\mathbb{P}^n$.

3 The Beauville–Bogomolov quadratic form

Let X be an irreducible holomorphic symplectic manifold of dimension $2n$. There is a quadratic form q_X on $\mathrm{H}^2(X, \mathbb{Z})$ known as the Beauville–Bogomolov quadratic form (see [2]). This form generalizes the intersection pairing on a K3 surface. We begin with some formulae involving q_X, which may be found in Huybrechts' notes in [8], for instance.

The Fujiki formula states that

$$q_X(\alpha)^n = \text{const.} \int_X \alpha^{2n} \tag{1}$$

for all $\alpha \in \mathrm{H}^2(X, \mathbb{Z})$, where the constant depends only on X. Fujiki also proved that if $\eta \in \mathrm{H}^{4j}(X, \mathbb{R})$ is of pure Hodge type $(2j, 2j)$ on X and on all small deformations of X then

$$q_X(\alpha)^{n-j} = \text{const.} \int_X \eta\alpha^{2(n-j)}$$

for all $\alpha \in \mathrm{H}^2(X, \mathbb{Z})$, where the constant depends only on η. In particular, the second Chern class $c_2(T_X)$ satisfies the hypothesis and thus

$$q_X(\alpha)^{n-1} = \text{const.} \int_X c_2\alpha^{2n-2}. \tag{2}$$

Writing out Eqs. (1) and (2) for α and $\beta \in \mathrm{H}^2(X, \mathbb{Z})$, we can eliminate $q_X(\alpha)$, $q_X(\beta)$, and both constants to obtain

$$\left(\int_X \alpha^{2n}\right)^{n-1} \left(\int_X c_2\beta^{2n-2}\right)^n = \left(\int_X \beta^{2n}\right)^{n-1} \left(\int_X c_2\alpha^{2n-2}\right)^n. \tag{3}$$

This equation will eventually yield a formula for the degree of the discriminant locus.

We return to the situation of the previous section. Thus we have a Lagrangian fibration $f : X \to \mathbb{P}^n$ with a divisor Y which restricts to the theta divisor on each smooth fibre and to the generalized theta divisor on a generic singular fibre (over Δ_{sm}). There is also a divisor L given by pulling back a hyperplane from $\mathbb{P}^n$. We denote the holomorphic symplectic form by σ. Substituting $\alpha = \sigma + t_1\bar{\sigma}$ and $\beta = Y + t_2L$ into Eq. (3), and then comparing coefficients of $(t_1t_2)^{n(n-1)}$ gives

$$\left(\int_X (\sigma\bar{\sigma})^n\right)^{n-1} \left(\int_X c_2Y^{n-1}L^{n-1}\right)^n = \left(\int_X Y^nL^n\right)^{n-1} \left(\int_X c_2(\sigma\bar{\sigma})^{n-1}\right)^n.$$

Note that we have used the fact that $q_X(L) = 0$, which implies that $t_2^{n(n-1)}$ is the highest power of t_2 appearing. Next we identify the terms appearing in this equation.

Lemma 1 *We have*

$$\int_X Y^nL^n = n!.$$

Proof Since L is the pullback of a hyperplane in $\mathbb{P}^n$, L^n must be the pullback of a point, i.e., a fibre F, which we assume is smooth. The restriction of Y to F is a theta divisor, and hence

$$\int_X Y^n L^n = \int_F \Theta^n = n!$$

since Θ is a principal polarization of F.

Lemma 2 *We have*

$$\frac{\left(\int_X c_2(\sigma\bar{\sigma})^{n-1}\right)^n}{\left(\int_X (\sigma\bar{\sigma})^n\right)^{n-1}} = \frac{24^n (n!)^2}{n^n}\sqrt{\hat{A}}[X],$$

where $\sqrt{\hat{A}}[X]$ is the characteristic number of X coming from the square root of the $\hat{A}$-polynomial.

Remark 2 Note that

$$\begin{aligned}\sqrt{\hat{A}} &= \left(1 + \hat{A}_1 + \hat{A}_2 + \cdots\right)^{1/2} \\ &= 1 + \frac{1}{2}\hat{A}_1 + \left(\frac{1}{2}\hat{A}_2 - \frac{1}{8}\hat{A}_1^2\right) + \cdots \\ &= 1 + \frac{1}{24}c_2 + \frac{1}{5760}(7c_2^2 - 4c_4) + \cdots\end{aligned}$$

In particular $\sqrt{\hat{A}}[X]$ does not mean $\left(\hat{A}[X]\right)^{1/2}$.

Proof The proof of the lemma is based on recognizing that the left-hand side is a *Rozansky–Witten invariant* of X. Following the notation of [10]

$$\begin{aligned}\int_X c_2(\sigma\bar{\sigma})^{n-1} &= \int_X \frac{1}{16\pi^2 n}[\Theta(\Phi)]\sigma^n\bar{\sigma}^{n-1} \\ &= \frac{1}{16\pi^2 n} c_\Theta \int_X (\sigma\bar{\sigma})^n,\end{aligned}$$

where Θ denotes the two-vertex trivalent graph and is unrelated to the theta divisor. Therefore

$$\begin{aligned}\frac{\left(\int_X c_2(\sigma\bar{\sigma})^{n-1}\right)^n}{\left(\int_X (\sigma\bar{\sigma})^n\right)^{n-1}} &= \frac{1}{(16\pi^2 n)^n} c_\Theta^n \int_X (\sigma\bar{\sigma})^n \\ &= \frac{n!}{2^n n^n} b_{\Theta^n}(X).\end{aligned}$$

The main result of Hitchin and Sawon [10] is that the Rozansky–Witten invariant $b_{\Theta^n}(X)$ can be written in terms of characteristic numbers

$$b_{\Theta^n}(X) = 48^n n! \sqrt{\hat{A}}[X]$$

which completes the proof.

