Math. Ann. (2007) 339:461–484
DOI 10.1007/s00208-007-0125-z

Mathematische Annalen

Maximum principle for fully nonlinear equations via the iterated comparison function method

Shigeaki Koike · Andrzej Święch

Received: 29 August 2006 / Revised: 16 February 2007 / Published online: 1 June 2007

Abstract We present various versions of generalized Aleksandrov–Bakelman–Pucci (ABP) maximum principle for L^p-viscosity solutions of fully nonlinear second-order elliptic and parabolic equations with possibly superlinear-growth gradient terms and unbounded coefficients. We derive the results via the "iterated" comparison function method, which was introduced in our previous paper (Koike and Święch in Nonlin. Diff. Eq. Appl. **11**, 491–509, 2004) for fully nonlinear elliptic equations. Our results extend those of (Koike and Święch in Nonlin. Diff. Eq. Appl. **11**, 491–509, 2004) and (Fok in Comm. Partial Diff. Eq. **23**(5–6), 967–983) in the elliptic case, and of (Crandall et al. in Indiana Univ. Math. J. **47**(4), 1293–1326, 1998; Comm. Partial Diff. Eq. **25**, 1997–2053, 2000; Wang in Comm. Pure Appl. Math. **45**, 27–76, 1992) and (Crandall and Święch in Lecture Notes in Pure and Applied Mathematics, vol. 234. Dekker, New York, 2003) in the parabolic case.

1 Introduction

In this paper, we are concerned with maximum principles of ABP type for fully nonlinear partial differential equations (PDEs) which may have terms with super-linear

S. Koike was supported by Grant-in-Aid for Scientific Research (no. 16340032) of Japan Society for the Promotion of Science. A. Święch was supported by NSF grant DMS 0500270.

Dedicated to Hitoshi Ishii on the occasion of his 60th birthday.

S. Koike (✉)
Department of Mathematics, Saitama University, Sakura, Saitama 338-8570, Japan
e-mail: skoike@rimath.saitama-u.ac.jp

A. Święch
School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332, USA
e-mail: swiech@math.gatech.edu

growth in the first space-derivatives of solutions, and unbounded coefficients. It is known that the maximum principle fails in general for such equations in the elliptic case. In [15] an example was given for the equation

$$-\triangle u - \mu|Du|^2 = C_0 \quad \text{in } B_1$$

with certain constants $\mu, C_0 > 0$, where $B_r = \{y \in \mathbb{R}^n \ : \ |y| < r\}$ for $r > 0$. Below we present a counter-example for which the maximum principle fails even when the PDE has any super-linear nonlinearity with respect to Du.

Example 1.1 (cf. [19]) For any $m > 1$, we define $u_0 \in C^2(B_2)$ by

$$u_0(x) = \begin{cases} -(2-|x|)^{-\alpha} & \text{for } 1 \le |x| < 2, \\ -\frac{\alpha^2}{8}|x|^4 - (\frac{\alpha}{2} + \frac{\alpha^2}{4})|x|^2 - 1 + \frac{\alpha}{2} + \frac{3\alpha^2}{8} & \text{for } |x| < 1. \end{cases}$$

It is easy to verify that

$$-\triangle u_0 - \mu|Du_0|^m \le 0 \quad \text{in } B_2 \setminus B_1$$

when $\alpha > (2-m)^+/(m-1)$ and $\mu = (\alpha + 2n - 1)\alpha^{1-m}$. Thus, putting $u_k(x) = u_0((1-\frac{1}{k})x)$ for $k > 2$, we verify

$$-\triangle u_k - \mu(1-\frac{1}{k})^{2-m}|Du_k|^m \le 0 \quad \text{in } B_2 \setminus B_{\frac{k}{k-1}}.$$

Hence, we find a constant $C > 0$, independent of k, such that $-\triangle u_k - \mu(1-\frac{1}{k})^{2-m}|Du_k|^m \le C$ in B_2. Finally, setting $v_k(x) = \mu^{1/(m-1)}(1-\frac{1}{k})^{(2-m)/m}\{u_k(x) + (\frac{k}{2})^\alpha\}$ for $k \ge 2$, we find $C' > 0$, independent of $k \ge 2$, such that

$$-\triangle v_k - |Dv_k|^m \le C' \quad \text{in } B_2,$$

and $v_k = 0$ on ∂B_2. However, $\sup_{B_2} v_k \ge v_k(0) \to \infty$ as $k \to \infty$.

In what follows, for fixed uniform ellipticity constants $0 < \lambda \le \Lambda$, we denote by $\mathcal{P}^-(X)$ the Pucci extremal operator defined as $\mathcal{P}^-(X) = \min\{-\text{trace}(AX) \ : \ \lambda I \le A \le \Lambda I, A \in S^n\}$, where S^n is the set of $n \times n$ symmetric matrices with the standard ordering. The other Pucci extremal operator $\mathcal{P}^+(X)$ is defined by $\mathcal{P}^+(X) = -\mathcal{P}^-(-X)$.

Throughout this paper, we consider PDEs in a bounded domain $\Omega \subset \mathbb{R}^n$ or $Q = \Omega \times (0, T]$ for parabolic problems. Moreover, for the sake of simplicity, we assume

$$\Omega \subset B_1 \text{ (i.e. diam}(\Omega) \le 2), \quad \text{and} \quad 0 < T \le 1.$$

It is easy to restate all of the results of the paper for a general domain by the standard scaling argument. The resulting estimates would then contain terms involving diam(Ω) or diam(Q) in the parabolic case.

We will present maximum principles for equations

$$\mathcal{P}^-(D^2u) - \mu(x)|Du|^m = f(x) \quad \text{in } \Omega, \tag{1.1}$$

and

$$\frac{\partial u}{\partial t} + \mathcal{P}^-(D^2u) - \mu(x,t)|Du|^m = f(x,t) \quad \text{in } Q = \Omega \times (0,T]. \tag{1.2}$$

Since these PDEs arise in the study of fully nonlinear PDEs with measurable terms and integrable right hand sides, it is natural to work with L^p-viscosity solutions which have been defined in [6]. The basis for our analysis is the iterated comparison function method which we introduced in [15]. We will show several results.

For the elliptic PDE (1.1) we will first consider the case when $m = 1$ and we will extend the result of [12] (see also [13]). When $m > 1$, we will show that the maximum principle holds provided that μ or f is small enough in a certain norm. This will generalize a result of [15].

For the parabolic PDE (1.2), we will first examine the case when $\mu \in L^\infty(Q)$. Here we will establish the maximum principle even when $m > 1$ with no smallness assumption on μ and f. This is precisely the case when the maximum principle fails for elliptic equation (1.1). Therefore, perhaps as it was expected, parabolic equations behave much better in this respect.

We will next establish the maximum principle in the case when $m = 1$ and $\mu \in L^q(Q)$ with $q > n+2$. Finally we will study the case when $m > 1$ and $\mu \in L^q(Q)$ with $q > n+2$. Here, as for elliptic PDEs, to prove the maximum principle we have to assume that either f or μ is small in some L^p norm.

The maximum principles obtained by the iterated comparison function method are slightly weaker than the classical ABP and Aleksandrov-Bakelman-Pucci-Krylov-Tso ones. The difference is in the fact that the L^p norms of f appearing there must be taken over the whole sets Ω and Q, not just over the contact sets. We refer the reader to [14,17,18] for more on this and to Aleksandrov [1], Bakelman [2] and Krylov [16] for pioneering works on the classical ABP maximum principles for elliptic and parabolic PDE.

Throughout the paper, when we discuss a function $g : U \to \mathbb{R}$ in a larger domain $\tilde{U}$, unless said otherwise, g will always denote the zero extension of itself to $\tilde{U}$. We will denote by $L^p_+(U)$ the set of all nonnegative functions in $L^p(U)$. Finally we will often write $\|\cdot\|_p$ $(1 \le p \le \infty)$ instead of $\|\cdot\|_{L^p(\Omega)}$ and $\|\cdot\|_{L^p(Q)}$.

2 Elliptic equations

We recall the definition of L^p-viscosity solution of

$$F(x, u, Du, D^2u) = f(x) \quad \text{in } \Omega, \tag{2.1}$$

where $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times S^n \to \mathbb{R}$ and $f : \Omega \to \mathbb{R}$ are given measurable functions, and F is continuous in the last three variables.

Definition 2.1 We call $u \in C(\Omega)$ an L^p-viscosity subsolution (resp., supersolution) of (2.1) if

$$ess \liminf_{y \to x}\{F(y, u(y), D\phi(y), D^2\phi(y)) - f(y)\} \le 0$$
$$\big(\text{respectively,}\quad ess \limsup_{y \to x}\{F(y, u(y), D\phi(y), D^2\phi(y)) - f(y)\} \ge 0\big)$$

whenever $\phi \in W^{2,p}_{\mathrm{loc}}(\Omega)$ and $x \in \Omega$ is a local maximum (resp., minimum) point of $u - \phi$.

The function $u \in C(\Omega)$ is an L^p-viscosity solution of (2.1) if it is an L^p-viscosity subsolution and an L^p-viscosity supersolution of (2.1).

We call $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$ an L^p-strong solution of (2.1) if u satisfies

$$F(x, u(x), Du(x), D^2u(x)) = f(x) \quad \text{a.e. in } \Omega.$$

Remark 2.2 We remark that in the above definition, we do not assume that $f \in L^p(\Omega)$. Moreover if u is an L^p-viscosity subsolution of (2.1), then it is also an L^q-viscosity subsolution of (2.1) provided $q \ge p$.

We first establish the maximum principle for (1.1) with $m = 1$, i.e. for

$$\mathcal{P}^-(D^2u) - \mu(x)|Du| = f(x) \quad \text{in } \Omega. \tag{2.2}$$

The following version of the classical ABP maximum principle can be easily deduced from its proof in [14] after the linearization of (2.2).

