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The multidirectional Neumann problem in $\mathbb{R}^4$

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Abstract. The Neumann problem as formulated in Lipschitz domains with L^p boundary data is solved for harmonic functions in any compact polyhedral domain of $\mathbb{R}^4$ that has a connected 3-manifold boundary. Energy estimates on the boundary are derived from new polyhedral Rellich formulas together with a Whitney type decomposition of the polyhedron into similar Lipschitz domains. The classical layer potentials are thereby shown to be semi-Fredholm. To settle the *onto* question a method of continuity is devised that uses the classical 3-manifold theory of E. E. Moise in order to untwist the polyhedral boundary into a Lipschitz boundary. It is shown that this untwisting can be extended to include the interior of the domain in local neighborhoods of the boundary. In this way the flattening arguments of B. E. J. Dahlberg and C. E. Kenig for the H^1_{at} Neumann problem can be extended to polyhedral domains in $\mathbb{R}^4$. A compact polyhedral domain in $\mathbb{R}^6$ of M. L. Curtis and E. C. Zeeman, based on a construction of M. H. A. Newman, shows that the untwisting and flattening techniques used here are unavailable in general for higher dimensional boundary value problems in polyhedra.

Key words. bi-Lipschitz, homology sphere, method of continuity, polyhedral Rellich, shelling, 3-manifold, layer potentials, atomic estimate

1. Introduction

Let Ω be the interior of a compact polyhedron sitting in $\mathbb{R}^n$. Suppose Ω is a domain and that the boundary $\partial\Omega$ is a topological $(n-1)$-manifold. When $n \geq 3$ such a domain need not be a Lipschitz domain, also variously known as a *strongly Lipschitz domain* [Mor66] p.72 or *Lipschitz graph domain*. When $n \leq 4$, however, it will necessarily satisfy the weaker definition of Lipschitz domain in terms of bi-Lipschitz mappings by [Mor66] pp.4,77. As will be made clear below, this is a consequence of the classical 3-manifold theory of E. E. Moise [Moi52]. It is a remarkable topological fact that when $n \geq 6$ such polyhedral domains need not satisfy Morrey's definition nor even the apparently weaker definition given by V. G. Maz'ja in terms of quasi-isometries [Maz85] pp.15–16,19. See §11.2.1.

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By now there is a vast literature on solutions to boundary value problems with corner and edge singularities. There are many different approaches to the subject. It is widely recognized that difficulties can arise when a boundary is not given as a graph in local coordinates. To cite just one example where work on such problems is done, we refer the reader to the book of Kozlov, Maz'ya and Rossman [KMR01] and its introductory discussion.

This paper and its companion [VV03] take the point of view that the approach to boundary value problems in Lipschitz domains that has been developed over the past few decades should be and can be extended to nongraph settings. It also indicates possible geometric and topological obstacles to such a project. Here the Neumann problem for Laplace's equation will be solved in compact polyhedra of $\mathbb{R}^4$. The domains, therefore, are not necessarily Lipschitz though they will satisfy the bi-Lipschitz definition given by Morrey. *The phrase Lipschitz polyhedron will mean a compact polyhedron with interior that is a Lipschitz domain (*pace *Morrey, strongly Lipschitz). Morrey's Lipschitz domains will be called bi-Lipschitz domains.* Our initial approach to boundary value problems is to decompose general polyhedra into Lipschitz polyhedra in which estimates are already known to exist.

The convex hull of a finite set of points that is not contained in any hyperplane of $\mathbb{R}^n$ is an n-polytope. Any finite union of n-polytopes that yields a domain for its interior will be termed a compact polyhedral domain. With neither loss nor gain of generality such a domain can be realized as a finite homogeneous simplicial complex. The geometric structure thus provided seems indispensable for our analysis of boundary value problems.

Consider, for example, a piecewise linear Jordan curve in the plane. The closure of the bounded component of its complement will be a polygon. Every polygon is a Lipschitz domain. If it is particularly twisted back upon itself it may take some argument to prove the fact that the polygon is piecewise linearly homeomorphic to a triangular disc. (See, for example, pp.20–21 [Bin83].) This is a global problem from the point of view of boundary value problems.

If, however, the polygon is coned to a point in $\mathbb{R}^3$ that is not in the plane, it is unlikely that the resulting polyhedron in $\mathbb{R}^3$ will be Lipschitz even though it will be piecewise linearly (PL) homeomorphic to a tetrahedron. The global difficulty becomes a local difficulty at the vertex.

A compact polyhedron in $\mathbb{R}^3$ that yields a Jordan domain, i.e. its boundary is homeomorphic to the 2-sphere $\mathbb{S}^2$, could be non-Lipschitz and also display some further global pathologies not possible in the plane. For example, it could be tied into a number of knots, or contain knotted tunnels as in a Furch knotted hole ball. When coned to a point in $\mathbb{R}^4$ its local and global pathologies both contribute to the non-Lipschitz nature of a domain that will nevertheless be PL-homeomorphic to a 4-simplex.

A further complication arises from the unsolved status of the PL-Schoenflies conjecture for $n \geq 4$. Compact polyhedral Jordan domains in $\mathbb{R}^4$ may exist that are not PL homeomorphic or even bi-Lipschitz homeomorphic to the 4-simplex.

Thus compact polyhedra in $\mathbb{R}^n$, $n \geq 3$, can not in general be approximated by Lipschitz domains either globally or locally with any control over the Lipschitz natures of the approximating domains. They do, however, admit a type of Whitney decomposition locally about the boundary into Lipschitz domains with uniform natures. After preliminary arguments in §2 such a decomposition is defined in §3 for polyhedra of $\mathbb{R}^4$. The uniformity is proved in Lemma 1 and the decomposition applied to get a priori estimates for both the Neumann and regularity (Dirichlet data in $W^{1,2}(\partial\Omega)$) problems in 4-dimensional polyhedra (Lemma 2). That the domains of the decomposition are Lipschitz is shown in the Appendix (§12) by a modification of an argument from [VV03].

The Neumann estimate used in each Lipschitz domain of the decomposition is from [JK81]. (See [Ver84] for a solution to the Neumann problem in non-starlike Lipschitz domains. See p. 1230 of [Bro94] for the kind of argument that yields the estimate on a disconnected boundary once local estimates are known.)

Theorem 1 (D. S. Jerison and C. E. Kenig). *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and let u be a harmonic function in Ω. If $N(\nabla u) \in L^2(\partial\Omega)$ then $\|N(\nabla u)\|_2 \leq C\|\frac{\partial u}{\partial N}\|_2$ with C depending only on the Lipschitz nature of Ω.*

Here $N(\nabla u)$ denotes the nontangential maximal function of ∇u, while N denotes the outer unit normal vector to $\partial\Omega$ and $\frac{\partial u}{\partial N}$ the normal derivative of u defined a.e. with respect to surface measure ds. The L^2 norms are with respect to surface measure. The regularity estimate of Lemma 2 is obtained by using R. M. Brown's mixed (regularity and Neumann) estimate [Bro94] in each Lipschitz domain of the decomposition. The insufficiency of the standard (unmixed) regularity estimate [JK81] is due to a dimensional difficulty resolved in §4. A bounded domain Ω is called a Lipschitz domain if for each point P of the boundary there is a rotation of the Euclidean coordinates of $\mathbb{R}^n$, a neighborhood O of P and a Lipschitz function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ so that

$$\Omega \cap O = \{x \mid x_n > \phi(x')\} \cap O$$

Suppose in addition $\partial\Omega = \mathcal{N} \cup \mathcal{D}$ a disjoint union. Suppose there exists a *finite* covering of $\partial\Omega$ by neighborhoods O_i as above each with an additional Lipschitz function $\psi_i : \mathbb{R}^{n-2} \to \mathbb{R}$ such that either (1) $\mathcal{N} \cap O_i = \emptyset$ or (2) $\mathcal{D} \cap O_i = \emptyset$ or (3) $\mathcal{N} \cap O_i = \{x \mid x_1 \geq \psi_i(x'')\} \cap \partial\Omega \cap O_i$ and $\mathcal{D} \cap O_i = \{x \mid x_1 < \psi_i(x'')\} \cap \partial\Omega \cap O_i$ where $x = (x', x_n) = (x_1, x'', x_n)$. Finally suppose there is a number $\delta > 0$ and, for each i in case (3), a constant vector field α_i such that $\alpha_i \cdot N \geq \delta$ on $\mathcal{N} \cap O_i$ and $\alpha_i \cdot N \leq -\delta$ on $\mathcal{D} \cap O_i$. Then Ω together with $\mathcal{N}$ and $\mathcal{D}$ will be called a *creased domain*. Because the next theorem is about estimates and not existence, the idea of creased domain is a bit more general than in [BS01]. See Remark 4.

Theorem 2 (R. M. Brown). *Let* $\Omega \subset \mathbb{R}^n$ *with* $\partial\Omega = \mathcal{N} \cup \mathcal{D}$ *be a creased domain and let u be a harmonic function in* Ω. *If* $N(\nabla u) \in L^2(\partial\Omega)$, *then*

$$\|N(\nabla u)\|_2 \leq C\left(\|\frac{\partial u}{\partial N}\|_{\mathcal{N}} + \|u\|_{\mathcal{D}}\right)$$

with C depending only on the Lipschitz natures of Ω, $\mathcal{N}$ *and* $\mathcal{D}$.

Here $\| \|_{\mathcal{N}}$ denotes the norm for $L^2(\mathcal{N})$ and $\| \|_{\mathcal{D}}$ denotes the norm for the Sobolev space $W^{1,2}(\mathcal{D})$ with respect to surface measure.

It is important that these two theorems apply to *all* harmonic functions with gradients in the nontangential L^2 class. Otherwise our solution to the Neumann problem in polyhedra might only be for data subject to a finite number of linear conditions depending on a given polyhedron rather than on just the one independent condition of mean value zero. Such a result would hardly be interesting given that a polyhedron with its finite number of corners and edges presumably presents only a finite number of difficulties. See Remark 5.

The Lipschitz domain estimates of Theorems 1 and 2 cannot suffice to establish comparable estimates in a non-Lipschitz polyhedron, however. Lemma 2 contains nontrivial error terms that arise from the data on the surfaces interior to the polyhedron that were introduced by the Whitney decomposition. Additional energy estimates specifically for polyhedral domains are used to control these error terms by either Neumann or regularity data now taken entirely on the polyhedral boundary. These are called *polyhedral Rellich formulas*. One for vertices is given in §2, while another for 1-dimensional edges (boundary 1-simplexes) is introduced in §4. Theorem 3 gives the completed a priori estimates for the Neumann and regularity problems in 4-dimensional polyhedra. Altogether §§2–4 extend and simplify the boundary energy methods of [Ver01].

In §5.1 the classical single layer potential supplies a large class of solutions to which Theorem 3 applies. Because the Neumann data then takes the familiar form of the identity operator plus a singular integral operator acting on $L^2(\partial\Omega)$, the singular integral operator being noncompact, Verchota's method of inverting layer potentials [Ver84] is used. This and Theorem 3 show that the boundary operator is semi-Fredholm (1:1 and closed range). Solvability of the Neumann problem for L^2 data is thereby reduced to showing that the boundary operator is onto. The boundary operator is known to be onto if it is defined on a Lipschitz boundary [Ver84]. A method of continuity relating the operator from a polyhedral boundary to one on the boundary of a Lipschitz polyhedron is therefore brought about in §§6-8. The version of the method of continuity used (see the beginning discussions in §§6 and 7) states that if a continuum of linear operators from a Banach space to itself is such that each operator in the continuum has a bounded inverse on its range, then if any one operator is onto so are the others.

Given a compact polyhedral domain Ω with 3-manifold boundary, a suitable Lipschitz polyhedron Ω_{Lip} is provided by Theorem 12 (Appendix). In §§6 and 8 it is shown by a classical *shelling* procedure that there is a way to pass from Ω to Ω_{Lip} through a finite sequence of polyhedra so that the successive boundaries are PL-homeomorphic. Moreover at each step there is a continuous and bi-Lipschitz (see Remark 8) deformation, i.e. isotopy, of one polyhedral boundary to the next. Lemma 4 then provides a continuum of operators at each step.

Disregarding the requirements of shelling makes it difficult to decide the Fredholm properties of the resulting boundary operators. This is demonstrated by example in §5.2. Indeed proofs of supporting lemmas for the method of continuity are not valid without the shelling hypotheses. For example, it can be shown that the boundary operators fail to form a continuum (Lemma 4) when shelling is violated in the manner of the example in §5.2.

The shelling procedure is also limited. For example, suppose the polyhedron $\overline{\Omega}$ is homeomorphic to the closed ball $\mathbb{B}^4$. While it is true that the shelling here shows that $\overline{\Omega}$ and $\overline{\Omega_{Lip}}$ are PL-homeomorphic (Theorem 5), it does not follow that one can continue to pass from polyhedron to polyhedron via PL-homeomorphisms until $\overline{\Omega}$ is shown to be PL-homeomorphic to a 4-simplex. This would resolve the PL-Schoenflies conjecture, which the authors are not about to do. A 3-dimensional shelling that is kept close to the original boundary is actually what is used here. And even it is justified only by the deep 3-manifold results of E. E. Moise [Moi52] together with a result on shelling convex bodies first proved by D. E. Sanderson [San57].

The bounds on the operator inverses, also termed the bound from below, are shown to be uniform by the results of §§5.1 and 6.2 together with either a known compactness argument or a further geometric argument presented in §7. Part of the justification of this second argument is that the above boundary PL-homeomorphisms are shown to extend inside as solid PL-homeomorphisms (Lemma 7). This then results in Lemma 8, which applies the method of continuity at each shelling stage, and also results in Theorem 5 and its corollary 3 which will be used in §10 to bi-Lipschitz flatten 4-dimensional polyhedra in local neighborhoods of the boundary.

Finally the boundary 1-skeleton (1 dimensional edges) of the given polyhedron is shelled off in §8. Lemma 8, applied at each step, then shows that the boundary operator from the original polyhedron is an isomorphism on the appropriate spaces (Theorem 4), settling the L^2-Neumann problem.

The results of §8 on the Neumann problem and the invertibility of the classical layer potentials are extended to infinite polyhedral cones in Theorems 6 and 7. This is in preparation for adapting to polyhedra B. E. J. Dahlberg and C. E. Kenig's method for obtaining estimates on the rate of decay of harmonic functions that have atomic Neumann data on a domain above a Lipschitz graph in $\mathbb{R}^n$ [DK87]. There solutions are extended by parallel projection to the region exterior to any

n-ball containing the support of the atomic data. The atomic data permits the use of the L^2 estimates. The extended solutions are then solutions to elliptic divergence equations for which decay estimates at infinity are known by the work of J. Serrin and H. F. Weinberger [SW66].

Altogether the theorems of §8 say that any compact polyhedral domain of $\mathbb{R}^4$ with a manifold boundary is a bi-Lipschitz domain in which the needed L^2 estimates can be obtained. Atomic estimates in this setting are therefore shown to hold in §10. Bi-Lipschitz flattening replaces parallel projection. The analysis closely follows that of [DK87] and the section can really only be read with a copy of that paper in hand. The geometric differences are identified and some alternative analytic arguments supplied.

In §11 Curtis and Zeeman's example of a compact polyhedral domain with manifold boundary in $\mathbb{R}^6$ is discussed in detail. Because its boundary is homeomorphic to the 5-sphere, by another deep topological result due to Cannon and Edwards, while its interior is not simply connected, it cannot satisfy Morrey's definition of Lipschitz domain. Without local bi-Lipschitz flattening at the boundary the method of [DK87] does not seem practicable. The example also indicates a lack of suitable boundary homeomorphisms for the method of continuity of §§6–8. Its relations to other notions of Lipschitz domain and to the difficult nature of higher dimensional polyhedra are also discussed. The Mazur manifold is briefly discussed.

The distinctive feature of boundary value problems is the relationship between the boundary and the domain. Without the graph hypothesis there is no single direction from the boundary of a polyhedron into its domain, even locally. The shape or location of the domain and its boundary becomes a concern which has as part of its resolution here a number of geometric constructions. Therefore we have included 20 Figures, most of which should be taken as schematics meant to summarize and clarify relations between elementary but technical geometric definitions.

In addition to the material on Lipschitz polyhedra, the Appendix contains some basic facts about Sobolev spaces on polyhedral boundaries and a subsection on notations and conventions.

Some readers might object that the paper is not longer. For example, the regularity problem for the Dirichlet problem with Sobolev data appears in several places but is never solved. This and perhaps some other omissions from the Lipschitz domain inventory will be done elsewhere.

As it is, we claim some coherence. A method for obtaining boundary energy estimates on polyhedra is given. In order to use it to solve boundary value problems, it is shown that certain geometric and topological techniques naturally come into play on the boundary. As established in classical geometric topology these techniques have extensions into the domain near the boundary, which here seem both indispensable and analogous to extending L^2 estimates to Hardy space estimates. We close with an example from the literature that illustrates why it is at least prudent to limit the method of solution presented here for solving the multidirectional Neumann problem to four dimensions.

Outline

2. A preliminary argument

Let $\Omega \subset \mathbb{R}^4$ be the interior of a finite homogeneous 4-complex K. Suppose Ω is a domain and that $\partial\Omega = \dot{K}$ is a 3-manifold. For any simplex $\kappa \in \dot{K}$ define the dyadic *arch*

$$A_j(\kappa) = \{x = (1-t)P + tQ \mid P \in \kappa,\ Q \in \mathrm{Lk}(\kappa, K),\ 2^{-j-1} \le t \le 2^{-j}\} \tag{2.1}$$

for $j = 0, 1, 2, \ldots$. When $\kappa^0 \in \dot{K}$ is a vertex the A_j are rescalings

$$A_j(\kappa^0) = \{2^{-j}(x - |\kappa^0|) + |\kappa^0| \mid x \in A_0(\kappa^0)\}$$

By Lemma 8.9 of [VV03] the interiors of these vertex arches are domains. They decompose $\mathrm{St}(\kappa^0, K)$ into geometrically similar domains from which a priori estimates can be transferred back to the star. This is necessary because in general the stars of boundary vertices are not Lipschitz domains (graph domains) for which the estimates are known.

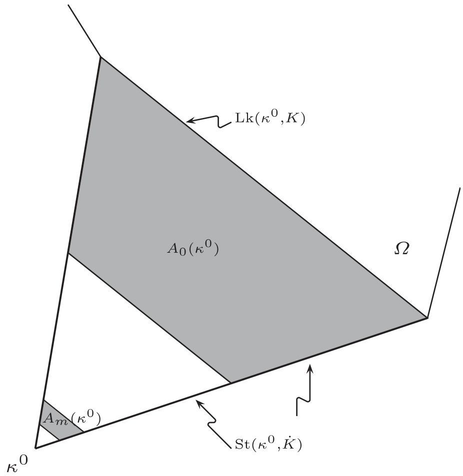

Fig. 1. $\mathrm{St}(\kappa^0, K)$

Let $\Delta u = 0$ in Ω and $N_\alpha(\nabla u) \in L^2(\partial\Omega)$. Suppose that on each $A_j(\kappa^0)$ we have the estimates

$$\int_{\partial A_j} |\nabla u|^2 ds \le C \int_{\partial A_j} \left(\frac{\partial u}{\partial N}\right)^2 ds \tag{2.2}$$

and

$$\int_{\partial A_j} |\nabla u|^2 ds \le C \left[\int_{\partial A_j} |\nabla_T u|^2 + u^2 ds\right] \tag{2.3}$$

where C is a constant depending only on $\kappa^0 \in K$. By Lemma 8.8 of [VV03] and the Poincaré inequality (2.3) may be replaced with

$$\int_{\partial A_j} |\nabla u|^2 ds \le C \int_{\partial A_j} |\nabla_T u|^2 ds \tag{2.4}$$

Take κ^0 to be the *origin*. By summing in j (2.2) would yield an a priori estimate on $\mathrm{St}(\kappa^0, \dot{K})$ except for the integrals over portions of the arch boundaries that are interior to Ω. However, (2.2) remains uniform when $u(x)$ is replaced by $u(\theta x)$, $\frac{1}{2} \leq \theta \leq 1$, so that by averaging one obtains

$$\int\limits_{\mathrm{St}(\kappa^0,\dot{K})\setminus A_0(\kappa^0)} |\nabla u|^2 ds \leq C\left[\int\limits_{\mathrm{St}(\kappa^0,\dot{K})} |\frac{\partial u}{\partial N}|^2 ds + \int\limits_{\mathrm{St}(\kappa^0,K)} |\nabla u(x)|^2 \frac{dx}{|x|}\right] \tag{2.5}$$

The solid integral with its singularity at the origin is the result of averaging the interior boundaries of (2.2). Similarly

$$\int\limits_{\mathrm{St}(\kappa^0,\dot{K})\setminus A_0(\kappa^0)} |\nabla u|^2 ds \leq C\left[\int\limits_{\mathrm{St}(\kappa^0,\dot{K})} |\nabla_T u|^2 ds + \int\limits_{\mathrm{St}(\kappa^0,K)} |\nabla u(x)|^2 \frac{dx}{|x|}\right] \tag{2.6}$$

The solid integral of (2.5) (and (2.6)) is bounded as follows. Letting $W = \dfrac{x}{|x|}$ for $x \in \mathbb{R}^n$, integration by parts (see §5 of [VV03] for polyhedra) yields the second equality in the identity

$$\begin{aligned}
&\int\limits_{\bigcup_{i=1}^m A_i(\kappa^0)} (n-3)|\nabla u|^2 + 2(W\cdot\nabla u)^2 \frac{dx}{|x|} \\
&\quad = \int\limits_{\bigcup_{i=1}^m A_i(\kappa^0)} divW|\nabla u|^2 - 2D_k W_j D_j u D_k u\; dx \\
&\quad = \int\limits_{\partial[\bigcup_{i=1}^m A_i(\kappa^0)]} N\cdot W|\nabla u|^2 - 2\frac{\partial u}{\partial N} W\cdot\nabla u\; ds
\end{aligned} \tag{2.7}$$

Now $N \cdot W = 0$ on $\partial\Omega$. Also the boundary term of (2.7) from $\partial A_m(\kappa^0)$ vanishes as $m \to \infty$. This follows because that term can be bounded by $\int_E N_\alpha(\nabla u)^2\; ds$ with the measure of E controlled by 2^{-m}. This in turn follows in polyhedra by Lemma 5.3 of [VV03] and the Carleson lemma (see §4 of [VV03]). In addition $W \cdot \nabla u$ on $\mathrm{St}(\kappa^0, \dot{K})$ is a tangential derivative. Thus by monotone convergence applied to (2.7) with $n = 4$, (2.5) becomes

$$\begin{aligned}
&\int\limits_{\mathrm{St}(\kappa^0,\dot{K})\setminus A_0(\kappa^0)} |\nabla u|^2 ds \\
&\quad \leq C\left[\int\limits_{\mathrm{St}(\kappa^0,\dot{K})} |\frac{\partial u}{\partial N}|^2 + |\nabla_T u||\frac{\partial u}{\partial N}| ds + \int\limits_{A_0(\kappa^0)} |\nabla u|^2 dx\right]
\end{aligned} \tag{2.8}$$

And similarly for (2.6) where C again depends only on κ^0, and the solid integral, now away from the vertex, can be considered lower order.

Remark 1. When $n = 2$ formula (2.7) fails to imply (2.8) from (2.5). It fails when it is not needed. Every polygonal Jordan domain is Lipschitz. When $n = 3$ it succeeds only when a more careful analysis than (2.5) is carried out, as was done in [Ver01]. Though not apparent here, a more careful analysis is also needed when $n \geq 4$ and will be begun in the following section. Compare (2.7) with (4.1).

Thus we need estimates (2.2) and (2.4) on the arches $A_j(\kappa^0)$, i.e. our boundary value problems must first be solvable on $A_0(\kappa^0)$. But neither is $A_0(\kappa^0)$ a Lipschitz domain.

3. Sperical Arches, sectors, crowns, stacks, pieces and slices

We modify the arches $A_j(\kappa^0)$ to be spherical. See Remark 5 at the end of this section. Given $\kappa^0 \in \dot{K}$ we may assume $\operatorname{dist}(\kappa^0, \operatorname{Lk}(\kappa^0, K)) = 2$, take κ^0 as the origin of $\mathbb{R}^4$, and define

$$\tilde{A}_j(\kappa^0) = \{x \in \operatorname{St}(\kappa^0, K) \mid 2^{-j-1} \leq |x| \leq 2^{-j}\} \quad \text{for } j = 0, 1, 2, \ldots$$

Each $\tilde{A}_j(\kappa^0)$ is a rescaling of $\tilde{A}_0(\kappa^0)$. Therefore by the arguments of §2, it will suffice to obtain estimates (2.2) and (2.4) on $\tilde{A}_0(\kappa^0)$. To do this we must now remove in a similar dyadic fashion any $\kappa^1 \in \dot{K}$ that contains κ^0.

If $\kappa^0 \in \kappa^1 \in \dot{K}$ first orient κ^1 along the positive x_4 axis. Next minimize the angles formed by the vectors $\vec{\kappa^1}$ and $\vec{\kappa^0 y}$ where y is any point in any simplex that contains κ^0 but not κ^1. Define the angle $\theta(\kappa^1, \kappa^0) > 0$ to be $\frac{1}{4}$ this minimum. Now define *sectors* of $\tilde{A}_0(\kappa^0)$ for angles $0 \leq \theta_1 < \theta_0 \leq \theta(\kappa^1, \kappa^0)$ by

$$\tilde{A}(\theta_1, \theta_0) = \tilde{A}(\kappa^1, \kappa^0; \theta_1, \theta_0) = \left\{x \mid \cos(\theta_0) \leq \frac{x_4}{|x|} \leq \cos(\theta_1)\right\} \cap \tilde{A}_0(\kappa^0)$$

By the definition of $\theta = \theta(\kappa^1, \kappa^0)$ each sector $\tilde{A}(\theta_1, \theta_0)$ is contained in $\operatorname{St}(\kappa^1, K)$. The $\operatorname{St}(\kappa^1, K)$ is barycenter connected, [VV03] Lemma 8.9. Therefore, given any two points x, y in Int $\tilde{A}(\theta_1, \theta_0)$ there is a path γ in Int $\operatorname{St}(\kappa^1, K)$ connecting them. Let $x = (x', x_4) = |x|(\frac{x'}{|x'|} \sin\phi, \cos\phi)$, $y = (y', y_4) = |y|(\frac{y'}{|y'|} \sin\psi, \cos\psi)$ and suppose that $\psi \geq \phi$. Any points of γ whose angle with the x_4-axis is greater than ψ can be projected onto the cone with angle ψ by using the point $(0, 3/4) \in \kappa^1$ as a starcenter for $\operatorname{St}(\kappa^1, K)$ and projecting radially from this starcenter onto the cone. This results in a new path, again called γ, from x to y in Int $\operatorname{St}(\kappa^1, K)$ so that the angle between any point on γ and the x_4-axis is at most $\psi < \theta_1$. Now those points of γ with norm greater $\max(|x|, |y|)$ are projected (radially from κ^0 the origin) onto the sphere with this radius and those points of γ with norm smaller than $\min(|x|, |y|)$ are projected onto the sphere with this radius. These projections result in a new path, again called γ, in Int $\mathrm{St}(\kappa^1, K)$. By the definition of θ and $\mathrm{dist}(\kappa^0, \mathrm{Lk}(\kappa^0, K)) = 2$, if $z = |z|(\frac{z'}{|z'|}\sin\nu, \cos\nu) \in \mathrm{St}(\kappa^1, K)$ satisfies $\nu < \theta$ then all points of the form $|z|(\frac{z'}{|z'|}\sin\eta, \cos\eta)$ for $0 < \eta < \theta$ are also in $\mathrm{St}(\kappa^1, K)$. Using this observation, any point $z \in \gamma$ with $\nu < \phi$ may be projected along a great circle (in the variable η) to the point $|z|(\frac{z'}{|z'|}\sin\phi, \cos\phi)$. This results in a path, again called γ, in $\mathrm{St}(\kappa^1, K)$ which is now seen to be in the interior of $\tilde{A}(\theta_1, \theta_0)$. Therefore, Int $\tilde{A}(\theta_1, \theta_0)$ is a domain.

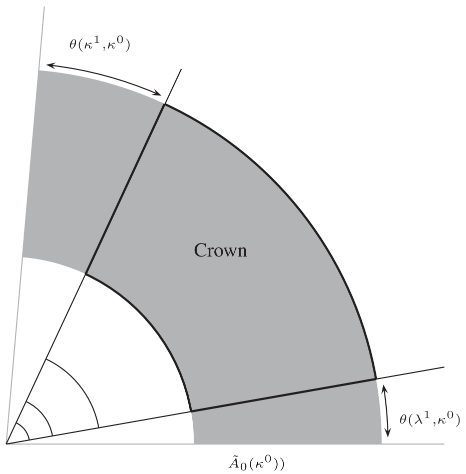

Fig. 2. Full sectors for $\kappa^1, \lambda^1 \in \mathrm{St}(\kappa^0, \dot{K})$ removed

A *crown* is defined by

$$\tilde{A}_0(\kappa^0) \setminus \bigcup_{\kappa^1 \in \mathrm{St}(\kappa^0, \dot{K})} \tilde{A}(\kappa^1, \kappa^0; 0, \theta(\kappa^1, \kappa^0)) \tag{3.1}$$

where here the sector for each κ^1 has a coordinate free definition. A crown is just a partially beveled $\mathrm{St}(\kappa^0, K)$ which is barycenter connected, therefore the interior of a crown is connected, see [VV03] section 5 and Lemma 8.9. Crowns are Lipschitz domains by Lemma 13.

