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The Ruelle-Sullivan map for actions of $\mathbb{R}^n$

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Abstract. The Ruelle Sullivan map for an $\mathbb{R}^n$-action on a compact metric space with invariant probability measure is a graded homomorphism between the integer Cech cohomology of the space and the exterior algebra of the dual of $\mathbb{R}^n$. We investigate flows on tori to illuminate that it detects geometrical structure of the system. For actions arising from Delone sets of finite local complexity, the existence of canonical transversals and a formulation in terms of pattern equivariant functions lead to the result that the Ruelle Sullivan map is even a ring homomorphism provided the measure is ergodic.

1. Introduction

We consider a variety of cohomology groups for a continuous $\mathbb{R}^n$-action φ on a compact metric space X. Among them are the Cech cohomology $\check{H}(X, \mathbb{Z})$ of X and the dynamical cohomology of the dynamical system (X, φ) by which we mean the Lie-algebra cohomology $H(\mathbb{R}^n, C^\infty(X, \mathbb{R}))$ of $\mathbb{R}^n$ with coefficients in the φ-smooth real functions on X. An analog of the Cech-de Rham complex provides us with a graded ring homomorphism between the two.

Cohomology captures topological information about the dynamical system. If we are also given an invariant Borel probability measure μ then we can capture some of the geometric structure which comes from the (euclidean) geometry of the $\mathbb{R}^n$-orbits. We do this by exploiting Ruelle-Sullivan currents associated with sub-actions. These yield linear functionals on dynamical cohomology which we combine into a graded homomorphism from the cohomology to the exterior algebra of the dual $\mathbb{R}^{n*}$ of $\mathbb{R}^n$. The Ruelle Sullivan map

$$\tau_{\varphi,\mu} : \check{H}(X, \mathbb{Z}) \to \Lambda\mathbb{R}^{n*} \tag{1}$$

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is defined as the composition of the two homomorphisms. A similar group homomorphism related to the above but between the K-group $K(C(X))$ and $\Lambda\mathbb{R}^{n*}$ can be found in [C080].

Our aim here is to study some properties of the Ruelle-Sullivan map and to provide interesting and computable examples which illuminate that the Ruelle-Sullivan map detects geometrical structure. For instance, it captures the continuous eigenvalues of the action (Theorem 14) and can be used to distinguish cut & projection sets of fixed dimension and codimension, which have the same cohomology group but project out of geometrically different lattices (Theorem 15).

A large part of the paper is devoted to the more special class of $\mathbb{R}^n$-actions arising from Delone sets of finite local complexity in $\mathbb{R}^n$. In this case $X = \Omega_P$ is the continuous hull of such a Delone set P. This system has the particular advantage that P also defines canonical local transversals for the $\mathbb{R}^n$-action. These transversals are used to define $H(\mathbb{R}^n, C^\infty_{tlc}(\Omega_P, \mathbb{R}))$, the transversally locally constant dynamical cohomology of Ω_P. It arises from a subcomplex of the Lie-algebra complex and is isomorphic to the Cech cohomology $\check{H}(\Omega_P, \mathbb{R})$ with real coefficients under the above mentionned ring homomorphism.

Two further groups, $\check{H}_P(\mathbb{R}^n)$ and $H_P(\mathbb{R}^n)$, the strongly and weakly P-equivariant cohomology groups of $\mathbb{R}^n$, are defined as cohomologies of subcomplexes of the de Rham complex for $\mathbb{R}^n$. $\check{H}_P(\mathbb{R}^n)$ is a more intuitive picture of $H(\mathbb{R}^n, C^\infty_{tlc}(\Omega_P, \mathbb{R}))$. Theorems 20,23 contain the proof of the result announced in [Ke03] that the (strongly) P-equivariant cohomology is isomorphic to the Cech cohomology (with real coefficients) of the continuous hull of P. $H_P(\mathbb{R}^n)$ is the closure of $\check{H}_P(\mathbb{R}^n)$ in an appropriate sense and is isomorphic to the dynamical cohomology $H(\mathbb{R}^n, C^\infty(\Omega_P, \mathbb{R}))$. We use techniques from differential topology to show that the Ruelle-Sullivan map is even a ring homomorphism for ergodic $\mathbb{R}^n$-actions associated with Delone sets of finite local complexity (Theorem 26).

