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The moduli space of surfaces with $K^2 = 6$ and $p_g = 4$

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0 Introduction

The motivation for our work stems from the questions posed by Enriques in Chapter VIII of his book "Le superficie algebriche" [10] about surfaces of general type with $p_g = 4$ and their moduli.

For these, $K^2 \geq 4$, and the cases $K^2 = 4, 5$ were completely classified by Enriques (cf. [10], section 2, chapter VIII, pp.268–271, and also [13]). Enriques also discussed at length the case $K^2 = 6$, which was later completely classified by Horikawa [14].

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The first question posed by Enriques was the following: for which value of K^2 does there exist a surface with $p_g = 4$ and birational canonical map? This existence question, posed by Enriques for $K^2 \geq 8$, was later solved by virtue of the contributions of several authors, and we now know that such surfaces exist for $7 \leq K^2 \leq 28$, (see [6,8], and see [15] for an example with $K^2 = 31$).

The answer to the question concerning classification and moduli is much harder, and a complete classification has been achieved up to now only for $K^2 \leq 7$; see for instance the monograph [1] for the case $K^2 = 7$.

The challenging open problem for $K^2 = 6, 7$ is to completely understand the structure of the moduli space, i.e., to determine the incidence correspondence of the various locally closed strata described in the classification.

Horikawa [12] showed that the moduli space for $K^2 = 5$ is connected, with two irreducible components of dimension 40 meeting along a divisor. He showed later [14] that for $K^2 = 6$ there are exactly four irreducible components and at most three connected components. He did so by first listing all possibilities for the canonical map, dividing the moduli space of surfaces with $K^2 = 6$ into 11 nonempty locally closed strata, and then analysing some of their local deformations.

More precisely, Horikawa named the 11 strata I_a, I_b, II, III_a, III_b, IV_{a_1}, IV_{a_2}, IV_{b_1}, IV_{b_2}, V_1, V_2 (see [14] or (1.3) below for precise definitions of each stratum). According to Horikawa's notation we define

Notation *Let A and B be two of the strata introduced above. The notation "$A \to B$" means that B intersects the closure of A, i.e., there is a deformation of a surface of type B to surfaces of type A (it suffices to have a flat family over a small disk $\Delta_\varepsilon \subset \mathbb{C}$ whose central fibre is of type B and whose general fibre is of type A).*

With this notation Horikawa summarized his results in the following diagram

$$\begin{tikzcd}
39 & & & & & \mathrm{III}_a \arrow[dl] \arrow[d] & \\
38 & & \mathrm{IV}_{a_1} \arrow[dl] \arrow[d] & \mathrm{I}_a \arrow[dl] \arrow[dd] & \mathrm{V}_1 \arrow[d] & \mathrm{III}_b \arrow[dl] & \mathrm{II} \\
37 & \mathrm{IV}_{a_2} \arrow[d] & \mathrm{IV}_{b_1} & & \mathrm{V}_2 & & \\
36 & \mathrm{IV}_{b_2} & & \mathrm{I}_b & & &
\end{tikzcd}$$

The numbers on the left are the dimensions of the strata in the corresponding row.

Our main result is:

Theorem 0.1 *The moduli space of surfaces with $p_g = 4, K^2 = 6$ has at most two connected components.*

In fact, II $\to$ III$_b$*, i.e., there is a deformation of surfaces of type* III$_b$ *to surfaces of type* II.

We would moreover like to pose the following

Question 0.2 *Is the above moduli space (for $p_g = 4, K^2 = 6$) disconnected?*

A possible reason for this could be that the surfaces of both connected components degenerate to surfaces with a genus two pencil. Therefore, if we view the corresponding algebraic surfaces as two respective symplectic four-manifolds (with the canonical symplectic structure defined in Catanese [7]), the former has a symplectic genus 2 Lefschetz pencil with transitive braid monodromy, the latter has a symplectic genus 2 Lefschetz pencil with nontransitive braid monodromy. Remains the problem to show whether a symplectic four–manifold underlying one of our surfaces does not admit two such different Lefschetz pencils.

The new idea that we exploit here is the following: the canonical models X of surfaces of type II are exactly the hypersurfaces of degree 9 in the weighted projective space $\mathbb{P}(1,1,2,3)$ (with rational double points as singularities). This remark was used in [5] to give a new explanation of the result of Horikawa that the moduli space is nonreduced on the open set II, and it implies that the canonical divisor is 2-divisible as a Weil divisor on X.

The same 2–divisibility occurs for type III$_b$, so for both type of surfaces we have a semicanonical ring $\mathcal{B} = \bigoplus H^0(\mathcal{O}(nL))$ (for some L with $2L \equiv K_X$), and what we do is to find a flat family of deformations of the semicanonical ring.

How to do this? The ring $\mathcal{B}$ is a Gorenstein ring, of codimension 1 in case II, of codimension 4 in case III$_b$, where X is embedded in $\mathbb{P}(1,1,2,3,4,5,6)$.

To describe the semicanonical ring and its deformations in case III$_b$, we use, as in our previous paper [2], the format of 4×4 Pfaffians of antisymmetric extrasymmetric 6×6 matrices.

This format applies to a codimension 2 subvariety of the stratum of surfaces of type III$_b$: these have a pencil of hyperelliptic curves of genus 3, and one has to lift (cf. [18]) this graded ring of dimension 1 to the semicanonical ring of the surface.

The deformation trick is similar to the one used in [2] for the canonical ring: filling entries of homogeneous degree 0 in the matrix with parameters (and respecting the extrasymmetric format). When these parameters are nonzero, three of the given Pfaffians can be used to eliminate the three variables of respective weights $4, 5, 6$. We obtain then a semicanonical ring of type II.

We want now to discuss briefly the cited method of extrasymmetric antisymmetric 6×6 matrices.

The main point here is the lack of a structure theorem for Gorenstein subvarieties of codimension 4 (for codimension 3 we have the celebrated theorem of Buchsbaum and Eisenbud [4]).