When $\theta_1 > 0$ for a given κ^1 the $\tilde{A}(\theta_1, \theta_0)$ are, by Corollary 6, Lipschitz domains but without the uniform Lipschitz natures which are needed for estimates. To begin to remedy this we create a *stack* of Lipschitz domains in each sector. There are other satisfactory ways of doing this, but the following produces a stack in which all domains in the stack are geometrically similar.

Let $\theta_1 > 0$ and divide $\tilde{A}(\theta_1, \theta_0)$ into 2^j $(j \geq 0)$ *pieces*

$$\tilde{A}_{j,l}(\theta_1, \theta_0) = \tilde{A}(\theta_1, \theta_0) \cap \{2^{-l2^{-j}} \leq |x| \leq 2^{-(l-1)2^{-j}}\}$$

for $l = 1, 2, \dots, 2^j$. For j fixed each piece scales to the preceding by a factor of 2^{-j}. The interior of each piece is a domain by the same argument used for the sectors.

Pieces from stacks from different sectors do not scale, but they almost do if we divide the *full sector* $\tilde{A}(\kappa^1, \kappa^0; 0, \theta(\kappa^1, \kappa^0))$ in the same proportions as used in producing the stacks. Define the numbers $\nu_j = \nu_0 \sum_{k=0}^{j}(2^{2^{-k}} - 1)$ for $j = 0, 1, 2, \dots$ where ν_0 is chosen so that $\nu_j \to 1$.

Remark 2.

$$2^{2^{-j}} - 1 = \prod_{l=1}^{j}(2^{2^{-l}} + 1)^{-1}$$

for $j = 1, 2, \dots$ and

$$(2^{2^{-j}} - 1)^{-1} \sum_{k=j+1}^{\infty} (2^{2^{-k}} - 1) \to 1$$

as $j \to \infty$.

Define $\nu_{-1} = 0$ and write $\theta = \theta(\kappa^1, \kappa^0)$. The full sector is then decomposed into the sectors

$$\tilde{A}((1 - \nu_j)\theta, (1 - \nu_{j-1})\theta) \quad \text{for } j = 0, 1, 2, \dots \tag{3.2}$$

each of which in turn is divided into 2^j similar pieces

$$\tilde{A}_{j,l}((1 - \nu_j)\theta, (1 - \nu_{j-1})\theta) \quad \text{for } \ l = 1, \dots, 2^j \tag{3.3}$$

forming the stack of similar Lipschitz domains in each sector $j = 0, 1, 2, \dots$. Thus it is enough to show that uniform Lipschitz estimates hold for the bottom members of each stack, vis. the

$$B_j = \tilde{A}_{j,2^j}((1 - \nu_j)\theta, (1 - \nu_{j-1})\theta) \quad \text{for } \ j = 0, 1, \dots \tag{3.4}$$

Each B_j is a distance $1/2$ from the origin, has a radial side length of $\frac{1}{2}(2^{2^{-j}} - 1)$ and is in a sector of angle $\nu_0\theta(2^{2^{-j}} - 1)$. This suggests that each B_j be rescaled by $(2^{2^{-j}} - 1)^{-1}$ and mapped to a *cylindrical* domain of $I \times I \times \mathbb{S}^2$.

Lemma 1. *Given $\kappa^0 \in \kappa^1 \in \dot{K}$, the angle $\theta(\kappa^1, \kappa^0)$ and the Lipschitz domains* (3.4) *there is a Lipschitz domain $\omega \subset \mathbb{S}^2$ so that*

$$B_j = \Big\{\rho(s \sin\phi, \cos\phi) \in \mathbb{R}^4 \mid 2^{-1} \leq \rho \leq 2^{-1+2^{-j}}, \\ s \in \bar{\omega} \textit{ and } (1 - \nu_j)\theta \leq \phi \leq (1 - \nu_{j-1})\theta\Big\} \tag{3.5}$$

for $j = 0, 1, \dots$.

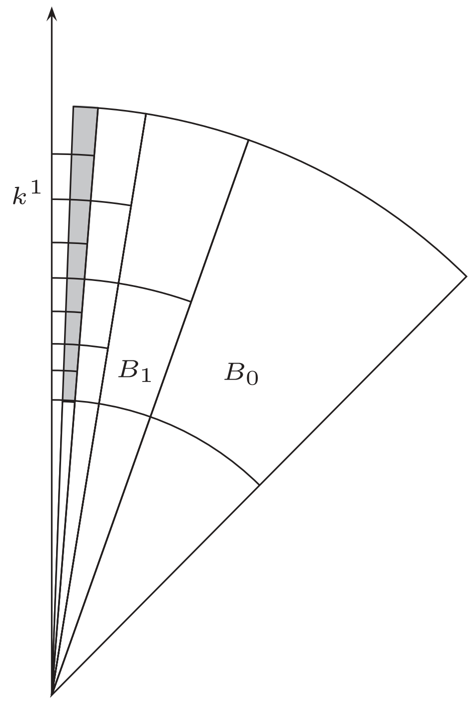

Fig. 3. The sector $\tilde{A}((1-\nu_3)\theta, (1-\nu_2)\theta)$ shaded and the bottom pieces of each stack

Moreover, the domains (3.3) *(and* (3.8)*,* (3.9) *below) have uniform Lipschitz natures over* $j = 0, 1, 2, \ldots$. *In particular the constants arising from applying the Lipschitz domain estimates of Theorems 1 and 2 are uniform over the domains* (3.3).

Proof. Fix j and fix any $x \in B_j$. Because each B_j is contained in $\mathrm{St}(\kappa^1, K)$ there are unique $P = (0, P_4) \in \kappa^1$, $Q \in \mathrm{Lk}(\kappa^1, K)$ and $0 < t < 1$ so that $x = (1-t)P + tQ$. Consequently, Q and then every point in the 2-simplex $\kappa^1 Q$ has the form

$$\rho\left(\frac{x'}{|x'|}\sin\phi, \cos\phi\right) \qquad \text{for some } \rho \geq 0 \text{ and } \phi \geq 0. \tag{3.6}$$

By definition the 1-simplex $\kappa^0 Q$ of $\kappa^1 Q$ makes an angle greater than θ with κ^1. The third 1-simplex of $\kappa^1 Q$ is contained in $\mathrm{Lk}(\kappa^0, K)$. Thus every point of the form (3.6) for ρ and ϕ restricted as in (3.5) is seen to be in B_j. Moreover, by the definition of θ and the assumption that $\mathrm{dist}(\kappa^0, \mathrm{Lk}(\kappa^0, K)) = 2$ all points of the form (3.6) with $\frac{1}{4} < \rho < \frac{5}{4}$ and $0 < \phi < \theta$ are in $\mathrm{St}(\kappa^1, K)$. Define $V = \mathrm{St}(\kappa^1, K) \cap \{x \mid \frac{1}{4} < |x| < \frac{5}{4}, \cos(\theta) < \frac{x_4}{|x|} < 1\}$, $Proj_1(x) = x'$, and $Proj_2(x') = x'/|x'|$ then

$$\bar{\omega} = Proj_2 \circ Proj_1(B_j) = Proj_2 \circ Proj_1(V)$$

Therefore $\bar{\omega}$ is independent of j. This establishes (3.5).

Dilating each B_j by $(2^{2^{-j}}-1)^{-1}$ each dilated B_j is the image under a smooth homeomorphism F_j from

$$U = \left\{ y \in \mathbb{R}^4 \mid 0 \le y_4 \le \frac{1}{2}, \frac{y'}{|y'|} \in \bar{\omega}, \frac{1}{2}\nu_0\theta \le |y'| \le \nu_0\theta \right\} \tag{3.7}$$

so that the differentials F_j' converge to the identity matrix as $j \to \infty$.

In fact

$$\begin{aligned} F_j(y) = {} & (y_4 + 2^{-1}(2^{2^{-j}}-1)^{-1}) \\ & \times \left(\frac{y'}{|y'|} \sin((2|y'| - \nu_0\theta)(2^{2^{-j}}-1) + (1-\nu_j)\theta), \right. \\ & \left. \cos((2|y'| - \nu_0\theta)(2^{2^{-j}}-1) + (1-\nu_j)\theta) \right) \end{aligned}$$

Set $\omega = Proj_2 \circ Proj_1(V \setminus \dot{K})$, then ω is open since $V \setminus \dot{K}$ is open. Furthermore, $\bar{\omega} = \omega \cup \partial\omega$, i.e. $s \in \mathrm{Int}(\bar{\omega})$ if and only if $s \in \omega$. It follows that $s \in \partial\omega$ if and only if there is an $x \in V \cap \dot{K}$ with $s = x'/|x'|$. B_j is a Lipschitz domain by Corollary 6. From the properties of the mappings F_j and the cylindrical nature of U it follows that ω is a Lipschitz domain. The number and shape of truncated cylinders used to cover the Lipschitz boundaries ∂B_j in the manner of [Dah79] together with the Lipschitz norms of the functions describing the boundary in each cylinder can be taken to be uniform in j. □

Remark 3. Given K^4 a finite homogeneous subcomplex of a combinatorial triangulation T of all $\mathbb{R}^4$ with $\dot{K}$ a 3-manifold it follows from the 3-manifold theory of Moise [Moi77] that for any $\kappa^1 \in \dot{K}$ the $\mathrm{Lk}(\kappa^1, \dot{K})$ is a PL 1-sphere. This PL 1-sphere separates the PL 2-sphere $\mathrm{St}(\kappa^1, T)$ into two PL 2-balls with the PL 1-sphere as common boundary (Schoenflies theorem for $n = 2$). Therefore, $\mathrm{Lk}(\kappa^1, K)$ is a PL 2-ball.

Now, with κ^0 the origin and κ^1 lying along the x_4-axis there cannot be two points $P, Q \in \mathrm{Lk}(\kappa^1, K)$ with $P - Q$ having the same direction as κ^1, since P, Q and κ^1 all lie in the same 2-dimensional affine hull. Similarly, it cannot be the case that $P'/|P'| = Q'/|Q'|$. Thus $\bar{\omega} = Proj_2 \circ Proj_1(\mathrm{Lk}(\kappa^1, K))$ and $Proj_2 \circ Proj_1$ is a bi-Lipschitz homeomorphism of a PL 2-ball.

The Lipschitz domains (3.3) can therefore be written as a union of *slices*

$$\begin{aligned} \tilde{A}_{j,l} &= \tilde{A}_{j,l}(\kappa^1, \kappa^0) \\ &= \bigcup_{s \in \bar{\omega}} \left\{ \rho(s \sin(\phi), \cos(\phi)) \mid 2^{-l2^{-j}} \le \rho \le 2^{-(l-1)2^{-j}}, \right. \\ & \qquad \left. (1-\nu_j)\theta \le \phi \le (1-\nu_{j-1})\theta \right\} \end{aligned} \tag{3.8}$$

$l = 1, \ldots, 2^j$, $j = 0, 1, 2, \ldots$.

In order to use the uniform estimates guaranteed by Lemma 1 it will be necessary to vary the domains (3.8) in the variables ρ and ϕ. Fix a center $(\rho_{j,l}, \phi_j)$ about which we dilate each slice of $\tilde{A}_{j,l}$ by the same amount $1 \leq t \leq \frac{3}{2}$ (will do) in the variables ρ and ϕ (i.e. x_4 and $|x'|$) only, thereby obtaining a continuum of Lipschitz domains

$$t\tilde{A}_{j,l} \subset |\,\mathrm{St}(\kappa^1, K)| \;\;,\;\; 1 \leq t \leq \frac{3}{2} \tag{3.9}$$

for each j and l. We will refer to

$$\frac{3}{2}\tilde{A}_{j,l} = \tilde{A}^*_{j,l} \tag{3.10}$$

as the *double* of $\tilde{A}_{j,l}$.

By the arguments of Lemma 1, Lipschitz estimates for boundary value problems will be uniform for the domains (3.9) over all the t, j, and l. The virtue of the next lemma is in the type of derivatives found in the solid integrals, the effect of the above geometric constructions. The inequalities should be compared to 2.5 and 2.6; see Remark 1.

Lemma 2. *Let the domain $\Omega \subset \mathbb{R}^4$ be the interior of a finite homogeneous complex K^4 with $\dot{K}$ a manifold. Let the harmonic function u satisfy $N_\alpha(\nabla u) \in L^2(\partial\Omega)$. Let $\kappa^0 \in \kappa^1 \in \dot{K}$ where κ^0 is taken to be the origin and κ^1 oriented on the positive x_4-axis. Write $x = (x', x_4)$ and denote the full sector of the arch $\tilde{A}_0(\kappa^0)$ by*

$$D = \tilde{A}(\kappa^1, \kappa^0; 0, \theta(\kappa^1, \kappa^0)) = \bigcup_{j=0}^{\infty} \bigcup_{l=1}^{2^j} \tilde{A}_{j,l}$$

Let $D^* = \bigcup_{j,l} \tilde{A}^*_{j,l}$ as in (3.10). *Then*

$$\int_{D\cap\partial\Omega} |\nabla u|^2 ds \leq C\left[\int_{D^*\cap\partial\Omega}\left(\frac{\partial u}{\partial N}\right)^2 ds + \int_{D*}\left(\frac{x'}{|x'|}\cdot\nabla u\right)^2 + \left(\frac{\partial u}{\partial x_4}\right)^2 \frac{dx}{|x'|}\right] \tag{3.11}$$

and

$$\int_{D\cap\partial\Omega} |\nabla u|^2 ds \leq C\left[\int_{D^*\cap\partial\Omega} u^2 + |\nabla_T u|^2 ds + \int_{D*}\left(\frac{x'}{|x'|}\cdot\nabla u\right)^2 + \left(\frac{\partial u}{\partial x_4}\right)^2 \frac{dx}{|x'|}\right] \tag{3.12}$$

Where C depends only on K.

Proof. Each $\tilde{A}_{j,l}$ has radial height comparable to $2^{2^{-j}} - 1$ and angular width comparable to this same number since $\frac{1}{2} \le 2^{-l2^{-j}} \le 1$ for all l. Further, each $\tilde{A}_{j,l}$ has distance to κ^1 comparable to this number. In particular $x \in \tilde{A}_{j,l}$ implies $|x'|$ is comparable to $2^{2^{-j}} - 1$. These statements remain true for the dilated domains (3.9) for all $1 \le t \le \frac{3}{2}$.

Applying the Neumann estimate of Theorem 1. to any $t\tilde{A}_{j,l}$ yields normal derivatives on $\partial\left(t\tilde{A}_{j,l}\right) \setminus \partial\Omega$ that are bounded combinations of precisely the directional derivatives $\dfrac{x'}{|x'|} \cdot \nabla u$ and $\dfrac{\partial u}{\partial x_4}$ because of the spherical and conical construction of the domain. Averaging in t and accounting for the distance to κ^1 results in (3.11) with D replaced by $\tilde{A}_{j,l}$. By Lemma 1 and summing, (3.11) follows.

R.M. Brown's theorem is used to establish (3.12). Neumann data is used on the surfaces interior to Ω while regularity data is used on $\partial\Omega$. In order to apply the theorem the geometric hypotheses of a creased domain need to be checked (see the paragraph before Theorem 2). Given the mappings F_j of B_j to U in the preceding proof it is enough to show that U satisfies the definition of a creased domain. Let $\mathcal{D}$ be the set of points $\{y \in U \mid 0 < y_4 < \frac{1}{2}, y'/|y'| \in \partial\omega, \frac{1}{2}\nu_0\theta < |y'| < \nu_0\theta\}$ (these are all points mapped by F_j to surfaces of ∂B_j from $\dot{K}$ and not from any spherical or conical surface of ∂B_j) and let $\mathcal{N}$ be the remaining boundary points of U (all those points mapped to spherical or conical surfaces of ∂B_j) so $y_4 = 0, \frac{1}{2}$ or $|y'| = \frac{1}{2}\nu_0\theta, \nu_0\theta$.

To see that (3) in the definition of creased domain holds, first note that the spherical surfaces in B_j correspond to $y_4 = 0$ or $y_4 = 1/2$ in $\mathcal{N}$, the conical surfaces correspond to $|y'| = \frac{1}{2}\nu_0\theta$ or $|y'| = \nu_0\theta$ in $\mathcal{N}$ and simplexes from $\dot{K}$ correspond to $y'/|y'| \in \partial\omega$ in $\mathcal{D}$. Consider the case of a boundary point y that is in all three types of boundary surfaces, for example $y_4 = 0$, $|y'| = \frac{1}{2}\nu_0\theta$, and $y'/|y'| \in \partial\omega$. In this case ∂U is a Lipschitz graph in the direction of

$$N_H = (-e_4) + (-y'/|y'|) + N_\omega(y)$$

where $N_\omega(y)$ provides a direction, perpendicular to y' and e_4, in which $\partial\omega$ is a Lipschitz graph. A vector field satisfying the definition of creased domain is given by

$$(-e_4) + (-y'/|y'|) - N_\omega(y)$$

Let H be the hyperplane through y with normal in the direction of N_H and H_ω be the hyperplane through y with normal N_ω. The crease in a neighborhood O of y consists of the points from $\{y \mid y'/|y'| \in \partial\omega, \rho = 0, \frac{1}{2}\nu_0\theta < |y'| < \nu_0\theta\}$ and $\{y \mid y'/|y'| \in \partial\omega, 0 < \rho < \frac{1}{2}, |y'| = \frac{1}{2}\nu_0\theta\}$ that are in O and is seen to be a Lipschitz graph in H_ω in the direction of $(-e_4) + (-y'/|y'|)$. The orthogonal projection $Proj_H$ from H_ω onto H shows that condition (3) of creased domain is satisfied with the x_1 direction in the direction of $Proj_H((-e_4) + (-y'/|y'|))$.

The remaining cases may all be similarly proved. □

Remark 4. In establishing Theorem 2, as Russell Brown kindly pointed out to the authors, the argument on p. 1230 [Bro94] does not require his creased domain condition (1.2) on p. 1219. The application here is actually to this more restricted class of domains, but this observation saves some work.

Remark 5. A geometrically better way to decompose $\mathrm{St}(\kappa^0, K)$ into Lipschitz polyhedra in the manner of a Whitney decomposition is to use barycentric coordinates rather than spherical. For example one can take $\kappa = \kappa^1$ in the unmodified definition of arch (2.1) and replace the sectors (3.2) with $A_j(\kappa^1) \cap A_0(\kappa^0)$ which then may be divided into 2^j pieces by intersecting with

$$\{(1-t)\kappa^0 + tQ \mid Q \in \mathrm{Lk}(\kappa^0, K)\ ,\ l-1 < (2t-1)2^j < l\}$$

$l = 1, 2, \ldots, 2^j$. The resulting Lipschitz polyhedra replace (3.3) and have the advantage that they are all geometrically similar even as j varies. This decomposition also generalizes easily to higher dimensions. The difficulty is that the boundary value estimate needed to obtain the solid integral of (3.11) must now come from an oblique derivative problem. In contrast to Theorems 1 and 2 it is not clear to the authors how large the space of solutions with zero oblique derivative might be. Compare, for example, the statements of Theorems 2.1 and 2.2 of [Nad86]. That the solid integrals of (3.11) and (3.12) are just so is used in the next section.

4. Polyhedral Rellich formulas

Let $\kappa^0 \in \kappa^1 \in \dot{K}$ and let D denote the full sector about κ^1 of the arch $\tilde{A}_0(\kappa^0)$ oriented with respect to the rectangular coordinates of $\mathbb{R}^4$ as in Lemma 2. The decomposition of D into Lipschitz domains necessarily led to estimates in terms of solid integrals with singular measure $|x'|^{-1}\, dx$, just as the decomposition of vertex stars in §2 resulted in the measures $|x|^{-1}\, dx$. Recall the Rellich formula (2.7). The appropriate vector field for D is now $W(x) = \dfrac{x'}{|x'|}$. Using this field in the central integral of (2.7) in $\mathbb{R}^n$, computing the left integral and integrating by parts to obtain the right yields the Rellich formula

$$\int_{\tilde{A}((1-\nu_j)\theta,(1-\nu_0)\theta)} (n-4)|\nabla' u|^2 + (n-2)(\frac{\partial}{\partial x_n}u)^2 + 2\left(\frac{x'}{|x'|}\cdot\nabla' u\right)^2 \frac{dx}{|x'|}$$
$$= \int_{\partial\tilde{A}((1-\nu_j)\theta,(1-\nu_0)\theta)} N\cdot\frac{x'}{|x'|}|\nabla u|^2 - 2\left(\frac{x'}{|x'|}\cdot\nabla' u\right)\frac{\partial u}{\partial N}\, ds \qquad (4.1)$$

for $j = 1, 2, \ldots$. Here $\nabla' = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{n-1}}, 0)$ and the integrals are over the union of j contiguous sectors 3.2.

The weakness of the $n = 3$ case of (2.7) mentioned in Remark 1 recurs here when $n = 4$. In all higher dimensions the $n = 3$ case similarly recurs because one must remove the $(n-3)$-skeleton from the boundary in order to obtain the Lipschitz polyhedra in which estimates are known. Thus a need for an analogous geometric analysis and analogous Lemma 2 will recur in higher dimensions.

Define a doubling of the arch $\tilde{A}_0(\kappa^0)$ by $\tilde{A}_0^* = \{x \in \mathrm{St}(\kappa^0, K) \mid \frac{1}{3} \le |x| \le \frac{3}{2}\}$.

Lemma 3. *With the same hypotheses as Lemma 2*

$$\int_D (\frac{x'}{|x'|} \cdot \nabla u)^2 + (\frac{\partial u}{\partial x_4})^2 \frac{dx}{|x'|} \le C \left[\int_{\partial\Omega \cap \tilde{A}_0^*} |\frac{\partial u}{\partial N}||\nabla_T u| ds + \int_{\tilde{A}_0^*} |\nabla u|^2 dx \right] \tag{4.2}$$

where C depends only on K.

Proof. By letting $j \to \infty$ the polyhedral Rellich formula (4.1) holds for the full sector D. In particular the boundary integral over surfaces that are converging to κ^1 vanish by the argument used after (2.7) for the surfaces ∂A_m.

The boundary $\partial D \cap \partial\Omega$ is contained in 3-simplexes each of which contains κ^1. Thus for x in that part of the boundary $0 = N \cdot x = N \cdot \kappa^1 = N \cdot x'$ and $x' \cdot \nabla u$ is a tangential derivative. Consequently the boundary integral of (4.2) dominates this part of (4.1).

Replacing D in (4.1) by domains tD defined in the same manner as (3.9), the integrals over the boundaries $\partial(tD) \cap \Omega$ may be arranged and dominated by the second integral on the right side. □

Theorem 3. *Let $K^4 \subset \mathbb{R}^4$ be a finite homogeneous simplicial complex such that $\Omega = \mathrm{Int}\,|K|$ is a domain and $\partial\Omega = |\dot{K}|$ is a connected manifold. Let u be a harmonic function defined in Ω with nontangential maximal function of its gradient in $L^2(\partial\Omega)$. Then there is a finite constant C depending only on K such that*

$$\int_{\partial\Omega} N_\alpha(\nabla u)^2 ds \le C \int_{\partial\Omega} (\frac{\partial u}{\partial N})^2 ds \le C^2 \int_{\partial\Omega} |\nabla_T u|^2 ds$$

Or if u is harmonic in the domain exterior to $|K|$, with mean value zero over $\partial\Omega$, with nontangential maximal function of its gradient in $L^2(\partial\Omega)$, and $u(x)$ vanishing as $|x| \to \infty$, then the inequalities also hold.

Proof. By removing the full sectors about each boundary 1-simplex of $\tilde{A}_0(\kappa^0)$ one obtains the Lipschitz domain (the crown) (3.1). Any integration of the gradient squared over portions of the crown boundary can be controlled by varying the crown, applying Theorem 1 and averaging. The result will be a solid integral of the gradient squared and a $\partial\Omega$-integral of the normal derivative squared, each over regions contained in $\tilde{A}_0^*$.

Combining this fact with (3.11) of Lemma 2 and Lemma 4.1 it follows that

$$\int_{\tilde{A}_0(\kappa^0)\cap\partial\Omega} |\nabla u|^2 ds \\ \le C\left[\int_{\tilde{A}_0^*(\kappa^0)\cap\partial\Omega} (\frac{\partial u}{\partial N})^2 + |\frac{\partial u}{\partial N}||\nabla_T u| ds + \int_{\tilde{A}_0^*(\kappa^0)} |\nabla u|^2 dx\right] \quad (4.3)$$

The regularity estimate

$$\int_{\tilde{A}_0(\kappa^0)\cap\partial\Omega} |\nabla u|^2 ds \\ \le C\left[\int_{\tilde{A}_0^*(\kappa^0)\cap\partial\Omega} u^2 + |\nabla_T u|^2 + |\frac{\partial u}{\partial N}||\nabla_T u| ds + \int_{\tilde{A}_0^*(\kappa^0)} |\nabla u|^2 dx\right]$$

is obtained in the same way using Jerison and Kenig's regularity estimate [JK81] on the crowns.

Estimate (4.3) and its companion scale to

$$\int\limits_{\tilde{A}_j(\kappa^0)\cap\partial\Omega} |\nabla u|^2 ds \le C\left[\int\limits_{\tilde{A}_j^*(\kappa^0)\cap\partial\Omega} (\frac{\partial u}{\partial N})^2 + |\frac{\partial u}{\partial N}||\nabla_T u| ds + \int\limits_{\tilde{A}_j^*(\kappa^0)} |\nabla u|^2 \frac{dx}{|x|}\right]$$

for $j = 0, 1, 2, \ldots$ when κ^0 is taken to be the origin. As in §2, the vertex Rellich formula (2.7) for $n = 4$ yields

$$\int\limits_{\mathrm{St}(\kappa^0,\dot{K})\cap\{|x|\le 1\}} |\nabla u|^2 ds \\ \le C\left[\int\limits_{\mathrm{St}(\kappa^0,\dot{K})} (\frac{\partial u}{\partial N})^2 + |\frac{\partial u}{\partial N}||\nabla_T u| ds + \int\limits_{\mathrm{St}(\kappa^0,K)} |\nabla u|^2 dx\right] \quad (4.4)$$

and similarly for the regularity estimate.

Estimate (4.4) can be obtained at boundary points other than vertices. Thus by the finiteness of the complex K and Young's inequality applied to the mixed boundary term on the right of (4.4)

$$\int_{\partial\Omega} |\nabla u|^2 ds \le C\left[\int_{\partial\Omega} (\frac{\partial u}{\partial N})^2 ds + \int_{\Omega} |\nabla u|^2 dx\right] \quad (4.5)$$

Now Green's first identity and the Poincaré inequality (see the Appendix) on the connected boundary can be used to remove the last integral.

The analogue of (4.4) for the regularity estimate is dealt with in similar fashion, the Poincaré inequality applied to two terms resulting in

$$\int_{\partial\Omega} |\nabla u|^2 ds \leq C \int_{\partial\Omega} |\nabla_T u|^2 ds \tag{4.6}$$

The left sides of (4.5) and 4.6 may be replaced by the nontangential maximal function $N_\alpha(\nabla u)$ by [VV03], Theorems 4.5 and 5.4.

When u is defined in the exterior domain, estimate (4.5) follows with $|K|^c$ in place of Ω because K is a subcomplex of a locally finite triangulation of $\mathbb{R}^4$ and the preceding estimates were of a local nature about the boundary. The rest of the argument is valid because both u and its gradient must vanish at least like $|x|^{-2}$ as $|x| \to \infty$ by the maximum principle and the mean value theorem. □

Theorem 3 provides the required a priori estimates for the Neumann and regularity problems with L^2-data. Uniqueness of solutions is immediate from Green's first identity. The question now is existence for the Neumann problem. We give an answer in dimension 4.

5. Global approximation of polyhedral domains

The a priori estimate of Theorem 3 shows that the Neumann problem is semi-Fredholm in 4-dimensional polyhedra. In order to show that solutions exist for a dense class of Neumann data, one usually constructs approximating domains with stronger geometric properties in which solutions are already known to exist. One requires that (1) the a priori estimate that establishes the semi-Fredholm property in the weaker geometry remains uniform in the approximating domains and that (2) the dense class of data is transferable under a homeomorphism between the geometrically stronger and weaker boundaries.

Smooth domains form a subclass of Lipschitz domains. Given a Lipschitz domain, one can prove the existence of smooth approximating domains that satisfy requirement (2) and have Lipschitz natures uniformly controlled by that of the Lipschitz domain so that requirement (1) is satisfied.

Polyhedra can be approximated by Lipschitz polyhedra as in [VV03]. However, it is not clear that the quantities of §2 and §3 describing the given polyhedron can be made uniform over a sequence of such approximating Lipschitz polyhedra, i.e. Theorem 3 may not be uniform over such approximations. This may be related to the fact in §11 that there are compact polyhedral domains with manifold boundary that cannot be homeomorphic to any approximating Lipschitz domain.

The requirement (2) cannot be met in general.

In dimension 4 the 3-manifold theory of Moise provides a way out of this dilemma. As explained in §§6, 7, and 8 a continuum of piecewise linearly (PL) homeomorphic polyhedral domains that are not generally Lipschitz can be constructed beginning with the given polyhedron and ending with a Lipschitz polyhedron that carry the estimates of Theorem 3 at each stage.