2. Ruelle-Sullivan currents

We consider (X, φ), a topological $\mathbb{R}^n$-dynamical system. That is, X is a compact metric space and φ is a continuous action of $\mathbb{R}^n$ on X. For each v in $\mathbb{R}^n$, $\varphi_v : X \to X$ is a homeomorphism of X and the map sending (x, v) to $\varphi_v(x)$ is jointly continuous. Moreover, we have $\varphi_v \circ \varphi_w = \varphi_{v+w}$, for all v, w in $\mathbb{R}^n$. We also call (X, φ) simply an $\mathbb{R}^n$-action.

We let μ be a φ-invariant Borel probability measure on X. Such measures exist always [G176]. In some cases we will assume that the system is ergodic; that is, a closed invariant set has either measure 1 or measure 0.

We let $C(X)$ denote the algebra of continuous complex functions on X. We call $f \in C(X)$ φ-differentiable if, given any $v \in \mathbb{R}^n$, the limits $\lim_{t\to 0} \frac{f(\varphi_{tv}(x))-f(x)}{t}$ exist for all $x \in X$ and hence define a function on X. If for each v this function is continuous then f is called continuously φ-differentiable. f is called $\varphi - smooth$

or just *smooth* if it is infinitely continuously φ-differentiable. We let $C^\infty(X)$ denote the set of smooth functions in $C(X)$. An easy adaptation of the usual arguments involving convolution with a smooth bump function shows that $C^\infty(X)$ is uniformly dense in $C(X)$.

More generally, if W is any finite dimensional real vector space, we let $C(X, W)$ denote the set of continuous W-valued functions on X. The definition of $C^\infty(X, W)$ extends easily.

We let $\mathbb{R}^{n*}$ denote the dual space of the vector space $\mathbb{R}^n$ and for $v \in \mathbb{R}^n$ and $w \in \mathbb{R}^{n*}$, we denote the pairing by $\langle v, w\rangle$. We let $\Lambda\mathbb{R}^{n*}$ denote the graded exterior algebra of $\mathbb{R}^{n*}$. We consider $C^\infty(X, \Lambda\mathbb{R}^{n*})$, which is also a graded ring under point-wise wedge-product, and define $d : C^\infty(X, \mathbb{R}) \to C^\infty(X, \mathbb{R}^{n*})$,

$$df(x)(v) = \lim_{t\to 0} \frac{f(\varphi_{tv}(x)) - f(x)}{t},$$

for $v \in \mathbb{R}^n$ and $x \in X$. This extends as usual to a differential

$$d : C^\infty(X, \Lambda^k\mathbb{R}^{n*}) \to C^\infty(X, \Lambda^{k+1}\mathbb{R}^{n*}),$$

i.e. a derivation of degree one with $d^2 = 0$. For concreteness, if $\{e_1, \dots, e_n\}$ is a basis for $\mathbb{R}^n$, we let $\{de_1, \dots, de_n\}$ denote the dual basis. Then every element of degree k is a linear combination of elements of the form

$$f de_{i_1} \wedge \cdots \wedge de_{i_k},$$

where $f \in C^\infty(X, \mathbb{R})$ and $i_j \in \{1, \dots, n\}$. Furthermore,

$$d(f de_{i_1} \wedge \cdots \wedge de_{i_k}) = \sum_{i=1}^{n} \partial_i f \, de_i \wedge de_{i_1} \wedge \cdots \wedge de_{i_k} \tag{2}$$

where $\partial_i f(x) = \lim_{t\to 0} \frac{f(\varphi_{te_i}(x)) - f(x)}{t}$.

Definition 1. *The Lie-algebra cohomology of* $\mathbb{R}^n$ *with values in* $C^\infty(X, \mathbb{R})$*,* $H(\mathbb{R}^n, C^\infty(X, \mathbb{R}))$*, is the cohomology of the complex* $(C^\infty(X, \Lambda\mathbb{R}^{n*}), d)$*. We call it also simply the dynamical cohomology of* (X, φ)*.*

The product on $C^\infty(X, \Lambda\mathbb{R}^{n*})$ induces a graded ring structure on $H(\mathbb{R}^n, C^\infty(X, \mathbb{R}))$. If φ is locally free, i.e. if for all $x \in X$ there is ϵ such that $|v| < \epsilon$ and $\varphi_v(x) = x$ imply $v = 0$, then X is a foliated space in the sense of [MS88] and the above definition coincides with their definition of tangential cohomology. However we do not require the local freeness of φ.

