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Unconditional families in Banach spaces

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Abstract It is shown that for every separable Banach space X with non-separable dual, the space X^{**} contains an unconditional family of size $|X^{**}|$. The proof is based on Ramsey Theory for trees and finite products of perfect sets of reals. Among its consequences, it is proved that every dual Banach space has a separable quotient.

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1 Introduction

The problem of the existence of an unconditional basic sequence in every, infinite dimensional, Banach space was a central one and remained open for many years. At the beginning of 1990s Gowers and Maurey [18] settled that problem in the negative. Their celebrated example led to the profound concept, introduced by Johnson, of Hereditarily Indecomposable (HI) spaces, which completely changed our understanding of the structure of Banach spaces. The class of HI spaces stands in the opposite of the class of spaces with an unconditional basis and Gowers' dichotomy [16], a Ramsey theoretic principle for Banach spaces, yields that every Banach space either is HI saturated, or contains an unconditional basic sequence. Further investigation, by several authors,

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has shown that HI spaces occur almost everywhere and this indicates the difficulty to obtain positive results concerning the existence of unconditional sequences.

The aim of the present work is to prove a theorem, of an unexpected generality, providing unconditional families in the second dual of a separable Banach space, and also, to present some of its consequences. More precisely the following is proved.

Theorem 1 *Let X be a separable Banach space not containing ℓ_1 and such that X^* is non-separable. Then there exists a bounded bi-orthogonal system $\{(z^*_\sigma, z^{**}_\sigma) : \sigma \in 2^{\mathbb{N}}\}$ in $X^* \times X^{**}$ such that the family $\{z^{**}_\sigma : \sigma \in 2^{\mathbb{N}}\}$ is 1-unconditional, weak* discrete and having 0 as the unique weak* accumulation point.*

A rather direct consequence is the following.

The second dual X^{**} of a separable space X with non-separable dual contains an unconditional family of size $|X^{**}|$.

We also obtain a trichotomy, answering affirmatively the "unconditionality or reflexivity problem", which is stated as follows.

Every separable space X either is reflexively saturated, or one of its second or third dual contains an unconditional family of cardinality equal to the size of the corresponding dual.

A third application concerns the classical "separable quotient problem", posed by Banach, and settles the problem for the class of spaces isomorphic to a dual. In particular, the following is shown.

Every dual Banach space has a separable quotient.

Let us point out that Theorem 1, as well as the aforementioned trichotomy, are sharp. Indeed there are separable spaces with separable dual and non-separable HI second dual [7]. Moreover, such a space X can be chosen so that X, X^* and X^{**} are all HI and not containing a reflexive subspace [2]. Let us also note the stability of the unconditionality constant obtained by Theorem 1 which remains the best possible for any equivalent norm on X. This could be compared to Maurey's theorem [27] concerning second dual types in separable Banach spaces containing ℓ_1. The Odell-Rosenthal theorem [32] permits us to lift structure from the 1-unconditional family $\{z^{**}_\sigma : \sigma \in 2^{\mathbb{N}}\}$ into the space X itself. This is the content of the following theorem which corresponds to Theorem 18 in the main text.

Theorem 2 *Let X be as in Theorem 1. Then there exists a Schauder tree basis $(w_t)_{t\in 2^{<\mathbb{N}}}$ in X such that the following are satisfied.*

(1) *For every $n \geq 1$ the finite family $\{w_t : t \in 2^n\}$ is $\left(1 + \frac{1}{n}\right)$-unconditional.*
(2) *For every $n, m \in \mathbb{N}$ with $1 \leq n < m$ and every $\{s_t : t \in 2^n\} \subseteq 2^m$ with $t \sqsubset s_t$ for all $t \in 2^n$, the families $\{w_t : t \in 2^n\}$ and $\{w_{s_t} : t \in 2^n\}$ are $\left(1 + \frac{1}{n}\right)$-equivalent under the natural correspondence.*
(3) *For every infinite chain $(t_n)_n$ of $2^{<\mathbb{N}}$ the sequence $(w_{t_n})_n$ is weak Cauchy and for every infinite antichain $(s_n)_n$ of $2^{<\mathbb{N}}$ the sequence $(w_{s_n})_n$ is weakly-null.*

This result reveals the generic character of the basis of the James Tree space JT, the first example of a separable Banach space not containing ℓ_1 and with non-separable dual [21]. For further applications of Theorem 1 we refer to [1].

The ingredients for proving Theorem 1 are mainly Ramsey theoretical. In particular, we use results concerning definable partitions of certain classes of antichains of the dyadic tree, which we call increasing and decreasing, as well as, definable partitions of finite products of perfect sets. Theorem 12, extracted from Stegall's fundamental construction [36] for separable Banach spaces with non-separable dual, also plays a key role. More precisely, using the Ramsey properties of increasing and decreasing antichains, proved in [3], we obtain the following extension of Stern's theorem [37] (see Sect. 2 for unexplained terminology).

Theorem 3 *Let X be a separable Banach space and $\Delta = \{x_t : t \in 2^{<\mathbb{N}}\}$ be a bounded family in X. Then there exists a regular dyadic subtree T of $2^{<\mathbb{N}}$ such that the following are satisfied.*

(1) *Either,* (i) *there exists $C > 0$ such that for every infinite chain $(t_n)_n$ of T the sequence $(x_{t_n})_n$ is C-equivalent to the standard basis of ℓ_1, or*
(ii) *for every infinite chain $(t_n)_n$ of T the sequence $(x_{t_n})_n$ is weak Cauchy.*

(2) *Either,* (i) *there exists $C > 0$ such that for every increasing antichain $(t_n)_n$ of T the sequence $(x_{t_n})_n$ is C-equivalent to the standard basis of ℓ_1, or*
(ii) *for every increasing antichain $(t_n)_n$ of T the sequence $(x_{t_n})_n$ is weak Cauchy. Moreover, for every pair $(t_n)_n$ and $(s_n)_n$ of increasing antichains of T with the same limit point in $2^{\mathbb{N}}$, the sequences $(x_{t_n})_n$ and $(x_{s_n})_n$ are both weak* convergent to the same element of X^{**}.*

(3) *Similar to* (2) *for decreasing antichains.*

We should point out that part (1)(i) of Theorem 3 does not necessarily imply part (2)(i), or conversely (see Remark 1 in the main text). Theorem 3 incorporates all the machinery of Ramsey theory for trees needed in the proof of Theorem 1, which proceeds as follows. For a separable Banach space X with non-separable dual, Theorem 12 yields that there exist a bounded family $\{x_t\}_{t\in 2^{<\mathbb{N}}}$ in X and a bounded family $\{x^*_\sigma : \sigma \in 2^{\mathbb{N}}\}$ in X^* such that for every $\sigma, \tau \in 2^{\mathbb{N}}$ and every weak* accumulation point x^{**}_σ of $\{x_{\sigma|n}\}_n$ we have $x^{**}_\sigma(x^*_\tau) = \delta_{\sigma\tau}$. Next applying Theorem 3 and taking into account that ℓ_1 does not embed into X, we obtain a regular dyadic subtree T of $2^{<\mathbb{N}}$ and to each σ in the body $[\hat{T}]$ of T a triplet $\{x^0_\sigma, x^+_\sigma, x^-_\sigma\}$ in X^{**} associated to the unique weak* limit points along subsequences of $\{x_t\}_{t\in T}$ determined by chains, increasing and decreasing antichains. The key observation is that the family $\{z^{**}_\sigma = x^0_\sigma - x^+_\sigma : \sigma \in [\hat{T}]\}$ is weak* discrete having 0 as the unique weak* accumulation point. Moreover, for every $\sigma, \tau \in [\hat{T}]$ we have $z^{**}_\sigma(x^*_\tau) = \delta_{\sigma\tau}$. The final step in the proof of Theorem 1 is the perfect unconditionality theorem, stated as follows.

Theorem 4 *Let X be a separable Banach space. Let also Q be a perfect subset of $2^{\mathbb{N}}$ and $\mathcal{D} = \{z^{**}_\sigma : \sigma \in Q\}$ be a bounded family in X^{**} which is weak* discrete and having 0 as the unique weak* accumulation point. Assume that the map $\Phi : Q \times (B_{X^*}, w^*) \to \mathbb{R}$ defined by $\Phi(\sigma, x^*) = z^{**}_\sigma(x^*)$ is Borel. Then there exists a perfect subset R of Q such that the family $\{z^{**}_\sigma : \sigma \in R\}$ is 1-unconditional.*

The construction of the perfect subset R in Theorem 4 is done by induction and by repeated applications of a partition theorem due to Galvin. Note that the Borelness of the function Φ is crucial for the proof, as it is used to show that certain partitions are definable. One could not expect a similar result for an arbitrary uncountable family as above. Indeed, there exists an uncountable weakly discrete family accumulating to 0 in a reflexive and HI saturated space X (see [6]).

2 The Ramsey theoretical background

The aim of this section is to review the Ramsey theoretical background needed in the sequel. There is a long history on the interaction between infinite dimensional Ramsey Theory and Banach space Theory, going back to Farahat's proof [13] of Rosenthal's ℓ_1 Theorem [35]. We refer the reader to the survey papers [31] and [17] for an account of related results.

The component of Ramsey Theory we will use, concerns partitions of infinite subsets of the dyadic tree and in particular partitions of chains and antichains. As it is well-known there is a complete Ramsey theory for partitions of infinite subsets of $\mathbb{N}$, as long as the colors are sufficiently definable (see [12]). On the other hand, the corresponding result for partitions of infinite dyadic subtrees of the Cantor tree fails, in the sense that if we color all dyadic subtrees of the Cantor tree into finitely many, say, open colors, then we cannot expect to find a dyadic subtree all of whose dyadic subtrees are monochromatic. This has been recognized quite early by Galvin. His conjecture about partitions of k-tuples of reals, settled by the affirmative by Blass [9], reflects this phenomenon.

So, it is necessary, in order to have Ramsey theorems for trees, to consider not all subsets of the dyadic tree but only those which are of a fixed "shape". By now, there are several partition theorems along this line, obtained in [37] for chains, in [28] for strong subtrees, in [26] for strongly increasing sequences of reals and in [23] for rapidly increasing subtrees.

It is well-known, and it is incorporated in the Abstract Ramsey Theory due to Carlson [10], that in order to obtain an infinite dimensional Ramsey result, one needs a Pigeon Hole Principle that corresponds to the finite dimensional one. In the case of partitions of infinite subsets of $\mathbb{N}$, this is the classical Pigeon Hole Principle. In the case of trees, this is the deep and fundamental Halpern-Läuchli Partition Theorem [20]. The original proof was using metamathematical arguments. The proof avoiding metamathematics was given in [4].

For a presentation of some of the partition theorems we use, we refer the reader to [6]. Applications of Ramsey theory for trees in Analysis and Topology can be found in [3,38].