Because the Neumann problem is solvable on the Lipschitz polyhedron, a method of continuity then yields solvability on the polyhedron that was the original object of study. Classical layer potentials, to which we now turn, are one way of organizing this procedure. The combinatorial geometric methods of §6, 7 and 8 that realize the boundary homeomorphism requirement (2) for these potentials can be motivated by the example in §5.2.

5.1. Layer potentials

Let $\Gamma(x) = -[(n-2)\omega_n]^{-1}|x|^{2-n}$ denote the fundamental solution for Laplace's equation in $\mathbb{R}^n$ $(n \geq 3)$ where ω_n is the surface area of $\mathbb{S}^{n-1}$. For $f \in L^p(\partial\Omega)$, $p > 1$, define the single layer potential by

$$\mathcal{S}f(x) = \int_{\partial\Omega} \Gamma(x-Q)f(Q)ds(Q) \text{ for } x \in \mathbb{R}^n$$

where now P and Q will denote points on $\partial\Omega$ while x and y will usually be in $\mathbb{R}^n$.

Because $\mathcal{S}f(x)$ is harmonic in x away from the support of f and because $\partial\Omega$ is a finite simplicial complex one may establish many of the standard norm and trace properties of $\mathcal{S}f$ by simply looking at two boundary simplexes at a time. In particular

$$\int_{\partial\Omega} N_\alpha(\nabla\mathcal{S}f)^p ds \leq C_p \int_{\partial\Omega} |f|^p ds \quad 1 < p < \infty \tag{5.1}$$

and

$$\lim_{x\in\Gamma_\alpha^\pm(P)\,,\,x\to P} \nabla\mathcal{S}f(x) = \pm\frac{1}{2}N_P f(P) + p.v.\int_{\partial\Omega} \nabla\Gamma(P-Q)f(Q)ds(Q) \quad a.e.ds(P) \tag{5.2}$$

where $p.v.\int_{\partial\Omega} = \lim_{\epsilon\to 0}\int_{|P-Q|>\epsilon}$, Γ_α^+ denotes nontangential cones in $\mathbb{R}^n \setminus \bar{\Omega}$, Γ_α^- denotes nontangential cones in Ω, and N is the *outer* pointing unit normal from Ω. When the inner product of $\nabla\mathcal{S}f$ with N_P is formed in (5.2) one obtains the bounded operators $\pm\frac{1}{2}I + D^*$ acting on $f \in L^p(\partial\Omega)$, $1 < p < \infty$. The double layer potential is defined by

$$\mathcal{D}f(x) = \int_{\partial\Omega} N_Q \cdot \nabla\Gamma(Q-x)f(Q)ds(Q) \quad x \in \mathbb{R}^n - \partial\Omega$$

and $\mathcal{D}f$ can replace $\nabla\mathcal{S}f$ in (5.1). Also

$$\lim_{x\in\Gamma_\alpha^\pm(P),\ x\to P} \mathcal{D}f(x) = \mp\frac{1}{2}f(P) + Df(P) \tag{5.3}$$

where D is the $p.v.$ operator adjoint to D^*. The operator D is bounded on L^p, $1 < p < \infty$.

In [Ver84] it was shown, in the case Ω is a bounded Lipschitz domain with connected boundary, that the operator $\mathcal{S}$ restricted to the boundary is invertible from $L^p(\partial\Omega)$ to the Sobolev space $W^{1,p}(\partial\Omega)$, $1 < p \le 2$, and that $\pm\frac{1}{2}I + D^*$ is invertible on $L^2(\partial\Omega)$ (or L^2 with mean value zero) Further, the operator $\pm\frac{1}{2}I + D$ was shown to be invertible on $W^{1,2}(\partial\Omega)$. These results were later extended by Dahlberg and Kenig [DK87] to all relevant p's because of their $p = 1$ Hardy space result.

Sobolev spaces on the boundary complexes of compact polyhedral domains are defined in the Appendix 12. Equivalent definitions, when the boundary is a manifold, and other basic theorems are also discussed there. Given these definitions the following additional properties of the layer potentials may be stated. When $f \in W^{1,p}(\partial\Omega)$

$$\int_{\partial\Omega} N_\alpha(\nabla\mathcal{D}f)^p ds \le C_p \int_{\partial\Omega} |\nabla_T f|^p ds \quad 1 < p < \infty$$

and $\nabla\mathcal{D}f$ will have pointwise limits a.e. on $\partial\Omega$. Thus $\pm\frac{1}{2}I + K$ maps $W^{1,p}(\partial\Omega)$ continuously to $W^{1,p}(\partial\Omega)$ and $\mathcal{S}$ maps $L^p(\partial\Omega)$ continuously to $W^{1,p}(\partial\Omega)$, $1 < p < \infty$.

Because the functions $\mathcal{S}f(x)$ satisfy (5.1) and decay at infinity, the polyhedral Rellich estimates of §4 apply to them. Thus they provide a large class of solutions to the Neumann and regularity problems.

Our main objective is the invertibility of the operators $\pm\frac{1}{2}I + D^*$ in $L^2(\partial\Omega)$ when $\partial\Omega$ satisfies the hypotheses of Theorem 3. Define $L_0^p(\partial\Omega) = \{f \in L^p(\partial\Omega) \mid \int_{\partial\Omega} f ds = 0\}$, $1 \le p \le \infty$. Then by the arguments in [Ver84] Theorem 3 implies

$$\frac{1}{2}I + D^* : L^2(\partial\Omega) \to L^2(\partial\Omega) \text{ satisfies } \|f\|_2 \le C\|(\frac{1}{2}I + D^*)f\|_2 \tag{5.4}$$

and thus has closed range and is one-to-one. And

$$-\frac{1}{2}I + D^* : L_0^2(\partial\Omega) \to L_0^2(\partial\Omega) \text{ satisfies } \|f\|_2 \le C\|(-\frac{1}{2}I + D^*)f\|_2 \tag{5.5}$$

and thus has closed range and is one-to-one.

Given (5.4) and (5.5) we are now set up for defining the method of continuity mentioned above. First a cautionary example.

5.2. Loss of operator index

Consider a Lipschitz polyhedron Ω in $\mathbb{R}^3$ that has a torus for its boundary. Suppose that in a neighborhood of the origin $\overline{\Omega}$ is the union of two wedges and a 3-simplex as, for example,

$$\{y \leq 0, x \geq 0, z \geq 0, x+z \leq 1\} \cup \{x \leq 0, y \geq 0, z \geq 0, y+z \leq 1\} \cup \sigma$$

where σ has vertices the origin, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.

The operators (5.4) and (5.5) will be isomorphisms. Feigning ignorance of this, suppose, in an effort to simplify, one wishes to change the domain by successively removing simplexes and then compare through a succession of comparisons the operators in the original domain to those in the simpler. If the simplex σ is first removed, two 2-simplexes of the boundary are thereby removed but then are replaced in the new boundary by two newly exposed 2-simplexes. Thus an isomorphism between Lebesgue spaces on the boundaries can be obtained by nicely mapping the one pair of 2-simplexes to the other pair. It only appears that no damage has been done.

The single layer potential maps Lebesgue spaces to Sobolev spaces. When a polyhedral boundary is a manifold, Sobolev spaces on the boundary can be equivalently described either as spaces of weak derivatives or as completions of the Lipschitz functions (Theorem 13, 14 below). Consider a Lipschitz function on $\partial\Omega$ that is identically zero on the two 2-simplexes of σ that will be removed, while nonzero in a boundary neighborhood of the point $(0, 0, 1/2)$. Such a Sobolev function will not be mapped by the above mapping between pairs of simplexes to the corresponding Sobolev space on the (nonmanifold) boundary of $\Omega \setminus \sigma$.

A more striking result comes by considering the space of weak derivatives $W^{1,2}$ on the boundary of $\Omega \setminus \sigma$. This is a larger space ((12.2) below) that includes, for example, functions equal to zero a.e. on the boundary of one wedge while equal to some other constant on the boundary of the other wedge. One observation is, like the last, that such functions cannot come from mapping the corresponding space as defined on $\partial\Omega$.

But another observation brings us back to isomorphisms between Lebesgue spaces. Denote by κ the boundary 1-simplex with vertices the origin and $(0, 0, 1)$. Denote by J the set of functions Lipschitz continuous everywhere on $\partial(\Omega \setminus \sigma) \setminus \kappa$ with continuous extensions on each wedge separately that do not agree on κ. By Lipschitz domain theory there is no $f \in L^2(\partial(\Omega \setminus \sigma))$ such that $\mathcal{S}f$ can equal a.e. any of the functions of J. This is because $\mathcal{S}f$ would necessarily be continuous across each wedge boundary in a neighborhood of κ. On the other hand classical solutions in $\Omega \setminus \sigma$ with Dirichlet data taken in J exist and Lipschitz domain estimates taken in Lipschitz subdomains that include only one wedge at a time show that the gradients of these solutions are nontangentially in L^2 of the boundary. They are solutions to the Neumann problem. But the classical Neumann uniqueness (modulo constants) argument still works in the domain $\Omega \setminus \sigma$ even though its boundary is not a manifold. Therefore one can now assert that the operator (5.5) necessarily misses all the L^2 Neumann data arising in this manner from the set of Dirichlet data J. It in fact misses an infinite dimensional subspace of L^2 and is no longer Fredholm. The operators for Ω and $\Omega \setminus \sigma$ do not compare well. Without the specific and detailed information of the example one can not conclude much about the original operator from the operator in the altered domain.

Removing a next simplex from $\Omega \setminus \sigma$ might result in a vertex as the only nonmanifold point. In this case the cokernels of the operators (5.5) for this new domain and Ω will differ by a single dimension. However, the kernels will not differ. Even in this case the index changes.

Removing the simplex σ resulted in the loss of the manifold condition because it violated the classical shelling method of combinatorial geometry. This method is explained and then adhered to in the following sections when mapping a non-Lipschitz polyhedron to a Lipschitz polyhedron. It is a method that in general does not work in higher dimensions, as indicated in §11, and has pitfalls in lower dimensions.

6. A method of continuity

Suppose X is a Banach space and (i) L_t are linear operators on X, $0 \le t \le 1$, with (ii) $t \mapsto L_t$ continuous in operator norm as a function of t and (iii) $\|x\| \le C\|L_t x\|$ with C independent of $x \in X$ and t. It follows that if any one of the operators is an isomorphism then all the operators are isomorphisms. This section prepares the boundary operators $\pm\frac{1}{2}I + D^*$ to satisfy (i) and (ii) when defined on a certain deformation of polyhedral boundaries. Hypothesis (iii) and the inequalities (5.4) and (5.5) are discussed in the next section.

Let $K^4 \subset \mathbb{R}^4$ be a finite homogeneous simplicial complex such that $\Omega = \operatorname{Int}|K|$ is a domain with connected boundary $\partial\Omega = |\dot{K}|$. Suppose further that $|\dot{K}|$ is a 3-manifold. Theorem 12 from the Appendix provides approximating Lipschitz domains that differ from Ω in tubular neighborhoods about the boundary 1-skeleton. By exploiting the 3-manifold theory of Moise we will now begin to show that Ω can be shelled back to one of these domains, and that $\partial\Omega$ can therefore be isotopically deformed to a Lipschitz boundary.

Let $\kappa^0 \in \dot{K}$ be a vertex. Because $\dot{K}$ is a *triangulated 3-manifold* Theorem 1 of [Moi52] pp. 96–97 says that it is a *combinatorial 3-manifold*, see also [Moi77] pp.247-252 and especially the comments on p. 252. This means that $\operatorname{Lk}(\kappa^0, \dot{K})$ is piecewise linearly homeomorphic to the boundary complex of a 3-simplex, i.e. is a PL 2-sphere. By the subdivision theorem (see Theorem I.2 on p.10, Ex. I.14 on p.8 and the lemma on p.19 of [Gla70]) K may be taken to be a subcomplex of a combinatorial triangulation of $\mathbb{R}^4$. Thus $\operatorname{Lk}(\kappa^0, \dot{K})$ is a PL 2-sphere that is a subcomplex of the PL 3-sphere $\operatorname{Lk}(\kappa^0, \mathbb{R}^4)$. (In general a homogeneous n-complex is said to be combinatorial if the link of every vertex is PL homeomorphic to the boundary complex of an n-simplex or to an $(n-1)$-simplex. Any complex that is PL homeomorphic to an n-simplex is also called a combinatorial n-ball and any complex PL homeomorphic to the boundary of an $(n+1)$-simplex is also called a combinatorial n-sphere. See p.18 of [Gla70].) Consequently, the 3-complex $\mathrm{Lk}(\kappa^0, K)$ is PL homeomorphic to a 3-simplex (i.e. is a PL 3-ball) by the PL Schoenflies theorem of Alexander, Graub and Moise (see p.161 of Bing's book [Bin83]). By Theorems 2 and 3 of D. E. Sanderson [San57] there is a subdivision of the ball $\mathrm{Lk}(\kappa^0, K)$ which can be shelled.

Remark 6. It follows that $|K^4| \subset \mathbb{R}^4$, with $\dot{K}$ a manifold, is a combinatorial 4-manifold. This is only a local property. For example, if $\dot{K}$ is homeomorphic to the 3-sphere it follows that it is a PL 3-sphere and further that K is then homeomorphic to the 4-ball (by M. Brown's generalized Schoenflies theorem). But it is still unknown whether or not K would then be a PL 4-ball, aka combinatorial 4-ball. This is the PL Schoenflies conjecture. See pp 7 and 47 of [RS72]

6.1. Shelling

Informally a finitely triangulated n-cell (alternative term for n-ball) can be shelled if the n-simplexes that form it can be given an order so that at each stage, as one n-simplex after another is removed, the remaining complex is homeomorphic to the n-ball. Thus a shellable n-cell is necessarily a *combinatorial ball*. The proof is by induction on the number of simplexes, the preservation of topological properties under induction being the purpose of shelling.

On the other hand a combinatorial n-ball in $\mathbb{R}^n$ is not necessarily shellable. The tetrahedron of M.E. Rudin [Rud58] triangulated into 41 subtetrahedra is non-shellable. Bing's house with two rooms [Bin83] and Furch's knotted hole ball are other examples. When $n = 3$ Sanderson's theorem says that a new triangulation that is a subdivision of the given triangulation may be produced yielding a shellable ball.

Remark 7. Suppose every $|K^4| \subset \mathbb{R}^4$ homeomorphic to a 4-ball has a subdivision that can be shelled. Then the PL Schoenflies conjecture would be resolved. On the other hand if K^4 exists that is not a combinatorial 4-ball in $\mathbb{R}^4$, it is resolved again, negatively. Here again the 3-manifold theory is used so that $\dot{K}$ is seen to satisfy the conjecture's hypothesis. However, every combinatorial n-ball does have a subdivision that can be shelled by [BM71]. See also Ex. 7 on p.47 of [Gla70].

Formally a finite homogeneous m-subcomplex S of a combinatorial m-manifold L can be shelled from L if there is an ordering of all the m-simplexes of S, $\sigma_1, \sigma_2, \ldots$ so that each σ_i is *free* in $L \setminus \bigcup_{j<i} \sigma_j$ (excepting the last in the case $S = L$). An m-simplex σ is free in an m-complex with boundary M if both $\sigma \cap \dot{M}$

and $\sigma \cap (M - \sigma)^{\cdot}$ are $(m-1)$-cells. I.e. $\dot{M}$ and $(M - \sigma)^{\cdot}$ are PL homeomorphic. If $S = L$ then L is necessarily an m-cell. See [RS72] pp.39–41.

Now the subdivision of $\mathrm{Lk}(\kappa^0, K)$ induces a subdivision of $\mathrm{St}(\kappa^0, K)$ by joining each 3-simplex of the subdivision to κ^0. $\mathrm{St}(\kappa^0, K)$ cannot necessarily be shelled from K as the first figure in Figure 4 shows. There $\mathrm{Lk}(\kappa^0, K)$ is a shellable 1-cell. But $\mathrm{St}(\kappa^0, K)$ cannot be shelled from K.

The purpose is not to shell K but is more modest, especially given Remark 7. It is to shell back to a beveled subdomain of K. For any fixed $0 < \epsilon < 1$ let $R_\epsilon = R_\epsilon(\kappa^0, K) = \{x = (1-t)\kappa^0 + tQ \mid Q \in \mathrm{Lk}(\kappa^0, K), 0 \le t < \epsilon\}$. Then the closure $\bar{R}_\epsilon$ is a rescaling of $\mathrm{St}(\kappa^0, K)$ and $\bar{R}_\epsilon \setminus R_\epsilon$ a rescaling of $\mathrm{Lk}(\kappa^0, K)$. Now the induced subdivision of $\bar{R}_\epsilon$ may be shelled from $|K|$ resulting in a finite sequence of domains ending with $|K| \setminus R_\epsilon$. There will be a PL-homeomorphism between each pair of successive boundaries and thus between $|\dot{K}|$ and $\mathrm{Bd}(|K| \setminus R_\epsilon)$. Moreover there will be PL-isotopies between successive boundaries.

6.2. Isotopy

By isotopy we essentially mean a homotopy that at each level is a PL-homeomorphism. Let I denote the closed unit interval [0, 1] and let X and Y denote triangulated spaces. Almost following [RS72] p.37 an *isotopy* of X in Y is an embedding $F : X \times I \to Y \times I$ so that for all $x \in X$, $F(x, t) \in Y \times t$ for each $t \in I$. Then embeddings $F_t : X \to Y$ are defined by $F(x, t) = (F_t(x), t)$.

When σ is a free 4-simplex in a 4-manifold-with-boundary M, let $X = \dot{M}$ and let Y be the nonhomogeneous complex $\dot{M} \cup \sigma$. Let F_0 be the identity $F_0(x) = id(x) = x$. $F_1 : \dot{M} \to (M \setminus \sigma)^{\cdot}$ will be defined below to be a PL homeomorphism so that $F_1|_{\dot{M}\setminus\sigma} = id$, and we will define $F_t = (1-t)F_0 + tF_1$. We will say that σ is the *support* of the isotopy.

Remark 8. The isotopy constructed here will not be PL with respect to the product spaces $X \times I \to Y \times I$, unlike that in [RS72], though it will be bi-Lipschitz and F_t will be PL for each t. A PL isotopy can be constructed by employing the

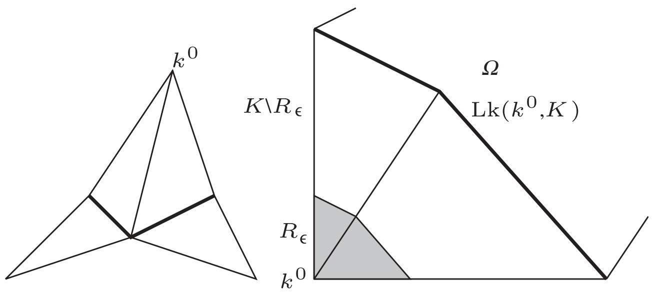

Fig. 4. $\mathrm{St}(\kappa^0, K)$ cannot be shelled from K. R_ϵ to be shelled from K

Alexander trick as on pp. 37–38 of [RS72], but will not be suitable for the method of continuity here. For example, consider the 2-simplex with vertices $A(-1, 0)$, $B(1, 0)$, and $C(0, 2)$ in the (x, y) plane and an isotopy deforming the base AB to the union of the sides $AC + CB$. Write $F_t(x, 0) = (x, f_t(x))$. The isotopy here yields $f_t(x) = 2t(1 - |x|)$ for $|x| \leq 1$ and $t \in I$. The Alexander trick yields $G_t(x, 0) = (x, g_t(x))$ with $g_t(x) = 1 - |x|$ for $|1 - 2t| \leq |x| \leq 1$, $t \in I$; $g_t(x) = 2t$ for $|x| \leq 1 - 2t$, $0 \leq t \leq \frac{1}{2}$; and $g_t(x) = 2(t - |x|)$ for $|x| \leq 2t - 1$, $\frac{1}{2} \leq t \leq 1$. Note that $\frac{d}{dx} f_t(x)$ can converge uniformly to either $f_0'(x)$ or $f_1'(x)$, $|x| \leq 1$. But $\frac{d}{dx} g_t(x)$ can converge uniformly to neither g_0' nor g_1'.

The homeomorphism F_1 will be defined as follows. Denote by M^4 the remainder of $|K|$ at some stage of the shelling of $\bar{R}_\epsilon$. M can be triangulated. $\dot{M}$ is a 3-manifold. When a 4-simplex σ is next to be shelled from M, $\sigma \cap \dot{M}$ will consist of $j + 1$ 3-simplexes $j = 0, 1, 2$ or 3, with $j = 3$ occurring only at the last. One writes $\sigma = \sigma^{3-j}\lambda^j$ with $\sigma^{3-j} \in \dot{M}$. It follows that $\mathrm{St}(\sigma^{3-j}, \dot{\sigma}) = \sigma \cap \dot{M}$ while $\mathrm{St}(\lambda^j, \dot{\sigma}) = \dot{\sigma} \setminus \dot{M}$. Then $F_1 : \sigma \cap \dot{M} \to \dot{\sigma} \setminus \dot{M}$ is defined by mapping the barycenter $\hat{\sigma}^{3-j}$ to the barycenter $\hat{\lambda}^j$, fixing the vertices of σ (the vertices other than σ^{3-j} when $j = 3$) and extending piecewise linearly.

Example 1. When $j = 1$ denote the vertices of σ^2 by v_1, v_2 and v_3, and the vertices of λ^1 by w_1, w_2, see Figure 5.

Then $\sigma^4 \cap \dot{M}$ consists of the two 3-simplexes $\sigma^2 w_1$ and $\sigma^2 w_2$. F_1 maps each of the six 3-simplexes $\hat{\sigma}^2 v_i v_j w_k$ linearly (affinely) to the corresponding $\hat{\lambda}^1 v_i v_j w_k$, $1 \leq i < j \leq 3$, $1 \leq k \leq 2$, see Figure 6.

If, for example, $x = a_1 v_2 + a_2 v_3 + a_3 w_2 + b\hat{\sigma}^2$ with $a_1 + a_2 + a_3 + b = 1$, then $F_1(x) = a_1 v_2 + a_2 v_3 + a_3 w_2 + b\hat{\lambda}^1 = x + b(\hat{\lambda}^1 - \hat{\sigma}^2)$ and $F_t(x) = x + tb(\hat{\lambda}^1 - \hat{\sigma}^2)$, parallel projections that fix $v_2 v_3 w_2$. All six triangles $v_i v_j w_k$ are fixed under F_1 and form the 2-sphere $\dot{\sigma}^2 \dot{\lambda}^1 \subset \dot{M}$ that is the common boundary in $\dot{\sigma}^4$ of $\sigma^4 \cap \dot{M}$ and $F_1(\sigma^4 \cap \dot{M})$.

In general F_1 projects $\sigma \cap \dot{M}$ along lines parallel to the direction determined by $\hat{\sigma}^{3-j}$ and $\hat{\lambda}^j$. In a rectangular coordinate system with the x_4-axis oriented in this

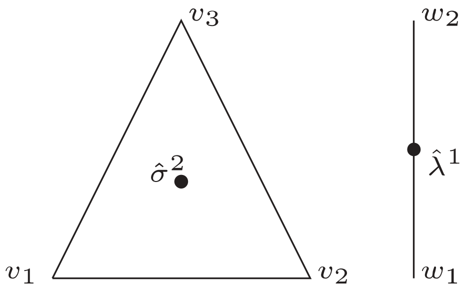

Fig. 5. The join $\sigma^4 = \sigma^2\lambda^1$

direction, each $F_t(\sigma \cap \dot{M})$, $t \in I$, is a graph over the region in $\mathbb{R}^3$ that has the projection of $\dot{\sigma}^{3-j}\dot{\lambda}^j$ into $\mathbb{R}^3$ as its boundary.

By elementary starring (§7.1 below) the barycenters $\hat{\sigma}^{3-j}$ and $\hat{\lambda}^j$ subdivide σ^4 into $(4-j)(j+1)$ 4-simplexes each of which we will denote by τ^4. Then the map $\tau \cap \dot{M} \to F_1(\tau \cap \dot{M})$ is a map of a single 3-simplex of τ onto a single 3-simplex of τ, and the isotopy $F_t(\tau \cap \dot{M})$, $t \in I$ takes place entirely in τ. Further $\tau \cap \dot{\sigma}^{3-j}\dot{\lambda}^j = \tau \cap \dot{M} \cap F_1(\tau \cap \dot{M})$.

Because $\dot{\sigma}^{3-j}\dot{\lambda}^j \subset \dot{M}$ is fixed by F_1, F_t can be extended to the identity on $\dot{M} \setminus \sigma$. Denote by $\dot{M}_t$ the 3-manifolds $F_t(\dot{M})$, $0 \le t \le 1$. Define isomorphisms $\Phi_t : L^p(\dot{M}_0) \to L^p(\dot{M}_t)$ by $\Phi_t f = f \circ F_t^{-1}$, $0 \le t \le 1$, $1 \le p \le \infty$. Let $T_t : L^p(\dot{M}_t) \to L^p(\dot{M}_t)$ for $1 < p < \infty$ denote any of the boundary layer potentials (5.2) and (5.3) defined above. Then $\Phi_t^{-1}T_t\Phi_t$ is bounded on $L^p(\dot{M}_0)$, $1 < p < \infty$ for all $t \in I$. Moreover we have the following version of an observation of A. McIntosh. See [MC97] or [Ken94] for example.

Lemma 4. *Using the preceding definitions, with σ^4 as the support of an isotopy F, the $\Phi_t^{-1}T_t\Phi_t$ are Lipschitz continuous as functions of $t \in I$ in the topology of the operator norms $\|\Phi_t^{-1}T_t\Phi_t\|_p$ for $1 < p < \infty$.*

Proof. When the maps F_t represent homotopies between boundaries given as graphs of Lipschitz functions with respect to the same rectangular coordinate system the lemma is proved by showing that the operators $U_t = \dfrac{\partial}{\partial t}(\Phi_t^{-1}T_t\Phi_t)$ are uniformly bounded on $L^p(\dot{M}_0)$. The proof generalizes to any sequence of graph boundaries if convergence of the boundaries is in the Lip norm. In these cases Theorem IX of [CMM82] settles the issue.

Since the $F_t(\sigma \cap \dot{M})$ converge in Lip norm, the operators U_t are uniformly bounded when restricted to be maps from $L^p(\dot{M} \cap \sigma)$ to itself. When mapping between disjoint boundary simplexes the U_t are compact and uniformly bounded. The last case occurs when $\kappa^3 \in \dot{M}$, $\kappa \cap \sigma \neq \emptyset$ is at most a 2-simplex and the U_t are restricted to be maps from $L^p(\dot{M} \cap \sigma)$ to $L^p(\kappa)$ or vice versa. In this case the $F_t(\dot{M} \cap \sigma) \cup \kappa$, $t \in I$ cannot be realized as graphs with respect to a rectangular coordinate system. However, it suffices to consider the singular integrals on

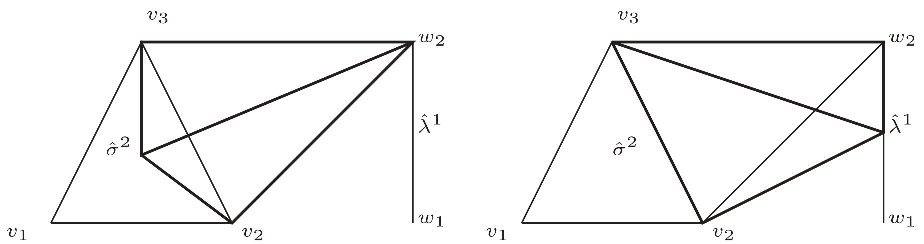

Fig. 6. The map F_1

$L^p(\tau \cap \dot{M})$ and $L^p(\kappa)$ where τ^4 is as above. This case is settled by the following geometric lemma.

Denote the *affine hull* of a j-simplex $\sigma^j \subset \mathbb{R}^n$ by aff σ^j. Then aff σ^j is the j-dimensional *affine set* ([Roc70] p3) of $\mathbb{R}^n$ that contains σ^j. If $j = n - 1$, an affine set is called a *hyperplane*. Two convex sets are *separated* by a hyperplane if one set is contained in one of the closed half-spaces determined by the hyperplane and the other set is contained in the opposite closed half-space. A *supporting hyperplane* to a convex set is the boundary of a closed half-space that contains the set so that the hyperplane contains a point of the boundary of the convex set. See [Roc70]

Lemma 5. *Let simplexes κ^3 and τ^4 form a (nonhomogeneous) complex in $\mathbb{R}^4$ so that $\emptyset \neq \kappa^3 \cap \tau^4 \in \delta^2$ where δ is a 2-face of τ. Consider points $P \in \mathrm{Lk}(\delta^2, \tau)$ and the 3-simplexes that are the joins $P\delta$. Then there is a vector $\vec{l}$ independent of P so that every line with direction $\vec{l}$ intersects the 3-complex $P\delta \cup \kappa$ at most once, i.e. all such 3-complexes are graphs with respect to the same rectangular coordinate system.*

Proof. Because κ and τ are simplexes it is possible to fix a separating hyperplane E^3 with the stronger property $E \cap |\kappa \cup \tau| = |\kappa \cap \tau|$. In the case $|\delta| \subset E$ any direction of E that is not a direction of aff δ will do.