We may use our invariant measure μ to define a map from this cohomology to $\Lambda\mathbb{R}^{n*}$.

Definition 2. *The Ruelle-Sullivan current C_μ associated with μ is the linear map*

$$\langle C_\mu, \cdot\rangle : C^\infty(X, \Lambda\mathbb{R}^{n*}) \to \Lambda\mathbb{R}^{n*},$$

defined by

$$\langle C_\mu, \omega\rangle = \int_X \omega(x) d\mu(x).$$

Lemma 3. *Let μ be an invariant probability measure for the action φ. Then*

$$\langle C_\mu, d\omega\rangle = 0,$$

for all ω in $C^\infty(X, \Lambda\mathbb{R}^{n*})$.

Proof. This follows from (2) together with the invariance of μ under φ. □

Corollary 4. *The pairing with C_μ descends to a map on cohomology,*

$$\tilde{\tau}_{\varphi,\mu} : H(\mathbb{R}^n, C^\infty(X, \mathbb{R})) \to \Lambda\mathbb{R}^{n*}.$$

Remark 5. All of this section can be easily generalized to the case of an action φ of an arbitrary Lie-group G on X. Since this would be the right framework for tilings which have finite local complexity w.r.t. a larger subgroup of the Euclidean group than the translation group we indicate the main changes. The role of $\mathbb{R}^{n*}$ is played by the dual $\mathfrak{g}^*$ of the Lie-algebra $\mathfrak{g}$ of G and the definition of φ-differentiable is expressed with the help of the Lie-algebra action involving the exponential map $\exp : \mathfrak{g} \to G$. Thus the differential $d : C^\infty(X, \Lambda^k\mathfrak{g}^*) \to C^\infty(X, \Lambda^{k+1}\mathfrak{g}^*)$ is defined by

$$\begin{aligned} df(x)(v) &= \lim_{t\to 0} \frac{f(\varphi_{\exp tv}(x)) - f(x)}{t}, \\ dw(v_1, v_2) &= -w([v_1, v_2]), \end{aligned}$$

for $f \in C^\infty(X, \Lambda^0\mathfrak{g}^*)$ and $w \in \mathfrak{g}^*$ viewed as constant smooth function in $C^\infty(X, \Lambda^1\mathfrak{g}^*)$, $v, v_1, v_2 \in \mathfrak{g}$. Finally, the Ruelle-Sullivan current takes values in $\Lambda\mathfrak{g}^*$ and satisfies $\langle C_\mu, d\omega\rangle = d\langle C_\mu, \omega\rangle$. Hence it descends to a map on the Lie-algebra cohomology:

$$\tilde{\tau}_{\varphi,\mu} : H(\mathfrak{g}, C^\infty(X, \mathbb{R})) \to H(\mathfrak{g}, \mathbb{R}).$$

3. The Cech-cohomology and the Ruelle-Sullivan map

The purpose of this section is to construct a morphism from the Cech cohomology $\check{H}(X, \mathbb{Z})$ of X to the dynamical cohomology of (X, φ) and to prolong $\tilde{\tau}_{\mu,\varphi}$ to a map $\tau_{\mu,\varphi} : \check{H}(X, \mathbb{Z}) \to \Lambda\mathbb{R}^{n*}$, the Ruelle-Sullivan map.

We begin by fixing a open cover of $X, \mathcal{U} = \{U_i\}_i$, and a (smooth) partition of unity ρ_i which is subordinate to this cover, i.e. $\mathrm{supp}\rho_i \subset U_i$. For $i_0 < i_1 < \cdots < i_j$, we let

$$U_{i_0 i_1 \cdots i_j} = U_{i_0} \cap U_{i_1} \cap \cdots U_{i_j}.$$

We note that for any open set U, any continuous function f on U and any i, the expression $\partial_i f$ still makes sense as a function on U (assuming the limit exists) even though the set U is not invariant under φ. So we define $C^\infty(U, \Lambda\mathbb{R}^{n*})$ as the smooth functions on U taking values in $\Lambda\mathbb{R}^{n*}$.

