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Flips and the Hilbert scheme over an exterior algebra

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Abstract We study the tangent space at a monomial point M in the Hilbert scheme that parameterizes all ideals with the same Hilbert function as M over an exterior algebra.

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1 Introduction

Throughout the paper k stands for an infinite field. Let E be the exterior algebra over k with basis $x_1, \ldots, x_n$.

In [9] we showed that the Hilbert scheme, that parameterizes all ideals with a fixed Hilbert function over E, is connected.Now, we introduce the notion of flips and show in Theorem 2.6 that the basic flips form a basis of the tangent space at a monomial point in the Hilbert scheme. We have proved that the same property holds for toric Hilbert schemes [10, Corollary 5.2].

Flips over polynomial rings were studied in [1]. Our results and proofs are completely different because it turns out that flips have different properties over exterior algebras than over polynomial rings. Some of the differences are listed below:

1. Over an exterior algebra, Theorem 3.4 says that the tangent space at a monomial point in the Hilbert scheme has a basis consisting of directions tangent to deformations built using Gröbner basis. Such a nice structure of the tangent space is surprising since it does not hold in the polynomial case.
2. The underlying reason for (1) is: over an exterior algebra, Theorem 3.3 yields that flips can be described either using Gröbner basis theory (3.1) or using

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homomorphisms (2.3). Example 3.6 shows that in the polynomial case there are flip homomorphisms that do not come from Gröbner flips.
3. There are fewer flips in the exterior algebra case. A flip over a polynomial ring may not be a flip over an exterior algebra. In Example 4.2 we give the striking example that (cd) is a flip of (ab) over a polynomial ring, but it is not a flip over an exterior algebra.
4. Over an exterior algebra, Theorem 2.6 says that the basic flips form a basis of the tangent space at a monomial point in the Hilbert scheme. This property does not hold in the polynomial case.

Thus, the properties of flips and the structure of the Hilbert scheme differ in the polynomial and the exterior algebra cases. In Sect. 5, we present an example of the structure of the Hilbert scheme over an exterior algebra on five variables.

Reeves and Stillman [11] proved that the lexicographic point is smooth on the Hilbert scheme in the polynomial case. There is no lexicographic point on the toric Hilbert scheme, but there is another distinguished point—the toric ideal; we proved in [10] that the toric ideal is a smooth point on the toric Hilbert scheme. The following problem is open:

Question *Is the lexicographic point smooth on the Hilbert scheme over an exterior algebra?*

We make use of Gröbner basis theory over exterior algebras, cf. [2]. In Sects. 3 and 4 we study flips from the points of view of Gröbner basis theory and Combinatorics. In Thoerem 4.4 we show that a monomial ideal (or its corresponding simplicial complex) has a flip if and only if the ideal is not a power of the maximal ideal; the analogues problem in the toric case "Find a criterion whether a given triangulation of the convex hull of a set of n points in $\mathbf{N}^d \setminus \mathbf{0}$ has a (toric) flip" is completely open.

2 Flips and tangent spaces

We use the following description of the tangent space:

Proposition 2.1 *Let I be a graded ideal in E. The tangent space at the point I to the Hilbert scheme is* $\mathrm{Hom}(I, E/I)_0$.

Proof Let $f_1, \ldots, f_r$ be a minimal set of graded generators of I. Consider the ideal

$$I_\epsilon = (f_1 + \epsilon g_1, \ldots, f_r + \epsilon g_r)$$

where $g_1, \ldots, g_r$ are graded polynomials in E such that $\deg(g_i) = \deg(f_i)$ for $1 \le i \le r$, and ϵ is a parameter. Such an ideal I_ϵ corresponds to an element in the tangent space if and only if $E[\epsilon]/I_\epsilon$ is flat over $k[\epsilon]/\epsilon^2$. We will study when $E[\epsilon]/I_\epsilon$ is flat over $k[\epsilon]/\epsilon^2$.

Since (ϵ) is the only nonzero graded ideal in $k[\epsilon]/\epsilon^2$, we have that $E[\epsilon]/I_\epsilon$ is flat over $k[\epsilon]/\epsilon^2$ if and only if $\mathrm{Tor}_1^A(k[\epsilon]/\epsilon, E[\epsilon]/I_\epsilon)$ vanishes. Let $u_1, \ldots, u_r$ be graded polynomials in E and $\sum_{1 \le i \le r} f_i u_i = 0$ be a graded syzygy. We need to check when the syzygy can be lifted, that is, when there exist graded polynomials $h_1, \dots, h_r$ such that $u_1 + \epsilon h_1, \dots, u_r + \epsilon h_r$ yield the syzygy

$$\sum_{1 \le i \le r} (f_i + \epsilon g_i)(u_i + \epsilon h_i) = 0\,.$$

We get that

$$\begin{aligned}\sum_{1 \le i \le r} (f_i + \epsilon g_i)(u_i + \epsilon h_i) &= \sum_{1 \le i \le r} f_i u_i + \epsilon \left(\sum_{1 \le i \le r} f_i h_i + \sum_{1 \le i \le r} g_i u_i \right) \\ &\quad + \epsilon^2 \left(\sum_{1 \le i \le r} g_i h_i \right) \\ &= \epsilon \left(\sum_{1 \le i \le r} f_i h_i + g_i u_i \right)\end{aligned}$$

vanishes if and only if $\sum_{1 \le i \le r} f_i h_i + g_i u_i = 0$. Therefore, the desired $h_1, \dots, h_r$ exist if and only if $\sum_{1 \le i \le r} g_i u_i \in I$.