Several explicit formats were proposed by Dicks, Reid and Papadakis ([11,16,17]).

The geometric roots (cf. [19]) for the format we use here lie in the fact that the Segre product $\mathbb{P}^2 \times \mathbb{P}^2$ is embedded in $\mathbb{P}^8$ as the variety of 3×3 matrices A of rank 1, hence defined there by nine quadratic equations, admitting 16 syzygies.

If however one writes $A = B + C$, with C symmetric, B antisymmetric, then one can form the antisymmetric 6×6 matrix:

$$D = \begin{pmatrix} B & C \\ -C & B \end{pmatrix}.$$

The matrix D has an extrasymmetry from which follows indeed that the 15 (4×4) Pfaffians of D are not independent, but exactly reduce to the above 9 quadratic equations.

Using a flat family of deformations of the above subvariety, and interpreting the entries of the matrix as indeterminates to be specialized, one obtains an easy construction of Gorenstein subvarieties of codimension 4.

We refer to Reid [19] for a thorough discussion of the problem of understanding Gorenstein rings in codimension 4. Our result shows that the "moduli space" of such rings could be rather complicated, since this format does not apply for a general surface of type III_b, and moreover since we obtain a deformation from codimension 4 to codimension 1, but we observe that one cannot pass through all the lower codimensions.

In the following section we shall determine which of the eleven Horikawa classes yield canonical models X where the canonical divisor is 2-divisible as a Weil divisor (we use here the classical notation $\equiv$ for linear equivalence).

1 Surfaces with 2-divisible canonical divisor

Throughout the whole paper the ground field is supposed to be the field of the complex numbers $\mathbb{C}$.

Let S be the minimal model of a surface with $K_S^2 = 6, p_g = 4$, and X its canonical model (obtained by contracting the curves C with $K_S \cdot C = 0$). We recall that by [9] these surfaces are automatically regular, since they are minimal with $K^2 < 2p_g$.

In this section we shall classify the closed set of the moduli space given by the surfaces for which

(2-div) there exists a Weil divisor L on X such that $2L \equiv K_X$ and $h^0(\mathcal{O}_X(L)) \geq 2$.

Observe that this hypothesis immediately implies that the image of the canonical map ϕ_{K_X} is a quadric cone $Q \subset \mathbb{P}^3$ and that $h^0(\mathcal{O}_X(L)) = 2$.

Recall now that, if $|K_S|$ has a nonempty fixed part Φ and we write $|K_S| = |M| + \Phi$, then, since K_S is nef,

$$6 = K_S^2 = K_S \cdot \Phi + M \cdot \Phi + M^2 \geq M \cdot \Phi + M^2 \geq 2 + 4,$$

the last inequality following from the 2-connectedness of canonical divisors, and the fact that the canonical system is not a pencil (as shown in Horikawa [14]).

Thus, if the fixed part $\Phi \neq \emptyset$ we have

$$M^2 = 4, \quad M \cdot \Phi = 2, \quad K_S \cdot \Phi = 0.$$

Therefore we conclude that K_X has no fixed part on the canonical model X, and only the following cases a priori occur:

(0) $|K_X|$ is base point free and φ is 3-to-1
(2a) $|K_X|$ has two smooth base points or
(2b) $|K_X|$ has a smooth base point plus necessarily an infinitely near one (in both the cases (2a) and (2b) φ should be 2-to-1; they are however shown by Horikawa not to occur)
(1) $|K_X|$ has a singular base point p on X and φ is 2-to-1.

The last case is the only one where there is a fixed part Φ on the minimal model. Since to the point $p \in \mathrm{Sing}(X)$ corresponds the fundamental cycle Z (the -2-cycle pullback of the maximal ideal $\mathcal{M}_p \subset \mathcal{O}_{X,p}$), we have $\Phi \geq Z$. Z has the property that for each divisor Ψ which lies in the inverse image of p one has $Z \cdot \Psi \leq 0$.

Write $\Phi = Z + \Psi$, and assume $\Psi > 0$. Observe then that $M \cdot \Phi = 2, K_S \cdot \Phi = 0$ implies $\Phi^2 = -2$. Then $-2 = \Phi^2 = Z^2 + 2Z \cdot \Psi + \Psi^2 \leq -2 + 0 - 2 = -4$ is a contradiction, so that Φ equals the fundamental cycle. Therefore the base point scheme of $|K_X|$ is exactly the point p. Blowing up the base points of ϕ_{K_X} we obtain a morphism $f^0\colon X^0 \to Q$. We observe that there are at most two irreducible exceptional curves of $X^0 \to X$.

In any case the blow-up formula yields

- $K_{X^0} = \pi^*(K_X)$ in cases (0) and (1)
- in case (2) there are two -1-curves E_1, E_2 such that $K_{X^0} - E_1 - E_2 = \pi^*(K_X)$.

Remark 1.1 In case (0) the canonical map Φ_X is a finite morphism, whence (2-div) holds if and only if $\mathrm{Im}(\Phi_X)$ is a quadric cone, i.e., if and only if Φ_X has degree 3 (as it is easy to see, cf. Lemma 4.1 of [14]).

In the other two cases we have $\deg(\Phi_X) = 2$.

Remark 1.2 Assume that we are in cases (2) or (1) and that the canonical image is a quadric cone Q.

Let L', L^0, L'' be the respective proper transforms on X, resp. X^0, resp. S of a general line $l \subset Q$.

Observe that the canonical divisor K_X is the pull back of a hyperplane divisor on Q, and its pullback to X^0 has as movable part the pullback H of a hyperplane divisor on Q.

Whence $H \equiv 2L^0 + W$, where W is the effective divisor on X^0 corresponding to the inverse image of the vertex $v \in Q$.

More precisely, W is the fixed part of $|H - 2L^0|$.

Assume that there is a Weil divisor L satisfying (2-div): then one immediately sees that $h^0(\mathcal{O}_X(L)) = 2$ and there is an effective divisor E' on X with $L \equiv L' + E'$, and where E' is the fixed part of the pencil $|L|$. Then the fixed part of $K_X - 2L'$ equals $2E'$, whence $\pi_*(W) = 2E'$.