We define the double complex

$$K^{j,k}(\mathcal{U}) = \bigoplus_{i_0 < i_1 < \cdots < i_j} C^\infty(U_{i_0 i_1 \cdots i_j}, \Lambda^k \mathbb{R}^{n*}),$$

where here and below the sum is only over $i_0 < i_1 < \cdots < i_j$ with non-empty $U_{i_0 i_1 \cdots i_j}$. The $i_0 \cdots i_j$-component of $\omega \in K^{j,k}(\mathcal{U})$ is denoted by $\omega_{i_0 \cdots i_j}$. For notational purposes, we also define $\omega_{i_0 \cdots i_j} = 0$ if $U_{i_0 i_1 \cdots i_j} = \emptyset$, and $\omega_{\sigma(i_0) \cdots \sigma(i_j)} = \mathrm{sgn}(\sigma)\omega_{i_0 \cdots i_j}$, for any permutation σ.

The double complex has two commuting differentials, the Lie-algebra differential $d : K^{j,k}(\mathcal{U}) \to K^{j,k+1}(\mathcal{U})$, defined above and the Cech differential

$$\delta : K^{j,k}(\mathcal{U}) \to K^{j+1,k}(\mathcal{U}),$$

given by

$$(\delta\omega)_{i_0 \cdots i_{j+1}} = \sum_{l=0}^{j+1} (-1)^l \omega_{i_0 \cdots \hat{i}_l \cdots i_{j+1}},$$

$\hat{i}_l$ meaning that i_l is omitted.

Using the subordinate partition of unity $\{\rho_i\}_i$ we define for $j > 0$, $h : K^{j,k}(\mathcal{U}) \to K^{j-1,k}(\mathcal{U})$,

$$(h\omega)_{i_0 \cdots i_{j-1}} = \sum_i \rho_i \omega_{i i_0 \cdots i_{j-1}}.$$

Then $h\delta + \delta h = 1$ and so for any k the complex (Mayer-Vietoris sequence)

$$0 \to C^\infty(X, \Lambda^k \mathbb{R}^{n*}) \xrightarrow{r} K^{0,k}(\mathcal{U}) \xrightarrow{\delta} \cdots K^{j,k}(\mathcal{U}) \xrightarrow{\delta} K^{j+1,k}(\mathcal{U}) \cdots \tag{3}$$

is exact, i.e. has trivial cohomology, where $r(\omega)_i$ is the restriction of ω to U_i.

We let $\check{C}(U_{i_0 i_1 \cdots i_j}, \mathbb{R})$ and $\check{C}(U_{i_0 i_1 \cdots i_j}, \mathbb{Z})$ denote the functions from $U_{i_0 i_1 \cdots i_j}$ to $\mathbb{R}$ and $\mathbb{Z}$, respectively, which are locally constant, and hence smooth. Then we set

$$\check{C}^j(\mathcal{U}, \mathbb{R}) = \bigoplus\nolimits_{i_0 < i_1 < \cdots < i_j} \check{C}(U_{i_0 i_1 \cdots i_j}, \mathbb{R}),$$
$$\check{C}^j(\mathcal{U}, \mathbb{Z}) = \bigoplus\nolimits_{i_0 < i_1 < \cdots < i_j} \check{C}(U_{i_0 i_1 \cdots i_j}, \mathbb{Z}).$$

It is clear that we have

$$\check{C}^j(\mathcal{U},\mathbb{Z}) \subset \check{C}^j(\mathcal{U},\mathbb{R}) \subset K^{j,0}(\mathcal{U}),$$

and together with the map δ, these form subcomplexes. We let i denote the inclusion of either of the first two in the last. The cohomology (in δ) of the subcomplexes no longer vanishes, because the corresponding maps h involve functions (the ρ_i) which are not locally constant. Then, $\check{H}(\mathcal{U},\mathbb{R})$ and $\check{H}(\mathcal{U},\mathbb{Z})$ are the Cech cohomologies of the covering $\mathcal{U}$ with coefficients in $\mathbb{R}$ and $\mathbb{Z}$, respectively. For notational convenience, we will use $\check{C}^j(\mathcal{U})$ and $\check{H}(\mathcal{U})$ to denote either of $\check{C}^j(\mathcal{U},\mathbb{R})$ or $\check{C}^j(\mathcal{U},\mathbb{Z})$ and the corresponding cohomology.