Proposition 2.3 (cf. [14]) *There exist $C_k = C_k(n, \lambda, \Lambda) > 0$ $(k = 1, 2)$ such that if $f, \mu \in L^n_+(\Omega)$, and $u \in C(\overline{\Omega}) \cap W^{2,n}_{\mathrm{loc}}(\Omega)$ is an L^n-strong subsolution of (2.2), then*

$$\sup_{\Omega} u \le \sup_{\partial\Omega} u + C_1 \exp(C_2\|\mu\|_n)\|f\|_n. \tag{2.3}$$

Remark 2.4 In the above statement, one can replace $\|f\|_n$ by $\|f\|_{L^n(\Gamma[u])}$, where $\Gamma[u]$ is the upper contact set of u in Ω. See [14] for the definition of $\Gamma[u]$. We also note that it is trivial to obtain from Proposition 2.3 the corresponding result for L^p-strong supersolutions of

$$\mathcal{P}^+(D^2u) + \mu(x)|Du| = f(x) \quad \text{in } \Omega$$

since $v = -u$ is a subsolution of

$$\mathcal{P}^-(D^2v) - \mu(x)|Dv| = -f(x) \quad \text{in } \Omega.$$

All results of the paper are only stated for L^p-viscosity subsolutions. The corresponding results for supersolutions can be derived by the above reduction.

It is known (see [3,10,11,13]) that there exists $p_0 = p_0(n, \Lambda/\lambda)$ satisfying $n/2 \le p_0 < n$ such that for $p > p_0$ there is a constant $C = C(n, p, \lambda, \Lambda)$ such that if $f \in L^p(\Omega)$ and $u \in C(\overline{\Omega}) \cap W^{2,p}_{\text{loc}}(\Omega)$ is an L^p-strong subsolution of

$$\mathcal{P}^-(D^2u) = f(x) \quad \text{in } \Omega,$$

then

$$\sup_{\Omega} u \le \sup_{\partial\Omega} u + C\|f^+\|_p. \tag{2.4}$$

The basis of the iterated comparison function method used in this paper is the following result about the solvability of extremal equations (see [6]).

Proposition 2.5 *Let $p > p_0$ and let Ω satisfy the uniform exterior cone condition. There exists $C = C(n, p, \lambda, \Lambda) > 0$ such that for $f \in L^p(\Omega)$, there is an L^p-strong solution $v \in C(\overline{\Omega}) \cap W^{2,p}_{\text{loc}}(\Omega)$ of*

$$\mathcal{P}^+(D^2v) = f(x) \quad \textit{in } \Omega$$

such that $v = 0$ on $\partial\Omega$, and

$$-C\|f^-\|_p \le v \le C\|f^+\|_p \quad \textit{in } \Omega.$$

Moreover, for each open set $\Omega' \Subset \Omega$, there is $C' = C'(n, p, \lambda, \Lambda, \text{dist}(\Omega', \partial\Omega)) > 0$ such that

$$\|v\|_{W^{2,p}(\Omega')} \le C'\|f\|_p.$$

We need to obtain Proposition 2.3 for viscosity solutions. To achieve this we first have to prove a result about strong solvability of extremal inequalities. The proposition below is a restatement of Lemma 2.11 of [12] even though the assumption that $\text{supp}\mu \Subset \Omega$ may be more restrictive than that of [12]. However we will need our version of the result in later sections of the paper. The proof of Proposition 2.6 in large parts repeats the arguments of the proof of Lemma 2.11 of [12], which in turn was just a modification of the proof of Lemma 3.1 of [6]. However we correct here some small mistakes made in the proof of Lemma 2.11 of [12].

Proposition 2.6 *Let Ω satisfy the uniform exterior cone condition, for*

$$q \ge p > n \quad \textit{or} \quad q > p = n, \tag{2.5}$$

$f \in L^p(\Omega)$, and let $\psi \in C(\partial\Omega)$. Let $\mu \in L^q_+(\Omega)$ satisfy $\text{supp}\mu \Subset \Omega$. Then there exist strong solutions $u, v \in C(\overline{\Omega}) \cap W^{2,p}_{\text{loc}}(\Omega)$ of

$$\mathcal{P}^-(D^2u) - \mu(x)|Du| \ge f(x), \quad \mathcal{P}^+(D^2v) + \mu(x)|Dv| \le f(x) \quad \text{in } \Omega$$

such that $u = v = \psi$ on $\partial\Omega$. Moreover we have

$$\|u\|_{L^\infty(\Omega)}, \|v\|_{L^\infty(\Omega)} \leq \|\psi\|_{L^\infty(\partial\Omega)} + C_1 \exp(C_2\|\mu\|_n)\|f\|_n, \tag{2.6}$$

where C_1 and C_2 are constants from Proposition 2.1, and for every $\Omega' \Subset \Omega$,

$$\|u\|_{W^{2,p}(\Omega')}, \|v\|_{W^{2,p}(\Omega')} \leq C(n, p, \lambda, \Lambda, \|\mu\|_{L^q(\Omega)}, \mathrm{dist}\,(\Omega', \partial\Omega))(\|\psi\|_{L^\infty(\partial\Omega)} + \|f\|_{L^p(\Omega)}). \tag{2.7}$$

Proof We will only prove the result for subsolutions as the proof for supersolutions is similar.

First, we suppose $q \geq p > n$ from (2.5). Let $\mu_j \in C(\Omega)$ be such that $\mu_j \to \mu$ in $L^q(\Omega)$ and pointwise a.e. Let $u_j \in C(\overline{\Omega}) \cap W^{2,p}_{\mathrm{loc}}(\Omega)$ be the unique strong solution of

$$\mathcal{P}^-(D^2u_j) - \mu_j(x)|Du_j| = f(x) \quad \text{in } \Omega \tag{2.8}$$

such that $u = \psi$ on $\partial\Omega$. The existence of such strong solutions follows for instance from Corollary 3.10 of [6] or Theorem 3.1 of [20]. By Proposition 2.3, (2.3) holds for u_j with μ replaced by μ_j. Since $\mu_j \to \mu$ in $L^q(\Omega)$, we can assume that it holds with μ.

Now, since we can cover Ω' by a finite number of balls having fixed radius R it is enough to show (2.7) for the u_j for B_R instead of Ω'. We will denote the measure of B_R by $|B_R| = \omega_n R^n$, where ω_n is the measure of B_1. Let $\rho \in (0, 1)$ and $\eta \in C^2_0(B_R)$ be such that $0 \leq \eta \leq 1$, $\eta = 1$ in $B_{\rho R}$, $\eta = 0$ for $|x| \geq \tilde{\rho}R$, where $\tilde{\rho} = (1+\rho)/2$, and

$$|D\eta| \leq \frac{4}{(1-\rho)R}, \quad \|D^2\eta\| \leq \frac{16}{(1-\rho)^2R^2}.$$

Then setting $v = \eta u_j \in W^{2,p}(B_R)$ (i.e. $\mathrm{supp} v \subset B_{\tilde{\rho}R}$), and therefore using the estimates of [5] (see also [4]), we have

$$\|\eta u_j\|_{W^{2,p}(B_{\tilde{\rho}R})} \leq C_1\|\mathcal{P}^-(D^2(\eta u_j))\|_{L^p(B_{\tilde{\rho}R})},$$

which implies

$$\|D(\eta u_j)\|_{L^\infty(B_{\tilde{\rho}R})} \leq C_2\|\eta u_j\|_{W^{2,p}(B_{\tilde{\rho}R})} \leq C_1C_2\|\mathcal{P}^-(D^2(\eta u_j))\|_{L^p(B_{\tilde{\rho}R})}. \tag{2.9}$$

Hence we have

$$\begin{aligned}\|D^2u_j\|_{L^p(B_{\rho R})} &\leq \|D^2(\eta u_j)\|_{L^p(B_{\tilde{\rho}R})} \leq C_1C_2\|\mathcal{P}^-(D^2(\eta u_j))\|_{L^p(B_{\tilde{\rho}R})}\\ &= C_1C_2\|\mathcal{P}^-(\eta D^2u_j) + 2D\eta \otimes Du_j + u_jD^2\eta\|_{L^p(B_{\tilde{\rho}R})}\\ &\leq C_3\left(\|\eta\mathcal{P}^-(D^2u_j)\|_{L^p(B_{\tilde{\rho}R})} + \frac{1}{(1-\rho)R}\|Du_j\|_{L^p(B_{\tilde{\rho}R})} + \frac{1}{(1-\rho)^2R^2}\|u_j\|_{L^p(B_{\tilde{\rho}R})}\right).\end{aligned} \tag{2.10}$$

It follows from the equation that

$$
\begin{aligned}
&C_3\|\eta\mathcal{P}^-(D^2u_j)\|_{L^p(B_{\tilde\rho R})} \le C_4\|f\|_{L^p(B_{\tilde\rho R})} + C_4\|\eta\mu_j Du_j\|_{L^p(B_{\tilde\rho R})}\\
&\le C_4\|f\|_{L^p(B_{\tilde\rho R})} + C_4\|\mu_j D(\eta u_j)\|_{L^p(B_{\tilde\rho R})} + C_4\|\mu_j\|_{L^p(B_{\tilde\rho R})}\frac{\|u_j\|_{L^\infty(\Omega)}}{(1-\rho)R}\\
&\le C_4\|f\|_{L^p(B_{\tilde\rho R})} + C_4\|\mu_j D(\eta u_j)\|_{L^p(B_{\tilde\rho R})} + C_5\|\mu_j\|_{L^p(B_{\tilde\rho R})}\frac{\|\psi\|_{L^\infty(\partial\Omega)}+\|f\|_{L^p(\Omega)}}{(1-\rho)R}.
\end{aligned}
\tag{2.11}
$$