We showed that $\mathrm{Tor}_1^A(k[\epsilon]/\epsilon, E[\epsilon]/I_\epsilon)$ vanishes if and only if $\sum_{1 \le i \le r} g_i u_i \in I$ for every choice of graded polynomials $u_1, \dots, u_r$ such that $\sum_{1 \le i \le r} f_i u_i = 0$ is a graded syzygy, if and only if the map

$$\begin{aligned}\varphi : \quad I &\longrightarrow E/I \\ f_i &\longmapsto g_i \quad \text{for } 1 \le i \le r\end{aligned}$$

is a homomorphism in $\mathrm{Hom}(I, E/I)_0$. □

The ring E is $\mathbf{N}^n$-graded (or multigraded) so that for each $1 \le i \le n$ the multidegree of x_i is the i'th standard vector. A monomial in the exterior algebra is a product $x_{i_1} \dots x_{i_r}$ with $1 \le i_1 < \cdots < i_r \le n$. For a monomial m denote by $\overline{m}$ its multidegree.

Throughout this section M stands for a monomial ideal generated by monomials $m_1, \dots, m_r$.

Definition 2.2 Let M be a monomial ideal generated by monomials $m_1, \dots, m_r$. We say that a non-trivial sequence $\mathbf{s} = \{s_1, \dots, s_r\}$ is a *flip* of M if the following conditions (2.2.1), (2.2.2), and (2.3) are satisfied:

(2.2.1) for each i the element s_i is either zero or a $\pm$monomial not in M.
(2.2.2) there exists a vector $\mathbf{v}$ such that $\overline{m}_i - \overline{s}_i = \mathbf{v}$ for each i such that $s_i \neq 0$.
(2.3) $\phi : M \longrightarrow E/M$ defined by $\phi(m_i) = s_i$ for each $1 \le i \le r$ is a homomorphism of (total) degree 0. (Here we think of the s_i's as monomials in E/M.)

In this case we say that ϕ is a *flip homomorphism*. A flip homomorphism corresponds to an element in the tangent space at M on the Hilbert scheme via the construction in the proof of Proposition 2.1. If we need to emphasize $\mathbf{v}$, then we say that the sequence $\mathbf{s}$ is a $\mathbf{v}$-flip.

Example 2.4 Consider the exterior algebra A with basis a, b, c, d and the ideal $M = (ab, bc, acd)$. We have the flip homomorphisms ϕ_1 and ϕ_2 defined by

$$\begin{aligned} \phi_1(ab) &= ad \quad & \phi_2(ab) &= 0 \\ \phi_1(bc) &= 0 \quad & \phi_2(bc) &= cd \\ \phi_1(acd) &= 0 \quad & \phi_2(acd) &= 0. \end{aligned}$$

Their sum $\phi = \phi_1 + \phi_2$ is defined by

$$\phi(ab) = ad, \quad \phi(bc) = cd, \quad \phi(acd) = 0$$

and is a flip homomorphism as well. Thus, the flips are not linearly independent in the tangent space at the point M on the Hilbert scheme.

Let $\mathbf{s} = \{s_1, \ldots, s_r\}$ and $\mathbf{s}' = \{s'_1, \ldots, s'_r\}$ be different flips. We say that $\mathbf{s} > \mathbf{s}'$ if for each $1 \le i \le r$ we have that s'_i is either s_i or 0. Assume that the monomials $m_1, \ldots, m_r$ are ordered by the degree-lex order so that $m_1 \succ \cdots \succ m_r$. We say that a flip $\mathbf{s}$ is a *basic flip* if it is a minimal element among the flips and the first non-zero s_i is a monomial (that is, its coefficient is 1 and not -1). Furthermore, we define the *support* of a flip $\mathbf{s}$ to be $\text{supp}(\mathbf{s}) = \{j \mid s_j \neq 0\}$. Note that a basic $\mathbf{v}$-flip has minimal support among the $\mathbf{v}$-flips.

The next result shows that the basic $\mathbf{v}$-flips form a basis of the vector space spanned by the $\mathbf{v}$-flips:

Proposition 2.5

1. *Every* $\mathbf{v}$*-flip is a linear combination with coefficients* ± 1 *of basic* $\mathbf{v}$*-flips with disjoint support.*
2. *If* $\mathbf{s}$ *and* $\mathbf{s}'$ *are basic* $\mathbf{v}$*-flips, then supp*$(\mathbf{s}) \cap$ *supp*$(\mathbf{s}') = \emptyset$*. The basic* $\mathbf{v}$*-flips are linearly independent.*

Proof 1. Let $\mathbf{s} = \{s_1, \ldots, s_r\}$ be a non-minimal element among the $\mathbf{v}$-flips. There exists a smaller $\mathbf{v}$-flip $\mathbf{s}' = \{s'_1, \ldots, s'_r\}$. Set $\mathbf{s}'' = \mathbf{s} - \mathbf{s}'$. Thus,

$$s''_i = \begin{cases} s_i & \text{if } s_i \neq s'_i, \text{ (i.e. } s'_i = 0) \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\mathbf{s}''$ satisfies (2.2.1) and (2.2.2). It also satisfies (2.3) because $\phi'' = \phi - \phi'$ is a homomorphism. Now, we have that $\mathbf{s} = \mathbf{s}' + \mathbf{s}''$ is a sum of $\mathbf{v}$-flips. Since both $\mathbf{v}$-flips $\mathbf{s}'$ and $\mathbf{s}''$ are smaller than $\mathbf{s}$, we can assume by induction hypothesis that the proposition holds for them. Note that $\text{supp}(\mathbf{s}') \cap \text{supp}(\mathbf{s}'') = \emptyset$. Therefore, the proposition holds for $\mathbf{s}$.