$\check{H}(\mathcal{U})$ carries a graded ring structure [BT82]. The product is induced from the map $\check{C}^j(\mathcal{U}) \times \check{C}^k(\mathcal{U}) \to \check{C}^{j+k}(\mathcal{U})$, $(\omega,\omega') \mapsto \omega\cdot\omega'$,

$$(\omega\cdot\omega')_{i_0\cdots i_{j+k}} = (-1)^{jk}\omega_{i_0\cdots i_j} \wedge \omega'_{i_{j+1}\cdots i_{j+k}}.$$

It is useful to have the following special case in mind. If φ is a free $\mathbb{R}^n$-action and X consists of a single orbit then the action induces on X the structure of a differentiable manifold and the Lie-algebra cohomology with coefficients in $C^\infty(X,\mathbb{R})$ agrees with its de Rham cohomology. The above double-complex is then the Cech-de Rham double complex used to construct a ring homomorphism between the Cech cohomology of X and its de Rham cohomology [BT82]. When Cech cohomology with real coefficients is considered this homomorphism becomes an isomorphism.

In our more general situation the same approach yields a ring homomorphism between the Cech cohomology of X and the dynamical cohomology of (X,φ), but it won't, in general, become an isomorphism when real coefficients are considered. This is the next result, whose proof goes exactly as in the de Rham theorem [BT82]. Note that h maps $K^{j,k}(\mathcal{U})$ to $K^{j-1,k}(\mathcal{U})$, while d maps $K^{j-1,k}(\mathcal{U})$ to $k^{j-1,k+1}(\mathcal{U})$. Hence, $(dh)^n$ will map $K^{n,0}(\mathcal{U})$ to $K^{0,n}(\mathcal{U})$, for all $n \geq 0$.

Theorem 6. *The maps*

$$(-1)^j r^{-1}(dh)^j i : \check{C}^j(\mathcal{U}) \to C^\infty(X, \Lambda^j\mathbb{R}^{n*}),$$

induce a graded ring homomorphism

$$\theta_{\varphi,\mathcal{U}} : \check{H}(\mathcal{U}) \to H(\mathbb{R}^n, C^\infty(X,\mathbb{R})),$$

with coefficients either $\mathbb{Z}$ or $\mathbb{R}$ in the Cech cohomology.

Recall that an open cover $\mathcal{U}'$ is a refinement of $\mathcal{U}$, if there is a map $\alpha : \mathcal{U}' \to \mathcal{U}$ such that $U \subset \alpha(U)$. Such a map induces a map at the level of cohomology $\check{H}(\mathcal{U}) \to \check{H}(\mathcal{U}')$ which is independent of the choice of α. The Cech cohomology of X, $\check{H}(X)$ is defined to be the inductive limit of the groups $\check{H}(\mathcal{U})$ over all open covers $\mathcal{U}$ of X. The morphisms $\theta_{\varphi,\mathcal{U}}$ furnish a graded ring homomorphism $\theta_\varphi : \check{H}(X) \to H(\mathbb{R}^n, C^\infty(X,\mathbb{R}))$, with coefficients either $\mathbb{Z}$ or $\mathbb{R}$ in the Cech cohomology.

Definition 7. *Let* (X, φ) *be an* $\mathbb{R}^n$*-action with* φ*-invariant measure* μ*. The Ruelle-Sullivan map* $\tau_{\varphi,\mu} : \check{H}(X, \mathbb{Z}) \to \Lambda\mathbb{R}^{n*}$ *is defined by*

$$\tau_{\varphi,\mu}(a) = \langle C_\mu, \theta_\varphi(a)\rangle.$$

In particular, if a is represented as an element of $\check{H}(\mathcal{U}, \mathbb{Z})$, where $\mathcal{U}$ is an open cover of X, then $\tau_{\varphi,\mu}(a) = \langle C_\mu, \theta_{\varphi,\mathcal{U}}(a)\rangle$.

The Ruelle-Sullivan map does not have to be injective. But it contains information coming from an invariant measure and therefore about the (euclidean) geometry of the orbits. The philosophy is that Cech cohomology together with the Ruelle-Sullivan map will furnish a better invariant for $\mathbb{R}^n$-actions than just Cech cohomology alone.