It is a standard fact that once one is willing to present results about trees one has to set up a, rather large, notational system. Below, we gather all the notations we need. We follow the conventions of [3] which are, more or less, standard.

2.1 Notations

We let $\mathbb{N} = \{0, 1, 2, \ldots\}$. By $[\mathbb{N}]$ we denote the set of all infinite subsets of $\mathbb{N}$, while for every $L \in [\mathbb{N}]$ by $[L]$ we denote the set of all infinite subsets of L.

A. By $2^{<\mathbb{N}}$ we denote the set of all finite sequences of 0s and 1s (the empty sequence is included). We view $2^{<\mathbb{N}}$ as a tree equipped with the (strict) partial order $\sqsubset$ of extension. If $t \in 2^{<\mathbb{N}}$, then the length $|t|$ of t is defined to be the cardinality of the set $\{s : s \sqsubset t\}$. If $s, t \in 2^{<\mathbb{N}}$, then by $s^\frown t$ we denote their concatenation. Two nodes s, t are said to be incomparable if neither $s \sqsubseteq t$ nor $t \sqsubseteq s$. A subset of $2^{<\mathbb{N}}$ consisting of pairwise incomparable nodes is said to be an *antichain*, while a subset of $2^{<\mathbb{N}}$ consisting of pairwise comparable nodes is called a *chain*. For every $x \in 2^{\mathbb{N}}$ and every $n \geq 1$ we set $x|n = (x(0), \dots, x(n-1)) \in 2^{<\mathbb{N}}$ while $x|0 = (\varnothing)$. For $x, y \in (2^{<\mathbb{N}} \cup 2^{\mathbb{N}})$ with $x \neq y$ we denote by $x \wedge y$ the $\sqsubset$-maximal node t of $2^{<\mathbb{N}}$ with $t \sqsubseteq x$ and $t \sqsubseteq y$. Moreover, we write $x \prec y$ if $w^\frown 0 \sqsubseteq x$ and $w^\frown 1 \sqsubseteq y$, where $w = x \wedge y$. We also write $x \preceq y$ if either $x = y$ or $x \prec y$. The ordering $\prec$ restricted on $2^{\mathbb{N}}$ is the usual lexicographical ordering of the Cantor set.

B. We view every subset of $2^{<\mathbb{N}}$ as a *subtree* with the induced partial ordering. A subtree T of $2^{<\mathbb{N}}$ is said to be *pruned* if for every $t \in T$ there exists $s \in T$ with $t \sqsubset s$. It is said to be *downwards closed* if for every $t \in T$ and every $s \sqsubset t$ we have that $s \in T$. For a subtree T of $2^{<\mathbb{N}}$ (not necessarily downwards closed), by $\hat{T}$ we denote the *downwards closure* of T, i.e. the set $\hat{T} = \{s : \exists t \in T \text{ with } s \sqsubseteq t\}$. If T is downwards closed, then the *body* $[T]$ of T is the set $\{x \in 2^{\mathbb{N}} : x|n \in T \ \forall n\}$.

C. Let T be a (not necessarily downwards closed) subtree of $2^{<\mathbb{N}}$. For every $t \in T$ by $|t|_T$ we denote the cardinality of the set $\{s \in T : s \sqsubset t\}$ and for every $n \in \mathbb{N}$ we set $T(n) = \{t \in T : |t|_T = n\}$. Moreover, for every $t_1, t_2 \in T$ by $t_1 \wedge_T t_2$ we denote the $\sqsubset$-maximal node w of T such that $w \sqsubseteq t_1$ and $w \sqsubseteq t_2$. Notice that $t_1 \wedge_T t_2 \sqsubseteq t_1 \wedge t_2$. Given two subtrees S and T of $2^{<\mathbb{N}}$, we say that S is a *regular* subtree of T if $S \subseteq T$ and for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $S(n) \subseteq T(m)$. For a regular subtree T of $2^{<\mathbb{N}}$, the *level set* L_T of T is the set $\{l_n : T(n) \subseteq 2^{l_n}\} \subseteq \mathbb{N}$. Notice that the infinite chains of T are naturally identified with the product $[\hat{T}] \times [L_T]$. A pruned subtree T of $2^{<\mathbb{N}}$ is said to be *dyadic* if every $t \in T$ has exactly two immediate successors in T. We observe that a subtree T of the Cantor tree is a regular dyadic subtree of $2^{<\mathbb{N}}$ if and only if there exists a (necessarily unique) bijection $i_T : 2^{<\mathbb{N}} \to T$ such that the following are satisfied.

(1) For all $t_1, t_2 \in 2^{<\mathbb{N}}$ we have $|t_1| = |t_2|$ if and only if $|i_T(t_1)|_T = |i_T(t_2)|_T$.
(2) For all $t_1, t_2 \in 2^{<\mathbb{N}}$ we have $t_1 \sqsubset t_2$ (respectively $t_1 \prec t_2$) if and only if $i_T(t_1) \sqsubset i_T(t_2)$ (respectively, $i_T(t_1) \prec i_T(t_2)$).

When we write $T = (s_t)_{t \in 2^{<\mathbb{N}}}$, where T is a regular dyadic subtree of $2^{<\mathbb{N}}$, we mean that $s_t = i_T(t)$ for all $t \in 2^{<\mathbb{N}}$. Finally we notice the following. If T is a regular dyadic subtree of $2^{<\mathbb{N}}$ and R is a regular dyadic subtree of T, then R is a regular dyadic subtree of $2^{<\mathbb{N}}$ too.

D. Let L be an infinite subset of $2^{<\mathbb{N}}$ and $\sigma \in 2^{\mathbb{N}}$. We say that L *converges* to σ if for every $k \in \mathbb{N}$ the set $L \setminus \{t \in 2^{<\mathbb{N}} : \sigma|k \sqsubseteq t\}$ is finite. The element σ will be called the *limit* of the set L. We write $L \to \sigma$ to denote that L converges to σ.

E. For every $L \subseteq 2^{<\mathbb{N}}$ infinite and every $\sigma \in 2^{\mathbb{N}}$ we write $L \prec \sigma$ (respectively, $\sigma \prec L$) if for every $t \in L$ we have $t \prec \sigma$ (respectively, for every $t \in L$ we have $\sigma \prec t$).

2.2 Chains

For a regular dyadic subtree T of $2^{<\mathbb{N}}$, denote by $[T]_{\text{chains}}$ the set of all infinite chains of T. By identifying every infinite chain of T with its characteristic function, i.e. an element of 2^T, it is easy to see that the set $[T]_{\text{chains}}$ is a G_δ (hence Polish) subspace of 2^T. The following result, essentially due to Stern [37] (see also [29,33]), includes the Ramsey property of $[T]_{\text{chains}}$ needed in the sequel.

Theorem 5 *Let T be a regular dyadic subtree of $2^{<\mathbb{N}}$ and A be an analytic subset of $[T]_{\text{chains}}$. Then there exists a regular dyadic subtree R of T such that either $[R]_{\text{chains}} \subseteq A$, or $[R]_{\text{chains}} \cap A = \varnothing$.*

2.3 Increasing and decreasing antichains

This subsection is devoted to the presentation of an analogue of Theorem 5 for infinite antichains of the Cantor tree. It not difficult to find an open partition of all infinite antichains $(t_n)_n$ of $2^{<\mathbb{N}}$ satisfying $t_n \prec t_{n+1}$ and $|t_n| < |t_{n+1}|$ for every $n \in \mathbb{N}$ and such that there is no dyadic subtree of $2^{<\mathbb{N}}$ for which all of its antichains of the above form are monochromatic. This explains the necessity of condition (2) in the following definition.

Definition 6 Let T be a regular dyadic subtree of the Cantor tree $2^{<\mathbb{N}}$. An infinite antichain $(t_n)_n$ of T will be called increasing if the following conditions are satisfied.

(1) For all $n, m \in \mathbb{N}$ with $n < m$, $|t_n|_T < |t_m|_T$.
(2) For all $n, m, l \in \mathbb{N}$ with $n < m < l$, $|t_n|_T \leq |t_m \wedge_T t_l|_T$.
(3I) For all $n, m \in \mathbb{N}$ with $n < m$, $t_n \prec t_m$.

The set of all increasing antichains of T will be denoted by $\mathrm{Incr}(T)$. Similarly, an infinite antichain $(t_n)_n$ of T will be called decreasing if (1) and (2) above are satisfied and (3I) is replaced by the following.

(3D) For all $n, m \in \mathbb{N}$ with $n < m$, $t_m \prec t_n$.

The set of all decreasing antichains of T will be denoted by $\mathrm{Decr}(T)$.

Below we gather the basic properties of increasing and decreasing antichains.

Proposition 7 *The following hold.*

(P1) *Let $(t_n)_n \in \mathrm{Incr}(T)$ and $L = \{l_0 < l_1 < \cdots\}$ be an infinite subset of $\mathbb{N}$. Then $(t_{l_n})_n \in \mathrm{Incr}(T)$. Similarly, if $(t_n)_n \in \mathrm{Decr}(T)$, then $(t_{l_n})_n \in \mathrm{Decr}(T)$.*
(P2) *Let $(t_n)_n$ be an infinite antichain of T. Then there exists $L = (l_n)_n \in [\mathbb{N}]$ such that either $(t_{l_n})_n \in \mathrm{Incr}(T)$ or $(t_{l_n})_n \in \mathrm{Decr}(T)$.*
(P3) *We have $\mathrm{Incr}(T) = \mathrm{Incr}(2^{<\mathbb{N}}) \cap 2^T$ and similarly for the decreasing antichains.*
(P4) *Let $(t_n)_n$ be an increasing (respectively, decreasing) antichain of $2^{<\mathbb{N}}$. Then $(t_n)_n$ converges to σ, where σ is the unique element of $2^{\mathbb{N}}$ determined by the chain $(c_n)_n$, where $c_n = t_n \wedge t_{n+1}$.*
(P5) *If L is an infinite subset of $2^{<\mathbb{N}}$ and $\sigma \in 2^{\mathbb{N}}$ are such that $L \to \sigma$ and $L \prec \sigma$ (respectively, $\sigma \prec L$), then every infinite subset of L contains an increasing (respectively, decreasing) antichain converging to σ.*
(P6) *Let $A_1 = (t_n^1)_n, A_2 = (t_n^2)_n$ be two increasing (respectively, decreasing) antichains of $2^{<\mathbb{N}}$ converging to the same $\sigma \in 2^{\mathbb{N}}$. Then there exists an increasing (respectively, decreasing) antichain $(t_n)_n$ of $2^{<\mathbb{N}}$ converging to σ such that $t_{2n} \in A_1$ and $t_{2n+1} \in A_2$ for all $n \in \mathbb{N}$.*

(P7) *Let* $(\sigma_n)_n$ *be a sequence in* $2^{\mathbb{N}}$ *converging to* $\sigma \in 2^{\mathbb{N}}$. *For every* $n \in \mathbb{N}$, *let* $N_n = (t^n_k)_k$ *be a sequence in* $2^{<\mathbb{N}}$ *converging to* σ_n. *If* $\sigma_n \prec \sigma$ *(respectively,* $\sigma_n \succ \sigma$*) for all* n*, then there exist an increasing (respectively, decreasing) antichain* $(t_m)_m$ *and* $L = \{n_m : m \in \mathbb{N}\}$ *such that* $(t_m)_m$ *converges to* σ *and* $t_m \in N_{n_m}$ *for all* $m \in \mathbb{N}$.