We now estimate

$$
\begin{aligned}
C_4\|\mu_j D(\eta u_j)\|_{L^p(B_{\tilde\rho R})} &\le C_4\|\mu_j\|_{L^p(B_{\tilde\rho R})}\|D(\eta u_j)\|_{L^\infty(B_{\tilde\rho R})}\\
&\le C_4C_6\|\mu_j\|_{L^p(B_{\tilde\rho R})}\|\mathcal{P}^-(D^2(\eta u_j))\|_{L^{\frac{n+p}{2}}(B_{\tilde\rho R})}\\
&\le C_4C_6(\omega_n R^n)^{\frac{(p-n)}{(n+p)p}}\|\mu_j\|_{L^p(B_{\tilde\rho R})}\|\mathcal{P}^-(D^2(\eta u_j))\|_{L^p(B_{\tilde\rho R})}.
\end{aligned}
\tag{2.12}
$$

Therefore by choosing R small enough, we can guarantee in (2.12) that

$$
C_4\|\mu_j D(\eta u_j)\|_{L^p(B_{\tilde\rho R})} \le \frac{C_1}{2}\|\mathcal{P}^-(D^2(\eta u_j))\|_{L^p(B_{\tilde\rho R})}. \tag{2.13}
$$

It now follows from (2.10), (2.11), and (2.13) that

$$
\begin{aligned}
\frac{C_1}{2}\|\mathcal{P}^-(D^2(\eta u_j))\|_{L^p(B_{\tilde\rho R})} \le C_4\|f\|_{L^p(\Omega)} + \frac{C_7}{(1-\rho)^2R^2}\Big((1-\rho)R(\|\psi\|_{L^\infty(\partial\Omega)}\\
+\|f\|_{L^p(\Omega)}+\|Du_j\|_{L^p(B_{\tilde\rho R})})+\|u_j\|_{L^p(B_{\tilde\rho R})}\Big).
\end{aligned}
$$

Hence, again from (2.10), we obtain

$$
\begin{aligned}
(1-\rho)^2R^2\|D^2u_j\|_{L^p(B_{\rho R})} &\le C_8(\|\psi\|_{L^\infty(\partial\Omega)}+\|f\|_{L^p(\Omega)})\\
&\quad + C_8\Big((1-\tilde\rho)R\|Du_j\|_{L^p(B_{\tilde\rho R})}+\|u_j\|_{L^p(B_{\tilde\rho R})}\Big).
\end{aligned}
\tag{2.14}
$$

If we introduce norms

$$
\Psi_k(v) = \sup_{0<\rho<1}(1-\rho)^kR^k\|D^kv\|_{L^p(B_{\rho R})}, \quad k=0,1,2,
$$

then (2.14) gives the inequality

$$
\Psi_2(u_j) \le C_8(\|\psi\|_{L^\infty(\partial\Omega)} + \|f\|_{L^p(\Omega)}) + C_8\left(\Psi_1(u_j)+\Psi_0(u_j)\right).
$$

The required estimate follows from the interpolation inequality

$$
\Psi_1 \le \varepsilon\Psi_2 + \frac{C}{\varepsilon}\Psi_0,
$$

which may be found in [14].

Therefore there exists $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$ such that $u_j \rightharpoonup u$ in $W^{2,p}_{\mathrm{loc}}(\Omega)$ as $j \to \infty$. Passing to a subsequence if necessary, we see that $Du_j \to Du$ a.e. (in this case, the Sobolev embedding yields the local uniform convergence). Thus this implies that $\mu_j|Du_j| \to \mu|Du|$ a.e. as $j \to \infty$. Since $\mathcal{P}^-$ is concave, we have for a.e. x,

$$\begin{aligned}\mathcal{P}^-(D^2u) &\geq \limsup_{j\to\infty} \mathcal{P}^-(D^2u_j)\\ &= \limsup_{j\to\infty}\left(\mathcal{P}^-(D^2u_j) - \mu_j(x)|Du_j| + \mu_j(x)|Du_j|\right)\\ &= f(x) + \lim_{j\to\infty} \mu_j(x)|Du_j| = f(x) + \mu(x)|Du|.\end{aligned}$$

Obviously u satisfies (2.6) and (2.7). It remains to show that $u \in C(\overline{\Omega})$. By the superadditivity of $\mathcal{P}^-$, we have for $i, j \geq 1$

$$\begin{cases}\mathcal{P}^-(D^2(u_i - u_j)) = \mu_i(x)|Du_i| - \mu_j(x)|Du_j| + f_i(x) - f_j(x) & \text{in } \Omega\\ u_i - u_j = 0 \quad \text{on } \partial\Omega\end{cases}$$

Since $\mathrm{supp}\mu \Subset \Omega$, we may assume that $\mathrm{supp}(\mu_j) \subset \Omega'$ for some $\Omega' \Subset \Omega$ and for all $j \geq 1$. It is enough to show that

$$\|\mu_i(x)|Du_i| - \mu_j(x)|Du_j|\|_{L^p(\Omega)} \to 0 \quad \text{as } i, j \to \infty \tag{2.15}$$

since the maximum principle will give us that $\sup(u_i - u_j) \to 0$ and we can obviously obtain a symmetrical estimate by switching u_i and u_j. But (2.15) is obvious since

$$\begin{aligned}&\|\mu_i|Du_i| - \mu_j|Du_j|\|_{L^p(\Omega)} \leq \|(\mu_i - \mu_j)|Du_i|\|_{L^p(\Omega')} + \|\mu_j|Du_i - Du_j|\|_{L^p(\Omega')}\\ &\leq \|\mu_i - \mu_j\|_{L^p(\Omega')}\|Du_i\|_{L^\infty(\Omega')} + \|\mu_j\|_{L^p(\Omega')}\|Du_i - Du_j\|_{L^\infty(\Omega')}\\ &\leq C\left(\|\mu_i - \mu_j\|_{L^p(\Omega)} + \|Du_i - Du_j\|_{L^\infty(\Omega')}\right) \to 0.\end{aligned} \tag{2.16}$$

This completes the proof.

Next, under the other condition (2.5), $q > p = n$, there arises only one major difference in the above argument. In fact, to obtain the estimate (2.12), we cannot use $\|D(\eta u_j)\|_{L^\infty(B_{\tilde{\rho}R})}$. Instead, recalling that for each $r > 1$, there is $C_r > 0$ such that

$$\|D(\eta u_j)\|_{L^r(B_{\tilde{\rho}R})} \leq C_r\|\mathcal{P}^-(D^2(\eta u_j))\|_{L^n(B_{\tilde{\rho}R})}, \tag{2.17}$$

we obtain

$$\|\mu_j D(\eta u_j)\|_{L^n(B_{\tilde{\rho}R})} \leq \|\mu\|_{L^q(B_{\tilde{\rho}R})}\|D(\eta u_j)\|_{L^{\hat{r}}(B_{\tilde{\rho}R})},$$

where $\hat{r} = nq/(q-n)$. Thus, fixing $r > \hat{r}$ and using (2.17), we find $s > 0$ such that

$$\|\mu_j D(\eta u_j)\|_{L^n(B_{\tilde{\rho}R})} \leq C_r(\omega_n R^n)^s\|\mu\|_{L^q(B_{\tilde{\rho}R})}\|\mathcal{P}^-(D^2(\eta u_j))\|_{L^n(B_{\tilde{\rho}R})}.$$

Therefore, we obtain (2.13) for small $R > 0$. We can then use the same argument with an obvious modification in (2.16) to get (2.15), which concludes the proof in this case. □

Remark 2.7 In the case when $q > n > p > p_0$, we can obtain the $W^{2,p}_{loc}$ estimate (2.12) if the L^∞-estimate on u_j is known. However, we have not yet established the ABP maximum principle for u_j under this hypothesis. We will discuss this case later.

Proposition 2.8 *Assume* (2.5). *There exist* $C_k = C_k(n, \lambda, \Lambda) > 0$ $(k = 1, 2)$ *such that if* $f \in L^p_+(\Omega)$, $\mu \in L^q_+(\Omega)$, *and* $u \in C(\overline{\Omega})$ *is an* L^p*-viscosity subsolution of* (2.2), *then*

$$\sup_\Omega u \le \sup_{\partial\Omega} u + C_1 \exp(C_2 \|\mu\|_n)\|f\|_n. \tag{2.18}$$

Proof Let $\varepsilon > 0$. By Proposition 2.6 we can find a strong solution $v \in C(\overline{B}_2) \cap W^{2,p}_{\text{loc}}(B_2)$ of

$$\mathcal{P}^+(D^2 v) + \mu(x)|Dv| \le -f(x) - \varepsilon \quad \text{in } B_2, \quad v = 0 \text{ on } \partial B_2$$

such that

$$\|v\|_{L^\infty(B_2)} \le C_1 \exp(C_2\|\mu\|_n)(\|f\|_n + \varepsilon).$$

Then $w = u + v$ is an L^p-viscosity subsolution of

$$\mathcal{P}^-(D^2 w) - \mu(x)|Dw| = -\varepsilon \quad \text{in } \Omega.$$

Hence, by the definition of L^p-viscosity solutions, we have

$$\sup_\Omega w \le \sup_{\partial\Omega} w.$$

This implies that

$$\sup_\Omega u \le \sup_{\partial\Omega} u + 2 \sup_\Omega |v|$$

and the result follows upon letting $\varepsilon \to 0$. □

2.1 Linear growth (i.e. $m = 1$)

Our first result extends that of [12] and [13].