Clearly, the Ruelle-Sullivan map is a graded homomorphism of groups. We will discuss in Section 6.1 a large class of examples for which it is even a graded ring homomorphism.

Remark 8. Let us comment on the terminology. This is really just for motivation, so we will not go into any details. Given a foliation and a transverse invariant measure, Ruelle and Sullivan constructed a current, usually called the Ruelle-Sullivan current. In our situation we have an action rather than just a foliation but if the action is locally free this yields a foliated space, in the sense of Moore and Schochet [MS88]. However, we have much more. If one selects a linearly independent set of k vectors, one can define an action of $\mathbb{R}^k$ by restricting the flow to the linear span of the vectors. Any invariant measure for the $\mathbb{R}^n$-action can be used to give an invariant measure for this action. Associated to this is a Ruelle-Sullivan current which yields a map from the cohomology of X to the real numbers. If one fixes the cohomology class and considers the set of vectors to vary, this is an element of degree k in the exterior algebra of $\mathbb{R}^{n*}$. Hence, the Ruelle-Sullivan construction can be viewed as giving map from the cohomology to the exterior algebra of $\mathbb{R}^{n*}$. This is our Ruelle-Sullivan map.

Remark 9. Also the double complex generalises to the case of an arbitrary Lie-group action on X. As a result one gets a Ruelle-Sullivan map

$$\tau_{\varphi,\mu} : \check{H}(X, \mathbb{Z}) \to H(\mathfrak{g}, \mathbb{R}).$$

Remark 10. Connes presents in [C080] a similar group homomorphism in non-commutative geometry. He starts with a Lie-group G action on a C^*-algebra A with invariant trace tr to construct a group homomorphism from the K-group of A to the cohomology of the Lie-algebra,

$$\mathrm{Ch}_{tr} : K_i(A) \to \bigoplus_k H^{i+2k}(\mathfrak{g}, \mathbb{R}).$$

When $A = C(X)$ with trace induced from an invariant measure μ and rational coefficients are considered, Connes' construction specialises to the one considered above; Ch_{tr} factors through the Chern character identifying $K(C(X)) \otimes \mathbb{Q}$ with $\check{H}(X, \mathbb{Q})$. The question when this identification can be made even over integer coefficients is a lot harder.

Coming back to $G = \mathbb{R}^n$ where the Connes-Thom isomorphism yields $K_i\,(C(X)) \cong K_{i-n}(C(X) \rtimes_\varphi \mathbb{R}^n)$ the components of the Ruelle Sullivan map may be also be related to functionals on $K(C(X) \rtimes_\varphi \mathbb{R}^n)$ which arise from pairings with certain cyclic cocycles of $C^\infty(X) \rtimes_\varphi \mathbb{R}^n$ [C94]. We do not elaborate on that apart from mentionning that for a minimal, uniquely ergodic dynamical system (X, φ) whose action is locally free and such that X has a totally disconnected transversal we have, at least if $\check{H}(X, \mathbb{Z})$ is torsion free, that $K_0(C(X) \rtimes_\varphi \mathbb{R}^n) \cong \bigoplus_k \check{H}^{n+2k}(X, \mathbb{Z})$ [FH99] and the state on $K_0(C(X) \rtimes_\varphi \mathbb{R}^n)$ defined by the unique measure on X coincides when restricted to $\check{H}^n(X, \mathbb{Z})$ with the restriction of $\tau_{\varphi,\mu}$ to $\check{H}^n(X, \mathbb{Z})$. For these systems the Ruelle-Sullivan map is therefore an extension of the gap-labelling.

We end this section with two results which will be useful for our computations.

Proposition 11. *Let (X, φ) and (Y, ψ) be $\mathbb{R}^n$-actions with invariant measures μ and ν, respectively. Suppose that*

$$\eta : X \to Y$$

is a continuous function which is measure preserving and equivariant; i.e. $\nu = \mu \circ \eta^{-1}$ and $\psi_v \circ \eta = \eta \circ \varphi_v$, for all v in $\mathbb{R}^n$. Then

$$\begin{array}{ccc} \check{H}^*(Y, \mathbb{Z}) & \xrightarrow{\tau_{\psi,\nu}} & \Lambda\mathbb{R}^{n*} \\ {\scriptstyle \eta^*}\big\downarrow & & \big\downarrow{\scriptstyle =} \\ \check{H}^*(X, \mathbb{Z}) & \xrightarrow{\tau_{\varphi,\mu}} & \Lambda\mathbb{R}^{n*} \end{array}$$

is a commutative diagram.