It follows that (the linear equivalence class of) $K_X \equiv 2L' + \pi_*(W)$ is 2-divisible as a Weil divisor if and only if $\pi_*(W)$ is 2-divisible as an effective divisor.

There are two possibilities for this: ${f^0}^{-1}(v)$ has dimension 0, or, in case where ${f^0}^{-1}(v)$ has dimension 1, we have $\pi_* {f^0}^{-1}(v) = 2E'$. In this last case, we have that f^0 factors through the Segre-Hirzebruch surface $\mathbb{F}_2$.

We recall now the case subdivision given by Horikawa in [14].

Definition 1.3 *Assume we are in case* (0) *(K_X has no base points): then we have type* I_a *if ϕ_{K_X} has degree* 1*, type* I_b *if ϕ_{K_X} has degree* 2*, type* II *if ϕ_{K_X} has degree* 3.

Otherwise we are in case (1) and ϕ_{K_X} has degree 2.

Type III is the case where there is no genus 2 pencil on X. There are two subcases: III_a, where the canonical image is a smooth quadric and we have two smooth base points, and III_b, where the canonical image is a quadric cone and we have one singular base point.

The two cases of type IV_{a_1}, IV_{a_2} have a smooth quadric as canonical image, the two cases of type IV_{b_1}, IV_{b_2} have a quadric cone as canonical image, X^0 is a double cover of $\mathbb{F}_2$, but the section Δ_∞ (inverse image of the singular point of Q) is not part of the branch locus.

Surfaces of type IV and V all have a genus 2 pencil. Surfaces of type V_1 and V_2 have a quadric cone as canonical image, X^0 is a double cover of $\mathbb{F}_2$, and the section at infinity is part of the branch locus. For type V_1 we are in case (2), for type V_2 we are in case (1).

Proposition 1.4 *The canonical model X of a surface with $K^2 = 6$, $p_g = 4$ satisfies condition (2-div) (there exists a Weil divisor L on X such that $2L \equiv K_X$ and $h^0(\mathcal{O}_X(L)) \geq 2$) if and only if it is of one of the following types:* II, III_b, V_1 *or* V_2.

Proof Since the canonical image must be a quadric cone Q, cases I_a, I_b, III_a, IV_{a_1}, IV_{a_2} are immediately excluded.

Assume that (2-div) holds for some surface of type IV_{b_1} or IV_{b_2}. We know that the section at infinity Δ_∞ is not part of the branch locus on $\mathbb{F}_2$.

Then, by our previous remark, the inverse image W_∞ of Δ_∞ on X^0 must be contracted by $\pi : X^0 \to X$.

In case (2), we observe that $|H|$ is base point free and $H \cdot E_i = 1$, whence E_i maps to a line and not to the vertex.

In case (1), again $|H|$ is base point free, and by Horikawa[14] Theorem 6.2 the base point of $|K_X|$ is an ordinary double point. Let F be the corresponding -2 curve: then $H \cdot F = 2$ and we have again a contradiction.

We have already seen that (2-div) holds for type II.

In case III_b the section Δ_∞ (cf. Theorem 5.2, ibidem) is isolated in the branch locus, whence its set theoretic inverse image is a smooth -1-curve W_∞, thus ${f^0}^{-1}(v)$ has dimension 0.

In cases $\mathrm{V}_1, \mathrm{V}_2$ again one shows that ${f^0}^{-1}(v)$ has dimension 0, using Theorems 6.1, resp. 6.2, ibidem. □

2 Surfaces of type II and III_b

In this section we want to concentrate on the surfaces of type II and III_b, that is on the surfaces for which (2-div) holds (cf. prop. 1.4), but there is no genus 2 pencil on S (as in cases V_1 and V_2).

The following lemma shows that the classes II and III_b are exactly those for which the pencil $|L|$ has no fixed part.

Lemma 2.1 *Assume that (2-div) holds and write $K_S \equiv 2\Lambda + Z$ where $K_S \cdot Z = 0$. Then the pencil $|L|$ is without fixed part (i.e., $L' \equiv L$, equivalently $\Lambda \equiv L''$) if and only if there is no genus 2 pencil on X. In this case, the general element in $|L''|$ is smooth irreducible of genus $g(L'') = 3$ and we have: $(L'')^2 = 1, L'' \cdot Z = 1$. It follows also that $Z^2 = -2$ and thus Z is the fundamental cycle of a singular point P_1 of X.*

Proof Write $|\Lambda| = |L''| + E''$ with $E'' > 0$. Since $K_S \equiv 2L'' + 2E'' + Z$, we have $L'' \cdot K_S + E'' \cdot K_S = 3$, and moreover $L'' \cdot K_S > 0, E'' \cdot K_S > 0$.

It is impossible that $L'' \cdot K_S = 1$, since then L''^2 is odd, and $L'' \cdot K_S \geq 2$ (L'' is nef), a contradiction.

Thus $L'' \cdot K_S = 2$ and $L''^2 = 0$, thus we have a genus 2 pencil.

Conversely, the curves of a genus 2 pencil map to the lines of the quadric cone Q, but if $|L|$ is without fixed part then $E'' = 0$, and we claim that $L'' = \Lambda$ is a pencil of genus 3 curves.

In fact, $L''K_S = 3 = 2(L'')^2 + L''Z$, thus $L''Z$ is odd. Since $L''Z$ is non negative and odd, while $L''K_S = 3$ implies that $(L'')^2$ is also odd, the only possibility is that $(L'')^2 = 1, L''Z = 1$. Whence, $p(L'') = 3$.

In particular, $|L''|$ has a unique smooth base point P_0 and the general curve in $|L''|$ is smooth by Bertini's theorem.