Most of the above properties are easily verified. We refer the reader to [3] for more information.

By property (P4) of the above proposition, we see that for every regular dyadic subtree T of $2^{<\mathbb{N}}$ and every increasing (respectively, decreasing) antichain $(t_n)_n$ of T there exists a unique $\sigma \in [\hat{T}]$ such that the sequence $(t_n)_n$ converges to σ. We call this σ as the *limit point* of $(t_n)_n$.

Let T be a regular dyadic subtree of $2^{<\mathbb{N}}$. As in the case of chains and by identifying every increasing antichain of T with its characteristic function, we see that the set $\mathrm{Incr}(T)$ is a G_δ subspace of 2^T. Respectively, the set $\mathrm{Decr}(T)$ is also a G_δ subspace of 2^T. The Ramsey properties of increasing and decreasing antichains are included in the following theorem.

Theorem 8 *Let* T *be a regular dyadic subtree of* $2^{<\mathbb{N}}$ *and* A *be an analytic subset of* $\mathrm{Incr}(T)$ *(respectively of* $\mathrm{Decr}(T)$*). Then there exists a regular dyadic subtree* R *of* T *such that either* $\mathrm{Incr}(R) \subseteq A$*, or* $\mathrm{Incr}(R) \cap A = \varnothing$ *(respectively, either* $\mathrm{Decr}(R) \subseteq A$*, or* $\mathrm{Decr}(R) \cap A = \varnothing$*).*

We will briefly comment on the proof, referring to [3] for a more detailed presentation. The method is to reduce the coloring of $\mathrm{Incr}(T)$ [respectively, $\mathrm{Decr}(T)$] to a coloring of a certain class of subtrees of the dyadic tree, for which it is known that it is Ramsey.

Let us argue for the case of increasing antichains as the case of decreasing antichains is similar. For every regular dyadic subtree T of $2^{<\mathbb{N}}$ we define a class $[T]_{\mathrm{Incr}}$ of regular subtrees of T, as follows. For notational convenience, let us assume that $T = 2^{<\mathbb{N}}$. Let $\sigma \in 2^{\mathbb{N}}$ not eventually zero. We select a sequence $(s_n)_n$ in $2^{<\mathbb{N}}$ such that $s_n \sqsubset s_n^\frown 1 \sqsubseteq s_{n+1} \sqsubset \sigma$ for every $n \in \mathbb{N}$ (this can be done as σ is not eventually zero). Now we select a sequence $(\sigma_n)_n$ in $2^{\mathbb{N}}$ such that $s_n^\frown 0 \sqsubset \sigma_n$ for every $n \in \mathbb{N}$. Let $L = \{l_0 < l_1 < \cdots\} \in [\mathbb{N}]$ where $l_n = |s_n|$ for all $n \in \mathbb{N}$. A tree S belongs to $[2^{<\mathbb{N}}]_{\mathrm{Incr}}$ if there exist $\sigma \in 2^{\mathbb{N}}$, a sequence $(s_n)_n$ in $2^{<\mathbb{N}}$ and a sequence $(\sigma_n)_n$ in $2^{\mathbb{N}}$ as described above, such that

$$S = \bigcup_{k \in \mathbb{N}} \{\sigma_n | l_k : n \leq k\}.$$

It is easy to see that S is a regular subtree and $[\hat{S}] = \{\sigma_n : n \in \mathbb{N}\} \cup \{\sigma\}$. Moreover, observe that the sequence $I_S = (\sigma_n | l_{n+1})_n$ is an increasing antichain of $2^{<\mathbb{N}}$ (which converges to σ). The map $\Phi : [T]_{\mathrm{Incr}} \to \mathrm{Incr}(T)$ defined by $\Phi(S) = I_S$ is easily seen to be continuous and onto. By the results in [23], the family $[T]_{\mathrm{Incr}}$ is Ramsey, i.e. for every analytic subset B of $[T]_{\mathrm{Incr}}$ there exists a regular dyadic subtree R of T such that either $[R]_{\mathrm{Incr}} \subseteq B$, or $[R]_{\mathrm{Incr}} \cap B = \varnothing$.

Now let T and A be as in Theorem 8 and consider the coloring $B = \Phi^{-1}(A)$ of $[T]_{\mathrm{Incr}}$. If R is any regular dyadic subtree of T such that $[R]_{\mathrm{Incr}}$ is monochromatic with respect to B, then it is easy to see that so is $\mathrm{Incr}(R)$ with respect to A.

We notice that Theorem 8 has been obtained independently by Todorčević with a different proof based on Milliken's theorem [39].

2.4 Partitions of perfect sets of reals

Recall that a subset M of a Polish space X is said to be meager (or of first category) if M is covered by a countable union of closed nowhere dense sets. A subset C of X is said to be co-meager if its complement is meager. Finally, a subset A of X is said to have the Baire property if there exist an open subset U of X and meager set M such that $A \triangle U = M$. It is classical fact that the family of all sets with the Baire property contains the σ-algebra generated by the analytic sets (see [24]). We will need the following partition theorem due to Galvin (see [24, Theorem 19.6]).

Theorem 9 *Let $X_1, \dots, X_n$ be perfect Polish spaces. Let also A be a subset of $X_1 \times \cdots \times X_n$ with the Baire property. If A is non-meager, then for every $i \in \{1, \dots, n\}$ there exists $P_i \subseteq X_i$ perfect such that $P_1 \times \cdots \times P_n \subseteq A$.*

3 An extension of Stern's theorem

Let us start with the proof of Theorem 3 stated in the introduction, which is implicitly contained in [3].

Proof of Theorem 3 Denote by $(e_n)_n$ the standard basis of ℓ_1. First, we argue as in [37] to homogenize the behavior of all subsequences of $\{x_t : t \in 2^{<\mathbb{N}}\}$ determined by chains. In particular, consider the following subsets of $[2^{<\mathbb{N}}]_{\text{chains}}$ defined by

$$\mathcal{X}_1 = \left\{(t_n)_n \in [2^{<\mathbb{N}}]_{\text{chains}} : (x_{t_n})_n \text{ is equivalent to } (e_n)_n\right\},$$
$$\mathcal{X}_2 = \left\{(t_n)_n \in [2^{<\mathbb{N}}]_{\text{chains}} : (x_{t_n})_n \text{ is weak Cauchy}\right\} \text{ and}$$
$$\mathcal{X}_3 = [2^{<\mathbb{N}}]_{\text{chains}} \setminus (\mathcal{X}_1 \cup \mathcal{X}_2).$$

It is easy to see that the set $\mathcal{X}_1$ is F_σ. On the other hand the set $\mathcal{X}_2$ is co-analytic (see [37] for a detailed explanation of this fact). Applying Theorem 5 successively three times, we get a regular dyadic subtree T_1 of $2^{<\mathbb{N}}$ such that for every $i \in \{1, 2, 3\}$ we have that either $[T_1]_{\text{chains}} \subseteq \mathcal{X}_i$ or $[T_1]_{\text{chains}} \cap \mathcal{X}_i = \varnothing$. By Rosenthal's ℓ_1 Theorem [35], we see that for every for every regular dyadic subtree R of $2^{<\mathbb{N}}$ we have that either $[R]_{\text{chains}} \cap \mathcal{X}_1 \neq \varnothing$ or $[R]_{\text{chains}} \cap \mathcal{X}_2 \neq \varnothing$. It follows that there exists $i \in \{1, 2\}$ such that $[T]_{\text{chains}} \subseteq \mathcal{X}_i$, i.e. either for every infinite chain $(t_n)_n$ of T_1 the sequence $(x_{t_n})_n$ is equivalent to the standard basis of ℓ_1, or for every infinite chain $(t_n)_n$ of T_1 the sequence $(x_{t_n})_n$ is weak Cauchy.

Now consider the following subsets of $\text{Incr}(T_1)$ defined by

$$\mathcal{C}_1 = \left\{(t_n)_n \in \text{Incr}(T_1) : (x_{t_n})_n \text{ is equivalent to } (e_n)_n\right\},$$
$$\mathcal{C}_2 = \left\{(t_n)_n \in \text{Incr}(T_1) : (x_{t_n})_n \text{ is weak Cauchy}\right\} \text{ and}$$
$$\mathcal{C}_3 = \text{Incr}(T_1) \setminus (\mathcal{C}_1 \cup \mathcal{C}_2).$$

Again we see that $\mathcal{C}_1$ is F_σ while the set $\mathcal{C}_2$ is co-analytic (this can be checked by similar arguments as in [37]). Applying Theorem 8 three times and arguing as before, we get a regular dyadic subtree T_2 of T_1 and $j \in \{1,2\}$ such that $\mathrm{Incr}(T_2) \subseteq \mathcal{C}_j$.

Finally, applying Theorem 8 for the decreasing antichains of T_2 and the colors

$$\begin{aligned}
\mathcal{K}_1 &= \left\{(t_n)_n \in \mathrm{Decr}(T_2) : (x_{t_n})_n \text{ is equivalent to } (e_n)_n\right\},\\
\mathcal{K}_2 &= \left\{(t_n)_n \in \mathrm{Decr}(T_2) : (x_{t_n})_n \text{ is weak Cauchy}\right\},\\
\mathcal{K}_3 &= \mathrm{Decr}(T_2) \setminus (\mathcal{K}_1 \cup \mathcal{K}_2)
\end{aligned}$$

we find a regular dyadic subtree T_3 of T_2 and $l \in \{1,2\}$ such that $\mathrm{Decr}(T_3) \subseteq \mathcal{K}_l$.