The remaining term $\int_X c_2 Y^{n-1} L^{n-1}$ will be calculated in the next section.

4 The second Chern class of X

On $f : X \to \mathbb{P}^n$ we have the inclusion $f^*\Omega^1_{\mathbb{P}^n} \to \Omega^1_X$, which is dual to the derivative $df : T_X \to f^*T_{\mathbb{P}^n}$ of f. The holomorphic symplectic form σ gives an isomorphism between Ω^1_X and T_X, so the two maps can be combined into a complex

$$0 \to f^*\Omega^1_{\mathbb{P}^n} \to \Omega^1_X \cong T_X \to f^*T_{\mathbb{P}^n}.$$

For a Lagrangian fibration with good singular fibres, let

$$\mathrm{Sing} = \cup_{t\in\Delta}\mathrm{Sing}(X_t)$$

be the union of the singular loci of all singular fibres of X, and let $\iota : \mathrm{Sing} \hookrightarrow X$ be the inclusion into X. Note that Sing is a fibration over Δ whose generic fibre (over a point of Δ_{sm}) is an abelian variety of dimension $n-1$. In particular, Sing is codimension two in X.

Lemma 3 *Let $f : X \to \mathbb{P}^n$ be a Lagrangian fibration with good singular fibres. Then*

$$0 \to f^*\Omega^1_{\mathbb{P}^n} \to \Omega^1_X \cong T_X \to f^*T_{\mathbb{P}^n} \to \iota_*\mathcal{F} \to 0$$

is exact over $\mathbb{P}^n \backslash \Delta_{\mathrm{sing}}$, where $\mathcal{F}$ is a sheaf on Sing *which is generically rank one.*

Proof Over smooth fibres and over smooth points of singular fibres our sequence comes from splicing the two exact sequences

$$\begin{array}{ccccccc} 0 \to & f^*\Omega^1_{\mathbb{P}^n} & \to & \Omega^1_X & \to & \Omega^1_{X/\mathbb{P}^n} \\ & & & \downarrow\cong & & \\ 0 \to & T_{X/\mathbb{P}^n} & \to & T_X & \to & f^*T_{\mathbb{P}^n}. \end{array}$$

The composition $T_{X/\mathbb{P}^n} \to T_X \overset{\sigma}{\cong} \Omega^1_X \to \Omega^1_{X/\mathbb{P}^n}$ is zero, since σ restricted to a (Lagrangian) fibre must vanish. This proves exactness away from Sing, where all of the above sheaves are locally free.

In a neighbourhood of Sing we do a local computation. Recall that f is given locally by

$$f : (z_1, \dots, z_n, w_1, \dots, w_n) \mapsto (z_1 w_1, z_2, \dots, z_n).$$

Therefore $f^*\Omega^1_{\mathbb{P}^n} \to \Omega^1_X$ is given by

$$\begin{pmatrix} w_1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ z_1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix},$$

the isomorphism $\Omega^1_X \overset{\sigma}{\cong} T_X$ is given by

$$\begin{pmatrix} 0 & \mathrm{Id}_{n\times n} \\ -\mathrm{Id}_{n\times n} & 0 \end{pmatrix},$$

and $df : T_X \to f^*T_{\mathbb{P}^n}$ is given by

$$df = \begin{pmatrix} w_1 & 0 & \dots & 0 & z_1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

It is now a simple matter to check that

$$0 \to f^*\Omega^1_{\mathbb{P}^n} \to \Omega^1_X \cong T_X \to f^*T_{\mathbb{P}^n}$$

is exact, and that df drops rank by one when $w_1 = z_1 = 0$, which are precisely the local equations for Sing. Thus the cokernel of df looks like $\iota_*\mathcal{F}$ where $\mathcal{F}$ is a generically rank one sheaf on Sing.

It follows immediately from the lemma that

$$c_1(T_X) = f^*c_1(\Omega^1_{\mathbb{P}^n}) + f^*c_1(T_{\mathbb{P}^n}) = 0$$

and

$$\begin{aligned} c_2(T_X) &= [\mathrm{Sing}] + f^* c_2(\Omega^1_{\mathbb{P}^n}) + f^* c_2(T_{\mathbb{P}^n}) + \mathrm{const.}[f^{-1}(\Delta_{\mathrm{sing}})] \\ &= [\mathrm{Sing}] + n(n+1)L^2 + \mathrm{const.}L^2 \end{aligned}$$

for some constant.

Remark 3 This formula for the second Chern class is the holomorphic analogue of a well-known formula relating the first Chern class and singular locus of a real Lagrangian fibration on a (real) symplectic manifold. It is really the key to Theorem 1 below, as [Sing] will lead directly to $\deg\Delta$, while we already saw that $c_2(T_X)$ leads to $\sqrt{\hat{A}}[X]$.

Lemma 4 *We have*

$$\int_X c_2 Y^{n-1} L^{n-1} = \int_X [\mathrm{Sing}] Y^{n-1} L^{n-1} = (n-1)! \deg\Delta.$$

Proof Firstly

$$\begin{aligned} \int_X c_2 Y^{n-1} L^{n-1} &= \int_X [\mathrm{Sing}] Y^{n-1} L^{n-1} + \mathrm{const.} Y^{n-1} L^{n+1} \\ &= \int_X [\mathrm{Sing}] Y^{n-1} L^{n-1} \end{aligned}$$

since $L^{n+1} = 0$ (L is the pull-back of a divisor from the n-dimensional base).

The locus Sing is supported over the discriminant locus Δ, while L^{n-1} is the pull-back of a line ℓ in $\mathbb{P}^n$. Since we can assume ℓ is generic, it will intersect Δ in precisely $\deg\Delta$ points, with each point in Δ_{sm}. In this way we reduce the lemma to computing an intersection number in a good singular fibre. This computation will be invariant under deformation, so we can assume that the good singular fibre is the compactified Jacobian $\bar{J}_0 C$ of a curve C with a single node.