Since $L''Z = 1$, follows that $Z^2 = -2$, and since Z is the only divisor in $2L'' + Z \equiv K_S$ exceptional for $S \to X$, follows that Z is a fundamental cycle. □

Definition 2.2 *Assume that (2-div) holds. Then the semicanonical ring of X is the graded ring*

$$\mathcal{B} := \bigoplus_{m=0}^{\infty} \mathcal{B}_m := \bigoplus_{m=0}^{\infty} H^0(\mathcal{O}_X(mL)).$$

Remark 2.3 Obviously,

$$\mathcal{B}_{2m} \cong H^0(\mathcal{O}_X(mK_X)) \cong H^0(\mathcal{O}_S(mK_S)), \mathcal{B}_{2m+1} \cong H^0(\mathcal{O}_S(mK_S + L'')).$$

Lemma 2.4 *Assume that $|L|$ is without fixed part on X. Then the sequence*

$$0 \to H^0(\mathcal{O}_S(mK_S)) \to H^0(\mathcal{O}_S(mK_S + L'')) \to H^0(\omega_{L''}((m-1)K_S)) \to 0$$

is exact. Moreover, $\dim \mathcal{B}_{2m} = 5+3m(m-1)$ *for* $m \geq 2$, $\dim \mathcal{B}_{2m+1} = 7+3(m+1)(m-1)$ *for* $m \geq 1$.

Proof We have an exact sequence of sheaves given by the adjunction formula and moreover $h^1(\mathcal{O}(mK_S)) = 0$ for each m since $q(S) = 0$. The rest follows from the previous remark and from the fact that $K_S \cdot L'' = 3$ and a general L'' is smooth of genus 3. □

Lemma 2.5 *Consider a basis $\{x_0, x_1\}$ of $H^0(\mathcal{O}_X(L))$ and pick $y_2 \in H^0(\mathcal{O}_X(2L))$ in order to complete $\{x_0^2, x_0x_1, x_1^2\}$ to a basis of $H^0(\mathcal{O}_X(2L))$. Then $\{x_0^3, x_0^2x_1, x_0x_1^2, x_1^3, x_0y_2, x_1y_2\}$ are linearly independent and there exists an element z_3 completing the previous set to a basis of $H^0(\mathcal{O}_X(3L))$.*

Proof If not, w.l.o.g. we may assume that we have a relation

$$x_0y_2 = G_3(x_0, x_1)$$

where G is homogeneous of degree 3 and divisible by x_1.

But then, setting $y_{00} := x_0^2, y_{11} := x_1^2, y_{01} := x_0x_1$, we have

$$G_3(x_0, x_1) = x_1L(y_{00}, y_{11}, y_{01}, y_2)$$

where L is a linear form. Whence, the canonical image of X satisfies three quadratic equations

$$y_{00}y_{11} = y_{01}^2,\ y_{00}y_2 = y_{01}L(y),\ y_{01}y_2 = y_{11}L(y).$$

But it is shown by Horikawa [14] that $|K_S|$ is not a pencil.

There remains only to observe that $h^0(\mathcal{O}_X(3L)) = 7$, by the previous lemma. □

Lemma 2.6 *Assume that $|L|$ is without fixed part on X. The following three conditions are equivalent:*

- *X is of type* II, *i.e., the canonical map has degree* 3
- *the general curve in $|L''|$ is nonhyperelliptic*
- *the sections x_0, x_1, y_2, z_3 provide a morphism ψ to the weighted projective 3-space $\mathbb{P}(1,1,2,3)$ which is birational onto a hypersurface Σ of degree* 9.

In case III_b *we have:*

- *every curve in* $|L''|$ *is hyperelliptic,*
- *the base point* P_0 *of* $|L''|$ *is a Weierstraß point of every curve* L''
- Z *is the fixed part of* K_S
- *the canonical map is a double covering of a quadric cone.*

Proof Since $|L''|$ has a simple base point P_0 the general curve in the pencil is smooth of genus 3 and its canonical system has no base points. Whence $|3L'' + Z| = |K_S + L''|$ has no base points on the general curve L'', and in particular P_0 is not a base point; thus the base points are contained in Z. Whence on X the only possible base point (where $x_0 = x_1 = z_3 = 0$) is P_1.

We observe that there is a base point of ψ if and only if y_2 vanishes on P_1, and this case will be denoted by (*).

In case (*) P_1 is a point where $x_0 = x_1 = y_2 = 0$, whence Z is in the fixed part of the canonical system, but $|K_S - Z| = |2L''|$ has no base point since $(K_S - Z)^2 = 4$. Hence, $|K_S - Z|$ yields a morphism $f\colon X \to \mathbb{P}^3$ which is a double cover of the quadric cone, and P_1 is the unique point where $x_0 = x_1 = y_2 = 0$, and Z is exactly the fixed part of the canonical system.

In general, by Lemma 2.5 follows that ψ is birational if and only if the general curve in $|L'|$ is non-hyperelliptic, and in any case the degree of ψ is at most 2.

If we are not in case (*), we simply observe that ψ is then a morphism and that $\deg(\psi)\deg(\Sigma) = 9 = 6(3/2)$, but we have already seen that if the map is not birational, its degree is 2.

Whence, it follows that ψ is a birational morphism (and obviously then $\deg(\Sigma) = 9$).

In case (*) every curve in $|L'|$ is a double cover of a line, whence all the curves in $|L'|$ are hyperelliptic. Since the point P_0 is invariant by the involution of S yielding the hyperelliptic involution on every curve in $|L'|$, it follows that P_0 is a Weierstraß point.

More precisely, the exact sequence

$$0 \to \mathcal{O}_S(L'' + Z) \to \mathcal{O}_S(K_S) \to \mathcal{O}_{L''}(K_S) \to 0$$

and the remark that $\mathcal{O}_{L''}(K_S) = \omega_{L''}(-P_0)$ shows that $|K_S|$ has P_0 as a base point on L'' if and only if L'' is hyperelliptic and P_0 is a Weierstraß point. □

Horikawa gave a very concrete description of surfaces of type III_b.

