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Hyperbolic equations with non-analytic coefficients

Tamotu Kinoshita · Sergio Spagnolo

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Abstract We prove that the Cauchy problem for a hyperbolic, homogeneous equation with $\mathcal{C}^\infty$ coefficients depending on time, is well posed in every Gevrey class, although in general it is not well-posed in $\mathcal{C}^\infty$, provided the characteristic roots satisfy the condition

$$\lambda_i(t)^2 + \lambda_j(t)^2 \leq M\,(\lambda_i(t) - \lambda_j(t))^2 \quad (i \neq j).$$

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1 Introduction

This paper is devoted to the Cauchy problem in $[0, T] \times \mathbb{R}$

$$\partial_t^m u + a_1(t)\,\partial_x \partial_t^{m-1} u + \cdots + a_m(t)\,\partial_x^m u = 0, \tag{1}$$

$$\partial_t^j u(0,x) = \varphi_j(x), \quad j = 0, \ldots, m-1, \tag{2}$$

where the coefficients of (1) are $\mathcal{C}^\infty$ functions, and the characteristic roots satisfy

$$\lambda_i(t)^2 + \lambda_j(t)^2 \leq M\,(\lambda_i(t) - \lambda_j(t))^2, \quad 1 \leq i < j \leq m, \ 0 \leq t \leq T. \tag{3}$$

T. Kinoshita
Institute of Mathematics, Tsukuba University, Tsukuba, Ibaraki 305-8571, Japan
e-mail: kinosita@math.tsukuba.ac.jp

S. Spagnolo (✉)
Department of Mathematics "L. Tonelli", University of Pisa, largo Pontecorvo 5,
56127 Pisa, Italy
e-mail: spagnolo@dm.unipi.it

We first recall some known results on (non-strictly) hyperbolic, homogeneous equations. The equation

$$\partial_t^m u + \sum_{j=1}^{m} \Big[\sum_{|\alpha|=j} a_{j,\alpha}(t,x)\partial_x^\alpha \Big] \partial_t^{m-j} u = 0, \quad t \in [0,T],\ x \in \mathbb{R}^n, \tag{4}$$

is *hyperbolic* if there are m real functions, $\lambda_1(t,x,\xi), \dots, \lambda_m(t,x,\xi)$, for which

$$\tau^m + \sum_{j=1}^{m} \sum_{|\alpha|=j} a_{j,\alpha}(t,x)\,\xi^\alpha \tau^{m-j} = \prod_{h=1}^{m} (\tau - \lambda_h(t,x,\xi)), \quad \xi \in \mathbb{R}^n. \tag{5}$$

In the constant coefficient case, the hyperbolicity is a necessary and sufficient condition for the wellposedness in $\mathcal{C}^\infty$ of the Cauchy problem, whereas it may not be sufficient when the coefficients are variable. Therefore, assuming that the $a_{j,\alpha}(t,x)$ are $\mathcal{C}^\infty$ functions, it is natural to try to characterize the Eq. (4) for which one has $\mathcal{C}^\infty$ wellposedness.

In spite of many important contributions, no general characterization of the wellposedness is still known, even limitedly to the class of second order equations in one space variable:

$$\partial_t^2 u + a_1(t,x)\partial_t\partial_x u + a_2(t,x)\partial_x^2 u = 0. \tag{6}$$

In particular, it has not been possible to state for which kind of functions $a(t,x) \geq 0$ the Cauchy problem

$$\partial_t^2 u - a(t,x)\partial_x^2 u = 0, \quad u(0,x) = \varphi_0(x), \quad \partial_t u(0,x) = \varphi_1(x), \tag{7}$$

is $\mathcal{C}^\infty$ well posed.

Most of the results on $\mathcal{C}^\infty$ wellposedness are concerned either with second order equations, or with equations having coefficients independent of t, or at least analytic in t. We recall briefly some of these results, restricting ourselves to the class of homogeneous equations.

(I) Oleinik [12] considering hyperbolic equations of second order with $\mathcal{C}^\infty$ coefficients, found a sufficient condition for the $\mathcal{C}^\infty$ wellposedness. Limitedly to the Eq. (7), with $a(t,x) \geq 0$, such a condition states that, for some constant C,

$$Ca(t,x) + \partial_t a(t,x) \geq 0.$$

This is true, in particular, if either $a = a(x)$, or $a = a(t)$ with $a'(t) \geq 0$.

(II) Nishitani [11] found a necessary and sufficient condition of $\mathcal{C}^\infty$ wellposedness for second order equations with analytic coefficients in one space variable. In particular, this condition is fulfilled by (7), with $a(t,x) \geq 0$

and analytic, while is not fulfilled by any one of the following (hyperbolic) equations:

$$\partial_t^2 u - 2x\,\partial_t\partial_x u + x^2\,\partial_x^2 u = 0, \tag{8}$$

$$\partial_t^2 u - 2t\,\partial_t\partial_x u + t^2\,\partial_x^2 u = 0. \tag{9}$$

(III) Colombini and Spagnolo [2] constructed a $\mathcal{C}^\infty$ function $a(t) \geq 0$, in such a way that the Cauchy problem for the equation

$$\partial_t^2 u - a(t)\,\partial_x^2 u = 0, \tag{10}$$

is not $\mathcal{C}^\infty$ well-posed. One of the properties of this function is that $a(t) = 0$ only at the initial time $t = 0$, and makes infinitely many oscillations as $t \to 0$.

(IV) Colombini and Orrú [4] studied the Cauchy problem for higher order hyperbolic equations of the form (1), assuming the coefficients $a_j(t)$ *analytic* functions on $[0, T]$ (or, more generally, smooth functions with zeroes of finite order), and found that (3) is a sufficient condition for the $\mathcal{C}^\infty$ wellposedness. If all the $a_j(t)$'s vanish at the initial time, the condition is also necessary. Note that (3) is trivially fulfilled by the Eq. (10) with $a(t) \geq 0$.

In view of these results, it is apparent that the $\mathcal{C}^\infty$ wellposedness is not an achievable goal for weakly hyperbolic equations with non-analytic coefficients. On the other hand we know, after the pioneering work of Ivrii (see [6]) that the Cauchy problem for any hyperbolic equation with sufficiently smooth coefficients is well posed in every Gevrey class γ^s with $1 \leq s < \bar{s}(m)$, for some $\bar{s}(m) > 1$. More precisely:

(V) Bronshtein [1] proved that every hyperbolic equation of order $\leq m$ (or, more generally, every equation with characteristics of multiplicity $\leq m$), with coefficients $\mathcal{C}^\infty$ in t and γ^s in x, enjoys γ^s wellposedness for

$$1 \leq s < 1 + \frac{1}{m-1}\,.$$

The upper bound $\bar{s} = m/(m-1)$ is sharp, for example, the Cauchy problem for the Eqs. (8) and (9) is ill posed in each γ^s with $s > 2$; however, it can be improved in particular cases. For some special classes of equations, we have γ^∞ wellposedness (i.e., γ^s wellposedness for all $s \geq 1$), a property which is not very far from $\mathcal{C}^\infty$ wellposedness considering that the most common $\mathcal{C}^\infty$ functions belong to the space

$$\gamma^\infty = \bigcup_{s>1} \gamma^s.$$

VI Colombini et al. [3] proved that, for every non-negative $a(t) \in \mathcal{C}^\infty$, the Cauchy problem for (10) is γ^∞ well posed. More precisely, if $a(t) \in \mathcal{C}^k$, the problem is well posed in γ^s for $s \leq 1 + k/2$, while for $a(t)$ analytic it is well posed in $\mathcal{C}^\infty$.