Theorem 2.9 *Let* $p_0 < p < n < q$ *and* $m = 1$. *There exist an integer* $N = N(n, p, q)$ *and* $C = C(n, \lambda, \Lambda, p, q) > 0$ *such that if* $f \in L^p_+(\Omega)$, $\mu \in L^q_+(\Omega)$, *and* $u \in C(\overline{\Omega})$ *is an* L^p*-viscosity subsolution of* (2.2), *then*

$$\sup_\Omega u \le \sup_{\partial\Omega} u + C \left\{ \exp(C\|\mu\|_n)\|\mu\|_q^N + \sum_{k=0}^{N-1} \|\mu\|_q^k \right\} \|f\|_p.$$

Proof We may assume that that $q < \infty$. (The case $q = \infty$ is well known, see for instance [9].) We set $\varepsilon = q - n > 0$, $q_0 = p$, and $q_k = nq_{k-1}q/((n - q_{k-1})q + q_{k-1}n)$ for $k \geq 1$. Since $q_k - q_{k-1} = \varepsilon q_{k-1}^2/(nq - \varepsilon q_{k-1}) \geq n\varepsilon/(4n + 2\varepsilon) > 0$ for $k \geq 1$ as long as $q_{k-1} < n$, there exists an integer $N \geq 1$ such that $q_{N-1} < n \leq q_N < q$.

Fix $R_1 > \cdots > R_N > 1$. By Proposition 2.5 we find an L^p-strong solution $v_1 \in C(\overline{B}_{R_1}) \cap W^{2,p}(B_{R_2})$ of

$$\mathcal{P}^+(D^2 v_1) = -f(x) \quad \text{in } B_{R_1}$$

such that $v_1 = 0$ on ∂B_{R_1}, $0 \leq -v_1 \leq C\|f\|_p$ in B_{R_1}. By the Sobolev embedding

$$\|Dv_1\|_{L^{p^*}(B_{R_2})} \leq C\|f\|_p. \tag{2.19}$$

Here and later, for $n > p > 1$,

$$p^* = \frac{np}{n-p} > 0.$$

We will also use $C > 0$ to denote various universal constants.

Since $\mathcal{P}^-(X + Y) \leq \mathcal{P}^-(X) + \mathcal{P}^+(Y)$ for $X, Y \in S^n$, by setting $w_1 = u + v_1$ in Ω, it is easy to see that w_1 is an L^p-viscosity subsolution of

$$\mathcal{P}^-(D^2 w_1) - \mu(x)|Dw_1| = \mu(x)|Dv_1(x)| =: f_2(x) \quad \text{in } \Omega.$$

Inequality (2.19) and the Hölder inequality yield

$$\|f_2\|_{L^{q_1}(B_{R_2})} \leq \|\mu\|_q \|Dv_1\|_{L^{p^*}(B_{R_2})} \leq C\|\mu\|_q \|f\|_p.$$

We next take the strong solution $v_2 \in C(\overline{B}_{R_2}) \cap W^{2,q_1}(B_{R_3})$ of

$$\mathcal{P}^+(D^2 v_2) = -f_2(x) \quad \text{in } B_{R_2}$$

such that $v_2 = 0$ on ∂B_{R_2}. Then $0 \leq -v_2 \leq C\|f_2\|_{L^{q_1}(B_{R_2})}$ in B_{R_2} and

$$\|Dv_2\|_{L^{q_1^*}(B_{R_3})} \leq C\|\mu\|_q \|f\|_p.$$

We then see that $w_2 := w_1 + v_2$ is an L^p-viscosity subsolution of

$$\mathcal{P}^-(D^2 w_2) - \mu(x)|Dw_2| = \mu(x)|Dv_2(x)| =: f_3(x) \quad \text{in } \Omega.$$

It is easy to verify that

$$\|f_3\|_{L^{q_2}(B_{R_3})} \leq \|\mu\|_q \|Dv_2\|_{L^{q_1^*}(B_{R_2})} \leq C\|\mu\|_q^2 \|f\|_p.$$

Hence we inductively choose strong solutions $v_k \in C(\overline{B}_{R_k}) \cap W^{2,q_{k-1}}(B_{R_{k+1}})$ of

$$\mathcal{P}^+(D^2 v_k(x)) = -\mu(x)|Dv_{k-1}(x)| =: f_k(x) \quad \text{in } B_{R_k}$$

such that $v_k = 0$ on ∂B_{R_k}. As before we have

$$0 \le -v_k \le C\|f_k\|_{L^{q_{k-1}}(B_{R_k})} \le C\|\mu\|_q^{k-1}\|f\|_p \quad \text{in } B_{R_k}, \tag{2.20}$$

and

$$\|\mu D v_k\|_{L^{q_k}(B_{R_{k+1}})} \le \|\mu\|_q \|Dv_k\|_{L^{q^*_{k-1}}(B_{R_{k+1}})} \le C\|\mu\|_q^k \|f\|_p.$$

Since $w_N := u + \sum_{k=1}^N v_k$ is an L^p-viscosity subsolution of

$$\mathcal{P}^-(D^2 w_N) - \mu(x)|Dw_N| = \mu(x)|Dv_N(x)| =: \hat{f}(x) \quad \text{in } \Omega$$

and $\hat{f} \in L^{q_N}(\Omega)$ for $q_N \ge n$, by Proposition 2.8 we have

$$\sup_{\Omega} w_N \le \sup_{\partial\Omega} w_N + C\exp(C\|\mu\|_n)\|\hat{f}\|_n.$$

Since $\|\hat{f}\|_{L^n(\Omega)} \le C\|\hat{f}\|_{q_N} \le C\|\mu\|_q^N \|f\|_p$, it then follows that

$$\sup_{\Omega} u \le \sup_{\partial\Omega} w_N + \sum_{k=1}^N \sup_{\Omega}(-v_k) + C\exp(C\|\mu\|_n)\|\mu\|_q^N \|f\|_p.$$

It now remains to use (2.20) to finish the proof. □

Remark 2.10 As pointed out in Remark 2.7, noting that we only used Proposition 2.8 in the proof above, if we replace (2.5) by

$$q > n > p > p_0,$$

we obtain the L^∞-estimate (2.3) for u_j in the proof of Proposition 2.6. The remaining $W^{2,p}_{loc}$-esitmate can be obtained by the same argument as in the proof of Proposition 2.6.

2.2 Super-linear growth (i.e. $m > 1$)

In this subsection, for a fixed $m > 1$, we consider the PDE

$$\mathcal{P}^-(D^2 u) - \mu(x)|Du|^m = f(x) \quad \text{in } \Omega. \tag{2.21}$$

In light of Example 1.1 in order to show the maximum principle for (2.21) we need some restrictions as in [15]. We first present a result which corresponds to Proposition 2.8.

Theorem 2.11 *Let $n < p$ and $m > 1$. There exist $\delta = \delta(n, \lambda, \Lambda, m, p) > 0$ and $C = C(n, \lambda, \Lambda, m, p) > 0$ such that if $f \in L^p_+(\Omega)$, $\mu \in L^p_+(\Omega)$,*

$$\|f\|_p^{m-1}\|\mu\|_p < \delta, \tag{2.22}$$

and $u \in C(\overline{\Omega})$ is an L^p-viscosity subsolution of (2.21), then

$$\sup_\Omega u \le \sup_{\partial\Omega} u + C\left(\|f\|_p + \|f\|_p^m\|\mu\|_p\right).$$

Proof Fix $1 < R_2 < R_1$. By Proposition 2.5 we can find an L^p-strong solution $v \in C(\overline{B}_{R_1}) \cap W^{2,p}(B_{R_2})$ of

$$\mathcal{P}^+(D^2 v_1) = -f(x) \quad \text{in } B_{R_1}$$

such that $v_1 = 0$ on ∂B_{R_1}, and $0 \le -v_1 \le C_1\|f\|_p$ in B_{R_1}. Moreover, by the Sobolev embedding,

$$\|Dv_1\|_{L^\infty(B_{R_2})} \le C_2\|f\|_p.$$

We notice that $w_1 := u + v_1$ is an L^p-viscosity subsolution of

$$\mathcal{P}^-(D^2 w_1) - 2^{m-1}\mu(x)|Dw_1|^m = 2^{m-1}\mu(x)\|Dv_1\|^m_{L^\infty(B_{R_1})} \quad \text{in } \Omega.$$

Next, for every $\varepsilon > 0$, we take the L^p-strong solution $\zeta_\varepsilon \in C(\overline{B}_{R_2}) \cap W^{2,p}(\Omega)$ of

$$\mathcal{P}^+(D^2\zeta_\varepsilon) = -(2^{m-1}C_2^m + 1)\|f\|_p^m\mu(x) - \varepsilon \quad \text{in } B_{R_2},$$

such that $\zeta_\varepsilon = 0$ on ∂B_{R_2}. Again $0 \le -\zeta_\varepsilon \le C_3(\|f\|_p^m\|\mu\|_p + \varepsilon)$ in B_{R_2}, and

$$\|D\zeta_\varepsilon\|_{L^\infty(\Omega)} \le C_4(\|f\|_p^m\|\mu\|_p + \varepsilon). \tag{2.23}$$

Thus, setting $W_\varepsilon = w_1 + \zeta_\varepsilon$, we verify that W_ε is an L^p-viscosity subsolution of

$$\mathcal{P}^-(D^2 W_\varepsilon) - 2^{2(m-1)}\mu(x)|DW_\varepsilon|^m = \mu(x)(2^{2(m-1)}|D\zeta_\varepsilon(x)|^m - \|f\|_p^m) - \varepsilon \quad \text{in } \Omega.$$

By (2.23), we find $C_5 > 0$ such that the right hand side of the above is estimated from above by

$$\mu(x)(C_5(\|f\|_p^m\|\mu\|_p + \varepsilon)^m - \|f\|_p^m) - \varepsilon.$$

Hence, for $\delta := 1/C_5^{1/m} > 0$, if (2.22) holds, we see that, for sufficiently small $\varepsilon > 0$, W_ε is an L^p-viscosity subsolution of

$$\mathcal{P}^-(D^2 W_\varepsilon) - 2^{2(m-1)}\mu(x)|DW_\varepsilon|^m = -\varepsilon \quad \text{in } \Omega.$$

This obviously implies that $W_\varepsilon \le \sup_{\partial\Omega} W_\varepsilon$ in Ω, and so we obtain that

$$\sup_\Omega u \le \sup_{\partial\Omega} W_\varepsilon + \sup_\Omega(-v_1) + \sup_\Omega(-\zeta_\varepsilon) \le \sup_{\partial\Omega} u + C(\|f\|_p + \|f\|_p^m \|\mu\|_p + \varepsilon).$$

Thus, the conclusion follows by letting $\varepsilon \to 0$. □

Following the argument used in the proof of Theorem 2.9, we can now extend Theorem 2.11 to the case when $p \in (p_0, n]$.