Theorem 2.7 (5.2 in [14]) *Let S be a surface of type* III_b. *Then S is birationally equivalent to a double covering of* $\mathbb{F}_2$ *whose branch locus B consists of the negative section* Δ_∞ *and of* $B_0 \in |7\Delta_\infty + 14\Gamma|$ *which has a quadruple point x and a* $(3,3)$*-point at y such that x and y belong to the same fibre* $\Gamma_0 \in |\Gamma|$. *Moreover, y may be infinitely near to x.*

3 On rings associated to curves of genus 3

To compute the semi-canonical ring we use ideas related to the hyperplane section principle introduced by Miles Reid (cf. page 218 of [18]).

The hyperplane section principle: Let $\mathcal{B}$ be a graded ring, and $x_0 \in \mathcal{B}$ a homogeneous nonzero divisor of degree $\deg x_0 > 0$; set $\overline{\mathcal{B}} = \mathcal{B}/(x_0)$. The hyperplane section principle says that quite generally, the generators, relations and syzygies of $\mathcal{B}$ reduce mod x_0 to those of $\overline{\mathcal{B}}$.

Proposition 3.1 *Let $\mathcal{B}$ be the semicanonical ring of X and fix an element x_0 of degree 1 in $\mathcal{B}$ whose divisor yields a smooth curve $C \in |L''|$. Then the quotient ring $\overline{\mathcal{B}} = \mathcal{B}/(x_0)$ satisfies*

$$\overline{\mathcal{B}}_{2m+1} = H^0(\omega_{L''}((m-1)K_S)), \quad \overline{\mathcal{B}}_{2m} = H^0(\mathcal{O}_{L''}(mK_S)).$$

Proof The first assertion follows immediately from Lemma 2.4. The second assertion is immediately verified for $m = 1$, while, for $m \geq 2$, it follows from the exact sequence

$$\begin{aligned} 0 &\to H^0(\mathcal{O}_S(mK_S - L'')) \to H^0(\mathcal{O}_S(mK_S)) \to H^0(\mathcal{O}_{L''}(mK_S)) \\ &\to H^1(\mathcal{O}_S(mK_S - L'')) \to 0 \end{aligned}$$

and the vanishing of

$$h^1(\mathcal{O}_S(mK_S - L'')) = h^1(\mathcal{O}_S(-[(m-2)K_S + L'' + Z])).$$

Here we use Serre duality plus the fact that, on a regular surface ($q = 0$) $H^1(\mathcal{O}_S(-D)) = 0$ if the divisor D is effective and numerically connected (cf. [3]). That $(m-2)K_S + L'' + Z$ is numerically connected can be proved directly, but follows more easily for case III_b when we observe that by Theorem 5.2 of [14], Z is an irreducible -2-curve. □

Let us compute the ring $\overline{\mathcal{B}}$ for surfaces of type III_b.

We see immediately from the previous proposition, and from Lemma 2.6 that the quotient ring $\overline{\mathcal{B}}$ is isomorphic to a ring of the type described in the following

Definition 3.2 *Let C be a smooth hyperelliptic curve of genus 3, $p \in C$ be a Weierstraß point, and X a section of $H^0(\mathcal{O}_C(p))$ with $div(X) = p$.*

Consider the ring $R(C,p) := \bigoplus_{d \geq 0} H^0(\mathcal{O}_C(p)^{\otimes n})$ and define $R(C, \frac{3}{2}p)$ as the graded subring with

$$\begin{cases} R(C, \frac{3}{2}p)_{2d} & := R(C,p)_{3d} \\ R(C, \frac{3}{2}p)_{2d+1} & := R(C,p)_{3d+1} \end{cases},$$

and with product defined, for homogeneous elements, a and b, by $ab := a \otimes b$, unless a, b have odd degree, in which case we define $ab := a \otimes b \otimes X$.

So, the ring $\overline{\mathcal{B}}$ for surfaces of type III_b being a subring of $R(C,p)$, we need first to describe the latter.

The ring of a Weierstraß point of a smooth hyperelliptic curve is well known in every genus: for the convenience of the reader we state and prove here the result in the case of genus 3.

Lemma 3.3 *Let C be a hyperelliptic curve of genus 3, $p \in C$ a Weierstraß point: then $R(C,p) \cong \mathbb{C}[X,Y,T]/(T^2 - P_{14}(X,Y))$ where $\deg(X,Y,T) = (1,2,7)$ and P_{14} is homogeneous of degree 14.*

Proof Let X be a section of $H^0(\mathcal{O}_C(p))$ with $div(X) = p$: the section X is antiinvariant and the divisor p is invariant under the hyperelliptic involution σ, such that $\phi\colon C \to C/\sigma \cong \mathbb{P}^1$ is branched on a divisor B of degree 8.

The morphism ϕ is given by a basis of $H^0(\mathcal{O}_C(2p))$, for instance let us take X^2 and a new element Y.

We have $\phi_*\mathcal{O}_C = \mathcal{O}_{\mathbb{P}^1} \oplus Z\mathcal{O}_{\mathbb{P}^1}(-4)$, where Z is an equation for the ramification divisor of ϕ.

The even part of our ring is thus $\bigoplus_{m=0}^{\infty} H^0(\phi_*\mathcal{O}_C(m)) = \mathbb{C}[X^2, Y] \oplus Z\mathbb{C}[X^2, Y]$.
Since $Z \in H^0(\mathcal{O}_C(8p))$, and X divides Z, we may write

$$Z = XT, T \in H^0(\mathcal{O}_C(7p)).$$

Observe that X, Z being antiinvariant, then T is invariant.

Consider now $H^0(\mathcal{O}_C((2m + 1)p)$: this space splits as the direct sum of the (± 1)-eigenspaces. By looking at the behaviour at the ramification points, we see immediately that the sections of $H^0(\mathcal{O}_C((2m+1)p)^+$ are divisible by T, the ones of $H^0(\mathcal{O}_C((2m + 1)p)^-$ are divisible by X, thus the odd part of our ring is $T\mathbb{C}[X^2, Y] \oplus X\mathbb{C}[X^2, Y]$.

It follows that our ring is $\mathbb{C}[X, Y] \oplus T\mathbb{C}[X, Y]$.