The result in (VI) relies upon a delicate lemma of real analysis which does not apply to higher order equations (see Remark 1 in Sect. 3). It suggests that, for equations with infinitely differentiable but non-analytic coefficients, the γ^∞ wellposedness is the natural substitute to the $\mathcal{C}^\infty$ wellposedness in the analytic case. Thus, going back to (IV), it is natural to expect that the condition (3) implies γ^∞ wellposedness for {(1),(2)}, in case of $\mathcal{C}^\infty$ coefficients. In fact, we prove:

Theorem 1 *If the coefficients* $a_j \in \mathcal{C}^\infty([0,T])$, *and the characteristic roots are real and satisfy* (3)*, the Cauchy problem* {(1),(2)} *is* γ^∞ *well posed. More precisely, if* $a_j \in \mathcal{C}^k([0,T])$ *for some* $k \geq 2$, *we have* γ^s *wellposedness for*

$$1 \leq s < 1 + \frac{k}{2\,(m-1)}.$$

When the a_j*'s are analytic on* $[0,T]$, *the problem is* $\mathcal{C}^\infty$ *well posed.*

Outline of the proof. The crucial point in our discussion is that, if (3) holds, then the Eq. (1) admits a quasi-symmetrizer which can only degenerate in a diagonal way. This allows us to extend to (1) the technique introduced in [3] for (10).

Entering into more details, we first reduce, by Fourier transform, (1) to the system

$$V'(t,\xi) = i\xi\, A(t)\, V(t,\xi),$$

where $A(t)$ is the $m \times m$ *Sylvester matrix* with eigenvalues $\{\lambda_j(t)\}$. Then, in view of the γ^∞ wellposedness, we try to prove that, for $|\xi| \geq 1$,

$$|V(t,\xi)| \leq |V(0,\xi)|\, e^{C(\delta)\,|\xi|^\delta}, \quad \forall\, \delta > 0,\ \ 0 \leq t \leq T. \tag{11}$$

To this end, we look for a *quasi-symmetrizer* of $A(t)$, that is, a family of coercive Hermitian matrices $Q_\varepsilon(t)$, $0 < \varepsilon \leq 1$, such that

$$\{Q_\varepsilon A - A^* Q_\varepsilon\} \to 0 \quad \text{as } \varepsilon \to 0\,.$$

We recall that in [5] and [8], it was constructed, for any Sylvester matrix $A(t)$, a smooth quasi-symmetrizer $Q_\varepsilon(t)$, which is a (matrix valued) polynomial in ε, and satisfies:

$$C^{-1}\varepsilon^{2(m-1)}|V|^2 \leq (Q_\varepsilon V, V) \leq C|V|^2, \quad |(\{Q_\varepsilon A - A^* Q_\varepsilon\}V, V)| \leq C\varepsilon\,(Q_\varepsilon V, V).$$

Using such a quasi-symmetrizer, we define the *approximate energy*

$$E_\varepsilon(t,\xi) = (Q_\varepsilon(t)V(t,\xi), V(t,\xi)),$$

and we estimate the ratio $E'_\varepsilon/E_\varepsilon$. The properties of Q_ε are sufficient to ensure the γ^s wellposedness for $s < m/(m-1)$ (as in Bronshtein's theorem), but, in order to get (11) for all $\delta > 0$, we need an additional property of Q_ε, namely that

$$\int_0^T \frac{(Q'_\varepsilon(t)V(t), V(t))}{(Q_\varepsilon(t)V(t), V(t))}\, dt \;\le\; C(\delta)\, \varepsilon^{-\delta}, \quad \forall\, \delta > 0, \tag{12}$$

for every continuous function $V : [0,T] \to \mathbb{C}^m$.

Now, (12) holds for any family $\{Q_\varepsilon(t)\}$ of diagonal matrices (see Lemma 1 of [3]), but it is not true in general. However, we prove here (Lemma 2) that (12) holds when $Q_\varepsilon(t)$ is *nearly diagonal* in the sense that there are two constants $c_0, C_0 > 0$, and a family of diagonal matrices $\Lambda_\varepsilon(t)$, for which

$$c_0\,(\Lambda_\varepsilon(t)V, V) \;\le\; (Q_\varepsilon(t)V, V) \;\le\; C_0\,(\Lambda_\varepsilon(t)V, V). \tag{13}$$

Consequently, if we succeed to find a quasi-symmetrizer $Q_\varepsilon(t)$ of this type for the matrix $A(t)$, we can conclude that our Cauchy problem is γ^∞ well posed. Now it turns out (Proposition 3) that, for a given Sylvester matrix $A(t)$, the condition (3) on the eigenvalues ensures the existence of a quasi-symmetrizer satisfying (13). Thus, we reach the conclusion of Theorem 1.

The plan of the paper is as follows: In Sect. 2, we make explicit the condition (3) for equations of order $m \le 3$; in Sect. 3, we state the definition and basic properties of "nearly diagonal matrices"; in Sect. 4, we develop the theory of quasi-symmetrizers; in Sect. 5, we prove Theorem 1; finally, in Sect. 6, we write down explicitly the quasi-symmetrizers for $m = 2, 3$.

2 Second order and third order equations

The condition (3) can be written under the equivalent form

$$\sum_{1\le i<j\le m} \Big[(\lambda_i(t)^2 + \lambda_j(t)^2) \prod_{\substack{1\le h<k\le m \\ (h,k)\ne(i,j)}} (\lambda_h(t) - \lambda_k(t))^2\Big] \;\le\; M\,\Delta(t), \tag{14}$$

for some new constant M, where $\Delta(t) = \prod_{i<j} (\lambda_i(t) - \lambda_j(t))^2$ is the *discriminant* of (1). The left hand side of (14) is a symmetric polynomial in $\lambda_1, \ldots, \lambda_m$, hence, by Newton's theorem, it is also a polynomial in the coefficients $a_1, \ldots, a_m$. Thus, (14) provides an expression of (3) in terms of the a_j's. In particular:

(i) ($m = 2$) For the equation

$$\partial_t^2 u + a_1(t)\partial_t\partial_x u + a_2(t)\partial_x^2 u = 0,$$

where $\Delta(t) = a_1^2 - 4a_2 \geq 0$, we have, for some constant $c > 0$,

$$(3) \iff \Delta(t) \geq c\, a_1(t)^2. \tag{15}$$

(i) ($m = 3$) For the equation

$$\partial_t^3 u + a_1(t)\partial_x\partial_t^2 u + a_2(t)\partial_x^2\partial_t u + a_3(t)\partial_x^3 u = 0, \tag{16}$$

where $\Delta(t) = -4a_2^3 - 27a_3^2 + a_1^2 a_2^2 - 4a_1^3 a_3 + 18a_1a_2a_3 \geq 0$, we have

$$(3) \iff \Delta(t) \geq c\,(a_1(t)a_2(t) - 9a_3(t))^2. \tag{17}$$

The equivalence (15) is trivial. To prove (17), we recall that

$$a_1 = -(\lambda_1 + \lambda_2 + \lambda_3), \quad a_2 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1, \quad a_3 = -\lambda_1\lambda_2\lambda_3,$$

and we appeal to the identity

$$\sum_{\substack{1\leq i<j\leq 3\\ l\neq i,j}} (\lambda_i^2 + \lambda_j^2)(\lambda_i - \lambda_l)^2(\lambda_j - \lambda_l)^2 = \frac{5}{2}\Delta + \frac{1}{2}(a_1a_2 - 9a_3)^2. \tag{18}$$

Such an identity can be proved by a direct computation, taking care that the left and the right sides of (18) are two polynomials of order 6 in $(\lambda_1, \lambda_2, \lambda_3)$ which coincide on the six hyperplanes $\{\lambda_i = 0\}$ and $\{\lambda_i = \lambda_j\}, 1 \leq i < j \leq 3$.

Particularly simple is the *traceless case,* i.e., the case when $a_1(t) \equiv 0$. Here:

$$a_2 = -\frac{1}{2}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) \leq 0, \qquad \Delta(t) = -4a_2^3 - 27a_3^2,$$

so that

$$(3) \iff \Delta(t) \geq -c\,a_2(t)^3 \iff \Delta(t) \geq c' a_3(t)^2.$$

In such a case, Theorem 1 was already proved in [14]. Note that the general case cannot be reduced to the traceless case by a change of variables, without going out of the class of Eq. (16). Another special case of Theorem 1 for the Eq. (16), namely the case when $a_3(t) \equiv 0$, was proved in [10].

3 Nearly diagonal matrices

Given a matrix $Q = [q_{ij}]_{i,j=1,\dots,m}$, we consider the *diagonal part* of Q, i.e.,

$$Q_\Delta = \begin{pmatrix} q_{11} & & 0 \\ & \ddots & \\ 0 & & q_{mm} \end{pmatrix}.$$

Then we state:

Definition 1 *A family* $\{Q_\alpha\}$ *of nonnegative Hermitian matrices is called* nearly diagonal *when one of the following equivalent conditions is fulfilled:*

(a) There is some positive constant c_0 for which

$$(Q_\alpha V, V) \geq c_0 \, (Q_{\alpha,\Delta} V, V), \quad \forall \alpha. \tag{19}$$

(b) There are some positive constants c_0, C_0 for which

$$c_0 \, (Q_{\alpha,\Delta} V, V) \leq (Q_\alpha V, V) \leq C_0 \, (Q_{\alpha,\Delta} V, V), \qquad \forall \alpha.$$

(c) There is a family $\{\Lambda_\alpha\}$ of real diagonal matrices, and $c_0, C_0 > 0$, such that

$$c_0 \, (\Lambda_\alpha V, V) \leq (Q_\alpha V, V) \leq C_0 \, (\Lambda_\alpha V, V), \quad \forall \alpha.$$

The equivalence between (a), (b) and (c) is an easy consequence of the estimate

$$(Q \, V, V) \leq m \, (Q_\Delta V, V), \tag{20}$$

which holds true for any $m \times m$ matrix such that $Q = Q^* \geq 0$. Indeed, we have

$$|\, q_{ij}| = |\, (Qe_{(i)}, e_{(j)})| \leq (Qe_{(i)}, e_{(i)})^{1/2} \, (Qe_{(j)}, e_{(j)})^{1/2} = \sqrt{q_{ii} \, q_{jj}}\,, \tag{21}$$

and hence (20) follows ready.

Now we prove a sufficient condition for the near diagonality, which will be used later.

Lemma 1 *Let* Q *be a non-negative Hermitian,* $m \times m$ *matrix such that*

$$\det Q > 0, \quad \det Q \geq c \, \det Q_\Delta \equiv c \, q_{11} \, q_{22} \cdots q_{mm}, \tag{22}$$

for some $c > 0$*. Then, we have*

$$(Q \, V, V) \geq c \, m^{1-m} \, (Q_\Delta V, V). \tag{23}$$

Consequently, a family $\{Q_\alpha\}$ *satisfying* (22) *with a constant c independent of* α*, is nearly diagonal.*

Proof From (20) it follows that the matrix $T_Q = (Q_\Delta)^{-1/2}\, Q\, (Q_\Delta)^{-1/2}$ is well defined and satisfies

$$0 \le (T_Q V, V) \le m\,|V|^2.$$

Hence, $\|T_Q\| \le m$. On the other hand, from (22) it follows that

$$\det T_Q = \frac{\det Q}{\det Q_\Delta} \ge c.$$

Consequently, if $0 < \mu_1 \le \cdots \le \mu_m$ are the eigenvalues of T_Q, we get

$$(T_Q\, V, V) \ge \mu_1\,|V|^2 = \frac{\mu_1 \cdots \mu_m}{\mu_2 \cdots \mu_m}\,|V|^2 \ge \frac{\det T_Q}{\|T_Q\|^{m-1}}\,|V|^2 \ge \frac{c}{m^{m-1}}\,|V|^2.$$

□

The following result on nearly diagonal matrices depending on two real parameters will be crucial to provide a suitable energy estimate for Theorem 1.

Lemma 2 *Let $\{Q_\varepsilon(t) : 0 < \varepsilon \le 1, 0 \le t \le T\}$ be a nearly diagonal family of coercive Hermitian $m \times m$ matrices of class $\mathcal{C}^k$ in $t \in [0, T], k \ge 1$. Then we have, for any continuous function $V : [0, T] \to \mathbb{C}^m$,*

$$\int_0^T \frac{|(Q'_\varepsilon(t)V(t), V(t))|}{(Q_\varepsilon(t)V(t), V(t))^{1-1/k}\,|V(t)|^{2/k}}\,\mathrm{d}t \le C_T\,\|Q_\varepsilon\|^{1/k}_{\mathcal{C}^k([0,T])}. \tag{24}$$

Remark 1 When $m = 1$, (24) is a special case of

$$\int_0^T \frac{|f'(t)|}{|f(t)|^{1-1/k}}\,\mathrm{d}t \le C_T \|f\|^{1/k}_{\mathcal{C}^k([0,T])} \tag{25}$$

for every complex function $f \in \mathcal{C}^k([0, T])$. This inequality was proved in [3] for a nonnegative $f(t)$, and was more recently extended by Tarama to every real function ([15], cf. also [13]). The case of a complex valued $f(t)$ can be easily derived from the real case, by applying (25) to $\Re f$ and $\Im f$. We note that, if $Q_\varepsilon(t)$ is a diagonal matrix, (24) follows directly by applying (25) to each of the entries $q_{ii}(t)$. Similarly, if $V(t) \equiv V$ is constant, we obtain (24) by applying (25) to the scalar function $f(t) = (Q_\varepsilon(t)V, V)$. However, to get (24) for a non-diagonal matrix $Q_\varepsilon(t)$, and non-constant $V(t)$, we must require that $Q_\varepsilon(t)$ satisfies some property of the type (19).