Theorem 2.12 *Let $p_0 < p \le n < q$ and $m > 1$. Denote $a_0 = 0$ and $a_k = 1 + m + \cdots + m^{k-1}$ for $k \ge 1$. There exist an integer $N = N(n, m, p, q) \ge 1$, $\delta = \delta(n, \lambda, \Lambda, m, p, q) > 0$ and $C = C(n, \lambda, \Lambda, m, p, q) > 0$ such that if $f \in L_+^p(\Omega)$, $\mu \in L_+^q(\Omega)$,*

$$p > \frac{nq(m-1)}{mq-n}, \tag{2.24}$$

$$\|f\|_p^{m^N(m-1)} \|\mu\|_q^{a_N(m-1)+1} < \delta, \tag{2.25}$$

and $u \in C(\overline{\Omega})$ is an L^p-viscosity subsolution of (2.21), then

$$\sup_\Omega u \le \sup_{\partial\Omega} u + C \sum_{k=0}^{N+1} \|\mu\|_q^{a_k} \|f\|_p^{m^k}.$$

Remark 2.13 When $1 < m \le 2 - n/q$, (2.24) is automatically satisfied.

Proof Without loss of generality we may assume as before that $q < \infty$. We define $q_0 = p$, and $q_k = nq_{k-1}q/\{n(q_{k-1} + mq) - mq_{k-1}q\}$ for $k \ge 1$. Note that $q_1 - p \ge p\{p(mq-n) - nq(m-1)\}/\{n(2^{-1}mq + p)\} > 0$ by (2.24). We can then inductively show that there is $\theta > 0$ such that $q_k - q_{k-1} \ge q_{k-1}\{q_{k-1}(mq-n) - nq(m-1)\}/\{n(2^{-1}mq + p)\} \ge \theta$ for $k \ge 1$ as long as $q_{k-1} < n$. Hence, we can find an integer $N \ge 1$ such that $q_{N-1} \le n < q_N$. If $q_{N-1} = n$ we set $q_N = (n+q)/2$.

We fix $R_1 > \cdots > R_N > 1$. We first find the L^p-strong solution $v_1 \in C(\overline{B}_{R_1}) \cap W^{2,p}(B_{R_2})$ of

$$\mathcal{P}^+(D^2 v_1) = -f(x) \quad \text{in } B_{R_1}$$

such that $v_1 = 0$ on ∂B_{R_1}. Then $0 \le -v_1 \le C\|f\|_p$ in B_{R_1}, and

$$\|Dv_1\|_{L^{p^*}(B_{R_2})} \le C\|f\|_p. \tag{2.26}$$

Setting $w_1 = u + v_1$ in Ω, we obtain that w_1 is an L^p-viscosity subsolution of

$$\mathcal{P}^-(D^2 w_1) - 2^{m-1}\mu(x)|Dw_1|^m = 2^{m-1}\mu(x)|Dv_1(x)|^m =: f_2(x) \quad \text{in } \Omega.$$

Moreover, Hölder inequality, together with (2.26), yields

$$\|f_2\|_{L^{q_1}(B_{R_2})} \le \|\mu\|_q \|Dv_1\|^m_{L^{p^*}(B_{R_2})} \le C\|\mu\|_q \|f\|^m_p.$$

We next find the strong solution $v_2 \in C(\overline{B}_{R_2}) \cap W^{2,q_1}(B_{R_3})$ of

$$\mathcal{P}^+(D^2 v_2) = -f_2(x) \quad \text{in } B_{R_2}$$

such that $v_2 = 0$ on ∂B_{R_2}. Again $0 \le -v_2 \le C\|f_2\|_{L^{q_1}(B_{R_2})} \le C\|\mu\|_q \|f\|^m_p$ in B_{R_2}, and

$$\|Dv_2\|_{L^{q_1^*}(B_{R_3})} \le C\|\mu\|_q \|f\|^m_p. \tag{2.27}$$

Then $w_2 := w_1 + v_2$ is an L^p-viscosity subsolution of

$$\mathcal{P}^-(D^2 w_2) - 2^{2(m-1)}\mu(x)|Dw_2|^m = 2^{2(m-1)}\mu(x)|Dv_2(x)|^m =: f_3(x) \quad \text{in } \Omega$$

and (2.27) implies

$$\|f_3\|_{L^{q_2}(B_{R_3})} \le C\|\mu\|_q \|Dv_2\|^m_{L^{q_1^*}(B_{R_3})} \le C\|\mu\|_q^{1+m} \|f\|_p^{m^2}.$$

Inductively, for $f_k := \mu|Dv_{k-1}|^m \in L^{q_{k-1}}(B_{R_k})$, we take the strong solution $v_k \in C(\overline{B}_{R_k}) \cap W^{2,q_{k-1}}(B_{R_{k+1}})$ of

$$P^+(D^2 v_k) = -f_k(x) \quad \text{in } B_{R_k}$$

such that $v_k = 0$ on ∂B_{R_k}, for which we have $0 \le -v_k \le C\|f_k\|_{L^{q_{k-1}}(B_{R_k})}$ in B_{R_k}, and

$$\|f_k\|_{L^{q_{k-1}}(B_{R_k})} \le C\|\mu\|_q^{a_{k-1}} \|f\|_p^{m^{k-1}}, \quad \|Dv_k\|_{L^{q_{k-1}^*}(B_{R_{k+1}})} \le C\|\mu\|_q^{a_{k-1}} \|f\|_p^{m^{k-1}},$$

where if $q_{N-1} = n$, q_{N-1}^* is replaced by any exponent less than $+\infty$. We eventually obtain that $w_N = u + \sum_{k=1}^N v_k$ is an L^p-viscosity subsolution of

$$\mathcal{P}^-(D^2 w_N) - 2^{N(m-1)}\mu(x)|Dw_N|^m = 2^{N(m-1)}\mu(x)|Dv_N(x)|^m =: \hat{f}(x) \quad \text{in } \Omega,$$

where $\hat{f} \in L^{q_N}(\Omega)$. Hence, by Theorem 2.11, if $\|\hat{f}\|^{m-1}_{L^{q_N}(\Omega)}\|\mu\|_q$ is small, then we have

$$\sup_\Omega w_N \le \sup_{\partial\Omega} w_N + C(\|\hat{f}\|_{L^{q_N}(\Omega)} + \|\hat{f}\|^m_{L^{q_N}(\Omega)}\|\mu\|_q).$$

Since $\|\hat{f}\|_{L^{q_N}(\Omega)} \le C\|\mu\|_q^{a_N}\|f\|_p^{m^N}$, the result follows. □

3 Parabolic equations

In this section we consider parabolic PDEs in $Q := \Omega \times (0, T]$, where $0 < T \leq 1$. For $1 \leq p \leq \infty$, the parabolic Sobolev space $W^{2,1,p}(Q)$ is defined as

$$W^{2,1,p}(Q) = \left\{u \in L^p(Q) : u_t, Du, D^2u \in L^p(Q)\right\}.$$

Throughout this paper, we denote the parabolic boundary by $\partial_p Q := \Omega \times \{0\} \cup \partial\Omega \times [0, T]$.

We will also be using the space $W^{2,1,p}_{\text{loc}}(Q) = \{u : Q \to \mathbb{R} : u \in W^{2,1,p}(Q')$ for all $Q' \Subset Q\}$. Above, $Q' \Subset Q$ means that $\text{dist}(Q', \partial_p Q) > 0$. The parabolic distance between (x, t) and (y, s) is defined by

$$\text{dist}((x, t), (y, s)) = (|x - y|^2 + |t - s|)^{\frac{1}{2}}.$$

We recall the definition of L^p-viscosity solution of general fully nonlinear parabolic PDEs.

Definition 3.1 We call $u \in C(Q)$ an L^p-viscosity subsolution (respectively, supersolution) of

$$u_t + F(x, t, u, Du, D^2u) = f(x, t) \quad \text{in } Q, \tag{3.1}$$

if

$$ess \liminf_{(y,s)\in Q \to (x,t)} \left\{\phi_t(y, s) + F(y, s, u(y, s), D\phi(y, s), D^2\phi(y, s)) - f(y, s)\right\} \leq 0$$

$$\left(\text{respectively, } ess \limsup_{(y,s)\in Q \to (x,t)} \left\{\phi_t(y, s) + F(y, s, u(y, s), D\phi(y, s), D^2\phi(y, s)) - f(y, s)\right\} \geq 0\right)$$

whenever $\phi \in W^{2,1,p}_{\text{loc}}(Q)$ and $(x, t) \in \Omega \times (0, T)$ is a local maximum (resp., minimum) point of $u - \phi$.

We call $u \in C(Q)$ an L^p-viscosity solution of (3.1) if it is an L^p-viscosity sub- and supersolution of (3.1).

As in the elliptic case, we call $u \in W^{2,1,p}_{\text{loc}}(Q)$ an L^p-strong solution of (3.1) if u satisfies

$$u_t(x, t) + F(x, t, u(x, t), Du(x, t), D^2u(x, t)) = f(x, t) \quad a.e. \text{ in } Q.$$

We will establish maximum principles for the parabolic PDE

$$u_t + \mathcal{P}^-(D^2u) - \mu(x, t)|Du|^m = f(x, t) \quad \text{in } Q, \tag{3.2}$$

where $m \geq 1$.

The following version of maximum principle can be derived from [21] (see also [17,18]).