2. Set $\mathcal{T} = \{ i \mid s_i = s'_i \neq 0 \}$. Define a sequence $\mathbf{s}''$ by

$$s''_i = \begin{cases} s_i & \text{if } s_i = s'_i \neq 0, \text{ (i.e. } i \in \mathcal{T}) \\ 0 & \text{otherwise.} \end{cases}$$

We will show that $\mathbf{s}''$ is a $\mathbf{v}$-flip. Clearly, $\mathbf{s}''$ satisfies (2.2.1) and (2.2.2). We have to check that (2.3) holds. Let u_p, u_q be $\pm$ monomials such that $u_q m_p - u_p m_q = 0$ is a graded syzygy. We have to check that $\phi''(u_q m_p - u_p m_q) = 0$ in E/M. Thus, we have to show that $u_q s''_p - u_p s''_q \in M$.

If $p, q \in \mathcal{T}$, then $u_q s''_p - u_p s''_q = u_q s_p - u_p s_q \in M$. If $p, q \notin \mathcal{T}$, then $u_q s''_p - u_p s''_q = 0 \in M$. Suppose that $p \in \mathcal{T}$ and $q \notin \mathcal{T}$. If $s_q = 0$, then $u_q s''_p - u_p s''_q = u_q s_p - u_p s_q \in M$. If $s'_q = 0$, then $u_q s''_p - u_p s''_q = u_q s'_p - u_p s'_q \in M$. Suppose that $s_q \neq 0$ and $s'_q \neq 0$. By (2.2.2) it follows that $s_q = -s'_q \neq 0$ and $\operatorname{char}(k) \neq 2$. Then

$$\begin{aligned} 2(u_q s''_p - u_p s''_q) &= 2u_q s_p = 2u_q s_p - u_p s_q + u_p s_q = 2u_q s_p - u_p s_q - u_p s'_q \\ &= (u_q s_p - u_p s_q) + (u_q s'_p - u_p s'_q) \in M. \end{aligned}$$

Thus, $\mathbf{s}''$ is a $\mathbf{v}$-flip. Since $\mathbf{s}$ and $\mathbf{s}'$ are basic $\mathbf{v}$-flips, and $\mathbf{s}''$ is a smaller $\mathbf{v}$-flip, it follows that $\mathcal{T} = \emptyset$.

Set $\mathcal{P} = \{ i \mid s_i = -s'_i \neq 0 \}$. Define a sequence $\tilde{\mathbf{s}}$ by

$$\tilde{s}_i = \begin{cases} s_i & \text{if } s_i = -s'_i \neq 0, \text{ (i.e. } i \in \mathcal{P}) \\ 0 & \text{otherwise.} \end{cases}$$

We will show that $\tilde{\mathbf{s}}$ is a $\mathbf{v}$-flip. Clearly, $\tilde{\mathbf{s}}$ satisfies (2.2.1) and (2.2.2). We have to check that (2.3) holds. Let u_p, u_q be $\pm$ monomials such that $u_q m_p - u_p m_q = 0$ is a graded syzygy. We have to check that $\tilde{\phi}(u_q m_p - u_p m_q) = 0$ in E/M. Thus, we have to show that $u_q \tilde{s}_p - u_p \tilde{s}_q \in M$.

If $p, q \in \mathcal{P}$, then $u_q \tilde{s}_p - u_p \tilde{s}_q = u_q s_p - u_p s_q \in M$. If $p, q \notin \mathcal{P}$, then $u_q \tilde{s}_p - u_p \tilde{s}_q = 0 \in M$. Suppose that $p \in \mathcal{P}$ and $q \notin \mathcal{P}$. If $s_q = 0$, then $u_q \tilde{s}_p - u_p \tilde{s}_q = u_q s_p - u_p s_q \in M$. If $s'_q = 0$, then $u_q \tilde{s}_p - u_p \tilde{s}_q = -u_q s'_p + u_p s'_q \in M$. Suppose that $s_q \neq 0$ and $s'_q \neq 0$. By (2.2.2) it follows that $s_q = s'_q \neq 0$ contradicting to $\mathcal{T} = \emptyset$.

Thus, $\tilde{\mathbf{s}}$ is a $\mathbf{v}$-flip. Since $\mathbf{s}$ and $\mathbf{s}'$ are basic $\mathbf{v}$-flips, and $\tilde{\mathbf{s}}$ is a smaller $\mathbf{v}$-flip, it follows that $\mathcal{P} = \emptyset$.

As both $\mathcal{T} = \emptyset$ and $\mathcal{P} = \emptyset$, the desired property $\operatorname{supp}(\mathbf{s}) \cap \operatorname{supp}(\mathbf{s}') = \emptyset$ holds. □

Theorem 2.6 *Let M be a monomial ideal. The basic flips form a basis of the tangent space at the point M on the Hilbert scheme.*

Note that in the toric case, considered in [8,10], all flips are basic flips. Thus, Theorem 2.6 is the analogue of Corollary 5.2 in [10], which shows that the (toric) flips form a basis of the tangent space at a monomial point on the toric Hilbert scheme.

Proof We will show that the flips span the tangent space $\operatorname{Hom}(M, E/M)_0$. Let M be generated by monomials $m_1, \ldots, m_r$. We will show that every element in

$\mathrm{Hom}(M, E/M)_0$ is a sum of flips. Since $\mathrm{Hom}(M, E/M)_0$ is multigraded, it suffices to show that if ϕ is a homomorphism of multidegree $-\mathbf{v}$, then it is a sum of $\mathbf{v}$-flips. For each $1 \leq i \leq r$, denote by s_i the standard monomial of degree $\bar{m}_i - \mathbf{v}$ (if it exists), or if no such standard monomial exists then set $s_i = 0$. Therefore, $\phi(m_i) = \alpha_i s_i$ for some $\alpha_i \in k$. We will show that

1. some of the coefficients α_i vanish
2. some of the coefficients α_i can take any value
3. the remaining coefficients can be arranged in sets, so that the coefficients in each set are equal up to sign and the signs are uniquely determined. We call such a set an e-set.