Otherwise, $E \cap \text{aff}\,\delta$ is 1-dimensional. Let U^3 denote a supporting hyperplane for τ with the property $U \cap |\tau| = |\delta|$. Then $E \cap \text{aff}\,\delta \subset E \cap U$ a 2-dimensional affine set. If $E \cap U \subset \text{aff}\,\kappa$ choose a distinct supporting hyperplane $\tilde{U}$ with the same properties. Since $U \cap \tilde{U} = \text{aff}\,\delta$, $(E \cap U) \cap (E \cap \tilde{U})$ is 1-dimensional so that $E \cap U \neq E \cap \tilde{U}$. If now also $E \cap \tilde{U} \subset \text{aff}\,\kappa$ then it would follow that aff $\kappa = E$. Thus it may be assumed $E \cap U$ is not contained in aff κ and $E \cap U \cap \text{aff}\,\kappa$ is 1-dimensional. Let $\vec{l}$ be any direction of $E \cap U$ that is not in the direction of $E \cap \text{aff}\,\delta$ nor in the direction of $E \cap U \cap \text{aff}\,\kappa$. Since $\vec{l}$ is a direction of E it is then not any direction of aff δ. Lines with direction $\vec{l}$ are transverse to aff κ and thus to κ. Because $\vec{l}$ is a direction of U they are transverse to $P\delta$. Because $\vec{l}$ is a direction of E, and E separates $P\delta$ and κ, no line with this direction intersects both except in $\kappa \cap \tau$.

This completes Lemma 4. □

7. The uniform bound from below

The bounds from below, (5.4) and (5.5), depend on the triangulation of the polyhedral domain. As suggested in the last section, when beveling a vertex, the set $\bar{R}_\epsilon$ has a triangulation and the rest of $|K|$ has a compatible triangulation. Each stage M in the shelling of $\bar{R}_\epsilon$ is a complex.

Given a simplex $\sigma^4 \in M$ next to be shelled, the boundaries of $\dot{M}_t$ were defined to be the 3-manifolds $F_t(\dot{M})$. Here corresponding 4-complexes M_t will be defined so that bounds from below that are uniform in t can be deduced from Theorem 3. More is required than using the isotopy to subdivide σ (and thus M) because the estimates of §4 were necessarily obtained in neighborhoods of vertices (and edges) that were kept away from the links of these vertices. Staying away from the link becomes problematic for the vertex $F_t(\hat{\sigma})$ as t increases, as illustrated in Figure 7.

On the other hand, each $M_t \quad (t < 1)$ could have a triangulation obtained by subdividing σ and M_1 could be taken to be $M \setminus \sigma$, insuring that there are constants $C_t < \infty$ for (5.4) and (5.5) for each $0 \leq t \leq 1$. This suffices to obtain condition (iii) in the method of continuity (stated at the beginning of the the previous section) by a functional analytic argument that then makes the constants C_t uniformly bounded, as long as conditions (i) and (ii) are known. See for example [Gri85] p.111. Conditions (i) and (ii) hold for the operators $\Phi_t^{-1} T_t \Phi_t$ by construction and Lemma 4. Consequently, for the purpose of inverting the layer potentials only the isotopic beveling off of the 1-skeleton needs to be done, and will be done in the next section.

Nevertheless, here the uniform bound from below will be obtained directly by geometric argument instead of relying on this functional analytic observation. One reason for this is that there are other problems of interest such as the biharmonic Dirichlet problem for which the setup of boundary integral equations and invertible linear operators does not seem available. It would therefore be advantageous to know that the analogues of the Rellich estimates of §4 could be made uniform over a deformation of polyhedral boundaries by purely geometric considerations. Figure 8 suggests that this might be done by a triangulation of M_t that is not from a subdivision of K. Figure 8 also indicates that the boundary PL homeomorphisms extend to solid PL homeomorphisms near the boundary. This makes explicit Remark 6. These local solid homeomorphisms will be used in §§9 and 10. In addition the results of this section will establish the global result that K^4 is PL homeomorphic to the Lipschitz polyhedron obtained by the beveling process.

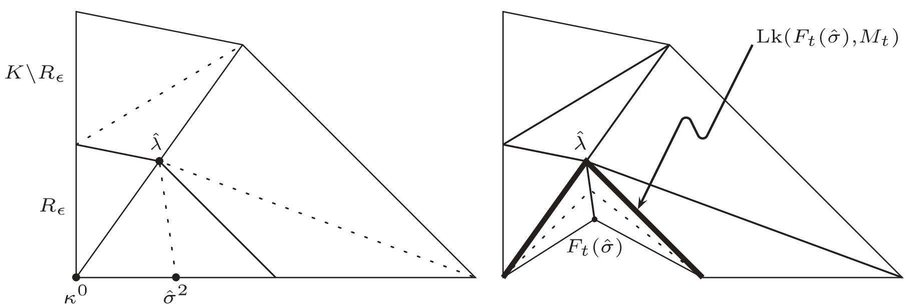

Fig. 7. Triangulating $\sigma^4 = \sigma^2\lambda$ and $K \setminus R_\epsilon$. Then isotopically shelling σ^4

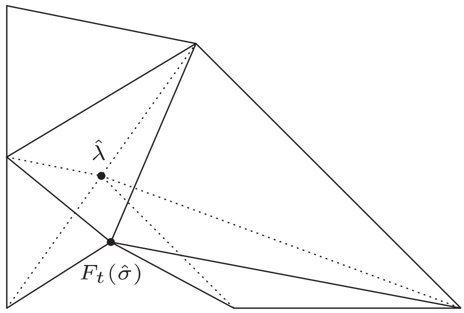

Fig. 8. Extending the isotopy inside by $\hat{\lambda} \to F_t(\hat{\sigma})$

7.1. Elementary Starring

Following [Gla70] pp.8,13,22–24, let A be any simplex of K and choose a point $a \in \operatorname{Int} A$. Then $\operatorname{St}(A, K)$ may be subdivided into what is called a *stellar subdivision* by forming the join $a * \dot{A} * \operatorname{Lk}(A, K)$. The change from $\operatorname{St}(A, K)$ to its stellar subdivision is called an *elementary starring*. Because no simplex of $\operatorname{Lk}(A, K)$ is subdivided in this procedure, K itself has been subdivided to yield a new simplicial complex L. This is called an elementary starring of K at $a \in \operatorname{Int} A$. A stellar subdivision of K is a subdivision of K obtained by a finite sequence of elementary starrings.

The regions R_ϵ originally meant to be beveled from $|K|$ and subsequently removed by shelling arise by a sequence of elementary starrings that produce a stellar subdivision of $\operatorname{St}(\kappa^0, K)$. This is illustrated here in Figure 9.

The points a_j are chosen in the interiors of 1-simplexes A_j for $j = 1, 2, 3$ and the new 1-simplexes arising from the starrings are labelled 1, 2, and 3. A different starring order produces the same $\bar{R}_\epsilon$ but a different triangulation of $\operatorname{St}(\kappa^0, K)$.

Lemma 6. *Fix* $0 < \epsilon < 1$ *and* $\kappa^0 \in \dot{K}$. *For each* $\kappa^1 \in \operatorname{St}(\kappa^0, K)$ *that contains* κ^0 *denote by* λ^0 *the 0-simplex such that* $\kappa^1 = \kappa^0 * \lambda^0$. *For each such* κ^1 *define the point* $a = a(\kappa^1) = (1 - \epsilon)\kappa^0 + \epsilon\lambda^0$. *Stellar subdivide* $\operatorname{St}(\kappa^0, K)$ *in any*

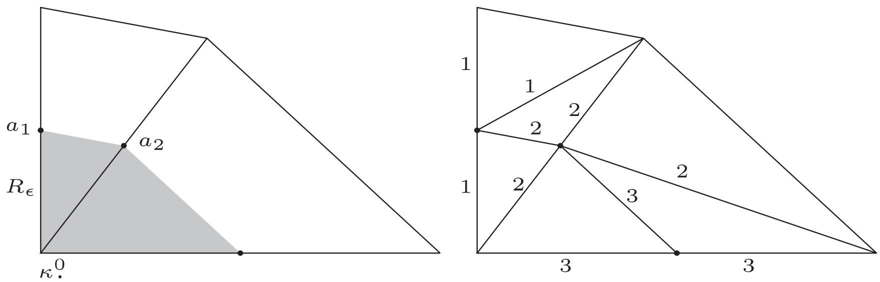

Fig. 9. $\operatorname{St}(\kappa^0, K)$ and R_ϵ with choice of points for starring. Stellar subdivision ordered by the choice of points

order with respect to the $a(\kappa^1)$ *obtaining a new triangulation* L *of* $|K|$. *Then* $|\operatorname{Lk}(\kappa^0, L)| = \bar{R}_\epsilon \setminus R_\epsilon = \{x = (1-\epsilon)\kappa^0 + \epsilon Q \mid Q \in \operatorname{Lk}(\kappa^0, K)\}$ *with a triangulation independent of the starring order, because* $\operatorname{St}(\kappa^0, L)$ *is isomorphic to* $\operatorname{St}(\kappa^0, K)$.

Proof. Let $\sigma^4 = \kappa^0 v_1 v_2 v_3 v_4 \in \operatorname{St}(\kappa^0, K)$ and let $a_j = a(\kappa^0 v_j)$ with the v_j enumerated in the order that the given starring sequence affects σ^4. It is enough to observe that the starring produces the sequence of 4-simplexes containing κ^0, vis. $\kappa^0 a_1 v_2 v_3 v_4$, $\kappa^0 a_1 a_2 v_3 v_4$, $\kappa^0 a_1 a_2 a_3 v_4$, and $\kappa^0 a_1 a_2 a_3 a_4$, the last being the rescaling of σ from the original definition of $\bar{R}_\epsilon$. □

7.2. Extending the boundary homeomorphism inside

Lemma 7. *Let* σ^4 *be free in* M^4. *Let* $\sigma = \sigma^{3-j}\lambda^j$ *for some* $j = 0, 1, 2$ *or* 3 *and* F_t *for* $t \in I$ *be defined as in §6.2. Then there exists an isotopy* $h : |M| \times I \to |M| \times I$ *and a subdivision* M_0 *of* M *so that*

(i) h *is supported in* $\operatorname{St}(\hat{\lambda}^j, M_0)$
(ii) The complexes defined by $M_t = h_t(M_0)$ *are isomorphic for all* $t \in I$
(iii) $h_t(\dot{M}_0) = F_t(\dot{M}_0) = \dot{M}_t$ *for* $t \in I$
(iv) The distance of a vertex v_t *to* $\operatorname{Lk}(v_t, M_t)$ *is uniformly bounded away from zero over all* t *and* $v_t \in \dot{M}_t$.

Proof. Star first with respect to $\hat{\sigma}^{3-j}$ and then $\hat{\lambda}^j$. The first starring affects only σ, while the second further subdivides $\operatorname{St}(\lambda^j, M)$ when $j \neq 0$. Denote this new triangulation by M_0.

Recall that $\operatorname{St}(\lambda^j, \dot{\sigma}^4) = \dot{\sigma}^4 \setminus \dot{M}$. Extend the boundary PL-homeomorphism $F_t \circ F_1^{-1} : \dot{\sigma}^4 \setminus \dot{M} \to F_t(\sigma \cap \dot{M})$ to a solid PL-homeomorphism $g_t : |\operatorname{St}(\hat{\lambda}^j, M_0)| \to |\operatorname{St}(\hat{\lambda}^j, M_0)|$ as the second figure in Figure 10 indicates, by mapping $\hat{\lambda}^j$ to $F_t(\hat{\sigma}^{3-j})$, fixing the vertices of $\operatorname{Lk}(\hat{\lambda}^j, M_0)$ and extending piecewise linearly. Do this for $t \in [b, 1]$ for some $0 < b < 1$ depending on σ and K. The number $b < 1$ always exists because $\operatorname{St}(\hat{\lambda}, M_0)$ is star convex with respect to any point in a small enough ball around $\hat{\lambda}$.

For $0 \le t \le b$ extend F_t to the solid homeomorphism h_t by extending to be the identity on $M \setminus \sigma$ and by extending piecewise linearly in σ^4. For $b \le t \le 1$ define $h_t = g_t \circ g_b^{-1} \circ h_b$. See Figure 11. □

Lemma 8. *Let* $M^4 \subset \mathbb{R}^4$ *be a finite homogeneous simplicial complex such that* $\operatorname{Int}|M|$ *is a domain and* $|\dot{M}|$ *is a connected manifold. Suppose* σ^4 *is free in* M. *Then the operators* $\pm\frac{1}{2}I + D^*$ *from* (5.4) *and* (5.5) *are isomorphisms on* $L^2(\dot{M})$ *and* $L^2_0(\dot{M})$ *respectively if and only if they are isomorphisms on* $L^2((M \setminus \sigma)^{\cdot})$ *and* $L^2_0((M \setminus \sigma)^{\cdot})$.

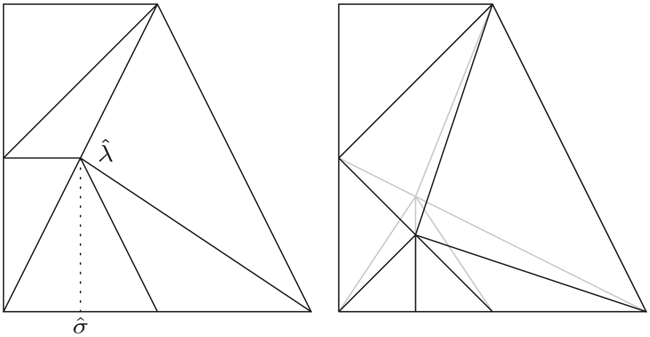

Fig. 10. M_0 by starring at $\hat{\sigma}$ and $\hat{\lambda}$. The solid homeomorphism g_b (black) and g_t(gray) for $b < t < 1$

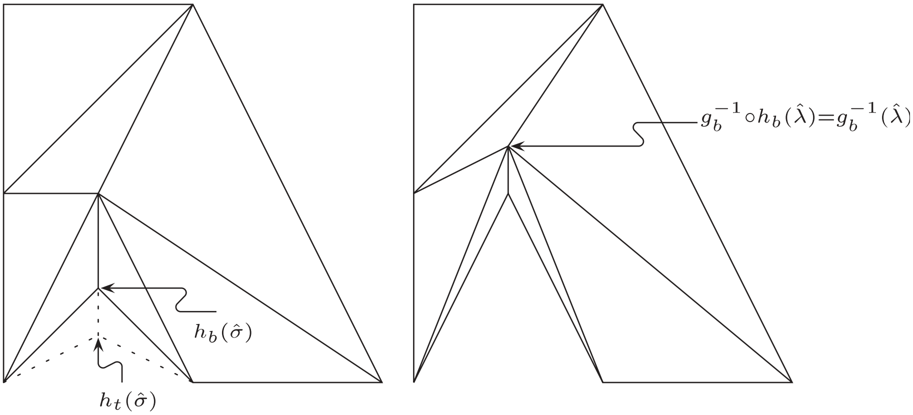

Fig. 11. $h_b(M_0)$ and $g_b^{-1} \circ h_b(M_0) = h_1(M_0) = M \setminus \sigma$

Proof. The isotopy h of Lemma 7 yields finite complexes M_t with $|M_0| = |M|$ and $|M_1| = |M \setminus \sigma|$ so that Int $|M_t|$ is a domain and $|\dot{M}_t|$ is a connected manifold for all $t \in I$. Thus the domain hypotheses of Theorem 3 are met by each M_t. By construction of h and (iv) of Lemma 7 the Lipschitz natures of the domains of Lemma 1, the Poincaré inequality, and the constants of Lemma 2 are uniform in t. Thus the constants of Theorem 3 and inequalities (5.4) and (5.5) are independent of t.

In the method of continuity stated at the beginning of §6 let L_t be the operators $\Phi_t^{-1} T_t \Phi_t$ of Lemma 4 with T_t the operators $\frac{1}{2}I + D^*$ as defined on $\dot{M}_t$ by (5.2) (or (5.3)) and $X = L^2(\dot{M})$. Condition (i) of the method is met by L_t and the above considerations show (iii) is met. Lemma 4 establishes (ii). Since the Φ_t are isomorphisms for L^2, the lemma follows for $\frac{1}{2}I + D^*$.

The operator $-\frac{1}{2}I + D^*$ maps L^2 into L_0^2. Restrict it to L_0^2 and redefine it on L^2 by $(-\frac{1}{2} + D^*)(A + f) = A + (-\frac{1}{2}I + D^*)f$ for $f \in L_0^2$ and constant functions A. The lemma now follows for the extended and then restricted operator. □

Remark 9. In sum, part (iv) of Lemma 7 insures that the geometry of the complexes M_t, when isotopically shelling a free simplex of $\overline{R}_\epsilon$ from the triangulation L of Lemma 6, leads to a uniform bound from below for the operators $\pm\frac{1}{2}I + D^*$; but once the free simplex is removed it is to be noted that one can dispense with triangulation M_1 and revert back to L now minus the free simplex.

8. Shelling the 1-skeleton

Let $\kappa^1 \in \dot{K}^4$ and suppose $\sigma^4 \in \mathrm{St}(\kappa^1, K)$. Let $\sigma^4 = \kappa^1 * \lambda^2$ with $\kappa^1 = v_0v_1$ and $\lambda^2 = v_2v_3v_4$ for vertices v_j $(j = 0, 1, 2, 3, 4)$. Every point $x \in |\sigma|$ has unique barycentric coordinates $(s_0, \ldots, s_4)$ determined by $x = s_0v_0 + \cdots + s_4v_4$, $s_0+\cdots+s_4 = 1$, and $s_j \geq 0$ for $j = 0, \ldots, 4$. Then $|\sigma| \setminus R_{\epsilon_0}(v_0, \sigma) \setminus R_{\epsilon_0}(v_1, \sigma)$ is a convex linear cell derived from the previous beveling of boundary vertices of K. Define $R_\epsilon(\kappa^1, K) = \{(1-t)P + tQ \mid P \in \kappa^1,\ Q \in \mathrm{Lk}(\kappa^1, K),\ 0 \leq t < \epsilon\}$. By choosing $0 < \epsilon_1 < \frac{1}{2}\epsilon_0 < \frac{1}{4}$ one insures that the set $\bar{R}_{\epsilon_1}(\kappa^1, K) \setminus R_{\epsilon_0}(v_0, K) \setminus R_{\epsilon_0}(v_1, K)$, to be removed about κ^1, has empty intersection with any other such set constructed for the purpose of removing the 1-skeleton of $\dot{K}$. For example, $x \in \bar{R}_{\epsilon_1}(\kappa^1, \sigma) \cap \bar{R}_{\epsilon_1}(v_0v_2, \sigma)$ implies $s_2 + s_3 + s_4 \leq \epsilon_1$ and $s_1 + s_3 + s_4 \leq \epsilon_1$ which imply $1 - \epsilon_0 < s_0$ so that $x \in R_{\epsilon_0}(v_0, \sigma)$.

For each 4-simplex $\sigma \in \mathrm{St}(\kappa^1, K)$ define

$$C(\sigma) = \bar{R}_{\epsilon_1}(\kappa^1, \sigma) \setminus R_{\epsilon_0}(v_0, \sigma) \setminus R_{\epsilon_0}(v_1, \sigma)$$

a convex linear 4-cell.

We wish to order the 4-simplexes of $\mathrm{St}(\kappa^1, K)$ so that the $C(\sigma)$ can be removed one at a time in a process known as *cellular shelling*. That is, we want an order $\sigma_1, \sigma_2, \ldots, \sigma_N$ so that

$$C(\sigma_i) \cap \left[|\,\mathrm{St}(\kappa, K)| \setminus R_{\epsilon_0}(v_0, K) \setminus R_{\epsilon_0}(v_1, K) \setminus \bigcup_{j<i} C(\sigma_j) \right]$$

is always a 3-ball.

By the manifold condition on $\dot{K}$ and [Moi52], $\mathrm{Lk}(\kappa^1, \dot{K})$ is a 1-sphere in the 2-sphere $\mathrm{Lk}(\kappa^1, \mathbb{R}^4)$. See Thm II.2 of [Gla70] p.19. Consequently $\mathrm{Lk}(\kappa^1, K)$ is a shellable 2-ball. See [Bin83] p.80. Let $\sigma_j = \kappa^1\lambda_j$ where $\{\lambda_j\}_{j=1}^N$ is a shelling order for $\mathrm{Lk}(\kappa^1, K)$. Then $\{C(\sigma_j)\}_{j=1}^N$ will be a cellular shelling. This will follow by inspecting a typical cell and its boundary when it is next to be shelled.

Hence, in the barycentric coordinates defined above, a $C(\sigma)$ is given by the additional restrictions

$$\begin{aligned} s_0 &\leq 1 - \epsilon_0 \\ s_1 &\leq 1 - \epsilon_0 \\ s_2 + s_3 + s_4 &\leq \epsilon_1 \end{aligned}$$

The boundary of $C(\sigma)$ consists of the six 3-cells:
in the hyperplanes that bevel v_0 and v_1

$$s_0 = 1 - \epsilon_0 \tag{8.1}$$

$$s_1 = 1 - \epsilon_0 \tag{8.2}$$

in the hyperplane that bevels κ^1

$$s_2 + s_3 + s_4 = \epsilon_1 \tag{8.3}$$

and in $\mathrm{St}(\kappa^1, \dot{\sigma})$

$$s_2 = 0 \tag{8.4}$$

$$s_3 = 0 \tag{8.5}$$

$$s_4 = 0 \tag{8.6}$$

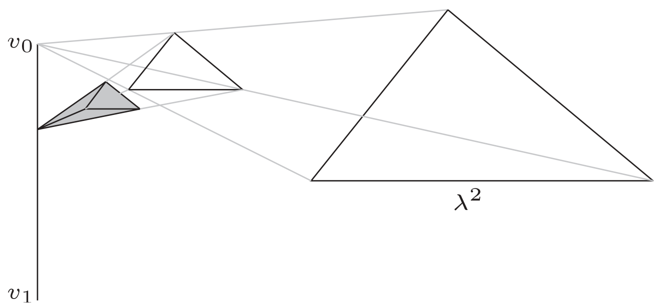

Fig. 12. Facet (8.1)

Facets (8.1) and (8.2) are tetrahedra, while the rest are triangular cylinders each with five sides. There are thus fourteen 2-cells total, each contained in two of the six 3-cells.

At any time before $C(\sigma)$ is shelled, facets (8.1) and (8.2) are always exposed, i.e. contained in the boundary, while (8.3) is never exposed. Call an exposed facet *free*. When $C(\sigma)$ is next to be shelled either one or two of (8.4),(8.5), and (8.6) are free (all three for $C(\sigma_N)$). Each of these last three intersects both (8.1) and (8.2) in a 2-cell. Each intersects (8.3) in a 2-cell. Consequently, when $C(\sigma)$ is next to be shelled, the union of its free facets is a 3-ball, as is the union of the nonexposed facets. Thus the shelling of $\mathrm{Lk}(\kappa^1, K)$ induces a cellular shelling of $\bar{R}_{\epsilon_1}(\kappa^1, K) \setminus R_{\epsilon_0}(v_0, K) \setminus R_{\epsilon_0}(v_1, K)$. Because the method of continuity has been set up with respect to simplicial shelling, this cellular shelling will now be reduced to simplicial shelling. This could be done by an explicit starring procedure, but the number of simplexes now appears to be getting large. Instead, the argument will

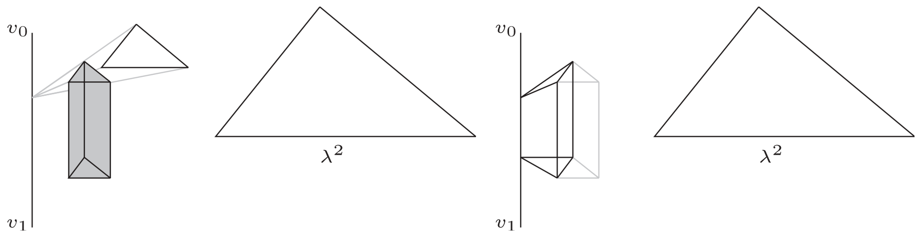

Fig. 13. Facet (8.3). Facet (8.5) intersecting facet (8.3)

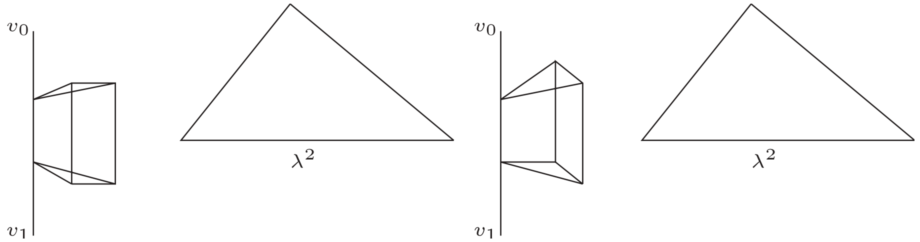

Fig. 14. Facets (8.6) and (8.4)

be based on classical subdivision theorems (in which the number of simplexes gets even larger).

Upon beveling off the boundary vertices each of the original 4-simplexes of K either remains the same, has only vertices removed, or has vertices and 1-simplexes removed. In the latter case what remains of a $|\sigma^4|$ may be decomposed into one or more of the convex cells $C(\sigma)$ together with a remaining piece that has closure that is again a convex linear cell. In this way the polyhedron $|K|$ with its boundary vertices beveled off can be considered a cell complex where each cell is convex linear ([Gla70] pp. 9–10). A first derived subdivision of this complex is produced by introducing new vertices in the interiors of each convex linear cell (the interior of a vertex is the vertex). By Ex. I.16 of [Gla70] the result is a simplicial complex. Denote this new complex by K_0. Then $|K_0|$ is $|K|$ with the original boundary vertices removed.

Suppose now that a $C(\sigma)$ is next to be cellularly shelled. Denote by $\hat{c}$ the new vertex in its interior and by B_f the triangulated 3-ball of free facets of $\dot{C}(\sigma)$ and by B_g the remaining 3-ball. Then $\mathrm{Lk}(\hat{c}, K_0) = B_f + B_g$. Again by [San57] there is a subdivision of B_f that shells. By proposition I.1(b) of [Gla70] this subdivision extends to B_g. Now $\hat{c}$ joined with these subdivisions is a subdivision of $C(\sigma)$ that is *still a star with respect to* $\hat{c}$. By [Gla70] this subdivision extends to a subdivision of K_0.

Remark 10. Again shelling is justified by [San57]. In higher dimensions the paper to read for shelling convex cells is [BM71] particularly Lemma 1 and Proposition 1. See also Ex. II.7 of [Gla70] p.47.

$B_f * \hat{c}$ may now be shelled from the subdivision of K_0. The same subdivision arguments now allow a shelling off of $B_g * \hat{c}$. Thus $C(\sigma)$ can be shelled simplicially. Now one can return to the subcomplex of K_0 that remains and apply the subdivision and shelling arguments to the next cell in the cellular shelling order. Consequently $\bar{R}_{\epsilon_1}(\kappa^1, K) \setminus R_{\epsilon_0}(v_0, K) \setminus R_{\epsilon_0}(v_1, K)$ can be shelled simplicially and Lemmas 7 and 8 apply to the beveling off of boundary 1-simplexes.

Theorem 4. *Let $|K^4| \subset \mathbb{R}^4$ be the polyhedron of a finite homogeneous simplicial complex with interior Ω that is a domain with boundary $\partial\Omega$ that is also a connected 3-manifold. Then the classical layer potentials satisfy*

(i) $\frac{1}{2}I + D^* : L^2(\partial\Omega) \to L^2(\partial\Omega)$ is an isomorphism and
(ii) $-\frac{1}{2}I + D^* : L^2_0(\partial\Omega) \to L^2_0(\partial\Omega)$ is an isomorphism

Proof. Fix $0 < \epsilon_0 < \frac{1}{4}$. For any $\kappa^0 \in \dot{K}$ apply Lemma 6 with $\epsilon = \sqrt{\epsilon_0}$. Apply it again to the resulting $\mathrm{St}(\kappa^0, L)$. Then $R_{\epsilon_0}(\kappa^0)$ can be shelled from $|K|$ with the associated isotopies of Lemma 8 supported in $R_{1/2}(\kappa^0)$. Because these supports are disjoint over all $\kappa^0 \in \dot{K}$ only local triangulations of K are needed. Repeated applications of Lemma 8 then show that the isomorphisms (i)–(ii) need only be established for $|K|$ with its boundary vertices beveled off.

