

Boundary Harnack principle for Brownian motions with measure-valued drifts in bounded Lipschitz domains

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Abstract Let $\mu = (\mu^1, \dots, \mu^d)$ be such that each μ^i is a signed measure on \mathbf{R}^d belonging to the Kato class $\mathbf{K}_{d,1}$. A Brownian motion in \mathbf{R}^d with drift μ is a diffusion process in \mathbf{R}^d whose generator can be informally written as $\frac{1}{2}\Delta + \mu \cdot \nabla$. When each μ^i is given by $U^i(x)dx$ for some function U^i , a Brownian motion with drift μ is a diffusion in \mathbf{R}^d with generator $\frac{1}{2}\Delta + U \cdot \nabla$. In Kim and Song (Ill J Math 50(3):635–688, 2006), some properties of Brownian motions with measure-value drifts in bounded smooth domains were discussed. In this paper we prove a scale invariant boundary Harnack principle for the positive harmonic functions of Brownian motions with measure-value drifts in bounded Lipschitz domains. We also show that the Martin boundary and the minimal Martin boundary with respect to Brownian motions with measure-valued drifts coincide with the Euclidean boundary for bounded Lipschitz domains. The results of this paper are also true for diffusions with measure-valued drifts, that is, when Δ is replaced by a uniformly elliptic divergence form operator $\sum_{i,j=1}^d \partial_i(a_{ij}\partial_j)$ with C^1 coefficients or a uniformly elliptic non-divergence form operator $\sum_{i,j=1}^d a_{ij}\partial_i\partial_j$ with C^1 coefficients.

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1 Introduction

The boundary Harnack principle in Lipschitz domains was first proved independently for the Laplacian Δ in [1, 14, 26]. Later the boundary Harnack principle was extended to uniformly elliptic operators with bounded coefficients in [8, 16] (see [4] for an extension to more general domains using probabilistic methods).

The main purpose of this paper is to prove a boundary Harnack principle in bounded Lipschitz domains for diffusions with measure-valued drifts. We will concentrate on Brownian motions with measure-valued drifts. The assumption on the drift is that each component of the drift belongs to the Kato class $\mathbf{K}_{d,1}$ (see below for the definition). Our main results generalize the corresponding results of [8, 16] where only bounded drifts were considered.

In this paper, we always assume that $d \geq 3$. Suppose $\mu = (\mu^1, \dots, \mu^d)$ is such that each μ^i is a signed measure on \mathbf{R}^d belonging to the Kato class $\mathbf{K}_{d,1}$, that is

$$\limsup_{r \downarrow 0} \sup_{x \in \mathbf{R}^d} \int_{|x-y| \leq r} \frac{|\mu^i|(dy)}{|x-y|^{d-1}} = 0, \quad 1 \leq i \leq d.$$

Informally, a Brownian motion in \mathbf{R}^d with drift μ is a diffusion process in \mathbf{R}^d with generator $\frac{1}{2}\Delta + \mu \cdot \nabla$. When each μ^i is given by $U^i(x)dx$ for some function U^i , a Brownian motion with drift μ is a diffusion in \mathbf{R}^d with generator $\frac{1}{2}\Delta + U \cdot \nabla$ and it is a solution to the SDE

$$dX_t = dW_t + U(X_t) \cdot dt.$$

To recall the precise definition of a Brownian motion with drift μ in $\mathbf{K}_{d,1}$, we fix a non-negative smooth radial function $\varphi(x)$ in \mathbf{R}^d with $\text{supp}[\varphi] \subset B(0, 1)$ and $\int \varphi(x)dx = 1$. For any positive integer n , we put $\varphi_n(x) = 2^{nd}\varphi(2^n x)$. For $1 \leq i \leq d$, define

$$U_n^i(x) = \int \varphi_n(x - y)\mu^i(dy).$$

Put $U_n(x) = (U_n^1(x), \dots, U_n^d(x))$. The following definition is taken from [6].

Definition 1.1 Suppose $\mu = (\mu^1, \dots, \mu^d)$ is such that each μ^i is a signed measure on \mathbf{R}^d belonging to the Kato class $\mathbf{K}_{d,1}$. A Brownian motion with drift μ is a family of probability measures $\{\mathbf{P}_x : x \in \mathbf{R}^d\}$ on $C([0, \infty), \mathbf{R}^d)$, the space of continuous \mathbf{R}^d -valued functions on $[0, \infty)$, such that under each \mathbf{P}_x we have

$$X_t = x + W_t + A_t$$

where

- (a) $A_t = \lim_{n \rightarrow \infty} \int_0^t U_n(X_s)ds$ uniformly in t over finite intervals, where the convergence is in probability;

(b) there exists a subsequence $\{n_k\}$ such that

$$\sup_k \int_0^t |U_{n_k}(X_s)| ds < \infty$$

almost surely for each $t > 0$;

(c) W_t is a standard Brownian motion in \mathbf{R}^d starting from the origin.

In this paper, we will fix a $\mu = (\mu^1, \dots, \mu^d)$ with each $\mu^i \in \mathbf{K}_{d,1}$ and use X to denote a Brownian motion with drift μ defined above. The existence and uniqueness of X were established in [6]. In fact, X was shown to be a Feller process.

Bass and Chen raised the following question in [6]: do the Harnack principle and the boundary Harnack principle hold for the positive harmonic functions of X ? Recall that a non-negative function f on a domain D is said to be harmonic in D with respect to X if for every relatively compact open subset B of D with $\bar{B} \subset D$,

$$f(x) = \mathbf{E}_x[f(X_{\tau_B})], \quad x \in D$$

where $\tau_B = \inf\{s > 0 : X_s \notin B\}$.

In [21], we showed that X has a density $q(t, x, y)$ which is continuous on $(0, \infty) \times \mathbf{R}^d \times \mathbf{R}^d$ and that there exist positive constants $C_i, i = 1, \dots, 9$ such that

$$C_1 e^{-C_2 t} t^{-\frac{d}{2}} e^{-\frac{C_3|x-y|^2}{2t}} \leq q(t, x, y) \leq C_4 e^{C_5 t} t^{-\frac{d}{2}} e^{-\frac{C_6|x-y|^2}{2t}} \quad (1.1)$$

and

$$|\nabla_x q(t, x, y)| \leq C_7 e^{C_8 t} t^{-\frac{d+1}{2}} e^{-\frac{C_9|x-y|^2}{2t}} \quad (1.2)$$

for all $(t, x, y) \in (0, \infty) \times \mathbf{R}^d \times \mathbf{R}^d$. We also showed that, for every bounded $C^{1,1}$ domain D , X^D (the process obtained by killing X upon exiting from D) has a density q^D which is continuous on $(0, \infty) \times D \times D$ and that for any $T > 0$, there exist positive constants $C_i, i = 10, \dots, 14$, such that

$$C_{10} \psi_D(t, x, y) t^{-\frac{d}{2}} e^{-\frac{C_{11}|x-y|^2}{t}} \leq q^D(t, x, y) \leq C_{12} \psi_D(t, x, y) t^{-\frac{d}{2}} e^{-\frac{C_{13}|x-y|^2}{t}} \quad (1.3)$$

and

$$|\nabla_x q^D(t, x, y)| \leq C_{14} \left(1 \wedge \frac{\rho(y)}{\sqrt{t}}\right) t^{-\frac{d+1}{2}} e^{-\frac{C_{13}|x-y|^2}{t}} \quad (1.4)$$

for all $(t, x, y) \in (0, T) \times D \times D$, where $a \wedge b := \min\{a, b\}$, $\rho(x)$ is the distance between x and ∂D and

$$\psi_D(t, x, y) := \left(1 \wedge \frac{\rho(x)}{\sqrt{t}}\right) \left(1 \wedge \frac{\rho(y)}{\sqrt{t}}\right).$$

(See Sect. 4 of [18] for estimates of heat kernels of the Schrödinger-type equations for X^D .) By using these estimates, we established in [21] that the Harnack principle holds

for the positive harmonic functions of X and that the boundary Harnack principle holds for the positive harmonic functions of X in bounded $C^{1,1}$ domains.

However, the boundary Harnack principle established in [21] is not scale invariant and the domains are assumed to be bounded $C^{1,1}$. The purpose of this paper is to prove a scale invariant version of the boundary Harnack principle for bounded Lipschitz domains and to identify the Martin boundary.

A nice framework for the potential theory of strong Markov processes without the duality assumption was proposed in [13]. In this paper, we first show that Brownian motions with measure-valued drifts fit nicely into this framework. Then we revisit the Green function estimates obtained in [21] and show that, in fact, the Green function for a Brownian motion with measure-valued drift is uniformly comparable to that of a Brownian motion in small sets. Using an argument similar to that of the proof of Theorem 2.2 in [9], we prove that the comparability of Green functions implies the comparability of harmonic measures.

Bass and Burdzy [3, 5] introduced a probabilistic method, the so-called box method, to prove the boundary Harnack principle. In this paper, by using this method and combining the uniform comparability of harmonic measures with the results of [21], we prove a Carleson type estimate and a scale invariant boundary Harnack principle for Brownian motions with measure-valued drifts in bounded Lipschitz domains.

Finally, we apply the Carleson type estimate and the boundary Harnack principle together with the results of [21] to identify the Martin boundary with respect to Brownian motions with measure-valued drifts in bounded Lipschitz domains. Unlike the case of symmetric processes, identifying the Martin boundary with the Euclidean boundary is very delicate in the present case. We show that if D is a bounded Lipschitz domain, there exists a homeomorphism between the Martin compactification and \bar{D} which is an identity map in D . Unlike [21], there are no sharp estimates on the Green functions of Brownian motions with measure-valued drifts in bounded Lipschitz domains. Instead we use the Carleson type estimate and the boundary Harnack principle. The scale invariant property of our boundary Harnack principle is critical in our arguments.

The content of this paper is organized as follows. In Sect. 2, we prove some basic properties of Brownian motion with measure-valued drift in arbitrary bounded open sets. In Sect. 3, we use the estimates of [21] to prove that the harmonic measures with respect to X are uniformly (under scaling and translation) comparable to the corresponding harmonic measures with respect to Brownian motion if the domain D is shrunk small enough. In Sect. 4, we prove a scale invariant boundary Harnack principle. Section 5 deals with Martin boundary and the Martin representation. We prove that the Martin boundary and the minimal Martin boundary of killed Brownian motion with measure-valued drift can all be identified with the Euclidean boundary if the domain is bounded and Lipschitz. In the final section we briefly indicate that all the results remain valid for diffusions with measure-valued drifts, that is, when Δ is replaced by a uniformly elliptic divergence form operator $\sum_{i,j=1}^d \partial_i (a_{ij} \partial_j)$ with C^1 coefficients or a uniformly elliptic non-divergence form operator $\sum_{i,j=1}^d a_{ij} \partial_i \partial_j$ with C^1 coefficients.

In this paper, we will use the following convention: the values of the constants M_1, M_2, \dots will remain the same throughout this paper, while the values of the

constants C_1, C_2, \dots might change from one appearance to another. The labeling of the constants C_1, C_2, \dots starts anew in the statement of each result. In this paper, we use “:=” to denote a definition, which is read as “is defined to be”.

2 Some properties of Brownian motions with measure-valued drifts in bounded domains

Throughout this paper we assume that $\mu = (\mu^1, \dots, \mu^d)$ is fixed with each μ^i being a signed measure on \mathbf{R}^d belonging to $\mathbf{K}_{d,1}$ and that X is a Brownian motion with drift μ . In this section, we will study some basic properties of X which will be needed later.

We know from (1.1) that there exist c_1, c_2 and c_3 such that for every positive t_0 and δ ,

$$\begin{aligned} \sup_{t \leq t_0, |x-y| > \delta} q(t, x, y) &\leq c_1 e^{c_2 t_0} \sup_{t \leq t_0, |x-y| > \delta} t^{-\frac{d}{2}} e^{-c_3 \frac{|x-y|^2}{t}} \\ &\leq c_1 e^{c_2 t_0} \sup_{t \leq t_0} t^{-\frac{d}{2}} e^{-c_3 \frac{\delta^2}{t}} < \infty. \end{aligned} \tag{2.1}$$

Since

$$\lim_{t_0 \downarrow 0} \sup_{t \leq t_0} t^{-\frac{d}{2}} e^{-c_3 \frac{\delta^2}{t}} = 0,$$

we have

$$\lim_{t_0 \downarrow 0} \sup_{t < t_0, |x-y| > \delta} q(t, x, y) = 0. \tag{2.2}$$

Moreover, using (2.1), we get

$$\begin{aligned} \sup_{t \leq t_0, x \in \mathbf{R}^d} \mathbf{P}_x(|X_t - x| \geq \delta) &\leq c_1 e^{c_2 t_0} \sup_{t \leq t_0, x \in \mathbf{R}^d} \int_{|x-y| \geq \delta} t^{-\frac{d}{2}} e^{-c_3 \frac{|x-y|^2}{t}} dy \\ &= c_4 e^{c_2 t_0} \sup_{t \leq t_0} \int_{\delta}^{\infty} t^{-\frac{d}{2}} r^{d-1} e^{-c_3 \frac{r^2}{t}} dr \leq c_5 e^{c_2 t_0} \int_{\frac{\delta}{\sqrt{t_0}}}^{\infty} u^{d-1} e^{-c_3 u^2} du \end{aligned}$$

for some $c_4 = c_4(d)$ and $c_5 = c_5(d)$. Thus

$$\lim_{t_0 \downarrow 0} \sup_{t \leq t_0, x \in \mathbf{R}^d} \mathbf{P}_x(|X_t - x| \geq \delta) = \lim_{t_0 \downarrow 0} \sup_{t \leq t_0, x \in \mathbf{R}^d} \mathbf{P}_x(X_t \in \mathbf{R}^d \setminus B(x, \delta)) = 0. \tag{2.3}$$

For any open set D , we use τ_D to denote the first exit time of D , i. e., $\tau_D = \inf\{t > 0 : X_t \notin D\}$. Using (2.3) we can easily prove the next lemma.

Lemma 2.1 For any $\delta > 0$, we have

$$\limsup_{s \downarrow 0} \sup_{x \in \mathbf{R}^d} \mathbf{P}_x(\tau_{B(x,\delta)} \leq s) = 0.$$

Proof For any $t > 0$ and any Borel set A in \mathbf{R}^d , we put $N_t(x, A) = \mathbf{P}_x(X_t \in A)$. Then by an extended version of the strong Markov property (see [7, pp. 43–44]), we have for every $x \in \mathbf{R}^d$,

$$\begin{aligned} \mathbf{P}_x(\tau_{B(x,\delta)} \leq s) &\leq \mathbf{P}_x\left(\tau_{B(x,\delta)} \leq s, X_s \in B\left(x, \frac{\delta}{2}\right)\right) + \mathbf{P}_x\left(X_s \in B\left(x, \frac{\delta}{2}\right)^c\right) \\ &\leq \mathbf{E}_x\left[N_{s-\tau_{B(x,\delta)}}\left(X_{\tau_{B(x,\delta)}}, B\left(x, \frac{\delta}{2}\right)^c\right); \tau_{B(x,\delta)} \leq s\right] + \mathbf{P}_x\left(X_s \in B\left(x, \frac{\delta}{2}\right)^c\right) \end{aligned}$$

Now the conclusion of the lemma follows from (2.3). □

Recall that, a point z on the boundary ∂D of an open set D is said to be a regular boundary point if $\mathbf{P}_x(\tau_D = 0) = 1$. An open set D is said to be regular if every point in ∂D is a regular boundary point.