Proposition 3.2 *Let* $m = 1$, $f \in L^{n+1}_+(Q)$ *and* $\mu \in L^{n+1}_+(Q)$. *There exist* $C_k = C_k(n, \lambda, \Lambda) > 0$ $(k = 1, 2)$ *such that if* $u \in C(\overline{Q}) \cap W^{2,1,n+1}_{\text{loc}}(Q)$ *is an* L^{n+1}*-strong subsolution of* (3.2), *then*

$$\sup_Q u \le \sup_{\partial_p Q} u + C_1 \exp(C_2 \|\mu\|_{n+1}) \|f\|_{n+1}.$$

One may also refine the above estimate using the upper contact set (see [21] for the details).

In the remaining part of the paper we fix p_1 to be the “parabolic" constant that gives the range of exponents for which the following generalized maximum principle holds. It is known (see [3,10,11]) that there exists an exponent $p_1 = p_1(n, \Lambda/\lambda)$ satisfying $(n+2)/2 \le p_1 < n+1$ with the following property: for $p > p_1$ there is a constant $C = C(n, \lambda, \Lambda, p)$ such that if $f \in L^p(Q)$ and $u \in C(\overline{Q}) \cap W^{2,1,p}_{\text{loc}}(Q)$ is an L^p-strong solution of $u_t + \mathcal{P}^-(D^2u) \le f(x,t)$ in Q, then

$$\sup_Q u \le \sup_{\partial_p Q} u + C\|f^+\|_p.$$

We recall results on solvability of extremal equations and on estimates of Du.

Proposition 3.3 (cf. Theorem 2.8 in [8]) *Let* $p>p_1$. *There exists* $C=C(n, \lambda, \Lambda, p)>0$ *such that for* $f \in L^p(Q)$, *there exists an* L^p*-strong solution* $u \in C(\overline{Q}) \cap W^{2,1,p}_{\text{loc}}(Q)$ *of*

$$u_t + \mathcal{P}^+(D^2u) = f(x,t) \quad \text{in } Q \tag{3.3}$$

such that $u = 0$ *on* $\partial_p Q$ *and*

$$-C\|f^-\|_p \le u \le C\|f^+\|_p \quad \text{in } Q.$$

Moreover, for each set $Q' \Subset Q$, *there exists* $C' = C'(n, \lambda, \Lambda, p, dist(Q', \partial_p Q)) > 0$ *such that*

$$\|u\|_{W^{2,1,p}(Q')} \le C'\|f\|_p.$$

Proposition 3.4 (cf. Theorem 7.3 in [8]) *Let* $p > p_1$. *For each set* $Q' \Subset Q$, *there exists* $C = C(n, \lambda, \Lambda, p, dist(Q', \partial_p Q)) > 0$ *such that if* $u \in C(\overline{Q}) \cap W^{2,1,p}_{\text{loc}}(Q)$ *is an* L^p*-strong solution of* (3.3), *then we have*

$$\|Du\|_{L^\infty(Q')} \le C(\|u\|_{L^\infty(\partial_p Q)} + \|f\|_p) \quad \text{if } p > n+2,$$
$$\|Du\|_{L^{p^\star}(Q')} \le C(\|u\|_{L^\infty(\partial_p Q)} + \|f\|_p) \quad \text{if } p \in (p_1, n+2).$$

The constant $p^\star$ above and in the rest of the paper is defined by

$$p^\star = \frac{p(n+2)}{n+2-p} \quad \text{for } p < n+2.$$

The result below is a parabolic equivalent of Proposition 2.6 and can be proved by a similar argument.

Proposition 3.5 *Let Ω satisfy the uniform exterior cone condition, let*

$$q \geq p > n+2 \quad \text{or} \quad q > p = n+2, \tag{3.4}$$

and let $f \in L^p_+(Q)$ and $\psi \in C(\partial_p Q)$. Let $\mu \in L^q_+(Q)$ satisfy $\operatorname{supp}\mu \Subset Q$. *Then there exist strong solutions $u, v \in C(\overline{Q}) \cap W^{2,p}_{\text{loc}}(Q)$ of*

$$u_t + \mathcal{P}^-(D^2u) - \mu(x,t)|Du| \geq f(x,t), \;\; v_t + \mathcal{P}^+(D^2v) + \mu(x,t)|Dv| \leq f(x,t) \;\; \text{in } Q$$

such that $u = v = \psi$ on $\partial_p Q$. Moreover, we have

$$\|u\|_{L^\infty(Q)}, \|v\|_{L^\infty(Q)} \leq \|\psi\|_{L^\infty(\partial_p Q)} + C_1 \exp(C_2\|\mu\|_{n+1})\|f\|_{n+1}, \tag{3.5}$$

where C_1 and C_2 are constants from Proposition 3.2, and for every $Q' \Subset Q$

$$\|u\|_{W^{2,1,p}(Q')}, \|v\|_{W^{2,1,p}(Q')} \leq C(n, p, \lambda, \Lambda, \|\mu\|_{L^q(Q)}, \operatorname{dist}(Q', \partial_p Q))(\|\psi\|_{L^\infty(\partial_p Q)} + \|f\|_{L^p(Q)}). \tag{3.6}$$

Repeating the arguments of the proof of Proposition 2.8, Proposition 3.5 allows us to obtain the following maximum principle for L^p-viscosity solutions.

Proposition 3.6 *Assume* (3.4) *and let $m = 1$. There exist $C_k = C_k(n, \lambda, \Lambda) > 0$ ($k = 1, 2$) such that if $f \in L^p_+(Q)$, $\mu \in L^q_+(Q)$, and $u \in C(\overline{Q})$ is an L^p-viscosity subsolution of (3.2), then*

$$\sup_Q \leq \sup_{\partial_p Q} u + C_1 \exp(C_2\|\mu\|_{n+1})\|f\|_{n+1}.$$

3.1 Bounded coefficients (i.e. $q = \infty$)

We first show that if $\mu \in L^\infty_+(Q)$ then, even for $m > 1$, we do not need to assume that $\|\mu\|_\infty$ or $\|f\|_p$ is small. Recall that such a restriction is necessary in the elliptic case as discussed in Sect. 2.2 and [15].

Theorem 3.7 *Let $n+2 < p$ and $m \geq 1$. There exixts $C = C(n, \lambda, \Lambda, p, m) > 0$ such that if $f \in L^p_+(Q)$, $\mu \in L^\infty_+(Q)$, and $u \in C(\overline{Q})$ is an L^p-viscosity subsolution of* (3.2), *then*

$$\sup_Q u \leq \sup_{\partial_p Q} u + C(\|f\|_p + \|\mu\|_\infty \|f\|^m_p).$$

Proof We set $Q_1 = B_2 \times (-1, T]$. In view of Proposition 3.3, we find the L^p-strong solution $v \in C(\overline{Q}_1) \cap W^{2,1,p}_{\text{loc}}(Q_1)$ of

$$v_t + \mathcal{P}^+(D^2 v) = -f(x,t) \quad \text{in } Q_1$$

such that $v = 0$ on $\partial_p Q_1$. We have $0 \le -v \le C_1 \|f\|_p$ in Q. Since Proposition 3.4 implies

$$\|Dv\|_{L^\infty(Q)} \le C_1 \|f\|_p, \tag{3.7}$$

we see that $w := u + v$ is an L^p-viscosity subsolution of

$$w_t + \mathcal{P}^-(D^2 w) - 2^{m-1}\mu(x,t)|Dw|^m = 2^{m-1}\mu(x,t)|Dv(x,t)|^m \quad \text{in } Q.$$

For $\varepsilon > 0$ we now set $U_\varepsilon(x,t) := w(x,t) - \alpha_\varepsilon t$, where $\alpha_\varepsilon = 2^{m-1} C_1^{m-1}\|\mu\|_\infty \|f\|_p^m + \varepsilon$. By using (3.7) it is easy to verify that U_ε is an L^p-viscosity subsolution of

$$(U_\varepsilon)_t + \mathcal{P}^-(D^2 U_\varepsilon) - 2^{m-1}\mu(x,t)|DU_\varepsilon|^m = -\varepsilon \quad \text{in } Q.$$

Thus, by the definition of L^p-viscosity solution, we obtain $\sup_Q U_\varepsilon \le \sup_{\partial_p Q} U_\varepsilon$. Therefore, we have

$$\sup_Q u \le \sup_{\partial_p Q} U_\varepsilon + \sup_Q(-v) + (C + \varepsilon)T\|\mu\|_\infty \|f\|_p^m,$$

which yields the desired conclusion upon sending $\varepsilon \to 0$. □

We next extend Theorem 3.7 to the case $p \in (p_1, n+2]$.

Theorem 3.8 *Let $p_1 < p \le n+2$ and $m \ge 1$. There exist an integer $N = N(n,p,m) \ge 1$ and $C = C(n, \lambda, \Lambda, p, m) > 0$ such that if $f \in L^p_+(Q)$, $\mu \in L^\infty_+(Q)$,*

$$p > \frac{(m-1)(n+2)}{m}, \tag{3.8}$$

and $u \in C(\overline{Q})$ is an L^p-viscosity subsolution of (3.2), then

$$\sup_Q u \le \sup_{\partial_p Q} u + C\left(\|f\|_p^m \sum_{k=0}^N \|\mu\|_\infty^k + \|\mu\|_\infty^{mN+1} \|f\|_p^{m^2} \right).$$

Remark 3.9 We remark that when $m \in [1,2]$, since $p_1 \ge (n+2)/2 \ge (m-1)(n+2)/m$, the restriction (3.8) is not necessary.

Proof The proof uses the iteration technique. Set $q_0 = p$. If $p = n+2$ we set $q_1 = n+3$, otherwise we set $q_1 = p^\star/m$ and then $q_k = q^\star_{k-1}/m$, for $k > 1$ as long as $q_{k-1} < n+2$, and $q_k = n+3$ if $q_{k-1} = n+2$. Notice that $q_1 - p \ge p\{pm - (m-1)(n+2)\}/\{m(n+2-p)\} > 0$ by (3.8). It is then easy to find $N \ge 1$ such that $q_{N-1} \le n+2 < q_N$.