If $[T_3]_{\mathrm{chains}}$, $\mathrm{Incr}(T_3)$ and $\mathrm{Decr}(T_3)$ avoid the colors $\mathcal{X}_1$, $\mathcal{C}_1$ and $\mathcal{K}_1$ respectively, then the tree T_3 is the desired one. If not, then we will pass to a further dyadic subtree T of T_3 in order to achieve uniformity. So, assume that $\mathrm{Incr}(T_3)$ is included in $\mathcal{C}_1$ (the other cases are similar). For every $k \in \mathbb{N}$ let

$$\mathcal{F}_k = \left\{(t_n)_n \in \mathrm{Incr}(T_3) : (x_{t_n})_n \text{ is } k-\text{equivalent to } (e_n)_n\right\}.$$

Clearly $\mathcal{F}_k$ is a closed subset $\mathrm{Incr}(T_3)$. Moreover, $\mathrm{Incr}(T_3) = \bigcup_k \mathcal{F}_k$. It follows that there exists $k_0 \in \mathbb{N}$ such that the set $\mathcal{F}_{k_0}$ has non-empty interior in $\mathrm{Incr}(T_3)$. Let $(t_n)_n \in \mathrm{Int}(\mathcal{F}_{k_0})$. There exists $n_0 \in \mathbb{N}$ such that if $(s_n)_n \in \mathrm{Incr}(T_3)$ and $s_n = t_n$ for every $n \leq n_0$, then $(s_n)_n \in \mathcal{F}_{k_0}$. Set $w = t_{n_0+1} \wedge_T t_{n_0+2}$ and let $T = \{t \in T_3 : w \sqsubseteq t\}$. Clearly T is a regular dyadic subtree of T_3. Moreover, it is easy to see that $\mathrm{Incr}(T) \subseteq \mathcal{F}_{k_0}$. That is, for every increasing antichain $(r_n)_n$ of T, the sequence $(x_{r_n})_n$ is k_0-equivalent to $(e_n)_n$. Thus, we have achieved the desired uniformity.

Finally, we notice that if $\mathrm{Incr}(T) \subseteq \mathcal{C}_2$, then for every $(t_n)_n$ and $(s_n)_n$ in $\mathrm{Incr}(T)$ with the same limit point in $[\hat{T}]$, the sequences $(x_{t_n})_n$ and $(x_{s_n})_n$ are both weak* convergent to the same element of X^{**}. For if not, then by property (P6) of Proposition 7 we would be able to construct an increasing antichain $(r_n)_n$ of T such that the sequence $(x_{r_n})_n$ is not weak Cauchy, contradicting the fact that $\mathrm{Incr}(T) \subseteq \mathcal{C}_2$. The case of decreasing antichains is similarly treated. The proof is completed. □

Remark 1 We notice that the behavior of the sequence $(x_t)_{t\in T}$ along chains of T is independent of the corresponding one along increasing antichains (and decreasing antichains, respectively). In particular, all subsequences of $(x_t)_{t\in T}$ determined by chains and increasing antichains can be weak* convergent while all subsequences determined by decreasing antichains are equivalent to the standard basis of ℓ_1. For example, let X be the completion of $c_{00}(2^{<\mathbb{N}})$ under the norm

$$\|x\| = \sup\left\{\sum_{n\in\mathbb{N}} |x(t_n)| : (t_n)_n \in \mathrm{Decr}(2^{<\mathbb{N}})\right\}.$$

Consider the standard Hamel basis $(e_t)_{t\in 2^{<\mathbb{N}}}$ of $c_{00}(2^{<\mathbb{N}})$. It is easy to see that for every sequence $(t_n)_n$ in $2^{<\mathbb{N}}$ which is either a chain or an increasing antichain, the sequence $(e_{t_n})_n$ is 1-equivalent to the standard basis of c_0. In particular, it is weakly-null. On the other hand, if $(t_n)_n$ is a decreasing antichain, then the sequence $(e_{t_n})_n$ is 1-equivalent to the standard basis of ℓ_1.

We will also need the following result, whose proof is based on Theorem 3 as well as on the properties of increasing and decreasing antichains described in Proposition 7.

Theorem 10 *Let X be a separable space not containing ℓ_1. Let also $\Delta = \{x_t : t \in 2^{<\mathbb{N}}\}$ be a bounded family in X. Then there exist a regular dyadic subtree T of $2^{<\mathbb{N}}$ and a family $\{y^0_\sigma, y^+_\sigma, y^-_\sigma : \sigma \in P\} \subseteq X^{**}$, where $P = [\hat{T}]$, such that for every $\sigma \in P$ the following are satisfied.*

(1) *The sequence $(x_{\sigma|n})_{n\in L_T}$ is weak* convergent to y^0_σ (recall that L_T stands for the level set of T).*
(2) *For every sequence $(\sigma_n)_n$ in P converging to σ such that $\sigma_n \prec \sigma$ for all $n \in \mathbb{N}$, the sequence $(y^{\varepsilon_n}_{\sigma_n})_n$ is weak* convergent to y^+_σ for any choice of $\varepsilon_n \in \{0, +, -\}$. If such a sequence $(\sigma_n)_n$ does not exist, then $y^+_\sigma = y^0_\sigma$.*
(3) *For every sequence $(\sigma_n)_n$ in P converging to σ such that $\sigma \prec \sigma_n$ for all $n \in \mathbb{N}$, the sequence $(y^{\varepsilon_n}_{\sigma_n})_n$ is weak* convergent to y^-_σ for any choice of $\varepsilon_n \in \{0, +, -\}$. If such a sequence $(\sigma_n)_n$ does not exist, then $y^-_\sigma = y^0_\sigma$.*
(4) *For every infinite subset L of T converging to σ with $L \prec \sigma$, the sequence $(x_t)_{t\in L}$ is weak* convergent to y^+_σ.*
(5) *For every infinite subset L of T converging to σ with $\sigma \prec L$, the sequence $(x_t)_{t\in L}$ is weak* convergent to y^-_σ.*

Moreover, the functions $0, +, - : P \times (B_{X^*}, w^*) \to \mathbb{R}$ defined by

$$0(\sigma, x^*) = y^0_\sigma(x^*),\ +(\sigma, x^*) = y^+_\sigma(x^*),\ -(\sigma, x^*) = y^-_\sigma(x^*)$$

are all Borel.

The family $\{y^0_\sigma, y^+_\sigma, y^-_\sigma : \sigma \in P\}$, obtained by Theorem 10, determines the weak* closure of the family $\{x_t : t \in T\}$. Theorem 10 appears in [3], where it is stated and proved in the broader frame of separable Rosenthal compacta. It is part of a finer analysis of the topological behavior of the family $\{x_t : t \in 2^{<\mathbb{N}}\}$, yielding a complete canonicalization of any such family as above.

Proof Applying Theorem 3 and invoking our hypotheses on the space X, we get a regular dyadic subtree T of $2^{<\mathbb{N}}$ such that, setting $P = [\hat{T}]$, the following are satisfied.

(i) For every $(t_n)_n \in \mathrm{Incr}(T)$, the sequence $(x_{t_n})_n$ is weak Cauchy.
(ii) For every $(t_n)_n \in \mathrm{Decr}(T)$, the sequence $(x_{t_n})_n$ is weak Cauchy.
(iii) For every $\sigma \in P$, the sequence $(x_{\sigma|n})_{n\in L_T}$ is weak Cauchy.

For every $\sigma \in P$ we define y^0_σ, y^+_σ and y^-_σ in X^{**} as follows. First we set y^0_σ to be the weak* limit of the sequence $(x_{\sigma|n})_{n\in L_T}$. If there exists an increasing antichain $(t_n)_n$ of T converging to σ, then we set y^+_σ to be the weak* limit of the sequence $(x_{t_n})_n$. By Theorem 3, y^+_σ is well-defined and independent of the choice of $(t_n)_n$. Otherwise we set $y^+_\sigma = y^0_\sigma$. Similarly, we define y^-_σ to be the weak* limit of the sequence $(x_{t_n})_n$, with $(t_n)_n$ a decreasing antichain of T converging to σ, if such an antichain exists. Otherwise we set $y^-_\sigma = y^0_\sigma$.

We claim that the tree T and the family $\{y^0_\sigma, y^+_\sigma, y^-_\sigma : \sigma \in P\}$ are as desired. First we notice that, by (P5) of Proposition 7, properties (i) and (ii) above are strengthened as follows.

(iv) For every $\sigma \in P$ and every infinite subset L of T with $L \to \sigma$ and $L \prec \sigma$, the sequence $(x_t)_{t\in L}$ is weak* convergent to y^+_σ.
(v) For every $\sigma \in P$ and every infinite subset L of T with $L \to \sigma$ and $\sigma \prec L$, the sequence $(x_t)_{t\in L}$ is weak* convergent to y^-_σ.

Hence, by (iii), (iv) and (v) we see that properties (1), (4) and (5) in the statement of the theorem are satisfied. We will only check that property (2) is satisfied (the argument for (3) is symmetric). We argue by contradiction. So assume that there exist a sequence $(\sigma_n)_n$ in P, $\sigma \in P$ and a sequence $(\varepsilon_n)_n$ in $\{0, +, -\}$ such that $\sigma_n \prec \sigma$ for all $n \in \mathbb{N}$, $\sigma_n \to \sigma$ while $(y^{\varepsilon_n}_{\sigma_n})_n$ is not weak* convergent to y^+_σ. Hence, there exist $L \in [\mathbb{N}]$, a weak* open neighborhood V of y^+_σ such that $y^{\varepsilon_n}_{\sigma_n} \notin \overline{V}^{w*}$ for every $n \in L$. For every $n \in L$ we select a sequence $(t^n_k)_k$ in T such that the following are satisfied.

(a) The sequence $N_n = (t^n_k)_k$ converges (as a subset of T) to σ_n.
(b) The sequence $(x_{t^n_k})_k$ is weak* convergent to $y^{\varepsilon_n}_{\sigma_n}$.
(c) For every $k \in \mathbb{N}$ we have $x_{t^n_k} \notin \overline{V}^{w*}$.

By property (P7) of Proposition 7, there exists a diagonal increasing antichain $(t_m)_m$ converging to σ. By (c) above, we see that $(x_{t_m})_m$ is not weak* convergent to y^+_σ, which is a contradiction by the definition of y^+_σ.

Finally we will check the Borelness of the maps 0, + and −. Let $\{l_0 < l_1 < \cdots\}$ be the increasing enumeration of the level set L_T of T. For every $n \in \mathbb{N}$ define $h_n : P \times (B_{X^*}, w^*) \to \mathbb{R}$ by $h_n(\sigma, x^*) = x^*(x_{\sigma|l_n})$. Clearly h_n is continuous. Notice that for every $(\sigma, x^*) \in P \times B_{X^*}$ we have

$$0(\sigma, x^*) = y^0_\sigma(x^*) = \lim_{n\in\mathbb{N}} h_n(\sigma, x^*).$$

Hence 0 is Borel (actually it is Baire class one). We will only check the Borelness of the function + (the argument for the map is symmetric). For every $n \in \mathbb{N}$ and every $\sigma \in P$ let $l_n(\sigma)$ be the lexicographically minimum of the closed set $\{\tau \in P : \sigma|l_n \sqsubset \tau\}$. Clearly $l_n(\sigma) \in P$. Moreover, observe that the function $P \ni \sigma \mapsto l_n(\sigma) \in P$ is continuous. Invoking the definition of y^+_σ and property (2) in the statement of the theorem we see that for all $(\sigma, x^*) \in P \times B_{X^*}$ we have

$$+(\sigma, x^*) = y^+_\sigma(x^*) = \lim_{n\in\mathbb{N}} y^0_{l_n(\sigma)}(x^*) = \lim_{n\in\mathbb{N}} 0\left(l_n(\sigma), x^*\right).$$

Thus + is a Borel map and the proof is completed. □

4 Perfect unconditional families

This section is devoted to the proof of Theorem 4 stated in the introduction. Let us recall that a family $\{x_i\}_{i\in I}$ in a Banach space X is said to be 1-unconditional if for every $F \subseteq G \subseteq I$ and every $(a_i)_{i\in G}$ in $\mathbb{R}^G$ we have

$$\left\| \sum_{i\in F} a_i x_i \right\| \leq \left\| \sum_{i\in G} a_i x_i \right\| .$$

As we have already mentioned, the construction of the perfect subset R in Theorem 4 is done by induction. The basic step for accomplishing the construction is described in the following lemma. Its proof is based on the partition theorem of Galvin (Theorem 9 above).