Let u_p, u_q be $\pm$ monomials or zero, and let $p > q$. We say that u_p, u_q is a syzygy pair if $u_q m_p - u_p m_q = 0$ is a graded syzygy of M. Note that ϕ is a homomorphism if and only if for every syzygy pair u_p, u_q we have

$$u_q \phi(m_p) - u_p \phi(m_q) = u_q \alpha_p s_p - u_p \alpha_q s_q \in M \,.$$

This property holds if and only if the following conditions are satisfied:

(i) Suppose that $u_q s_p = \pm\, u_p s_q \notin M$. Then $\alpha_p = \pm\, \alpha_q$ and the sign is uniquely determined.

(ii) Suppose that (i) does not hold. If $u_p s_q \notin M$ then $\alpha_q = 0$. If $u_p s_q \in M$ then α_q can take any value. If $u_q s_p \notin M$ then $\alpha_p = 0$. If $u_q s_p \in M$ then α_p can take any value.

Therefore, the coefficients α_i satisfy (1), (2), and (3). Hence the vector space $\mathrm{Hom}(M, E/M)_{-\mathbf{v}} \cong \{\, (\alpha_1, \dots, \alpha_r) \in k^r \mid \phi \text{ is a homomorphism} \,\}$ is spanned by all the tuples $(\alpha_1, \dots, \alpha_r)$ of one of the following two types:

(a) Choose a coefficient α_i that can take any value by (ii). Set $\alpha_i = 1$ and set all other α's to be zero.

(b) Fix an e-set. Set one coefficient in this e-set to be 1. Then the α's in this e-set are ± 1 and are uniquely determined by (i). Set all other α's to be zero.

Note that a homomorphism $\phi \in \mathrm{Hom}(M, E/M)_{-\mathbf{v}}$ is a $\mathbf{v}$-flip if an only if each α_i is ± 1 or 0. Therefore, each homomorphism obtained by (a) or (b) is a $\mathbf{v}$-flip. We have shown that $\mathrm{Hom}(M, E/M)_{-\mathbf{v}}$ is spanned by the $\mathbf{v}$-flips.

By Proposition 2.5, it follows that the basic flips form a basis of $\mathrm{Hom}(M, E/M)_0$. □

Example 2.7 We illustrate how Theorem 2.6 works in a simple example. Consider the exterior algebra A with basis a, b, c, d and the ideal $L = (ab, ac, ad)$. If $\phi \in \mathrm{Hom}(L, A/L)_0$, then there exist coefficients $\beta_j \in k$ such that

$$\begin{aligned}\phi(ab) &= \beta_1 bd + \beta_2 cd + \beta_3 bc\\ \phi(ac) &= \beta_4 bd + \beta_5 cd + \beta_6 bc\\ \phi(ad) &= \beta_7 bd + \beta_8 cd + \beta_9 bc\,.\end{aligned}$$

The syzygies $aab = 0$, $aac = 0$, and $aad = 0$ yield no conditions on the coefficients. The syzygy $bab = 0$ yields $\beta_2 = 0$. The syzygy $cac = 0$ yields $\beta_4 = 0$. The syzygy $dad = 0$ yields $\beta_9 = 0$.

Furthermore, the syzygy $bac+cab = 0$ yields $\beta_1 = \beta_5$. The syzygy $bad+dab = 0$ yields $\beta_3 = -\beta_8$. The syzygy $cad + dac = 0$ yields $\beta_6 = \beta_7$. Thus, the tangent space is defined by the linear equations:

$$\beta_2 = 0, \quad \beta_4 = 0, \quad \beta_9 = 0, \quad \beta_1 - \beta_5 = 0, \quad \beta_6 - \beta_7 = 0, \quad \beta_3 + \beta_8 = 0\,.$$

Set $\gamma = \beta_1 = \beta_5$, $\nu = \beta_3 = -\beta_8$, and $\mu = \beta_6 = \beta_7$. We get that

$$\begin{aligned} \phi(ab) &= \gamma bd + \nu bc \\ \phi(ac) &= \gamma cd + \mu bc \\ \phi(ad) &= \mu bd - \nu cd \end{aligned}$$

Choosing $\gamma = 1, \nu = \mu = 0$, we obtain the flip ϕ_1 defined by $\phi_1(ab) = bd$, $\phi_1(ac) = cd$, $\phi_1(ad) = 0$. Choosing $\nu = 1, \gamma = \mu = 0$, we obtain the flip ϕ_2 defined by $\phi_2(ab) = bc$, $\phi_2(ac) = 0$, $\phi_2(ad) = -cd$. Choosing $\mu = 1, \nu = \gamma = 0$, we obtain the flip ϕ_3 defined by $\phi_3(ab) = 0$, $\phi_3(ac) = bc$, $\phi_3(ad) = bd$.

Thus, $\phi = \gamma\phi_1 + \nu\phi_2 + \mu\phi_3$ is a linear combination of basic flips.

The tangent space at L has a basis consisting of the flips corresponding to ϕ_1, ϕ_2, ϕ_3. In particular, the tangent space is three dimensional. □

3 Flips from the point of view of Gröbner basis theory

We use the notation introduced in the previous section.

Definition Let $\mathbf{s} = \{s_1, \ldots, s_r\}$ be a sequence that satisfies conditions (2.2.1) and (2.2.2). Consider the binomial ideal

$$J = \left(m_i - s_i \mid 1 \le i \le r\right).$$

We say that J is a *Gröbner flip* if the following property holds:

(3.1) J is graded (by total degree) and M is an initial ideal of J.