We now fix $R_1 > \cdots > R_N > 1$, and set $Q_k = B_{R_k} \times (-N-1+k, T]$. We first find the L^p-strong solution $v_1 \in C(\overline{Q}_1) \cap W^{2,1,p}(Q_2)$ of

$$(v_1)_t + \mathcal{P}^+(D^2 v_1) = -f(x,t) \quad \text{in } Q_1$$

such that $v_1 = 0$ on $\partial_p Q_1$. We have $0 \le -v_1 \le C_1 \|f\|_p$ in Q_1, and

$$\||Dv_1|^m\|_{L^{q_1}(Q_2)} \le C_1 \|f\|_p^m.$$

Note that $w_1 = u + v_1$ is an L^p-viscosity subsolution of

$$\begin{aligned}(w_1)_t + \mathcal{P}^+(D^2 w_1) - 2^{m-1}\mu(x,t)|Dw_1|^m &= 2^{m-1}\|\mu\|_\infty |Dv_1(x,t)|^m \\ &=: -f_2(x,t) \quad \text{in } Q,\end{aligned}$$

and

$$\|f_2\|_{L^{q_1}(Q_2)} \le C_1 \|\mu\|_\infty \|f\|_p^m.$$

We inductively find the strong solutions $v_k \in C(\overline{Q}_k) \cap W^{2,1,q_{k-1}}(Q_{k+1})$, for $k \ge 2$, of

$$(v_k)_t + \mathcal{P}^+(D^2 v_k) = -2^{(k-1)(m-1)}\|\mu\|_\infty |Dv_{k-1}(x,t)|^m =: f_k(x,t) \quad \text{in } Q_k$$

such that $v_k = 0$ on $\partial_p Q_k$. They satisfy $0 \le -v_k \le C\|f_k\|_{L^{q_{k-1}}(Q_k)}$ in Q_k, and

$$\|f_k\|_{L^{q_{k-1}}(Q_k)} \le C\|\mu\|_\infty^{k-1} \|f\|_p^m.$$

Hence, we see that $w_N =: u + \sum_{k=1}^N v_k$ is an L^p-viscosity subsolution of

$$\begin{aligned}(w_N)_t + \mathcal{P}^-(D^2 w_N) - 2^{N(m-1)}\mu(x,t)|Dw_N|^m &= 2^{N(m-1)}\|\mu\|_\infty |Dv_N(x,t)|^m \\ &=: \hat{f}(x,t) \quad \text{in } Q.\end{aligned}$$

Since $\hat{f} \in L^{q_N}(Q)$ with $q_N > n+2$, we obtain by Theorem 3.7 that

$$\sup_Q w_N \le \sup_{\partial_p Q} w_N + C(\|\hat{f}\|_{L^{q_N}(Q)} + \|\mu\|_\infty \|\hat{f}\|_{L^{q_N}(Q)}^m).$$

Since $\|\hat{f}\|_{q_N} \le C\|\mu\|_\infty^N \|f\|_p^m$, the result follows. □

3.2 Linear growth (i.e. $m = 1$)

In this section we discuss the case when $m = 1$ in (3.2) but $\mu \in L^q(Q)$ with $q > n+2$.

Theorem 3.10 *Let $p_1 < p \le n+2 < q$ and $m = 1$. There exist an integer $N = N(n, p, q) \ge 1$ and $C = C(n, \lambda, \Lambda, p, q) > 0$ such that if $f \in L^p_+(Q)$, $\mu \in L^q_+(Q)$, and $u \in C(\overline{Q})$ is an L^p-viscosity subsolution of* (3.2)*, then*

$$\sup_Q u \le \sup_{\partial_p Q} u + C\left\{\exp(C\|\mu\|_{n+1})\|\mu\|_q^N + \sum_{k=0}^{N-1}\|\mu\|_q^k\right\}\|f\|_p.$$

Proof Set $q_0 = p$ and inductively $q_k = p_{k-1}q(n+2)/\{q_{k-1}(n+2)+q(n+2-q_{k-1})\}$ for $k \ge 1$. Since $q_k - q_{k-1} \ge q_{k-1}^2(q-n-2)/\{q(n+2-p_0)+q_{k-1}(n+2)\} \ge \theta > 0$ for some $\theta > 0$ and all $k \ge 1$ as long as $q_{k-1} \le n+2$, there exists $N = N(n, p, q) \ge 1$ such that $q_{N-1} < n+2 \le q_N$.

We now fix $R_1 > \cdots > R_N > 1$ and set $Q_k = B_{R_k} \times (-N+k-1, T]$. Let $v_1 \in C(\overline{Q}_1) \cap W^{2,1,p}_{\text{loc}}(Q_1)$ be the L^p-strong solution of

$$(v_1)_t + \mathcal{P}^+(D^2 v_1) = -f(x,t) \quad \text{in } Q_1$$

such that $v_1 = 0$ on $\partial_p Q_1$. Then $0 \le -v_1 \le C_1\|f\|_{L^p(Q)}$ in Q_1, and

$$\|\mu D v_1\|_{L^{q_1}(Q_2)} \le C_1\|\mu\|_{L^q(Q)}\|f\|_{L^p(Q)}.$$

We easily check that $w_1 := u + v_1$ is an L^p-viscosity subsolution of

$$(w_1)_t + \mathcal{P}^+(D^2 w_1) - \mu(x,t)|Dw_1| = \mu(x,r)|Dv_1(x,t)| := f_2(x,t) \quad \text{in } Q.$$

We inductively choose for $k \ge 2$ the strong solutions $v_k \in C(\overline{Q}_k) \cap W^{2,1,p}_{\text{loc}}(Q_k)$ of

$$(v_k)_t + \mathcal{P}^+(D^2 v_k) = -f_k(x,t) \quad \text{in } Q_k$$

such that $v_k = 0$ on $\partial_p Q_k$, where $f_k = \mu|Dv_{k-1}|$. Again $0 \le -v_k \le C_k\|f_k\|_{L^{q_{k-1}}(Q)}$ in Q_k, and

$$\|\mu D v_k\|_{L^{q_k}(Q_{k+1})} \le C\|\mu\|_{L^q(Q)}\|f_k\|_{L^{q_{k-1}}(Q)}.$$

It is then easy to see that

$$\|f_k\|_{L^{q_{k-1}}(Q)} \le C\|\mu\|^{k-1}_{L^q(Q)}\|f\|_{L^p(Q)}. \tag{3.9}$$

Hence, setting $w_N = u + \sum_{k=1}^N v_k$, as in the proof of Theorem 2.9, we have

$$\sup_Q w_N \le \sup_{\partial_p Q} w_N + C\exp(C\|\mu\|_{L^{n+1}(Q)})\|\mu D v_N\|_{L^{n+1}(Q)}.$$

Therefore, we conclude the proof using (3.9). □

3.3 Superlinear growth (i.e. $m > 1$)

In this section, we give sufficient conditions under which the maximum principle for (3.2) with $m > 1$ holds true. The first result corresponds to Theorem 2.11 for elliptic PDEs.

Theorem 3.11 *Let $n + 2 < p$ and $m > 1$. There exist $\delta = \delta(n, \lambda, \Lambda, m, p) > 0$ and $C = C(n, \lambda, \Lambda, m, p) > 0$ such that if $f \in L^p_+(Q)$, $\mu \in L^p_+(Q)$,*

$$\|f\|_p^{m-1}\|\mu\|_p < \delta, \tag{3.10}$$

and $u \in C(\overline{Q})$ is an L^p-viscosity subsolution of (3.2), then

$$\sup_Q u \le \sup_{\partial_p Q} u + C(\|f\|_p + \|\mu\|_p\|f\|_p^m).$$

Proof We fix $R_1 > R_2 > 1$ and set $Q_k = B_{R_k} \times (-3 + k, T]$ for $k = 1, 2$. Let $v \in C(\overline{Q}_1) \cap W^{2,1,p}(Q_2)$ be the L^p-strong solution of

$$v_t + \mathcal{P}^+(D^2 v) = -f(x, t) \quad \text{in } Q_1$$

such that $v = 0$ on $\partial_p Q_1$. Then $0 \le -v \le C_1\|f\|_p$ in Q_1, and, by Proposition 3.4,

$$\|Dv\|_\infty \le C_2\|f\|_p.$$

Thus, we see that $w := u + v$ is an L^p-viscosity subsolution of

$$w_t + \mathcal{P}^-(D^2 w) - 2^{m-1}\mu(x, t)|Dw| = 2^{m-1}C_2^m\mu(x, t)\|f\|_p^m \quad \text{in } Q_1.$$

Next, for $\varepsilon > 0$ we find the L^p-strong soluion $\zeta_\varepsilon \in C(\overline{Q}_2) \cap W^{2,1,p}(Q)$ of

$$(\zeta_\varepsilon)_t + \mathcal{P}^+(D^2\zeta_\varepsilon) = -(2^{m-1}C_2^m + 1)\|f\|_p^m\mu(x, t) - \varepsilon \quad \text{in } Q_2$$

such that $\zeta_\varepsilon = 0$ on $\partial_p Q_2$. Then $0 \le -\zeta_\varepsilon \le C_3(\|f\|_p^m\|\mu\|_p + \varepsilon)$ in Q_2, and

$$\|D\zeta\|_{L^\infty(Q)} \le C_4(\|f\|_p^m\|\mu\|_p + \varepsilon).$$

We now see that $W_\varepsilon := w + \zeta_\varepsilon$ is an L^p-viscosity subsolution of

$$\begin{aligned}(W_\varepsilon)_t + \mathcal{P}^-(D^2 W_\varepsilon) - 2^{2(m-1)}\mu(x, t)|DW_\varepsilon|^m \\ = \mu(x, t)(2^{2(m-1)}|D\zeta|^m - \|f\|_p^m) - \varepsilon \quad \text{in } Q.\end{aligned}$$

Hence, taking $\delta = 2^{-2(m-1)/m} C_4^{-1} > 0$, we see that if (3.10) holds, then W is an L^p-viscosity subsolution of

$$(W_\varepsilon)_t + \mathcal{P}^-(D^2 W_\varepsilon) - 2^{2(m-1)}\mu(x,t)|DW_\varepsilon|^m = -\varepsilon \quad \text{in } Q$$

for small $\varepsilon > 0$. Therefore the definition of viscosity solution implies that $\sup_Q W \le \sup_{\partial_p Q} W$, which completes the proof. □

Our last result extends Theorem 3.11 to the case of $p > p_1$.