Proposition 2.2 Suppose D is an open subset of \mathbf{R}^d and $z \in \partial D$. If there is a cone A with vertex z such that $A \cap B(z, r) \subset D^c$ for some $r > 0$, then z is a regular boundary point of D .

Proof Without loss of generality, we may assume that $z = 0$. For $n \geq 1$, put $r_n = r/n$. Under \mathbf{P}_0 , we have

$$\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{X_{r_n} \in A \cap B(0, r)\} \subset \{\tau_D = 0\}.$$

Hence

$$\begin{aligned} \mathbf{P}_0(\tau_D = 0) &\geq \mathbf{P}_0\left(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{X_{r_n} \in A \cap B(0, r)\}\right) \\ &\geq \limsup_{n \rightarrow \infty} \mathbf{P}_0(X_{r_n} \in A \cap B(0, r)) \\ &\geq \limsup_{n \rightarrow \infty} c_1 \int_{A \cap B(0,r)} r_n^{-\frac{d}{2}} e^{-\frac{c_2|x|^2}{n}} dx \\ &\geq \limsup_{n \rightarrow \infty} c_1 \int_{A \cap B(0,n)} e^{-c_2|y|^2} dy > 0. \end{aligned}$$

The assertion of the proposition now follows from Blumenthal’s zero-one law (Proposition I.5.17 in [7]). □

This result implies that all bounded Lipschitz domains, and in particular, all bounded $C^{1,1}$ domains, are regular. Repeating the argument in the second part of the proof of Theorem 1.23 in [11], we immediately get the following result.

Proposition 2.3 *Suppose that D is a domain in \mathbf{R}^d and f is a bounded Borel function on ∂D . If z is a regular boundary point of D and f is continuous at z , then*

$$\lim_{D \ni x \rightarrow z} \mathbf{E}_x[f(X_{\tau_D}); \tau_D < \infty] = f(z).$$

Proof We omit the details. □

Given an open set $D \subset \mathbf{R}^d$, we define $X_t^D(\omega) = X_t(\omega)$ if $t < \tau_D(\omega)$ and $X_t^D(\omega) = \partial$ if $t \geq \tau_D(\omega)$, where ∂ is a cemetery state. The process X^D is called a killed Brownian motion with drift μ in D . Throughout this paper, we use the convention $f(\partial) = 0$.

Let

$$k^D(t, x, y) := \mathbf{E}_x[q(t - \tau_D, X_{\tau_D}, y) : \tau_D < t]$$

and $q^D(t, x, y) := q(t, x, y) - k^D(t, x, y).$

Then $q^D(t, x, y)$ is the transition density of X^D i.e., for every $t > 0$ and Borel set A ,

$$\mathbf{P}_x(X_t^D \in A) = \int_A q^D(t, x, y) dy. \tag{2.4}$$

For the proof of (2.4), see [3, pp. 121–122] where only the strong Markov property was used.

Using some standard arguments (for example, [3, 11]), we can show the following.

Theorem 2.4 *For every $t > 0$, $q^D(t, x, y)$ is jointly continuous on $D \times D$. For every $t > 0$ and $s > 0$,*

$$q^D(t + s, x, y) = \int_D q^D(t, x, z) q^D(s, z, y) dz. \tag{2.5}$$

For every $(t, x, y) \in (0, \infty) \times D \times D$, $q^D(t, x, y)$ is strictly positive. If z is a regular boundary point of D , then for any $t > 0$ and $y \in D$,

$$\lim_{D \ni x \rightarrow z} q^D(t, x, y) = 0. \tag{2.6}$$

Proof First, we show that $k^D(t, x, \cdot)$ is continuous in D for each $(t, x) \in (0, \infty) \times D$. Suppose $(t_0, x_0, y_0) \in (0, \infty) \times D \times D$ and y_n converges to y_0 in D . Let $\delta_0 := \text{dist}(y_0, \partial D)$. Choose n_0 large such that $\text{dist}(y_n, \partial D) > \frac{1}{2}\delta_0$ for every $n \geq n_0$. Given $\varepsilon > 0$, using (2.2), we can choose $s > 0$ small such that

$$\sup_{n \geq n_0} \mathbf{E}_{x_0} [q(t_0 - \tau_D, X_{\tau_D}, y_n) : t_0 - s < \tau_D < t_0] < \frac{\varepsilon}{2}.$$

So using (2.1), the continuity of q and the bounded convergence theorem, we get

$$\lim_{n \rightarrow \infty} \mathbf{E}_{x_0} [q(t_0 - \tau_D, X_{\tau_D}, y_n) : t_0 - s \geq \tau_D] = 0.$$

Now assume $(t_0, x_0, y_0) \in (0, \infty) \times D \times D$ and x_n converges to x_0 in D . Let

$$h(s, x) := \mathbf{E}_x [\mathbf{E}_{X_s} [q(t_0 - s - \tau_D, X_{\tau_D}, y_0) : \tau_D < t_0 - s]], \quad s < t_0.$$

We know from (2.1) that $\mathbf{E}_{X_s} [q(t_0 - s - \tau_D, X_{\tau_D}, y_0) : \tau_D < t_0 - s]$ is bounded. So by using the continuity of q , $h(s, \cdot)$ is continuous in D for every $s < t_0$. Now using the Markov property, we have

$$\begin{aligned} & \left| k^D(t_0, x, y_0) - h(s, x) \right| \\ &= \left| \mathbf{E}_x [q(t_0 - \tau_D, X_{\tau_D}, y_0) : \tau_D \leq s] \right. \\ & \quad \left. + \mathbf{E}_x [\mathbf{E}_{X_s} [q(t_0 - s - \tau_D, X_{\tau_D}, y_0) : \tau_D < t_0 - s]; s < \tau_D < t] - h(s, x) \right| \\ &\leq \mathbf{E}_x [q(t_0 - \tau_D, X_{\tau_D}, y_0) : \tau_D \leq s] \\ & \quad + \mathbf{E}_x [\mathbf{E}_{X_s} [q(t_0 - s - \tau_D, X_{\tau_D}, y_0) : \tau_D < t_0 - s]; \tau_D \leq s], \end{aligned}$$

which is less than

$$2c_1 e^{c_2 t_0} \left(\sup_{t \leq t_0} t^{\frac{d}{2}} e^{-c_3 \frac{\delta_0^2}{t}} \right) \mathbf{P}_x(\tau_D \leq s).$$

By Lemma 2.1, the quantity above converges to zero uniformly on any compact subset K of D as $s \rightarrow 0$. Thus, given ε , by choosing s_0 small first and then choosing n_0 large, we get for $n \geq n_0$

$$\begin{aligned} & \left| k^D(t_0, x_n, y_0) - k^D(t_0, x_0, y_0) \right| \\ &\leq 2 \sup_{m \geq 0} \left| k^D(t_0, x_m, y_0) - h(s_0, x_m) \right| + |h(s_0, x_n) - h(s_0, x_0)| < \varepsilon. \end{aligned}$$

Since $q(t, x, y)$ is continuous, we can now conclude that $q^D(t, x, y)$ is continuous in $x \in D$ and in $y \in D$. (2.5) can be proved using the Markov property and the continuity of $q^D(t, x, \cdot)$ (see [11, p. 35]).

We will prove the joint continuity of $q^D(t, \cdot, \cdot)$ by using (2.5) and (1.1). Assume $(t_0, x_0, y_0) \in (0, \infty) \times D \times D$ and (x_n, y_n) converges to (x_0, y_0) in $D \times D$. By (2.5), we have

$$q^D(t_0, x_n, y_n) = \int_D q^D\left(\frac{1}{2}t_0, x_n, z\right) q^D\left(\frac{1}{2}t_0, z, y_n\right) dz.$$

It follows from (1.1) that $q^D\left(\frac{1}{2}t_0, x_n, z\right) q^D\left(\frac{1}{2}t_0, z, y_n\right)$ is bounded. So by the bounded convergence theorem, we have the joint continuity of $q^D(t, \cdot, \cdot)$.

Finally, for $(t_0, x_0, y_0) \in (0, \infty) \times D \times D$, choose a smooth domain G such that $x_0, y_0 \in G \subset D$. Then $q^D(t_0, x_0, y_0) \geq q^G(t_0, x_0, y_0) > 0$ by (1.4).

By Proposition 2.3, the last assertion of the theorem can be proved using the argument in the last paragraph of the proof of Theorem 2.4 of [11]. We omit the details. □

In the remainder of this section we assume that D is a bounded domain in \mathbf{R}^d . The next lemma is basically Lemma 6.1 in [21].

Lemma 2.5 *There exist positive constants C_1 and C_2 depending on D only via its diameter such that*

$$q^D(t, x, y) \leq C_1 e^{-C_2 t}, \quad (t, x, y) \in (1, \infty) \times D \times D.$$

Proof With the help of (1.1), one can repeat the argument in the proof of Lemma 6.1 in [21] to arrive at the result with the constants depending on D only via its diameter. We omit the details. □

Combining the result above with (1.1) we know that there exist positive constants C_1 and C_2 depending on D via its diameter such that for any $(t, x, y) \in (0, \infty) \times D \times D$,

$$q^D(t, x, y) \leq C_1 t^{-\frac{d}{2}} e^{-\frac{C_2|x-y|^2}{2t}}. \tag{2.7}$$

Therefore the Green function

$$G_D(x, y) := \int_0^\infty q^D(t, x, y) dt$$

is finite for $x \neq y$ and

$$G_D(x, y) \leq C_3 \frac{1}{|x - y|^{d-2}} \tag{2.8}$$

for some $C_3 = C_3(\text{diam}(D)) > 0$.

Theorem 2.6 *$G_D(x, y)$ is strictly positive and jointly continuous on $(D \times D) \setminus \{(x, y) : x = y\}$. $G_D(x, y)$ is infinite if and only if $x = y$. For any $y \in D$,*

$$\lim_{x \rightarrow y} G_D(x, y) = \lim_{x \rightarrow y} G_D(y, x) = \infty. \tag{2.9}$$

Moreover, if z is a regular boundary point of D , then for any $y \in D$,

$$\lim_{D \ni x \rightarrow z} G_D(x, y) = 0.$$

Proof Suppose (x_n, y_n) converge to (x_0, y_0) with $x_0 \neq y_0$ and there exist disjoint compact sets K_1 and K_2 such that $x_n \in K_1$ and $y_n \in K_2$. Let $\delta := \text{dist}(K_1, K_2)$. Then

by (2.7), there exist C_1 and C_2 depending on D only via its diameter such that for any $t \in (0, \infty)$

$$q^D(t, x_n, y_n) \leq C_1 t^{-\frac{d}{2}} e^{-\frac{C_2 \delta^2}{2t}} =: f(t).$$

Since $f(t)$ is in $L^1((0, \infty))$, the dominated convergence theorem and Theorem 2.4 give the joint continuity on $(D \times D) \setminus \{(x, y) : x = y\}$. The strict positivity of $G_D(x, y)$ follows from the strict positivity of $q^D(t, x, y)$.

Fix $x_0 \in D$ and choose a smooth domain G such that $B(x_0, \frac{1}{2}\rho(x_0)) \subset G \subset D$. Then by (1.3), there exists positive constants c_1 and c_2 such that

$$q^D(t, x_0, y) \geq q^G(t, x_0, y) \geq c_1 \left(1 \wedge \frac{\rho(x_0)^2}{t}\right) t^{-\frac{d}{2}} e^{-\frac{c_2|x_0-y|^2}{t}},$$

$$t < 1 \text{ and } y \in B\left(x_0, \frac{1}{4}\rho(x_0)\right).$$

So

$$G_D(x_0, x_0) \geq c_1 \int_0^{\rho(x_0)^2 \wedge 1} t^{-\frac{d}{2}} dt = \infty.$$

Also, by Fatou’s lemma,

$$\begin{aligned} \liminf_{y \rightarrow x_0} G_D(x_0, y) &\geq \liminf_{y \rightarrow x_0} c_1 \int_0^{\rho(x_0)^2 \wedge 1} t^{-\frac{d}{2}} e^{-\frac{c_2|x_0-y|^2}{t}} dt \\ &\geq c_1 \int_0^{\rho(x_0)^2 \wedge 1} t^{-\frac{d}{2}} \liminf_{y \rightarrow x_0} e^{-\frac{c_2|x_0-y|^2}{t}} dt \\ &= c_1 \int_0^{\rho(x_0)^2 \wedge 1} t^{-\frac{d}{2}} dt = \infty. \end{aligned}$$

The other claim in (2.9) can be proved similarly.

The last assertion follows easily from the last assertion of Theorem 2.4. □

Therefore G_D satisfies the condition (i), (ii), (iii) and (vi) in [12] and hence the Riesz representation (see [12, p. 28] and [13, pp. 186–187]) holds. To make the statement above precise, we first recall some definitions.

Definition 2.7 Suppose that U is an open subset of \mathbf{R}^d . A Borel function u defined on \mathbf{R}^d is said to be

- (1) harmonic in U with respect to X if

$$u(x) = \mathbf{E}_x \left[u(X_{\tau_B}^U) \right], \quad x \in B, \quad (2.10)$$

for every bounded open set B with $\overline{B} \subset U$;

- (2) regular harmonic in U with respect to X if it is harmonic for X^U and for each $x \in U$,

$$u(x) = \mathbf{E}_x \left[u(X_{\tau_U}) \right];$$

- (3) superharmonic in U with respect to X if u is non-negative and

$$u(x) \geq \mathbf{E}_x \left[u(X_{\tau_B}^U) \right], \quad x \in B,$$

for every bounded open set B with $\overline{B} \subset U$;

- (4) excessive for X^U if u is non-negative and

$$\begin{aligned} u(x) &\geq \mathbf{E}_x \left[u(X_t^U) \right] = \mathbf{E}_x \left[u(X_t) : \tau_U > t \right] \quad \text{and} \\ u(x) &= \lim_{t \downarrow 0} \mathbf{E}_x \left[u(X_t^U) \right], \quad t > 0, x \in U, \end{aligned}$$

- (5) a potential for X^U if it is excessive for X^U and for every sequence $\{U_n\}_{n \geq 1}$ of open sets with $\overline{U_n} \subset U_{n+1}$ and $\cup_n U_n = U$,

$$\lim_{n \rightarrow \infty} \mathbf{E}_x \left[u(X_{\tau_{U_n}}^U) \right] = 0, \quad \text{for every } x \in U \text{ with } u(x) < \infty.$$

It is well known that u is excessive for X^U if and only if f is lower-semicontinuous in U and superharmonic in U with respect to X . (See Theorem 4.5.3 in [10] for the Brownian motion case, and the proof there can be adapted easily to the present case.)

Harmonic functions with respect to X are continuous if they are locally bounded.