Proof Let $Q_\varepsilon(t) = [q_{\varepsilon,ij}(t)],\ V = (v_1, \ldots, v_m)^t \in \mathbb{C}^m$. By (21), it follows

$$|q_{\varepsilon,ij}(t)||v_i||v_j| \le \sqrt{q_{\varepsilon,ii}(t)q_{\varepsilon,jj}(t)}\,|v_i||v_j| \le \sum_h^{1,m} q_{\varepsilon,hh}(t)|v_h|^2,$$

and hence, by (19),

$$|(Q_\varepsilon(t)V, V)| \ge c_0\,|q_{\varepsilon,ij}(t)||v_i||v_j|, \quad \forall\, i,j \in \{1,\ldots,m\}. \tag{26}$$

Thus, noting that $|V|^2 \ge |v_i||v_j|$ for all i, j, we obtain

$$\begin{aligned}
\int_0^T \frac{|(Q'_\varepsilon(t)V(t), V(t))|}{(Q_\varepsilon(t)V(t), V(t))^{1-1/k}\,|V(t)|^{2/k}}\,\mathrm{d}t &\le \int_0^T \sum_{i,j}^{1,m} \frac{|\,q'_{\varepsilon,ij}(t)|\,|v_i(t)|\,|v_j(t)|}{(Q_\varepsilon(t)V(t), V(t))^{1-1/k}\,|V(t)|^{2/k}}\,\mathrm{d}t \\
&\le \int_0^T \sum_{i,j}^{1,m} \frac{|\,q'_{\varepsilon,ij}(t)|\,|v_i(t)|\,|v_j(t)|}{\{c_0\,|q_{\varepsilon,ij}(t)|\,|v_i(t)|\,|v_j(t)|\}^{1-1/k}\,\{|v_i(t)|\,|v_j(t)|\}^{1/k}}\,\mathrm{d}t \\
&= c_0^{-(1-1/k)} \sum_{i,j}^{1,m} \int_0^T \frac{|\,q'_{\varepsilon,ij}(t)|}{|q_{\varepsilon,ij}(t)|^{1-1/k}}\,\mathrm{d}t.
\end{aligned}$$

By applying (25) to each one of the scalar functions $q_{\varepsilon,ij}(t)$, we get (24). □

4 The quasi-symmetrizer

We first recall, from [5] (cf. [7],[8],[9]), the construction of a quasi-symmetrizer for a hyperbolic matrix of *Sylvester type*, then we show that this quasi-symmetrizer is nearly diagonal when the eigenvalues satisfy (3).

Let $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m$; the $m \times m$ Sylvester matrix with eigenvalues $\{\lambda_i\}$ is

$$A(\lambda) \equiv A^{(m)}(\lambda) = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ -\sigma_m^{(m)}(\lambda) & & \ldots & & -\sigma_1^{(m)}(\lambda) \end{pmatrix}, \tag{27}$$

where the last row is formed by the *elementary symmetric functions*

$$\sigma_h^{(m)}(\lambda) = (-1)^h \sum_{1\le i_1<\cdots<i_h\le m} \lambda_{i_1}\ldots\lambda_{i_h}, \quad 1 \le h \le m. \tag{28}$$

Denoting by $\mathcal{P}_m$ the class of permutations on $\{1,\ldots,m\}$, we put, for $\lambda \in \mathbb{R}^m$,

$$\lambda_\rho = (\lambda_{\rho_1}, \ldots, \lambda_{\rho_m}), \quad \text{where } \rho \in \mathcal{P}_m.$$

We also put

$$\pi_i\lambda = (\lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots, \lambda_m), \quad \lambda' = \pi_m\lambda = (\lambda_1, \dots, \lambda_{m-1}).$$

Therefore, the quasi-symmetrizer defined in [5], is the Hermitiam matrix

$$Q_\varepsilon^{(m)}(\lambda) = \sum_{\rho\in\mathcal{P}_m} P_\varepsilon^{(m)}(\lambda_\rho)^* P_\varepsilon^{(m)}(\lambda_\rho), \quad 0 < \varepsilon \le 1, \tag{29}$$

where

$$P_\varepsilon^{(m)}(\lambda) = H_\varepsilon^{(m)} P^{(m)}(\lambda), \quad H_\varepsilon^{(m)} = \mathrm{diag}\{\varepsilon^{m-1}, \dots, \varepsilon, 1\}, \tag{30}$$

and where the $m \times m$ matrix $P^{(m)}(\lambda)$ is defined inductively by

$$P^{(1)}(\lambda) = 1, \quad P^{(m)}(\lambda) = \begin{pmatrix} & & & 0 \\ & P^{(m-1)}(\lambda') & & \vdots \\ & & & 0 \\ \sigma_{m-1}^{(m-1)}(\lambda') & \dots & \sigma_1^{(m-1)}(\lambda') & 1 \end{pmatrix}. \tag{31}$$

Actually, $P^{(m)}(\lambda)$ is depending only on $\lambda' = (\lambda_1, \dots, \lambda_{m-1})$.

The following properties of the quasi-symmetrizer follow directly from the definition:

Proposition 1 ([5]) *The matrix defined in* (29) *satisfies:*

$$Q_\varepsilon^{(m)}(\lambda) = Q_0^{(m)}(\lambda) + \varepsilon^2 Q_1^{(m)}(\lambda) + \dots + \varepsilon^{2(m-1)} Q_{m-1}^{(m)}(\lambda), \tag{32}$$

$$C_m(\lambda)^{-1} \varepsilon^{2(m-1)} |V|^2 \le (Q_\varepsilon^{(m)}(\lambda)V, V) \le C_m(\lambda)|V|^2, \tag{33}$$

$$\left| \left(\left\{ Q_\varepsilon^{(m)}(\lambda)A(\lambda) - A(\lambda)^* Q_\varepsilon^{(m)}(\lambda) \right\} V, V \right) \right| \le C_m(\lambda)\, \varepsilon\, (Q_\varepsilon(\lambda)V, V), \tag{34}$$

for some $C_m(\lambda)$ *bounded for* $|\lambda|$ *bounded.*

Moreover, $Q_1^{(m)}(\lambda), \dots, Q_{m-1}^{(m)}(\lambda)$ *are nonnegative Hermitian matrices with entries symmetric polynomials in* $\lambda_1, \dots, \lambda_m$.

