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Global homeomorphisms and covering projections on metric spaces

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Abstract For a large class of metric spaces with nice local structure, which includes Banach–Finsler manifolds and geodesic spaces of curvature bounded above, we give sufficient conditions for a local homeomorphism to be a covering projection. We first obtain a general condition in terms of a path continuation property. As a consequence, we deduce several conditions in terms of path-liftings involving a generalized derivative, and in particular we obtain an extension of Hadamard global inversion theorem in this context. Next we prove that, in the case of quasi-isometric mappings, some of these sufficient conditions are also necessary. Finally, we give an application to the existence of global implicit functions.

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1 Introduction

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map with everywhere nonvanishing Jacobian. A natural question is to ask under which conditions we can assure that f is a global diffeomorphism (or, equivalently, a global homeomorphism). This problem was

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first considered by Hadamard [8], who obtained a sufficient condition in terms of the growth of $\|[\mathrm{d}f(x)]^{-1}\|$, by means of his celebrated *integral condition*. Namely, f is a global diffeomorphism provided

$$\int_0^\infty \inf_{|x|=t} \|[\mathrm{d}f(x)]^{-1}\|^{-1}\, \mathrm{d}t = \infty.$$

This result was extended to the infinite-dimensional setting by Lévy [14], who considered the case of a C^1 mapping f between Hilbert spaces.

Some years later, Cacciopoli [5] and Banach and Mazur [2] obtained a purely topological condition of global inversion for a local homeomorphism f between Banach spaces. Namely, they proved that f is a global homeomorphism if, and only if, f is a proper map. Concerning properness, it should be noted that every continuous, open and proper map from a metric space to a connected metric space is always onto. Properness was also used by Palais [17, Sect. 4], in the context of locally compact spaces and (finite-dimensional) manifolds. The properness condition was relaxed to closedness by Browder [4] in a more general setting.

Later on, Plastock [21] extended the Hadamard–Lévy theorem to the case of C^1 mappings between Banach spaces. In fact, Plastock obtained a more general result as follows. He introduced a limiting property for lines, called *condition* (L), which is analogous to the continuation property used by Rheinboldt [26]. Then, for a local homeomorphism between Banach spaces, Plastock proved that f satisfies condition (L) if, and only if, f is a covering projection (or, equivalently, a global homeomorphism). Finally, he showed that properness, closedness or the Hadamard integral condition all imply condition (L). Since then, this condition (L) has proved to be quite useful in global inversion theorems, and it has been widely used, as can be seen for instance in [10,22,25] or [20].

The question of global invertibility has been also studied from the point of view of nonsmooth analysis. In this sense, the problem of giving analytic conditions of global inversion for a nonsmooth mapping f between Banach spaces was first considered by F. John [11]. In particular, John obtained an extension of the Hadamard integral condition in this setting, in terms of the lower scalar Dini derivative of f. For the proof, he used the prolongation of local inverses of f along lines. Further results in this direction were given by Ioffe [10] in terms of the so-called *surjection constant* of the mapping f, making use of the aforementioned condition (L). In the finite-dimensional case, analogous results were obtained by Pourciau [22,23] by means of Clarke generalized jacobian of f. The surjection constant of a mapping f was also considered by Katriel [12] in order to obtain global inversion theorems in certain metric spaces. The methods of Katriel came from critical point theory, and in particular are based on a suitable Mountain Pass theorem.

Global inversion problems have been less studied in the context of smooth mappings between Riemannian or Finsler manifolds. In the Riemannian case, a classical result due to Ambrose [1] gives conditions for a local isometry to be a global diffeomorphism. This was extended by Wolf and Griffiths [29], who obtained more general conditions under which a local diffeomorphism is a covering projection. On the other hand, we have a quite general result due to Rabier [24] for Finsler manifolds, which establishes the global inversion of f using a growth condition for $\|[df(x)]^{-1}\|$ that does not require integrals.

Our aim in this paper is twofold. On one hand, we provide an extension of the above mentioned results to the framework of metric spaces. On the other hand, we present them in a unified and systematic way, where the ideas and methods of line-lifting play a central role, which leads to a clarification and simplification of proofs. To this end, we introduce in Sect. 2 a fairly general class of path-connected metric spaces with nice local structure, namely the class of metric spaces which are $\mathcal{P}$-connected and locally $\mathcal{P}$-contractible spaces, which we will define below. These include Banach spaces and Banach manifolds, as well as many other "singular" spaces, as for example geodesic metric spaces of curvature bounded above. If X and Y belong to this class of spaces, our main goal is to find conditions for a local homeomorphism $f : X \to Y$ to be a covering projection. We first obtain in Theorem 2.6 a general condition, in terms of a continuation property. This Theorem is the key of our presentation, since every further result will be derived from it. Next, in order to give analytical conditions in this nonsmooth setting, we consider in Sect. 3 the upper and lower scalar Dini derivatives of the mapping f at x, denoted respectively by $D_x^+ f$ and $D_x^- f$ (in the smooth case, these quantities reduce to $\|df(x)\|$ and $\|[df(x)]^{-1}\|^{-1}$). Then we obtain in Proposition 3.8 and Theorem 3.9 two mean value inequalities in this context, which are going to be quite useful in the sequel. In Sect. 4 we introduce a bounded path-lifting condition in terms of $D_x^- f$, and in Theorem 4.1 we see that it is a sufficient condition for f to be a covering projection. We derive some consequences, and in particular we obtain a version of Hadamard integral condition (see Theorem 4.6) in our setting. In Sect. 5 we provide a more complete result under some extra regularity conditions on f. More precisely, for a quasi-isometric mapping f we give in Theorem 5.2 several conditions which are necessary and sufficient for f to be a covering projection or a global homeomorphism.

2 Continuation property on metric spaces

Our purpose in this section is to give a general condition for a local homeomorphism between metric spaces to be a covering projection. This will be achieved by means of a continuation property, much in the spirit of Rheinboldt [26] and Plastock [21]. Our result will apply to a wide class of path-connected metric spaces, which we introduce now.

Let Y be a metric space, and let $\mathcal{P}$ be a family of continuous paths in Y. We say that Y is *$\mathcal{P}$-connected* if the following conditions hold:

(1) If the path $p : [a, b] \to Y$ belongs to $\mathcal{P}$, then the reverse path $\bar{p}$, defined by $\bar{p}(t) = p(a - t + b)$, also belongs to $\mathcal{P}$.
(2) Every two points in Y can be joined by a path in $\mathcal{P}$.

We say that Y is *locally $\mathcal{P}$-contractible* if every point $y_0 \in Y$ has an open neighborhood U which is *$\mathcal{P}$-contractible*, in the sense that there exists a (continuous) homotopy $H : U \times [0,1] \to U$ satisfying:

(3) (a) $H(y_0, t) = y_0$, for all $t \in [0,1]$.
(b) $H(y, 0) = y_0$ and $H(y, 1) = y$, for all $y \in U$.
(c) For every $y \in U$, the path $p_y(t) := H(y,t)$ belongs to $\mathcal{P}$.

Next we give some general examples of spaces satisfying these conditions.

Example 2.1 It is clear that every normed vector space V is $\mathcal{L}$-connected and locally $\mathcal{L}$-contractible, where $\mathcal{L}$ is the family of all lines in V. The same is true for any convex subset of V.

Example 2.2 Let M be a connected paracompact Banach manifold of class C^k, for $0 \leq k \leq \infty$, and let $\mathcal{P}^k$ denote the family of all C^k-paths on M. Since M is paracompact, then it is metrizable (see [18]). It is easy to see that, with any equivalent metric, M is $\mathcal{P}^k$-connected and locally $\mathcal{P}^k$-contractible.

Example 2.3 Let M be a connected n-dimensional Lipschitz manifold (see e.g. [15] or [6]) and suppose that M is endowed with a metric which is locally Lipschitz equivalent to the Euclidean one. If $\mathcal{P}_L$ denotes the family of all Lipschitz paths in M, it can be seen as before that M is $\mathcal{P}_L$-connected and locally $\mathcal{P}_L$-contractible.