Proposition 2.8 *Assume that U is an open subset of \mathbf{R}^d . Then any locally bounded harmonic function u in U is continuous. In particular, for any bounded Borel function f in ∂U , $\mathbf{E}_x[f(X_{\tau_U})]$ is continuous and regular harmonic in U .*

Proof Fix $x \in U$ and choose smooth compact open subsets D_1 and D_2 with $x \in D_1 \subset \overline{D_1} \subset D_2 \subset \overline{D_2} \subset U$. Let $L := \|u\|_{L^\infty(\overline{D_2})} < \infty$. We assume that $\{x_n\}_{n \geq 1} \subset D_1$ and

$x_n \rightarrow x$. By (2.10) and the Markov property,

$$\begin{aligned}
 |u(x_n) - u(x)| &= \left| \mathbf{E}_{x_n}[u(X_{\tau_{D_2}})] - \mathbf{E}_x[u(X_{\tau_{D_2}})] \right| \\
 &\leq 2L \sup_{z \in D_1} \mathbf{P}_z(\tau_{D_2} \leq t) + \left| \mathbf{E}_{x_n} \left[\mathbf{E}_{X_t}[u(X_{\tau_{D_2}})] : \tau_{D_2} > t \right] \right. \\
 &\quad \left. - \mathbf{E}_x \left[\mathbf{E}_{X_t}[u(X_{\tau_{D_2}})] : \tau_{D_2} > t \right] \right| \\
 &= 2L \sup_{z \in D_1} \mathbf{P}_z(\tau_{D_2} \leq t) + \left| \mathbf{E}_{x_n} \left[\mathbf{E}_{X_t^{D_2}}[u(X_{\tau_{D_2}})] \right] - \mathbf{E}_x \left[\mathbf{E}_{X_t^{D_2}}[u(X_{\tau_{D_2}})] \right] \right| \\
 &\leq 2L \sup_{z \in D_1} \mathbf{P}_z(\tau_{D_2} \leq t) + L \int_{D_2} \left| q^{D_2}(t, x_n, y) - q^{D_2}(t, x, y) \right| dy
 \end{aligned}$$

Given $\varepsilon > 0$, by Lemma 2.1, there exists $t_0 > 0$ such that

$$\sup_{z \in D_1} \mathbf{P}_z(\tau_{D_2} \leq t_0) < \frac{\varepsilon}{8L}.$$

By (1.3),

$$\begin{aligned}
 &\int_{D_2} \left| q^{D_2}(t_0, x_n, y) - q^{D_2}(t_0, x, y) \right| dy \\
 &= \left(\int_{D_2 \cap \{\text{dist}(y, \partial D_2) < \delta\}} + \int_{D_2 \setminus \{\text{dist}(y, \partial D_2) \geq \delta\}} \right) \left| q^{D_2}(t_0, x_n, y) - q^{D_2}(t_0, x, y) \right| dy \\
 &\leq c_1 |D_2| \delta t_0^{-\frac{d+1}{2}} + \int_{D_2 \cap \{\text{dist}(y, \partial D_2) \geq \delta\}} \left| q^{D_2}(t_0, x_n, y) - q^{D_2}(t_0, x, y) \right| dy,
 \end{aligned}$$

for some c_1 . Now we choose δ small so that $c_1 |D_2| L \delta t_0^{-\frac{d+1}{2}} < \frac{\varepsilon}{4}$. The convergence of the second term in the last equation above follows from the uniform continuity of $q^{D_2}(t_0, \cdot, \cdot)$ on $\overline{D_1} \times (D_2 \cap \{\text{dist}(y, \partial D_2) \geq \delta\})$ (see Theorem 3.1 in [21]). Thus we have proved the Proposition. \square

Now we state some properties of Green functions. Since X^D is a transient diffusion, it satisfies the conditions in [23]. Thus, combining Theorem 1 in [23], the results in [12, p. 28], Corollary 2 to Theorem 2, Proposition 11 and Theorems 5–6 in [13], we have

Theorem 2.9 (1) *For each $y, x \rightarrow G_D(x, y)$ excessive for X^D . Moreover, for every open subset U of D , we have*

$$\mathbf{E}_x[G_D(X_{T_U}^D, y)] = G_D(x, y), \quad (x, y) \in D \times U \tag{2.11}$$

where $T_U := \inf\{t > 0 : X_t^D \in U\}$. In particular, for every $y \in D$ and $\varepsilon > 0$, $G_D(\cdot, y)$ is regular harmonic in $D \setminus B(y, \varepsilon)$ with respect to X^D .

- (2) For every Radon measure ν on D ,

$$G_D \nu(x) := \int_D G_D(x, y) \nu(dy)$$

is a potential for X^D . Conversely, if u is a potential for X^D , then there exists a unique Radon measure ν on D such that $u = G_D \nu$.

- (3) If f is an excessive function for X^D which is not identically zero, then there exists a unique Radon measure ν on D and a harmonic function h for X^D such that $f = G_D \nu + h$.

The proof of the next proposition can be found in the proofs of Theorems 2–3 in [23]. We put the proof here for reader's convenience.

Proposition 2.10 *If h is a nonnegative harmonic for X^D and U is an open subset of D with $\bar{U} \subset D$, then there exists a Radon measure ν supported on ∂U such that $h = G_D \nu$ in U . In particular, every nonnegative harmonic function for X^D is continuous.*

Proof Note that from Proposition 2.8, we know that every bounded harmonic function is continuous. Since h is the increasing limit of $\mathbf{E}_x[h(X_{\tau_V}^D) \wedge k]$ for every open subset V of D with $\bar{V} \subset D$, h is lower semicontinuous. Thus h is excessive. Let $T_U := \inf\{t > 0 : X_t^D \in U\}$. Since h is excessive, Corollary 1 to Theorem 2 in [13] implies that there exists a Radon measure ν supported on \bar{U} such that $\mathbf{E}_x[h(X_{T_U}^D)] = G_D \nu(x)$ for all $x \in D$. Since

$$G_D \nu(x) = \int_U G_D(x, y) \nu(dy) + \int_{\partial U} G_D(x, y) \nu(dy) =: h_1(x) + h_2(x), \quad x \in D$$

and h_1 and h_2 are excessive (Theorem 2.9), h_1 and h_2 must be harmonic with respect to X in U . Let K be a compact subset of U . By the harmonicity of h_1 , we have

$$\mathbf{E}_x[h_1(X_{T_K^D}^D)] = \int_{\partial U} G_D(x, y) \nu(dy).$$

But, by Corollary 1 to Theorem 2 in [13], ν can not charge the interior of K . Since K is an arbitrary compact subset of U , we get that h_1 is identically zero and ν is supported by ∂U . Therefore we have shown $h(x) = \mathbf{E}_x[h(X_{T_U}^D)] = G_D \nu(x)$ for $x \in U$. Now the continuity of h follows from the continuity of G_D . \square

3 Green function estimates and comparison of harmonic measures

In this section, we assume that D is a bounded $C^{1,1}$ domain. Recall that a bounded domain D in \mathbf{R}^d is said to be a $C^{1,1}$ domain if there is a localization radius $r_0 > 0$

and a constant $\Lambda > 0$ such that for every $Q \in \partial D$, there is a $C^{1,1}$ function $\phi = \phi_Q : \mathbf{R}^{d-1} \rightarrow \mathbf{R}$ satisfying $\phi(0) = |\nabla\phi(0)| = 0$, $\|\nabla\phi\|_\infty \leq \Lambda$, $|\nabla\phi(x) - \nabla\phi(z)| \leq \Lambda|x - z|$, and an orthonormal coordinate system CS_Q with origin at Q such that

$$B(Q, r_0) \cap D = B(Q, r_0) \cap \{y = (y_1, \dots, y_{d-1}, y_d) \\ := (\tilde{y}, y_n) \text{ in } CS_Q : y_d > \phi(\tilde{y})\}.$$

The pair (r_0, Λ) is called the characteristics of the $C^{1,1}$ domain D . The main objective of this section is to show that the harmonic measures with respect to X are uniformly (under scale and translation) comparable to the corresponding harmonic measures with respect to Brownian motion if the domain D is shrunk small enough.

Let $p^D(t, x, y)$ be the density of Brownian motion killed upon exiting D . It is known that for any $T > 0$, there exist positive constants $M_i, i = 1, \dots, 5$, depending on T and D such that

$$M_1\psi_D(t, x, y)t^{-\frac{d}{2}}e^{-\frac{M_2|x-y|^2}{t}} \leq p^D(t, x, y) \leq M_3\psi_D(t, x, y)t^{-\frac{d}{2}}e^{-\frac{M_4|x-y|^2}{t}} \tag{3.1}$$

and

$$|\nabla_x p^D(t, x, y)| \leq M_5t^{-\frac{d+1}{2}}e^{-\frac{M_4|x-y|^2}{t}} \tag{3.2}$$

for all $(t, x, y) \in (0, T) \times D \times D$. (3.1) was proved in [15,28] and (3.2) was proved in [17]. By taking M_4 smaller if necessary, we can assume that $M_4 \leq 1/2$. Differentiating with respect to x in the equation

$$p^D(t, x, y) = \int_D p^D\left(\frac{t}{2}, x, z\right) p^D\left(\frac{t}{2}, z, y\right) dz$$

and using the above estimates on $p^D(t, x, y)$ and $\nabla_x p^D(t, x, y)$ we get

$$|\nabla_x p^D(t, x, y)| \leq 2^{d+1}M_3M_5 \int_D t^{-\frac{d+1}{2}}e^{-\frac{2M_4|x-z|^2}{t}} \rho(y)t^{-\frac{d+1}{2}}e^{-\frac{2M_4|z-y|^2}{t}} dz \\ \leq 2^{d+1}M_3M_5\rho(y) \int_{\mathbf{R}^d} t^{-\frac{d+1}{2}}e^{-\frac{2M_4|x-z|^2}{t}} t^{-\frac{d+1}{2}}e^{-\frac{2M_4|z-y|^2}{t}} dz \\ := M_6\rho(y)t^{-\frac{d+2}{2}}e^{-\frac{M_4|x-y|^2}{t}}.$$

In the last equality above, we have used the semigroup property of the Gaussian kernel. Combining this with (3.2) we see that, for any $T > 0$, there exists a positive constant M_7 such that

$$|\nabla_x p^D(t, x, y)| \leq M_7 \left(1 \wedge \frac{\rho(y)}{\sqrt{t}}\right) t^{-\frac{d+1}{2}}e^{-\frac{M_4|x-y|^2}{t}}, \tag{3.3}$$

for all $(t, x, y) \in (0, T] \times D \times D$. Since $p^D(t, x, y) \leq p(t, x, y)$, there exists a positive constant M_8 depending only on d such that

$$p^D(t, x, y) \leq M_8 t^{-\frac{d}{2}} e^{-M_4 \frac{|x-y|^2}{t}}, \quad (t, x, y) \in (0, \infty) \times D \times D. \tag{3.4}$$

By combining this with (3.2), Lemma 2.5 (works for p^D) and the semigroup property, we can easily get that there exists positive constant M_9 depending on D such that

$$|\nabla_x p^D(t, x, y)| \leq M_9 t^{-\frac{d+1}{2}} e^{-M_4 \frac{|x-y|^2}{t}}, \quad (t, x, y) \in (0, \infty) \times D \times D. \tag{3.5}$$

For any $z \in \mathbf{R}^d$ and $r \in (0, 1)$, put $D_r^z := z + rD$ and let $\rho_{D_r^z}(x)$ be the distance between x and ∂D_r^z . By the translation and scaling property of p^D , we see that the constants in (3.4)–(3.5) are invariant under translation and Brownian scaling. Thus we have

$$p^{D_r^z}(t, x, y) \leq M_8 t^{-\frac{d}{2}} e^{-M_4 \frac{|x-y|^2}{t}}, \quad (t, x, y) \in (0, \infty) \times D_r^z \times D_r^z \tag{3.6}$$

and

$$|\nabla_x p^{D_r^z}(t, x, y)| \leq M_9 t^{-\frac{d+1}{2}} e^{-M_4 \frac{|x-y|^2}{t}}, \quad (t, x, y) \in (0, \infty) \times D_r^z \times D_r^z. \tag{3.7}$$

We will suppress indices above from D_r^z when it is clear from the context.

By taking scale and translation invariance into consideration in the proof of Theorem 4.2 of [21], we have observed at the beginning of Sect. 5 of [21] that for any $T > 0$, there exist positive constants t_0 and $M_j, 10 \leq j \leq 14$, independent of z and r and depending on μ only via the rate at which $\max_{1 \leq i \leq d} M_{\mu^i}^1(r)$ goes to zero, such that

$$M_{10} t^{-\frac{d}{2}} \psi_{D_r^z}(t, x, y) e^{-\frac{M_{11}|x-y|^2}{2t}} \leq q^{D_r^z}(t, x, y) \leq M_{12} t^{-\frac{d}{2}} \psi_{D_r^z}(t, x, y) e^{-\frac{M_{13}|x-y|^2}{2t}} \tag{3.8}$$

and

$$|\nabla_x q^{D_r^z}(t, x, y)| \leq M_{14} \left(1 \wedge \frac{\rho_{D_r^z}(y)}{\sqrt{t}} \right) t^{-\frac{d+1}{2}} e^{-\frac{M_{13}|x-y|^2}{2t}} \tag{3.9}$$

for all $(t, x, y) \in (0, t_0 \wedge (r^2 T)] \times D_r^z \times D_r^z$. We now show the scale and translation invariant version of the Green function estimate of X^D when D is a bounded $C^{1,1}$ domain.

First we show the lower bound of $G_{D_r^z}$, which is a direct consequence of the lower estimates of $q^{D_r^z}$ in (3.8). In the next theorem we will take $T = \text{diam}(D)^2$, and let t_0 be the corresponding constant in (3.8). Recall from [21] that

$$M_v(r) := \sup_{x \in \mathbf{R}^d} \int_{|x-y| \leq r} \frac{|v|(dy)}{|x-y|^{d-1}}, \quad N_v(t) := \sup_{x \in \mathbf{R}^d} \int_0^t \int_{\mathbf{R}^d} s^{-\frac{1}{2}} p(s, x, y) |v|(dy) ds.$$

Theorem 3.1 *Let D be a bounded $C^{1,1}$ domain in \mathbf{R}^d . There exists a constant $C_1 = C_1(D, \mu) > 0$ depending on μ only via the rate at which $\max_{1 \leq i \leq d} M_{\mu^i}(r)$ goes to zero such that for $r \leq \frac{\sqrt{t_0}}{\text{diam}(D)}$, $z \in \mathbf{R}^d$ and $x, y \in D_r^z$,*

$$G_{D_r^z}(x, y) \geq C_1 \left(1 \wedge \frac{\rho_{D_r^z}(x)}{|x - y|}\right) \left(1 \wedge \frac{\rho_{D_r^z}(y)}{|x - y|}\right) \frac{1}{|x - y|^{d-2}}.$$

Proof It is easy to see that $\text{diam}(D_r^z)^2 = r^2 \text{diam}(D)^2$. For $r \leq \frac{\sqrt{t_0}}{\text{diam}(D)}$, by (3.8), we have

$$\begin{aligned} G_{D_r^z}(x, y) &\geq \int_0^{\text{diam}(D_r^z)^2} q^{D_r^z}(t, x, y) dt \\ &\geq M_{10} \int_0^{\text{diam}(D_r^z)^2} \left(1 \wedge \frac{\rho_{D_r^z}(x)}{\sqrt{t}}\right) \left(1 \wedge \frac{\rho_{D_r^z}(y)}{\sqrt{t}}\right) t^{-\frac{d}{2}} e^{-\frac{M_{11}|x-y|^2}{2t}} dt \\ &\geq \frac{M_{10}}{|x - y|^{d-2}} \int_1^\infty u^{\frac{d-4}{2}} \left(1 \wedge \frac{\sqrt{u}\rho_{D_r^z}(x)}{|x - y|}\right) \left(1 \wedge \frac{\sqrt{u}\rho_{D_r^z}(y)}{|x - y|}\right) e^{-\frac{1}{2}M_{11}u} du \\ &\geq C_1 \left(1 \wedge \frac{\rho_{D_r^z}(x)}{|x - y|}\right) \left(1 \wedge \frac{\rho_{D_r^z}(y)}{|x - y|}\right) \frac{1}{|x - y|^{d-2}} \end{aligned}$$

where $C_1 = M_{10} \int_1^\infty u^{\frac{d-4}{2}} e^{-\frac{1}{2}M_{11}u} du$ □

The following lemmas will be needed in proving the upper bound of the Green functions.