Proof. This is a straightforward substitution in the integral. □

We also want to understand how our map is affected by linear reparametrizations of the flow.

Proposition 12. *Let (X, φ) be an $\mathbb{R}^M$-action and $A : \mathbb{R}^n \to \mathbb{R}^M$ be a linear map. Define an $\mathbb{R}^n$ action, ψ, on X by $\psi_v = \varphi_{Av}$. Then*

$$\begin{array}{ccc} \check{H}(X, \mathbb{Z}) & \xrightarrow{\tau_{\varphi,\mu}} & \Lambda\mathbb{R}^{M*} \\ {\scriptstyle =}\big\downarrow & & \big\downarrow{\scriptstyle \Lambda A^*} \\ \check{H}(X, \mathbb{Z}) & \xrightarrow{\tau_{\psi,\mu}} & \Lambda\mathbb{R}^{n*} \end{array}$$

is a commutative diagram where ΛA^* denotes the extension of the dual map $A^* : \mathbb{R}^{M*} \to \mathbb{R}^{n*}$ to the exterior algebras.

Proof. For clarity we denote the differentials by d_φ and d_ψ. Let $v \in \mathbb{R}^M$. Then by (2) $\langle v, A^* d_\varphi f\rangle = \langle Av, d_\varphi f\rangle = \langle v, d_\psi f\rangle$. Hence $A^* d_\varphi f = d_\psi f$ which implies $\theta_\psi^j = \Lambda^j A^* \theta_\varphi^j$ and therefore $\tau_{\psi,\mu} = \Lambda A^* \tau_{\varphi,\mu}$. □

4. Flows on tori

We discuss a variety of examples which are all related to flows on tori.

Let Γ denote a regular lattice in $\mathbb{R}^M$; that is, Γ is a cocompact discrete subgroup of $\mathbb{R}^M$. We consider the torus $X_\Gamma = \mathbb{R}^M/\Gamma$ with $q : \mathbb{R}^M \to X_\Gamma$ the quotient map. Let $A : \mathbb{R}^n \to \mathbb{R}^M$ be a linear map. We define

$$\varphi_v^A(q(x)) = \varphi_{Av}(q(x)), \quad \varphi_v(q(x)) = q(x+v) \tag{4}$$

for all x in $\mathbb{R}^M$ and v in $\mathbb{R}^n$. This is our basic dynamical system. We call it the flow on the torus defined by (A, Γ). If it has one dense orbit then it is minimal and uniquely ergodic, the measure being the normalised Lebesgue measure. If the system is not minimal there are other invariant measures. We note that φ^A is locally free whenever A is injective. In that case we may call it a generalized Kronecker flow.

Of course, for all choices of A and Γ, $\check{H}(X_\Gamma, \mathbb{Z})$ is isomorphic as a graded ring to the exterior ring $\Lambda\mathbb{Z}^M$. What we are after here is the image of the Ruelle-Sullivan map, which will, as it contains geometrical data, distinguish between different choices for Γ and A.

Theorem 13. *Consider the flow on the torus defined by* (A, Γ) *as above. Let* μ *be the normalised Lebesque measure on* X_Γ. *The corresponding Ruelle-Sullivan map is a surjective graded ring homomorphism from* $\check{H}(X_\Gamma, \mathbb{Z})$ *onto* $\Lambda(A^*\Gamma^*)$. *Here* A^* *is the dual of* A *and* Γ^* *the dual (or reciprocal) lattice. If the restriction of* A^* *to* Γ^* *is injective then the Ruelle-Sullivan map is also injective.*

Proof. We begin with the case that $n = M$, $A = \mathrm{id}$, the identity, and $\Gamma = \mathbb{Z}^M$ the standard cubic lattice generated by an orthonormal basis $\{e_i\}_i$. Then the generators of $\theta_\varphi(\check{H}(X_{\mathbb{Z}^M}, \mathbb{Z}))$ have representatives given by forms of the type $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ and we see immediately that the latter has image under the Ruelle-Sullivan map $de_{i_1} \wedge \cdots \wedge de_{i_k}$. Hence the Ruelle-Sullivan map is a graded ring isomorphism between $\check{H}(X_{\mathbb{Z}^M}, \mathbb{Z})$ and $\Lambda\mathbb{Z}^{M*}$.