The restriction of Sing to $\bar{J}_0 C$ is of course the singular locus s which comes from identifying s_0 and s_∞. The restriction of Y to $\bar{J}_0 C$ is the generalized theta divisor Θ. In the Jacobian $J_0 C$ of a smooth curve C, Θ^{n-1} is cohomologous to $(n-1)!C$, with C embedded in $J_0 C$ by the Abel–Jacobi map. In fact this relation remains true for a curve with a single node, which can also be embedded in its compactified Jacobian by a generalization of the Abel–Jacobi map. Then C intersects the singular locus s at precisely one point, the node of C.

Combining the above observations we find

$$\int_X [\mathrm{Sing}] Y^{n-1} L^{n-1} = \deg\Delta \int_{\bar{J}_0 C} [s] \Theta^{n-1} = (n-1)! \deg\Delta.$$

Remark 4 One could also observe that the restriction of the generalized theta divisor Θ to the singular locus s induces a principal polarization on s, and thus

$$\int_{X_t}[s]\Theta^{n-1}=\int_{s}(\Theta|_s)^{n-1}=(n-1)!.$$

There is then no need to mention compactified Jacobians.

These calculations now yield a formula for the degree of Δ.

Theorem 1 *Let $X\to\mathbb{P}^n$ be a Lagrangian fibration by principally polarized abelian varieties, by which we mean that there is a divisor Y on X which restricts to (a multiple of) a principal polarization on the generic fibre. If X has good singular fibres then*

$$\begin{aligned}\deg\Delta&=\frac{1}{2}b_{\Theta^n}(X)^{\frac{1}{n}}\\&=24\left(n!\sqrt{\hat{A}}[X]\right)^{\frac{1}{n}}.\end{aligned}$$

Proof We simply substitute the results of Lemmas 1, 2, and 4 into the equation preceding Lemma 1. Note that even if Y restricts to a non-trivial multiple $m\Theta$ of a theta divisor on each fibre, the factor m will ultimately cancel out.

Remark 5 The hypotheses imply that X is projective, as $Y+kL$ will be ample for sufficiently large k. However, we expect that the formula will hold more generally, when the generic fibre is only abstractly a principally polarized abelian variety, without any reference to a global divisor on X. The reason is that there are ways to deform a Lagrangian fibration until it admits a section (see [24,25]) without changing the local structure of the fibration, and in particular, without changing the discriminant locus Δ. Now a Lagrangian fibration is projective if and only if it admits a rational section or multi-section (Proposition 3.2 of [21]). In particular, our Lagrangian fibration with a section will contain an ample divisor Y, which should then induce the principal polarization of the generic fibre.

5 The Beauville–Mukai system

In this section we verify our formula for the Beauville–Mukai integrable system [3] described in Sect. 2, whose total space is a deformation of the Hilbert scheme $S^{[n]}$ of n points on a K3 surface S. In [22] the author calculated various Rozansky–Witten invariants; in particular

$$b_{\Theta^n}(S^{[n]})=12^n(n+3)^n.$$

Applying Theorem 1, the discriminant locus of a fibration on $S^{[n]}$ (or on any deformation of $S^{[n]}$) should therefore have degree

$$\deg \Delta = 6(n+3).$$

For $n = 1$ it is well-known that a generic elliptic K3 surface has exactly 24 singular fibres. For $n \geq 2$ we have the Beauville–Mukai system coming from a genus n curve C contained in S, which is a fibration over $|C| \cong \mathbb{P}^n$. There is a map $S \to (\mathbb{P}^n)^\vee$ which for generic S is an embedding (or branched double cover when $n = 2$). The discriminant locus $\Delta \subset |C|$ parametrizes singular curves in the linear system, i.e., it parametrizes hyperplanes in $(\mathbb{P}^n)^\vee$ whose intersection with S is singular. In other words, $\Delta \subset \mathbb{P}^n$ is the variety dual to $S \subset (\mathbb{P}^n)^\vee$ (or dual to the branch curve of $S \to (\mathbb{P}^2)^\vee$ when $n = 2$).

Consider a pencil of hyperplanes $H_t \subset (\mathbb{P}^n)^\vee$, with $t \in \mathbb{P}^1$. Generically there will be $\deg \Delta$ singular hyperplane sections of S in this pencil, and each one will have a single node. The union $\cup_{t\in\mathbb{P}^1} H_t \cap S$ of these hyperplane sections gives a divisor in $S \times \mathbb{P}^1$ whose corresponding line bundle is $\mathcal{O}(C, 1)$. If this divisor is given locally by $f = 0$, then the singularities of $H_t \cap S$ are given by $f = 0$ and $df = 0$, where the derivative is taken only in the direction of S. Globally, we have a section of the rank three vector bundle

$$\mathcal{O}(C, 1) \oplus T^*S(C, 1)$$

which vanishes precisely at the singular points. Therefore

$$\begin{aligned}\deg \Delta &= c_3(\mathcal{O}(C, 1) \oplus T^*S(C, 1))[S \times \mathbb{P}^1] \\ &= 6(n+3),\end{aligned}$$

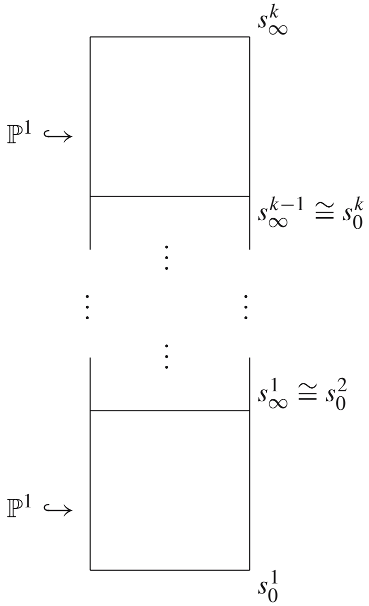
where we have used the fact that $C^2 = 2n-2$. Thus we have a verification of Theorem 1 in this case.