Along the paper, we will focus on the family $\mathcal{R}$ of all *rectifiable* paths on a given metric space, and we will consider accordingly the class of *locally $\mathcal{R}$-contractible* spaces. Recall that, for a metric space Y, the *length* of a path $p : [a,b] \to Y$ is defined by

$$\ell(p) = \sup_{a=t_0 \leq t_1 \leq \cdots \leq t_n = b} \sum_{i=0}^{n-1} d(q(t_i), q(t_{i+1})),$$

where the supremum is taken over all partitions $a = t_0 \leq t_1 \leq \cdots \leq t_n = b$ (no bound on n). The path p is said to be *rectifiable* when $\ell(p) < \infty$. It is not difficult to see that if Y is path-connected and locally $\mathcal{R}$-contractible then Y is also $\mathcal{R}$-connected.

Example 2.4 Now we describe some classes of metric spaces which are locally $\mathcal{R}$-contractible.

(1) *Normed spaces.*
(2) *Finsler manifolds.* Let M be a Banach manifold of class C^1 modeled on a Banach space $(E, |\cdot|)$. A functional $\|\cdot\| : TM \to [0,\infty)$ is said to be a *Finsler structure* for TM (according to [19,24]) if:
   (a) If $\phi : U_\alpha \to E$ is a chart, then for every $x \in U_\alpha$, the map $\|\cdot\|\big|_{T_xM}$ is a norm for T_xM such that $\|\mathrm{d}\phi^{-1}(\phi(x))(\cdot)\|$ defines a norm equivalent to $|\cdot|$ in E.

(b) For every $p \in M$ and $k > 1$, there exists a chart $\phi : U_\alpha \to E$ (where U_α depends on k) such that

$$\frac{1}{k}\|\mathrm{d}\phi^{-1}(\phi(p))w\| \leq \|\mathrm{d}\phi^{-1}(\phi(x))w\| \leq k\|\mathrm{d}\phi^{-1}(\phi(p))w\|$$

for all $x \in U_\alpha$ and $w \in E$.

If there exists such a functional for TM, we say that M is a *Finsler manifold*. In this case, the *Finsler length* of a C^1 path $p : [a,b] \to M$ is defined as:

$$\ell_F(p) = \int_a^b \|p'(t)\|\mathrm{d}t.$$

On the other hand, if M is connected, for every $x, y \in M$, there exists a C^1 path joining x to y, and we can define the *Finsler distance* by

$$d_F(x,y) = \inf\{\ell_F(p) : p \text{ is a } C^1 \text{ path from } x \text{ to } y\}.$$

We always consider that M is endowed with this metric, which is compatible with the topology of M (see [19]). Therefore, every C^1-path in M is rectifiable. As a consequence, we obtain as in Example 2.2 that every connected C^1 Finsler manifold is $\mathcal{R}$-connected and locally $\mathcal{R}$-contractible. In particular this includes connected Riemannian manifolds, both in the finite-dimensional and infinite-dimensional cases (see e.g. [13]).

(3) *Lipschitz manifolds.* Every Lipschitz path in a metric space is rectifiable. Therefore if M is a connected n-dimensional Lipschitz manifold endowed with a metric which is locally Lipschitz equivalent to the Euclidean one, then M is $\mathcal{R}$-connected and locally $\mathcal{R}$-contractible.

(4) *Geodesic spaces*. Recall that a path $g : [0,1] \to Y$ in a metric space Y is said to be a (constant speed) *geodesic* if there exists $L > 0$ such that $d(g(t), g(t')) = L \cdot |t - t'|$, for all $t, t' \in [0,1]$. Note that in this case g is rectifiable and $\ell(g) = L$. We say that Y is a *geodesic space* if every two points in Y can be joined by a geodesic.

Now suppose that Y is a geodesic space such that every point $y_0 \in Y$ has an open neighborhood U verifying: (a) for each $y \in U$ there exists a unique geodesic g_y in U from y_0 to y, and (b) if $y_n \to y$ in U, then $g_{y_n}(t) \to g_y(t)$ uniformly on $[0,1]$. Then we can apply Lemma 2.5 below to the mapping $H : U \times [0,1] \to U$ given by $H(y,t) = g_y(t)$ and we obtain that Y is $\mathcal{R}$-connected and locally $\mathcal{R}$-contractible.

In the book by Bridson and Haefliger [3] we can find several classes of geodesic spaces satisfying the above requirements. For example, this is the case of proper geodesic spaces which are locally uniquely geodesic (see [3, I.3.13]). It is also the case of geodesic spaces of curvature $\leq \kappa$ (see [3, II.1.4]). Spaces of this kind include a large class of polyhedral complexes (see [3, II.5.5]; see also [3, I.7.57] and the comments before [3, I.7.57]).

The proof of the following lemma is easy and it is left to the reader.

Lemma 2.5 *Let U be an open set in a metric space Y, consider a mapping $H : U \times [0,1] \to U$, and set $p_y(t) := H(y,t)$. The following statements are equivalent*:

(1) *The map H is continuous on $U \times [0,1]$.*
(2) *For every $y \in U$ and $\varepsilon > 0$, there exists $\delta > 0$ such that if $z \in B_\delta(y) \subset U$ then*

$$d(p_z(t), p_y(t)) < \varepsilon, \quad \forall t \in [0,1].$$

(3) *For all $y \in U$, if $y_n \to y$ in U, then $p_{y_n}(t) \to p_y(t)$ uniformly on $[0,1]$. In this case, we say that the paths $\{p_y : y \in U\}$ vary continuously with their endpoints.*

Let X and Y be metric spaces, and let $p : [0,1] \to Y$ be a path in Y. We will say that a continuous map $f : X \to Y$ has the *continuation property for* p if, for every $b \in (0,1]$ and every continuous path $q : [0,b) \to X$ such that $f \circ q = p$ over $[0,b)$, there exists a sequence $\{t_n\}$ in $[0,b)$ convergent to b and such that $\{q(t_n)\}$ converges in X. Recall that a continuous map $f : X \to Y$ is called a *covering projection* if every $y \in Y$ has an open neighborhood U such that $f^{-1}(U)$ is the disjoint union of open subsets of X each of which is mapped homeomorphically onto U by f.

Theorem 2.6 *Let $f : X \to Y$ be a local homeomorphism between metric spaces and suppose that Y is $\mathcal{P}$-connected and locally $\mathcal{P}$-contractible for some family $\mathcal{P}$ of paths. Then f is a covering projection if and only if f has the continuation property for every path in $\mathcal{P}$.*

Proof The sufficiency follows from general properties of covering projections. If f is a covering projection, then f lifts paths and has the unique-path-lifting property (see for example [28, Sect. 2.2]). That is, for every path $p : [0,1] \to Y$ in $\mathcal{P}$, there exists a unique path $q_0 : [0,1] \to X$ such that $f \circ q_0 = p$ over $[0,1]$. Thus every partial lifting $q : [0,b) \to X$ of p with $0 < b \leq 1$ must coincide with q_0 over $[0,b)$, and therefore f has the continuation property for p.