Lemma 3.2 *Let* $\mathbf{s} = \{s_1, \ldots, s_r\}$ *be a sequence that satisfies conditions* (2.2.1) *and* (2.2.2). *Let* u_p, u_q *be* $\pm$ *monomials such that* $u_q m_p - u_p m_q = 0$ *is a graded syzygy.*

1. *If* $s_q \neq 0$ *and* $s_p \neq 0$, *then* $u_q s_p = \pm\, u_p s_q$.
2. *If* $u_q s_p \neq u_p s_q$ *and* $\mathbf{s}$ *satisfies either* (2.3) *or* (3.1), *then* $u_q s_p,\ u_p s_q \in M$.

Proof First, we will show that if one of s_p and s_q vanishes, $u_q s_p \neq u_p s_q$, and $\mathbf{s}$ satisfies either (2.3) or (3.1), then $u_q s_p,\ u_p s_q \in M$. By symmetry, we may assume that $s_q = 0$. If (2.3) holds, then $\phi(0) = \phi(u_q m_p - u_p m_q) = u_q s_p \in M$. If (3.1) holds, then $u_q(m_p - s_p) - u_p(m_q - s_q) = -u_q s_p \in J$, so $u_q s_p$ is in the initial ideal M.

For the rest of the proof, suppose that $s_q \neq 0$ and $s_p \neq 0$.

1. We will prove that $u_q s_p = \pm u_p s_q$. Since $u_q m_p - u_p m_q = 0$, we get $\overline{u}_q + \overline{m}_p = \overline{u}_p + \overline{m}_q$, so $\overline{u}_q - \overline{u}_p = \overline{m}_q - \overline{m}_p$. By (2.2.2) we have that $\overline{m}_q - \overline{s}_q = \overline{m}_p - \overline{s}_p$. Therefore, $\overline{m}_q - \overline{m}_p = \overline{s}_q - \overline{s}_p$. Hence,

$$\overline{u}_q - \overline{u}_p = \overline{m}_q - \overline{m}_p = \overline{s}_q - \overline{s}_p.$$

We conclude that $\overline{u}_q + \overline{s}_p = \overline{u}_p + \overline{s}_q$. Therefore, $u_q s_p = \pm\, u_p s_q$.

2. Now, suppose that $u_q s_p = -u_p s_q \neq 0$ and $\{s_1, \ldots, s_r\}$ satisfy either (2.3) or (3.1). Since $u_q s_p \neq u_p s_q$ by assumption, it follows that $\mathrm{char}(k) \neq 2$.

First, assume that (2.3) holds. In this case,

$$\phi(0) = \phi(u_q m_p - u_p m_q) = u_q s_p - u_p s_q = 2u_q s_p \in M.$$

Therefore, $u_q s_p = -u_p s_q \in M$ as desired.

Now, assume that (3.1) holds. We have that

$$u_q(m_p - s_p) - u_p(m_q - s_q) = -u_q s_p + u_p s_q \in J\,,$$

so its initial term is in the initial ideal M. It follows that $u_q s_p = -u_p s_q \in M$ as desired. □

Theorem 3.3 *Let M be a monomial ideal generated by monomials $m_1, \ldots, m_r$. Let $\mathbf{s} = \{s_1, \ldots, s_r\}$ be a sequence of zeros and $\pm$ monomials not in M. Consider the binomial ideal $J = \big(m_i - s_i \mid 1 \leq i \leq r\big)$ and $\phi : M \longrightarrow E/M$ defined by $\phi(m_i) = s_i$ for each i. The ideal J is a Gröbner flip if and only if ϕ is a flip homomorphism.*

Proof First, we assume that J is a Gröbner flip and we will prove that ϕ is a well-defined homomorphism. For every $\pm$monomials u_p, u_q such that $u_q m_p - u_p m_q = 0$ is a graded syzygy we have that $\phi(u_q m_p - u_p m_q) = u_q s_p - u_p s_q \in M$ by Lemma 3.2. It follows that $\phi \in \mathrm{Hom}(M, E/M)_0$. Conditions (2.2.1) and (2.2.2) are clearly satisfied. Therefore, ϕ is a flip homomorphism.

Now, we assume that ϕ is a flip homomorphism and we will prove that J is a Gröbner flip. Properties (2.2.1) and (2.2.2) hold. Let $\mathbf{v}$ be the vector considered in (2.2.2). We have to prove (3.1). Choose a vector $\mathbf{w}$ such that $\mathbf{v} \cdot \mathbf{w} > 0$. Moreover, since for each i the monomials m_i and s_i have the same total degree, it follows that we can choose the vector $\mathbf{w}$ with strictly positive integer coordinates. Consider the weight order $\prec_{\mathbf{w}}$ defined by $\mathbf{w}$. Then, for each i we have $in_{\prec_{\mathbf{w}}}(m_i - s_i) = m_i$. Hence, $M \subseteq in_{\prec_{\mathbf{w}}}(J)$. We will show that $\big\{ m_i - s_i \mid 1 \leq i \leq r \big\}$ is a Gröbner basis of J. For every $\pm$monomials u_p, u_q such that $u_q m_p - u_p m_q = 0$ is a graded syzygy we have that the following holds. By Lemma 3.2 it follows that

$$u_q(m_p - s_p) - u_p(m_q - s_q) = -u_q s_p + u_p s_q$$

either vanishes, or is equal to a multiple of $u_q s_p$ if $s_p \neq 0$, or is equal to a multiple of $u_p s_q$ if $s_q \neq 0$. By symmetry, we can assume that $u_q(m_p - s_p) - u_p(m_q - s_q)$

is a multiple of $u_q s_p$ and $s_p \neq 0$. We want to show that $u_q s_p$ reduces to zero. By Lemma 3.2, we have that $u_q s_p \in M$. Let m_t divide $u_q s_p$.