6 Non-principal polarizations

Let us illustrate how to modify our theorem for non-principal polarizations. Let X be an irreducible holomorphic symplectic manifold fibred over $\mathbb{P}^n$, and let Y be a divisor on X which on a generic fibre restricts to a polarization of type $(d_1, \ldots, d_n)$ with $d_1|d_2|\cdots|d_n$. We first generalize the notion of a good singular fibre to this case: in fact there is more than one model.

A singular fibre X_t, with $t \in \Delta_{\mathrm{sm}}$, should look like a generic semi-stable degeneration of an abelian variety with polarization of type $(d_1, \ldots, d_n)$. In other words, X_t should be a semi-stable degeneration that occurs over a generic point of the boundary of an Igusa [13] and Mumford [18,19] compactification of the moduli space of abelian varieties. For non-principal polarizations, the boundary consists of several irreducible (codimension one) components, thus we expect to find several different models which we now describe explicitly.

The normalization $\tilde{X}_t$ of X_t will look like a collection of k $\mathbb{P}^1$-bundles over an abelian variety of dimension $n-1$. The singular fibre itself is obtained by gluing the zero and infinity sections in a chain, as shown (with s_0^1 also glued to s_∞^k, with a translation).



Note that the singular locus $\mathrm{Sing}(X_t)$ consists of k irreducible components, each isomorphic to the abelian variety of dimension $n-1$. Moreover the polarization of a nearby smooth fibre, which is of type $(d_1, \ldots, d_n)$, will degenerate to a divisor Y_t in X_t. Suppose that Y_t induces a polarization of type $(d'_1, \ldots, d'_{n-1})$ on each irreducible component of $\mathrm{Sing}(X_t)$. Compatibility requires that $d_i | d'_i$ for $i = 1, \ldots, n-1$, and

$$d_1 d_2 \cdots d_{n-1} d_n = d'_1 d'_2 \cdots d'_{n-1} k.$$

In particular, this implies that k must divide d_n. For example, in the case of abelian surfaces with polarization of type $(1, p)$, with p prime, there are two possible degenerations: one is irreducible whereas the other consists of p irreducible components (see Propositions 4.5 and 4.7 in [11]).

Definition 2 We say a Lagrangian fibration $X \to \mathbb{P}^n$ by abelian varieties with polarization of type $(d_1, \ldots, d_n)$ has *good* singular fibres if the generic singular fibre X_t for $t \in \Delta_{\mathrm{sm}}$ looks like the picture described above. Note that Δ may consist of several irreducible components and the model for the generic singular fibre X_t may differ over each component (e.g. k and $(d'_1, \ldots, d'_{n-1})$ need not be the same over every component).

Let L be the pullback of a hyperplane in $\mathbb{P}^n$, and Y the relative theta divisor. As before, we have

$$\left(\int_X (\sigma\bar{\sigma})^n\right)^{n-1}\left(\int_X c_2 Y^{n-1}L^{n-1}\right)^n = \left(\int_X Y^n L^n\right)^{n-1}\left(\int_X c_2(\sigma\bar{\sigma})^{n-1}\right)^n.$$

Lemma 1 becomes

$$\int_X Y^n L^n = n! d_1 d_2 \cdots d_{n-1} d_n.$$

Lemma 2 remains unchanged

$$\frac{\left(\int_X c_2(\sigma\bar{\sigma})^{n-1}\right)^n}{\left(\int_X (\sigma\bar{\sigma})^n\right)^{n-1}} = \frac{24^n (n!)^2}{n^n}\sqrt{\hat{A}}[X].$$

The exact sequence of Lemma 3 also remains unchanged, because although the singular locus $\mathrm{Sing}(X_t)$ of each generic singular fibre now consists of k irreducible components, the local description of $f : X \to \mathbb{P}^n$ near these singularities does not change. Therefore our expression for the second Chern class of X is still valid, and Lemma 4 becomes

$$\begin{aligned}\int_X c_2 Y^{n-1}L^{n-1} &= \int_X [\mathrm{Sing}]Y^{n-1}L^{n-1}\\ &= \deg\Delta \int_{X_t} [\mathrm{Sing}(X_t)]Y_t^{n-1}\\ &= k(n-1)! d_1' d_2' \cdots d_{n-1}' \deg\Delta\\ &= (n-1)! d_1 d_2 \cdots d_n \deg\Delta\end{aligned}$$

because $\mathrm{Sing}(X_t)$ consists of k irreducible components, each isomorphic to an abelian variety of dimension $n-1$, and Y_t intersects each component in a polarization of type $(d_1', \ldots, d_{n-1}')$.

Combining these formulae we obtain the following result.

Theorem 2 *Let $X \to \mathbb{P}^n$ be a Lagrangian fibration by abelian varieties with polarization of type $(d_1, \ldots, d_n)$, by which we mean that there is a divisor Y on X which restricts to a polarization of this type on the generic fibre. If X has good singular fibres then*

$$\begin{aligned}\deg\Delta &= \frac{1}{2}\left(\frac{b_{\Theta^n}(X)}{d_1\cdots d_n}\right)^{\frac{1}{n}}\\ &= 24\left(\frac{n!\sqrt{\hat{A}}[X]}{d_1\cdots d_n}\right)^{\frac{1}{n}}.\end{aligned}$$

Remark 6 One could always change the polarization of X to mY with $m \geq 2$, and this would multiply all the d_i by the factor m. Our formula then appears to be inconsistent; however, our models for singular fibres implicitly assume that Y is a primitive divisor. This suggests that we should assume $d_1 = 1$. Indeed if $d_1 > 1$ then Y is not primitive when restricted to a fibre, and in some circumstances one can use the methods described in [24,25] to deform $X \to \mathbb{P}^n$ so that $Y = d_1 Y'$ globally, without changing the fibration locally. Changing to the new polarization Y', we could then assume that $d_1 = 1$.