For the necessity, suppose that f has the continuation property for each path in $\mathcal{P}$. We first prove that f lifts every path $p : [0,1] \to Y$ in $\mathcal{P}$. Indeed, let $y = p(0) \in f(X)$ and consider $x \in f^{-1}(y)$. Since f is a local homeomorphism, there exist $\varepsilon > 0$ and a path $q : [0,\varepsilon) \to X$ beginning at x such that $f \circ q = p$ over $[0,\varepsilon)$. Let b the supremum of the set of numbers $s \in [0,1]$ for which q can be extended to a continuous path such that $f \circ q = p$ over $[0,s)$. There exists a sequence $\{t_n\}$ in $[0,b)$ convergent to b and such that $\{q(t_n)\}$ converges to some point $z \in X$. By continuity $f(z) = p(b)$. Now let V be a neighborhood of z such that $f|_V$ is a homeomorphism. Then there exists some $n_0 \in \mathbb{N}$ such that $q(t_n) \in V$ for $n \geq n_0$. Also, there exists $\delta > 0$ and a path $q_1 : (b-\delta, b+\delta) \to X$ with $q_1(t_m) = q(t_m)$, where $m \geq n_0$ and $b - \delta < t_m < b$, and such that $f \circ q_1 = p$ over

$(b-\delta, b+\delta)$. Since local homeomorphisms have the unique-lifting property, q can be extended to a continuous path (call it again q) over $[0, b+\delta)$, beginning at x and such that $f \circ q = p$ over $[0, b+\delta)$. This contradicts the maximality of $[0, b)$. Therefore $b = 1$ and the same reasoning shows that q can be extended to $[0, 1]$.

Now let $y \in Y, y' \in f(X)$ and $p : [0,1] \to Y$ be a path in $\mathcal{P}$ joining y' to y. Then there exists q such that $f \circ q = p$ over $[0, 1]$; in particular $f(q(1)) = p(1) = y$. Therefore, f is onto. Next we are going to show that f is a covering projection.

Let $y_0 \in Y, x \in f^{-1}(y_0)$ and let U be a $\mathcal{P}$-contractible neighborhood of y_0. For every $y \in U$, let p_y as in the definition of locally $\mathcal{P}$-contractible space. Let q_y be the unique lifting of p_y, with $q_y(0) = x$ and such that $f \circ q_y = p_y$. Now consider:

$$O_x = \{q_y(1) : y \in U\}.$$

We will prove that $f|_{O_x} : O_x \to U$ is a homeomorphism and that $f^{-1}(U)$ is the disjoint union of open sets $O_x, x \in f^{-1}(y_0)$.

It is easy to see that $f|_{O_x} : O_x \to U$ is bijective. Since f is a local homeomorphism, in order to see that $f|_{O_x}$ is a homeomorphism we just have to prove that O_x is open in X. Let $x' \in O_x$ and $y \in U$ such that $x' = q_y(1)$. For every $u \in \operatorname{Im} q_y$ there exist an open neighborhood U^u and an open ball $B_{r_u}(f(u))$, such that $f|_{U^u} : U^u \to B_{r_u}(f(u))$ is a homeomorphism. Let $V^u \subset U^u$ be an open set such that $f(V^u) = B_{\frac{r_u}{2}}(f(u))$. By compactness, there exist $u_1, \ldots, u_m \in \operatorname{Im} q_y$ such that $\operatorname{Im} q_y \subset V^{u_1} \cup \cdots \cup V^{u_m}$. For $k = 1, 2, \ldots, m$, let us denote $V_k = V^{u_k}$, $U_k = U^{u_k}$, $y_k = f(u_k)$ and $r_k = r_{u_k}$. Then,

$$\operatorname{Im} p_y \subset \bigcup_{k=1}^{m} B_{\frac{r_k}{2}}(y_k) = \bigcup_{k=1}^{m} B_k.$$

Let $s_k : B_{r_k}(y_k) \to U_k$ be the inverse of $f|_{U_k}$. Let $\rho > 0$ be the Lebesgue's number of $[0, 1]$ for the finite covering $\left\{q_y^{-1}(V_k \cap \operatorname{Im} q_y)\right\}_{k=1}^{m}$ and let $0 = t_0 < t_1 < \cdots < t_n = 1$ be a partition of $[0, 1]$ such that, for every $j = 1, \ldots, n$, the diameter of $[t_{j-1}, t_j]$ is less than ρ. Then for each $j = 1, \ldots, n$ there exists $k_j \in \{1, \ldots, m\}$ such that $q_y[t_{j-1}, t_j] \subset V_{k_j}$. For each $j = 1, \ldots, n$, let $\tilde{u}_j = q_y(t_j) \in V_{k_j} \cap V_{k_{j+1}} = \tilde{V}_j$. Since $\tilde{V}_j$ is open, then $f(\tilde{V}_j)$ is open in Y and contains $\tilde{y}_j = p_y(t_j)$. Also, $f|_{\tilde{V}_j} : \tilde{V}_j \to f(\tilde{V}_j)$ is a homeomorphism and $s_{k_j} \equiv s_{k_{j+1}}$ over $f(\tilde{V}_j)$. Let $\delta_j > 0$ be such that $B_{\delta_j}(\tilde{y}_j) \subset f(\tilde{V}_j)$; in particular $s_{k_j} \equiv s_{k_{j+1}}$ over $B_{\delta_j}(\tilde{y}_j)$. Let $\varepsilon > 0$ be such that

$$0 < \varepsilon < \operatorname{dist}(\operatorname{Im} p_y; Y \backslash (B_1 \cup \cdots \cup B_m))$$

and $0 < \varepsilon < \min\{r_1, \ldots, r_m; \delta_1, \ldots, \delta_n\}$. Therefore, there exists $\delta > 0$ such that, if $z \in B_\delta(y)$ and $B_\delta(y) \subset B_{r_{k_n}}(y_{k_n})$, then $d(p_z(t), p_y(t)) < \frac{\varepsilon}{2}$, $\forall t \in [0, 1]$. For all

$j = 1, \ldots, n$; if $t \in [t_{j-1}, t_j]$ then $d(p_y(t), y_{k_j}) < \frac{r_{k_j}}{2}$, therefore:

$$d(p_z(t), y_{k_j}) \le d(p_z(t), p_y(t)) + d(p_y(t), y_{k_j}) < \frac{\varepsilon}{2} + \frac{r_{k_j}}{2} < r_{k_j}.$$

In other words, $p_z[t_{j-1}, t_j] \subset B_{r_{k_j}}(y_{k_j})$ where the local inverse s_{k_j} is defined. Furthermore, for all $j = 1, \ldots, n$:

$$d(p_z(t_j), p_y(t_j)) = d(p_z(t_j), \tilde{y}_j) < \varepsilon < \delta_j$$

so, $p_z(t_j) \in B_{\delta_j}(\tilde{y}_j)$ where we know that $s_{k_j} \equiv s_{k_{j+1}}$. Therefore, $s_{k_j}(p_z(t_j)) = s_{k_{j+1}}(p_z(t_j))$, for all $j = 0, \ldots, n-1$. Then, the lifting q_z of p_z with $q_z(0) = x$ satisfies $q_z(t) = s_{k_j} \circ p_z(t)$, for all $t \in [t_{j-1}, t_j]$. Therefore, the set $s_{k_n}(B_\delta(y))$ is an open set containing $x' = q_y(1)$ and contained in O_x.

We will prove that $f^{-1}(U) = \bigcup_{x \in f^{-1}(y_0)} O_x$. Let $x \in f^{-1}(y_0)$ and $x' \in O_x$. Then $x' = q_y(1)$ for some $y \in U$. Therefore $f(x') = p_y(1) = y$, so $x' \in f^{-1}(U)$. On the other hand, let $x' \in f^{-1}(U)$; there exists $y \in U$ such that $f(x') = y$. Let $\bar{p}(t) = p_y(1-t)$. Since f lifts paths in $\mathcal{P}$, there exists $\bar{q}$ such that $f \circ \bar{q} = \bar{p}$ with $\bar{q}(0) = x'$. Setting $q_y(t) = \bar{q}(1-t)$ we get $f \circ q_y = p_y$, with $q_y(1) = x'$. Therefore $x' \in O_{q_y(0)}$.