Its ring structure is easily obtained when we observe that T^2 is the pull back of the equation of the seven remaining branch points: thus we have a relation of the form $T^2 = P_{14}(X, Y)$ and our claim is proven. □

Proposition 3.4 *Let C be a hyperelliptic curve of genus 3, $p \in C$ be a Weierstraß point. Then $R(C, \frac{3}{2}p) \cong \mathbb{C}[x,y,z,w,v,u]/I$, where $\deg(x,y,z,w,v,u) = (1,2,3,4,5,6)$ and the ideal I is generated by the 4×4 Pfaffians of the antisymmetric 'extrasymmetric' 6×6 matrix*

$$M = \begin{pmatrix} 0 & 0 & z & v & y & x \\ & 0 & w & u & z & y \\ & & 0 & \tilde{P}_9 & u & v \\ & & & 0 & w^2 & zw \\ & & & & 0 & 0 \\ -\mathrm{sym} & & & & & 0 \end{pmatrix},$$

where $\tilde{P}_9$ is a homogeneous polynomial of degree 9 in the variables x, y, z, w.

Proof Observe preliminarily that by our definition of $R(C, \frac{3}{2}p)$, X and T lie in the ring, but Y does not. However XY, YT, Y^2, Y^3 lie in the ring.

It is easy to verify that the following 6 elements

$$\begin{aligned} x &= X \quad y = XY \quad z = Y^2 \\ w &= Y^3 \quad v = T \quad u = YT \end{aligned}$$

generate $R(C, \frac{3}{2}p)$; note that, by the definition of the above six elements, the 2×2 minors of the matrix

$$\begin{pmatrix} x & y & z & v \\ y & z & w & u \end{pmatrix} \tag{1}$$

belong to the ideal I of relations (beware: the product is taken in the ring $R(C, \frac{3}{2}p)$).

The other relations for these generators come from the equation $T^2 - P_{14}$; we note that the coefficient of Y^7 in the polynomial P_{14} of Lemma 3.3 cannot vanish (or the eight branch points of the hyperelliptic map ϕ would not be distinct), and therefore we can assume w.l.o.g. that this coefficient is 1. We write then the relation as $T^2 - Y^7 - XP_{13}$. Let $\tilde{P}_9$ be a homogeneous polynomial of degree 9 in the variables (x, y, z, w) such that $\tilde{P}_9(X, XY, Y^2, Y^3) = P_{13}(X, Y)$ (we checked that it exists and is uniquely determined modulo the 2×2 minors of the matrix (1)).

The relation $T^2 - P_{14}$ is a relation in degree 14 in $R(C, p)$, and $R(C, p)_{14}$ is not contained in the subring $R(C, \frac{3}{2}p)$: but multiplying it by suitable monomials we obtain the following relations

$$\begin{aligned} XT^2 &- XY^7 - X^2P_{13}(X, Y) \quad \text{in degree 15} \\ YT^2 &- Y^8 - XYP_{13}(X, Y) \quad \text{in degree 16} \\ Y^2T^2 &- Y^9 - XY^2P_{13}(X, Y) \quad \text{in degree 18,} \end{aligned}$$

which can be rewritten as polynomials in the variables x, y, z, w, v, u (uniquely determined modulo the 2×2 minors of (1)) as

$$\begin{aligned} v^2 &- z^2w - x\tilde{P}_9 \quad \text{in degree 10} \\ vu &- zw^2 - y\tilde{P}_9 \quad \text{in degree 11} \\ u^2 &- w^3 - z\tilde{P}_9 \quad \text{in degree 12.} \end{aligned}$$

The ideal I generated by these three last equations and the 2×2 minors of (1) is the ideal of the 4×4 Pfaffians of the matrix in the statement.

That there are no other relations follows since the Hilbert function of $R(C, \frac{3}{2}p)$ is equal to the one of $\mathbb{C}[x, y, z, w, v, u]/I$. □

4 The family of deformations

The hyperplane section principle gives a strategy to reconstruct the ring $\mathcal{B}$ for every surface of type III_b: take C a smooth element of the pencil $|L''|$ and x_0 corresponding section, the 'hyperplane section' quotient $\overline{\mathcal{B}} = \mathcal{B}/(x_0)$ equals the ring $R(C, \frac{3}{2}p)$ of Proposition 3.4.

$\mathcal{B}$ is obtained from $R(C, \frac{3}{2}p)$ by adding the generator x_0 and deforming the 9 equations, adding suitable multiples of x_0 in such a way that all the syzygies also deform.

To compute all possible 'extensions' as above is in general very difficult. For these problems it is in general useful to use a 'flexible format' (i.e., with free parameters) as the one we are going to recall.

The extrasymmetric format. Let A be a polynomial ring.

An antisymmetric matrix M is defined to be *extrasymmetric* in [19], remark 6.6, if it has the form

$$\begin{pmatrix} 0 & a & b & c & d & e \\ & 0 & f & g & h & d \\ & & 0 & i & pg & pc \\ & & & 0 & qf & qb \\ & & & & 0 & pqa \\ -\mathrm{sym} & & & & & 0 \end{pmatrix}$$

for suitable elements $a, b, c, d, e, f, g, h, i, p, q \in A$. For example, if $p = q = 1$ and $a, b, c, d, e, f, g, h, i$ are general linear forms in $\mathbb{P}^8$, we get the ideal of the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8$ mentioned in the introduction.

The ideal I generated by the fifteen 4×4 Pfaffians of an antisymmetric extrasymmetric matrix is in fact generated by 9 of them; moreover, if the entries are general enough, this Pfaffian ideal has exactly 16 independent syzygies, *which can all be explicitly written as functions of the entries of the matrix* (the computation is done in [18] in a slightly more special case, and it extends to this more general case).

This implies that, if we have a ring that can be written in this form, and the ring has no further syzygies (except the 16 we know), deforming the entries of the matrix (preserving the symmetries) we get at the same time a deformation of the ideal and a deformation of the syzygies (i.e. a *flat* deformation).