We will show that $s_t = 0$, so $u_q s_p$ reduces to zero as desired. Suppose that $s_t \neq 0$. Then, by (2.3) we have that $\mathbf{v} = \overline{m}_t - \overline{s}_t = \overline{m}_p - \overline{s}_p$. Set $m = \prod_{\{j|v_j>0\}} x_j^{v_j}$. Since m divides m_t, it follows that m divides $u_q s_p$. However, $\gcd(m, s_p) = 1$. Hence, m divides u_q. Note that $u_q = \dfrac{m_q}{\gcd(m_q, m_p)}$. As m divides both u_q and m_p, it follows that m^2 divides m_q. This contradicts the fact that $m_q \neq 0$. Therefore, $s_t = 0$. □

Set $\tilde{E} = E \otimes k[t]$. Let $\tilde{M}$ be an ideal in $\tilde{E}$ such that $\tilde{E}/\tilde{M}$ is flat as a $k[t]$-module. For $\alpha \in k$, the quotient $\tilde{E}/\tilde{M} \otimes (k[t]/(t-\alpha))$ is denoted $(\tilde{E}/\tilde{M})_\alpha$ and is called the *fiber over* α. For any $\alpha, \beta \in k$ we say that the fibers $(\tilde{E}/\tilde{M})_\alpha$ and $(\tilde{E}/\tilde{M})_\beta$ are *connected by a deformation over* $\mathbf{A}^1_k$. Two ideals M and M' in E are *connected by a sequence of deformations over* $\mathbf{A}^1_k$ if E/M and E/M' are connected by a sequence of deformations over $\mathbf{A}^1_k$. Initial ideals and Gröbner basis theory can be used to built deformations, cf. [4, Sect. 15]. An immediate corollary of Theorems 2.6 and 3.3 is the following result:

Theorem 3.4 *The tangent space at a monomial point in the Hilbert scheme* (*over an exterior algebra*) *has a basis consisting of directions tangent to deformations built using Gröbner basis.*

It should be noted that unfortunately this result does not hold over polynomial rings.

Example 3.5 Consider the exterior algebra A with basis a, b, c, d and the ideal $L = (ab, ac, ad)$ from Example 2.7. The sequence $\{bc, 0, -cd\}$ is a flip because L is the initial ideal of the binomial ideal $(ab - bc, ac, ad + cd)$ with respect to the lexicographic order. We have the flip homomorphism ϕ defined by

$$\phi(ab) = bc\,, \quad \phi(ac) = 0\,, \quad \phi(ad) = -cd.$$

Example 3.6 The definitions of a flip homomorphism and a Gröbner flip can be introduced over a polynomial ring (over k) along the lines of [1]. We remark that the same proof as in Theorem 3.3 shows (in the polynomial ring case) that if J is a Gröbner flip, then ϕ is a flip homomorphism. However, there is a simple example that the property in Theorem 3.3 does not hold in a polynomial ring if we do not require that the monomials $m_1, \ldots, m_r$ are square-free. Let S be the polynomial ring $k[x, y, z]$ and $M = (xy^2, y^4)$. Consider ϕ defined by $\phi(xy^2) = xz^2$ and $\phi(y^4) = 0$. This is a well-defined homomorphism because there is only one minimal syzygy on the generators, and we have that $\phi(y^2xy^2 - xy^4) = y^2xz^2 - 0 = xy^2z^2$ vanishes in S/M. Now, consider the binomial ideal $J = (xy^2 - xz^2, y^4)$. Let $\prec$ be a monomial order such that $in_\prec(xy^2 - xz^2) = xy^2$. Considering the s-pair $xy^2 - xz^2,\ y^4$ we get

$$y^2(xy^2 - xz^2) - xy^4 + z^2(xy^2 - xz^2) = -xz^4 \in J\,,$$

but $xz^4 \notin M$. Thus, M is not an initial ideal of J. In this case, the homomorphism flip ϕ does not come from a Gröbner flip. □

Theorem 3.7 *Let $J = \big(m_i - s_i \mid 1 \le i \le r\big)$ be a Gröbner flip. Fix a monomial order $\prec$ such that $s_i \succ m_i$ for each i with $s_i \neq 0$. Denote by M_{flip} the initial ideal of J with respect to this order. There exist only two monomial ideals that are initial ideals of J, and these two initial ideals are M and M_{flip}. The ideal J has a universal Gröbner basis consisting of $\{m_i - s_i \mid 1 \le i \le r$ and $s_i \neq 0\}$ and some monomials.*

Proof Let $\mathcal{T}$ be the set of all vectors $\mathbf{w}$ such that $\mathbf{w} \cdot \mathbf{v} > 0$, and let $\mathcal{P}$ be the set of all vectors $\mathbf{w}'$ such that $\mathbf{w}' \cdot \mathbf{v} < 0$.

Let $\mathbf{w} \in \mathcal{T}$. With respect to the order defined by the weight vector $\mathbf{w}$ we have that the set $\{ m_i - s_i \mid 1 \le i \le r \}$ is a Gröbner basis since M is the initial ideal by (3.1).