Theorem 3.12 *Let $p_1 < p \le n+2 < q$ and $m > 1$. Denote $a_0 = 0$ and $a_k = 1 + m + \cdots + m^{k-1}$ for $k \ge 1$. There exist an integer $N = N(n,m,p,q) \ge 1$, $\delta = \delta(n,\lambda,\Lambda,m,p,q) > 0$ and $C = C(n,\lambda,\Lambda,m,p,q) > 0$ such that if $f \in L^p_+(Q)$, $\mu \in L^q_+(Q)$,*

$$p > \frac{(m-1)q(n+2)}{mq-n-2}, \tag{3.11}$$

and $u \in C(\overline{Q})$ is an L^p-viscosity subsolution of (3.2),

$$\|f\|_p^{m^N(m-1)} \|\mu\|_q^{a_N(m-1)+1} < \delta, \tag{3.12}$$

then

$$\sup_Q u \le \sup_{\partial_p Q} u + C\left\{\sum_{k=0}^{N+1} \|\mu\|_q^{a_k} \|f\|_p^{m^k}\right\}.$$

Remark 3.13 If $1 < m < 2 - (n+2)/q$, the restriction (3.11) is not necessary.

Proof We again employ the iteration process. We set $q_0 = p$ and $q_k = q_{k-1}q(n+2)/\{q_{k-1}(n+2) + q(n+2-q_{k-1})\}$ for $k \ge 1$. Since $q > n+2 > (n+2)/m$, by (3.11), we can find $\theta > 0$ such that $q_k - q_{k-1} \ge \theta$ for $k \ge 1$ as long as $q_{k-1} \le n+2$. Thus, we can select an integer $N \ge 1$ such that $q_{N-1} \le n+2 < q_N$. If $q_{N-1} = n+2$ we set $q_N = (n+2+q)/2$.

We now fix $R_1 > \cdots > R_N > 1$ and set $Q_k = B_{R_k} \times (-N-1+k, T]$ for $k = 1, \ldots, N$. We first take the L^p-strong solution $v_1 \in C(\overline{Q}_1) \cap W^{1,2,p}(Q_2)$ of

$$(v_1)_t + \mathcal{P}^+(D^2 v_1) = -f(x,t) \quad \text{in } Q_1$$

such that $v_1 = 0$ on $\partial_p Q_1$. We have $0 \le -v_1 \le C_1\|f\|_p$ in Q_1, and

$$\|Dv_1\|_{L^{p^\star}(Q_2)} \le C_1\|f\|_p.$$

Thus, we see that $w_1 := u + v_1$ is an L^p-viscosity subsolution of

$$\begin{aligned}(w_1)_t + \mathcal{P}^-(D^2 w_1) - 2^{m-1}\mu(x,t)|Dw_1|^m &= 2^{m-1}\mu(x,t)|Dv_1(x,t)|^m \\ &=: f_2(x,t) \text{ in } Q_2.\end{aligned}$$

Note that

$$\|f_2\|_{L^{q_1}(Q_2)} \leq C\|\mu\|_q \|f\|_p^m.$$

Inductively, for $k \geq 2$ we can find strong solutions $v_k \in C(\overline{Q}_k) \cap W^{1,2,q_{k-1}}(Q_{k+1})$ of

$$(v_k)_t + \mathcal{P}^+(D^2 v_k) = -f_k(x,t) \quad \text{in } Q_k,$$

where $f_k(x,t) = 2^{(k-1)(m-1)}\mu(x,t)|Dv_{k-1}(x,t)|^m$, such that $v_k = 0$ on $\partial_p Q_k$. The v_k satisfy $0 \leq -v_k \leq C_k\|f_k\|_{L^{q_{k-1}}(Q_k)}$ in Q_k, and

$$\|Dv_k\|_{L^{q^\star_{k-1}}(Q_{k+1})} \leq C_k\|f_k\|_{L^{q_{k-1}}(Q_k)} \leq 2^{(k-1)(m-1)} C_k \|\mu\|_q^{a_{k-1}} \|f\|_p^{m^{k-1}}.$$

If $q_{N-1} = n+2$ we need to replace $q^\star_{N-1} = +\infty$ by a sufficiently big exponent. Hence, setting $w_N = u + \sum_{k=1}^N v_k$ we see that w_N is an L^p-viscosity subsolution of

$$\begin{aligned}(w_N)_t + \mathcal{P}^-(D^2 w_N) - 2^{N(m-1)}\mu(x,t)|Dw_N|^m \\ = 2^{N(m-1)}\mu(x,t)|Dv_N(x,t)|^m =: \hat{f}(x,t) \quad \text{in } Q.\end{aligned}$$

By Proposition 3.4, we note that $\|\hat{f}\|_{L^{q_N}(Q)} \leq C\|\mu\|_q \|Dv_N\|^m_{L^{q^\star_{N-1}}(Q)}$. Thus, in view of Theorem 3.11, there is $\hat{\delta} > 0$ such that if $\|\hat{f}\|_{q_N}^{m-1}\|\mu\|_q < \hat{\delta}$, then

$$\sup_Q w_N \leq \sup_{\partial_p Q} w_N + C(\|\hat{f}\|_{q_N} + \|\mu\|_q \|\hat{f}\|_{q_N}^m).$$

Since $\|\hat{f}\|_{q_N} \leq C\|\mu\|_q^{a_N}\|f\|_p^{m^N}$, the assertion follows. □

References

1. Aleksandrov, A.D.: Majorization of solutions of second-order linear equations, Vestnik Lenningrad Univ. 21, (1966) 5–25 English Translation in Am. Math. Soc. Transl. **68**, 120–143 (1968)
2. Bakelman, I.Y.: Theory of quasilinear elliptic equations. Siberian Math. J. **2**, 179–186 (1961)
3. Cabré, X.: On the Alexandroff-Bakelman-Pucci estimate and the reversed Hölder inequality for solutions of elliptic and parabolic equations. Comm. Pure Appl. Math. **48**, 539–570 (1995)
4. Caffarelli, L.A., Cabré, X.: Fully nonlinear elliptic equations. American Mathematical Society, Providence (1995)
5. Caffarelli, L.A.: Interior a priori estimates for solutions of fully non-linear equations. Ann. Math. **130**, 189–213 (1989)
6. Caffarelli, L.A., Crandall, M.G., Kocan, M., Święch, A.: On viscosity solutions of fully nonlinear equations with measurable ingredients. Comm. Pure Appl. Math. **49**, 365–397 (1996)
7. Crandall, M.G., Fok, K., Kocan, M., Święch, A.: Remarks on nonlinear uniformly parabolic equations. Indiana Univ. Math. J. **47**(4), 1293–1326 (1998)
8. Crandall, M.G., Kocan, M., Święch, A.: L^p-Theory for fully nonlinear uniformly parabolic equations. Comm. Partial Diff. Eq. **25**, 1997–2053 (2000)

9. Crandall, M.G., Święch, A.: A note on generalized maximum principles for elliptic and parabolic PDE, Evolution equations, 121–127, Lecture Notes in Pure and Appl. Math., 234, Dekker, New York (2003)
10. Escauriaza, L.: $W^{2,n}$ a priori estimates for solutions to fully non-linear equations. Indiana Univ. Math. J. **42**, 413–423 (1993)
11. Fabes, E.B., Stroock, D.W.: The L^p-integrability of Green's functions and fundamental solutions for elliptic and parabolic equations. Duke Math. J. **51**(4), 997–1016 (1984)
12. Fok, P.: Some maximum principles and continuity estimates for fully nonlinear elliptic equations of second order, Ph.D. Thesis, UCSB (1996)
13. Fok, K.: A nonlinear Fabes-Stroock result. Comm. Partial Diff. Eq. **23**(5-6), 967–983 (1998)
14. Gilbarg, D., Trudinger, N.S.: Elliptic partial differential equations of second order, 2nd edn. Springer, New York (1983)
15. Koike, S., Święch, A.: Maximum principle and existence of L^p-viscosity solutions for fully nonlinear uniformly elliptic equations with measurable and quadratic terms. Nonlin. Diff. Eq. Appl. **11**, 491–509 (2004)
16. Krylov, N.V.: Sequences of convex functions and estimates of the maximum of the solution of a parabolic equation. Siberian Math J. **17**, 226–236 (1976)
17. Krylov, N.V.: Nonlinear elliptic and parabolic equations of the second order. Reidel Pub. Co., Dordrecht (1987)
18. Lieberman, G.M.: Second order parabolic differential equations. World Scientific, Singapore (1996)
19. Shirai, T.: On fully nonlinear uniformly elliptic PDEs with superlinear growth nonlinearity. Master Thesis, Saitama University (2002)
20. Święch, A.: (1997) $W^{1,p}$-Interior estimates for solutions of fully nonlinear, uniformly elliptic equations **2**(6), 1005–1027
21. Tso, K.: On an Aleksandrov-Bakelman type maximum principle for second-order parabolic equations. Comm. Partial Diff. Eq. **10**(5), 543–553 (1985)
22. Wang, L.: On the regularity of fully nonlinear parabolic equations: I. Comm. Pure Appl. Math. **45**, 27–76 (1992)