Returning to the triangulation K, $|K|$ with its boundary vertices removed is, as discussed above, a cell complex that can be shelled so as to remove the remainder of the 1-skeleton of $\dot{K}$ with Lemma 8 applying at each stage. By Theorem 12, $|K|$ with its boundary 1-skeleton removed is a Lipschitz domain for which the theorem is known [Ver84]. □

A. P. Calderon's corollary on singular integral isomorphisms [Cal85] now applies to yield.

Corollary 1. *Given $K^4 \subset \mathbb{R}^4$ as in Theorem 4 there exists an $\epsilon > 0$ depending on K so that $\pm\frac{1}{2}I + D^*$ are isomorphisms on $L^p(\partial\Omega)$ and $L^p_0(\partial\Omega)$ respectively for $|p-2| < \epsilon$.*

Corollary 2. *The L^p-Neumann problem, for p as in Corollary 1, is solvable in the interiors and exteriors of the polyhedral domains of Theorem 4.*

Proof. Harmonic u with $N(\nabla u) \in L^p(\partial\Omega)$ will also have L^p Dirichlet boundary values [Car62], [HW70], [Dah79]. Consequently, when $p \geq 2$ uniqueness follows by Green's first identity, while for $p < 2$ Green's representation formula leads to $(\pm\frac{1}{2} + D)u = 0$ (5.3) from which it follows by Theorem 4 that u is identically zero outside $|K|$ or constant inside. □

Theorem 5. *Let $K^4 \subset \mathbb{R}^4$ be a finite simplicial complex with interior a domain and boundary a manifold. Then $|K|$ is piecewise linearly homeomorphic to the closure of a Lipschitz domain $\Omega_{Lip} \subset \mathrm{Int}\,|K|$. In fact, $|K|$ isotopically deforms to Ω_{Lip}*

Proof. Let Ω_{Lip} be $|K|$ with its 1-skeleton beveled off and repeatedly apply Lemma 7 as in the proof of Theorem 4, obtaining a (finite) composition of PL homeomorphisms. A single subdivision of K and another of Ω_{Lip} exists, making the composition a PL homeomorphism, by Theorem I.6 of [Gla70] p.15. □

Remark 11. As in Remark 6, if $\bar{\Omega}_{Lip}$ is homeomorphic to $\mathbb{B}^4$ Theorem 5 does not then imply that it is a PL 4-ball or even bi-Lipschitz homeomorphic to $\mathbb{B}^4$. See §11.2 below.

The following corollary will be useful in section 10.

Corollary 3. *Let $\kappa^0 \in \dot{K}$, take κ^0 to be the origin in $\mathbb{R}^4$, and let $\{\sigma_j\}_{j=1}^N$ be a shelling of $\bar{R}_\epsilon(\kappa^0)$ as described in §6. Then there is a PL-homeomorphism $H : |K| \to |K| \setminus \bigcup_{j=1}^{N-1} |\sigma_j|$ so that H has the following homogeneity property: There exists a ball B centered at κ^0 so that for all $x \in |K| \cap B$ and $s \in I$, $H(sx) = sH(x)$.*

Proof. Denote by H_j the PL homeomorphism h_1 (i.e. $t = 1$) of Lemma 7 that corresponds to the 4-simplex σ_j, $j = 1, \ldots, N-1$. Each H_j maps κ^0 to κ^0 and therefore has the homogeneity property by piecewise linearity. As does H, the composition of these maps. □

9. Extension of L^2 Estimates to infinite cones

Given a point $v \in \mathbb{R}^n$ and a set $X \subset \mathbb{R}^n$ the infinite cone of X with vertex v is

$$\mathcal{C}(v, X) = \{v + t(x - v) : t \geq 0, x \in X\} \tag{9.1}$$

Estimates for the L^2-Neumann problem in infinite cones are deduced from the completed bounded case.

First a geometric localization lemma.

Lemma 9. *Let $L_0^{n-1} \subset \mathbb{R}^n$ be a finite homogeneous $(n-1)$-complex such that its join with the origin is defined. Let $\mathcal{C}$ be the infinite cone of L_0 with vertex the origin. Let $L_j = \{2^j x \mid x \in L_0\}$ for integers $-\infty < j < \infty$ and let A_j be the closed region of $\mathcal{C}$ between L_j and L_{j+1}. Let w be a Borel measurable function supported on L_0 and let $N = N_\alpha$ denote a nontangential maximal function for* Int $\mathcal{C}$ *then*

$$\int_{L_0} |w|^p ds \leq C_{p,\alpha} \int_{\dot{\mathcal{C}} \cap (A_{-1} \cup A_0 \cup A_1)} \mathcal{M}(N(w))^p ds, \quad 0 \leq p < \infty$$

where $\mathcal{M}$ is the Hardy-Littlewood maximal function on $\dot{\mathcal{C}}$.

Proof. The inequality holds without $\mathcal{M}$ and with the right hand integral over all of $\dot{\mathcal{C}}$ because surface measure of L_0 is a Carleson measure of $\mathcal{C}$ (see for example Lemma 4.2 of [VV03] which holds here in the infinite case because $\mathcal{C}$ is a cone). Let $j < -1$, and for $Q \in A_j \cap \dot{\mathcal{C}}$ define $T(Q) = 2^{-j} Q \in A_0 \cap \dot{\mathcal{C}}$. By definition of nontangential region and $j < -1$, there is a constant $d > 0$ that depends only on L_0 and α so that $y \in \Gamma_\alpha(Q) \cap L_0$ implies that $\operatorname{dist}(y, \dot{\mathcal{C}})$ and d are comparable. If $N_\alpha(w)(Q) > 0$ choose $y \in \Gamma_\alpha(Q) \cap L_0$ so that $w(y) \approx N_\alpha(w)(Q)$. By Corollary 5.5 of [VV03] there is a set $F(y) \in \dot{\mathcal{C}}$ with (1) measure greater than $C^{-1} d^{n-1}$, (2) $y \in \Gamma_\alpha(P)$ for all $P \in F(y)$ and (3) the distance of each $P \in F(y)$ to Q and therefore $T(Q)$ is less than $C\, d$. Consequently, using the doubling of surface measure on $\dot{\mathcal{C}}$, (e.g. (4.4) of [VV03]), to pass from Q to $T(Q)$

$$N(w)(Q) \le \fint_{F(y)} N(w) ds \le C' \fint_{B(Q, Cd)} N(w) ds \le C'' \mathcal{M}(N(w))(T(Q))$$

Now by change of variables

$$\int_{A_j \cap \dot{\mathcal{C}}} N(w)^p ds \le 2^{j(n-1)} \int_{A_0 \cap \dot{\mathcal{C}}} \mathcal{M}(N(w))^p ds \quad (j < -1)$$

which sums.

For $j > 1$ there is a number J depending only on $\mathcal{C}$ and α so that nontangential regions based at $Q \in A_j$ for $j > J$ have empty intersection with L_0 so that the same argument leads to a finite sum. □

Theorem 6. *Let $K \subset \mathbb{R}^4$ be a finite homogeneous 4-complex such that $\dot{K}$ is a manifold. Let $\kappa^0 \in \dot{K}$, take κ^0 to be the origin and let $\mathcal{C} = \mathcal{C}(\kappa^0) = \mathcal{C}(\kappa^0, \operatorname{Lk}(\kappa^0, K))$. Take $g \in L^p(\dot{\mathcal{C}})$ compactly supported in $\operatorname{St}(\kappa^0, \dot{K})$. Then there is a unique harmonic function u in $\operatorname{Int} \mathcal{C}$ decaying to zero at infinity so that $\frac{\partial u}{\partial N} = g$ a.e. on $\dot{\mathcal{C}}$ and there is a constant C depending only on K so that*

$$\int_{\dot{\mathcal{C}}} N(\nabla u)^p ds \le C \int_{\dot{\mathcal{C}}} |g|^p ds \tag{9.2}$$

where N denotes a nontangential maximal function for $\mathcal{C}$ whenever $|p - 2| < \epsilon$, $\epsilon > 0$ depending on K.

Proof. Let $\Omega_0 = \operatorname{Int} \operatorname{St}(\kappa^0, K)$ and define nested domains $\Omega_j = \{2^j x \mid x \in \Omega_0\}$, $j = 1, 2, \ldots$. First consider $g \in L_0^p(\dot{\mathcal{C}})$. By Corollary 2 and scale invariance of the Ω_j there is a harmonic function u_j in Ω_j so that $\frac{\partial u_j}{\partial N} = g$ a.e. on $\partial \Omega_j$ and

$$\int_{\partial \Omega_j} N_j(\nabla u_j)^p ds \le C \int_{\dot{\mathcal{C}}} |g|^p ds \tag{9.3}$$

uniformly in $j = 1, 2, \ldots$ where N_j denotes the nontangential maximal function over approach regions of Ω_j. By properties of nontangential regions (see [VV03]) for the polyhedral setting), 9.3, and exhausting by compact subdomains (see Lemma 4.11 of [JK82]), one can pass to a subsequence of the $u_j + c_j$, where c_j is an appropriate constant, that converges uniformly on all compact subdomains to a harmonic function u, and so that 9.2 holds by monotone convergence as the subdomains expand.

By (9.2) and properties of nontangential regions it then follows that

$$|\nabla u(x)| \leq C \operatorname{dist}^{-3/p}(x, \dot{\mathcal{C}}) \tag{9.4}$$

so that by the fundamental theorem of calculus $\lim_{t\to\infty} u(tx)$ exists for any $x \in \operatorname{Int}\mathcal{C}$. That this limit is independent of x follows by the fundamental theorem and (9.4) used over paths (see Theorem 8.1 of [VV03]) connecting tx with ty so that $|u(tx) - u(ty)| \leq Ct^{1-3/p} \operatorname{dist}(x, y)$. Consequently by subtracting a constant, u decays at infinity. To show that u has the correct Neumann data define $v_k = u - u_k$. It suffices to show that for any Ω_l

$$\lim_{k\to\infty} \int\limits_{\bar{\Omega}_l \cap \dot{\mathcal{C}}} N^p(\nabla v_k)\, ds = 0 \tag{9.5}$$

Denote by $[\nabla v_k]_j$ the restriction of ∇v_k to $\partial\Omega_j \cap \operatorname{Int}\mathcal{C}$. Let J denote the number from the proof of Lemma 9. By scale invariance of Lemma 9

$$\begin{aligned} I_{j,k} := \int_{\partial\Omega_j \cap \operatorname{Int}\mathcal{C}} |\nabla v_k|^p ds &\leq C'_p \int_{\dot{\mathcal{C}} \cap \bar{\Omega}_{j+1} \setminus \bar{\Omega}_{j-1}} \mathcal{M}^p(N([\nabla v_k]_j)) ds \\ &\leq C_p \int_{\dot{\mathcal{C}} \cap \bar{\Omega}_{j+1} \setminus \bar{\Omega}_{j-1}} \mathcal{M}^p(N_k(\nabla v_k)) ds \end{aligned} \tag{9.6}$$

for $k > j + J$. By the Hardy-Littlewood maximal function theorem, (9.2) and (9.3) (with $j = k$), $\sum_{j=l+1}^{k-J}$ is uniformly bounded in $k > J + l$. Consequently there exists $j = j(k)$ with $l < j \leq k - J$ so that $I_{j,k}$ vanishes as k increases, i.e. the Neumann data for v_k on $\partial\Omega_j$, $j = j(k)$, decreases to zero. By the scale invariance of the Ω_j, scale invariance of Neumann estimates and Corollary 2, (9.5) and therefore pointwise limits for u follow. If now u is a solution in $\mathcal{C}$ with zero Neumann data and nontangential maximal function of the gradient in L^p, then the same argument shows that the gradient must vanish everywhere.

To obtain solutions in L^p it now suffices to consider $g = 1$ on $\operatorname{St}(\kappa^0, \dot{K})$, $g = 0$ elsewhere on $\dot{\mathcal{C}}$, and let u_j be solutions in Ω_j with Neumann data g on $\bar{\Omega}_j \cap \dot{\mathcal{C}}$ and Neumann data a constant on $\partial\Omega_j \cap \operatorname{Int}\mathcal{C}$. (Actually any Neumann data on these latter sets that yields mean value zero and has L^p norm less than that of g.) Then (9.3) will hold as well as the rest of the existence argument. □

A consequence of uniqueness is invertibility of the classical layer potentials for the cones $\mathcal{C}(\kappa^0)$.

Theorem 7. *With $\mathcal{C}(\kappa^0)$ as in Theorem 6 and $2-\epsilon < p < 2+\epsilon$, the layer potentials $\pm\frac{1}{2}I + D^*$ are isomorphisms on $L^p(\dot{\mathcal{C}})$. The single layer potential provides the unique solution decaying at infinity to the Neumann problem with L^p-data and L^p-nontangential maximal function of the gradient.*

Proof. Let $g \in L^p(\dot{\mathcal{C}})$ be compactly supported and let u be the corresponding unique solution from Theorem 7. Using the proof of that theorem and its notations it follows there are solutions u_j in a subsequence of the Ω_j that converge pointwise to u and so that (9.3) holds. Each $u_j = \mathcal{S}_j f_j$ by the manifold condition and Theorem 4 where $\mathcal{S}_j$ denotes the single layer potential defined on Ω_j and $f_j \in L^p_0(\partial\Omega_j)$. By scale invariance and Theorem 4

$$||f_j||_p \leq C||g||_p \tag{9.7}$$

uniformly. Fixing any $x \in \operatorname{Int}\mathcal{C}$ and writing $u_j = v_j + w_j$ where w_j is obtained by restricting $\mathcal{S}_j$ to $\partial\Omega_j \cap \operatorname{Int}\mathcal{C}$, $w_j(x) \to 0$. Consequently

$$v_j(x) = \int_{\dot{\mathcal{C}}} \Gamma(x-Q) f_j(Q) ds(Q) \tag{9.8}$$

converges to $u(x)$ also. Since the restriction of the f_j to $\dot{\mathcal{C}}$ still satisfies (9.7) there is a weakly convergent subsequence in L^p, $f_j \rightharpoonup f$ with f also satisfying the inequality of (9.7). By (9.8) u is the single layer potential of f taken over $\dot{\mathcal{C}}$.

For noncompactly supported g define g_j to be the restriction of g to $\dot{\mathcal{C}} \cap \bar{\Omega}_j \setminus \bar{\Omega}_{j-1}$. There will be corresponding $f_j \in L^p(\dot{\mathcal{C}})$ so that (9.7) holds with g_j in place of g. Therefore the partial sums of the f_j will converge to an f in L^p norm, which will suffice to imply that $\mathcal{S}f$ will have Neumann data g. Thus $-\frac{1}{2}I + D^*$ and similarly $\frac{1}{2}I + D^*$ are onto. $\mathcal{S}f(x)$ decays at infinity by Hölder's inequality.

To show one-to-one, suppose $f \in L^p(\dot{\mathcal{C}})$ and $(-\frac{1}{2}I + D^*)f \equiv 0$. Then by the uniqueness of Theorem 7 $\mathcal{S}f \equiv 0$ in $\mathcal{C}$. Applying Lemma 9 as in the proof of Theorem 7 shows $\int_{\partial\Omega_j \cap \mathcal{C}^c} |\nabla \mathcal{S}f|^p ds$ vanishes as j increases where here the Ω_j are defined in the complement of $\mathcal{C}$. (The manifold condition insures this complementary cone meets the same conditions as $\mathcal{C}$.) When $p = 2$ therefore, the second inequality of Theorem 3, the vanishing of $\mathcal{S}f$ on $\dot{\mathcal{C}}$, and the scale invariance of the Ω_j imply that $\nabla\mathcal{S}f$ vanishes outside $\mathcal{C}$ as well. It follows that f is identically zero and $-\frac{1}{2}I + D^*$ is an isomorphism on $L^2(\dot{\mathcal{C}})$. Similarly for $\frac{1}{2}I + D^*$ and another use of Calderon's Corollary finishes the proof. □

10. Estimates by flattening

By definition Lipschitz domains can be locally flattened. Dahlberg and Kenig [JK82] used this property, together with Serrin and Weinberger's [SW66] decay estimates on solutions to divergence form elliptic equations, in order to reduce the study of harmonic functions in bounded domains to a study in unbounded annular regions of $\mathbb{R}^n$. Theorem 5 above says that a compact polyhedral domain of $\mathbb{R}^4$ can be locally flattened by a PL-homeomorphism. Recall the infinite cone of X with vertex v

$$\mathcal{C}(v, X) = \{v + t(x - v) \mid t \geq 0, x \in X\} \tag{10.1}$$

from section 9. The next lemma expresses what is meant here by locally flattening a polyhedron $|K^4| \subset \mathbb{R}^4$.

Lemma 10. *Let* $\kappa^0 \in \dot{K}^4$ *(a 3-manifold) and take* $X = \mathrm{St}(\kappa^0, K)$ *in* (10.1). *Then there is a PL-homeomorphism* $h : \mathcal{C}(\kappa^0, X) \to \overline{\mathbb{R}^4_+}$ *which is moreover bi-Lipschitz, i.e. there is a number* $m > 0$ *so that* $m|x-y| \leq |h(x)-h(y)| \leq m^{-1}|x-y|$ *for all* $x, y \in \mathcal{C}(\kappa^0, X)$.

Proof. Take κ^0 to be the origin and let the ball B, the homeomorphism H and the shelling $\{\sigma_j\}_{j=1}^N$ of $\bar{R}_\epsilon(\kappa^0)$ be as in Corollary 3. Then H has the homogeneity property and is a homeomorphism from $B \cap X$ onto a subset of σ_N that contains $\sigma_N \cap B'$ for a smaller concentric ball B'. Extend H to $H : \mathcal{C}(\kappa^0, X) \to \mathcal{C}(\kappa^0, \sigma_N)$ by $H(x) = t^{-1}H(tx)$ for t small enough depending on x. Since the boundary of $\mathcal{C}(\kappa^0, \sigma_N)$ can be realized as a graph over $\mathbb{R}^3$, projection parallel to the x_4-axis yields the PL-homeomorphism h from $\mathcal{C}(\kappa^0, X)$ to $\mathbb{R}^4{}_+$. The bi-Lipschitz property follows from that of H (which follows from the finiteness of K). □

Remark 12. By the preceding constructions it is not hard to see that h actually extends to a PL bi-Lipschitz homeomorphism of all of $\mathbb{R}^4$.

Recall the definition of atom. A bounded measurable function a defined on $\partial\Omega = |\dot{K}|$ is called an *atom* if

(i) it is supported in a surface ball $B(Q, r) \cap \partial\Omega$
(ii) $||a||_\infty \leq \left[\int_{B(Q,r)\cap\partial\Omega} ds\right]^{-1}$ and
(iii) $\int_{\partial\Omega} a ds = 0$.

The atomic Hardy space is defined by

$$H^1_{at}(\partial\Omega) = \{g \mid g = \sum_{j=1}^{\infty} \lambda_j a_j, a_j \ \text{ is an atom } \ j = 1, 2, \ldots \ \text{ and } \sum |\lambda_j| < \infty\}$$

with the norm

$$||g||_{at} = \inf\left\{\sum |\lambda_j| \mid g = \sum \lambda_j a_j, a_j \text{ an atom } j = 1, 2, \ldots\right\}$$

By (ii) H^1_{at} is a subspace of $L^1(\partial\Omega)$. If a harmonic function u has atomic Neumann data $\dfrac{\partial u}{\partial N} = a$ a.e. on $\partial\Omega$ then the key estimate to obtain is

$$||N(\nabla u)||_{L^1} \leq C$$

with C independent of a. When r in (i) and (ii) is larger than some fixed $r_0 > 0$, then the estimate follows by the L^2-theory above, C depending also on r_0. Thus it suffices to take r_0 so small that it may always be assumed $\operatorname{supp} a \subset \operatorname{St}(\kappa^0, \dot{K})$ for some $\kappa^0 \in \dot{K}$ with

$$\operatorname{dist}(\operatorname{supp} a, \operatorname{Lk}(\kappa^0, \dot{K})) > \frac{1}{3} \operatorname{dist}(\kappa^0, \operatorname{Lk}(\kappa^0, \dot{K}))$$

In this setting and following [DK87] we first obtain the key estimate in the corresponding infinite cones $\mathcal{C}(\kappa^0) = \mathcal{C}(\kappa^0, \operatorname{Lk}(\kappa^0, K))$.

Let u be any harmonic function in a cone $\mathcal{C}(\kappa^0)$ from Theorem 6 and let h be the biLipschitz map for $\mathcal{C}(\kappa^0)$ of Lemma 10. Define $v = u \circ h^{-1}$ in $\mathbb{R}^4_+$. Then (for any $2 - \epsilon < p < 2 + \epsilon$) v is in the space $W^{1,2}_{loc}(\mathbb{R}^4_+)$ and is a (weak) solution to

$$div(A \nabla v) = 0 \tag{10.2}$$

where the matrix $A = [a^{ij}]$ is formed from the Jacobian determinant Jh and the differential matrix h' by $A(x) = [(Jh)^{-1} h' h\prime^T] \circ h^{-1}(x)$ for $x \in \mathbb{R}^4_+$. Here $h\prime^T$ denotes the transpose matrix. Define the reflection $x^* = (x_1, x_2, x_3, -x_4)$. We also define A for $x \in \mathbb{R}^4_-$ by extending the definition of the a^{ij} by $a^{ij}(x) = a^{ij}(x^*)$ when $1 \le i, j \le 3$ or $i = j = 4$ and $a^{ij}(x) = -a^{ij}(x^*)$ otherwise. With these definitions it follows that the function $\tilde{v}(x) = v(x^*)$ satisfies $div(A \nabla \tilde{v}) = 0$ for $x \in \mathbb{R}^4_-$. The matrix A in $\mathbb{R}^4_+ \cup \mathbb{R}^4_-$ is symmetric, has bounded (piecewise constant) entries and satisfies the ellipticity conditions

$$\langle A(x)\xi, \xi \rangle \ge C(K)|\xi|^2 > 0$$

$\xi \in \mathbb{R}^4 \setminus \{0\}$, i.e. satisfies any condition imposed in [SW66].

Lemma 11. *Let $\mathcal{C}(\kappa^0)$ be as in Theorem 6, $p \ge 3/2$, and suppose u is harmonic in $\mathcal{C}(\kappa^0)$ with $N(\nabla u) \in L^p(\dot{\mathcal{C}})$. Then $u \in W^{1,2}_{loc}(\bar{\mathcal{C}})$.*

Proof. Let $v = u \circ h^{-1}$ as above where $h : \mathcal{C} \to \mathbb{R}^4_+$ is the bi-Lipschitz homeomorphism of Lemma 10. Let $D \subset \overline{\mathbb{R}^4_+}$ be any closed unit cube with one face contained in $\{x_4 = 0\}$ and let D_t be its translation by $t > 0$ in the x_4 direction. By the divergence theorem, justified by (9.2), and Hölder's inequality

$$\begin{aligned}\int_{D_t} |\nabla v|^2 dx &\le C \int_{D_t} \nabla v \cdot A \nabla v \, dx = C \int_{\partial D_t} vN \cdot A \nabla v \, ds \\ &\le C^2 \left(\int_{\partial D_t} |\nabla v|^p \right)^{1/p} \left(\int_{\partial D_t} |v|^{p'} \right)^{1/p'} \end{aligned} \tag{10.3}$$

Here C depends only on the ellipticity of A. For each t, v can be adjusted by an additive constant and Sobolev embedding invoked to bound (independently of t by the congruence of the ∂D_t) the p-norm by the p-norm of ∇v. When these latter norms are independent of t, the lemma follows by monotone convergence and the fact that h is bi-Lipschitz.

The p-norms of (10.3) are uniformly bounded by the p-norm of a nontangential maximal function of ∇v. This maximal function at each boundary point $h(Q)$ is dominated by $N_\alpha(\nabla u)(Q)$, if α is chosen small enough depending on the bi-Lipschitz constant. By an argument of [FS72] adapted to the polyhedral setting (Lemma 4.4 of [VV03]), the functions $N_\alpha(\nabla u)$ for all $0 < \alpha < 1$ are in L^p if any one is. □

Remark 13. The above standard argument is given here because it depends on the existence of h. In higher dimensions the result is not so clear. See §11.

Corollary 4. *Let u be as in Lemma 11 and let $G \subset \partial\mathbb{R}^4_+$ be a nonempty open set of $\mathbb{R}^3$. Suppose $\frac{\partial u}{\partial N}$ vanishes a.e. on $h^{-1}(G)$. Then $v = u \circ h^{-1}$, extended to be $\tilde{v}$ in $\mathbb{R}^4_-$, is a solution of* (10.2) *in the domain $\mathbb{R}^4_+ \cup \mathbb{R}^4_- \cup G$.*

The next theorem is the analogue of Theorem 2.12 of [DK87]. Because polyhedral domains have no local uniform Lipschitz nature it will be necessary to again exploit their property of being decomposable into geometrically similar domains.

Theorem 8. *Let $\mathcal{C} = \mathcal{C}(\kappa^0) = \mathcal{C}(\kappa^0, \mathrm{Lk}(\kappa^0, K))$ be as in Theorem 6 and $g \in H^1_{at}(\dot{\mathcal{C}})$. Then there is a unique harmonic function u modulo constants in* $\mathrm{Int}\,\mathcal{C}$ *so that $N(\nabla u) \in L^1(\dot{\mathcal{C}})$ and $\dfrac{\partial u}{\partial N} = g$ a.e. nontangentially on $\dot{\mathcal{C}}$. Moreover*

$$||N(\nabla u)||_{L^1(\dot{\mathcal{C}})} \leq C||g||_{H^1_{at}(\dot{\mathcal{C}})} \tag{10.4}$$

Proof. In order to obtain (10.4) it is enough to take $g = a$ an atom. By dilation invariance of estimates it suffices to consider $\mathrm{supp}(a) \subset B_1(Q) \cap \dot{\mathcal{C}}$ for some $Q \in \dot{\mathcal{C}}$ and $||a||_\infty \leq 1$. Let $R_0 = \mathrm{dist}(\mathrm{supp}(a), \kappa^0)$. Given $\kappa^0 \in \kappa^j \in \mathrm{St}(\kappa^0, \dot{K})$ let, by an abuse of notation, κ^j also denote $\mathcal{C}(\kappa^0, \kappa^j)$, i.e. a j-dimensional edge (j-face) of $\dot{\mathcal{C}}$. Suppose $Q \in \dot{\mathcal{C}}$ and let κ^1 be the closest 1-face to Q. It may be assumed for any $Q \in \dot{\mathcal{C}}$, not the vertex, that $\mathrm{dist}(Q, \kappa^0)$ is much larger than $\mathrm{dist}(Q, \kappa^1)$ by introducing a new triangulation (independent of Q) if necessary. There is a number d_0 depending only on $\mathcal{C}(\kappa^0)$ (and not $\mathrm{supp}(a)$) so that $R_0 > d_0$ implies that the distance of $\mathrm{supp}(a)$ to each 1-face other than κ^1 is greater than 1000.

We describe four geometric cases in which nested domains $\Omega_j \subset \Omega_{j+1}$ are defined so that in each case $\partial\Omega_0 \cap \dot{\mathcal{C}}$ contains the support of the atom, and the rings $\Omega_{j+1} \setminus \Omega_j$ are approximately a distance 2^j from Ω_0, i.e. a standard set up for atomic estimates is realized in each case.
Case I: $R_0 \leq d_0$.
Define $\Omega_j = \{x \in \mathrm{Int}\,\mathcal{C} \mid \mathrm{dist}(\kappa^0, x) < 2^{j+1}\max(R_0, 4)\}$, $j = 0, 1, 2, \ldots$.

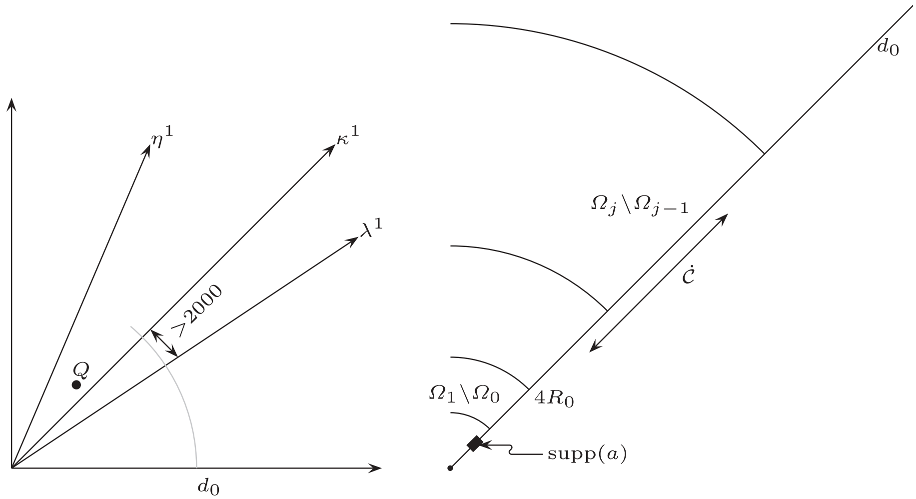

Fig. 15. 1-faces of $\dot{\mathcal{C}}$ with $d_0 > R_0 \approx \text{dist}(Q, \kappa^0) >> \text{dist}(Q, \kappa^1)$. Case I with $4 < R_0 < d_0$.