Let $x' \in O_{x_1} \cap O_{x_2}$. For $i = 1, 2$, there exists $y_i \in U$, the paths p_{y_i} and their liftings q_{y_i} such that $q_{y_i}(0) = x_i$, $q_{y_i}(1) = x'$ and $f \circ q_{y_i} = p_{y_i}$. If $\bar{q}_{y_i}(t) = q_{y_i}(1-t)$ and $\bar{p}_{y_i}(t) = p_{y_i}(1-t)$, we have $\bar{q}_{y_i}(0) = x'$, $f \circ \bar{q}_{y_i} = \bar{p}_{y_i}$. Because $f(x') = \bar{p}_{y_i}(0) = y_i$, then $y_1 = y_2$. Since the lifting is unique (see for example Chap. 2 of [28]), we obtain that $\bar{q}_{y_1} = \bar{q}_{y_2}$; in particular $x_1 = x_2$. □

Let $f : X \to Y$ be a covering projection between path-connected metric spaces, and consider the associated morphism between their fundamental groups $f_* : \pi_1(X) \to \pi_1(Y)$. It is well known that f is a homeomorphism onto Y if and only if $f_*[\pi_1(X)] = \pi_1(Y)$ (see e.g. [28, Chap. 2]). Thus we obtain at once the following corollary. In particular, if X and Y are Banach spaces, a local homeomorphism has the continuation property for lines if and only if it is a global homeomorphism. In this way we obtain the desired generalization of the result of Plastock [21, Theorem 1.2].

Corollary 2.7 *Let $f : X \to Y$ be a local homeomorphism between path-connected metric spaces, where Y is $\mathcal{P}$-connected and locally $\mathcal{P}$-contractible for some family $\mathcal{P}$ of paths. Suppose that either Y is simply connected or $\pi_1(X) = \pi_1(Y)$ is finite. Then f is a global homeomorphism if and only if f has the continuation property for every path in $\mathcal{P}$.*

Recall that a continuous map $f : X \to Y$ between topological spaces is said to be *proper* if $f^{-1}(K)$ is a compact set in X whenever K is a compact set in Y. More generally, we say that f is *weakly proper* if, for every compact subset K of Y, each connected component of $f^{-1}(K)$ is compact in X.

Corollary 2.8 *Let X and Y be metric spaces, and suppose that Y is path-connected and locally contractible. Then every weakly proper local homeomorphism $f : X \to Y$ is a covering projection.*

Proof Let $p : [0, 1] \to Y$ be a continuous path, consider $0 < b \leq 1$, and suppose that $q : [0, b) \to X$ satisfies $f \circ q = p$ over $[0, b)$. Then we have that $\operatorname{Im} q$ is relatively compact in X, since it is contained in a connected component of set $f^{-1}(\operatorname{Im} p)$. If we now choose a sequence $\{t_n\}$ in $[0, b)$ convergent to b, there exists a subsequence (t_{n_k}) such that $\{q(t_{n_k})\}$ is convergent in X. □

3 Mean value theorem on metric spaces

Let X and Y be metric spaces. If $f : X \to Y$ is a continuous map and $x \in X$ is not an isolated point of X, we define the *lower and upper scalar derivatives* of f at x by

$$D_x^- f = \liminf_{z \to x} \frac{d(f(z), f(x))}{d(z, x)}, \quad D_x^+ f = \limsup_{z \to x} \frac{d(f(z), f(x))}{d(z, x)},$$

where z is restricted to points of X different from x.

We introduce the above quantities motivated by the work of John [11] in a Banach space context. The next lemma follows immediately from the definition:

Lemma 3.1 *Let V and W be open sets in the metric spaces X and Y, respectively, and suppose that $g = f|_V : V \to W$ is a homeomorphism. If $x \in V$ is not an isolated point and $y = f(x) \in W$, we have that $D_y^+(g^{-1}) = (D_x^- g)^{-1}$ and $D_y^-(g^{-1}) = (D_x^+ g)^{-1}$.*

When X and Y are Banach spaces and $f : X \to Y$ is differentiable at x, F. John obtained in [11] that $D_x^+ f = \|f'(x)\|$ and, if in addition $f'(x)$ is invertible, $D_x^- f = \|f'(x)^{-1}\|^{-1}$. The same statement holds for smooth mappings between connected and complete Riemannian manifolds:

Example 3.2 Let $f : M \to N$ be a C^1 map between connected and complete Riemannian manifolds. Then, for every $x \in M$ we have that $D_x^+ f = \|\mathrm{d}f(x)\|$. If in addition $\mathrm{d}f(x) \in \operatorname{Isom}\left(T_x M; T_{f(x)} N\right)$, *then $D_x^- f = \|[\mathrm{d}f(x)]^{-1}\|^{-1}$.*

Proof Our proof will work both in the finite-dimensional and infinite-dimensional cases. Let $x \in M$ and consider $\varepsilon > 0$. Since the map $\|\mathrm{d}f(\cdot)\|$ is continuous on M, there exists $r > 0$ such that $|\, \|\mathrm{d}f(x)\| - \|\mathrm{d}f(y)\| \,| < \varepsilon$ for every $y \in B_{2r}(x)$. Now for each $z \in B_r(x)$ with $z \neq x$ there exists a C^1 path $\gamma_{z,\varepsilon} : [0, 1] \to M$ such that $\ell(\gamma_{z,\varepsilon}) \leq d(x, z) + \min\{r; \varepsilon; \varepsilon d(x, z)\}$. Thus for every $y \in \operatorname{Im} \gamma_{z,\varepsilon}$ we have that $d(x,y) \leq \ell(\gamma_{z,\varepsilon}) \leq \min\{2r;(1+\varepsilon)d(x,z)\}$, and $\|\mathrm{d}f(y)\| \leq \|\mathrm{d}f(x)\| + \varepsilon$. Then

$$\begin{aligned} d(f(x),f(z)) &\leq \ell(f\circ\gamma_{z,\varepsilon}) = \int_0^1 \|(f\circ\gamma_{z,\varepsilon})'(t)\|\mathrm{d}t \\ &\leq \sup\{(\|\mathrm{d}f(y)\| : y \in \mathrm{Im}\gamma_{z,\varepsilon}\} \cdot \int_0^1 \|(\gamma_{z,\varepsilon})'(t)\|\mathrm{d}t \\ &\leq (\|\mathrm{d}f(x)\| + \varepsilon)(1+\varepsilon)d(x,z). \end{aligned}$$

In this way we obtain that $D_x^+ f \leq \|\mathrm{d}f(x)\|$.

For the reverse inequality, consider $v \in T_xM$ with $\|v\| = 1$. Let $\gamma(t) = \exp_x(tv)$ be the unique geodesic in M such that $\gamma(0) = x$ and $\gamma'(0) = v$, and define $\sigma(t) = \exp_{f(x)}^{-1} \circ f \circ \exp_x(tv)$. Note that the path $\sigma : (-r,r) \to T_{f(x)}N$ is defined for some small interval $(-r,r)$ and satisfies $\sigma(0) = f(x)$ and $\sigma'(0) = \mathrm{d}f(x)(v)$. For $|t|$ small enough , we have by [13, Theorem VIII.6.4] that

$$|t| = \|tv\| = d(\exp_x(tv), x) = d(\gamma(t), x)$$

and also

$$\|\sigma(t)\| = d(\exp_{f(x)}(\sigma(t)), f(x)) = d(f(\gamma(t)), f(x)).$$

Therefore,

$$\|\mathrm{d}f(x)(v)\| = \|\sigma'(0)\| = \lim_{t\to 0} \frac{\|\sigma(t)\|}{|t|} = \lim_{t\to 0} \frac{d(f(\gamma(t)), f(x))}{d(\gamma(t), x)} \leq D_x^+ f.$$

Suppose now that $\mathrm{d}f(x) \in \mathrm{Isom}\left(T_xM; T_{f(x)}N\right)$. Then there exist V and W, open neighborhoods of x and $f(x)$ respectively, such that $f|_V : V \to W$ is a diffeomorphism. Using Lemma 3.1 we obtain that $(D_x^- f)^{-1} = D_{f(x)}^+ (f|_V)^{-1} = \|d(f|_V)^{-1}(f(x))\| = \|[\mathrm{d}f(x)]^{-1}\|$. □

The first part of the proof in the above example also works in the case of Finsler manifolds. Thus we obtain the following:

Example 3.3 Let $f : M \to N$ be a C^1 map between connected and complete C^1 Finsler manifolds. Then, for every $x \in M$ we have that $D_x^+ f \leq \|\mathrm{d}f(x)\|$. If in addition $\mathrm{d}f(x) \in \mathrm{Isom}\left(T_xM; T_{f(x)}N\right)$*, then $D_x^- f \geq \|[\mathrm{d}f(x)]^{-1}\|^{-1}$.*

Remark 3.4 Comparison of $D_x^- f$ with the Ioffe–Katriel surjection constant. The surjection constant was introduced by Ioffe [10] for non differentiable maps between Banach spaces to establish global inversion theorems. Katriel [12] also works with the surjection constant in order to give global homeomorphism theorems in certain metric spaces. It is defined as follows. If $f : X \to Y$ a continuous mapping between metric spaces we set, for $x \in X$ and $t > 0$:

$$\mathrm{Sur}(f,x)(t) = \sup\{r \geq 0 : B_r(f(x)) \subset f(B_t(x))\},$$
$$\mathrm{sur}(f,x) = \liminf_{t\to 0}\ t^{-1}\mathrm{Sur}(f,x)(t).$$

Then $\mathrm{sur}(f,x)$ is called the *surjection constant of f at x*. In general, $\mathrm{sur}(f,x)$ does not always coincide with the lower scalar derivative $D_x^- f$. A simple example of this is the inclusion map $i : \mathbb{R} \to \mathbb{R}^2$; it is easy to calculate $D_0^- i = 1$ and $\mathrm{sur}(i,0) = 0$. Nevertheless, *if $f : X \to Y$ is a local homeomorphism then, for all $x \in X$*

$$D_x^- f = \mathrm{sur}(f,x).$$

In order to obtain mean value inequalities in terms of the lower and upper scalar derivatives, we will prove two very simple, but useful lemmas. If q is a path defined on $[\alpha,\beta]$, we will denote by $q_{t,s}$ the restriction of q over $[t,s] \subseteq [\alpha,\beta]$.

Lemma 3.5 *Let $f : X \to Y$ be a continuous map between metric spaces. Suppose that $q : [\alpha,\beta] \to X$ is a rectifiable path such that $\ell(q_{t,s}) > 0$ for all $t,s \in [\alpha,\beta]$ with $t < s$, and denote $p = f \circ q$. Then, for every $t \in (\alpha,\beta)$ we have*

$$\frac{d(p(\alpha),p(\beta))}{\ell(q_{\alpha,\beta})} \leq \max\left\{\frac{d(p(\alpha),p(t))}{\ell(q_{\alpha,t})}, \frac{d(p(t),p(\beta))}{\ell(q_{t,\beta})}\right\}.$$

Proof Let $t \in (\alpha,\beta)$ be fixed and suppose first that

$$\ell(q_{t,\beta})d(p(\alpha),p(t)) \leq \ell(q_{\alpha,t})d(p(t),p(\beta)).$$

Then

$$\begin{aligned}\ell(p_{t,\beta})d(p(\alpha),p(\beta)) &\leq \ell(p_{t,\beta})[d(p(\alpha),p(t)) + d(p(t),p(\beta))]\\ &\leq \ell[(q_{\alpha,t}) + \ell(p_{t,\beta})]d(p(t),p(\beta)) = \ell(q_{\alpha,\beta})d(p(t),p(\beta)).\end{aligned}$$

If on the other hand

$$\ell(q_{t,\beta})d(p(\alpha),p(t)) \geq \ell(q_{\alpha,t})d(p(t),p(\beta)),$$

we obtain in the same way that

$$\ell(q_{\alpha,t})d(p(\alpha),p(\beta)) \leq \ell(q_{\alpha,\beta})d(p(\alpha),p(t)).$$

□

With the same proof of Lemma 3.5 we have the following result:

Lemma 3.6 *Let $f : X \to Y$ be a continuous map between metric spaces. Let $q : [\alpha, \beta] \to X$ be a path such that $q(\alpha) \neq q(\beta)$, and suppose that $p = f \circ q$ is rectifiable. Then, for every $t \in (\alpha, \beta)$ we have*

$$\frac{\ell(p_{\alpha,\beta})}{d(q(\alpha), q(\beta))} \geq \min\left\{\frac{\ell(p_{\alpha,t})}{d(q(\alpha), q(t))}, \frac{\ell(p_{t,\beta})}{d(q(t), q(\beta))}\right\}.$$

Remark 3.7 Note that, under the hypothesis of Lemma 3.6, $d(q(\alpha), q(t))$ and $d(q(t), q(\beta))$ cannot be 0 simultaneously. If for example $t \in (\alpha, \beta)$ is such that $q(t) = q(\alpha)$, we understand that $\frac{\ell(p_{\alpha,t})}{d(q(\alpha),q(t))} := \infty$. In this case $q(t) \neq q(\beta)$ and we have

$$\frac{\ell(p_{\alpha,\beta})}{d(q(\alpha), q(\beta))} \geq \frac{\ell(p_{t,\beta})}{d(q(t), q(\beta))}.$$

Proposition 3.8 *Let $f : X \to Y$ be a continuous map between metric spaces, let $q : [a, b] \to X$ be a rectifiable path, and denote $p = f \circ q$. Then*

(1) *There exists $\tau \in [a, b]$ such that $d(p(a), p(b)) \leq D^+_{q(\tau)} f \cdot \ell(q)$.*
(2) *We have $\ell(p) \leq \sup_{x \in \operatorname{Im} q} D^+_x f \cdot \ell(q)$.*

Proof In order to prove part (1), first note that using a suitable reparametrization of q (for example, the reparametrization by arc length, see [3, I.1.20]) we may assume with no lost of generality that $\ell(q_{t,s}) > 0$ for all $t, s \in [a, b]$ with $t < s$. Then we denote

$$\Delta_{t,s} f := \frac{d(p(t), p(s))}{\ell(q_{t,s})}.$$

Let $\alpha_0 = a$, $\beta_0 = b$ and consider the midpoint $m_0 \in [\alpha_0, \beta_0]$. By Lemma 3.5 we have

$$\Delta_{\alpha_0,\beta_0} f \leq \max\{\Delta_{\alpha_0,m_0} f, \Delta_{m_0,\beta_0} f\} := \Delta_{\alpha_1,\beta_1} f.$$

Now take the midpoint $m_1 \in [\alpha_1, \beta_1]$, and again by Lemma 3.5, we get

$$\Delta_{\alpha_1,\beta_1} f \leq \max\{\Delta_{\alpha_1,m_1} f, \Delta_{m_1,\beta_1} f\} := \Delta_{\alpha_2,\beta_2} f.$$

By proceeding in this way, we construct a nested sequence of intervals $[\alpha_n, \beta_n]$ with $\alpha_n < \beta_n$ and $(\beta_n - \alpha_n) \to 0$, satisfying:

$$\Delta_{a,b} f \leq \Delta_{\alpha_1,\beta_1} f \leq \cdots \leq \Delta_{\alpha_n,\beta_n} f \leq \cdots$$

Then there exists a point $\tau \in [a,b]$ such that $\bigcap_n [\alpha_n, \beta_n] = \{\tau\}$. In the case that $\alpha_n < \tau < \beta_n$ for every $n \in \mathbb{N}$, we have

$$\limsup_{n\to\infty} \Delta_{\alpha_n,\tau} f \leq \limsup_{n\to\infty} \frac{d(f(q(\alpha_n)), f(q(\tau)))}{d(q(\alpha_n), q(\tau))} \leq D^+_{q(\tau)} f$$

and

$$\limsup_{n\to\infty} \Delta_{\tau,\beta_n} f \leq \limsup_{n\to\infty} \frac{d(f(q(\tau)), f(q(\beta)))}{d(q(\tau), q(\beta_n))} \leq D^+_{q(\tau)} f$$

so we obtain that

$$\Delta_{a,b} f \leq \limsup_{n\to\infty} \Delta_{\alpha_n,\beta_n} f \leq \limsup_{n\to\infty} \max\{\Delta_{\alpha_n,\tau} f, \Delta_{\tau,\beta_n} f\} \leq D^+_{q(\tau)} f.$$

In the same way, we also obtain that $\Delta_{a,b} f \leq D^+_{q(\tau)} f$ if either $\tau = \alpha_n$ for some n or $\tau = \beta_n$ for some n.