Lemma 11 *Let X, Q and $\mathcal{D}$ be as in Theorem 4. Let $n \in \mathbb{N}$ and $Q_0, \dots, Q_n$ pairwise disjoint perfect subsets of Q. Then for every $i \in \{0, \dots, n\}$ there exists a perfect subset R_i of Q_i such that the following hold. For every $(\sigma_0, \dots, \sigma_n) \in R_0 \times \cdots \times R_n$ the family $\{z^{**}_{\sigma_0}, \dots, z^{**}_{\sigma_n}\}$ is 1-unconditional.*

Proof For every $k \in \mathbb{N}$ and every $P_0, \dots, P_k$ pairwise disjoint perfect subsets of Q we set

$$U(P_0, \dots, P_k) = \left\{(\sigma_0, \dots, \sigma_k) \in P_0 \times \cdots \times P_k : \{z^{**}_{\sigma_0}, \dots, z^{**}_{\sigma_k}\} \text{ is 1-unconditional}\right\}.$$

Let $n \in \mathbb{N}$ and $Q_0, \dots, Q_n$ be as in the statement of the lemma. For every $F \subseteq \{0, \dots, n\}$ non-empty, every rational $\varepsilon > 0$ and every $(a_i)_{i=0}^n \in \mathbb{Q}^{n+1}$ we define $D = D\left(F, \varepsilon, (a_i)_{i=0}^n\right)$ by

$$D = \left\{(\sigma_0, \dots, \sigma_n) \in Q_0 \times \cdots \times Q_n : \left\| \sum_{i\in F} a_i z^{**}_{\sigma_i} \right\| < (1+\varepsilon) \left\| \sum_{i=0}^{n} a_i z^{**}_{\sigma_i} \right\| \right\}.$$

Clearly we have that

$$U(Q_0, \dots, Q_n) = \bigcap_{F, \varepsilon, (a_i)_{i=0}^n} D\left(F, \varepsilon, (a_i)_{i=0}^n\right). \tag{1}$$

Claim 1 *The set $D = D\left(F, \varepsilon, (a_i)_{i=0}^n\right)$ has the Baire property in $Q_0 \times \cdots \times Q_n$.*

Proof of the claim By our assumptions, we see that the maps $\Phi_n, \Phi_F : Q_0 \times \cdots \times Q_n \times (B_{X^*}, w^*) \to \mathbb{R}$, defined by $\Phi_n(\sigma_0, \dots, \sigma_n, x^*) = \sum_{i=0}^n a_i z^{**}_{\sigma_i}(x^*)$ and $\Phi_F(\sigma_0, \dots, \sigma_n, x^*) = \sum_{i\in F} a_i z^{**}_{\sigma_i}(x^*)$ respectively, are both Borel. Notice that

$$\begin{aligned}(\sigma_0, \dots, \sigma_n) \in D \Leftrightarrow\ & \exists p \in \mathbb{Q} \left(\left\| \sum_{i\in F} a_i z^{**}_{\sigma_i} \right\| \leq p \text{ and } \frac{p}{1+\varepsilon} < \left\| \sum_{i=0}^{n} a_i z^{**}_{\sigma_i} \right\| \right) \\ \Leftrightarrow\ & \exists p \in \mathbb{Q} \Big[\left(\forall x^* \in B_{X^*} \text{ we have } \Phi_F(\sigma_0, \dots, \sigma_n, x^*) \leq p\right) \text{ and} \\ & \left(\exists x^* \in B_{X^*} \text{ with } \frac{p}{1+\varepsilon} < \Phi_n(\sigma_0, \dots, \sigma_n, x^*)\right)\Big]\end{aligned}$$

Hence, D belongs to the σ-algebra generated by the analytic sets. Finally, we recall that the σ-algebra generated by the analytic sets is included in the σ-algebra of all sets with Baire property. The claim is proved. □

Claim 2 *For every $P_0, \dots, P_n$ perfect subsets of $Q_0, \dots, Q_n$, there exists $(\sigma_0, \dots, \sigma_n) \in D\left(F, \varepsilon, (a_i)_{i=0}^n\right) \cap (P_0 \times \cdots \times P_n)$.*

Proof of the claim For every $i \in \{0, \dots, n\}$ we fix $\tau_i \in P_i$. Let $x_0^* \in B_{X^*}$ such that

$$\left\| \sum_{i \in F} a_i z_{\tau_i}^{**} \right\| < (1 + \varepsilon) \sum_{i \in F} a_i z_{\tau_i}^{**}(x_0^*).$$

The family $\{z_\sigma^{**} : \sigma \in Q\}$ accumulates to 0 in the weak* topology. Hence, $z_\sigma^{**}(x_0^*) = 0$ for all but countable many $\sigma \in Q$. For every $i \in \{0, \dots, n\} \setminus F$, we may select $\sigma_i' \in P_i$ with $z_{\sigma_i'}^{**}(x_0^*) = 0$. Finally, for every $i \in \{0, \dots, n\}$ we define $\sigma_i = \tau_i$ if $i \in F$ and $\sigma_i = \sigma_i'$ otherwise. Then $(\sigma_0, \dots, \sigma_n) \in P_0 \times \cdots \times P_n$ and moreover

$$\begin{aligned} \left\| \sum_{i \in F} a_i z_{\sigma_i}^{**} \right\| &= \left\| \sum_{i \in F} a_i z_{\tau_i}^{**} \right\| < (1 + \varepsilon) \sum_{i \in F} a_i z_{\tau_i}^{**}(x_0^*) \\ &= (1 + \varepsilon) \sum_{i=0}^{n} a_i z_{\sigma_i}^{**}(x_0^*) \le (1 + \varepsilon) \left\| \sum_{i=0}^{n} a_i z_{\sigma_i}^{**} \right\|. \end{aligned}$$

Thus $(\sigma_0, \dots, \sigma_n) \in D\left(F, \varepsilon, (a_i)_{i=0}^n\right) \cap (P_0 \times \cdots \times P_n)$ and the claim in proved. □

By Claim 1, for every F, ε and $(a_i)_{i=0}^n$ the set $D\left(F, \varepsilon, (a_i)_{i=0}^n\right)$ has the Baire property in $Q_0 \times \cdots \times Q_n$. We claim that the set $D\left(F, \varepsilon, (a_i)_{i=0}^n\right)$ must be co-meager in $Q_0 \times \cdots \times Q_n$. Indeed, if not, then by Theorem 9 there would existed $P_0, \dots, P_n$ perfect subsets of $Q_0, \dots, Q_n$ such that $D\left(F, \varepsilon, (a_i)_{i=0}^n\right) \cap (P_0 \times \cdots \times P_n) = \varnothing$, which clearly contradicts Claim 2. It follows that $D\left(F, \varepsilon, (a_i)_{i=0}^n\right)$ is co-meager. By (1), so is the set $U(Q_0, \dots, Q_n)$. Invoking Theorem 9 once more, we see that there exist $R_0, \dots, R_n$ perfect subsets of $Q_0, \dots, Q_n$ such that $R_0 \times \cdots \times R_n \subseteq U(Q_0, \dots, Q_n)$ and the proof is completed. □

We are ready to proceed to the proof of Theorem 4.

Proof of Theorem 4 By recursion on the length of finite sequences in $2^{<\mathbb{N}}$ we shall construct a family $(R_t)_{t \in 2^{<\mathbb{N}}}$ of perfect subsets of Q such that the following are satisfied.

(C1) For every $t \in 2^{<\mathbb{N}}$ we have $\mathrm{diam}(R_t) \le \frac{1}{2^{|t|}}$.

(C2) For every $t \in 2^{<\mathbb{N}}$, $R_{t^\frown 0}, R_{t^\frown 1} \subseteq R_t$ and $R_{t^\frown 0} \cap R_{t^\frown 1} = \varnothing$.

(C3) For every $n \ge 1$, every $t \in 2^n$ and every $\sigma_t \in R_t$, the family $\{z_{\sigma_t}^{**} : t \in 2^n\}$ is 1-unconditional.

Assuming that the construction has been carried out, we set

$$R = \bigcup_{\sigma \in 2^{\mathbb{N}}} \bigcap_{n \in \mathbb{N}} R_{\sigma|n}.$$

Clearly, R is a perfect subset of Q. Moreover, using condition (C2) above, it is easy to see that the family $\{z_\sigma^{**} : \sigma \in R\}$ is 1-unconditional.

We proceed to the construction. We set $R_{(\varnothing)} = Q$. Assume that for some $n \geq 1$ the family $(R_t)_{t \in 2^{n-1}}$ has been constructed. For every $t \in 2^{n-1}$ and every $i \in \{0, 1\}$ we select $Q_{t^\frown i}$ perfect subset of R_t with $\mathrm{diam}(Q_{t^\frown i}) \leq \frac{1}{2^n}$ and such that $Q_{t^\frown 0} \cap Q_{t^\frown 1} = \varnothing$. Let $\{t_0 \prec \cdots \prec t_{2^n - 1}\}$ be the $\prec$-increasing enumeration of 2^n. We apply Lemma 11 to the family of perfect sets $Q_{t_0}, \ldots, Q_{t_{2^n-1}}$ and we get for every $t \in 2^n$ a perfect subset R_t of Q_t such that for every $(\sigma_t)_{t \in 2^n} \in \prod_{t \in 2^n} R_t$ the family $\{z_{\sigma_t}^{**} : t \in 2^n\}$ is 1-unconditional. Clearly, the family $(R_t)_{t \in 2^n}$ satisfies (C1)–(C3) above. This completes the construction and the proof is completed. ◻

Remark 2 We notice that the existence of a subset of X^{**} of the size of the continuum which is weak* discrete and having 0 as the unique weak* accumulation point can be obtained by the results of Todorčević in [38], after observing that $(B_{X^{**}}, w^*)$ is a separable Rosenthal compact containing 0 as a non-G_δ point. His remarkable proof uses, among others, forcing arguments and absoluteness. This result has been strengthened and extended to a wider class of Rosenthal compacta in [3], with a proof avoiding metamathematics.