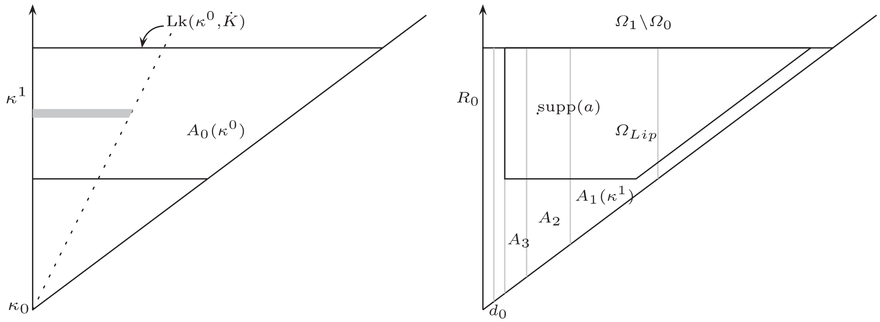

Fig. 16. Rescaled $\text{St}(\kappa^0, \dot{K})$ with range of supp(a) (gray) when $R_0 > d_0$.
Case II: $\text{St}(\kappa^0, K) = \Omega_0 \supset \Omega_{Lip} \supset \text{supp}(a)$ when $R_0 >> d_0$

When $R_0 > d_0$ redefine $\text{St}(\kappa^0, K)$ by dilating it so that $\text{supp}(a) \cap \{\frac{1}{4}\kappa^0 + \frac{3}{4}Q \mid Q \in \text{Lk}(\kappa^0, \dot{K})\} \neq \emptyset$. Recall $A_0(\kappa^0) = \{(1-t)\kappa^0 + tQ \mid Q \in \text{Lk}(\kappa^0, K) \text{ and } 2^{-1} \leq t \leq 1\}$ and the arches $A_j(\kappa^1) = \{x = (1-t)P + tQ \mid P \in \kappa^1, Q \in \text{Lk}(\kappa^1, K), 2^{-j-1} \leq t \leq 2^{-j}\}$, $j = 0, 1, 2, \ldots$. Let J denote the smallest j such that $\text{supp}(a) \cap A_j(\kappa^1) \neq \emptyset$.

Case II: $R_0 > d_0$ and $J = 0, 1,$ or 2.

Define $\Omega_0 = \text{Int}\,\text{St}(\kappa^0, K)$ and $\Omega_j = \{\kappa^0 + 2^j(x - \kappa^0) \mid x \in \Omega_0\}$, $j = 1, 2, \ldots$. Define $\Omega_{Lip} \subset A_0(\kappa^0)$ by beveling off each 1-simplex of the boundary containing κ^0 at level $t = 2^{-4}$ (as in Lemma 13).

When $J > 2$ define

$$\Omega_0 = \Omega_0(J) = \left\{(1-t)\kappa^0 + tQ \mid Q \in \mathrm{Lk}(\kappa^0, K), |t - \frac{3}{4}| < 2^{-J+1}\right\}$$
$$\cap \ \mathrm{Int} \bigcup_{l=J-1}^{\infty} A_l(\kappa^1) \tag{10.5}$$

geometrically similar domains as J varies.
Case III: $R_0 > d_0$, $J > 2$ *and* $\mathrm{dist}(\mathrm{supp}(a), \kappa^1) \le 10$.
Define Ω_j $(j = 1, \ldots, J-4)$ by (10.5) with $J - j$ in place of J. Define $\Omega_{J-3} = \mathrm{Int\,St}(\kappa^0, K)$ and Ω_j $(j \ge J-2)$ by dyadic scalings of $\mathrm{St}(\kappa^0, K)$.

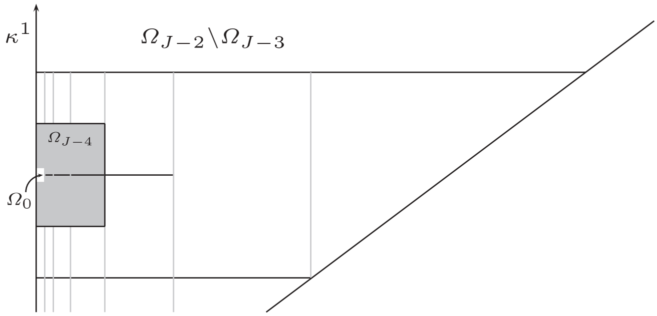

Fig. 17. Case III: $\mathrm{supp}(a) \subset \Omega_0$ and $\Omega_{J-3} = \mathrm{St}(\kappa^0, K)$ with $|\,\mathrm{supp}(a)| \sim |\Omega_0|$

Case IV: $R_0 > d_0$, $J > 2$ *and* $\mathrm{dist}(\mathrm{supp}(a), \kappa^1) > 10$.
Define the Ω_j as in Case III and define $\Omega_{Lip} = \Omega_0 \setminus \bigcup_{l=J+2}^{\infty} A_l(\kappa^1)$. Letting r_j denote the distance from $\Omega_{j+1} \setminus \bar{\Omega}_j$ to $\mathrm{supp}(a)$ and $\nu > 0$ the constant of [SW66] the estimate

$$\int_{\Omega_{j+1} \setminus \Omega_j} N(\nabla u)^2 ds \le C r_j^{1-n-2\nu} \tag{10.6}$$

follows in each of the cases with C independent of R_0 and $\mathrm{dist}(\mathrm{supp}(a), \kappa^1)$ as on p.445 of [DK87] (see also the remarks following this proof). Here the L^2 estimate in compact polyhedra of Theorem 3 is used. Corollary 4 justifies the use of [SW66]. In addition, for Cases II and IV the bounded domain estimates of [DK87] are used in the domains Ω_{Lip}. The complete proof follows pp.443–445, p.447 and the first two sentences of p.448 of [DK87]. □

We make some comments on the proof of [DK87] in order to isolate estimates that will be useful again, and to indicate alternate arguments that will be useful when solving problems in polyhedra for which invertible potentials may not be available.

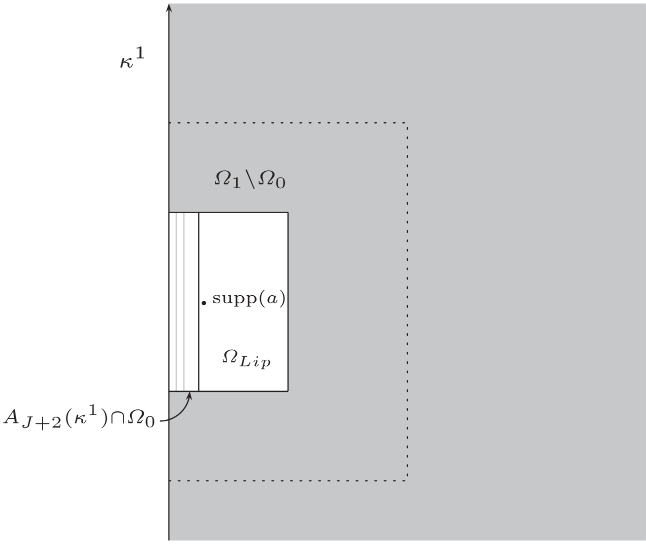

Fig. 18. Case IV: Ω_0 (white) with $|\operatorname{supp}(a)| << |\Omega_{Lip}| \sim |\Omega_0|$

C1 The standard Lipschitz domain device of estimating

$$|\nabla u(x)| \leq \fint_{B \cap \dot{\mathcal{C}}} N(\nabla u) ds \tag{10.7}$$

where B is centered on the boundary with radius like $R = \operatorname{dist}(x, \dot{\mathcal{C}})$, can be used in polyhedra (see [VV03] Lemma 4.4, for example). Thus when u is a solution with unit Neumann data, $|\nabla u(x)| \leq CR^{-3/2}$ by Jensen's inequality and Theorem 6. Integrating to infinity in the direction $\vec{\kappa^0 x}$ and subtracting from u the constant yields $|u(x)| \leq CR^{-1/2}$. The subtracted constant can be shown to be independent of x. Consequently $|u(x)| \leq C$ for all x with $\operatorname{dist}(x, \dot{\mathcal{C}}) \geq 1$ as on p. 443 of [DK87] without the single layer representation. (This also works for $\mathbb{R}^3$ by using $p < 2$ as in Theorem 6). The sub-mean value argument of [DK87] p. 443 now holds for the operator $L = div(A\nabla)$ where A is defined in $\mathbb{R}^4$ as above. See also the second and third inequalities on p.447.

C2 The analysis of the Serrin-Weinberger estimate on p.444 of [DK87] goes through for the operator L above. The sequence of six equalities on p.444 can be pulled back to $\mathcal{C}$ and the approximating domains of Lemma 5.3 [VV03] used in place of the parallel graph domains.

C3 For the uniqueness part of the argument on p.447, let w be harmonic in $\operatorname{Int} \mathcal{C}$ with $N(\nabla w) \in L^1$ and with vanishing normal derivative. Then there is no translation of w in $\operatorname{Int} \mathcal{C}$ that remains a solution in $\operatorname{Int} \mathcal{C}$. To proceed, designate some point on $\dot{\mathcal{C}}$ to be the origin of $\mathbb{R}^4$ and let ψ denote a smooth cutoff function $\psi(x) \equiv 1$ for $|x| < R$, $\psi \equiv 0$ for $|x| > 2R$, and $R|\nabla u| + R^2|\nabla\nabla u| \leq C$ where C can always be taken independent of R. For $y \in \mathbb{R}^4_+$ define $W(y) = W(R; y) = \psi \circ h^{-1}(y)\ w \circ h^{-1}(y)$ with h as in Lemma 10. For $t > 0$ and $y_4 \geq 0$ define $W_t(y) = W(y', y_4 + t)$ where $y' = (y_1, y_2, y_3)$. Define the matrix $A_t(y)$ for $y_4 \geq 0$ as W_t was defined. Next extend A_t to $y_4 < 0$ by the same similarity transformation used before Lemma 11 to extend A. Define $L_t = div(A_t\nabla)$ in $\mathbb{R}^4$ and denote by $G_t(x, y) \geq 0$ the fundamental solution for L_t in $\mathbb{R}^4$ [LSW63]. Extend by reflection $W_t(y) = W_t(y^*)$ for $y_4 < 0$.

Now by definition of fundamental solution, the Lipschitz continuity of W_t and changes of variables

$$
\begin{aligned}
&(w \circ h^{-1})_t(x) \\
&\quad = \int_{\mathbb{R}^4} A_t(y)\nabla_y G_t(x, y) \cdot \nabla W_t(y)\, dy \\
&\quad = \int_{\{y_4=0\}} \left[G_t(x, \cdot) + G_t(x^*, \cdot)\right] \\
&\qquad \times \left[(\psi \circ h^{-1})_t \frac{\partial}{\partial \nu_t}(w \circ h^{-1})_t + \frac{\partial}{\partial \nu_t}(\psi \circ h^{-1})_t (w \circ h^{-1})_t\right] ds(y) \\
&\qquad - \int_{\{y_4>0\}} \left[G_t(x, \cdot) + G_t(x^*, \cdot)\right] \\
&\qquad \times \left[(Jh)_t^{-1}((\Delta\psi) \circ h^{-1})_t (w \circ h^{-1})_t + 2A_t\nabla(\psi \circ h^{-1})_t \cdot \nabla(w \circ h^{-1})_t\right] dy
\end{aligned}
\tag{10.8}
$$

where $x \in h(B_R(0) \cap \operatorname{Int} \mathcal{C})$ and new uses of the subscript t stand for translation and ν_t represents the conormal derivative with respect to A_t in $\mathbb{R}^4_+$.

Because h is bi-Lipschitz, distances and volumes are essentially preserved so that the analysis of [DK87] holds as R goes to infinity. For the same reason the nontangential maximal function again justifies taking limits in t. The ellipticity constants of the L_t are uniform in t so that away from the poles the G_t are uniformly bounded. See [GW82]. A related lemma will be useful below.

Lemma 12. *Let $G_t(x, y)$ be the fundamental solutions for the operators $L_t = div(A_t\nabla)$ and let $G(x, y)$ be the fundamental solution for $L = div(A\nabla)$. Then there is a sequence of t's converging to zero so that $G_t(x, y)$ converges to $G(x, y)$ pointwise and weakly in $W^{1,p}(B_R(x))$ for all $R > 1$ and $1 \leq p < 4/3$.*

Proof. By the uniform ellipticity of the L_t, the resulting uniform bounds of [GW82] ((1.6), (1.12) and (1.38) pp. 3,4,9) and a diagonalization argument there is convergence of a sequence to some $G(x, y)$ in the manners described. That G is the fundamental solution for L follows first because for any C_0^∞ function

$$\phi(x) = \int A_t(y)\nabla_y G_t(x, y) \cdot \nabla\phi(y)\, dy \to \int A(y)\nabla G(x, y) \cdot \nabla\phi(y)\, dy$$

by the weak L^p convergence and the fact that A_t converges to A pointwise a.e. and in any L^q_{loc} $(q < \infty)$. Next, the pointwise convergence of the G_t shows $G \geq 0$, and the fact that G is in the above $W^{1,2}$ spaces allow one to conclude by uniqueness that G is the fundamental solution for L. □

C4 The following corollary to the proof of Theorem 8 will be useful.

Corollary 5. *Let* $\Omega = \operatorname{Int}\operatorname{St}(\kappa^0, K)$, *identify* κ^0 *with the origin, let a be an atom with the property that* $Q \in \operatorname{supp}(a)$ *implies* $2Q \in \operatorname{St}(\kappa^0, \dot{K})$. *Then there is a harmonic function u in* Ω *so that* $\frac{\partial u}{\partial N} = a$ *nontangentially and*

$$\int_{\partial\Omega} N(\nabla u)ds \leq C \tag{10.9}$$

with C independent of a.

Proof. By finiteness of K we take Ω to be of unit size. Let $\mathcal{C} = \mathcal{C}(\kappa^0)$ and let v be the solution with Neumann data a from Theorem 8. Let w be the L^2 solution with data $\frac{\partial v}{\partial N}$ on $\operatorname{Lk}(\kappa^0, K)$ and zero elsewhere on $\partial\Omega$. Define $u = v - w$. Estimate (10.9) follows because

$$\int_{\operatorname{Lk}(\kappa^0,K)} |\nabla v|^2 ds \leq C$$

independently of a. To see this latter estimate, choose $\theta > 0$ depending only on Ω so that the integral over points a distance greater than θ from $\dot{\mathcal{C}}$ is controlled by (10.7) and the $L^1(\dot{\mathcal{C}})$ bound on $N(\nabla u)$. The remainder of the integral is controlled by an integral of $N(\nabla u)^2$ over points of $\dot{\mathcal{C}}$ that are approximately a distance no more than θ from $\operatorname{Lk}(\kappa^0, \dot{\mathcal{C}})$, thus a unit distance from $\operatorname{supp}(a)$. This integral is just a scaling of estimate (10.6). □

Theorem 9. *Let* $K^4 \subset \mathbb{R}^4$ *be a finite homogeneous complex with connected boundary* $\dot{K}$ *with that is a 3-manifold. Let* $\Omega = \operatorname{Int}|K|$. *Given* $g \in H^1_{at}(\partial\Omega)$ *there is, up to constants, a unique harmonic function u in* Ω *such that*

(i) $\frac{\partial u}{\partial N} = g$ *a.e. nontangentially*
(ii) $N(\nabla u) \in L^1(\partial\Omega)$.

Further, there is a constant $C = C(K)$ *independent of g so that*

(iii) $\|N(\nabla u)\|_1 \leq C\|g\|_{H^1_{at}}$

Proof. To establish (iii) it suffices to consider $g = a$ an atom such that for any $\kappa^0 \in \dot{K}$ either: I. $\operatorname{supp}(a) \subset \{\kappa^0 + \frac{2}{3}(x - \kappa^0) \mid x \in \operatorname{St}(\kappa^0, \dot{K})\}$ or II. $\operatorname{supp}(a) \cap \{\kappa^0 + \frac{1}{2}(x - \kappa^0) \mid x \in \operatorname{St}(\kappa^0, \dot{K})\} = \emptyset$. Now the proof consisting of p.459 and most of p.460 from [DK87] may be adapted to use these two cases and establish (iii). See additional comments below.

For uniqueness in the bounded domain case here, we supply a different proof based on the representation in the middle of p.447 of [DK87] that was used for uniqueness in the unbounded case. See comment **C3** above.

Suppose u satisfies (i) and (ii) with $g = 0$. Let $\kappa^0 \in \dot{K}$ and take h as in Lemma 10. Let $L = div(A\nabla)$ be the elliptic operator in all of $\mathbb{R}^4$ derived from h as described above and let $v(x) = u \circ h^{-1}(x)$ for $x \in h(\mathrm{Int}\,\mathrm{St}(\kappa^0, K))$. Thus we may take v to be a $W^{1,2}_{loc}$ solution to $Lv = 0$ in a neighborhood of the origin of $\mathbb{R}^4_+$. We may suppose this neighborhood to be large. Let $G(x, y)$ denote the fundamental solution for L in $\mathbb{R}^4$. We may suppose the origin to be on $\mathrm{St}(\kappa^0, \dot{K})$ and $\mathrm{Lk}(\kappa^0, K)$ to be far from the origin. Let ψ denote a cutoff function as in comment **C3** above with $R = 1$. Define $W = \psi \circ h^{-1} u \circ h^{-1}$ and the remaining quantities of comment **C3**. Then by Lemma 12 the limit in t may be taken yielding

$$\begin{aligned} u \circ h^{-1}(x) = &\int_{\{y_4=0\}} \left[G(x,\cdot) + G(x^*,\cdot)\right] \frac{\partial}{\partial \nu}(\psi \circ h^{-1})\, u \circ h^{-1}\, ds \\ &- \int_{\{y_4>0\}} \left[G(x,) + G(x^*,)\right] \left[(Jh)^{-1}(\Delta\psi \circ h^{-1}) u \circ h^{-1}\right. \\ &\left. + 2A\nabla(\psi \circ h^{-1}) \cdot \nabla(u \circ h^{-1})\right]\, dy \end{aligned} \tag{10.10}$$

for $x \in h(B_1(0) \cap \Omega)$. From which it follows, by the bounds uniform in the poles of G [GW82] and support properties, that $u \circ h^{-1}$ is $W^{1,2}$ in a neighborhood of $h(0)$ in $\overline{\mathbb{R}^4_+}$. Now reflection yields a $W^{1,2}$ solution for $Lv = 0$ in a neighborhood of $\mathbb{R}^4$ so that $u \circ h^{-1}$ is Hölder continuous [DG57] [Nas58] there, so that u must be Hölder continuous in $\overline{\Omega}$. By the strong maximum principle for L, [GT77] p.198, applied in an open neighborhood of a maximum of $u \circ h^{-1}$ and the strong maximum principle for harmonic functions in Ω it follows that u is a constant. □

C5 The polyhedral L^2-Neumann theory of section 8 and integration by parts as in section 5 of [VV03] justify the integrations of pp. 459-460 [DK87] here.

C6 When proving Case II of Theorem 9 as on p.459 of [DK87], (10.10) and the bounds $G(x, y) \le C|x - y|^{2-n}$ yield

$$\sup_{B_1} |u| \le \int_{B_2} |u| + |\nabla u|$$

This coupled with the Cacciopolli calculation

$$\int_{B_{1/2}} |\nabla u|^2 \le \int_{B_1} |u||\nabla u|$$

and Poincaré, readily yields the reverse Hölder inequality on the bottom of p.459. (The boundary integral of (10.10) may be converted to solid integrals of ∇u and u by the fundamental theorem of calculus and averaging.)

C7 The case I argument of p.460 follows here from Corollary 5.

11. An example of Curtis and Zeeman.

In [CZ60] a certain 4-manifold with boundary $\overline{X}$ is identified that embeds as a finite simplicial complex in $\mathbb{R}^4$. This manifold is then doubly suspended into $\mathbb{R}^6$. Consequences for the polyhedral version of the Schoenflies theorem, under the assumption that a homology $(n-2)$-sphere doubly suspends to a topological sphere (homeomorphic to $\mathbb{S}^n$), are then discussed. The space $\overline{X}$ and many of its properties were originally investigated by M. H. A. Newman [New48] with further consequences and explanations in Curtis and Wilder [CW59]. The assumed double suspension theorem was anticipated then, but is now established by Edwards and Cannon [Can79]. Thus the double suspension of X meets the definition of polyhedral domain considered here. The purpose here is to make more explicit Newman's construction and to elaborate on its consequences for boundary value theory in compact polyhedra in dimensions greater than 4.

11.1. Newman's construction

To begin, a homogeneous 2-complex P^2 with a nontrivial fundamental group $\pi_1(P)$ is constructed in $\mathbb{R}^4$ as follows (see the top of p. 155 in [CW59]). In the xy-plane let A denote the triangle (1-complex without boundary) with vertices and orientation $(0,0)$, $(-1,1)$ and $(-1,-1)$. Let A^{-1} denote the opposite orientation. Let B similarly denote $(0,0)$, $(1,1)$ and $(1,-1)$. See Figure 20. Next the

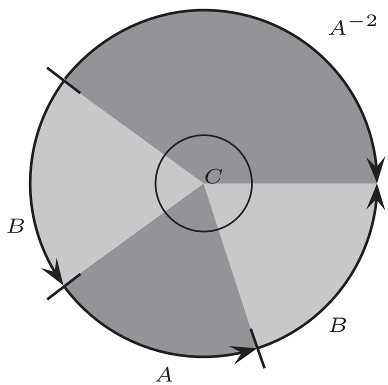

Fig. 19. The boundary of the disc identified to A and B

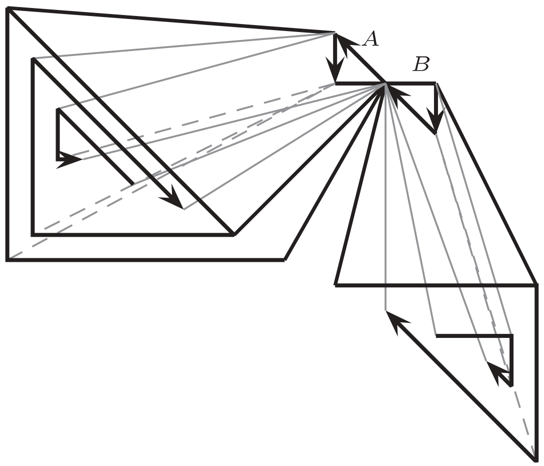

Fig. 20. The image of the torn annular region afixed to the figure eight formed by A and B

boundary of a 2-disc is divided into 5 arcs and each in succession is identified to A or B by the formula $A^{-2}BAB$ reading left to right. See Figure 19. A mapping (continuous) of the disc into $\mathbb{R}^4$ under these identifications is realized in several stages. It is helpful to first consider an annular subset of the disc that contains the disc boundary and then later to plug the hole. Also, one constructs as much as possible in $\mathbb{R}^3$ and then uses the 4th dimension to repair tears.

To these ends consider the oriented 1-complex (with six 1-simplexes) given by the successive vertices $(0, 0, 1)$, $(-7, -7, 1)$, $(-7, 6, 1)$, $(-1, 0, 1)$, $(-6, -5, 1)$, $(-6, 4, 1)$ and $(-2, 0, 1)$ in $\mathbb{R}^3$. Form the convex hull of each of these 1-simplexes with the side of the triangle A to which it is parallel. The result is a 2-complex of six 4-gons attached to A twice in A^{-1} orientation. Similarly attach the complex of three 1-simplexes $(1, 0, 1)$, $(6, 5, 1)$, $(6, -4, 1)$, $(2, 0, 1)$ to B; $(-3, 0, 1)$, $(-5, 2, 1)$, $(-5, -1, 1)$, $(4, 0, 1)$ to A; and $(3, 0, 1)$, $(5, 2, 1)$, $(5, -1, 1)$, $(4, 0, 1)$ to B. See Figure 20. Each of the 4-gons can be subdivided into two 2-simplexes. Next each of the four 1-complexes defined above in the plane $z = 1$ is joined to the point $C = (0, 0, 2)$ resulting in fifteen more 2-simplexes. Altogether there is now a complex of 45 2-simplexes that can be considered to be contained in the image of the 2-disc under the proposed identification map, with C the image of center of the 2-disc. This complex would be the image of the 2-disc except that it is torn in four places. The eight edges of these tears have all been put in the xz-plane. Let O denote the origin. The first 1-complex defined above in the z = 1 plane has lead to a subcomplex of eighteen 2-simplexes. Its beginning edge is O,$(0, 0, 1)$,C and its ending edge is O,$(-2, 0, 1)$,C. This last edge is to be connected to the beginning edge O,$(1, 0, 1)$,C of the subcomplex that was first attached to B. This can be done in $\mathbb{R}^4$ without unwanted intersections by joining these two edges to the point $(0, 0, 1, 1)$ in (x, y, z, w)-coordinates. The edges

O,(2, 0, 1),C and O,(−3, 0, 1),C are joined to (0, 0, 1, 2); O,(−4, 0, 1),C and O,(3, 0, 1),C to (0, 0, 1, 3). The last edge O,(4, 0, 1),C and O,(0, 0, 1),C must be joined in the $w < 0$ half-space. The point (0, 0, 1, −1) serves the purpose.

The resulting complex of 61 2-simplexes P_+ is the desired image of the disc under the identification of its boundary by $A^{-2}BAB$. Except for its boundary P_+ resides in the half space $z > 0$ of $\mathbb{R}^4$. In the same way a complex of 101 2-simplexes P_- is realized in $z \leq 0$ as the image of another 2-disc under the identification $B^{-5}ABAB$. The union $P_+ \cup P_-$ is the 2-complex P.

Let $\langle A, B\rangle$ denote the free group on the two generators A and B (see pp.199–203 of [Lee00]). Let R denote the smallest normal subgroup of $\langle A, B\rangle$ containing the relators $A^{-2}BAB$ and $B^{-5}ABAB$. Then the quotient group $\langle A, B\rangle/R$ is what is meant by the group presentation $\langle A, B \mid A^{-2}BAB, B^{-5}ABAB\rangle$ (p.201 [Lee00]). It is the Seifert-VanKampen theorem ([Lee00] Chapter 10, [Mas91] Chapter IV, [Bre93]) that shows that $\pi_1(P^2)$ has precisely this group presentation. As elements of the quotient group the relators satisfy the relations $A^{-2}BAB = 1$, and $B^{-5}ABAB = 1$, or equivalently

$$A^3 = B^5 = (AB)^2 \tag{11.1}$$

Newman remarks then that $\pi_1(P)$ must be nontrivial because these relations are a subset of relations that can be used to present the alternating group A_5 (icosahedral group), i.e. the argument is $0 \neq A_5 \subset \pi_1(P)$. Indeed, according to [CM57] p.14, A_5 is presented by the relations $A^3 = B^5 = (AB)^2 = 1$ where A is the permutation cycle (245) and B the cycle (12345). In fact the group with relations (11.1) is precisely the group of order 120 that is the fundamental group of Poincaré's original homology sphere [Sav02] p.2.

Thus P is a finite homogeneous 2-complex realized in $\mathbb{R}^4$ with nontrivial fundamental group. By subdivision, P may be considered to be a subcomplex of a combinatorial triangulation of $\mathbb{R}^4$.

Next homology groups for P are calculated. Because P is path connected $H_0(P) = \mathbb{Z}$ the integers ([Mas91] p.148). To compute $H_1(P)$ [New48] and [CW59] observe that $\pi_1(P)$ is a perfect group. This means that it is its own *commutator subgroup*. If G is a group its commutator subgroup $[G, G]$ is the smallest subgroup that contains $\{xyx^{-1}y^{-1} \mid x, y \in G\}$. It is a theorem that $[G, G]$ is normal and that $G/[G, G]$ is abelian, the *abelianization* of G. Therefore, if G is perfect its abelianization is the trivial group. That $\pi_1(P)$ is perfect is seen in terms of group presentation by including, as Newman does, the commutator relation $AB = BA$ to (11.1). It follows that $A = B = 1$ so that the abelianization of $\pi_1(P)$ is the trivial group. By a theorem of W. Hurewicz for path connected spaces ([Bre93] p.174, [Lee00] p.305) $H_1(P)$ is the abelianization of $\pi_1(P)$, i.e. trivial.

That $H_2(P)$ is trivial is the next observation. Newman argues this directly by showing that the group of 2-cycles $Z_2(P)$ is trivial. ($H_2(P)$ is the quotient group

$Z_2(P)/B_2(P)$ where $B_2(P)$ is the group of 2-dimensional *bounding cycles*, i.e. boundaries of singular 3-chains in P, [Mas91] pp.158-160. $Z_2(P)$ would be all singular 2-chains with empty boundary.) First he claims that a 2-chain can only be a cycle if it is of the form $mP_+ + nP_-$ for $m, n \in \mathbb{Z}$. The boundary of such a chain is then $m(-A+2B)+n(-3B+2A)$. (Here the boundary of P_+ is written additively $-2A+B+A+B = -A+2B$ and similarly for P_-.) The only solution for m and n making this sum empty is $m = n = 0$. Thus $Z_2(P)$ and $H_2(P)$ are trivial.

Remark 14. The above argument appears to be a repetition of the $H_1(P) = 0$ argument. This can be seen by considering an alternative argument using the machinery of CW-complexes. See [Mas91] p.226 for a precise description of CW constructions by the attaching of open n-cells. Here P is obtained from the figure-eight $A \cup B$ by attaching two open 2-cells (the interiors of the two 2-discs). The important theorem is Theorem 2.1 on p.227 of [Mas91]. It states that if a space X^* is obtained from a space X by attaching a number, say k, of n-cells, then the *relative homology groups* $H_q(X^*, X)$ (pp.169–170 [Mas91]) are (1) trivial when $q \neq n$ and (2) isomorphic to $\oplus^k \mathbb{Z}$ when $q = n$. This allows one to compute homology groups from the *homology sequence of the pair* (X^*, X). It is a theorem that every such sequence is *exact* (the image of each map is identical to the kernel of the following map). See p.171 [Mas91] for the definition and theorem. In the case here the relevant part of the homology sequence is

$$0 \rightarrow H_2(A \cup B) \rightarrow H_2(P) \rightarrow H_2(P, A \cup B) \rightarrow H_1(A \cup B) \rightarrow H_1(P) \rightarrow 0$$

The zeros arise from conclusion (1) above. Conclusion (2) yields $\mathbb{Z} \oplus \mathbb{Z}$ for the relative homology. It is known that $H_1(A \cup B) = \mathbb{Z} \oplus \mathbb{Z}$ and that $H_2(A \cup B) = 0$. (Since $A \cup B$ is obtained from a single point by attaching two open 1-cells Massey's Theorem 2.1 also yields these two conclusions by the same procedure. See also Lemma 4.1 on p.233 of [Mas91] for the last.) It has been established above that $H_1(P) = 0$. Thus the sequence reduces to

$$0 \rightarrow H_2(P) \rightarrow \mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z} \rightarrow 0$$

By exactness the third map is onto, so by linear algebra it is also 1-1. Thus by exactness the second map is the zero map and by exactness again $H_2(P) = 0$. Nothing here has anything to do with *how* the discs were attached to $A \cup B$ except for the previous fact $H_1(P) = 0$.