7 Generalized Kummer varieties

The generalized Kummer varieties K_n were introduced by Beauville [2]. Debarre [5] exhibited a fibration on K_n; see also Example 3.8 in [23]. The fibres have polarization of type $(1, \dots, 1, n+1)$ and this fibration has good singular fibres. In [22] the author calculated

$$b_{\Theta^n}(K_n) = 12^n (n+1)^{n+1}.$$

Theorem 2 therefore gives

$$\deg \Delta = 6(n+1).$$

For $n = 1$ this gives twelve. This is correct because the Kummer K3 surface K_1 will be an elliptic fibration whose singular fibres each consist of two irreducible components; more precisely, they are of Kodaira type I_2 and so there will indeed be twelve of them.

For $n \geq 2$ one begins with an abelian surface A with polarization of type $(1, n+1)$. Thus A is polarized by a genus $n+2$ curve C with $C^2 = 2(n+1)$. The relative Jacobian of the family of curves linear equivalent to C is a fibration over $|C| \cong \mathbb{P}^n$ whose generic fibre is an abelian variety of dimension $n+2$. There is a map from the total space of this fibration to A (the Albanese map), and the kernel of this map gives a fibration on K_n. More precisely, the kernel is isomorphic to the generalized Kummer variety $K_n(\hat{A})$ constructed from the dual abelian surface $\hat{A}$, and it inherits the map to $\mathbb{P}^n$ which makes it a Lagrangian fibration.

As with the Beauville–Mukai system, $\Delta \subset \mathbb{P}^n$ parametrizes hyperplanes in $(\mathbb{P}^n)^\vee$ whose intersection with $A \subset |C|^\vee \cong (\mathbb{P}^n)^\vee$ is singular (with the obvious modifications for small n, when A is not necessarily embedded). We can therefore use the same method to calculate the degree of Δ, and we obtain

$$\begin{aligned} \deg \Delta &= c_3(\mathcal{O}(C,1) \oplus T^*A(C,1))[A \times \mathbb{P}^1] \\ &= 6(n+1) \end{aligned}$$

which agrees with the value obtained from Theorem 2.

8 Fibrations on fourfolds

Suppose $X \to \mathbb{P}^2$ is an irreducible holomorphic symplectic fourfold which admits a Lagrangian fibration by abelian surfaces with polarization of type (d_1, d_2), and write $d_2 = d_1 d$. Moreover, let's follow Remark 6 and assume $d_1 = 1$. In this dimension, if the base is smooth then Matsushita's results imply it must be $\mathbb{P}^2$. If the fibration has good singular fibres then Theorem 2 yields

$$\deg \Delta = \frac{1}{2}\left(\frac{b_{\Theta^n}(X)}{d}\right)^{\frac{1}{2}} = \left(\frac{1152\sqrt{\hat{A}}[X]}{d}\right)^{\frac{1}{2}}.$$

We will use Guan's bounds [9] on the Betti numbers of X to restrict the possible values of d and $\deg\Delta$.

Theorem 3 (Guan [9]) *Let X be an irreducible holomorphic symplectic fourfold. The Betti numbers of X are bounded and can only take the following values*:

- $b_2 = 23$ *and* $b_3 = 0$,
- $b_2 = 8$ *and* $b_3 = 0$,
- $b_2 = 7$ *and* $b_3 = 0$ *or* 8,
- $b_2 = 6$ *and* $b_3 = 0, 4, 8, 12,$ *or* 16,
- $b_2 = 5$ *and* $b_3 = 0, 4, 8, \ldots$ *or* 36,
- $b_2 = 4$ *and* $b_3 = 0, 4, 8, \ldots$ *or* 60,
- $b_2 = 3$ *and* $b_3 = 0, 4, 8, \ldots$ *or* 68.

The fourth Betti number is determined by Salamon's relation

$$b_4 = 46 + 10b_2 - b_3$$

and therefore

$$c_4[X] = \chi(X) = 48 + 12b_2 - 3b_3.$$

The relation

$$\hat{A}[X] = \frac{1}{720}(3c_2^2[X] - c_4[X]) = \chi(\mathcal{O}_X) = 3$$

between the Chern numbers allows us to write $\sqrt{\hat{A}}[X]$ solely in terms of $c_4[X]$, giving

$$1152\sqrt{\hat{A}}[X] = 1008 - \frac{1}{3}c_4[X] = 992 - 4b_2 + b_3.$$

We can use this to bound the degree of the discriminant locus.

Theorem 4 *Let X be an irreducible holomorphic symplectic fourfold which admits a Lagrangian fibration* $X \to \mathbb{P}^2$ *by abelian surfaces with polarization of type* $(1, d)$, *by which we mean that there is a divisor Y on X which restricts to a polarization of type* $(1, d)$ *on the generic fibre. If X has good singular fibres then* $\deg\Delta$ *is at most* 32 *and d is at most* 1036.