Next, we prove some additional properties of $Q_\varepsilon^{(m)}(\lambda)$, and more specifically of the principal term $Q_0^{(m)}(\lambda)$. To this end, we define the *row vectors*

$$W_i^{(m)}(\lambda) = \left(\sigma_{m-1}^{(m-1)}(\pi_i\lambda), \dots, \sigma_1^{(m-1)}(\pi_i\lambda), 1\right), \quad 1 \le i \le m, \tag{35}$$

and the matrix

$$\mathcal{W}^{(m)}(\lambda) = \begin{pmatrix} W_1^{(m)}(\lambda) \\ \vdots \\ W_m^{(m)}(\lambda) \end{pmatrix} \equiv \left[\sigma_{m-j}^{(m-1)}(\pi_i\lambda)\right]_{i,j=1,\dots,m}. \tag{36}$$

Noting that $\sigma^{(m)}_{j+1}(\lambda) = \sigma^{(m-1)}_{j+1}(\pi_i\lambda) - \lambda_i\,\sigma^{(m-1)}_{j}(\pi_i\lambda)$, we see that $W^{(m)}_i(\lambda)$ is a *left eigenvector* of $A^{(m)}(\lambda)$ with eigenvalue λ_i (actually, up to a dilatation, it is the only of such eigenvectors). Consequently, the matrix

$$\mathcal{W}^{(m)}(\lambda)^*\,\mathcal{W}^{(m)}(\lambda) \;\equiv\; \left[\sum_{h=1}^{m} \sigma^{(m-1)}_{m-i}(\pi_h\lambda)\,\sigma^{(m-1)}_{m-j}(\pi_h\lambda)\right]_{i,j=1,\ldots,m} \tag{37}$$

is a (possibly singular) *exact symmetrizer* for $A^{(m)}(\lambda)$.

Then we define, for any $(m-1)\times(m-1)$ matrix T, the $m \times m$ matrix

$$T^\sharp \;=\; \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix}, \tag{38}$$

and, using the discriminant

$$\Delta(\lambda) \;=\; \prod_{1\le i<j\le m} (\lambda_i - \lambda_j)^2, \tag{39}$$

we prove:

Proposition 2 *The quasi-symmetrizer* (29) *satisfies*

$$Q^{(m)}_\varepsilon(\lambda) \;=\; Q^{(m)}_0(\lambda) \;+\; \varepsilon^2 \sum_{i=1}^{m} Q^{(m-1)}_\varepsilon(\pi_i\lambda)^\sharp, \tag{40}$$

$$Q^{(m)}_0(\lambda) \;=\; (m-1)!\,\mathcal{W}^{(m)}(\lambda)^*\,\mathcal{W}^{(m)}(\lambda), \tag{41}$$

$$\det\, Q^{(m)}_0(\lambda) \;=\; (m-1)!\,\Delta(\lambda), \tag{42}$$

$$q^{(m)}_{0,11}(\lambda)\cdots q^{(m)}_{0,mm}(\lambda) \;\le\; C_m \prod_{1\le i<j\le m} (\lambda_i^2 + \lambda_j^2). \tag{43}$$

Proof of (40) By (30), (31), and (35), it follows

$$P^{(m)}_\varepsilon(\lambda) \;=\; P^{(m)}_0(\lambda) \;+\; \varepsilon\, P^{(m-1)}_\varepsilon(\lambda')^\sharp, \tag{44}$$

and hence, writing $W^{(m)}_m(\lambda) = (W'(\lambda), 1)$ where $W'(\lambda)$ is an $(m-1)-$row vector,

$$P^{(m)}_0(\lambda) \;=\; \begin{pmatrix} 0 & 0 \\ W'(\lambda) & 1 \end{pmatrix}, \quad P^{(m-1)}_\varepsilon(\lambda')^\sharp \;=\; \begin{pmatrix} P^{(m-1)}_\varepsilon(\lambda') & 0 \\ 0 & 0 \end{pmatrix}. \tag{45}$$

Consequently, we see that

$$P^{(m)}_0(\lambda)^*\,P^{(m-1)}_\varepsilon(\lambda')^\sharp \;=\; \left\{P^{(m-1)}_\varepsilon(\lambda')^\sharp\right\}^*\,P^{(m)}_0(\lambda) \;=\; 0,$$

and hence, by (44),

$$P_\varepsilon^{(m)}(\lambda)^* P_\varepsilon^{(m)}(\lambda) \;=\; P_0^{(m)}(\lambda)^* P_0^{(m)}(\lambda) \;+\; \varepsilon^2 \left[P_\varepsilon^{(m-1)}(\lambda')^* P_\varepsilon^{(m-1)}(\lambda')\right]^\sharp .$$

Going back to (29), we obtain (40).

Proof of (41) Here we omit, for the sake of brevity, the superscripts (m), writing $Q_0\,, P_0\,, W_i\,, \mathcal{W}$, in place of $Q_0^{(m)}, P_0^{(m)}, W_i^{(m)}, \mathcal{W}^{(m)}$. By (45) it follows

$$Q_0(\lambda) \;=\; \sum_{\rho\in\mathcal{P}_m} P_0(\lambda_\rho)^* P_0(\lambda_\rho) \;=\; \sum_{\rho\in\mathcal{P}_m} W_m(\lambda_\rho)^* W_m(\lambda_\rho).$$

On the other hand, for each fixed index i, we see that $W_m(\lambda_\rho) = W_i(\lambda)$ for each one of the $(m-1)!$ permutations ρ with $\rho_m = i$. Hence, recalling (36), we obtain

$$Q_0(\lambda) \;=\; (m-1)! \sum_{i=1}^{m} W_i(\lambda_\rho)^* W_i(\lambda_\rho) \;=\; (m-1)!\, \mathcal{W}(\lambda)^* \mathcal{W}(\lambda).$$

Proof of (42) Let us consider the *column vectors* $V(\lambda_j) = (1, \lambda_j, \ldots, \lambda_j^{m-1})^t$, with $j = 1, \ldots, m$, and the $m \times m$ matrix

$$\mathcal{V}(\lambda) \;=\; \big(V(\lambda_1)\; V(\lambda_2)\; \ldots\; V(\lambda_m)\big) \;\equiv\; [\lambda_j^{i-1}]_{i,j=1,\ldots,m}\,.$$

Up to a dilatation, $V(\lambda_i)$ is the only *right eigenvector* of $A(\lambda)$ with eigenvalue λ_i. Recalling (35), we see that

$$W_i(\lambda)\, V(x) \;=\; \sum_{1\le h\le m} \sigma_{m-h}^{(m-1)}(\pi_i\lambda)\, x^{h-1} \;=\; \prod_{1\le h\le m,\; h\ne i} (x-\lambda_h), \quad \forall x \in \mathbb{R},$$