In sum P is a finite homogeneous 2-complex realized as a subcomplex of a combinatorial triangulation T of $\mathbb{R}^4$ that satisfies $\pi_1(P) \neq 0 = H_1(P) = H_2(P)$ and $H_0(P) = \mathbb{Z}$.

The 4-complex $\overline{X}^4$ is constructed as in [CW59] to be a *regular neighborhood* of P. (Newman [New48] uses the same method but in $\mathbb{R}^5$.) This means that $\overline{X}$

will be a finite homogeneous 4-complex in $\mathbb{R}^4$ that contains P in its interior, will be a PL 4-manifold with boundary M^3 and will have P as a (PL) deformation retract (strong deformation retract). See the definition of regular neighborhood on p.20 and Lemma 1.6.2 on p.18 of [Rus83]. Since P will be contained in the interior X of $\overline{X}$ it will also be a deformation retract for X. Therefore it will follow by [Mas91] p.45 that the fundamental groups of $\overline{X}$ and X are both isomorphic to the fundamental group of P. In addition, the deformation retraction induces, via the inclusion map, isomorphisms of the homology groups $H_n(\overline{X})$ and $H_n(X)$ with $H_n(P)$ for $n = 0, 1, 2, \ldots$ [Mas91] p.168. A more precise description of the construction of $\overline{X}$ will make the calculations of the corresponding groups for M accessible.

Whitehead's theorem [Rus83] p.20 states that $\overline{X}$ can be constructed by first taking a barycentric (or derived) subdivision T' of T in order to ensure that any simplex of T' with all its vertices in the induced subdivision P' of P is contained in P' (P' is said to be *full* in T'). Taking a second derived subdivision T'', $\overline{X}$ is defined to be the minimal subcomplex of T'' containing all simplexes of T'' that intersect P'' (i.e. $|P|$).

An alternative to taking a second subdivision (see chapter 3 [RS72]) is to define a simplicial map f from T' to the unit interval [0, 1] by defining $f = 0$ on all vertices of P', $f = 1$ on all other vertices of T' and then extending piecewise linearly. Because P' is full in T' $f^{-1}(0) = P'$. Using this [New48] and [CW59] set $\overline{X} = f^{-1}([0, 1/2])$, $M = f^{-1}(1/2)$, introduce new vertices on the 1-simplexes of T' where they intersect M and employ elementary starring as in Lemma (6) to obtain a subdivision T'' in which $\overline{X}$ again has all the properties mentioned above that characterize a regular neighborhood of P.

The required retraction $\phi : \overline{X} \to P$ and homotopy to the identity ([Mas91] p.45) are induced by f. This is done by observing that every point $x \in \overline{X} \setminus P$ is determined by T' to be in the join of a unique smallest simplex from P and a unique smallest simplex from $f^{-1}(1)$ so that there are unique $p \in P, q \in f^{-1}(1)$ and $0 < t \leq 1/2$ so that $x = (1-t)p + tq$. Then $\phi(x) = p$ and $\phi(p) = p$. Also if $\sigma \in T'$ (not T'') then $\phi(|\sigma| \cap |\overline{X}|) = |\sigma \cap P'|$.

As in Proposition I.2 of [CW59] it can now be shown that $\pi_1(M) \neq 0$. Consider a closed path γ in P'. For each edge (1-simplex) σ_P in γ one may choose an edge σ_M in M so that $\phi(\sigma_M) = \sigma_P$. This is possible because by fullness there are at least two vertices in $\mathrm{Lk}(\sigma_p, T') \cap f^{-1}(1)$ that join with σ_P to give rise to an edge in M. Let σ_P, τ_P be edges in γ with $\sigma_P \cap \tau_P = v$ a vertex. These edges in M might be disconnected. Let v_σ be the vertex of σ_M that is mapped by ϕ to v and similarly define v_τ. By construction of T'' both v_σ and v_τ are contained in $\mathrm{Lk}(v, T'')$ a PL 3-sphere. Because P is also full in T'' no 2-simplex of P is contained in this sphere, i.e. P cannot separate the sphere. Therefore v_σ and v_τ can be connected by a path γ_v in $\mathrm{Lk}(v, T'') \subset \overline{X}$ that does not intersect P'. In fact this path is in M since vertices of $\mathrm{Lk}(v, T'')$ are either in P' or M.

Now ϕ does not necessarily map γ_v to v. However, every vertex of γ_v comes from $|\,\mathrm{St}(v, T')|$ which ϕ maps to $\mathrm{St}(v, P')$. This latter star contains $\phi(\gamma_v)$ and is contractible to v. In this way the edges σ_M and τ_M are connected by a γ_v in M so that the image of the union under ϕ homotopically deforms in P to $\sigma_P \cup \tau_P$. It has therefore been shown that every path γ in P gives rise to a path γ_M in M so that $\phi(\gamma_M)$ homotopically deforms to γ in P. Thus the homomorphism, induced by ϕ, between the fundamental groups of M and P is onto. Therefore $\pi_1(M)$ is at least as large as $\pi_1(P)$, i.e. $\pi_1(M) \neq 0$.

The homology calculations for M^3 are done as in [CW59] by applying the Mayer-Vietoris exact sequence (Theorem 5.1 on p.207 of [Mas91]), Alexander duality (Theorem 6.6 on p.374 of [Mas91]), and some simplifying observations about cohomology. By regular neighborhood theory (see parts 2. and 3. of Theorem 1.6.4 [Rus83]) $\overline{X}$ is compactly contained in the interior of another regular neighborhood, $f^{-1}([0, 3/4])$ for example, $\overline{X'}$ and it in turn in the interior of another $\overline{X''}$, all homeomorphic and having the same relations and properties with respect to P. One now considers these embedded in $\mathbb{S}^4$ and defines Y to be the complement of $\overline{X}$. Then M' (homeomorphic to M) is a deformation retract for $Y \cap X''$. In this way the requirements in the Mayer-Vietoris theorem for open sets and nonempty intersections can be met even as one writes

$$\to H_{n+1}(\mathbb{S}^4) \to H_n(M) \to H_n(X) \oplus H_n(Y) \to H_n(\mathbb{S}^4) \to$$

By the known homologies of the sphere, exactness and the above computations for X it follows that $H_n(M) \approx H_n(Y)$ for $n = 1, 2$. Alexander duality yields $H_1(Y) \approx H^2(X)$ and $H_2(Y) \approx H^1(X)$ (see the top of p.372 [Mas91] for a clarifying note). These last are cohomologies rather than the homologies already computed. It is the purpose of the Universal Coefficient Theorem for cohomologies (Theorem 4.4 and Corollary 4.6, pp. 314–315 [Mas91]) to express them back in terms of homologies. Here this yields the (split) exact sequence

$$0 \to Ext(H_{n-1}(X), \mathbb{Z}) \to H^n(X) \to Hom(H_n(X), \mathbb{Z}) \to 0 \qquad (11.2)$$

Hom indicates the group of homomorphisms from $H_n(X)$ into $\mathbb{Z}$. For $n = 1$ or 2 this must be the trivial group. Again Massey makes the helpful comment (p.313) that if the first group in the Ext functor is free abelian then the group indicated by Ext is trivial also. It follows that $H_1(M) = H_2(M) = 0$. (See also page 100 of [RS72]). Since M is connected $H_0(M) = \mathbb{Z}$. A general result is that any compact, connected, orientable n-manifold N has $H_n(N) = \mathbb{Z}$ and $H_q(N) = 0$ for all $q > n$ (see comment (e) on p.148 [Mas91]) which finishes the homology computations for M. The above general result follows from Poincaré duality (Theorem 4.1, p.360 [Mas91] and bottom p.358) and the (split) exact sequence (11.2) again.

In sum, the homology of the 3-manifold M is the homology of $\mathbb{S}^3$, but M is not simply connected. Such manifolds are variously termed homology spheres or Poincaré manifolds. The double suspension $\Sigma^2 M^3$ that was the object of study in

[CZ60] can be constructed in $\mathbb{R}^6$ as a finite homogeneous 5 complex by first joining each simplex of M to the points $(0, 0, 0, 0, \pm 1)$ in $\mathbb{R}^5$ (the suspension points) and the repeating the procedure in $\mathbb{R}^6$. The same can be done to $\overline{X}^4$ so that $\Sigma^2 M$ is the boundary of the finite homogeneous 6-complex $\Sigma^2 \overline{X}$. If the suspension points of $\Sigma \overline{X}$ are removed one is left with an open ended cylinder that has $\overline{X}$ as a deformation retract. Therefore, the interior of $\Sigma \overline{X}$ has X as a deformation retract. Likewise, by two stages, the interior of $\Sigma^2 \overline{X}$ has X as a deformation retract, and consequently shares its properties of being connected but not simply connected.

However, the boundary $\Sigma^2 M$ is homeomorphic to $\mathbb{S}^5$. This fact follows from theorems of J. W. Cannon in this case because X^4 is not contractible. (See p.108 of [Can79] where the theorem is proved and the differences between the cases considered by R. D. Edwards and Cannon are discussed.)

Remark 15. ΣM is not a manifold. If it were, then a neighborhood of a suspension point would be homeomorphic to $\mathbb{R}^4$. This neighborhood could then be taken to be contained in the star of the suspension point by considering the image of a small ball of $\mathbb{R}^4$ under the homeomorphism. Removing the suspension point from the neighborhood would result in a simply connected domain since this is true for $\mathbb{R}^4$. Therefore the star with the suspension point removed would also be simply connected because any path can be rescaled in order to fit into the neighborhood. But the link of the suspension point is a deformation retract of the star sans suspension point. Therefore the link would be simply connected. But the link here is M which is not simply connected. More generally, n-manifold implies simply connected vertex links when $n > 2$. See the parenthetic remark on p.121 of [Thu97].

Putting $K^6 = \Sigma^2 \overline{X}$, K is a finite homogeneous 6-complex in $\mathbb{R}^6$. Its interior is a domain and its boundary is a manifold, in fact a sphere. But the domain is not homeomorphic to the open ball because it is not simply connected; neither is $|K|$ homeomorphic to the closed ball. Therefore K cannot be a manifold with boundary whereas all such constructions in $\mathbb{R}^4$ by [Moi52] must be. The failure of K to be manifold with boundary under these circumstances follows from a consequence of M. Brown's generalized Schoenflies Theorem due to Cantrell ([Rus83] p.49 and [Bin83] p.61). (Manifold with boundary implies local collar implies collar implies ball).

11.2. Some consequences for boundary value problems

11.2.1. Obtaining atomic estimates in compact polyhedral domains of $\mathbb{R}^n$ that have manifold boundaries becomes not so clear when $n \geq 5$. The local flattening, as in §10, cannot be done in general, given the example in §11.1, by definition of manifold with boundary.

When failing to satisfy the topological manifold with boundary condition such domains a fortiori cannot be bi-Lipschitz domains. They are however NTA domains because that property follows when the boundary is a manifold [VV03].

A definition of Lipschitz domain that generalizes Morrey's bi-Lipschitz definition is found in [Maz85] p.19. There the defining local homeomorphisms of the domain have a much more tenuous relation with the boundary. Each boundary point of a bounded domain Ω is required to have a neighborhood U of $\mathbb{R}^n$ so that $U \cap \Omega$ maps quasi-isometrically onto an open half-ball. A quasi-isometric map $T : \Omega_1 \to \Omega_2$ between domains is defined to have a Jacobian that preserves its sign in Ω_1, be a homeomorphism onto Ω_2 , and to satisfy

$$\limsup_{x\to y} \frac{|x-y|}{|Tx-Ty|} + \limsup_{x\to y} \frac{|Tx-Ty|}{|x-y|} \le L \tag{11.3}$$

for each $y \in \Omega_1$ and some L independent of y. Such a domain can be called *infinitesimally Lipschitz*. An infinitesimally Lipschitz domain need not have a manifold boundary. The slit domain $\Omega = \{z \in \mathbb{C} \mid |z| < 1, -\pi < arg(z) < \pi\}$ and the map $T(z) = |z|^{1/2} z^{1/2}$ provide an example. Any open set containing $\overline{\Omega}$ serves as the neighborhood U. T does not satisfy a Lipschitz condition on Ω while it does satisfy (11.3).

Infinitesimally Lipschitz domains that do have manifold boundaries need not be NTA or, more particularly, uniform domains. The above example can be modified by requiring $|arg(z)| < \pi - f(|z|) < \pi$ with, for example, f' continuous and $f'(0) = f(0) = 0$ so that and inward pointing cusp can replace the slit.

On the other hand, while compact polyhedral domains with manifold boundary are NTA domains, the Curtis-Zeeman example shows they need not be infinitesimally Lipschitz. No neighborhood of a suspension point of the Curtis-Zeeman polyhedron when intersected with the interior can be simply connected. (Any nontrivial path in the nonsimply connected interior can be homotopically mapped to the suspension point neighborhood.) More generally, the infinitesimally Lipschitz and bi-Lipschitz conditions are equivalent in this context.

Theorem 10. *Let the domain $\Omega \subset \mathbb{R}^n$ be the interior of a finite homogeneous simplicial complex K^n with a boundary that is a manifold. If Ω is an infinitesimally Lipschitz domain then it is a bi-Lipschitz domain and therefore K is a manifold with boundary.*

Proof. Let U be a neighborhood of the boundary point P so that $T : U \cap \Omega \to \operatorname{Int} \mathbb{B}^n$ is a quasi-isometry. T^{-1} immediately extends to a Lipschitz map from $\mathbb{B}^n$ onto the closure of $U \cap \Omega$ because of the convexity of $\mathbb{B}^n$. By showing that T extends continuously to a subset of $\partial\Omega$, the extended T^{-1} will be shown to be 1:1 and a bi-Lipschitz homeomorphism will be obtained.

Because Ω is a uniform domain [VV03] there is a constant C depending only on K so that whenever $x, y \in \Omega$ there is a piecewise linear arc in Ω with endpoints x and y that has arc length bounded by $C|x - y|$. This together with

(11.3) and the Rademacher-Stepanov theorem is enough to show that whenever $x_j \to Q \in U \cap \partial\Omega$, arcs connecting x_j and x_k, for j and k large enough, are contained in U and consequently $T(x_j)$ will converge to a unique $T(Q)$. □

Thus compact polyhedral domains with manifold boundaries but with closures that are not manifolds with boundary are neither Lipschitz, nor bi-Lipschitz, nor infinitesimally Lipschitz domains.

11.2.2. Working with just the boundary of a compact polyhedral domain also becomes problematic because of the above example. Defining a method of continuity as in §6 by homeomorphically mapping a polyhedral boundary piecewise linearly to the boundary of a Lipschitz polyhedron is not possible in general.

To see this suppose $h : \Sigma^2 M \to \partial\Omega_{Lip}$ is a PL-homeomorphism onto the boundary of a Lipschitz polyhedron. By radial projection the new link of the suspension point v, in the new triangulation induced by h, is homeomorphic to ΣM, i.e. the link of v is, by Remark 15, not a manifold. However, h maps this link and star of v isomorphically ([Gla70] pp.12–13) onto the star of $h(v)$ in $\partial\Omega_{Lip}$. Therefore this homeomorphic image is given as a graph over $\mathbb{R}^5$ and can be parallel projected homeomorphically to a starlike image in $\mathbb{R}^5$ with respect to the point $h(v)$. Thus ΣM is seen to be homeomorphic to $\mathbb{S}^4$ by radial projection in $\mathbb{R}^5$, a contradiction. Another way of saying this is that the boundary of any Lipschitz polyhedron is a combinatorial manifold, while $\Sigma^2 M$ is not.

It is interesting to note that attempting the method of continuity by mapping a polyhedral boundary to the boundary of a Lipschitz polyhedron with Lipschitz maps instead of PL is also not possible. By a theorem of L. C. Siebenmann and D. P. Sullivan ([SS79] p.504), any locally finite simplicial complex that is a Lipschitz n-manifold with respect to its barycentric metric has the property that the link of any simplex has the same homotopy as a sphere of the appropriate dimension. Here $\Sigma^2 M$ is finite and its barycentric metric is uniformly equivalent to the induced metric from $\mathbb{R}^6$. (See the remark on p.95 of [Gla70].) Lipschitz n-manifold means that every point admits a neighborhood that can be mapped homeomorphically to $\mathbb{R}^n$ by a locally Lipschitz map (or to Int $\mathbb{B}^n$ by a uniformly Lipschitz map). If $\Sigma^2 M$ could be mapped as proposed then it would be a Lipschitz manifold. The link of any 1-simplex from the suspension circle is M which does not have the homotopy of $\mathbb{S}^3$.

11.2.3. Juha Heinonen directed us to a lemma of O. Martio, S. Rickman and J. Väisälä [MRV71] p.9 which when applied here says that if a closed subset of zero Hausdorff $(n-2)$-measure is removed from $\mathbb{S}^n$ then the remaining space is still simply connected. Consequently the Cannon-Edwards homeomorphism from $\Sigma^2 M$ onto $\mathbb{S}^5$ maps the suspension circle (a finite 1-complex) onto a set of positive Hausdorff 3-measure! This is because $\Sigma^2 M$ without the suspension circle is the cylinder $M \times (-1, 1)^2$ which is not simply connected. Again this shows that the homeomorphism cannot be Lipschitz.

11.2.4. As a contrast to Theorem 5, in dimensions greater than 4 it cannot be expected that the approximating Lipschitz domains of Theorem 12 or [VV03] have boundaries that are topologically homeomorphic to the boundaries of the polyhedral domains they approximate. As in 11.1 the interior of $\Sigma^2\overline{X}$ is not simply connected and thus contains a closed path that cannot be shrunk inside that domain. The beveling process that produces the approximating Lipschitz domain Ω_{Lip} can be done so as to contain the same path. That path cannot be shrunk inside the smaller domain. But Lipschitz boundaries can be collared, so if $\partial\Omega_{Lip}$ is homeomorphic to $\mathbb{S}^5$ it will follow by M. Brown's generalized Schoenflies that Ω_{Lip} is an open ball, a contradiction.

11.2.5. A perhaps more appealing example, from the point of view of analysis, may be derived from the Mazur 4-manifold (with boundary) [Maz61] which B. Mazur denotes W_Γ. W_Γ is a finite homogeneous 4-complex. By part(3) of Corollary 1 on p.224 of [Maz61] it embeds piecewise linearly in $\mathbb{R}^4$. Its boundary, denoted M_Γ, is a nonsimply connected homology 3-sphere. Among the ways in which it differs from Newman's construction, however, is the fact that $W_\Gamma \times I$ is piecewise linearly homeomorphic to I^5. Consequently $\Omega_\Gamma = \text{Int}\, \Sigma^2 W_\Gamma$ is topologically an open 6-ball while $\partial\Omega_\Gamma = \Sigma^2 M_\Gamma$ is again topologically a 5-sphere, but from these two facts alone it does not follow that $\Sigma^2 W_\Gamma$ is a 6-ball. Nor can the argument in the last paragraph of §11.1 be applied to show that it is not a 6-ball. Remark 15 holds for ΣM_Γ. The arguments of §§11.2.2 and 11.2.3 apply also to $\Sigma^2 M_\Gamma$. Thus $\partial\Omega_\Gamma$ is noncombinatorial and Ω_Γ is not any kind of Lipschitz domain. If one were to first bevel off the suspension circle the result would be the combinatorial ball $W_\Gamma \times I^2$. Consequently one could complete the beveling process by shelling in the manner of §§6 and 8, justified by Bruggesser and Mani [BM71], and the result, unlike §11.2.4, would be an approximating Lipschitz domain with boundary at least homeomorphic to $\partial\Omega_\Gamma$.

11.2.6. Our understanding is that it is unknown whether or not the double suspension of the Mazur manifold is a manifold with boundary. In addition we know of no example of a noncombinatorial manifold with boundary that can be constructed as a *finite* homogeneous n-complex in $\mathbb{R}^n$. (The Alexander horned 2-sphere together with its bounded complementary domain in $\mathbb{R}^3$ is a manifold with boundary with wild boundary. See [Bin83] p.41.) From the point of view of boundary value problems either an example or a nonexistence theorem would be welcome. By Remark 6 nonexistence holds for $n \leq 4$.

12. Appendix

12.1. Approximations by Lipschitz polyhedra

Here we provide two slight modifications of the Lipschitz approximations to polyhedral domains used in [VV03] Theorem 6.1. First we have a modification to beveling which provides a way to bevel off a different amount from the various boundary skeletons.

Let K be a finite homogeneous n-complex geometrically realized in $\mathbb{R}^n$ with $\Omega = \operatorname{Int}|K|$ a domain. Let $\epsilon = (\epsilon_0, \ldots, \epsilon_{n-2})$ with $0 < \epsilon_{n-2} < \epsilon_{n-3} < \cdots < \epsilon_1 < \epsilon_0 < 1/n$. For $\sigma^n \in K$ and $\sigma \cap \dot{K} = \emptyset$ we define $C_\epsilon(\sigma) = \sigma$. For $\sigma^n \in K$ and $\kappa^j \in \sigma \cap \dot{K}^{n-2}$ let $\lambda = \operatorname{Lk}(\kappa, \sigma)$. Define

$$R_\epsilon(\kappa, \sigma) = \{(1-t)P + tQ \mid P \in \kappa, Q \in \lambda, 0 \le t < \epsilon_j\}$$

and define $C_\epsilon(\sigma)$ to be σ with all such $R_\epsilon(\kappa, \sigma)$ removed. Then $C_\epsilon(\sigma)$ is a convex n-cell whose boundary consists of portions of $\dot{\sigma}$ along with points contained in the union over κ of the sets

$$h_\epsilon(\kappa, \sigma) = \{(1-\epsilon_j)P + \epsilon_j Q \mid P \in \kappa, Q \in \lambda, \}$$

The surface measures of these sets are $\epsilon_j^{n-1-j}\|\lambda^{n-1-j}\|(1-\epsilon_j)^j\|\kappa^j\|$, so no more than order ϵ_0. Set $\Omega_\epsilon = \operatorname{Int}\left(\bigcup_{\sigma\in K} C_\epsilon(\sigma)\right)$. Then $\partial\Omega_\epsilon \cap \Omega$ is contained in the union of the sets $h_\epsilon(\kappa, \sigma)$ above. To see this suppose that $\sigma^n, \tau^n \in K$ and $x \in \sigma \cap \tau = \eta$ has been removed in $R_\epsilon(\kappa^j, \sigma)$, we argue that x is removed from τ. If $x \in \dot{K}$ then $x \in \dot{K}^{n-2}$ and is removed. Otherwise $\eta \cap \kappa \neq \emptyset$ and $\eta \cap \lambda \neq \emptyset$ since $x \notin \lambda$ by construction and now $x \notin \dot{K}$ so $x \notin \kappa$. τ is the join of $\eta \cap \kappa$ and $\operatorname{Lk}(\eta \cap \kappa, \tau)$ and this link contains $\eta \cap \lambda$ so that x has the same unique representation in $R_\epsilon(\kappa^j, \sigma)$ as it does in $R_\epsilon(\eta \cap \kappa, \tau)$. The former representation yields $x = (1-a)P + aQ$ for $a < \epsilon_j$ which must also be the unique representation in the latter. Since $\eta \cap \kappa = \gamma^l$ with $l \le j$ we have $\epsilon_j \le \epsilon_l$ so $a < \epsilon_l$ and x is removed in $R_\epsilon(\eta \cap \kappa, \tau)$.

Theorem 11. *Let K be a finite homogeneous n-complex geometrically realized in $\mathbb{R}^n$ with $\Omega = \operatorname{Int}|K|$ a domain. Then Ω_ϵ defined above is a Lipschitz domain.*

Proof. The proof is the same as in [VV03] Theorem 6.1. □

In case $|\dot{K}|$ is a manifold we need not bevel off $\kappa^{n-2} \in \dot{K}$ since such simplexes κ are shared by exactly two $(n-1)$-simplexes from $\dot{K}$. Thus any point $x \in \kappa \setminus |\dot{K}^{n-3}|$ already has a neighborhood in which the boundary is a Lipschitz graph.

Theorem 12. *Let K be a finite homogeneous n-complex $(n \ge 3)$ geometrically realized in $\mathbb{R}^n$ with $\Omega = \operatorname{Int}|K|$ a domain and $|\dot{K}|$ a manifold. Then Ω_ϵ obtained by beveling off all $\kappa \in \dot{K}^{n-3}$, as above but now $\epsilon = (\epsilon_0, \ldots, \epsilon_{n-3})$, is a Lipschitz domain.*

Proof. As in [VV03], if $x \in \sigma^n$ is removed in $R_\epsilon(\kappa, \sigma)$ and $x \in \tau^n$ then also x is removed from τ. The only boundary points $x \in \partial\Omega_\epsilon$ for which a local Lipschitz representation of the boundary is not found by the argument in the proof of [VV03] Theorem 6.1 are those points $x \in \partial\Omega_\epsilon \cap (\dot{K}^{n-2} \setminus \dot{K}^{n-3})$ which are also in some bevel surface $h_\epsilon(\kappa, \nu^n)$. Now $x \in (\dot{K}^{n-2} \setminus \dot{K}^{n-3})$ implies that there is a unique $\gamma^{n-2} \in \dot{K}^{n-2}$ containing x and thus any n-simplex $\sigma^n \in K$ containing x must also have $\gamma \in \sigma$. The beveling process then produces a convex $(n-2)$-cell $C_\epsilon(\gamma)$ with nonempty interior containing x in its boundary. Let N_1 be a direction from x into the interior of $C_\epsilon(\gamma)$, for any $h_\epsilon(\kappa, \sigma)$ containing x let the normal to $h_\epsilon(\kappa, \sigma^n)$ into σ be $N(\kappa, \sigma)$ then $N_1 \cdot N(\kappa, \sigma) > \delta > 0$ for some fixed $\delta = \delta(K)$ depending on K. Let $\sigma^{n-1}, \tau^{n-1} \in \dot{K}$ be the two $(n-1)$-simplexes from $\dot{K}$ that contain γ^{n-2}, they exist by the manifold condition, and let their interior pointing normals be N_σ and N_τ. Then the direction of the sum $N_2 = N_\sigma + N_\tau$ provides a direction from x into Ω_ϵ. Since $N_1 \cdot N_\sigma = 0$ and $N_1 \cdot N_\tau = 0$ we have $N_1 \cdot N_2 = 0$, therefore $\frac{1}{\delta}N_1 + N_2$ provides a direction into Ω_ϵ from any point in $\partial\Omega_\epsilon$ near x. That the boundary is a locally a graph then follows as in the proof of [VV03] Theorem 6.1. □

The next lemma shows that the crowns of §3 are Lipschitz domains.

Lemma 13. *For $\kappa^0 \in \dot{K}$ fixed, the interior of the crown, see* (3.1)*,*

$$A = \tilde{A}_0(\kappa^0) \setminus \bigcup_{\kappa^1 \in \mathrm{St}(\kappa^0, \dot{K})} \tilde{A}(\kappa^1, \kappa^0; 0, \theta(\kappa^1, \kappa^0))$$

is a Lipschitz domain.

Proof. Let $x \in \partial A$ and let κ^0 be the origin. Recall that $A \subset \mathrm{St}(\kappa^0, K)$. Write $\theta = \theta(\kappa^1, \kappa^0)$. If $x \in \mathrm{Int}\,\mathrm{St}(\kappa^0, K)$ then we have the cases

i) $|x| = 1$ or $|x| = 1/2$ and x is not in any conical surface around any κ^1 (i.e. in local coordinates about any κ^1, $\dfrac{x_4}{|x|} < \cos\theta$). Then in a small ball centered at x the boundary of A is simply a spherical cap.

ii) $1/2 < |x| < 1$. Then a small ball centered at x intersects the boundary in a single conical surface. (By definition of the angles θ, all conical surfaces in a crown are a positive distance apart.)

iii) $|x| = 1/2$ or $|x| = 1$ and x is in a conical surface. Let N_x be the normal to the cone at x (points toward κ^1), and N_r the normal to the sphere that is also outer to A. The sum provides a direction that is transverse to the boundary in a neighborhood of x so that the boundary is locally a Lipschitz graph in this direction.

When $x \in \partial A \cap \dot{K}$, we have the cases

1) x is not in any conical or spherical surface. Then either $x \in \mathrm{Int}\,\sigma^3$ for a unique $\sigma^3 \in \dot{K}$, or there are exactly two simplexes $\sigma^3, \tau^3 \in \mathrm{St}(\kappa^0, \dot{K})$ with

$\sigma^3 \cap \tau^3 = \gamma^2$ and $x \in \operatorname{Int} \gamma^2$. For if x were in three 3-simplexes of $\operatorname{St}(\kappa^0, \dot{K})$ then the manifold condition (see Lemma 8.7 [VV03]) would imply that x is in a 1-simplex containing κ^0. But all of these have been removed. Likewise when $x \in \gamma^2$ it is in the interior of γ^2 because two of the 1-simplexes of $\dot{\gamma}^2$ contain κ^0 and are removed while the third is in $\operatorname{Lk}(\kappa^0, K)$ and has been removed by the sphere. So in this latter case the sum of the normals to σ^3 and τ^3 provides a direction in which the boundary is locally a Lipschitz graph. In the first case the boundary is locally flat.