The first example of a ring presented through the 4×4 Pfaffians of an antisymmetric extrasymmetric matrix was produced by D. Dicks and M. Reid in [17]. This more general form appeared in [19]; see also [2] for another application of it. M. Reid has recently found a further generalisation (seen by the authors at a conference in Utrecht in june 2004) which we do not explain here since we do not need it for our present purposes.

In Proposition 3.4 we wrote the ring $R(C, \frac{3}{2}p) = \overline{\mathcal{B}}$ in extrasymmetric format; therefore, if we add a variable x_0 in degree 1 and lift M to a matrix N that is still antisymmetric and extrasymmetric, its Pfaffians should define a surface of type III_b.

Note that, since the format is not complete in the sense of Kodaira-Spencer and Kuranishi, the family which we write may be (and in fact is) smaller than the whole family of surfaces of type III_b.

We add then the variable x_0 and rename the old variable x by x_1.

Theorem 4.1 *Consider the ring* $\mathbb{C}[x_0, x_1, y, z, w, v, u]$ *with variables of respective degrees* $(1, 1, 2, 3, 4, 5, 6)$.

Consider an antisymmetric extrasymmetric matrix

$$M' = \begin{pmatrix} 0 & 0 & z & v & y & x_1 \\ & 0 & w & u & P_3 & y \\ & & 0 & P_9 & u & v \\ & & & 0 & wP_4 & zP_4 \\ & & & & 0 & 0 \\ -\mathrm{sym} & & & & & 0 \end{pmatrix}.$$

where the P_i *are homogeneous polynomials of degree i in the first* 5 *variables of the ring, and let J be the ideal generated by the* 4×4 *Pfaffians of M.*

Then, for general choice of the polynomials P_i, $\mathbb{C}[x_0, x_1, y, z, w, v, u]/J$ *is the semi-canonical ring of a surface of type* III_b.

We obtain in this way a family of codimension 2 *(and therefore of dimension* 36*) in the stratum of the moduli space of surfaces with* $p_g = 4$, $K^2 = 6$ *corresponding to surfaces of type* III_b.

Proof J defines a subvariety X in $\mathbb{P}(1, 1, 2, 3, 4, 5, 6)$.

We first show that, for general choice of P_3, P_4 and P_9, X has only rational double points as singularities.

We denote by V the threefold (containing X) defined by the 2×2 minors of the submatrix of M'

$$B = \begin{pmatrix} z & v & y & x_1 \\ w & u & P_3 & y \end{pmatrix}.$$

Since the condition of having rational double points is well known to be open, it is sufficient to consider the case $P_3 = z$.

V is a cone over a quasismooth scroll in $\mathbb{P}(1, 2, 3, 4, 5, 6)$ (observe that the variable x_0 does not appear)): it is therefore quasismooth outside the vertex $p_0 := (1, 0, 0, 0, 0, 0, 0)$.

X is defined in V by the remaining three equations

$$v^2 = z^2 P_4 + x_1 P_9$$
$$vu = zwP_4 + yP_9$$
$$u^2 = w^2 P_4 + zP_9.$$

The three above equations describe (as we vary P_4 and P_9) an open set of a linear system of Weil divisors on V without other base points than p_0; by Bertini's theorem, for general choice of the coefficients of P_4 and P_9 the surface X is quasismooth outside p_0. We claim that the singular points other than p_0 must satisfy $x_0 = x_1 = 0$: in fact if, say, $x_1 \neq 0$, we may set $x_1 = 1$ and smoothness follows from quasismoothness.

If $x_0 = x_1 = 0$ the equations of V force $y = z = 0$, and consequently by the first of the above further three equations we get the vanishing of the coordinate v.

We are left with at most two nonzero coordinates, u and w, but consider the last equation ($u^2 = \cdots$): for general P_4 we have (up to a rescaling) $P_4 = w + \cdots$, thus we get only one point, exactly the point $(0, 0, 0, 0, 1, 0, 1)$.

This point is in fact a singular point of the ambient space, since the $\mathbb{C}^*$ action has in the corresponding point in $\mathbb{C}^7$ a nontrivial stabilizer $\cong \mathbb{Z}/2\mathbb{Z}$. Therefore in this point X has an isolated singularity locally isomorphic to the quotient of a smooth surface by a $\mathbb{Z}/2\mathbb{Z}$ action: in other words, a singular point of type A_1.

We are left with the vertex p_0 of the cone V. This point is smooth for the ambient space $\mathbb{P}(1, 1, 2, 3, 4, 5, 6)$: we set $\{x_0 = 1\}$ and work in affine coordinates.

For general choice of the coefficients P_9 is invertible at p_0: since all other entries of M do vanish in p_0 we get that all equations vanish in p_0 and only the Pfaffians including P_9 are smooth in it.

We get that the Zariski tangent space of X at p_0 has dimension 3, and more precisely it is $\{x_1 = y = z = 0\}$. We eliminate then (we have set $\{x_0 = 1\}$) x_1, y and z. We obtain the equation $wv = P_9^{-1} u(u^2 - w^2 P_4)$: a rational double point of type A_2.

We have thus shown that, for general choice of the coefficients, X has only rational double points as singularities.

The projection from V to the quadric cone $\mathbb{P}(1, 1, 2)$ given by the first three variables x_0, x_1, y has $\mathbb{P}^1$ as general fibre: for $x_1 \neq 0$ the equations of V can be explicitly solved, and yield $z = y^2/x_1$, $w = y^3/x_1^2$.

The remaining three equations cut clearly two points on the general $\mathbb{P}^1$ fibre of the projection to $\mathbb{P}(1, 1, 2)$: therefore the induced map $X \to \mathbb{P}(1, 1, 2)$ has degree 2.

We have the following recipe to obtain the branch curve:

- start with the polynomial $z^2 P_4 + x_1 P_9$;
- substitute $z \mapsto y^2/x_1$;
- substitute $w \mapsto y^3/x_1^2$;
- multiply the result by x_1^4.