To prove part (2), we may assume that $K := \sup_{x \in \operatorname{Im} q} D^+_x f < \infty$ (since otherwise the result holds trivially). Consider a partition $t_0 = a \leq t_1 \leq \cdots \leq t_n = b$ of the interval $[a,b]$. For each $i = 1, \ldots, n$, applying part (1) to the interval $[t_{i-1}, t_i]$ we have that

$$d(p(t_{i-1}), p(t_i)) \leq K \cdot \ell\left(q|_{[t_{i-1},t_i]}\right).$$

Therefore,

$$\sum_{i=1}^{n} d(p(t_{i-1}), q(t_i)) \leq K \sum_{i=1}^{n} \ell\left(q|_{[t_{i-1},t_i]}\right) = K \cdot \ell(q).$$

Taking the supremum over all partitions of $[a,b]$ we conclude the proof. □

Next we give an analogous result for the lower scalar derivative:

Proposition 3.9 *Let $f : X \to Y$ be a continuous map between metric spaces, let $q : [a,b] \to X$ be a path, and suppose that $p = f \circ q$ is rectifiable. Then*:

(1) *If $q(a) \neq q(b)$, there exists $\tau \in [a,b]$ such that $\ell(p) \geq D^-_{q(\tau)} f \cdot d(q(a), q(b))$.*
(2) *If $0 < \inf_{x \in \operatorname{Im} q} D^-_x f < \infty$, we have that $\ell(p) \geq \inf_{x \in \operatorname{Im} q} D^-_x f \cdot \ell(q)$.*

Proof Using Lemma 3.6 we can construct a nested sequence of intervals $[\alpha_n, \beta_n] \subset [a,b]$ with $\alpha_n < \beta_n$, $g(\alpha_n) \neq g(\beta_n)$ and $(\beta_n - \alpha_n) \to 0$, satisfying:

$$\frac{\ell(p_{a,b})}{d(q(a), q(b))} \geq \frac{\ell(p_{\alpha_1,\beta_1})}{d(q(\alpha_1), q(\beta_1))} \geq \cdots \geq \frac{\ell(p_{\alpha_1,\beta_1})}{d(q(\alpha_n), q(\beta_n))} \geq \cdots$$

Then we consider $\bigcap_n [\alpha_n, \beta_n] = \{\tau\}$, and we proceed as in the proof of Theorem 3.8. □

4 Covering projections via the bounded path-lifting property

Let $f : X \to Y$ be a continuous map between path-connected metric spaces, and let $p : [0,1] \to Y$ be a path in Y. We will say that f has the *bounded path-lifting property for* p if, for every $b \in (0,1]$ and every $q : [0,b) \to X$ such that $f \circ q = p$ over $[0,b)$, there exists $\alpha > 0$ such that:

$$\inf\{D_x^- f : x \in \operatorname{Im} q\} \geq \alpha.$$

Proposition 4.1 *Let $f : X \to Y$ a continuous map between metric spaces, and suppose that X is complete. If f has the bounded path-lifting property for rectifiable paths, then f has the continuation property for rectifiable paths. As a consequence, if Y is path-connected and locally $\mathcal{R}$-contractible, and f is a local homeomorphism, then f is a covering projection.*

Proof Let $p : [0,1] \to Y$ be a rectifiable path, and consider $b \in (0,1]$ and $q : [0,b) \to X$ such that $f \circ q = p$ over $[0,b)$. If f has the bounded path-lifting property for p, there exists $\alpha > 0$ such that $\inf\{D_x^- f : x \in \operatorname{Im} q\} \geq \alpha$. By using Proposition 3.9 we obtain that, for every $s, t \in [0,b)$ with $s < t$:

$$\ell\left(p|_{[s,t]}\right) \geq \alpha \cdot d(q(s), q(t)).$$

Now let $\{t_n\}$ be an increasing sequence in $[0,b)$ convergent to b. Then for all $m > n$, we have

$$d(q(t_n), q(t_m)) \leq \frac{1}{\alpha}\,\ell\left(p|_{[t_n,t_m]}\right) \leq \frac{1}{\alpha}\,\ell\left(p|_{[t_n,b]}\right).$$

Since p is rectifiable, the map $t \mapsto \ell\left(p|_{[t,b]}\right)$ is continuous (see, for example, [3, 1.20(5)]). This implies that $\{q(t_n)\}$ is a Cauchy sequence, and therefore convergent since X is complete. So, f has the continuation property for p. Now, since every path-connected and locally $\mathcal{R}$-contractible space is $\mathcal{R}$-connected, the last part of the result follows directly from Theorem 2.6. □

Our next Corollary extends a classical result due to Ambrose for smooth mappings between Riemannian manifolds (see [1, Theorem A] and see also [13, Theorem VIII.6.9]).

Corollary 4.2 *Let $f : X \to Y$ be a local homeomorphism between metric spaces, where X is complete and Y is path-connected and locally $\mathcal{R}$-contractible. If there exists $\alpha > 0$ such that $D_x^- f \geq \alpha$ for all $x \in X$, then f is a covering projection.*

Another easy consequence of Proposition 4.1 is the following.

Corollary 4.3 *Let $f : X \to Y$ be a local homeomorphism between metric spaces, where X is complete and Y is path-connected and locally $\mathcal{R}$-contractible.*

Suppose that:

(1) *For every bounded subset B of X, we have* $\inf_{x\in B}(D_x^- f) > 0$.
(2) *For some $y_0 \in Y$ and $x_0 \in X$, we have $d(f(x), y_0) \to \infty$ as $d(x, x_0) \to \infty$.*

Then, f is a covering projection.

Proof Let $p : [0,1] \to Y$ be a rectifiable path, let $b \in (0,1]$ and let $q : [0,b) \to X$ be such that $f \circ q = p$ over $[0,b)$. Consider $R := \max\{d(p(t), y_0) : 0 \le t \le 1\}$. There exists $r > 0$ such that $d(f(x), y_0) > R$ whenever $d(x, x_0) > r$. Then

$$\begin{aligned}\inf\{D_x^- f : x \in \operatorname{Im} q\} &\ge \inf\{D_x^- f : d(f(x), y_0) \le R\}\\ &\ge \inf\{D_x^- f : d(x, x_0) \le r\} > 0.\end{aligned}$$

□

The conditions of Corollaries 4.2 and 4.3 appear frequently in global inversion theorems. For example, Corollary 4.2 extends the classical well known result in [27, Theorem 1.22] concerning C^1 mappings between Banach spaces, and also extends the analogous result of F. John (see [11], Corollary in p. 87) in the context of nonsmooth mappings between Banach spaces. On the other hand, Corollary 4.3 is an extension of [30, Corollary 3.3] by Zampieri. Similar global inversion results for metric spaces (with more complicated topological hypothesis over X and Y) have been obtained by Katriel [12, Theorem 6.1] and [12, Theorem 6.2].