5 The main results

In this section we present the proof of Theorem 1 stated in the introduction. We also state and prove some of its consequences. As we have mentioned, the proof is based on the following fundamental construction due to Stegall [36]. A variation of Stegall's construction has been presented by Godefroy and Talagrand [15] in the more general context of representable Banach spaces (see also [14]). We refer the reader to [5] for a full account of related results.

Theorem 12 *Let X be a separable Banach space with non-separable dual. Then for every $\varepsilon > 0$ there exist a family $\Delta_\varepsilon = \{x_t : t \in 2^{<\mathbb{N}}\}$ in $(1+\varepsilon)B_X$ and a subset $D_\varepsilon = \{x_\sigma^* : \sigma \in 2^{\mathbb{N}}\}$ in the sphere of X^* which is weak* homeomorphic to the Cantor set $2^{\mathbb{N}}$ via the map $\sigma \mapsto x_\sigma^*$ and such that for every $\sigma \in 2^{\mathbb{N}}$ and every $t \in 2^{<\mathbb{N}}$ we have*

$$|x_\sigma^*(x_t) - \delta_{\sigma t}| < \frac{1}{2^{|t|}}$$

where $\delta_{\sigma t} = 1$ if $t \sqsubset \sigma$ and $\delta_{\sigma t} = 0$ otherwise.

Although the above statement is not explicitly isolated in [36], it is the precise content of the proof.

We notice the following property of the sets Δ_ε and D_ε obtained by Theorem 12. For every $\sigma \in 2^{\mathbb{N}}$ let x_σ^{**} be any weak* accumulation point of the family $\{x_{\sigma|n} : n \in \mathbb{N}\}$. Then the family $\{(x_\sigma^*, x_\sigma^{**}) : \sigma \in 2^{\mathbb{N}}\} \subseteq X^* \times X^{**}$ forms a bi-orthogonal system, and so, the set $\{x_\sigma^{**} : \sigma \in 2^{\mathbb{N}}\}$ is weak* discrete.

We are ready to proceed to the proof of Theorem 1.

Proof of Theorem 1 We apply Theorem 12 for $\varepsilon = 1$ and we get a family $\Delta_1 = \{x_t : t \in 2^{<\mathbb{N}}\}$ in $2B_X$ and a family $D_1 = \{x_\sigma^* : \sigma \in 2^{\mathbb{N}}\}$ in the sphere of X^* as described in Theorem 12. Now we apply Theorem 10 for the family $\Delta = \{x_t/2 : t \in 2^{<\mathbb{N}}\}$ and we get a regular dyadic subtree T of $2^{<\mathbb{N}}$ and a family $\{y_\sigma^0, y_\sigma^+, y_\sigma^- : \sigma \in P\} \subseteq B_{X^{**}}$, where $P = [\hat{T}]$. Notice that the set $\{(y_\sigma^0, 2x_\sigma^*) : \sigma \in P\}$ forms a bi-orthogonal system. We fix a perfect subset Q of P with the following property. For every $\tau \in Q$ there exists a sequence $(\tau_n)_n$ in P with $\tau_n \prec \tau$ for all $n \in \mathbb{N}$ and such that $\tau_n \to \tau$. This condition guarantees that the function y_τ^+ is not trivially equal to y_τ^0. For every $\tau \in Q$ we set $z_\tau^{**} = y_\tau^0 - y_\tau^+$ and $z_\tau^* = 2x_\tau^*$.

Claim The following hold.

(1) For every $\tau \in Q$, $z_\tau^{**} \neq 0$.
(2) The family $\{(z_\tau^*, z_\tau^{**}) : \tau \in Q\}$ forms a bounded bi-orthogonal system in $X^* \times X^{**}$.
(3) The family $\{z_\tau^{**} : \tau \in Q\}$ is weak* discrete having 0 as the unique weak* accumulation point.
(4) The function $\Phi : Q \times (B_{X^*}, w^*) \to \mathbb{R}$ defined by $\Phi(\tau, x^*) = z_\tau^{**}(x^*)$ is Borel.

Granting the claim, we complete the proof as follows. By (3) and (4) above, we see that Theorem 4 can be applied to the family $\mathcal{D} = \{z_\tau^{**} : \tau \in Q\}$. Hence, there exists a further perfect subset R of Q such that the family $\{z_\tau^{**} : \tau \in R\}$ is 1-unconditional. By (2) above and identifying R with $2^{\mathbb{N}}$, we conclude that the family $\{(z_\tau^*, z_\tau^{**}) : \tau \in R\}$ is as desired.

So it only remains to prove the claim. First we argue for (1). Fix $\tau \in Q$ and pick a sequence (τ_n) in P with $\tau_n \to \tau$ and such that $\tau_n \prec \tau$ for every $n \in \mathbb{N}$. By property (2) of Theorem 10, we see that $y_{\tau_n}^0(x^*) \to y_\tau^+(x^*)$ for all $x^* \in B_{X^*}$. By the bi-orthogonality of the family $\{(y_\sigma^0, 2x_\sigma^*) : \sigma \in P\}$ we see that

$$0 = y_{\tau_n}^0(x_\tau^*) \to y_\tau^+(x_\tau^*)$$

and so $z_\tau^{**}(z_\tau^*) = 2y_\tau^0(x_\tau^*) = 1$. Hence $z_\tau^{**} \neq 0$. With identical arguments, we get that for every $\tau, \tau' \in Q$ with $\tau \neq \tau'$ we have $z_\tau^{**}(z_{\tau'}^*) = 0$. Thus the family $\{(z_\tau^*, z_\tau^{**}) : \tau \in Q\}$ forms a bi-orthogonal system in $X^* \times X^{**}$, i.e. (2) is satisfied. To see (3), it is enough to show that for every sequence $(\tau_n)_n$ in Q with $t_n \neq t_m$ if $n \neq m$, the sequence $(z_{\tau_n}^{**})_n$ has a subsequence weak* convergent to 0. So, let $(\tau_n)_n$ be one. By passing to a subsequence we may assume that there exists $\tau \in Q$ such that $\tau_n \to \tau$ and either $\tau_n \prec \tau$ for all $n \in \mathbb{N}$ or vice versa. We will treat the first case (the argument is symmetric). By property (2) of Theorem 10 we see that both $(y_{\tau_n}^0)_n$ and $(y_{\tau_n}^+)_n$ are weak* convergent to y_τ^+. Hence

$$z_{\tau_n}^{**} = y_{\tau_n}^0 - y_{\tau_n}^+ \overset{w*}{\to} y_\tau^+ - y_\tau^+ = 0.$$

This shows that $\{z_\tau^{**} : \tau \in Q\}$ is weak* discrete having 0 as the unique weak* accumulation point. Finally, the Borelness of the map Φ is an immediate consequence of the Borelness of the maps 0 and + obtained by Theorem 10. This completes the proof of the claim, and so, the entire proof is completed.

Consequences. Below we state and prove some consequences of Theorem 1. We start with the following.

Theorem 13 *Let X be a separable Banach space with non-separable dual. Then X^{**} contains an unconditional family of size $|X^{**}|$.*

Proof If $\ell_1(\mathbb{N})$ embeds into X, then $\ell_1(2^{\mathfrak{c}})$ embeds into X^{**}. Hence X^{**} contains an unconditional family of size $2^{\mathfrak{c}} = |X^{**}|$. If $\ell_1(\mathbb{N})$ does not embed into X, then the cardinality of X^{**} is equal to the continuum (see [32]). By Theorem 1, the result follows. □

The following trichotomy provides the first positive answer to the "reflexivity or unconditionality problem".

Theorem 14 *Let X be a separable Banach space. Then one of the following holds.*

(a) *The space X is saturated with reflexive subspaces.*
(b) *There exists an unconditional family in X^{**} of size $|X^{**}|$.*
(c) *There exists an unconditional family in X^{***} of size $|X^{***}|$.*

Proof Let X be a separable Banach space. If X^{**} is separable, then by a result stated in [30] and proved in [22] (see also [11, Theorem 4.1] for a somewhat more general result), we get that the space X is reflexive saturated, i.e. part (a) holds. So assume that X^{**} is non-separable. If X^* is non-separable, then by Theorem 13 we see that (b) is satisfied. Finally, if X^* is separable, then invoking again Theorem 13 we conclude that (c) holds. The proof is completed. □

We close this section with the following result which provides a positive answer for the class of dual spaces to Banach's classical "separable quotient problem".

Theorem 15 *Let X be a Banach space which is isomorphic to a dual Banach space. Then one of the following holds.*

(i) *The space X has the Radon–Nikodym property.*
(ii) *The space X has a separable quotient with an unconditional basis.*

Thus, every dual Banach space has a separable quotient.

For the proof of Theorem 15 we need the following well-known result [19]. We include the proof for completeness.

Proposition 16 *Let X be a Banach space. If X^* contains an unconditional basic sequence, then X has a separable quotient with an unconditional basis.*

Proof Let $(x_n^*)_n$ be an unconditional basic sequence in X^* and set $R = \overline{\text{span}}\{x_n^* : n \in \mathbb{N}\}$. Then, by a classical result of James (see [25]), either R is reflexive, or ℓ_1 embeds into

R, or c_0 embeds into R. If R is reflexive, then the weak and weak* topologies on R coincide. Hence R is weak* linearly homeomorphic to a subspace of X^*, which yields that X maps onto R^*. Let also observe that if ℓ_1 embeds into X, then $L^1[0,1]$ embeds into X^* (see [34]). Hence ℓ_2 embeds into X^* which implies that ℓ_2 is a quotient of X.

From now on we assume that ℓ_1 does not embed into X. By [8], we conclude that c_0 does not embed into X^*. Hence R does not contain c_0. What remains is to treat the case where ℓ_1 embeds into R. Since ℓ_1 does not embed into X, by [19], we conclude that there exists a weak* null sequence $(z_n^*)_n$ in R equivalent to the usual basis of ℓ_1. Denote by $T : \ell_1 \to \overline{\text{span}}\{z_n^* : n \in \mathbb{N}\} \hookrightarrow X^*$ the natural isomorphism and let $T^* : X^{**} \to \ell_\infty$ the dual onto operator. Observe that $T^*|_X$ maps X to c_0, hence, it is weak*-weak continuous. It follows that T^* maps X onto c_0. This completes the proof. □

We are ready to proceed to the proof of Theorem 15.

Proof of Theorem 15 Let Y be a Banach space such that X is isomorphic to Y^*. Assume that (i) does not hold. It follows that there exists a separable subspace Z of Y such that Z^* is non-separable (see [36]). By Theorem 13, we get that Z^{**} contains an unconditional basic sequence. Hence, so does X^*. By Proposition 16, we conclude that X has a separable quotient with an unconditional basis and the result follows. □

Let us mention that Todorčević has shown that there exists a model of Set Theory where the Continuum Hypothesis fails and in which every Banach space of density character $\aleph_1$ has a separable quotient [40].

6 Tree bases in Banach spaces

We start with the following.