Lemma 3.2 *For any $a, r > 0$, $x_0 \in \mathbf{R}^d$ and measure μ on \mathbf{R}^d , there exists constant $C_1 = C_1(a, d)$ such that*

$$\sup_{u \in B(x_0, r)} \int_0^\infty \int_{B(x_0, r)} s^{-\frac{d+1}{2}} e^{-\frac{a|u-z|^2}{4s}} \mu(dz) ds \leq C_1(a, d) N_\mu \left(\frac{r^2}{a}\right).$$

Proof Fix $a, r > 0$, $x_0 \in \mathbf{R}^d$ and a measure μ on \mathbf{R}^d . Since

$$\sup_{u \in B(x_0, r)} \int_0^{\frac{r^2}{2}} \int_{B(x_0, r)} s^{-\frac{d+1}{2}} e^{-\frac{a|u-z|^2}{4s}} \mu(dz) ds \leq \left(\frac{2}{a}\right)^{\frac{d-1}{2}} (2\pi)^{\frac{d}{2}} N_\mu \left(\frac{r^2}{a}\right),$$

we only need to show that

$$\sup_{u \in B(x_0, r)} \int_{\frac{r^2}{2}}^{\infty} \int_{B(x_0, r)} s^{-\frac{d+1}{2}} e^{-\frac{a|u-z|^2}{4s}} \mu(dz) ds \leq c(a, d) N_{\mu} \left(\frac{r^2}{a} \right).$$

Since $\{z : |x_0 - z| < r\} \subset \{z : |u - z| < 2r\}$ for every $u \in B(x_0, r)$, we have

$$\begin{aligned} & \sup_{u \in B(x_0, r)} \int_{\frac{r^2}{2}}^{\infty} \int_{B(x_0, r)} s^{-\frac{d+1}{2}} e^{-\frac{a|u-z|^2}{4s}} \mu(dz) ds \\ & \leq \sup_{u \in B(x_0, r)} \int_{\frac{r^2}{2}}^{\infty} \int_{|u-z| < 2r} s^{-\frac{d+1}{2}} e^{-\frac{a|u-z|^2}{4s}} \mu(dz) ds \\ & \leq \int_{\frac{r^2}{2}}^{\infty} \sup_{u \in B(x_0, r)} \int_{|u-z| < 2r} s^{-\frac{d+1}{2}} e^{-\frac{a|u-z|^2}{4s}} \mu(dz) ds \leq \left(\int_{\frac{r^2}{2}}^{\infty} s^{-\frac{d+1}{2}} ds \right) (2r)^{d-1} M_{\mu}(2r) \end{aligned}$$

In the last inequality above, we used the fact that for any signed measure ν ,

$$\sup_{x \in \mathbf{R}^d} |\nu|(\overline{B(x, r)}) \leq r^{d-1} M_{\nu}(r)$$

(see (2.1) in [21]). Using the change of variable $l = \frac{r^2}{s}$, we have

$$\int_{\frac{r^2}{2}}^{\infty} s^{-\frac{d+1}{2}} ds = r^{-d+1} \int_0^2 l^{\frac{d-3}{2}} dl.$$

On the other hand, we have

$$\begin{aligned} & \int_0^{\frac{r^2}{a}} \int_{\mathbf{R}^d} s^{-\frac{1}{2}} p(s, x, y) \mu(dy) ds = \int_{\mathbf{R}^d} \int_0^{\frac{r^2}{a}} s^{-\frac{1}{2}} p(s, x, y) ds \mu(dy) \\ & \geq \int_{|x-y| \leq 2r} 2^{\frac{d-1}{2}} (2\pi)^{-\frac{d}{2}} |x-y|^{-d+1} \int_{\frac{a|x-y|^2}{2r^2}}^{\infty} u^{\frac{d-3}{2}} e^{-u} du \mu(dy) \\ & \geq 2^{\frac{d-1}{2}} (2\pi)^{-\frac{d}{2}} \left(\int_{2a}^{\infty} u^{\frac{d-3}{2}} e^{-u} du \right) \int_{|x-y| \leq 2r} |x-y|^{-d+1} \mu(dy). \end{aligned}$$

Thus

$$M_\mu(2r) \leq C_0(a, d)N_\mu\left(\frac{r^2}{a}\right).$$

Therefore

$$\left(\int_{\frac{r^2}{2}}^\infty s^{-\frac{d+1}{2}} ds\right) (2r)^{d-1} M_\mu(2r) \leq C_1(a, d)N_\mu\left(\frac{r^2}{a}\right).$$

□

The proof of the next lemma is almost identical to the proof of Lemma 3.1 of [27], so we skip the proof.

Lemma 3.3 *For any $a > 0$, there exist positive constants C_1, C_2 depending only on a and d such that for any measure μ on \mathbf{R}^d*

$$\begin{aligned} & \int_0^t \int_D s^{-\frac{d}{2}} e^{-\frac{a|x-z|^2}{2s}} (t-s)^{-\frac{d+1}{2}} e^{-\frac{a|z-y|^2}{t-s}} \mu(dz) ds \\ & \leq C_1 t^{-\frac{d}{2}} e^{-\frac{a|x-y|^2}{2t}} \sup_{u \in D} \int_0^t \int_D s^{-\frac{d+1}{2}} e^{-\frac{a|u-z|^2}{4s}} \mu(dz) ds \end{aligned}$$

and

$$\begin{aligned} & \int_0^t \int_D s^{-\frac{d+1}{2}} e^{-\frac{a|x-z|^2}{2s}} (t-s)^{-\frac{d+1}{2}} e^{-\frac{a|z-y|^2}{t-s}} \mu(dz) ds \\ & \leq C_2 t^{-\frac{d+1}{2}} e^{-\frac{a|x-y|^2}{2t}} \sup_{u \in D} \int_0^t \int_D s^{-\frac{d+1}{2}} e^{-\frac{a|u-z|^2}{4s}} \mu(dz) ds \end{aligned}$$

for all $(t, x, y) \in (0, \infty) \times D \times D$.

Combining Lemmas 3.2 and 3.3, we have the following lemma.

Lemma 3.4 *For any $a > 0$, there exist positive constants L_3, L_4 depending only on a and d such that for any $r, t > 0$, measure μ on \mathbf{R}^d and an open subset D with $\text{diam}(D) < 2r$,*

$$\int_0^t \int_D s^{-\frac{d}{2}} e^{-\frac{a|x-z|^2}{2s}} (t-s)^{-\frac{d+1}{2}} e^{-\frac{a|z-y|^2}{t-s}} \mu(dz) ds \leq L_3 N_\mu\left(\frac{r^2}{a}\right) t^{-\frac{d}{2}} e^{-\frac{a|x-y|^2}{2t}} \tag{3.10}$$

and

$$\int_0^t \int_D s^{-\frac{d+1}{2}} e^{-\frac{a|x-z|^2}{2s}} (t-s)^{-\frac{d+1}{2}} e^{-\frac{a|z-y|^2}{t-s}} \mu(dz) ds \leq L_4 N_\mu \left(\frac{r^2}{a}\right) t^{-\frac{d+1}{2}} e^{-\frac{a|x-y|^2}{2t}} \tag{3.11}$$

Using Lemma 3.4 and the estimates (3.6)–(3.7), we can show the global upper estimate of q^D if the diameter of D is sufficiently small.

Theorem 3.5 *There exists constant $r_0 > 0$ depending on M_4, M_8 and M_9 such that for any $r \leq r_0$ and $z \in \mathbf{R}^d$*

$$q^{D_r^z}(t, x, y) \leq 2 M_8 t^{-\frac{d}{2}} e^{-\frac{M_4|x-y|^2}{2r}} \tag{3.12}$$

and

$$|\nabla_x q^{D_r^z}(t, x, y)| \leq 2 M_9 t^{-\frac{d+1}{2}} e^{-\frac{M_4|x-y|^2}{2r}} \tag{3.13}$$

for all $(t, x, y) \in (0, \infty) \times D_r^z \times D_r^z$.

Proof By Theorems 4.5 of [21], without loss of generality, we may assume that $\mu(dx) = U(x)dx$ where U is smooth and bounded. By the translation and scaling property, we can also assume that $z = 0$ and the diameter of D is 2, so that there exist x_1 such that $D \subset B(x_1, 1)$. Let $D_r = rD$ and $B_r = B(rx_1, r)$ so that $D_r \subset B_r$. By Theorem 6 of [2] we get that for every $(t, x, y) \in (0, \infty) \times D_r \times D_r$,

$$q^{D_r}(t, x, y) = p^{D_r}(t, x, y) + \int_0^t \int_{D_r} q^{D_r}(s, x, z) U(z) \cdot \nabla_z p^{D_r}(t-s, z, y) dz ds.$$

Following the same argument in the proof of Theorem 4.2 of [21], using (3.6)–(3.7) and Lemma 3.4 above instead of using (4.1)–(4.3) and Lemma 4.1 of [21], we get that for $(t, x, y) \in (0, \infty) \times D_r \times D_r$

$$q^{D_r}(t, x, y) \leq M_8 t^{-\frac{d}{2}} e^{-\frac{M_4|x-y|^2}{2r}} \sum_{k=0}^\infty \left(M_8 L_3(d, M_4) \sum_{i=1}^d N_{U_i}^1 \left(\frac{r^2}{M_4}\right) \right)^k$$

and

$$|\nabla_x q^{D_r}(t, x, y)| \leq M_9 t^{-\frac{d+1}{2}} e^{-\frac{M_4|x-y|^2}{2r}} \sum_{k=0}^\infty \left(M_9 L_4(d, M_4) \sum_{i=1}^d N_{U_i}^1 \left(\frac{r^2}{M_4}\right) \right)^k.$$

Choose r_0 small such that

$$M_8 L_3(d, M_4) \sum_{i=1}^d N_{U_i}^1 \left(\frac{r_0^2}{M_4}\right) < \frac{1}{2} \quad \text{and} \quad M_9 L_4(d, M_4) \sum_{i=1}^d N_{U_i}^1 \left(\frac{r_0^2}{M_4}\right) < \frac{1}{2}.$$

Then for any $r \leq r_0$, we have

$$q^{D_r}(t, x, y) \leq 2M_8 t^{-\frac{d}{2}} e^{-\frac{M_4|x-y|^2}{2t}} \quad \text{and} \quad |\nabla_x q^{D_r}(t, x, y)| \leq 2M_9 t^{-\frac{d+1}{2}} e^{-\frac{M_4|x-y|^2}{2t}}.$$

□

Using the above estimates we can prove the upper bound on the Green functions when D is shrunk small enough.

Theorem 3.6 *Let D be a bounded $C^{1,1}$ domain in \mathbf{R}^d . Then there exists a constant $r_1 = r_1(D, \mu) > 0$ depending on μ only via the rate at which $\max_{1 \leq i \leq d} M_{\mu^i}(r)$ goes to zero such that for $r \leq r_1, z \in \mathbf{R}^d, x, y \in D_r^z$,*

$$G_{D_r^z}(x, y) \leq C_1 \left(1 \wedge \frac{\rho(x)}{|x-y|}\right) \left(1 \wedge \frac{\rho(y)}{|x-y|}\right) \frac{1}{|x-y|^{d-2}}$$

where C_1 only depends on $d, \text{diam}(D)$ and the constants in (3.8) and (3.12), and

$$|\nabla_x G_{D_r^z}(x, y)| \leq C_2 \left(1 \wedge \frac{\rho(x)}{|x-y|}\right) \left(1 \wedge \frac{\rho(y)}{|x-y|}\right) \frac{1}{|x-y|^{d-1}}$$

where C_2 only depends on $d, \text{diam}(D)$ and the constants in (3.9) and (3.13).

Proof Recall that r_0 is the constant r_0 from Theorem 3.5, and t_0 is the constant t_0 used in (3.8) and (3.9). Let $r_1 := \sqrt{t_0} \wedge r_0$ and $l := \text{diam}(D)$. Note that for every $x, y \in D_r^z, |x-y| \leq rl$. We fix $r \leq r_1$ and suppress indices. Using (3.8), we have

$$\begin{aligned} \int_0^{r^2} q^D(t, x, y) dt &\leq M_{12} \int_0^{r^2} \left(1 \wedge \frac{\rho(x)}{\sqrt{t}}\right) \left(1 \wedge \frac{\rho(y)}{\sqrt{t}}\right) t^{-\frac{d}{2}} e^{-\frac{M_{13}|x-y|^2}{2t}} dt \\ &= \frac{M_{12}}{|x-y|^{d-2}} \int_{\frac{|x-y|^2}{r^2}}^\infty u^{\frac{d-4}{2}} \left(1 \wedge \frac{\sqrt{u}\rho(x)}{|x-y|}\right) \left(1 \wedge \frac{\sqrt{u}\rho(y)}{|x-y|}\right) e^{-\frac{1}{2}M_{13}u} du \\ &\leq \frac{M_{12}}{|x-y|^{d-2}} \left(1 \wedge \frac{\rho(x)}{|x-y|}\right) \left(1 \wedge \frac{\rho(y)}{|x-y|}\right) \int_0^\infty u^{\frac{d-4}{2}} (u \vee 1) e^{-\frac{1}{2}M_{13}u} du. \end{aligned} \tag{3.14}$$

On the other hand, by the Chapman–Kolmogorov equation, (3.8) and Theorem 3.5, if $t > r^2$ we have

$$\begin{aligned}
 q^D(t, x, y) &= \int_D \int_D q^D\left(\frac{r^2}{2}, x, z\right) q^D(t - r^2, z, w) q^D\left(\frac{r^2}{2}, w, y\right) dz dw \\
 &\leq 2M_{12}^2 M_8 \psi_D(r^2, x, y) \int_D \int_D \left(\frac{r^2}{2}\right)^{-d} \\
 &\quad \times (t - r^2)^{-\frac{d}{2}} e^{-\frac{M_{13}|x-z|^2}{r^2}} e^{-\frac{M_4|z-w|^2}{2(t-r^2)}} e^{-\frac{M_{13}|w-y|^2}{r^2}} dz dw \\
 &\leq 2M_{12}^2 M_8 l^2 \psi_D(|x - y|^2, x, y) \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} \left(\frac{r^2}{2}\right)^{-d} \\
 &\quad \times (t - r^2)^{-\frac{d}{2}} e^{-\frac{M_{13}|x-z|^2}{r^2}} e^{-\frac{M_4|z-w|^2}{2(t-r^2)}} e^{-\frac{M_{13}|w-y|^2}{r^2}} dz dw \\
 &\leq C_0 M_{12}^2 M_8 l^2 \psi_D(|x - y|^2, x, y) t^{-\frac{d}{2}} e^{-\frac{C_1|x-y|^2}{2r}}
 \end{aligned}$$

where $C_0 = C_0(d)$ and $C_1 = C_1(M_4, M_{13})$. Since

$$\int_{r^2}^{\infty} t^{-\frac{d}{2}} e^{-\frac{C_1|x-y|^2}{2r}} dt = \frac{1}{|x - y|^{d-2}} \int_0^{\frac{|x-y|^2}{r^2}} u^{\frac{d-4}{2}} e^{-\frac{1}{2}C_1 u} du,$$

we have

$$\begin{aligned}
 \int_{r^2}^{\infty} q^D(t, x, y) dt &\leq C_0 M_{12}^2 M_8 l^2 \frac{1}{|x - y|^{d-2}} \left(1 \wedge \frac{\rho(x)}{|x - y|}\right) \\
 &\quad \times \left(1 \wedge \frac{\rho(y)}{|x - y|}\right) \int_0^{\infty} u^{\frac{d-4}{2}} e^{-\frac{1}{2}C_1 u} du. \tag{3.15}
 \end{aligned}$$

Combining (3.14) and (3.15), we have proved the upper estimate for the Green functions.