What we get is a polynomial (in fact in the last step we get rid of the denominators) of degree 14 in the variables x_0, x_1, y (remember that we have assumed v, u not to appear in P_9), therefore the branch curve is a general curve of degree 14 in $\mathbb{P}(1, 1, 2)$ contained in the ideal $(y^7, x_1 y^6, x_1^2 y^4, x_1^3 y^3, x_1^4 y^2, x_1^5)$; these curves form a linear system on $\mathbb{P}(1, 1, 2)$ with base point $(1, 0, 0)$. Its general element has a point of multiplicity 5 in $(1, 0, 0)$ and is smooth elsewhere.

We blow up the point $(1,0,0)$, take the complete transform and remove 4 times the exceptional divisor. Then we get a triple point with tangent cone three times the direction of the exceptional divisor.

Blowing up again and removing the new exceptional divisor twice from the complete transform of the branch curve, we then obtain a 4-tuple point with (in general) three different tangent directions: after a last blow up we remain with at most non essential singularities.

Summarizing, our branch curve of degree 14 has a 5-tuple point in $(1,0,0)$ with an infinitely near $(3,3)$-point; by [14], thm. 5.2., the desingularisation of X is a surface of type III_b.

In order to count the number of moduli of this subfamily of surfaces of type III_b, we observe that the two singular points are infinitely near (one condition), the 'infinitely near' triple point is in the direction of the exceptional divisor (one condition).

Moreover, we have the further condition that the coefficient of the monomial yx_1^4 (in the equation of the branch curve) vanishes: in fact a straightforward computation shows that the two first conditions force the equation of the branch curve to be only in the ideal

$$(y^7, x_1y^6, x_1^2y^4, x_1^3y^3, x_1^4y^2, yx_1^4, x_1^5).$$

Under the assumption that $P_3 = z$, we have a $38-3=35$ dimensional family of surfaces of type III_b; in general we can (by row and column operations plus change of coordinates of the form $z \mapsto z + \cdots$) only assume $P_3 = z + \lambda x_0^3$. The projection on $\mathbb{P}(1,1,2)$ has clearly still degree 2 and we have a similar recipe for computing the branch curve. We still have a curve as described in Theorem 2.7, still the two base points are infinitely near, but the singular point can have multiplicity 4 (instead of 5), so are not in the subfamily and the dimension of the whole family is one more, i.e., 36. □

Remark 4.2 In the previous proof, to show that the general surface in our 36-dimensional family has only rational double points as singularities, we showed that the general surface in a subfamily of codimension 1 given by the assumption $P_3 = z$ has at most a A_1 singularity in $x_0 = x_1 = 0$, and a further singularity of type A_2.

For the general case we will always find a A_1 singularity in $x_0 = x_1 = 0$, since we know that the semicanonical pencil has a singular base point. The other singularity is only a consequence of the assumption $P_3 = z$ and disappear for a general choice of the P_i as for example (MAGMA code provided by the referee)

$$P_3 = z + x_0^3 \quad P_4 = w + 4y^2 + x_0x_1^3$$
$$P_9 = x_0^9 + 3x_0^2x_1^7 + x_1^9 + x_1y^4.$$

Remark 4.3 It is clear from the previous proof that the above family given by the Pfaffians of M' (having dimension 36, cf. 4.1) is a proper subfamily of the

37-dimensional family of surfaces of type III_b, such that the two essential singularities of the branch curve of the canonical double cover are infinitely near (cf. [14]). We do not have any geometric characterization for this subfamily.

We also do not know whether there is a larger family of surfaces of type III_b which deform to surfaces of type II.

Theorem 4.4 *Let* $(x_0, x_1, y, z, w, v, u)$ *be variables of respective degrees* $(1, 1, 2, 3, 4, 5, 6)$. *Let* M *be the* 6×6 *antisymmetric matrix*

$$M = \begin{pmatrix} 0 & t & z & v & y & x_1 \\ & 0 & w & u & P_3 & y \\ & & 0 & P_9 & u & v \\ & & & 0 & wP_4 & zP_4 \\ & & & & 0 & tP_4 \\ -\mathrm{sym} & & & & & 0 \end{pmatrix}.$$

where the P_i*'s are homogeneous polynomials of degree* i *in the above variables and* t *is the parameter on a small disk* $\Delta_\varepsilon \subset \mathbb{C}$.

For general choice of P_3, P_4 *and* P_9 *the* 4×4 *Pfaffians of* M *define a variety* $X \subset \Delta_\varepsilon \times \mathbb{P}(1, 1, 2, 3, 4, 5, 6)$ *whose projection on* Δ_ε *is flat, with central fibre a surface of type* III_b *and with general fibre a surface of type* II.

Proof The flatness of the above family (for general entries) follows directly from the flexibility of the format. For $t = 0$ the above matrix equals the matrix M' in thm. 4.1.

Assume now that $t \neq 0$.

Note that the Pfaffians Pf_{1235} and Pf_{1236} are of the form $tu - \cdots$ and $tv - \cdots$, and that for a general choice of P_4, the Pfaffian Pf_{1256} can be written as $t^2 w - ....$ Therefore, for $t \neq 0$, we can eliminate the variables u, v, w, and $R \cong \mathbb{C}[x_0, x_1, y, z]/J$ for a suitable ideal J; a straightforward calculation shows that J is a principal ideal generated by the equation obtained from Pf_{1234} after eliminating the variables u, v, w using Pf_{1235}, Pf_{1236} and Pf_{1256}.

This is a polynomial of degree 9, so the surface is birational to a hypersurface of degree 9 in $\mathbb{P}(1, 1, 2, 3)$, whence we obtain a surface of type II. □

Our main theorem (0.1) follows right away from the above. The referee has observed that, if you take the three quantities P_3, P_4 and P_9 in theorem 4.4 as independent variables, you get a parametrized families of Fano 5-folds defining a small deformation of a Fano of index 18 [containing a surface of type III_b as a complete intersection of degree $(3, 4, 9)$] to a hypersurface of degree 9.

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