Let $\mathbf{w}' \in \mathcal{P}$. Consider the order defined by the weight vector $\mathbf{w}'$. We will show that every new element produced by the Buchberger algorithm is a monomial. First, note that if an s-pair consists of a monomial and a binomial, and if the reduction set consists of monomials and binomials, then the remainder is either zero or a (scalar multiple of a) monomial. Now consider an s-pair that consists of two binomials $m_p - s_p$ and $m_q - s_q$; let $u_q s_p - u_p s_q$ be the graded syzygy for this s-pair. We would like to apply Lemma 3.2. Set $m'_i = s_i$ for each i such that $s_i \neq 0$, and set $m'_i = m_i$ for each i such that $s_i = 0$. Let M' be the monomial ideal generated by $m'_1, \dots, m'_r$. Furthermore, set $s'_i = m_i$ for each i such that $s_i \neq 0$, and set $s'_i = 0$ for each i such that $s_i = 0$. We can apply Lemma 3.2 to the set $\{s'_1, \dots, s'_r\}$ since this set satisfies (2.2.1) and (2.2.2) for the monomials $\{m'_1, \dots, m'_r\}$. Now,

$$u_q(m_p - s_p) - u_p(m_q - s_q) = u_q(s'_p - m'_p) - u_p(s'_q - m'_q)$$

either vanishes or is $\pm 2u_p m'_q$ by Lemma 3.2(1); when we reduse this using the reduction set of monomials and binomials we get a remainder that is either zero or a (scalar multiple of a) monomial. Thus, every new element produced by the Buchberger algorithm is a monomial. Therefore, the initial ideal of J is the same with respect to every weight vector in $\mathcal{P}$. Furthermore, the Gröbner basis has the desired form.

We showed that M and M_{flip} are all the initial ideals of J, and that J has a universal Gröbner basis of the desired form. □

Remark 3.8 In [9] we prove that the Hilbert scheme, that parameterizes all ideals with a fixed Hilbert function over E, is connected. The binomial ideals used in our constructions are Gröbner flips. □

4 Flips from a combinatorial point of view

In this section we consider flips as operations on simplicial complexes via the Stanley–Reisner theory. We use the notation introduced in the previous sections. Let M be a monomial ideal generated by monomials $m_1, \dots, m_r$. Let $\mathbf{s} = \{s_1, \dots, s_r\}$ be a flip. Consider the binomial ideal $J = \big(m_i - s_i \mid 1 \le i \le r\big)$ which is a Gröbner flip. Fix a monomial order $\prec$ such that $s_i \succ m_i$ for each i with $s_i \neq 0$. Denote by M_{flip} the initial ideal of J with respect to this order. We say that the monomial ideal M_{flip} is a *flip* of M. Let Δ and Δ_{flip} be the corresponding Stanley–Reisner simplicial complexes of the ideals M and M_{flip}. We say that Δ_{flip} is a *flip* of Δ.

Example 4.1 Consider the exterior algebra A with basis a, b, c, d and the ideal $M = (abc)$. The corresponding Stanley–Reisner simplicial complex Δ is obtained by removing the triangle abc from the tetrahedra on vertices a, b, c, d. Any simplicial complex obtained from this tetrahedra by removing one triangle is a flip of Δ. Other examples of flips are given in Examples 2.4 and 3.5. □

Example 4.2 A similar concept is introduced in [1], but Altmann and Sturmfels work over a polynomial ring. Their definition of a flip is different: a flip by the definition in [1] may not be a flip according to our definition. We present a simple example. It is well known that two square-free monomial ideals have the same Hilbert function over the exterior algebra A on n variables $x_1, \dots, x_n$ if and only if they have the same Hilbert function over the polynomial ring $B = k[x_1, \dots .x_n]$. However, we will show that the flips over A and over B do not coincide for a square-free monomial ideal. Let Δ be a simplicial complex on n vertices, and I_Δ be its Stanley–Reisner monomial ideal. By (2.3), it follows that if $\{s_1, \dots, s_r\}$ is a flip over A, then $\{|s_1|, \dots, |s_r|\}$ is a flip over B (here, $|\cdot|$ means that we remove the sign). However, a flip over B may not lead to a flip over A. For example, if $I_\Delta = (ab)$, then cd is a flip over B, but it is not a flip over A. The point is that a monomial ideal has some new minimal syzygies over the exterior algebra in addition to the minimal syzygies over a polynomial ring. □

Proposition 4.3 *Let Δ be a simplicial complex with Stanley–Reisner monomial ideal M. Let M_{flip} be a flip of M. The ideal M is a flip of M_{flip}; equivalently, Δ is a flip of Δ_{flip}.*

Proof After renumbering if necessary, we can assume that $s_i \neq 0$ for $1 \leq i \leq q$ and $s_i = 0$ for $q < i \leq r$. Apply Theorem 3.7: let $t_{q+1} = m_{q+1}, \dots, t_r = m_r, t_{r+1}, \dots, t_p$ be monomials such that

$$\{ m_i - s_i \mid 1 \leq i \leq q \} \cup \{ t_{q+1}, \dots, t_p \}$$

is a universal Gröbner basis of J. It follows that $M = (m_1, \dots, m_q, t_{q+1}, \dots, t_p)$ and $M_{\text{flip}} = (s_1, \dots, s_q, t_{q+1}, \dots, t_p)$ are the two initial ideals of J. Conditions (2.2.1) and (2.2.2) are clearly satisfied. Hence, M is a flip of M_{flip}. □

Theorem 2.6 leads to Theorem 4.4 which provides a criterion on which simplicial complexes are flippable. The analogues problem in the toric case "Find a criterion whether a given triangulation of the convex hull of a set of n points in $\mathbf{N}^d \setminus \mathbf{0}$ has a (toric) flip" is completely open; in particular, the problem is open for unimodular triangulations.

Theorem 4.4 *Let M be a monomial ideal. It has a flip if and only if M is not a power of the maximal ideal $(x_1, \dots, x_n)$.*

Proof By Theorem 2.6 it follows that M does not have a non-trivial flip if and only if M is an isolated point on the Hilbert scheme. By [9], the Hilbert scheme is connected. Therefore, M does not have a non-trivial flip if and only if M is the only point on the Hilbert scheme.