Hence, taking $x = \lambda_j$, and $x = \lambda_i$, it follows that

$$W_i(\lambda)\, V(\lambda_j) \;=\; \begin{cases} 0, & \text{if } i \ne j, \\ \prod_{1\le h\le m,\; h\ne i} (\lambda_i - \lambda_h), & \text{if } i = j, \end{cases}$$

so that

$$\det\,(\mathcal{W}(\lambda)\, \mathcal{V}(\lambda)) \;=\; \prod_{i=1}^{m} W_i(\lambda)\, V(\lambda_i) \;=\; \prod_{i\ne j} (\lambda_i - \lambda_j) \;=\; \Delta(\lambda). \tag{46}$$

But $\det \mathcal{V}(\lambda) = \pm\sqrt{\Delta(\lambda)}$, since $\mathcal{V}(\lambda)$ is the *Vandermonde* matrix of $(\lambda_1, \ldots, \lambda_m)$, hence, by (46) we conclude that $\det \mathcal{W}(\lambda) = \pm\sqrt{\Delta(\lambda)}$. Thus (41) gives (42).

Proof of (43) Setting $\alpha_i = q_{0,ii}^{(m)}(\lambda)$, $\theta_i = \lambda_i^2$, we have, by (41) and (37),

$$\alpha_{m-h} = \frac{1}{(m-1)!}\sum_{j=1}^{m}\sigma_h^{(m-1)}(\pi_j\lambda)^2 = \frac{1}{(m-1)!}\sum_{j=1}^{m}\left\{\sum_{\substack{1\le i_1<\cdots<i_h\le m\\ i_p\neq j}}\lambda_{i_1}\cdots\lambda_{i_h}\right\}^2$$
$$\le C_m \sum_{1\le i_1<\cdots<i_h\le m}\theta_{i_1}\cdots\theta_{i_h}.$$

Hence

$$\alpha_1\cdots\alpha_m \le C_m \sum \theta_i\cdot(\theta_{j_1}\theta_{j_2})\cdots(\theta_{p_1}\cdots\theta_{p_h})\cdots(\theta_{q_1}\cdots\theta_{q_{m-1}}),$$

the sum being extended to all the sets of indices among 1 and m such that

$$j_1 < j_2\,;\ \ldots\ ;\ p_1 < \cdots < p_h\,;\ \ q_1 < \cdots < q_{m-1}. \tag{47}$$

To get (43), we prove the inequality

$$\Theta \equiv \theta_i\cdot(\theta_{j_1}\theta_{j_2})\cdots(\theta_{p_1}\cdots\theta_{p_h})\cdots(\theta_{q_1}\cdots\theta_{q_{m-1}}) \le \prod_{1\le i<j\le m}(\theta_i+\theta_j), \tag{48}$$

for all $(\theta_1,\ldots,\theta_m)\in\mathbb{R}_+^m$, and all the choices of indices satisfying (47). Now, assuming $\theta_1 \ge \cdots \ge \theta_m$, we have $\theta_{i_1}\cdots\theta_{i_h} \le \theta_1\cdots\theta_h$, since the i_p's are different from each other, thus

$$\Theta \le \theta_1\cdot(\theta_1\theta_2)\cdot(\theta_1\theta_2\theta_3)\cdots(\theta_1\theta_2\cdots\theta_{m-1}) = \theta_1^{m-1}\theta_2^{m-2}\cdots\theta_{m-1}.$$

But, the term on the right hand side of the last equality is one of the terms of the development of the product $\prod_{i<j}(\theta_i+\theta_j)$, hence we get (48).

This concludes the proof of Proposition 2. □

We are now in a position to prove that the quasi-symmetrizer of a Sylvester matrix with eigenvalues satisfying (3), is nearly diagonal.

Proposition 3 *For each $M > 0$, let us define the set*

$$\mathcal{S}_M = \left\{\lambda\in\mathbb{R}^m : \lambda_i^2+\lambda_j^2 \le M(\lambda_i-\lambda_j)^2, \quad 1\le i<j\le m\right\}. \tag{49}$$

Then, the family of matrices $\left\{Q_\varepsilon^{(m)}(\lambda) : 0<\varepsilon\le 1,\ \lambda\in\mathcal{S}_M\right\}$ is nearly diagonal.

Proof By Definition 1, we easily see that, if $\{T_\alpha\}$ is a nearly diagonal family of $(m-1)\times(m-1)$ matrices, then also $\{T_\alpha^\sharp\}$ (see (38)) is nearly diagonal. Moreover, if $\{Q_\alpha\},\{Q'_\alpha\}$ are nearly diagonal families, then the sum $\{Q_\alpha+Q'_\alpha\}$

is so. Hence, recalling (40) and using an inductive argument, it will be sufficient to prove that $\{Q_0^{(m)}(\lambda)\}_{\lambda\in\mathcal{S}_M}$ is nearly diagonal.

To this end, we split $\mathcal{S}_M$ into the subsets $\{\Delta(\lambda)\neq 0\}$ and $\{\Delta(\lambda)=0\}$, where $\Delta(\lambda)$ is the discriminant. On the first set we can apply Lemma 1; indeed, by (42), (43), and (49), there is some constant $c_M>0$ for which

$$\det Q_0^{(m)}(\lambda) \;\equiv\; \Delta(\lambda) \;\geq\; c_M\, q_{0,11}(\lambda)\cdots q_{0,mm}(\lambda), \quad \forall\,\lambda\in\mathcal{S}_M. \tag{50}$$

Let now take $\lambda\in\mathcal{S}_M$ with $\Delta(\lambda)=0$. By (49) it follows that the only multiple eigenvalue is $\lambda_i=0$, hence we can assume, up to a rearrangement, that

$$\lambda_1=\cdots=\lambda_{k-1}=0, \qquad 0<|\lambda_k|<|\lambda_{k+1}|<\cdots<|\lambda_m|\,.$$

Then we construct a sequence $\{\lambda^{(h)}\}$, belonging to $\mathcal{S}_{M'}$ for some $M'=M'(m,M)$, in such a way that $\Delta(\lambda^{(h)})\neq 0$, and $\{\lambda^{(h)}\}\to\lambda$ as $h\to\infty$. For instance, we can take $\lambda_j^{(h)}=j\lambda_k/hk$ if $1\leq j<k$, and $\lambda_j^{(h)}=\lambda_j$ if $k\leq j\leq m$. Thus, by an argument of continuity, we conclude that the condition (19) holds true also on $\mathcal{S}_M\cap\{\Delta(\lambda)=0\}$. □

5 Proof of Theorem 1

By the theorem of Bony and Schapira, we know that $\{(1),(2)\}$ is well posed in the analytic class γ^1. Thus, we prove Theorem 1 only for $s>1$.