Next, we are going to see that the bounded path-lifting property can be equivalently defined in terms of a weight. Here, be a *weight* we mean a nondecreasing map (not necessarily continuous) $\omega : [0,\infty) \to (0,\infty)$ such that:

$$\int_0^\infty \frac{\mathrm{d}t}{\omega(t)} = \infty.$$

Now let $f : X \to Y$ be a continuous map between path-connected metric spaces. If $p : [0,1] \to Y$ is a path, we will say that f has the *bounded path-lifting property for p with respect to the weight ω* if, for every $b \in (0,1]$ and $q : [0,b) \to X$ such that $f \circ q = p$ over $[0,b)$, there exist $x_0 \in X$ and $\alpha > 0$ such that:

$$\inf\{D_x^- f \cdot \omega(d(x, x_0)) : x \in \operatorname{Im} q\} \ge \alpha.$$

Lemma 4.4 *Let $f : X \to Y$ be a continuous map between metric spaces, and let $p : [0,1] \to Y$ be a rectifiable path in Y. Then, f has the bounded path lifting property for p if and only if f has the bounded path lifting property for p with respect to some weight ω.*

Proof The sufficiency follows trivially by choosing $\omega(t) \equiv 1$. We are going to prove the necessity. Let $b \in (0,1]$ and $q : [0,b) \to X$ such that $f \circ q = p$

over $[0, b)$. There exist a weight ω, some $x_0 \in X$ and some $\alpha > 0$ such that $\inf\{D_x^- f \cdot \omega(d(x, x_0)) : x \in \operatorname{Im} q\} \geq \alpha$. With no loss of generality, we can suppose that $x_0 = q(0)$, since otherwise we could consider the alternative weight $\bar{\omega}(t) := \omega(d(x_0, q(0)) + t)$. Now define the map $\xi : [0, b) \to \mathbb{R}$ by

$$\xi(t) = \max_{\tau \in [0,t]} d(q(\tau), x_0).$$

It is clear that ξ is continuous and non-decreasing. Before going further, we are going to show that:

$$\frac{\xi(t) - \xi(t')}{\omega(\xi(t))} \leq \frac{1}{\alpha} \cdot \ell\left(p|_{[t',t]}\right), \quad \forall\, t' \leq t. \tag{4.1}$$

Indeed, let $t' < t$ in $[0, b)$, and consider the interval $[t', t]$. Taking into account that $\omega(d(q(\tau), x_0) \leq \omega(\xi(t))$ for every $\tau \in [t', t]$, and using Proposition 3.9, we obtain that

$$\ell\left(p|_{[t',t]}\right) \geq \frac{\alpha}{\omega(\xi(t))} \cdot d(q(t'), q(t)).$$

Therefore

$$d(q(t), x_0) \leq d(q(t'), x_0) + \frac{\omega(\xi(t))}{\alpha} \cdot \ell\left(p|_{[t',t]}\right) \leq \xi(t') + \frac{\omega(\xi(t))}{\alpha} \cdot \ell\left(p|_{[t',t]}\right).$$

In order to establish (4.1), note that the inequality is clear if $\xi(t') = \xi(t)$. On the other hand, if $\xi(t') < \xi(t)$, there exists $t^* \in (t', t]$ such that $\xi(t) = d(q(t^*), x_0)$. In this case, by applying the above argument to $[t', t^*]$, we obtain that

$$\begin{aligned}\xi(t) = d(q(t^*), x_0) &\leq \xi(t') + \frac{\omega(\xi(t^*))}{\alpha} \cdot \ell\left(p|_{[t',t^*]}\right) \\ &\leq \xi(t') + \frac{\omega(\xi(t))}{\alpha} \cdot \ell\left(p|_{[t',t]}\right).\end{aligned}$$

Now let $0 < \delta < b$ be fixed. Given a partition $0 = t_0 < t_1 < \cdots < t_n = \xi(\delta)$ of $[0, \xi(\delta)]$, since ξ is continuous and non-decreasing we can find $0 = s_0 < s_1 < \cdots < s_n = \delta$ such that $s_i = \xi(t_i)$, for $i = 0, \ldots, n$. Then, by inequality (4.1), we have

$$\sum_{i=1}^{n} \frac{s_i - s_{i-1}}{\omega(s_i)} \leq \frac{1}{\alpha} \cdot \sum_{i=1}^{n} \ell\left(p|_{[t_i, t_{i-1}]}\right) = \frac{1}{\alpha} \cdot \ell\left(p|_{[0,\delta]}\right) \leq \frac{1}{\alpha} \cdot \ell(p).$$

Therefore, for every $\delta \in [0, b)$ we obtain that

$$\int_0^{\xi(\delta)} \frac{\mathrm{d}t}{\omega(t)} \leq \frac{1}{\alpha} \cdot \ell(p) < \infty.$$

Since ω is a weight, we conclude that there exists some $r > 0$ such that $\xi(\delta) \leq r$ for every $\delta \in [0, b)$. As a consequence, for every $x \in \operatorname{Im} q$ we have that $\omega(d(x, x_0)) \leq \omega(r)$ and since $D_x^- f \cdot \omega(d(x, x_0)) \geq \alpha$ we finally have that

$$\inf\{D_x^- f : x \in \operatorname{Im} q\} \geq \frac{\alpha}{\omega(r)} > 0.$$

□

We will say that $f : X \to Y$ *satisfies the Hadamard integral condition* if, for some $x_0 \in X$,

$$\int_0^\infty \inf_{x \in \overline{B_t(x_0)}} D_x^- f \, \mathrm{d}t = \infty.$$

Note that the Hadarmard integral condition is satisfied for some $x_0 \in X$, if and only if, it is satisfied for all $x \in X$.

Lemma 4.5 *Let $f : X \to Y$ be a map between metric spaces. Then f satisfies the Hadamard integral condition if and only if there exist $x_0 \in X$ and a weight ω such that $D_x^- f \cdot \omega(d(x, x_0)) \geq 1$, for every $x \in X$.*

Proof Suppose that f satisfies the Hadamard integral condition for some $x_0 \in X$. Then, for every $t \geq 0$:

$$\inf\{D_x^- f : x \in \overline{B_t(x_0)}\} > 0.$$

If we define $\omega(t) = [\inf\{D_x^- f : x \in \overline{B_t(x_0)}\}]^{-1}$, it is clear that ω is a weight and $D_x^- f \cdot \omega(d(x, x_0)) \geq 1$, for all $x \in X$.

Conversely, suppose that there exist $x_0 \in X$ and a weight ω, such that $D_x^- f \cdot \omega(d(x, x_0)) \geq 1$, for all $x \in X$. For every $t \geq 0$, and every $0 \leq r \leq t$, we have

$$\frac{1}{\omega(t)} \leq \frac{1}{\omega(r)} \leq \inf\{D_x^- f : d(x, x_0) = r\}.$$

Then $\omega(t)^{-1} \leq \inf\{D_x^- f : x \in \overline{B_t(x_0)}\}$ for all $t \geq 0$, and therefore f satisfies the Hadamard integral condition. □

Using Proposition 4.1 and Lemmas 4.4 and 4.5, we deduce at once the following result. This gives the desired extension of Hadamard theorem to our context, and extends also the analogous results of John (see [11], Corollary, p. 91) and Ioffe (see [10, Theorem 2]).

Theorem 4.6 *Let $f : X \to Y$ be a local homeomorphism between metric spaces. Suppose that X is complete and Y is path-connected locally $\mathcal{R}$-contractible. If f satisfies the Hadamard integral condition, then f is a covering projection.*

As a direct application, using Example 3.3, we obtain the following result (compare with [7, Corollary 3.4]).

Corollary 4.7 *Let $f : M \to N$ a C^1 map between connected C^1 Finsler manifolds, M complete. Suppose that* $\mathrm{d}f(x)$ *is invertible, for every $x \in M$. If*

$$\int_0^\infty \inf_{x \in \overline{B_t(x_0)}} \|[\mathrm{d}f(x)]^{-1}\|^{-1} \, \mathrm{d}t = \infty,$$

for some $x_0 \in M$, then f is a covering projection.

To finish this section, we note that Hadamard condition is not necessary for a local homeomorphism to be a covering projection, even in very simple cases. For instance, the map $f(x, y) = (x + y^3, y)$ is a global homeomorphism from $\mathbb{R}^2$ to $\mathbb{R}^2$, but it does not satisfy the Hadamard integral condition (see [16, Example 1.7]).