The upper estimate for $\nabla_x G_{D_r^z}(x, y)$ is similar. So we skip the proof here. □

Let G_D^0 be the Green function of the Brownian motion W in D . By the Green function estimates for Brownian motions, Theorem 3.1 and 3.6 imply that:

Theorem 3.7 *Let D be a bounded $C^{1,1}$ domain in \mathbf{R}^d . Then there exist constants $r_1 = r_1(D, \mu) > 0$ and $C = C(D, \mu) > 1$ depending on μ only via the rate at which $\max_{1 \leq i \leq d} M_{\mu^i}(r)$ goes to zero such that for $r \leq r_1, z \in \mathbf{R}^d, x, y \in D_r^z$,*

$$C^{-1} G_{D_r^z}^0(x, y) \leq G_{D_r^z}(x, y) \leq C G_{D_r^z}^0(x, y).$$

Now we show that the uniform comparability of Green functions (Theorem 3.7) implies the uniform comparability of the corresponding harmonic measures.

Theorem 3.8 *Let D be a bounded $C^{1,1}$ domain in \mathbf{R}^d . Then there exist constants $R = R(D, \mu) > 0$ and $C = C(D, \mu) > 1$ depending on μ only via the rate at which $\max_{1 \leq i \leq d} M_{\mu^i}(r)$ goes to zero such that for $r \leq R, z \in \mathbf{R}^d$ and $x \in D_r^z$*

$$C^{-1} \mathbf{P}_x \left(W_{\tau_{D_r^z}} \in dy \right) \leq \mathbf{P}_x \left(X_{\tau_{D_r^z}} \in dy \right) \leq C \mathbf{P}_x \left(W_{\tau_{D_r^z}} \in dy \right).$$

In particular, there exist constants $r_2 = r_2(d, \mu) > 0$ and $M_{15} = M_{15}(d, \mu) > 1$ depending only on μ only via the rate at which $\max_{1 \leq i \leq d} M_{\mu^i}(r)$ goes to zero such that for $r \leq r_2, z \in \mathbf{R}^d$ and $x \in B_r^z := B(z, r)$

$$M_{15}^{-1} \mathbf{P}_x \left(W_{\tau_{B_r^z}} \in dy \right) \leq \mathbf{P}_x \left(X_{\tau_{B_r^z}} \in dy \right) \leq M_{15} \mathbf{P}_x \left(W_{\tau_{B_r^z}} \in dy \right). \tag{3.16}$$

Proof The idea of the proof is similar to that of Theorem 2.2 in [9]. Fix a bounded $C^{1,1}$ domain D and constants $R := r_1 = r_1(D, \mu) > 0$ and $c = c(D, \mu) > 1$ in Theorem 3.7. We also fix $r \leq R$ and $z \in \mathbf{R}^d$ and suppress indices on D_r^z . Let $\varphi \geq 0$ is a continuous function on ∂D and let

$$u(x) := \mathbf{E}_x[\varphi(X_{\tau_D})],$$

which is harmonic for X^D and continuous on \overline{D} by Propositions 2.3 and 2.8. Choose increasing smooth domains D_n with $\overline{D_n} \subset D_{n+1}$ and $\cup_n D_n = D$. Denote $T_n := \inf\{t > 0 : X_t \in D_n\}$. Let $u_n(x) := \mathbf{E}_x[u(X_{T_n}^D)]$. Then by Proposition 2.10, there exist Radon measures ν_n supported on ∂D_n such that

$$u_n(x) = G_D \nu_n(x) = \int_{\partial D_n} G_D(x, y) \nu_n(dy).$$

Recall that G_D^0 is the Green function of the killed Brownian motion in D . Let

$$\nu_n(x) := G_D^0 \nu_n(x) = \int_{\partial D_n} G_D^0(x, y) \nu_n(dy).$$

Then by Theorem 3.7,

$$c^{-1} \nu_n(x) \leq u_n(x) \leq c \nu_n(x), \quad x \in D_n.$$

Since D is regular for X (see Proposition 2.2), using Proposition 2.3, the remainder of the argument is almost identical to the corresponding argument on the proof of Theorem 2.2 in [9] (with $b_1 \equiv 0$ there). So we omit it. □

It is well known that the harmonic measure $\mathbf{P}_x(W_{\tau_D} \in dy)$ is mutually absolutely continuous with respect to the surface measure on ∂D if D is a bounded $C^{1,1}$ domain. So immediately we have the following from Theorem 3.8.

Corollary 3.9 *The harmonic measure $\mathbf{P}_x(X_{\tau_D} \in dy)$ is mutually absolutely continuous with respect to the surface measure on ∂D if D is a bounded $C^{1,1}$ domain.*

4 Boundary Harnack principle in bounded Lipschitz domains

In this section, we will use the comparability of harmonic measures obtained in the last section to prove a scale invariant version of the boundary Harnack principle for X in bounded Lipschitz domains. Recall that a bounded domain D is said to be Lipschitz if there is a localization radius $r_0 > 0$ and a constant $\Lambda > 0$ such that for every $Q \in \partial D$, there is a Lipschitz function $\phi_Q : \mathbf{R}^{d-1} \rightarrow \mathbf{R}$ satisfying $|\phi_Q(x) - \phi_Q(z)| \leq \Lambda|x - z|$, and an orthonormal coordinate system CS_Q with origin at Q such that

$$B(Q, r_0) \cap D = B(Q, r_0) \cap \{y = (y_1, \dots, y_{d-1}, y_d) =: (\tilde{y}, y_d) \text{ in } CS_Q : y_d > \phi_Q(\tilde{y})\}.$$

The pair (r_0, Λ) is called the characteristics of the Lipschitz domain D .

One of the properties of Lipschitz domain is that it satisfies the uniform exterior cone condition. Recall that an open set D in \mathbf{R}^d is said to satisfy the uniform exterior cone condition if there exist constants $\eta > 0$, r and a cone

$$\mathcal{C} := \{x = (x_1, \dots, x_d) \in \mathbf{R}^d : x_d < 0, (x_1^2 + \dots + x_{d-1}^2)^{\frac{1}{2}} < \eta|x_d|\} \quad (4.1)$$

such that for every $Q \in \partial D$, there is a cone \mathcal{C}_Q with vertex Q , isometric to \mathcal{C} , such that $\mathcal{C}_Q \cap B(Q, r) \subset D^c$. If D is a bounded Lipschitz domain with its characteristics (r_0, Λ) , then it satisfies the uniform exterior cone condition with $\eta = \eta(d, \Lambda)$, $r = r_0$. Moreover, by choosing r_0 smaller if necessary, one can choose cones $\mathcal{C}_z := z + \mathcal{C}$ in CS_Q for each $Q \in \partial D$ and $z \in B(Q, r_0) \cap \partial D$ such that $\mathcal{C}_z \cap B(Q, r_0) \subset D^c \cap B(Q, r_0)$.

In the remainder of this section, we assume D is a bounded Lipschitz domain and fix its characteristics (r_0, Λ) and the constant η . The constant r_1 will be the constant from Corollary 5.8 in [21] and the constant r_2 will be the constant from Theorem 3.8. Without loss of generality, we also assume that $\Lambda \geq 1$.

For every $Q \in \partial D$ and $x \in B(Q, r_0) \cap D$, we define

$$\delta_Q(x) := x_d - \phi_Q(\tilde{x}),$$

where (\tilde{x}, x_d) is the coordinate of x in CS_Q . Since D is Lipschitz, there exists a constant $c = c(d, \Lambda) \geq 1$ such for every $Q \in \partial D$ and $x \in B(Q, r_0) \cap D$ we have

$$c^{-1} \delta_Q(x) \leq \rho(x) \leq \delta_Q(x) \quad (4.2)$$

Using (3.16), we can obtain the following result.

Lemma 4.1 *Suppose that $Q \in \partial D$. If $l \geq 1$, then for every $x \in D \cap B(Q, r_0/2)$ with $4l\delta_Q(x) < r_0 \wedge r_2$,*

$$\mathbf{P}_x \left(X_{\tau_{B(x, 2l\delta_Q(x))}} \in D \right) \leq c < 1$$

where $c = c(d, \Lambda)$.

Proof Fix $Q \in \partial D$ and an orthonormal coordinate system CS_Q with origin at Q such that

$$B(Q, r_0) \cap D = B(Q, r_0) \cap \{y = (y_1, \dots, y_{d-1}, y_d) =: (\tilde{y}, y_d) \text{ in } CS_Q : y_d > \phi_Q(\tilde{y})\}.$$

For any $x \in B(Q, r_0) \cap D$, we have $x = (\tilde{x}, \phi_Q(\tilde{x}) + \delta_Q(x))$ in CS_Q . Note that for any $x \in D \cap B(Q, r_0)$ with $4l\delta_Q(x) < r_0 \wedge r_2$ we have $B(x, 2l\delta_Q(x)) \cap D \subset B(Q, r_0) \cap D$. Let $Q_x := (\tilde{x}, \phi_Q(\tilde{x}))$ in CS_Q . Since $4l\delta_Q(x) < r_0$, there exists a cone $\mathcal{C}_{Q_x} := Q_x + \mathcal{C}$ with vertex Q_x satisfying

$$\mathcal{C}_{Q_x} \cap \partial B(x, 2l\delta_Q(x)) \subset \mathcal{C}_{Q_x} \cap B(Q_x, r_0) \subset D^c.$$

So by Theorem 3.8 (note that $4l\delta_Q(x) < r_2$),

$$\begin{aligned} \mathbf{P}_x \left(X_{\tau_{B(x, 2l\delta_Q(x))}} \notin D \right) &\geq \mathbf{P}_x \left(X_{\tau_{B(x, 2l\delta_Q(x))}} \in \partial B(x, 2l\delta_Q(x)) \cap \mathcal{C}_{Q_x} \right) \\ &\geq M_{15}^{-1} \mathbf{P}_x \left(W_{\tau_{B(x, 2\delta_Q(x))}} \in \partial B(x, 2\delta_Q(x)) \cap \mathcal{C}_{Q_x} \right) \\ &= M_{15}^{-1} \mathbf{P}_0 \left(W_{\tau_{B(0, 2)}} \in \partial B(0, 2) \cap \mathcal{C}_{(0, \dots, 0, -1)} \right), \end{aligned}$$

which is strictly positive since

$$|\partial B(0, 2)| \leq c |\partial B(0, 2) \cap \mathcal{C}_{(0, \dots, 0, -1)}|.$$

□

The above lemma and the Harnack principle (Corollary 5.8 of [21]) imply the following Carleson type estimate.

Theorem 4.2 *There exists $c_0 > 0$ such that for any $Q \in \partial D$ and any $x = Q + (0, \delta_Q(x)) \in D$ in CS_Q with $\delta_Q(x) < \frac{1}{10} \left(\frac{r_0}{\Lambda} \wedge r_1 \wedge r_2 \right)$, and any nonnegative function u which is harmonic with respect to X in $B(x, 4\delta_Q(x)) \cap D$ and vanishes continuously on $B(x, 4\delta_Q(x)) \cap \partial D$, we have*

$$u(y) \leq c_0 u(x), \text{ for every } y \in B(x, 2\delta_Q(x)) \cap D.$$

Proof The proof of this theorem is similar to that of Theorem III.1.7 in [3]. Fix $Q \in \partial D$ and an orthonormal coordinate system CS_Q with origin at Q such that

$$B(Q, r_0) \cap D = B(Q, r_0) \cap \{y = (y_1, \dots, y_{d-1}, y_d) =: (\tilde{y}, y_d) \text{ in } CS_Q : y_d > \phi_Q(\tilde{y})\}.$$

Also we fix $x = Q + (0, \delta_Q(x)) \in D$ in CS_Q with $\delta_Q(x) < \frac{1}{6} (\frac{r_0}{\Lambda} \wedge r_1 \wedge r_2)$ and a nonnegative function u which is harmonic with respect to X in $B(x, 4\delta_Q(x)) \cap D$ and vanishes continuously on $B(x, 4\delta_Q(x)) \cap \partial D$. Note that

$$B(x, 4\delta_Q(x)) \cap D \subset B(Q, r_0) \cap D.$$

For each $y \in B(x, 2\delta_Q(x)) \cap D$, let $\xi_y := (\tilde{y}, \phi_Q(\tilde{y}) + \delta_Q(x))$ in CS_Q . Since

$$|\tilde{y}|^2 + |\phi_Q(\tilde{y}) + \delta_Q(x)|^2 \leq 6\delta_Q(x)^2 + 8\Lambda^2\delta_Q(x)^2 < r_0^2,$$

ξ_y is in $B(Q, r_0) \cap D$. Since $\rho(y) \leq |y - x| + \delta_Q(x) < 3\delta_Q(x) < r_1$, using the Harnack principle (Corollary 5.8 of [21]), (4.2) and a standard chain argument, we can show that there exist $\sigma > 0$ such that

$$u(y) \leq c_1 \left(\frac{\delta_Q(y)}{\delta_Q(x)} \right)^{-\sigma} u(\xi_y) \tag{4.3}$$

(see Exercise 4 in Chap. 3 of [3]). Also, since $|x - \xi_y| \leq |x - y| + |y - \xi_y| \leq 5\delta_Q(x) < r_1$ and $\delta_Q(x) = \delta_Q(\xi_y)$, using the Harnack principle (Corollary 5.8 of [21]) and (4.3), we have

$$u(y) \leq c_2 \left(\frac{\delta_Q(y)}{\delta_Q(x)} \right)^{-\sigma} u(x).$$

So

$$\delta_Q(y) \leq c_3 \left(\frac{u(y)}{u(x)} \right)^{-\frac{1}{\sigma}} \delta_Q(x). \tag{4.4}$$

We will prove this lemma by contradiction. Lemma 4.1 implies that for every $y \in B(x, 2\delta_Q(x)) \cap D$, there exists $\varepsilon < 1$ such that

$$u(y) \leq \mathbf{E}_y \left[u(X_{\tau_{B(y, 2\delta_Q(y))}}) : X_{\tau_{B(y, 2\delta_Q(y))}} \in D \right] \leq \left(\sup_{\partial B(y, 2\delta_Q(y))} u \right) \varepsilon.$$