Setting $U=(\partial_x^{m-1}u,\partial_x^{m-2}\partial_t u,\ldots,\partial_t^{m-1}u)^t$, and

$$A(t) \;=\; \begin{pmatrix} 0 & 1 & 0 & \ldots & \ldots \\ & 0 & 1 & & \vdots \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ a_m(t) & \ldots & & a_2(t) & a_1(t) \end{pmatrix}, \tag{51}$$

we transform the scalar Eq. (1) into the first order system

$$U_t \;=\; A(t)\,U_x. \tag{52}$$

Then, effecting the Fourier transform, $V(t,\xi)=\mathcal{F}_x U(t,x)$, we get

$$V'(t,\xi) \;=\; i\,\xi A(t)\,V(t,\xi). \tag{53}$$

We associate to the matrix $A(t)$ the quasi-symmetrizer $Q_\varepsilon(t)=Q_\varepsilon^{(m)}(\lambda(t))$ introduced in §4, recalling that $Q_\varepsilon^{(m)}(\lambda)$ is a symmetric polynomial in the eigenvalues $\lambda_1,\ldots,\lambda_m$, hence is also a polynomial in the coefficients $a_1,\ldots,a_m$. In conclusion, $Q_\varepsilon(t)$ is a smooth, nonnegative Hermitian matrix with the following properties:

$$Q_\varepsilon(t) \;=\; Q_0(t) \,+\, \varepsilon^2\, Q_1(t) \,+\, \cdots + \varepsilon^{2(m-1)}\, Q_{m-1}(t)\,, \tag{54}$$

$$C^{-1}\, \varepsilon^{2(m-1)}\, |V|^2 \;\le\; (Q_\varepsilon(t)V, V) \;\le\; C\, |V|^2, \tag{55}$$

$$|\, (\{Q_\varepsilon(t)A(t) - A(t)^*\, Q_\varepsilon(t)\}\, V,\, V\,)\,| \;\le\; C\,\varepsilon\, (Q_\varepsilon(t)V, V)\,, \tag{56}$$

$$\text{the family } \{Q_\varepsilon(t) : 0 < \varepsilon \le 1,\ t \in [0, T]\} \text{ is nearly diagonal.} \tag{57}$$

Moreover, the matrices $Q_0(t), \dots, Q_m(t)$ have the same regularity in t as $A(t)$. Consequently, we obtain the a priori estimate

$$\begin{aligned} E'_\varepsilon \;&=\; (Q'_\varepsilon(t)V, V) \,+\, i\,\xi\, (\{Q_\varepsilon A - A^* Q_\varepsilon\}V, V) \\ &\le\; (K_\varepsilon(t,\xi) \,+\, C\varepsilon\, |\xi|)\, E_\varepsilon, \end{aligned} \tag{58}$$

where

$$E_\varepsilon(t,\xi) \;=\; (Q_\varepsilon(t)V(t,\xi), V(t,\xi)), \tag{59}$$

and

$$K_\varepsilon(t,\xi) \;=\; \frac{|\, (Q'_\varepsilon(t)V(t,\xi), V(t,\xi))\,|}{(Q_\varepsilon(t)V(t,\xi), V(t,\xi))}. \tag{60}$$

Now, in view of Gronwall's Lemma, we estimate the integral of K_ε.

(i) Let $A(t) \in \mathcal{C}^k([0, T])$. Therefore, also $Q_\varepsilon(t)$ belongs to $\mathcal{C}^k([0, T])$ so that

$$\|Q_\varepsilon\|^{1/k}_{\mathcal{C}^k([0,T])} \;\le\; M_T \;<\; \infty.$$

Thanks to (57), we can apply Lemma 2. By (24) and (55) we obtain, for all ξ,

$$\int_0^T K_\varepsilon(t,\xi)\, \mathrm{d}t \;=\; \int_0^T \frac{|(Q'_\varepsilon(t)\, V(t,\xi), V(t,\xi))|}{(Q_\varepsilon(t)V(t,\xi), V(t,\xi))}\, \mathrm{d}t \;\le\; C_T\, \varepsilon^{-2(m-1)/k},$$

hence, going back to (58) and applying Gronwall's Lemma, we find

$$E_\varepsilon(t,\xi) \;\le\; \left\{ C_T\, \varepsilon^{-2(m-1)/k} \;+\; Ct\,\varepsilon\, |\xi| \right\} E_\varepsilon(0,\xi).$$

Choosing $\varepsilon^{1+2(m-1)/k} = |\xi|^{-1}$, and recalling (55), we get the a priori estimate

$$|V(t,\xi)| \;\le\; C\, |\xi|^{k/(2\sigma)}\, e^{C|\xi|^{1/\sigma}}\, |V(0,\xi)|, \qquad \text{with } \sigma = 1 + k/[2(m-1)],$$

which implies that (52) is well posed in γ^s for $s < \sigma$. Indeed, a compactly supported function $U(x)$ belongs to γ^s if and only if, for some $C, \delta, \nu > 0$,

$$|\hat{U}(\xi)| \leq C\,|\xi|^\nu\, e^{-\delta\,|\xi|^{1/s}}.$$

(ii) Assume now that $A(t)$ is analytic on $[0, T]$. In this case, the entries of $Q_\varepsilon(t)$ are analytic functions of the form

$$q_{\varepsilon,ij}(t) = q_{0,ij}(t) + \varepsilon^2\, q_{1,ij}(t) + \cdots + \varepsilon^{2(m-1)}\, q_{m-1,ij}(t).$$

Hence we can find a partition of $[0, T]$, independent of ε, say

$$0 = \tau_0 < \tau_1 < \cdots < \tau_N = T, \tag{61}$$

in such a way that, for $h = 0, \ldots, N$, one has

$$|q'_{\varepsilon,ij}(t)| \leq C_1 \left(\frac{1}{t-\tau_h} + \frac{1}{\tau_{h+1}-t}\right) |q_{\varepsilon,ij}(t)| \quad \text{on } (\tau_h\,, \tau_{h+1}). \tag{62}$$

Indeed, let $f(t)$ be any one of the analytic functions $q_{k,ij}(t)$. Excluding the trivial case when $f \equiv 0, f(t)$ has a finite number of zeroes in $[0, T]$, and hence we can take the partition (61) such that $f(t) \neq 0$ in the interior of each of the intervals (τ_h, τ_{h+1}). Working first in $(0, \tau_1)$, we can write

$$f(t) \;=\; t^{\nu_0}\,(\tau_1 - t)^{\nu_1}\, g(t),$$

where ν_0, ν_1, are integers ≥ 0, and $g(t)$ is an analytic function never vanishing in $[0, \tau_1]$. Therefore, using the identity

$$f'(t) \;=\; \left(\nu_0 t^{\nu_0-1}(\tau_1 - t)^{\nu_1} - \nu_1 t^{\nu_0}(\tau_1 - t)^{\nu_1-1}\right) g(t) \;+\; t^{\nu_0}(\tau_1 - t)^{\nu_1} g'(t),$$

and noting that $|tg'(t)/g(t)|$ is bounded, we find a constant C_1 such that

$$t\,|f'(t)| \;=\; \left| f(t)\left(\nu_0 - \frac{\nu_1 t}{\tau_1 - t} + \frac{t\,g'(t)}{g(t)}\right)\right| \;\leq\; C_1\,|f(t)|, \quad \text{on } [\,0, \tau_1/2\,].$$

Similarly, we prove that

$$(\tau_1 - t)\,|f'(t)| \;\leq\; C_1\,|f(t)|, \qquad \text{on } [\,\tau_1/2\,, \tau_1],$$

hence we get (62) on $(0, \tau_1)$. We proceed in the same way on the intervals (τ_j, τ_{j+1}).