If M is a power of the maximal ideal $(x_1, \dots, x_n)$, then clearly M is the only point on the Hilbert scheme.

Suppose that M is the only point on the Hilbert scheme. Let p be the minimal degree in which M has a minimal monomial generator. Changing coordinates and then taking initial ideal, we obtain a monomial ideal with the same Hilbert function, so it has to coincide with M. It follows that M contains any monomial of degree p. Hence $M = (x_1, \dots, x_n)^p$. □

The following result is a corollary of our proof in [9].

Theorem 4.5 *Every two simplicial complexes with the same f-vector* (*equivalently, the corresponding Stanley–Reisner monomial ideals have the same Hilbert function*) *are connected by a sequence of flips and algebraic shiftings.*

In the spirit of the Baues conjecture, one could consider the question whether every two simplicial complexes with the same f-vector are connected by a sequence of flips.

Example 4.6 Consider the exterior algebra A with basis a, b, c, d. The ideals (ac) and (bd) are not connected by a flip by Remark 4.2. However, they are connected by the following sequence of two flips: (ab) is a flip of (ac), and (bd) is a flip of (ab). □.

In view of Theorem 4.5, the question whether every two simplicial complexes with the same f-vector are connected by a sequence of flips comes to:

Question 4.7 *Is it true that every monomial ideal is connected by a sequence of flips to its generic initial ideal* (*with respect to a fixed order*)*?*

5 An example

In this section we present an example computed using the computer algebra system *Macaulay 2* [7]. We plan to write an expository paper which will include the techniques used in these computations, as well as other examples.

Let A be the exterior algebra over an infinite field k on letters a, b, c, d, e. We consider the Hilbert scheme H parametrizing all ideals of A with the same Hilbert function as a generic quadric. This Hilbert function is $1, 5, 9, 5, 0$. There are 210 monomial ideals on $\mathcal{H}$. Up to S_5-action, there are only four monomial ideals:

$$\begin{aligned} L &= (ab, acd, ace, bcde) \\ B &= (ab, acd, bcd) \\ C &= (ab, acd, bce) \\ D &= (ab, acd, cde). \end{aligned}$$

The ideal L is lexicographic, B is Borel-fixed, the orbits of C and D contain no Borel-fixed ideals. There are 60 ideals in each of the orbits of L, C, and D. There are 30 ideals in the orbit of B. Each of the ideals L, C, D has a 16 dimensional tangent space, and B has a 17 dimensional tangent space. So each of L, C, D has 16 basic flips, and B has 17 basic flips.

The Hilbert scheme $\mathcal{H}$ has two components X_1 and X_2, of dimensions $\dim X_1 = 16$, and $\dim X_2 = 9$. Both X_1 and X_2 are smooth and rational, and the intersection $X_1 \cap X_2$ has dimension 8.

Every ideal on X_2 which is not on the intersection $X_1 \cap X_2$ is generated by a quadric which is not a product of linear forms. In particular, the general element of X_2 is generated by such a quadric. It follows that any monomial ideal on X_2 is in the intersection $X_1 \cap X_2$.

The ideals L, C, D are smooth points on $\mathcal{H}$, which lie on X_1, but not on X_2. Thus, there are 180 smooth monomial ideals on X_1, not on X_2.

The ideal B is on the intersection $X_1 \cap X_2$. Thus, there are 30 monomial ideals on $X_1 \cap X_2$.

The ideal B has 5 basic flips which lie in $X_1 \cap X_2$, and 12 basic flips which do not lie on X_2. Each of L, C, D has 2 basic flips which lie in $X_1 \cap X_2$, and 14 basic flips which do not lie on X_2.

In each of the orbits of L, C, D, we have that every two monomial ideals in the orbit are connected by a sequence of flips. This property does not hold for the orbit of B. The ideals in this orbit break into five groups with six ideals in each group, so that every two monomial ideals in the same group are connected by a sequence of flips.

Every monomial ideal has a flip that is an ideal in the orbit of L. Therefore, the total set of monomial ideals on $\mathcal{H}$ is connected by flips.

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References

1. Altmann, K., Sturmfels, B.: The graph of a monomial ideal. J. Pure Appl. Alg. (to appear)
2. Aramova, A., Herzog, J., Hibi, T.: Gotzmann theorems for exterior algebras and combinatorics. J. Algebra **191**, 353–378 (1998)
3. Bayer, D.: The division algorithm and the Hilbert scheme. Ph.D. Thesis, Harvard University (1982)
4. Eisenbud, D.: Commutative algebra with a view towards algebraic geometry. Springer, New York (1995)
5. Eisenbud, D., Fløystad, G., Schreyer, F.-O.: Sheaf cohomology and free resolutions over exterior algebras . Trans. Am. Math. Soc. **355**, 4397–4426 (2003)
6. Eisenbud, D., Schreyer, F.-O., Weyman, J.: Resultants and Chow forms via Exterior Syzygies. J. Am. Math. Soc. **16**, 537–579 (2003)
7. Grayson, D., Stillman, M.: MACAULAY 2—a system for computation in algebraic geometry and commutative algebra (1997). Available from http://www.math.uiuc.edu/Macaulay2/
8. Maclagan, D., Thomas, R.: Combinatorics of the toric Hilbert scheme. Discrete Comput. Geom. **27**, 249–264 (2002)
9. Peeva, I., Stillman, M.: Connectedness of Hilbert schemes. J. Algebraic Geom. (to appear)
10. Peeva, I., Stillman, M.: Toric Hilbert schemes. Duke Math. J. **111**, 419–449 (2002)
11. Reeves, A., Stillman, M.: Smoothness of the lexicographic point. J. Algebraic Geom. **6**, 235–246 (1997)