Therefore, if there exists $x_1 \in B(x, 2\delta_Q(x)) \cap D$ such that $u(x_1) \geq Mu(x)$ (M will be chosen later so that $|x_n - Q| < r_0/2$), there exists $x_2 \in \partial B(x_1, 2\delta_Q(x_1))$ such that

$$u(x_2) = \sup_{\partial B(x_1, 2\delta_Q(x_1))} u \geq \frac{u(x_1)}{\varepsilon} \geq \frac{M}{\varepsilon} u(x).$$

Recursively one can choose a sequence $\{x_n\}_{n \geq 1}$ with $|x_{n+1} - x_n| = 2\delta_Q(x_n)$ and

$$u(x_n) \geq \frac{M}{\varepsilon^{n-1}} u(x), \tag{4.5}$$

which implies that $u(x_n) \rightarrow \infty$. But, by (4.4)–(4.5),

$$\begin{aligned} |x_n - x| &\leq |x_n - x_1| + |x_1 - x| \leq \sum_{j=1}^{\infty} 2\delta_Q(x_j) + 2\delta_Q(x) \\ &\leq \left(\sum_{j=1}^{\infty} 2c_3 \left(\frac{u(x_j)}{u(x)} \right)^{-\frac{1}{\sigma}} + 2 \right) \delta_Q(x) \leq \left(2c_3 M^{-\frac{1}{\sigma}} \sum_{j=1}^{\infty} \varepsilon^{\frac{j-1}{\sigma}} + 2 \right) \delta_Q(x), \end{aligned}$$

which is less than $4\delta_Q(x)$ if we choose a M with

$$M > \left(c_3 \sum_{j=1}^{\infty} \varepsilon^{\frac{j-1}{\sigma}} \right)^{\sigma}.$$

The above argument also implies that $\delta_Q(x_n) \rightarrow 0$. This contradicts the fact that u vanishes continuously on $B(x, 4\delta_Q(x)) \cap \partial D$. □

To prove a scale invariant version of the boundary Harnack principle, we shall follow the “box method” of Bass and Burdzy [3,5]. We adopt the following notation from [3,5]: for every $Q \in \partial D$ with an orthonormal coordinate system CS_Q with origin at Q such that $B(Q, r_0) \cap D = B(Q, r_0) \cap \{y \text{ in } CS_Q : y_d > \phi_Q(\tilde{y})\}$, we let

$$\begin{aligned} \Delta_Q(a, R) &:= \{y \text{ in } CS_Q : \phi_Q(\tilde{y}) + a > y_d > \phi_Q(\tilde{y}), |\tilde{y}| < R\}, \\ \partial^s \Delta_Q(a, R) &:= \{y \text{ in } CS_Q : \phi_Q(\tilde{y}) + \frac{a}{2} > y_d \geq \phi_Q(\tilde{y}), |\tilde{y}| = R\}, \\ \partial^u \Delta_Q(a, R) &:= \{y \text{ in } CS_Q : \phi_Q(\tilde{y}) + a = y_d, |\tilde{y}| < R\} \\ &\quad \cup \{y \text{ in } CS_Q : \phi_Q(\tilde{y}) + a \geq y_n \geq \phi_Q(\tilde{y}) + \frac{a}{2}, |\tilde{y}| = R\}. \end{aligned}$$

The proofs of the next three lemmas are similar to those of Lemmas 3.1.5–6 and Theorem 3.1.7 in [3] (also see [4,5]). But one has to be more careful since we do not have the scaling property and we use Theorem 3.8 only for small balls. We spell out the details of the proofs of these lemmas for the reader’s convenience.

Lemma 4.3 *There exist $c_1 = c_1(D)$ and $c_2 = c_2(D) > 0$ such that for any $r, R, a < \frac{1}{4}(r_0 \wedge r_1 \wedge r_2)$, $Q \in \partial D$, and $y \in \Delta_Q(a, r)$, we have*

$$\mathbf{P}_y \left(X_{\tau_{\Delta_Q(a,R)}} \in \partial^s \Delta_Q(a, R) \right) \leq c_1 e^{-\frac{c_2(R-r)}{a}}$$

Proof We split into two cases. First we assume that $R - r > 12a$. We define $\{S_n\}_{n \geq 1}$, a sequence of stopping times:

$$S_1 := \inf\{t > 0 : |X_t - X_0| \geq 2a\} \quad \text{and} \quad S_{n+1} := S_n + S_1 \circ \theta_{S_n}.$$

Fix $y \in \Delta_Q(a, r)$. Since

$$\mathbf{P}_y \left(X_{\tau_{\Delta_Q(a,R)}} \in \partial^s \Delta_Q(a, R) \right) \leq \mathbf{P}_y \left(\tau_{\Delta_Q(a,R)} > S_{\lfloor \frac{R-r}{4a} \rfloor} \right),$$

it is enough to show that there exists $\epsilon \in (0, 1)$ such that for every $n \leq \frac{1}{4a}(R - r)$,

$$\mathbf{P}_y \left(\tau_{\Delta_Q(a,R)} > S_{n+1} \right) \leq \epsilon \mathbf{P}_y \left(\tau_{\Delta_Q(a,R)} > S_n \right). \tag{4.6}$$

In fact, (4.6) implies that there exist $c_1 = c_1(D)$ and $c_2 = c_2(D) > 0$ such that

$$\mathbf{P}_y \left(X_{\tau_{\Delta_Q(a,R)}} \in \partial^s \Delta_Q(a, R) \right) \leq c\epsilon^n \leq c_1 e^{-\frac{c_2(R-r)}{a}}.$$

Now we prove (4.6). By the strong Markov property,

$$\begin{aligned} & \mathbf{P}_y \left(\tau_{\Delta_Q(a,R)} > S_{n+1} \right) \\ & \leq \mathbf{E}_y \left[\mathbf{P}_{X_{S_n}} \left(\tau_{\Delta_Q(a,R)} > S_1 \right) : \tau_{\Delta_Q(a,R)} > S_n \right]. \end{aligned}$$

If $n \leq \frac{1}{4a}(R - r)$ and $\tau_{\Delta_Q(a,R)} > S_n$, then $X_{S_n} \in \Delta_Q \left(a, \frac{R+r}{2} \right)$. If $z \in \Delta_Q \left(a, \frac{R+r}{2} \right)$, with $l := \frac{a}{\delta_Q(z)} \geq 1$ we have $4l\delta_Q(z) = 4a < (r_0 \wedge r_1 \wedge r_2)$. So by Lemma 4.1, we have

$$\mathbf{P}_z \left(\tau_{\Delta_Q(a,R)} > S_1 \right) \leq \mathbf{P}_z \left(X_{S_1} \in D \right) \leq \epsilon < 1.$$

Therefore (4.6) is true.

If $R - r \leq 12a$, choose c_1 larger if necessary so that $c_1 e^{-\frac{c_2(R-r)}{a}} \geq 1$. □

Lemma 4.4 *There exist $c = c(D) > 0$ and $\sigma = \sigma(D) > 0$ such that for every $r < \frac{1}{4}(r_0 \wedge r_1 \wedge r_2)$, $a < 2r$, $Q \in \partial D$, and $y \in \Delta_Q(r, a)$, we have*

$$\mathbf{P}_y \left(X_{\tau_{\Delta_Q(2r,8r)}} \in \partial^u \Delta_Q(2r, 8r) \right) \geq c \left(\frac{\delta_Q(y)}{r} \right)^\sigma.$$

Proof Fix $Q \in \partial D$, $r < \frac{1}{4}(r_0 \wedge r_1 \wedge r_2)$, $a < 2r$ and $y \in \Delta_Q(r, a)$. (4.2) says that there exists a constant $c_1 = c_1(D) \geq 1$ such that $c_1^{-1}\delta_Q(x) \leq \rho(x) \leq \delta_Q(x)$ for all $x \in D \cap B(Q, r_0)$. Let $y_0 = y$ and choose y_i above y_0 in CS_Q such that $|y_i - y_{i-1}| = \rho(y_{i-1})/4$. Let n and m be the smallest integers such that $\delta_Q(y_n) > 2r$ and $m \geq 8c_1$, respectively. We see that

$$\delta_Q(y_m) = \delta_Q(y_0) + |y_0 - y_1| + \dots + |y_{m-1} - y_m| \geq 2\delta_Q(y_0).$$

By induction, we get $\delta_Q(y_{km}) \geq 2^k \delta_Q(y_0)$. Let k be the smallest integer such $2^k \delta_Q(y_0) > 2r$. Then by the choices of n, m and k , we get

$$n \leq mk \leq (8c_1 + 1) \log_2 \left(\frac{4r}{\delta_Q(y_0)} \right) \leq -\frac{9c_1}{\ln 2} \ln \frac{\delta_Q(y_0)}{4r}. \tag{4.7}$$

On the other hand,

$$\delta_Q(y_n) = \delta_Q(y_{n-1}) + |y_{n-1} - y_n| \leq 2r + \frac{\rho(y_{n-1})}{4} \leq 2r + \frac{\delta_Q(y_{n-1})}{4} \leq \frac{5r}{2}.$$

So $2r \leq \delta_Q(y_n) \leq \frac{5r}{2}$. Let

$$A_j := \bigcup_{i=0}^j B \left(y_i, \frac{\rho(y_i)}{4} \right).$$

Since $a + \delta_Q(y_n) < 5r$.

$$\mathbf{P}_y \left(X_{\tau_{\Delta_Q(2r, 8r)}} \in \partial^u \Delta_Q(2r, 8r) \right) \geq \mathbf{P}_y \left(X_{\tau_{A_n}} \in \partial A_n \setminus \Delta_Q(2r, 8r) \right).$$

By Theorem 3.8, we also have

$$\begin{aligned} & \mathbf{P}_{y_n} \left(X_{\tau_{A_n}} \in \partial A_n \setminus \Delta_Q(2r, 8r) \right) \\ & \geq \mathbf{P}_{y_n} \left(X_{\tau_{B(y_n, \frac{1}{4}\rho(y_n))}} \in \partial B \left(y_n, \frac{1}{4}\rho(y_n) \right) \setminus (A_{n-1} \cup \Delta_Q(2r, 8r)) \right) \\ & \geq M_{15}^{-1} \mathbf{P}_{y_n} \left(W_{\tau_{B(y_n, \frac{1}{4}\rho(y_n))}} \in \partial B \left(y_n, \frac{1}{4}\rho(y_n) \right) \setminus (A_{n-1} \cup \Delta_Q(2r, 8r)) \right) \geq c_2 > 0. \end{aligned}$$

Now, since $\delta_Q(y_n) < \frac{5}{2}r < r_1$, using the Harnack principle (Corollary 5.8 of [21]) and a standard chain argument we can show that

$$\mathbf{P}_y \left(X_{\tau_{A_n}} \in \partial A \setminus \Delta_Q(2r, 8r) \right) \geq c_2 c_3^n.$$

Therefore, with (4.7) we conclude that

$$\mathbf{P}_y \left(X_{\tau_{\Delta_Q(2r, 8r)}} \in \partial^u \Delta_Q(2r, 8r) \right) \geq c_2 c_3^n \geq c \left(\frac{\delta_Q(y)}{r} \right)^\sigma.$$

□

Lemma 4.5 *There exists $c = c(D) > 0$ such that for every $r < \frac{1}{4}(r_0 \wedge r_1 \wedge r_2)$, $Q \in \partial D$ and $y \in \Delta_Q(r, r)$, we have*

$$\mathbf{P}_y \left(X_{\tau_{\Delta_Q(2r, 8r)}} \in \partial^s \Delta_Q(2r, 8r) \right) \leq c \mathbf{P}_y \left(X_{\tau_{\Delta_Q(2r, 8r)}} \in \partial^u \Delta_Q(2r, 8r) \right).$$

Proof We fix $r < \frac{1}{4}(r_0 \wedge r_1 \wedge r_2)$ and $Q \in \partial D$. Let

$$\Omega_1 := \left\{ X_{\tau_{\Delta_Q}(2r, 8r)} \in \partial^s \Delta_Q(2r, 8r) \right\}, \quad \Omega_2 := \left\{ X_{\tau_{\Delta_Q}(2r, 8r)} \in \partial^u \Delta_Q(2r, 8r) \right\}$$

and $\Delta := \Delta_Q(2r, 8r)$. We define a decreasing sequence $\{r_i\}_{i \geq 1}$ by

$$r_i := 2r \left(\frac{3}{4} - \frac{1}{40} \sum_{j=1}^i \frac{1}{j^2} \right) > r.$$

Note that $r < \dots < r_{i+1} < r_i < \dots < r_1 < \frac{3r}{2}$. We also define

$$A_i := \Delta_Q(2^{-i+1}r, r_i) \setminus \Delta_Q(2^{-i}r, r_i) \quad \text{and} \quad l_i := \sup_{y \in A_i} \frac{\mathbf{P}_y(\Omega_1)}{\mathbf{P}_y(\Omega_2)}.$$

We claim that $\sup_{i \geq 1} l_i < \infty$. First, by Lemma 4.4, we get $l_1 < \infty$. Let $\tau_i := \tau_{\Delta_Q(2^{-i+1}r, r_{i-1})}$ and $y \in A_i$. We then have:

$$\begin{aligned} \mathbf{P}_y(\Omega_1) &\leq \mathbf{P}_y(X_{\tau_i} \in \partial^s \Delta_Q(2^{-i+1}r, r_{i-1})) \\ &\quad + \mathbf{P}_y(X_{\tau_i} \in \partial^u \Delta_Q(2^{-i+1}r, r_{i-1}); \Omega_1) := J_{i,1} + J_{i,2}. \end{aligned}$$

Since $r_{i-1} - r_i \leq \frac{r}{20i^2}$, by Lemma 4.3, $J_{i,1}$ is bounded above by

$$c_1 e^{-c_2 \frac{r/(20i^2)}{2^{-i}r}} \leq c_1 e^{-c_3 \frac{2^i}{i^2}} \leq c_4 \frac{2^{-i\sigma}}{i^2} \leq \frac{c_5}{i^2} \left(\frac{\delta_Q(y)}{r} \right)^\sigma,$$

which is, in turn, bound above by $\frac{c}{i^2} \mathbf{P}_y(\Omega_2)$ by Lemma 4.4. On the other hand, using the strong Markov property (twice), $J_{i,2}$ is bounded above by

$$\begin{aligned} &\mathbf{E}_y \left[\mathbf{P}_{X_{\tau_i}}(\Omega_1) : \partial^u \Delta_Q(2^{-i+1}r, r_{i-1}) \right] \\ &\leq l_{i-1} \mathbf{P}_y \left(X_{\tau_i} \in \partial^u \Delta_Q(2^{-i+1}r, r_{i-1}); \Omega_2 \right) \leq l_{i-1} \mathbf{P}_y(\Omega_2). \end{aligned}$$

Therefore

$$\mathbf{P}_y(\Omega_1) \leq \frac{c}{i^2} \mathbf{P}_y(\Omega_2) + l_{i-1} \mathbf{P}_y(\Omega_2).$$

Divide the above by $\mathbf{P}_y(\Omega_2)$ and take the supremum over $y \in A_i$. We get

$$l_i \leq \frac{c}{i^2} + l_{i-1},$$

which implies that $\sup_{i \geq 1} l_i < \infty$. □

The following is the main result of this section.