5 Locally quasi-isometric maps

In this section, we obtain a more complete result that relates the lifting concepts that we use in this paper, but we need an extra assumption on regularity. We shall say that a map $f : X \to Y$ between metric spaces is a *quasi-isometry* if f is a homeomorphism and there exist $0 < \alpha \leq \beta < \infty$ such that:

$$\alpha \leq \inf_{x \in X} D_x^- f \leq \sup_{x \in X} D_x^+ f \leq \beta.$$

In the same way, we will say that $f : X \to Y$ is a *local quasi-isometry* if for every $x \in X$ there exist open neighborhoods V of x and W of $f(x)$ such that $f|_V : V \to W$ is a quasi-isometry.

It is clear that if a map between metric spaces is locally a bi-Lipschitz homeomorphism, then it is a local quasi-isometry. More generally, using Example 3.3 we obtain that every local C^1-diffeomorphism between Banach spaces or between Finsler manifolds is also a local quasi-isometry.

Recall that, for a metric space X, the *length* of a path $p : [a, b) \to X$ defined on a semi-open interval is defined by

$$\ell(p) := \lim_{t \to b^-} \ell\left(p|_{[a,t]}\right).$$

In this case we also say that the path p is *rectifiable* when $\ell(p) < \infty$. We will need the following simple Lemma.

Lemma 5.1 *Let $q : [a, b) \to X$ be a rectifiable path on a metric space. Then for every sequence $\{t_n\} \subset [a, b)$ converging to b, the sequence $\{q(t_n)\}$ is a Cauchy sequence in X.*

Now we can easily derive our main result in this Section. This extends [7, Theorem 3.5]. We note that Condition 3 was the key of the original argument used by Hadamard [8]. On the other hand, Condition (8) was introduced by Rabier [24, Theorem 5.3] in the context of Finsler manifolds.

Theorem 5.2 *Let $f : X \to Y$ be a local quasi-isometry between complete metric spaces, and suppose that Y is path-connected and locally $\mathcal{R}$-contractible. Then the following statements are equivalent*:

(1) *f is a covering projection.*
(2) *f has the continuation property for rectifiable paths.*
(3) *For every path $q : [a,b) \to X$, we have $\ell(q) < \infty$ whenever $\ell(f \circ q) < \infty$.*
(4) *f has the bounded-path lifting property for rectifiable paths.*
(5) *f has the bounded-path lifting property for rectifiable paths with respect to some weight.*

If in addition, we assume that either Y is simply connected or $\pi_1(X) = \pi_1(Y)$ is finite, then the previous conditions are also equivalent to the following:

(6) *f is a homeomorphism.*
(7) *f is a proper map.*
(8) *For every compact subset $K \subset Y$, there is a constant $\alpha_K > 0$ such that $D_x^- f \geq \alpha_K$, for every $x \in f^{-1}(K)$.*

Proof (5) $\Leftrightarrow$ (4) It is given in Lemma 4.4.
(4) $\Rightarrow$ (3) Let $q : [a,b) \to X$ be a path in X, and suppose that $\ell(f \circ q) < \infty$. By Lemma 5.1 we have that, for every sequence $\{t_n\} \in [a,b)$ converging to b, the sequence $\{f \circ q\ (t_n)\}$ is convergent in Y. Then there exists a continuous path $p : [a,b] \to Y$ such that $f \circ q = p$ over $[a,b)$. In addition p is rectifiable since

$$\ell(p) = \lim_{t \to b^-} \ell\left(p|_{[a,t]}\right) = \ell(f \circ q) < \infty.$$

Therefore, there exists $\alpha > 0$ such that $\inf\{D_x^- f : x \in \operatorname{Im} q\} \geq \alpha$. Thus by Proposition 3.9 we get $\ell(q) < \ell(p) \cdot \alpha^{-1} < \infty$.
(3) $\Rightarrow$ (2) Let $p : [0,1] \to Y$ be a rectifiable path, $b \in (0,1]$ and $q : [0,b) \to X$ such that $f \circ q = p$ over $[0,b)$. Since $\ell\left(p|_{[0,b]}\right) < \infty$, by the hypothesis we obtain that $\ell(q) < \infty$. Therefore, by Lemma 5.1, for every sequence $\{t_n\} \in [0,b)$ converging to b we have that $\{q(t_n)\}$ is convergent in X.
(2) $\Rightarrow$ (1) It follows from Theorem 2.6.
(1) $\Rightarrow$ (4) Let $p : [0,1] \to Y$ be a rectifiable path, $b \in (0,1]$ and $q : [0,b) \to X$ such that $f \circ q = p$ over $[0,b)$. If f is a covering projection then f lifts paths and has the unique-path-lifting property (see for example [28, Sect. 2.2]). Therefore q can be continuously extended to $[0,1]$, and in particular the closure of $\operatorname{Im} q$ is compact in X. Now since f is a local quasi-isometry, a simple compactness argument gives that there exists $\alpha > 0$ such that $\inf\{D_x^- f : x \in \operatorname{Im} q\} \geq \alpha$.
Finally, in the case that either Y is simply connected or $\pi_1(X) = \pi_1(X)$ is finite, it is clear that (1) $\Leftrightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (8) $\Rightarrow$ (4). □

We finish with an application to the existence of a global implicit function. Our next result generalizes [9, Theorem 5] by Ichiraku, who considered finite dimensional spaces and constant weight.

Corollary 5.3 *Let E, F and W be Banach spaces, let* $f : E \times F \to W$ *be a* C^1 *map and consider the set*:

$$Z_0 := \{(x, y) \in E \times F : f(x, y) = 0\}.$$

Suppose that $\partial_y f(x, y) \in \mathrm{Isom}\,(F; W)$ *for all* $(x, y) \in Z_0$ *and that for some continuous weight* $\omega : [0, \infty) \to (0, \infty)$:

$$\|\partial_y f(x, y)^{-1}\| \cdot \|\partial_x f(x, y)\| \le \omega(\|y\|), \quad \forall (x, y) \in Z_0.$$

If Z_0 *is connected, then there exists a unique continuous map* $g : E \to F$ *such that* $f(x, g(x)) = 0$*, for all* $x \in E$.

Proof Since $\partial_y f(x, y) \in \mathrm{Isom}\,(F; F)$ for all $(x, y) \in Z_0$, there exists a local implicit map for f, that is, the natural projection $\pi : Z_0 \to E$ is a local homeomorphism. It is enough to show that $\pi : Z_0 \to E$ has the continuation property for every line. Let p be a line in E, $b \in (0, 1]$ and $y : [0, b) \to F$ such that $(p(t), y(t)) \in Z_0$ en $[0, b)$. Since p is a line, $p'(t)$ is constant, call it v. Therefore,

$$y'(t) = -\partial_y f(p(t), y(t))^{-1} \partial_x f(p(t), y(t))v.$$

On the other hand,

$$\|y(b) - y(0)\| \le \ell\left(y|_{[0,b]}\right) = \int_0^b \|y'(t)\|\mathrm{d}t \le \|v\| \int_0^b \omega(\|y(t)\|)\mathrm{d}t.$$

By [25, Lemma 2.1], we have

$$\int_{\|y(0)\|}^{\|y(s)\|} \frac{\mathrm{d}t}{\omega(t)} \le \int_0^s \|v\|\mathrm{d}t = \|v\|s \le \|v\|, \quad \forall s \in [0, b)$$

And, since ω is a weight, $\sup\{\|y(t)\| : t \in [0, b)\} < \infty$. Because ω is nondecreasing, there exists $\alpha > 0$ such that $\omega(\|y(t)\|) \le \alpha$, for every $t \in [0, b)$. Then, for $s \in [0, b)$,

$$\ell\left(y|_{[0,s]}\right) \le \|v\| \int_0^s \omega(\|y(t)\|)\mathrm{d}t \le \|v\| \int_0^s \alpha\mathrm{d}t \le \|v\|\alpha s \le \|v\|\alpha.$$

Therefore $\ell(y) < \infty$. By Lemma 5.1 and the completeness of F, π has the continuation property for lines. □

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