Proof Firstly, b_2 must be at least four since L corresponds to an isotropic element of $\mathrm{H}^2(X, \mathbb{Z})$ with respect to the Beauville–Bogomolov form, and this is a lattice of signature $(3, b_2 - 3)$. We substitute the possible values of b_2 and b_3 (as allowed by Guan's Theorem) into

$$1152\sqrt{\hat{A}}[X] = 992 - 4b_2 + b_3.$$

The largest value is 1036 when $b_2 = 4$ and $b_3 = 60$. Moreover, $1152\sqrt{\hat{A}}[X]$ is always an integer and the formula for $\deg\Delta$ shows that it must be divisible by d. Thus d is at most 1036. Moreover

$$\deg\Delta = \left(\frac{1152\sqrt{\hat{A}}[X]}{d}\right)^{\frac{1}{2}} \leq \left(\frac{1036}{d}\right)^{\frac{1}{2}} \leq \sqrt{1036} < 33.$$

Remark 7 In our two examples we have $d = 1$ and $\deg\Delta = 30$ for the Beauville–Mukai system on $S^{[2]}$, and $d = 3$ and $\deg\Delta = 18$ for the generalized Kummer fourfold K_2.

Next we explain how a certain assumption on the polarization Y leads to much stronger restrictions on $\deg\Delta$ and d. The divisor Y can be thought of as a relative theta divisor on the family of abelian surfaces $X \to \mathbb{P}^2$, and we write Y_t for the restriction of Y to the fibre X_t, for $t \in \mathbb{P}^2$. In a generic smooth fibre X_t, Y_t will be a smooth curve of genus $d + 1$. Our assumption on Y will be that Y_t is *always* smooth for $t \notin \Delta$.

Remark 8 In the principally polarized case $d = 1$ this means that no smooth fibre X_t is a product of elliptic curves, as otherwise the corresponding theta divisor Y_t would consist of two elliptic curves joined at a node. In the moduli space of principally polarized abelian surfaces $\mathcal{A}^\circ(1, 1)$ there is a divisor corresponding to products of elliptic curves. Our assumption then means that the image of $\mathbb{P}^2\backslash\Delta$ in $\mathcal{A}^\circ(1, 1)$ does not meet this divisor. Although this is a strong restriction, it is satisfied for a generic Beauville–Mukai integrable system, for in that case $\{Y_t | t \in \mathbb{P}^2\}$ is a complete linear system of curves in a generic K3 surface and hence every curve Y_t must be irreducible.

A similar interpretation is possible for $d > 1$. Each smooth abelian surface X_t contains a $(d-1)$-dimensional linear system of theta divisors. Hence there is a $\mathbb{P}^{d-1}$-bundle over $\mathcal{A}^\circ(1, d)$ whose total space P parametrizes pairs consisting of an abelian surface and a theta divisor. There is a divisor in P corresponding to pairs for which the theta divisor is singular, and our assumption is that the image of $\mathbb{P}^2\backslash\Delta$ in P does not meet this divisor. The author does not know whether the generic Lagrangian fibration on the generalized Kummer fourfold satisfies this hypotheses.

Remark 9 Without this assumption on Y, we would need to find some way to understand and control the intersection of the image of $\mathbb{P}^2\backslash\Delta$ with the divisors in the corresponding moduli spaces. This appears to be a difficult problem.

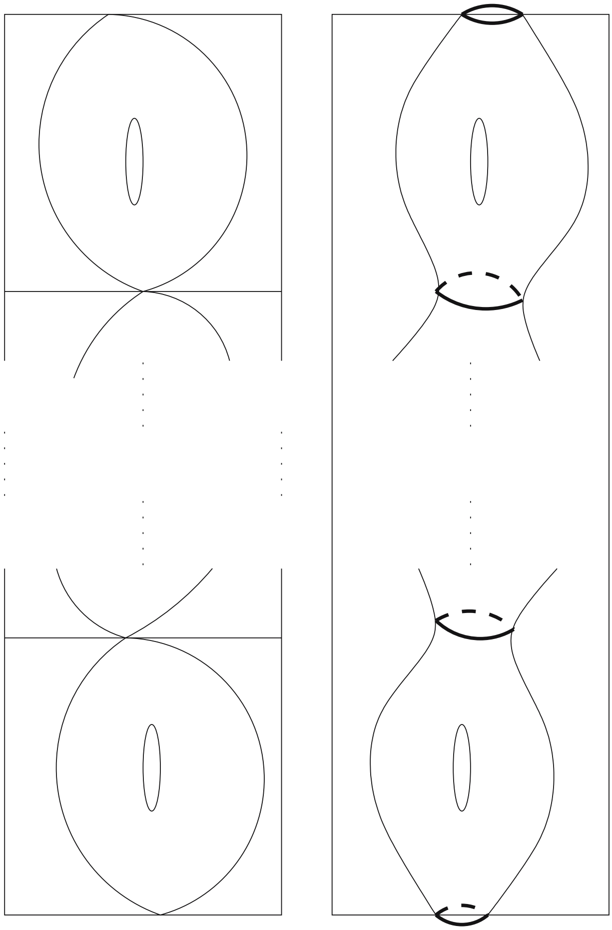


Fig. 1 Good singular fibre (*left*) and smooth fibre (*right*)

Now Y_t will always be singular for $t \in \Delta$; we can describe the generic situation. For $t \in \Delta_{\mathrm{sm}}$, X_t is a good singular fibre and Y_t will be a curve with d nodes. This singular curve Y_t could have one or more irreducible components. One possibility is shown in Fig. 1, with Y_t consisting of d irreducible components, each of genus one, in a cyclic configuration (recall that the 'top' of X_t is also glued to the 'bottom' of X_t). Another possibility is shown in Fig. 2, with Y_t irreducible. Note that the d nodes come from d vanishing cycles in the smooth fibre, as indicated in Figs. 1 and 2. If d is not a prime number, other configurations are possible (with the number of irreducible components dividing d), but the singular curve will always have d nodes.

Recall that L is the pullback under $X \to \mathbb{P}^2$ of a generic line in $\mathbb{P}^2$. Let Z be the surface in X given by the complete intersection of the divisors Y and L. Then Z is a fibration by genus $d+1$ curves over $\mathbb{P}^1$, with $\deg \Delta$ singular fibres which look like the singular curves Y_t described above (for $t \in \Delta_{\mathrm{sm}}$).