Theorem 4.6 (Scale invariant boundary Harnack principle) *There exist constants $M, c > 1$ and $r_3 > 0$, depending on μ only via the rate at which $\max_{1 \leq i \leq d} M_{\mu^i}(r)$ goes to zero such that for every $Q \in \partial D, r < r_3$ and any nonnegative functions u and v which are harmonic with respect to X in $D \cap B(Q, Mr)$ and vanish continuously on $\partial D \cap B(Q, Mr)$, we have*

$$\frac{u(x)}{v(x)} \leq c \frac{u(y)}{v(y)} \quad \text{for any } x, y \in D \cap B\left(Q, \frac{r}{1+\Lambda}\right). \tag{4.8}$$

Proof Let $A := 10\sqrt{1+\Lambda^2}, M := 10A$ and $r_3 := \frac{1}{M}(r_0 \wedge r_1 \wedge r_2)$. Fix $r < r_3$ and $Q \in \partial D$ with an orthonormal coordinate system CS_Q with origin at Q such that $B(Q, r_0) \cap D = B(Q, r_0) \cap \{y \text{ in } CS_Q : y_d > \phi_Q(\tilde{y})\}$. Note that if $y \in B\left(Q, \frac{r}{1+\Lambda}\right) \cap D$, then

$$y_d - \phi_Q(\tilde{y}) \leq y_d + \Lambda|\tilde{y}| < \frac{r}{1+\Lambda} + \Lambda \frac{r}{1+\Lambda} = r.$$

So $B\left(Q, \frac{r}{1+\Lambda}\right) \cap D \subset \Delta_Q(r, r)$. Let $\Delta := \Delta_Q(2r, 8r), \partial^s \Delta := \partial^s \Delta_Q(2r, 8r)$ and $\partial^u \Delta := \partial^u \Delta_Q(2r, 8r)$. Assume $x, y \in \Delta_Q(r, r)$ and let $a = Q + (\tilde{0}, Ar)$. By the Harnack principle (Corollary 5.8 of [21]),

$$\sup_{\partial^u \Delta} u \leq c_1 u(a) \quad \text{and} \quad \inf_{\partial^u \Delta} v \geq c_2 v(a). \tag{4.9}$$

If $z \in \bar{\Delta}$,

$$\begin{aligned} |a - z| &\leq |a - Q| + |Q - z| \leq Ar + |(\tilde{z}, \phi_Q(\tilde{z})) - (\tilde{Q}, 0)| + 2r \\ &\leq Ar + 8r\sqrt{1+\Lambda^2} + 2r < 2Ar. \end{aligned}$$

So $\bar{\Delta} \subset B(a, 2Ar)$. Clearly $B(a, 4Ar) \subset B(Q, Mr)$. Thus by Theorem 4.2,

$$\sup_{\partial^s \Delta} u \leq c_3 u(a). \tag{4.10}$$

Therefore by (4.9), (4.10) and Lemma 4.5, for every $x \in \Delta_Q(r, r)$,

$$\begin{aligned} u(x) &= \mathbf{E}_x [u(X_{\tau_\Delta}) : X_{\tau_\Delta} \in \partial^s \Delta] + \mathbf{E}_x [u(X_{\tau_\Delta}) : X_{\tau_\Delta} \in \partial^u \Delta] \\ &\leq c_3 u(a) \mathbf{P}_x (X_{\tau_\Delta} \in \partial^s \Delta) + c_1 u(a) \mathbf{P}_x (X_{\tau_\Delta} \in \partial^u \Delta) \\ &\leq c_4 \frac{u(a)}{v(a)} v(a) \mathbf{P}_x (X_{\tau_\Delta} \in \partial^u \Delta) \\ &\leq c_5 \frac{u(a)}{v(a)} \mathbf{E}_x [v(X_{\tau_\Delta}) : X_{\tau_\Delta} \in \partial^u \Delta] \leq c_5 \frac{u(a)}{v(a)} v(x). \end{aligned}$$

The above argument also implies that

$$\frac{v(y)}{u(y)} \leq c_5 \frac{v(a)}{u(a)}.$$

Therefore

$$\frac{u(x)}{v(x)} \leq c_5 \frac{u(a)}{v(a)} \leq c_5^2 \frac{u(y)}{v(y)}.$$

□

5 Martin kernel and Martin boundary

By using Green function estimates, we proved in [21] that, for any bounded $C^{1,1}$ domain D , both the Martin boundary and the minimal Martin boundary with respect to X in D coincide with the Euclidean boundary. In this section, we will extend this result to bounded Lipschitz domains by using the results of the previous sections.

Fix $x_0 \in D$ throughout this section and define

$$M_D(x, y) := \begin{cases} \frac{G_D(x, y)}{G_D(x_0, y)}, & \text{if } x \in D \text{ and } y \in D \setminus \{x_0\}, \\ 1_{\{x_0\}}(x), & \text{if } y = x_0. \end{cases}$$

By Theorem 2.9 (1), we know that for each $y \in D \setminus \{x_0\}$ and $\varepsilon > 0$, $M_D(\cdot, y)$ is a harmonic function with respect to X in $D \setminus B(y, \varepsilon)$ and

$$M_D(x, y) = \mathbf{E}_x \left[M_D(X_{\tau_{D \setminus B(y, \varepsilon)}^D}^D, y) \right], \quad x \in D \setminus B(y, \varepsilon). \quad (5.1)$$

Using the Riesz decomposition theorem (Theorem 2.9 (2)–(3)), Proposition 2.10, the Harnack inequality (Corollary 5.8 in [21]) and the Hölder continuity of harmonic functions (Theorem 5.5 in [21]), one can follow the arguments in [22] (see also Sect. 2.7 of [3] or [25]) to show that the process X^D has a Martin boundary $\partial_M D$ satisfying the following properties.

- (M1) $D \cup \partial_M D$ is a compact metric space;
- (M2) D is open and dense in $D \cup \partial_M D$ and its relative topology coincides with its original topology;
- (M3) $M_D(x, \cdot)$ can be extended to $\partial_M D$ uniquely in such a way that, $M_D(x, y)$ converges to $M_D(x, w)$ as $y \rightarrow w \in \partial_M D$, the function $M_D(x, w)$ is jointly continuous on $D \times \partial_M D$, and $M_D(\cdot, w_1) \neq M_D(\cdot, w_2)$ if $w_1 \neq w_2$;

The Harnack inequality (Corollary 5.8 in [21]) and the harmonicity of $M_D(\cdot, y)$ in $D \setminus \{y\}$ imply the harmonicity of $M_D(\cdot, w)$ in D for $w \in \partial_M D$.

Proposition 5.1 *For every $w \in \partial_M D$, $x \mapsto M(x, w)$ is harmonic with respect to X in D .*

Proof Fix $w \in \partial_M D$ and relatively compact open sets $U \subset \bar{U} \subset U_1 \subset \bar{U}_1$ in D . Let $\delta := \frac{1}{2} \text{dist}(U, \partial U_1)$. Choose a sequence $\{y_n\}_{n \geq 1}$ in $D \setminus \bar{U}_1$ converging to w in $D \cup \partial_M D$ so that

$$M_D(x, w) = \lim_{n \rightarrow \infty} M_D(x, y_n).$$

Since $M_D(\cdot, y_n)$ is harmonic in a neighborhood of U for every $n \geq 1$, we have

$$\mathbf{E}_x \left[M_D(X_{\tau_U}^D, y_n) \right] = M_D(x, y_n), \quad x \in U.$$

Using the Harnack inequality (Corollary 5.8 in [21]) and a Harnack chain argument, we have for every $z \in \partial U$,

$$M_D(z, y_n) = \frac{G_D(z, y_n)}{G_D(x_0, y_n)} \leq c_1 \frac{G_D(x_0, y_n)}{G_D(x_0, y_n)} = c_1, \quad n \geq 1,$$

for some $c_1 = c_1(\delta, D) > 0$. Thus by the bounded convergence theorem,

$$\lim_{n \rightarrow \infty} \mathbf{E}_x \left[M_D(X_{\tau_U}^D, y_n) \right] = \mathbf{E}_x \left[M_D(X_{\tau_U}^D, w) \right] = M_D(x, w), \quad x \in U.$$

□

Since $M_D(x_0, y) = 1$ for every $y \in D \cup \partial_M D \setminus \{x_0\}$, using Theorem 5.5 and Corollary 5.8 in [21] we can show that, for any compact subset K of D , the family $\{M_D(\cdot, w) : w \in \partial_M D\}$ is uniformly bounded and equicontinuous on K . One can then apply the Ascoli-Arzelà theorem to prove the existence of minimal Martin boundary $\partial_m D$. We omit the details since the proof will be almost identical to the classical case (see Sect. 2.7 of [3]). Thus, by combining the above argument with Theorem 2.9, we have that for every excessive function f of X^D , there are a unique Radon measure ν_1 on D and a unique finite measure ν_2 on $\partial_m D$ such that

$$f(x) = \int_D G_D(x, y) \nu_1(dy) + \int_{\partial_m D} M_D(x, z) \nu_2(dz), \tag{5.2}$$

and f is harmonic in D with respect to X if and only if $\nu_1 = 0$. When f is harmonic in D with respect to X , the measure ν_2 above is called the Martin measure of f .

From now on we shall assume that D is a bounded Lipschitz domain. First, we will use Carleson estimate (Theorem 4.2) to show that there exists a continuous map from the Martin compactification $\partial_M D \cup D$ onto the Euclidean closure \bar{D} .

Lemma 5.2 *There exists a continuous map ι from $\partial_M D \cup D$ onto \bar{D} which is an identity map in D .*

Proof Note that $D \cup \partial_M D$ is a compact metric space. We will show that if a sequence $\{y_n\}_{n \geq 1}$ in D converges to a point w in $\partial_M D$, it converges in \bar{D} . Assume that a subsequence $\{y_{n_k}\}_{k \geq 1}$ of $\{y_n\}_{n \geq 1}$ converges to $y_0 \in \bar{D}$. Let U and U_0 be relatively compact

open sets with $y_0 \in U_0 \subset \overline{U_0} \subset U \subset \overline{U} \subset D \setminus \{x_0\}$. Choose k_0 large so that $y_{n_k} \in U_0$ for every $k \geq k_0$ and let $\delta := \text{dist}(U_0, \partial U)$. Suppose $x \in D \setminus \overline{U}$. By the harmonicity of $G_D(\cdot, y_{n_k})$ on $D \setminus U_0$ and Theorem 2.6, we can use the Carleson estimate (Theorem 4.2) and a Harnack chain argument (Corollary 5.8 in [21]) with $G_D(\cdot, y_{n_k})$ on $D \setminus U_0$. Thus for every $a \in D \cap \partial U$,

$$M_D(a, y_{n_k}) = \frac{G_D(a, y_{n_k})}{G_D(x_0, y_{n_k})} \leq c_1 \frac{G_D(x_0, y_{n_k})}{G_D(x_0, y_{n_k})} = c_1, \quad k \geq k_0, \quad (5.3)$$

for some $c_1 = c_1(\delta, D)$. Therefore, using (5.1) and the bounded convergence theorem, we have for every $x \in D \setminus U$

$$\begin{aligned} M_D(x, w) &= \lim_{k \rightarrow \infty} M_D(x, y_{n_k}) = \lim_{k \rightarrow \infty} \mathbf{E}_x \left[M_D(X_{\tau_{D \setminus U}}^D, y_{n_k}) \right] \\ &= \mathbf{E}_x \left[M_D(X_{\tau_{D \setminus U}}^D, w) \right]. \end{aligned}$$

(5.3) implies that $M_D(a, w) \leq c_1$ for $a \in D \cap \partial U$. So we have for every $y \in \partial D \setminus \overline{U}$,

$$\lim_{x \rightarrow y} M_D(x, w) \leq c_1 \lim_{x \rightarrow y} \mathbf{P}_x(\tau_{D \setminus U} < \tau_D) = 0, \quad y \in \partial D \setminus \overline{U}.$$

If there exists a subsequence of $\{y_n\}_{n \geq 1}$ converging to a point in \overline{D} different from y_0 , the above argument says that the Martin kernel would vanish continuously on ∂D . But that implies that $M_D(\cdot, w) \equiv 0$ by the maximum principle (Lemma 7.2 in [21]), which is impossible. Therefore $y_0 \in \overline{D}$ must be unique. Moreover, y_0 must be in ∂D otherwise we could choose U in D and also argue that the Martin kernel would vanish continuously on ∂D .

Since $\{y_n\}_{n \geq 1}$ is bounded in \overline{D} , the above argument shows that every subsequence of $\{y_n\}_{n \geq 1}$ has a further subsequence converging to a unique point in \overline{D} . So the map ι defined by $\iota(w) = y_0$ is continuous.

Now we show that ι is onto. Fix a point $z_0 \in \partial D$ and choose a sequence $\{y_n\}_{n \geq 1}$ in D converging to a z_0 in \overline{D} . Since $\{y_n\}_{n \geq 1}$ is a sequence in the compact metric space $D \cup \partial_M D$, there exists a subsequence $\{y_{n_k}\}_{k \geq 1}$ of $\{y_n\}_{n \geq 1}$ converges to a $w_0 \in D \cup \partial_M D$. By the continuity of ι , $\iota(w_0) = z_0$. \square

From the proof of the above lemma, we have the following corollary.

Corollary 5.3 *Suppose $w_0 \in \partial_M D$. For every open subset $V \ni \iota(w_0)$, we have,*

$$M_D(x, w_0) = \mathbf{E}_x \left[M_D(X_{\tau_{D \setminus V}}^D, w_0) \right].$$

If $z \in \partial D$ is different from $\iota(w_0)$, then

$$\lim_{D \ni x \rightarrow z} M_D(x, w_0) = 0.$$

Lemma 5.4 *For each $w_0 \in \partial_M D$, the support of the Martin measure ν for $M_D(\cdot, w_0)$ is contained in $\iota^{-1}(\iota(w_0))$.*

Proof Let $z_0 := \iota(w_0)$ and ν be the Martin measure ν for $M_D(\cdot, w_0)$. For any closed subset U of $\partial_M D$ such that $U \cap \iota^{-1}(z_0) = \emptyset$, define

$$h(x) = \int_U M_D(x, w)\nu(dw) \quad \text{for } x \in D.$$

We will show that $h \equiv 0$, which implies that the support of ν is contained in $\iota^{-1}(z_0)$.

If $z \in \partial D$ is different from $z_0 = \iota(w_0)$, then by Corollary 5.3,

$$h(x) \leq \int_{\partial_M D} M_D(x, w)\nu(dw) = M_D(x, w_0) \rightarrow 0 \quad \text{as } x \rightarrow z. \tag{5.4}$$

On the other hand, for any $w \in U$, by Corollary 5.3, $\lim_{x \rightarrow z_0} M_D(x, w) = 0$. Let $\delta := \text{dist}(z_0, \iota(U))$. By the Carleson estimate (Theorem 4.2) and a Harnack chain argument (Corollary 5.8 in [21]) we get that for every $x \in B(z_0, \frac{1}{3}\delta)$ and $y \in \{z \in D : \text{dist}(z, \iota(U)) < \frac{1}{3}\delta\}$,

$$\frac{G_D(x, y)}{G_D(x_0, y)} \leq c_1 \frac{G_D(x_0, y)}{G_D(x_0, y)} = c_1$$

for some $c_1 = c_1(\delta, D)$. Thus, for any $w \in U$ and $x \in B(z_0, \frac{1}{3}\delta)$, $M_D(x, w)$ is bounded above by c_1 . Therefore by the bounded convergence theorem, we get

$$\lim_{x \rightarrow z_0} h(x) = \lim_{x \rightarrow z_0} \int_U M_D(x, w)\nu(dw) = 0. \tag{5.5}$$

Combining (5.4), (5.5) and Maximum principle, we conclude that $h \equiv 0$ □

The next lemma is a consequence of the scale invariant version of the boundary Harnack principle (Theorem 4.6).