Theorem 17 *Let X be a separable Banach space not containing ℓ_1 and such that X^* is non-separable. Then there exists a seminormalized family $(e_t)_{t\in 2^{<\mathbb{N}}}$ such that the following are satisfied.*

(1) *For every $\sigma \in 2^{\mathbb{N}}$ the sequence $(e_{\sigma|n})_n$ is weak* convergent to an element $z_\sigma^{**} \in X^{**}$.*
(2) *For every antichain A of $2^{<\mathbb{N}}$ the sequence $(e_t)_{t\in A}$ is weakly-null.*
(3) *The family $\{z_\sigma^{**} : \sigma \in 2^{\mathbb{N}}\}$ is weak* discrete having 0 as the unique weak* accumulation point.*

Theorem 17 follows by the general structural result obtained in [3] and concerning the behavior of non-G_δ points in a large class of Rosenthal compacta. The proof, however, given in [3] uses deep results from the theory of Rosenthal compacta and it is rather involved. The one we present below is based on Stegall's construction as well as on the analysis behind the proof of Theorem 1.

Proof of Theorem 17 First we argue as in the proof of Theorem 1. Specifically, applying Theorem 12 for $\varepsilon = 1$ we get $\Delta_1 = (x_t)_{t\in 2^{<\mathbb{N}}}$ and $D_1 = \{x_\sigma^* : \sigma \in 2^{\mathbb{N}}\}$. Now we apply Theorem 10 for the obtained family Δ_1 and we get a regular dyadic subtree T of $2^{<\mathbb{N}}$ and a family $\{y_\sigma^0, y_\sigma^+, y_\sigma^- : \sigma \in P\}$, where $P = [\hat{T}]$, as described in Theorem 10. Without loss of generality and by re-enumerating if necessary (which can be done as the tree T is regular dyadic), we may assume that $T = 2^{<\mathbb{N}}$ and so $P = 2^{\mathbb{N}}$.

We fix a regular dyadic subtree $R = (r_t)_{t\in 2^{<\mathbb{N}}}$ of $2^{<\mathbb{N}}$ with the following property.

(P) For every $t \in R$ we have that $t^\frown 0 \notin \hat{R}$ while $t^\frown 1 \in \hat{R}$.

A possible choice can be as follows. For every $t = (\varepsilon_0, \dots, \varepsilon_k) \in 2^{<\mathbb{N}}$ let $r_t = (1, \varepsilon_0, 1, \varepsilon_1, \dots, 1, \varepsilon_k)$ if $t \neq (\varnothing)$ and $r_{(\varnothing)} = (\varnothing)$. It is easy to see that $R = (r_t)_{t\in 2^{<\mathbb{N}}}$ satisfies (P) above. We denote by Q the body of $\hat{R}$. For every $\sigma \in 2^{\mathbb{N}}$ we let $\tau_\sigma = \bigcup_n r_{\sigma|n} \in Q$. The map $2^{\mathbb{N}} \ni \sigma \mapsto \tau_\sigma \in Q$ is a homeomorphism. We isolate the following properties of R and Q.

(a) If $t \in 2^{<\mathbb{N}}$ and $\sigma \in 2^{\mathbb{N}}$ with $r_t \prec \tau_\sigma$ (respectively $\tau_\sigma \prec r_t$), then $r_t^\frown 0 \prec \tau_\sigma$ (respectively $\tau_\sigma \prec r_t^\frown 0$).
(b) If $(t_n)_n$ is a sequence in $2^{<\mathbb{N}}$ and $\sigma \in 2^{\mathbb{N}}$ are such that $r_{t_n} \to \tau_\sigma$, then $r_{t_n}^\frown 0 \to \tau_\sigma$.
(c) For every $\sigma \in 2^{\mathbb{N}}$, the sequence $(r_{\sigma|n}^\frown 0)_n$ is an increasing antichain converging to τ_σ.

For every $t \in 2^{<\mathbb{N}}$ we define

$$e_t = x_{r_t} - x_{r_t^\frown 0}.$$

We claim that the family $(e_t)_{t\in 2^{<\mathbb{N}}}$ is the desired one. Using (c) above and properties (1) and (4) of Theorem 10, we see that for every $\sigma \in 2^{\mathbb{N}}$ the sequence $(e_{\sigma|n})_n$ is weak* convergent to the element

$$z_\sigma^{**} = y_{\tau_\sigma}^0 - y_{\tau_\sigma}^+ \in X^{**}.$$

With identical arguments as in the proof of Theorem 1, we get that the family $\{z_\sigma^{**} : \sigma \in 2^{\mathbb{N}}\}$ is weak* discrete having 0 as the unique weak* accumulation point. Hence (1) and (3) in the statement are satisfied. Let us see that (2) is also satisfied. Notice that it is enough to prove that for every infinite antichain A of $2^{<\mathbb{N}}$ there exists $B \subseteq A$ infinite such that the sequence $(e_t)_{t\in B}$ is weakly-null. So let A be one. There exist $\sigma \in 2^{\mathbb{N}}$ and an infinite subset B of A such that $B \to \sigma$ and either $t \prec \sigma$ for all $t \in B$ or vice versa. Assume that the first case occurs (the argument is symmetric). Observe that $r_t \prec \tau_\sigma$ for every $t \in B$. By (a) and (b) above and property (4) in Theorem 10 we get

$$\mathrm{w}^* - \lim_{t\in B} e_t = \mathrm{w}^* - \lim_{t\in B}(x_{r_t} - x_{r_t^\frown 0}) = y_{\tau_\sigma}^+ - y_{\tau_\sigma}^+ = 0.$$

So, the sequence $(e_t)_{t\in B}$ is weakly-null and the proof is completed. □

Actually, we can considerably strengthen the properties of the sequence $(e_t)_{t\in 2^{<\mathbb{N}}}$ obtained by Theorem 17, as follows.

Theorem 18 *Let X be a separable Banach space not containing ℓ_1 and with non-separable dual. Then there exists a family $(w_t)_{t\in 2^{<\mathbb{N}}}$ in X and a family $\{w^{**}_\sigma : \sigma \in 2^{\mathbb{N}}\}$ in X^{**} satisfying* (1), (2) *and* (3) *of Theorem* 17 *as well as the following.*

(i) *The family $(w_t)_{t\in 2^{<\mathbb{N}}}$ is Schauder basic when it is enumerated appropriately.*
(ii) *The family $\{w^{**}_\sigma : \sigma \in 2^{\mathbb{N}}\}$ is 1-unconditional.*
(iii) *For every n, if $\{t_1 \prec \cdots \prec t_{2^n}\}$ is the $\prec$-increasing enumeration of 2^n, then for every $\{\sigma_1, \ldots, \sigma_{2^n}\} \subseteq 2^{\mathbb{N}}$ with $t_i \sqsubset \sigma_i$ the families $\{w_{t_i}\}_{i=1}^{2^n}$ and $\{w^{**}_{\sigma_i}\}_{i=1}^{2^n}$ are $\left(1+\frac{1}{n}\right)$-equivalent.*
(iv) *For every n, the family $\{w_t : t \in 2^n\}$ is $\left(1+\frac{1}{n}\right)$-unconditional.*

The proof of Theorem 18 is based on Theorem 17, as well as, on the following lemmas. In the first one we use Theorem 9 in a similar way as in the proof of Lemma 11.

Lemma 19 *Let X be a separable Banach space. Let also Q be a perfect subset of $2^{\mathbb{N}}$ and $\{z^{**}_\sigma : \sigma \in Q\}$ be a bounded family in X^{**}. Assume that the map $\Phi : Q \times (B_{X^*}, w^*) \to \mathbb{R}$ defined by $\Phi(\sigma, x^*) = z^{**}_\sigma(x^*)$ is Borel. Let $n \in \mathbb{N}$ and $Q_0, \ldots, Q_n$ pairwise disjoint perfect subsets of Q. Then, for every $\varepsilon > 0$ there exist $P_0, \ldots, P_n$ perfect subsets of $Q_0, \ldots, Q_n$, respectively, such that*

$$\left| \left\| \sum_{i=0}^{n} \lambda_i z^{**}_{\sigma_i} \right\| - \left\| \sum_{i=0}^{n} \lambda_i z^{**}_{\tau_i} \right\| \right| < \varepsilon$$

for every $(\sigma_i)_{i=0}^n$ and $(\tau_i)_{i=0}^n$ in $P_0 \times \cdots \times P_n$ and every $(\lambda_i)_{i=0}^n$ in $[-1, 1]^{n+1}$.

Proof Let $\delta > 0$ sufficiently small, which will be determined later. Let $\Lambda \subseteq [-1, 1]$ and $N \subseteq [0, (n+1)M]$ be finite δ-nets, where $M > 0$ is such that $\|z^{**}_\sigma\| \leq M$ for all $\sigma \in Q$. For every $(a_i)_{i=0}^n$ in Λ^{n+1} and every $a \in N$, let

$$D(a_0, \ldots, a_n, a) = \left\{ (\sigma_0, \ldots, \sigma_n) \in Q_0 \times \cdots \times Q_n : a - \delta < \left\| \sum_{i=0}^{n} a_i z^{**}_{\sigma_i} \right\| < a + \delta \right\}.$$

Arguing as in Claim 1 in the proof Lemma 11, it is easy to verify that the set $D(a_0, \ldots, a_n, a)$ belongs to the σ-algebra generated by the analytic sets, hence, it has the Baire property in $Q_0 \times \cdots \times Q_n$. It is easy to see that for every $(a_i)_{i=0}^n$ in Λ^{n+1} we have

$$Q_0 \times \cdots \times Q_n = \bigcup_{a \in N} D(a_0, \ldots, a_n, a).$$

Applying successively Theorem 9 for every $(a_0, \ldots, a_n) \in \Lambda^{n+1}$, we obtain $P_0, \ldots, P_n$ prefect subsets of $Q_0, \ldots, Q_n$ respectively such that the following is satisfied. For every $(a_0, \ldots, a_n) \in \Lambda^{n+1}$ there exists a unique $a \in N$ such that $P_0 \times \cdots \times P_n \subseteq D(a_0, \ldots, a_n, a)$. We claim that $P_0, \ldots, P_n$ satisfy the conclusion of the lemma for a sufficiently small δ. Indeed, for every $(\sigma_i)_{i=0}^n$ and $(\tau_i)_{i=0}^n$ in $P_0 \times \cdots \times P_n$ and every $(a_i)_{i=0}^n$ in Λ^{n+1} we have

$$-2\delta \leq \left\| \sum_{i=0}^n a_i z_{\sigma_i}^{**} \right\| - \left\| \sum_{i=0}^n a_i z_{\tau_i}^{**} \right\| \leq 2\delta.$$

Using this, it is easy to check that for every $(\lambda_i)_{i=0}^n$ in $[-1, 1]^{n+1}$ we have

$$\left| \left\| \sum_{i=0}^n \lambda_i z_{\sigma_i}^{**} \right\| - \left\| \sum_{i=0}^n \lambda_i z_{\tau_i}^{**} \right\| \right| \leq 2(n+1)\delta M + 2\delta.$$

Choosing $\delta > 0$ so that $2(n+1)\delta M + 2\delta < \varepsilon$, the lemma is proved. □

Lemma 20 *Let X be a separable Banach space not containing ℓ_1. Let $n \in \mathbb{N}$ and $(z_i^{**})_{i=0}^n$ in X^{**}. For every $i \in \{0, \ldots, n\}$ let also $(e_k^i)_k$ be a sequence in X which is weak* convergent to z_i^{**}. Then, for every $\varepsilon > 0$ there exist $w_0, \ldots, w_n$ finite convex combinations of $(e_k^0)_k, \ldots, (e_k^n)_k$, respectively, such that*

$$\left| \left\| \sum_{i=0}^n \lambda_i w_i \right\| - \left\| \sum_{i=0}^n \lambda_i z_i^{**} \right\| \right| < \varepsilon$$

for every $(\lambda_i)_{i=0}^n$ in $[-1, 1]^{n+1}$.