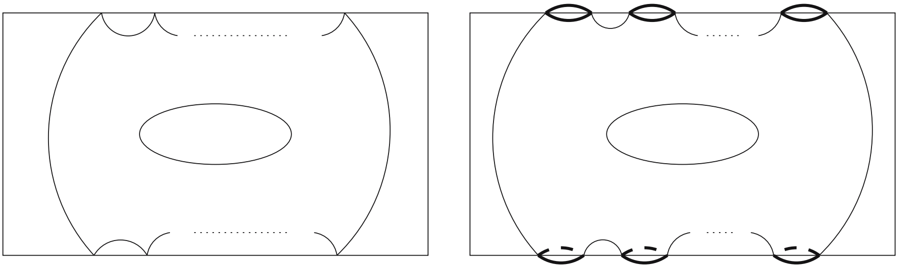


Fig. 2 Good singular fibre (*left*) and smooth fibre (*right*)

Lemma 5 *The holomorphic Euler characteristic of Z is given by the formula*

$$\chi(\mathcal{O}_Z) = \frac{d(\deg\Delta - 6)}{12}.$$

In particular, 12 *must divide* $d(\deg\Delta - 6)$.

Proof Since Z is the complete intersection of Y and L we can resolve its structure sheaf on X

$$0 \to \mathcal{O}_X(-Y-L) \to \mathcal{O}_X(-Y) \oplus \mathcal{O}_X(-L) \to \mathcal{O}_X \to \mathcal{O}_Z \to 0.$$

Riemann–Roch now gives

$$\begin{aligned}\chi(\mathcal{O}_Z) &= \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-Y) \oplus \mathcal{O}_X(-L)) + \chi(\mathcal{O}_X(-Y-L)) \\ &= \int_X (1 - \exp(-Y) - \exp(-L) + \exp(-Y-L))\mathrm{Td}_X \\ &= \int_X \left(YL - \frac{Y^2L + YL^2}{2} + \frac{4Y^3L + 6Y^2L^2 + 4YL^3}{24}\right)\mathrm{Td}_X.\end{aligned}$$

The degree four part of Td_X is irrelevant, and we can use Lemma 3 to write

$$\begin{aligned}\mathrm{Td}_X &= 1 + \frac{1}{12}c_2 + \cdots \\ &= 1 + \frac{1}{12}[\mathrm{Sing}] + \mathrm{const.}L^2 + \cdots\end{aligned}$$

Substituting this and using $L^3 = 0$ and $Y^2L^2 = 2d$ gives

$$\begin{aligned}\chi(\mathcal{O}_Z) &= \int_X \left(\frac{1}{12}[\mathrm{Sing}]YL + \frac{1}{6}Y^3L + \frac{1}{4}Y^2L^2\right) \\ &= \frac{1}{12}d.\deg\Delta + \frac{1}{6}Y^3L + \frac{1}{2}d.\end{aligned}$$

On the other hand, we can compute directly on Z using Noether's formula

$$\chi(\mathcal{O}_Z) = \left(\frac{c_1^2 + c_2}{12}\right)[Z].$$

By adjunction

$$c_1^2[Z] = K_Z^2 = (\mathcal{O}(Y + L)|_Z)^2 = (Y + L)^2 YL = Y^3 L + 4d$$

where we have once again used $L^3 = 0$ and $Y^2 L^2 = 2d$. The topological Euler characteristic $c_2[Z]$ can be computed from the fact that Z is a genus $d+1$ fibration over $\mathbb{P}^1$ with $\deg\Delta$ singular fibres. Each singular fibre has d nodes, so its Euler characteristic is d greater than a smooth fibre, and thus

$$c_2[Z] = 2(2 - 2(d + 1)) + d.\deg\Delta = -4d + d.\deg\Delta.$$

Therefore

$$\chi(\mathcal{O}_Z) = \frac{1}{12} d.\deg\Delta + \frac{1}{12} Y^3 L.$$

Comparing our two formulae for $\chi(\mathcal{O}_Z)$ gives $Y^3 L = -6d$, which can be substituted back in to complete the proof.

Our final restriction arises because the fibration $Z \to \mathbb{P}^1$ cannot have too few singular fibres.

Lemma 6 *The degree of the discriminant locus cannot be one or two, i.e.,*

$$\deg\Delta \geq 3.$$

Proof Beauville [4] proved a generalization of the Szpiro inequality: a non-trivial family $S \to B$ of semi-stable genus g curves with s singular fibres must have less than

$$(4g + 2)(s + 2g(B) - 2)$$

critical points. Applying this to $Z \to \mathbb{P}^1$ gives

$$d.\deg\Delta < (4(d + 1) + 2)(\deg\Delta - 2)$$

and rearranging gives

$$\deg\Delta > \frac{8d + 12}{3d + 6} = 2 + \frac{2d}{3d + 6} > 2.$$

We can now state our final result.

Theorem 5 *Let X be an irreducible holomorphic symplectic fourfold which admits a Lagrangian fibration $X \to \mathbb{P}^2$ by abelian surfaces with polarization of type $(1, d)$, by which we mean that there is a divisor Y on X which restricts to a polarization of type $(1, d)$ on the generic fibre. Moreover, assume that X has good singular fibres and that Y_t is smooth for $t \notin \Delta$. Then the only possible values for the pair $(d, \deg\Delta)$ are*

- $(1, 30)$, $(4, 15)$, $(9, 10)$, $(25, 6)$, $(36, 5)$, *or* $(100, 3)$ *when* $(b_2, b_3) = (23, 0)$,
- $(60, 4)$ *when* $(b_2, b_3) = (8, 0)$,
- $(3, 18)$, $(12, 9)$, $(27, 6)$, *or* $(108, 3)$ *when* $c_4[X] = 108$ (*which implies that* $(b_2, b_3) = (7, 8)$, $(6, 4)$, *or* $(5, 0)$),
- $(28, 6)$ *or* $(112, 3)$ *when* $c_4[X] = 0$ (*which implies that* $(b_2, b_3) = (5, 36)$ *or* $(4, 32)$).