2) $|x| = 1$ or $|x| = 1/2$, x is not in any conical surface and either $x \in \sigma^3$ or $x \in \sigma^3 \cap \tau^3 = \gamma^2$. In the latter case, the normal to the sphere at x is in the affine hull of γ^2 (since $\kappa^0 \in \gamma^2$) and so is perpendicular to the normals for σ^3 and τ^3. The sum of the three normals provides a direction in which the boundary is locally a Lipschitz graph. The former case is simpler.
3) x is in one of the spherical surfaces, one of the conical surfaces and either $x \in \operatorname{Int} \sigma^3$ or $x \in \operatorname{Int}(\sigma^3 \cap \tau^3) = \operatorname{Int} \gamma^2$. Let the conical surface correspond to κ^1. Let N_c be the normal to the conical surface, N_s the normal to the spherical surface, N_σ the normal to σ and N_τ the normal to τ. Because x is on the conical surface, any simplex that contains both x and κ^0 must also contain κ^1 by the way θ was defined. Therefore in the latter case $\kappa^1 \in \gamma^2$. The triangle $x\kappa^1$ is contained in γ^2 so that N_c and N_s are in the affine hull of γ^2. They are also perpendicular to each other. Now, N_σ and N_τ are perpendicular to both N_c and N_s so that the sum of all four of these normals provides a direction in which the boundary is locally the graph of a Lipschitz function in a neighborhood of x. The first is case simpler.
4) x is as in case 3 but not on a spherical surface. The argument is as for case 3.

This completes the proof. □

The same arguments also establish.

Corollary 6. *Each sector (see §3)* $\tilde{A}(\kappa^1, \kappa^0; \theta_1, \theta_0)$ *is a Lipschitz domain when* $\theta_1 > 0$.

12.2. Sobolev spaces

When Ω is the domain above the graph of a compactly supported Lipschitz function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ ($|\phi(x) - \phi(y)| \leq M|x - y|$ for all $x, y \in \mathbb{R}^{n-1}$), Sobolev spaces $W^{1,p}(\partial\Omega)$ of functions with weak first derivatives in $L^p(\partial\Omega)$, $1 \leq p \leq \infty$, can be defined by flattening. One can say that $f \in W^{1,p}(\partial\Omega)$ if and only if

$$\tilde{f}(x) = f(x, \phi(x)) \in W^{1,p}(\mathbb{R}^{n-1})$$

The later space is defined by the requirement that there exist functions $\tilde{g}_j \in L^p(\mathbb{R}^{n-1})$ so that for all $\tilde{\psi} \in C_0^\infty(\mathbb{R}^{n-1})$

$$\int_{\mathbb{R}^{n-1}} \frac{\partial}{\partial x_j} \tilde{\psi} \tilde{f} dx = -\int_{\mathbb{R}^{n-1}} \tilde{\psi} \tilde{g}_j dx \quad 1 \leq j \leq n-1 \tag{12.1}$$

In this case an equivalent definition without flattening is that $f \in L^p(\partial\Omega)$ is in $W^{1,p}(\partial\Omega)$ if there exist functions $g_{j,k} \in L^p(\partial\Omega)$ so that for all $\psi \in C_0^\infty(\mathbb{R}^n)$

$$\int_{\partial\Omega} (N_j D_k - N_k D_j)\psi f ds = -\int_{\partial\Omega} \psi g_{j,k} ds \quad 1 \leq j < k \leq n \tag{12.2}$$

Here $N = (N_1, \dots, N_n)$ is the outer unit normal vector to Ω and $(D_1, \dots, D_n)$ is the gradient operator for $\mathbb{R}^n$. In terms of Lipschitz functions ϕ

$$\sqrt{1 + |\nabla\phi(x)|^2} N = (\nabla\phi(x), -1) \text{ and } ds = \sqrt{1 + |\nabla\phi(x)|^2} dx$$

To show that (12.1) and (12.2) yield the same function spaces one flattens (12.2) and makes use of the fact that the Lipschitz functions ϕ can be approximated uniformly by a sequence of smooth functions ϕ_m (mollifications of ϕ) so that $\nabla\phi_m \to \nabla\phi$ pointwise a.e. and in L^p norm ($p < \infty$), and $\|\nabla\phi_m\|_\infty \leq M$ for all m. It follows that a.e. $-N_n \tilde{g}_j = g_{j,n}$ for $j = 1, \dots, n-1$ and that the $\binom{n}{2}$ functions $g_{j,k}$ satisfy $\binom{n-1}{2}$ compatibility conditions

$$g_{j,k} = \frac{\partial\phi}{\partial x_j} g_{k,n} - \frac{\partial\phi}{\partial x_k} g_{j,n} \quad 1 \leq j < k \leq n-1$$

An isomorphism between the respective normed spaces follows.

Now let $K^n \subset \mathbb{R}^n$ be a finite homogeneous n-complex so that Int $K = \Omega$ is a domain and $\dot{K}$ is a manifold. We define $W^{1,p}(\dot{K})$ for $1 \leq p \leq \infty$ by using condition (12.2). With the compatibility

$$N_l g_{j,k} = N_k g_{j,l} - N_j g_{k,l} \quad j, k, l = 1, \dots, n$$

and the norm

$$\|f\|_{1,p} = \|f\|_p + \sum_{1 \leq j < k \leq n} \|g_{j,k}\|_p \tag{12.3}$$

$W^{1,p}(\dot{K})$ is a Banach space. Let $Lip(1, \dot{K})$ denote the Banach space of Lipschitz functions f defined on $\dot{K}$ for which there is an $M < \infty$ so that (i) $\|f\|_\infty \leq M$ and (ii) $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in \dot{K}$. The smallest M, the Lipschitz constant, that works in (i) and (ii) may be taken to be the norm of f. The Whitney extension theorem (which holds for any closed set of $\mathbb{R}^n$) [JW84] p.47, says that $Lip(1, \dot{K})$ extends continuously to $Lip(1, \mathbb{R}^n)$ and that $Lip(1, \mathbb{R}^n)$ restricts to $Lip(1, \dot{K})$ (i.e. $Lip(1, \mathbb{R}^n)|_{\dot{K}} = Lip(1, \dot{K})$) with the bounds on both the extension and restriction operators independent of $\dot{K}$. In addition $Lip(1, \mathbb{R}^n)$

is isomorphic to $W^{1,\infty}(\mathbb{R}^n)$ and any function in the latter space can be modified on a set of measure zero to belong to the former, [Ste70] p.173. The directional derivatives $N_j D_k - N_k D_j$ are tangential derivatives in the interior of each $(n-1)$-simplex $\sigma \in \dot{K}$. Denote the standard unit basis vectors of $\mathbb{R}^n$ by $e_1, \dots, e_n$. Define $T_{jk} = T^{\sigma}_{jk} = -N_k e_j + N_j e_k$ where $N = N(\sigma)$ is the above normal direction to σ in $\mathbb{R}^n$. If a function f is differentiable in the $(n-1)$-dimensional Euclidean space $\text{Int}\,\sigma$ at a point x then we will write $T_{jk} \cdot \nabla f(x)$ for its directional derivative and $\nabla_T f$ for its gradient in $\text{Int}\,\sigma$. With K as above

Theorem 13. *$Lip(1, \dot{K})$ and $W^{1,\infty}(\dot{K})$ are isomorphic when $\dot{K}$ is a manifold.*

Proof. First let $f \in Lip(1, \dot{K})$ and consider any simplex $\sigma^{n-1} \in \dot{K}$. Let $h(\sigma)$ denote the $(n-1)$-dimensional hyperplane of $\mathbb{R}^n$ containing σ. By extending f to $\mathbb{R}^n$ and restricting to $h(\sigma)$ it follows by the Whitney extension theorem and the theorem of Denjoy, Rademacher and Stepanov [Ste70] p.250 that f is differentiable a.e. $(\mathbb{R}^{n-1})$ on $h(\sigma)$ with $\|\nabla_T f\|_\infty \le M$. Our goal is then to show that g_{jk} in (12.2) may be supplied by $T_{jk} \cdot \nabla f$.

By mollifying f in $h(\sigma)$ as above, the Gauss divergence theorem is justified in each $(n-1)$-simplex of $\dot{K}$ so that by dominated convergence in σ and uniform convergence in $\dot{\sigma}$

$$\int_{\dot{K}} T_{jk} \cdot \nabla\psi f\, ds = -\int_{\dot{K}} \psi T_{jk} \cdot \nabla f\, ds + \sum_{\sigma^{n-1} \in \dot{K}} \int_{\dot{\sigma}^{n-1}} T^{\sigma}_{jk} \cdot N(\dot{\sigma}) \psi f\, ds_{n-2}$$

Here ds_{n-2} is Lebesgue measure on $\dot{\sigma}$ and $N(\dot{\sigma})$ is the outer unit normal to the domain $\text{Int}(\sigma)$ in $h(\sigma)$. Thus $f \in W^{1,\infty}(\dot{K})$ if the summation vanishes.

Since $\dot{K}$ is a manifold each $(n-2)$-simplex $\kappa \in \dot{K}$ appears precisely twice in the summation. (See Lemma 8.7 [VV03] for example. More generally as long as K is a homogeneous n-complex it follows (p.17 [Gla70]) that the mod-2 boundary of $\dot{K}$ is empty, i.e. each κ appears an *even* number of times in the summation.) Let σ and τ denote the two $(n-1)$-simplexes of $\dot{K}$ so that $\sigma \cap \tau = \kappa$. Then with the normals restricted to κ

$$T^{\sigma}_{jk} \cdot N(\dot{\sigma}) + T^{\tau}_{jk} \cdot N(\dot{\tau}) = 0 \tag{12.4}$$

To see this confine κ to the set $\{(0, x_2, \dots, x_{n-1}, 0)\} \subset \mathbb{R}^n$. Let $\sigma = \sigma^0\kappa$ where σ^0 has coordinates $(s_1, s_2, \dots, s_{n-1}, 0)$ with $s_1 > 0$, and let $\tau = \tau^0\kappa$ where τ^0 has coordinates $(t_1, \dots, t_n)$ with $t_n = 0$ only if $t_1 < 0$. Then we may take $N(\sigma) = e_n$ and $N(\tau) = (t_1^2 + t_n^2)^{-1/2}(t_n e_1 - t_1 e_n)$, while $N(\dot{\sigma})$ restricted to κ is $-e_1$ and $N(\dot{\tau})$ restricted to κ is $-(t_1^2 + t_n^2)^{-1/2}(t_1 e_1 + t_n e_n)$. Now the T_{jk} may be computed and (12.4) follows.

Next suppose $f \in W^{1,\infty}(\dot{K})$. Then (12.2) implies that f satisfies the requirements to be in the Sobolev space $W^{1,\infty}(\text{Int}\,\sigma)$ for each $\sigma^{n-1} \in \dot{K}$, [Ste70] p.180, where $\text{Int}\,\sigma$ is a domain of $\mathbb{R}^{n-1} = h(\sigma)$. (Any compactly supported test function for Int σ can be extended to a test function for $\mathbb{R}^n$ with support still intersecting $\dot{K}$ only in σ.) Thus by Stein's $p = \infty$ result [Ste70] p.181, f extends to $W^{1,\infty}(\mathbb{R}^{n-1})$ and so can be modified on a set of $(n-1)$-dimensional measure zero to be in $Lip(1, \mathbb{R}^{n-1})$ thence $Lip(1, \sigma)$. Call the modifications $\tilde{f}(\sigma)$.

If $\sigma^{n-1} \cap \tau^{n-1} = \kappa^{n-2}$ then the respective modifications agree on the interior points of κ. This follows first because the pointwise derivatives of the modifications must agree a.e. $(\mathbb{R}^{n-1})$ with the respective g_{jk} on the interiors of the σ and τ by uniqueness of weak derivatives in Euclidean space. Second, given any point $x \in \operatorname{Int}\kappa$ the manifold condition implies that for all r small enough the balls $B(x, r)$ intersect $\dot{K}$ only in σ and τ. Thus for any T_{jk} transverse to κ integration by parts as above, (12.2) and (12.4) yield

$$0 = \int_\kappa [\tilde{f}(\sigma) - \tilde{f}(\tau)]\psi \, ds_{n-2}$$

for all ψ supported in these balls, establishing the claim. Now by continuity and the fact that all boundary stars are barycenter connected [VV03] the modifications must agree at all points in common yielding a modification on all of $\dot{K}$ which we again call f. The Lipschitz constant M for f will be bounded by a constant depending on K (e.g. see the quantities of section 8 [VV03]) times $\|f\|_{1,\infty}$ because of the finiteness of K. □

Remark 16. The inclusion $W^{1,\infty} \subset Lip(1, \dot{K})$ requires the manifold condition and cannot be obtained by using only that the mod-2 boundary of $\dot{K}$ is empty. For example, if 4 facets have intersection κ the definition (12.2) and the above argument lead to the existence of 2 continuous functions each defined and continuous on a pair of the facets but not necessarily equal on κ. See §5.2.

Another natural way to define Sobolev spaces $1 \le p < \infty$ on $\dot{K}$ is to define them as the completion in the norms (12.3) of $Lip(1, \dot{K})$. The two methods yield the same spaces when $\dot{K}$ is a manifold.

Lemma 14. *Let K^n be as above with $\dot{K}$ a manifold and $f \in W^{1,p}(\dot{K})$, $1 \le p \le \infty$. Let $\epsilon > 0$. Then there exists a constant C depending only on K such that for a.e. $y, t \in \dot{K}$ satisfying $|y - t| < \epsilon$*

$$|f(y) - f(t)| \le C \left[\int_{B_{C\epsilon}(y)\cap\dot{K}} \frac{|\nabla_T f(z)|}{|z-y|^{n-2}} ds(z) + \int_{B_{C\epsilon}(t)\cap\dot{K}} \frac{|\nabla_T f(z)|}{|z-t|^{n-2}} ds(z) \right]$$

Proof. It suffices to take ϵ very small. Then there is a simplex $\kappa \in \dot{K}$ such that $y, t \in \operatorname{St}(\kappa, \dot{K})$, κ is a simplex of largest dimension for which this is true, and $\operatorname{dist}(y, \kappa)$ is of order ϵ. (See, for example, the quantities of §8 [VV03].) Suppose y is contained in the $(n-1)$-simplex σ and likewise t in another τ. Consider only the case when $\sigma \cap \tau$ is not an $(n-2)$-simplex. By the manifold condition there is a shortest sequence of $(n-1)$-simplexes of $\operatorname{St}(\kappa, \dot{K})$, $\sigma_1, \ldots, \sigma_N$, such that

$\sigma_j \cap \sigma_{j+1}$, $\sigma \cap \sigma_1$, and $\sigma_N \cap \tau$ are all $(n-2)$-simplexes (see Lemma 8.8 [VV03] for example). There is a shortest piecewise linear path through this sequence of simplexes connecting the barycenters of each of the $(n-2)$-simplexes with the barycenters of the $N+2$ $(n-1)$-simplexes. Choose the *origin* in κ so that it is of order ϵ from y. Then the barycenter path may be scaled inside $\mathrm{St}(\kappa, \dot{K})$ to be in an approximately ϵ-neighborhood of the origin. An $(n-2)$-disc with radius like the distance of y to the rescaled barycenter of σ can be centered at the latter point so the line segments from y to the center and the barycentric path both meet the disc at an angle at least $\pi/4$. An $(n-2)$-disc with radius like ϵ can be centered at the rescaled barycenter of $\sigma \cap \sigma_1$ and contained in that simplex. Similar discs of about ϵ radius can be placed at all the rescaled barycenters except possibly that of τ. Now connect y by the line segments to the first disc and then each point of that disc, in the nicest diffeomorphic manner, to the next (it does not matter at this first stage that the resulting paths might cross), etc. until each path is similarly connected to t. $\dot{K}$ is always locally Lipschitz (a wedge) about the union of these paths since they stay away from the $(n-3)$-skeleton. Consequently the Sobolev function, in a neighborhood of the paths, admits classical smooth approximations by locally flattening to $\mathbb{R}^{n-1}$. Applying the fundamental theorem of calculus along each path, integrating the results over a disc of radius ϵ and taking the limit of the approximations yields the lemma. The constant depends on the fact that there are only a finite number of stars, unscaled barycenter paths, etc. □

Remark 17. The proof of the preceding lemma relies on the properties: (a) each $(n-2)$-simplex of $\dot{K}$ is shared by exactly two $(n-1)$-simplexes from $\dot{K}$ and (b) $\mathrm{St}(\kappa, \dot{K})$ is barycenter connected for every simplex $\kappa \in \dot{K}$, see Lemma 8.8 [VV03]. Therefore the lemma is also true for some polyhedra K^n with nonmanifold boundaries $\dot{K}$, as the single suspension $\Sigma M = \dot{K}$ in Remark 15.

The family of paths constructed in the proof can almost be derived from the curve families on S. Semmes's generalization of simplicial complexes. See Theorem B.6 [Sem96] pp. 274–275. There it is assumed that upper gradients in the manner of Heinonen and Koskela [Hei01] exist (Theorem B.10 pp. 275-276). For boundary value problems it is still the definition of Sobolev space in terms of weak derivatives (12.2) and (12.3) that is most useful, as §5.2 indicates. That our paths are kept away from the $(n-3)$-skeleton is therefore used to show these weak derivatives form an upper gradient.

Theorem 14. *Let K be as above with $\dot{K}$ a manifold. Then $Lip(1, \dot{K})$ is dense in $W^{1,p}(\dot{K})$, $1 \le p \le \infty$.*

Proof. By Theorem 13 $Lip(1, \dot{K}) \subset W^{1,p}(\dot{K})$. Given a function in the latter space it is possible to regularize it. Choose a radial decreasing function $\psi \in C_0^\infty(\mathbb{R}^n)$ supported in the unit ball with integral equal to 1 over any hyperplane through the origin. For $\epsilon > 0$ put

$$\psi_\epsilon(x) = \epsilon^{1-n}\psi(x/\epsilon) \tag{12.5}$$

For all x and y in an approximately ϵ-neighborhood of $\dot{K}$ define

$$\psi_\epsilon^K(x, y) = \frac{\psi_\epsilon(x - y)}{\int_{\dot{K}} \psi_\epsilon(x - t)ds(t)}$$

This function is Lipschitz near the boundary and allows x and y derivatives to be interchanged modulo a nice term when one defines the *regularization* of f

$$f_\epsilon(x) = \int_{\dot{K}} \psi_\epsilon^K(x, y) f(y)ds(y)$$

for $x \in \dot{K}$. The (j, k)-derivative (12.2) of f_ϵ becomes the regularization of $g_{j,k}$ plus a remainder term

$$\int_{\dot{K}} (T_{j,k}^x \cdot \nabla_x + T_{j,k}^y \cdot \nabla_y)\psi_\epsilon^K(x, y) f(y)ds(y) \tag{12.6}$$

Using the definition of ψ_ϵ^K for the x-derivatives and Theorem 13 for the y-derivatives, it is seen that the integral over $\dot{K}$ in the variable y of the integral kernel (12.6) is zero. The integral kernel vanishes when both x and y are taken on $\dot{K}$ outside an ϵ-neighborhood of the boundary $(n-2)$-skeleton. For each fixed $x \in \dot{K}$ it vanishes in y outside an ϵ-ball centered at x. For $\epsilon = 1$ the kernel may be dominated by $\Psi(x - y)$ where Ψ is defined like ψ so that in general the kernel is dominated by $\epsilon^{-1}\Psi_\epsilon(x - y)$ where Ψ_ϵ is as in (12.5). Denote by $\dot{K}(n - 2, \epsilon)$ an ϵ-neighborhood of the $(n - 2)$-skeleton of $\dot{K}$. Then (12.6) is dominated by

$$\epsilon^{-1} \int_{\dot{K}(n-2,\epsilon)} \Psi_\epsilon(x - y) \left| f(y) - \fint_{B_\epsilon(x)\cap\dot{K}} f(t)ds(t) \right| ds(y)$$

The ball may be doubled and centered at y, Lemma 14 applied and the integrals in t taken, resulting in

$$\int_{\dot{K}(n-2,\epsilon)} \Psi_\epsilon(x - y) \Bigg[\epsilon^{-1} \int_{B_{C\epsilon}(y)\cap\dot{K}} \frac{|\nabla_T f(z)|}{|z - y|^{n-2}} ds(z) + \fint_{B_{C\epsilon}(y)\cap\dot{K}} |\nabla_T f(z)|ds(z) \Bigg] ds(y)$$

Raising this to the $p < \infty$, using Jensen's inequality and integrating in x and then y yields

$$C \int_{\dot{K}(n-2,C\epsilon)} |\nabla_T f(z)|^p ds(z)$$

which vanishes in ϵ.

Since the regularizations of f and the $g_{j,k}$ converge as usual the theorem follows. □

Local and global Poincaré inequalities on $\dot{K}$ a manifold follow from Lemma 14. The global one also follows by the standard contradiction argument.

Corollary 7. *Let K^n be as above $\dot{K}$ a manifold and $f \in W^{1,p}(\dot{K})$ for $1 \le p < \infty$ then for $\epsilon > 0$ (with a new constant C depending only on K)*

$$\left(\fint_{B_\epsilon(x)\cap\dot{K}} \left| f(y) - \fint_{B_\epsilon(x)\cap\dot{K}} f(t)dt \right|^p dy \right)^{1/p} \le C\epsilon \left(\fint_{B_{C\epsilon(x)}\cap\dot{K}} |\nabla_T f(y)|^p dy \right)^{1/p}$$

and

$$\left(\fint_{\dot{K}} \left| f(y) - \fint_{\dot{K}} f(t)dt \right|^p dy \right)^{1/p} \le C\epsilon \left(\fint_{\dot{K}} |\nabla_T f(y)|^p dy \right)^{1/p}$$

12.3. Notations and Conventions

Domains of $\mathbb{R}^n$ will generally be denoted by Ω, $\partial\Omega$ denoting the boundary. Points of $\mathbb{R}^n$ are denoted by x, y, P and Q with P and Q usually on the boundary of a domain.

$x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$.
$\langle x, y\rangle = x \cdot y = x_j y_j$ denotes the inner product. Repeated indices indicate summation over $1 \le j \le n$.
$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_n > 0\}$.
$\mathbb{B}^n = \{x \in \mathbb{R}^n \mid |x| \le 1\}$.
$\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\}$.
$B(x, r) = \{y \in \mathbb{R}^n \mid |x - y| < r\}$.
$I = [0, 1]$ the closed unit interval.
The complement of a set $A \subset \mathbb{R}^n$ is denoted by A^c. The closure by $\overline{A}$. The interior by Int A. $A \setminus B = A \cap B^c$.
$\mathrm{dist}(x, y) = |x - y|$ and $\mathrm{dist}(A, B)$ denotes the infimum of distances between points from the sets A and B.
dx denotes Lebesgue measure in $\mathbb{R}^n$.
ds will denote $(n - 1)$-dimensional surface measure, i.e. Lebesgue measure.
a.e. means almost everywhere with respect to Lebesgue measure.
L^p spaces will be with respect to these Lebesgue measures, $\|\,\|_p$ denotes the norm.
L^p_0 denotes the subspace of L^p consisting of functions with mean value zero.
Sobolev spaces of functions with distributional gradients (∇ or ∇_T) in L^p will be denoted $W^{1,p}$.
I also denotes the identity on these Banach spaces.

$\displaystyle\fint_A$ denotes the integral average over the measurable set A.
$D_j = \dfrac{\partial}{\partial x_j}$ denote partial derivatives.
$\nabla = (D_1, \dots, D - n)$ the gradient operator, div is the divergence operator $\nabla\cdot$, and $div\nabla = \Delta$ the Laplacian.
$\Gamma(x) = (2-n)^{-1}\omega_n^{-1}|x|^{2-n}$ denotes the fundamental solution for Δ; ω_n the surface measure of $\mathbb{S}^{n-1}$.
Generally u or v will denote solutions, usually harmonic functions.
$N = N_Q$ the outer unit normal vector to a domain at the boundary point Q, when it exists.
$\dfrac{\partial u}{\partial N}(Q)$ the normal derivative at Q.
$\nabla_T u(Q)$ the tangential components of the gradient at a boundary point Q.

Generally a domain will be the interior of a finite homogeneous 4-complex in $\mathbb{R}^4$. Simplexes realized in $\mathbb{R}^n$ will be denoted by σ, τ, κ etc. with σ^j denoting the dimension $0 \le j \le n$, the exponent often suppressed when the dimension is understood. The convex hull of any set of $n+1$ points not contained in any $(n-1)$-plane (hyperplane) of $\mathbb{R}^n$ determines an n-simplex σ^n. Any subset of $j+1$ of these points likewise determines a j-simplex or j-face σ^j, $0 \le j \le n$. One writes $\sigma^j \in \sigma^n$. The 0-faces are also called the vertices and will also be denoted by letters v, w etc. If $\{v_0, \dots v_j\} \subset \mathbb{R}^n$ are the vertices of σ^j, every point x in the convex hull of these vertices has unique *barycentric coordinates* determined by $x = \sum_{k=0}^j c_k v_k$, $0 \le c_k \le 1$. The *barycenter* of σ^j is defined by $\hat{\sigma}^j = \sum_{k=0}^j \dfrac{1}{j+1} v_k$.

A *homogeneous* m*-complex* $K = K^m$ will be a finite collection of simplexes in $\mathbb{R}^n$ so that

1. If $\sigma \in K$ and τ is a face of σ then $\tau \in K$.
2. If $\sigma, \tau \in K$ then $\sigma \cap \tau \neq \emptyset$ is a face of both σ and τ.
3. Every simplex of K is contained in some m-simplex of K.

K is variously thought of as a set of sets or a set of points. In the latter case one writes $|K|$ called the *geometric realization or polyhedron* with the relative topology inherited from $\mathbb{R}^n$.

If K is an n-complex in $\mathbb{R}^n$ then $\dot{K}$ denotes the $(n-1)$-complex that is the boundary complex of K, i.e. the subcollection of $(n-1)$-simplexes that are each precisely in one n-simplex of K. If $\Omega = \operatorname{Int}|K|$, then $\partial\Omega$ and $|\dot{K}|$ coincide.

The j-skeleton of a complex is the subcomplex formed by the j-simplexes of the complex.

Let K be a complex and $\sigma \in K$. The *star* of σ in K is written $\operatorname{St}(\sigma, K)$ and is the complex of all simplexes of K containing σ together with their faces. The *link* of σ in K is the complex $\operatorname{Lk}(\sigma, K) = \{\tau \in \operatorname{St}(\sigma, K) \mid \tau \cap \sigma = \emptyset\}$.

The *join* of two simplexes is written $\sigma * \tau$ or $\sigma\tau$ especially when the simplexes are vertices as in $v_0 v_1 \cdots v_n = \sigma^n$. Joins $K * L$ betweens complexes are essentially convex combinations between the simplexes of K^k and L^l that result in a well defined complex of dimension $k + l + 1$ containing K and L as subcomplexes (see [Gla70] p.6). As examples, a σ^4 can be realized from its faces as a $\sigma^0 * \sigma^3$ or $\sigma^1\sigma^2$. If $\sigma \in K$, a homogeneous complex, then $\mathrm{St}(\sigma, K) = \sigma * \mathrm{Lk}(\sigma, K)$. $\emptyset * K = K$.

An *m-manifold* M^m is a separable metric space such that every point is contained in an open set homeomorphic to $\mathbb{R}^m$. If every point is contained in an open set either homeomorphic to $\mathbb{R}^m$ or to the closed half space $\overline{\mathbb{R}^m_+}$ then M is a *manifold with boundary*.

$M \subset \mathbb{R}^n$ will be called *triangulated* if there is a complex K^m so that $|K| = M$.

A complex L is a *subdivision* of K if $|L| = |K|$ and every simplex of L is contained in a simplex of K.

A piecewise linear (PL) homeomorphism h from one complex K to another L is a homeomorphism from K to L so that there are subdivisions K' of K and L' of L with h linear (affine) on each $\sigma \in K'$ and so that $h(\sigma) \in L'$. The identity homeomorphism will be denoted by id.

For $0 < \alpha < 1$ fixed the *nontangential approach region* in a domain Ω at a boundary point Q is the set $\Gamma_\alpha(Q) = \{x \in \Omega \mid \mathrm{dist}(x, \partial\Omega) > \alpha|x - Q|\}$. Given a polyhedral domain Ω it is possible to fix α small enough so that each Q is in the closure of the corresponding $\Gamma_\alpha(Q)$.

The *nontangential maximal function* of a function F in Ω at a boundary point Q is

$$N_\alpha(F)(Q) = \sup_{x \in \Gamma_\alpha(Q)} |F(x)|$$

and the *nontangential limit* of F at Q is

$$F(Q) = \lim_{\substack{x \to Q \\ x \in \Gamma_\alpha(Q)}} F(x)$$

if it exists. For more details on the above see [Gla70], [Rus83], [VV03].

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