Next, we effect two different kinds of energy estimates: a Kovalewskian-type estimate near the endpoints $\{\tau_j\}$ of our partition and a hyperbolic-type estimate in the rest of $[0, T]$, where we argue as in (i), but using (62) in place of (25). Considering, for instance, the first interval $[0, \tau_1]$, we define, for $\varepsilon < \tau_1/2$,

$$E_\varepsilon(t,\xi) = \begin{cases} |V(t,\xi)|^2, & \text{on } [0,\varepsilon] \cup [\tau_1 - \varepsilon, \tau_1], \\ (Q_\varepsilon(t)V(t,\xi), V(t,\xi)), & \text{on } [\varepsilon, \tau_1 - \varepsilon]. \end{cases}$$

Thus, by (53) it follows

$$|E'_\varepsilon(t,\xi)| = |\xi\,((A(t) - A^*(t))V, V)| \leq 2\alpha\,|\xi|\,E_\varepsilon(t,\xi),$$

with $\alpha = \max_{0 \leq t \leq T} \|A(t)\|$, and hence

$$E_\varepsilon(t,\xi) \leq \begin{cases} e^{2\alpha\varepsilon\,|\xi|}\,E_\varepsilon(0,\xi) & \text{on } [0,\varepsilon], \\ e^{2\alpha\varepsilon\,|\xi|}\,E(\tau_1 - \varepsilon, \xi) & \text{on } [\tau_1 - \varepsilon, \tau_1]. \end{cases} \tag{63}$$

On the other hand, on $[\varepsilon, \tau_1 - \varepsilon]$ we have, by (26) and (62),

$$\begin{aligned} \int_\varepsilon^{\tau_1-\varepsilon} \frac{|(Q'_\varepsilon(t)V(t), V(t))|}{(Q_\varepsilon(t)V(t), V(t))}\,\mathrm{d}t &\leq c_0{}^{-1} \int_\varepsilon^{\tau_1-\varepsilon} \sum_{i,j}^{1,m} \frac{|q'_{\varepsilon,ij}(t)|}{|q_{\varepsilon,ij}(t)|}\,\mathrm{d}t \\ &\leq C_2 \int_\varepsilon^{\tau_1-\varepsilon} \Big(\frac{1}{t} + \frac{1}{\tau_1 - t}\Big)\,\mathrm{d}t \\ &= C_2 \log \frac{\tau_1 - \varepsilon}{\varepsilon} \leq C_2 \log \frac{T}{\varepsilon}, \end{aligned}$$

for some constant C_2. Thus, going back to (60) and (58), it follows

$$E_\varepsilon(t,\xi) \leq E_\varepsilon(\varepsilon,\xi)\,e^{C_T\,(\log(1/\varepsilon) + \varepsilon|\xi|)} \quad \text{on } [\varepsilon, \tau_1 - \varepsilon]. \tag{64}$$

Putting together (63) and (64), and recalling that $C^{-1}E_\varepsilon \leq |V|^2 \leq C\varepsilon^{-2(m-1)}E_\varepsilon$, we conclude, by (55), that

$$|V(t,\xi)| \leq C\,\varepsilon^{-(m-1)}e^{C_T\,(\log(1/\varepsilon) + \varepsilon|\xi|)}\,|V(0,\xi)| \quad \text{on } [0, \tau_1].$$

If we apply the same technique on the intervals $[\tau_1, \tau_2], \ldots, [\tau_{N-1}, T]$, we obtain

$$|V(t,\xi)| \leq C\,\varepsilon^{-N(m-1)}\,e^{NC_T\,[\log(1/\varepsilon) + \varepsilon|\xi|]}\,|V(0,\xi)| \quad \text{on } [0, T].$$

Taking $\varepsilon = |\xi|^{-1}$, this gives, for some $\nu = \nu(m, N, T)$,

$$|V(t,\xi)| \leq C\,|\xi|^\nu\,|V(0,\xi)|, \quad \text{on } [0, T], \tag{65}$$

hence, by the theorem of Paley-Wiener, we prove the $\mathcal{C}^\infty$ wellposedness of (52). □

6 Appendix: quasi-symmetrizers for m ≤ 3

Let us recall (27), (36), and (40). Then, for $m = 2$ we have:

$$A^{(2)}(\lambda) = \begin{pmatrix} 0 & 1 \\ -\lambda_1\lambda_2 & \lambda_1+\lambda_2 \end{pmatrix}$$

$$\mathcal{W}^{(2)}(\lambda) = \begin{pmatrix} -\lambda_2 & 1 \\ -\lambda_1 & 1 \end{pmatrix}$$

$$Q_\varepsilon^{(2)}(\lambda_1,\lambda_2) = \begin{pmatrix} \lambda_1^2+\lambda_2^2 & -(\lambda_1+\lambda_2) \\ -(\lambda_1+\lambda_2) & 2 \end{pmatrix} + 2\,\varepsilon^2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

and for $m = 3$ we have:

$$A^{(3)}(\lambda) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \lambda_1\lambda_2\lambda_3 & -(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1) & \lambda_1+\lambda_2+\lambda_3 \end{pmatrix}$$

$$\mathcal{W}^{(3)}(\lambda) = \begin{pmatrix} \lambda_2\lambda_3 & -(\lambda_2+\lambda_3) & 1 \\ \lambda_3\lambda_1 & -(\lambda_3+\lambda_1) & 1 \\ \lambda_1\lambda_2 & -(\lambda_1+\lambda_2) & 1 \end{pmatrix}$$

$$\begin{aligned} Q_\varepsilon^{(3)}(\lambda_1,\lambda_2,\lambda_3) &= 2\sum_{1\le i<j\le 3} \begin{pmatrix} (\lambda_i\lambda_j)^2 & -\lambda_i\lambda_j(\lambda_i+\lambda_j) & \lambda_i\lambda_j \\ -\lambda_i\lambda_j(\lambda_i+\lambda_j) & (\lambda_i+\lambda_j)^2 & -(\lambda_i+\lambda_j) \\ \lambda_i\lambda_j & -(\lambda_i+\lambda_j) & 1 \end{pmatrix} \\ &\quad + 2\,\varepsilon^2 \sum_{1\le i\le 3} \begin{pmatrix} \lambda_i^2 & -\lambda_i & 0 \\ -\lambda_i & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + 6\,\varepsilon^4 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

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