Proof As in the proof of Theorem 4, b_2 must be at least four. We substitute the possible values of b_2 and b_3 (as allowed by Guan's Theorem) into

$$1152\sqrt{\hat{A}}[X] = 992 - 4b_2 + b_3.$$

Since

$$\deg\Delta = \left(\frac{1152\sqrt{\hat{A}}[X]}{d}\right)^{\frac{1}{2}}$$

we need only consider integers d which divide $1152\sqrt{\hat{A}}[X]$ and give a quotient which is a perfect square. Lemmas 5 and 6 can then be used to eliminate many of the possibilities (the author used Maple for this), leaving the values stated in the theorem.

Remark 10 In our two examples we have $(d, \deg\Delta, b_2, b_3) = (1, 30, 23, 0)$ for the Beauville–Mukai system on $S^{[2]}$, and $(d, \deg\Delta, b_2, b_3) = (3, 18, 7, 8)$ for the generalized Kummer fourfold K_2, although as stated earlier, the author does not know whether K_2 satisfies the hypotheses that Y_t be smooth for $t \notin \Delta$.

We suspect that further work will eliminate many (perhaps all) of the other values above. In particular, it is hard to imagine there could be any examples with d large.

Acknowledgments The author would like to thank Ciro Ciliberto, Manfred Lehn, Ivan Smith, and Richard Thomas for useful conversations, and the Max–Planck–Institut für Mathematik (Bonn) for hospitality. This work was also supported by NSF grant number 0305865.

References

1. Ash, A., Mumford, D., Rapoport, M., Tai, Y.: Smooth Compactification of Locally Symmetric Varieties. Math. Sci. Press, Brookline (1975)
2. Beauville, A.: Variétés Kähleriennes dont la première classe de Chern est nulle. J. Diff. Geom. **18**, 755–782 (1983)
3. Beauville, A.: Counting rational curves on K3 surfaces. Duke Math. J. **97**(1), 99–108 (1999)
4. Beauville, A.: The Szpiro inequality for higher genus fibrations, in Algebraic geometry, de Gruyter, pp. 61–63 (2002)

5. Debarre, O.: On the Euler characteristic of generalized Kummer varieties. Amer. J. Math. **121**(3), 577–586 (1999)
6. D'Souza, C.: Compactification of generalized Jacobians. Proc. Indian Acad. Sci. A **88**, 419–457 (1979)
7. Esteves, E.: Very ampleness for theta on the compactified Jacobian. Math. Z. **226**(2), 181–191 (1997)
8. Gross, M., Huybrechts, D., Joyce, D.: Calabi-Yau manifolds and related geometries. Springer Universitext, Berlin (2002)
9. Guan, D.: On the Betti numbers of irreducible compact hyperkähler manifolds of complex dimension four. Math. Res. Lett. **8**(5–6), 663–669 (2001)
10. Hitchin, N., Sawon, J.: Curvature and characteristic numbers of hyperkähler manifolds. Duke Math. J. **106**(3), 599–615 (2001)
11. Hulek, K., Kahn, C., Weintraub, S.: Moduli spaces of abelian surfaces: compactification, degenerations, and theta functions. de Gruyter Expos. Math. **12** (1993)
12. Igusa, J.: Fibre systems of Jacobian varieties. Amer. J. Math. **78**, 171–199 (1956)
13. Igusa, J.: A desingularization problem in the theory of Siegel modular functions. Math. Ann. **168**, 228–260 (1967)
14. Matsushita, D.: On fibre space structures of a projective irreducible symplectic manifold. Topology **38**(1), 79–83 (1999). Addendum, Topology **40**(2), 431–432 (2001)
15. Matsushita, D.: Equidimensionality of Lagrangian fibrations on holomorphic symplectic manifolds. Math. Res. Lett. **7**(4), 389–391 (2000)
16. Matsushita, D.: On singular fibres of Lagrangian fibrations over holomorphic symplectic manifolds. Math. Ann. **321**(4), 755–773 (2001)
17. Mukai, S.: Moduli of vector bundles on K3 surfaces and symplectic manifolds. Sugaku Expos. **1**(2), 139–174 (1988)
18. Mumford, D.: An analytic construction of degenerating abelian varieties over complete rings. Compositio Math. **24**, 239–272 (1972)
19. Mumford, D.: A New Approach to Compactifying Locally Symmetric Varieties. Discrete Subgroups of Lie Groups and Applications to Moduli (Bombay 1973), pp. 211–224. Oxford University Press (1975), Oxford
20. Oda, T., Seshadri, C.: Compactifications of the generalized Jacobian variety. Trans. Amer. Math. Soc. **253**, 1–90 (1979)
21. Oguiso, K.: Shioda-Tate formula for an abelian fibred variety and applications (preprint) math.AG/0703245
22. Sawon, J.: Rozansky–Witten invariants of hyperkähler manifolds. Cambridge Ph.D. thesis (2000) (preprint) math.DG/0404360
23. Sawon, J.: Abelian fibred holomorphic symplectic manifolds. Turkish J. Math. **27**(1), 197–230 (2003)
24. Sawon, J.: Derived equivalence of holomorphic symplectic manifolds. Algebraic structures and moduli spaces, CRM Proc. Lecture Notes 38. AMS (2004), 193–211
25. Sawon, J.: Twisted Fourier-Mukai transforms for holomorphic symplectic four-folds (preprint) math.AG/0509222
26. Todorov, A.: Large radius limit and SYZ fibrations of hyperkähler manifolds (preprint) math.SG/0308210