Lemma 5.5 *There exists constant $c_1 > 1$ such that for every $Q \in \partial D$ and any non-negative functions u and v which are harmonic with respect to X in D and vanish continuously on $\partial D \setminus \{Q\}$, we have*

$$\frac{u(x)}{u(x_0)} \leq c_1 \frac{v(x)}{v(x_0)} \quad \text{for any } x \in D. \tag{5.6}$$

Proof Recall the constants $M, c > 1$ and $r_3 > 0$ in Theorem 4.6 and the Lipschitz constant Λ . We fix $M, c > 1, r_3 > 0, \Lambda$ and a point Q on ∂D throughout this

proof. We also fix an orthonormal coordinate system CS_Q with origin at Q such that $B(Q, r_0) \cap D = B(Q, r_0) \cap \{y \text{ in } CS_Q : y_d > \phi(\tilde{y})\}$. Let $r < \frac{r_3}{4}$, $Q_r := \left(\tilde{0}, \frac{r}{2(1+\Lambda)}\right)$,

$$\begin{aligned} \Delta_r &:= \left\{ y \text{ in } CS_Q : \phi(\tilde{y}) + \frac{r}{2(1+\Lambda)} > y_d > \phi(\tilde{y}), |\tilde{y}| < 2Mr \right\}, \\ \partial^s \Delta_r &:= \left\{ y \text{ in } CS_Q : \phi(\tilde{y}) + \frac{r}{2(1+\Lambda)} \geq y_d > \phi(\tilde{y}), |\tilde{y}| = 2Mr \right\}, \\ \partial^u \Delta_r &:= \left\{ y \text{ in } CS_Q : \phi(\tilde{y}) + \frac{r}{2(1+\Lambda)} = y_d, |\tilde{y}| \leq 2Mr \right\}. \end{aligned}$$

Choose r smaller if necessary so that $x_0 \notin \overline{\Delta_r}$. For $|\tilde{y}| = 2Mr$, we have $|\langle \tilde{y}, \phi(\tilde{y}) \rangle| > Mr$. So u and v are harmonic with respect to X in $D \cap B((\tilde{y}, \phi(\tilde{y})), Mr)$ and vanish continuously on $\partial D \cap B((\tilde{y}, \phi(\tilde{y})), Mr)$ where $|\tilde{y}| = 2Mr$. Therefore by Theorem 4.6,

$$\frac{u(x)}{v(x)} \leq c \frac{u(y)}{v(y)} \quad \text{for any } x, y \in \partial^s \Delta_r \text{ with } \tilde{x} = \tilde{y}. \tag{5.7}$$

We claim that there exists c_3 such that for every $x \in \partial^s \Delta_r \cup \partial^u \Delta_r$

$$c_3^{-1} \frac{u(x)}{u(Q_r)} \leq \frac{v(x)}{v(Q_r)} \leq c_3 \frac{u(x)}{u(Q_r)}. \tag{5.8}$$

If $x \in \partial^u \Delta_r$, the Harnack principle (Corollary 5.8 in [21]) implies that there exists constant $c_2 > 1$ such that

$$c_2^{-1} < \frac{u(x)}{u(Q_r)}, \frac{v(x)}{v(Q_r)} < c_2. \tag{5.9}$$

Thus (5.8) is true for every $x \in \partial^u \Delta_r$. Now we assume that $x \in \partial^s \Delta_r$. Let $x_r := \left(\tilde{x}, \phi(\tilde{x}) + \frac{r}{2(1+\Lambda)}\right)$. Since $x_r \in \partial^s \Delta_r$, (5.7) implies that

$$c^{-1} \frac{u(x)}{u(x_r)} \leq \frac{v(x)}{v(x_r)} \leq c \frac{u(x)}{u(x_r)}. \tag{5.10}$$

On the other hand, x_r is also in $\partial^u \Delta_r$. Thus, by (5.9),

$$c_2^{-1} < \frac{u(x_r)}{u(Q_r)}, \frac{v(x_r)}{v(Q_r)} < c_2. \tag{5.11}$$

(5.10)–(5.11) imply that (5.8) is true for each $x \in \partial^s \Delta_r$ and c_3 is independent of the choice of $x \in \partial^s \Delta_r$. Therefore (5.8) is true for every $x \in \partial^s \Delta_r \cup \partial^u \Delta_r$. Moreover, by applying the maximum principle (Lemma 7.2 in [21]), (5.8) is true for every $x \in D \setminus \Delta_r$. In particular, $\frac{u(x_0)}{u(Q_r)} \geq c_3^{-1} \frac{v(x_0)}{v(Q_r)}$. Therefore

$$\frac{u(x)}{u(x_0)} = \frac{u(x)}{u(Q_r)} \frac{u(Q_r)}{u(x_0)} \leq c_3^2 \frac{v(x)}{v(Q_r)} \frac{v(Q_r)}{v(x_0)} = c_3^2 \frac{v(x)}{v(x_0)}, \quad x \in D \setminus \Delta_r.$$

Since c_3 is independent of r , by letting $r \downarrow 0$, the above inequality is true for every $x \in D$. □

Lemma 5.6 ι is one to one. Moreover $\partial_m D = \partial_M D$.

Proof Fix $z_0 \in \partial D$. (5.2) and Lemma 5.4 imply that for each $w \in \iota^{-1}\{z_0\}$, there is a unique Martin measure ν_w for $M_D(\cdot, w)$ such that

$$M_D(x, w) = \int_{\iota^{-1}(z_0) \cap \partial_m D} M_D(x, a) \nu_w(da).$$

Therefore $\iota^{-1}(z_0) \cap \partial_m D \neq \emptyset$. Fix $w_0 \in \iota^{-1}(z_0) \cap \partial_m D$ and let $w_1, w_2 \in \iota^{-1}(z_0)$. Since $M_D(x, w_1)$ and $M_D(x, w_0)$ are harmonic with respect to X in D (Proposition 5.1) and vanish continuously on $\partial D \setminus \{z_0\}$ (Corollary 5.3), from Lemma 5.5 we know that there exists c_3 such that

$$M_D(x, w_1) \leq c_3 \frac{M_D(x_0, w_1)}{M_D(x_0, w_0)} M_D(x, w_0) = c_3 M_D(x, w_0).$$

The minimal harmonicity of $M_D(\cdot, w_0)$ implies that $M_D(x, w_1) = c_3 M_D(x, w_0)$. But they agree on $x = x_0$. So $M_D(\cdot, w_1) = M_D(\cdot, w_0)$. The same argument shows that $M_D(\cdot, w_2) = M_D(\cdot, w_0)$, thus $M_D(\cdot, w_2) = M_D(\cdot, w_1)$, which implies that $w_1 = w_2$. Therefore ι is one to one. The above argument also says that every $w \in \iota^{-1}(\partial D)$ is a minimal Martin boundary point. Since ι is onto and $\iota(\partial_M D) = \partial D$ (Lemma 5.2), every $w \in \partial_M D$ is a minimal Martin boundary point of X^D . Therefore $\partial_m D = \partial_M D$ □

Lemmas 5.2 and 5.6 imply there exists a homeomorphism between $\partial_m D \cup D$ and \overline{D} which is an identity map in D . Therefore we arrive the following result.

Theorem 5.7 *There is a one-to-one correspondence between the minimal Martin boundary $\partial_m D$ and the Euclidean boundary ∂D .*

6 Extensions to non-symmetric diffusions with measure-valued drifts

In this section, we will briefly indicate that the results of this paper remain valid for non-symmetric diffusions with measure-valued drifts. To give a precise definition of a diffusion with drift μ in $\mathbf{K}_{d,1}$, we let $\varphi(x)$ and $\varphi_n(x)$ be as in Section 1. For $1 \leq i \leq d$ and $n = 1, 2, \dots$, define

$$U_n^i(x) = \int \varphi_n(x - y) \mu^i(dy).$$

Put $U_n(x) = (U_n^1(x), \dots, U_n^d(x))$. Let L be either L_1 or L_2 where

$$L_1 := \frac{1}{2} \sum_{i,j=1}^d \partial_i(a_{ij} \partial_j) \quad \text{and} \quad L_2 := \frac{1}{2} \sum_{i,j=1}^d a_{ij} \partial_i \partial_j$$

with $\mathbf{A} := (a_{ij})$ being C^1 and uniformly elliptic.

Definition 6.1 Suppose $\mu = (\mu^1, \dots, \mu^d)$ is such that each μ^i is a signed measure on \mathbf{R}^d belonging to the Kato class $\mathbf{K}_{d,1}$. A diffusion with drift μ is a family of probability measures $\{\mathbf{P}_x : x \in \mathbf{R}^d\}$ on $C([0, \infty), \mathbf{R}^d)$, the space of continuous \mathbf{R}^d -valued functions on $[0, \infty)$, such that under each \mathbf{P}_x we have

$$Z_t = x + Y_t + A_t$$

where

- (a) $A_t = \lim_{n \rightarrow \infty} \int_0^t U_n(Z_s) ds$ uniformly in t over finite intervals, where the convergence is in probability;
- (b) there exists a subsequence $\{n_k\}$ such that

$$\sup_k \int_0^t |U_{n_k}(Z_s)| ds < \infty$$

almost surely for each $t > 0$;

- (c) Y_t is a diffusion in \mathbf{R}^d starting from the origin with generator L .

When A is symmetric, the existence and uniqueness of Z were established in [6] (see Remark 6.1 in [6]). In [21], we discussed Brownian motions with the drifts $\mu \in \mathbf{K}_{d,1}$. when A is symmetric, all of results can be extended to diffusions with the drift μ due to the estimates on the density and its gradients for the diffusion Y (see [24] for divergence case and [17] for non-divergence case) and the continuities of the density and its gradients. We state the extensions of the main results in [21] without proofs since they will be almost the same.

Z has a transition density $r(t, x, y)$ which is continuous on $(0, \infty) \times \mathbf{R}^d \times \mathbf{R}^d$ and that there exist positive constants $C_i, i = 1, \dots, 9$ such that

$$C_1 e^{-C_2 t} t^{-\frac{d}{2}} e^{-\frac{C_3 |x-y|^2}{2t}} \leq r(t, x, y) \leq C_4 e^{C_5 t} t^{-\frac{d}{2}} e^{-\frac{C_6 |x-y|^2}{2t}} \tag{6.1}$$

and

$$|\nabla_x r(t, x, y)| \leq C_7 e^{C_8 t} t^{-\frac{d+1}{2}} e^{-\frac{C_9 |x-y|^2}{2t}} \tag{6.2}$$

for all $(t, x, y) \in (0, \infty) \times \mathbf{R}^d \times \mathbf{R}^d$.

In fact, when we deal with Z^D (the process obtained by killing Z upon exiting from D) for a bounded domain D , the symmetry assumption is unnecessary by the following simple reduction; Let $b_{ij} := \frac{1}{2}(a_{ij} + a_{ji})$ and note that

$$\sum_{i,j=1}^d a_{ij} \partial_i \partial_j = \sum_{i,j=1}^d b_{ij} \partial_i \partial_j = \sum_{i,j=1}^d \partial_i (b_{ij} \partial_j) - \sum_{i,j=1}^d (\partial_i b_{ij}) \partial_j$$

and

$$\begin{aligned} \sum_{i,j=1}^d \partial_i(a_{ij} \partial_j) &= \sum_{i,j=1}^d a_{ij} \partial_i \partial_j + \sum_{i,j=1}^d (\partial_i a_{ij}) \partial_j \\ &= \sum_{i,j=1}^d \partial_i(b_{ij} \partial_j) + \sum_{i,j=1}^d \frac{1}{2} \partial_i(a_{ij} - a_{ji}) \partial_j. \end{aligned}$$

Let Y_t be the diffusion in \mathbf{R}^d with generator

$$\frac{1}{2} \sum_{i,j=1}^d \partial_i(b_{ij} \partial_j).$$

Since, for any bounded domain D ,

$$\begin{aligned} &\left(\sum_{i=1}^d \frac{1}{2} \partial_i b_{i1} |_{D}, \dots, \sum_{i=1}^d \frac{1}{2} \partial_i b_{id} |_{D} \right) \text{ and} \\ &\left(\sum_{i=1}^d \frac{1}{4} \partial_i(a_{i1} - a_{1i}) |_{D}, \dots, \sum_{i=1}^d \frac{1}{4} \partial_i(a_{id} - a_{1d}) |_{D} \right) \end{aligned}$$

are in $\mathbf{K}_{d,1}$, we construct Z_t with a drift which is either

$$\left(\mu^1 + \frac{1}{2} \sum_{i=1}^d \partial_i b_{i1} |_{D} dx, \dots, \mu^d + \frac{1}{2} \sum_{i=1}^d \partial_i b_{id} |_{D} dx \right)$$

or

$$\left(\mu^1 + \frac{1}{4} \sum_{i=1}^d \partial_i(a_{i1} - a_{1i}) |_{D} dx, \dots, \mu^d + \frac{1}{4} \sum_{i=1}^d \partial_i(a_{id} - a_{1d}) |_{D} dx \right)$$

as in the Definition 6.1. Then the generator of the killed diffusion process Z^D in D can be informally written as $L + \mu \cdot \nabla$ where L is either

$$\frac{1}{2} \sum_{i,j=1}^d \partial_i(a_{ij} \partial_j) \quad \text{or} \quad \frac{1}{2} \sum_{i,j=1}^d a_{ij} \partial_i \partial_j.$$

with $\mathbf{A} := (a_{ij})$ being C^1 and uniformly elliptic but not necessarily symmetric.

Using the estimates on the density and its gradients for the diffusion Y^D for divergence case (see [24]) and the continuities of the density and its gradients, the proof of the following estimates is the same as the one in [21]: for every bounded $C^{1,1}$ domain

D , Z^D has a density r^D which is continuous on $(0, \infty) \times D \times D$ and that for any $T > 0$, there exist positive constants C_i , $i = 10, \dots, 14$, such that

$$C_{10}\psi_D(t, x, y)t^{-\frac{d}{2}}e^{-\frac{C_{11}|x-y|^2}{t}} \leq r^D(t, x, y) \leq C_{12}\psi_D(t, x, y)t^{-\frac{d}{2}}e^{-\frac{C_{13}|x-y|^2}{t}} \quad (6.3)$$

and

$$|\nabla_x r^D(t, x, y)| \leq C_{14} \left(1 \wedge \frac{\rho(y)}{\sqrt{t}} \right) t^{-\frac{d+1}{2}} e^{-\frac{C_{13}|x-y|^2}{t}} \quad (6.4)$$

for all $(t, x, y) \in (0, T] \times D \times D$.

With these estimates (6.1)–(6.4) in hand, the arguments of this paper can be generalized to the present setting with obvious modifications. We omit all the details.

Remark 6.2 In the forthcoming papers [19, 20], we will discuss on the intrinsic ultracontractivity of the semigroup of Z^D and the dual process of Z^D , respectively.

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