Proof We will need the following.

Claim Let $d \in \mathbb{N}$ and $y_0^{**}, \ldots, y_d^{**}$ in X^{**}. For every $j \in \{0, \ldots, d\}$ let also $(y_k^j)_k$ be a sequence in X which is weak* convergent to y_j^{**}. Then, for every $\theta > 0$ there exist $k_0 \in \mathbb{N}$ and $\mu_0, \ldots, \mu_{k_0}$ in $[0, 1]$ with $\sum_{k=0}^{k_0} \mu_k = 1$ and such that

$$\left| \left\| \sum_{k=0}^{k_0} \mu_k y_k^j \right\| - \left\| y_j^{**} \right\| \right| < \theta \tag{2}$$

for every $j = 0, \ldots, d$.

Granting the claim, we proceed as follows. Let $\delta > 0$ sufficiently small, which we will determine later, and Λ be a finite δ-net in $[-1, 1]$. Let also $\{(a_i^0)_{i=0}^n, \ldots, (a_i^d)_{i=0}^n\}$ be an enumeration of the set Λ^{n+1}. For every $j \in \{0, \ldots, d\}$ and every $k \in \mathbb{N}$ we set $y_j^{**} = \sum_{i=0}^n a_i^j z_i^{**}$ and $y_k^j = \sum_{i=0}^n a_i^j e_k^i$. Notice that the sequence $(y_k^j)_k$ is weak* convergent to y_j^{**} for every $j \in \{0, \ldots, d\}$. We apply the above claim for $\theta = \frac{\varepsilon}{2}$ and we get $k_0 \in \mathbb{N}$ and $\mu_0, \ldots, \mu_{k_0}$ in $[0, 1]$ satisfying inequality (2) above. For every $i \in \{0, \ldots, n\}$ we set

$$w_i = \sum_{k=0}^{k_0} \mu_k e_k^i.$$

Notice that for every $j \in \{0, \ldots, d\}$ we have

$$\sum_{k=0}^{k_0} \mu_k y_k^j = \sum_{k=0}^{k_0} \mu_k \left(\sum_{i=0}^{n} a_i^j e_k^i \right) = \sum_{i=0}^{n} a_i^j \left(\sum_{k=0}^{k_0} \mu_k e_k^i \right) = \sum_{i=0}^{n} a_i^j w_i .$$

Hence, inequality (2) is reformulated as follows. For every $j \in \{0, \ldots, d\}$ we have

$$\left| \left\| \sum_{i=0}^{n} a_i w_i \right\| - \left\| \sum_{i=0}^{n} a_i z_i^{**} \right\| \right| < \frac{\varepsilon}{2}.$$

Let $M > 0$ be such that $\|e_k^i\| \leq M$ and $\|z_i^{**}\| \leq M$ for every $i \in \{0, \ldots, n\}$ and every $k \in \mathbb{N}$. It follows that for every $(\lambda_i)_{i=0}^n$ in $[-1, 1]^{n+1}$ we have

$$\left| \left\| \sum_{i=0}^{n} \lambda_i w_i \right\| - \left\| \sum_{i=0}^{n} \lambda_i z_i^{**} \right\| \right| \leq 2(n+1)\delta M + \frac{\varepsilon}{2}.$$

Hence, by choosing δ sufficiently small, the result follows.

It remains to prove the claim. For every $j \in \{0, \ldots, d\}$ we select $x_j^* \in X^*$ with $\|x_j^*\| = 1$ and such that $\|y_j^{**}\| - \frac{\theta}{4} < y_j^{**}(x_j^*)$. By [32], for every $j \in \{0, \ldots, d\}$ we may select a sequence $(x_k^j)_k$ in X satisfying the following.

(a) The sequence $(x_k^j)_k$ is weak* convergent to y_j^{**}.
(b) For every $k \in \mathbb{N}$, $\|x_k^j\| \leq \|y_j^{**}\|$.
(c) For every $k \in \mathbb{N}$, $|x_j^*(x_k^j) - y_j^{**}(x_j^*)| < \frac{\theta}{4}$.

Notice that for every convex combination w of $(x_k^j)_k$ we have that

$$\|y_j^{**}\| - \frac{\theta}{2} \leq \|w\| \leq \|y_j^{**}\|. \tag{3}$$

For every $j \in \{0, \ldots, d\}$ and every $k \in \mathbb{N}$ we set $d_k^j = y_k^j - x_k^j$. Observe that the sequence $(d_k^j)_k$ is weakly-null. Applying successively Mazur's theorem (for every j), we obtain $k_0 \in \mathbb{N}$ and $\mu_0, \ldots, \mu_{k_0}$ in $[0, 1]$ with $\sum_{k=0}^{k_0} \mu_k = 1$ and such that for every $j \in \{0, \ldots, d\}$ we have

$$\left\| \sum_{k=0}^{k_0} \mu_k d_k^j \right\| < \frac{\theta}{4}. \tag{4}$$

As

$$\left| \left\| \sum_{k=0}^{k_0} \mu_k y_k^j \right\| - \left\| \sum_{k=0}^{k_0} \mu_k x_k^j \right\| \right| \leq \left\| \sum_{k=0}^{k_0} \mu_k d_k^j \right\|$$

by inequalities (3) and (4) above, the proof of the claim follows and the lemma is proved. □

We recall that a subset I of $2^{<\mathbb{N}}$ is said to be a (finite) segment if there exist $s, t \in 2^{<\mathbb{N}}$ with $s \sqsubseteq t$ and such that $I = \{w : s \sqsubseteq w \sqsubseteq t\}$. If $I = \{w : s \sqsubseteq w \sqsubseteq t\}$ is a segment, then we set $\min I = s$ and $\max I = t$. By $\phi : 2^{<\mathbb{N}} \to \mathbb{N}$ we denote the unique bijection satisfying $\phi(s) < \phi(t)$ if either $|s| < |t|$, or $|s| = |t|$ and $s \prec t$ for all $s, t \in 2^{<\mathbb{N}}$. For every $t \in 2^{<\mathbb{N}}$ by V_t we denote the clopen subset $\{\sigma : t \sqsubset \sigma\}$ of $2^{\mathbb{N}}$. We are ready to proceed to the proof of Theorem 18.

Proof of Theorem 18 First, we start with the families $(e_t)_{t\in 2^{<\mathbb{N}}}$ and $\{z^{**}_\sigma : \sigma \in 2^{\mathbb{N}}\}$ obtained by Theorem 17. Using Theorem 4 and by passing to regular dyadic subtree if necessary, we may assume that the family $\{z^{**}_\sigma : \sigma \in 2^{\mathbb{N}}\}$ is 1-unconditional. We observe the following. For every $t \in 2^{<\mathbb{N}}$ there exists an infinite antichain $(s_n)_n$ of $2^{<\mathbb{N}}$ such that $t \sqsubset s_n$ for every $n \in \mathbb{N}$. By property (2) of Theorem 17, we see that $(e_{s_n})_n$ is weakly-null. Hence, considering the space X as a subspace of $C[0, 1]$, using a standard sliding hump argument and by passing to a dyadic (but not necessarily regular) subtree of $2^{<\mathbb{N}}$, we may assume the following.

(i) If $(t_n)_n$ is the enumeration of $2^{<\mathbb{N}}$ according to ϕ, then the sequence $(e_{t_n})_n$ is Schauder basic.

Let $(\varepsilon_n)_n$ be a decreasing sequence of positive reals converging sufficiently fast to zero. By recursion on the length of finite sequences in $2^{<\mathbb{N}}$ we shall construct the following.

(C1) A Cantor scheme $(P_t)_{t\in 2^{<\mathbb{N}}}$ of perfect subsets of $2^{\mathbb{N}}$,
(C2) a family $(I_t)_{t\in 2^{<\mathbb{N}}}$ of segments of $2^{<\mathbb{N}}$ and
(C3) a family $(w_t)_{t\in 2^{<\mathbb{N}}}$ of convex combinations of $(e_t)_{t\in 2^{<\mathbb{N}}}$.

The construction is done so that for every $t \in 2^{<\mathbb{N}}$ the following are satisfied.

(P1) w_t is a convex combination of $\{e_s : s \in I_t\}$.
(P2) $P_t \subseteq V_{\max I_t}$.
(P3) For every $\epsilon \in \{0, 1\}$ we have $\max I_t^\frown \epsilon \sqsubseteq \min I_{t^\frown \epsilon}$.
(P4) For every $s, t \in 2^{<\mathbb{N}}$ we have $|s| < |t|$ if and only if $|\max I_s| < |\min I_t|$.
(P5) For every $n \in \mathbb{N}$ and every $(\sigma_t)_{t\in 2^n}$ and $(\tau_t)_{t\in 2^n}$ in $\prod_{t\in 2^n} P_t$ we have
  (a) $(z^{**}_{\sigma_t})_{t\in 2^n}$ is $(1 + \varepsilon_n)$-equivalent to $(z^{**}_{\tau_t})_{t\in 2^n}$ and
  (b) $(z^{**}_{\sigma_t})_{t\in 2^n}$ is $(1 + \varepsilon_n)$-equivalent $(w_t)_{t\in 2^n}$.

Using Lemma's 19 and 20 one can easily realize that such a construction can be carried out.

For every $\sigma \in 2^{\mathbb{N}}$ let τ_σ be the unique element of $2^{\mathbb{N}}$ determined by the infinite chain $\bigcup_n I_{\sigma|n}$. Clearly the sequence $(w_{\sigma|n})_n$ is weak* convergent to $w^{**}_\sigma = z^{**}_{\tau_\sigma}$. It is easy to check using properties (P3), (P4) and (P5) above that the families $(w_t)_{t\in 2^{<\mathbb{N}}}$ and $\{w^{**}_\sigma : \sigma \in 2^{\mathbb{N}}\}$ are as desired. The proof is completed. □

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