Next we consider the case $\Gamma = \mathbb{Z}^M$ but general A which yields by Proposition 12 that the Ruelle-Sullivan map is a surjective graded ring homomorphism between $\check{H}(X_{\mathbb{Z}^M}, \mathbb{Z})$ and $\Lambda A^*\mathbb{Z}^{M*}$.

Finally we consider the case that $\Gamma = B\mathbb{Z}^M$ for some invertible matrix B and general A. Hence $B : \mathbb{R}^M \to \mathbb{R}^M$ induces a map $X_{\mathbb{Z}^M} \to X_\Gamma$ intertwining the action φ^A with φ^{BA}. From Proposition 11 it follows that the Ruelle-Sullivan map is a surjective graded ring homomorphism between $\check{H}(X_\Gamma, \mathbb{Z})$ and $\Lambda(A^*B^*\mathbb{Z}^{M*})$. $B^*\mathbb{Z}^{M*} = \Gamma^*$ is the dual lattice.

If the restriction of A^* to Γ^* is injective then, by multiplicativity, this is also the case for the restriction of ΛA^* to $\Lambda\Gamma^*$. This implies by Proposition 12 the injectivity of $\tau_{\varphi,\mu}$. □

4.1. *Eigenvalues of minimal actions*

We consider the above example but in the extreme case that $n \geq M = 1$, $\Gamma = \mathbb{Z}$, and A is surjective. We look at $X_{\mathbb{Z}} \cong S^1$ as the unit circle in the complex plane. There is a non-zero vector w in $\mathbb{R}^{n*}$ such that $Av = \langle v, w\rangle$. The $\mathbb{R}^n$-action on the complex unit circle is thus given by

$$\psi_v(z) = e^{2\pi i\langle v,w\rangle} z, \tag{5}$$

for v in $\mathbb{R}^n$, and z in S^1. Lebesgue measure on the circle is the unique invariant probability measure for this action. By Theorem 13 the Ruelle Sullivan map identifies the generator of $\check{H}^1(S^1, \mathbb{Z}) \cong \mathbb{Z}$ with w.

Now let φ be a minimal action of $\mathbb{R}^n$ on a compact space X with invariant probability measure μ. An element $w \in \mathbb{R}^{n*}$ is called a continuous eigenvalue of the action if there is a continuous non-zero function $f : X \to \mathbb{C}$ such that

$$f \circ \varphi_v = e^{2\pi i\langle v,w\rangle} f,$$

for all v in $\mathbb{R}^n$. Solomyak characterises the continuous eigenvalues of a minimal $\mathbb{R}^n$-action on a compact metric space (X, φ) with the help of its metric D as follows [So98]: For $\delta > 0$, $x \in X$ let $\Theta_\delta(x) = \{v \in \mathbb{R}^n | D(\varphi_v(x), x) < \delta\}$. Then w is a continuous eigenvalue if and only if for some $x \in X$

$$\lim_{\delta\to 0} \sup_{v\in\Theta_\delta(x)} |e^{2\pi i\langle v,w\rangle} - 1| = 0.$$

Theorem 14. *Let φ be a minimal action of $\mathbb{R}^n$ on a compact metric space X with invariant measure μ. $\tau_{\varphi,\mu}\check{H}^1(X, \mathbb{Z})$ contains all continuous eigenvalues of φ.*

Proof. Let w be a continuous eigenvalue with corresponding function f. By rescaling f we may assume it has absolute value 1 at some point. From the eigenvalue condition, the set of points where it has absolute value 1 is φ-invariant, and since the flow is minimal, this must be all of X. The map f provides an $\mathbb{R}^n$-equivariant map from X to S^1, the latter with action as given in (5). $\nu = \mu \circ f^{-1}$ is hence the normalised Lebegue measure on S^1. Applying Proposition 11 with $\eta = f$ one obtains $\tau_{\varphi,\mu}(\check{H}^1(X, \mathbb{Z})) \supset \tau_{\psi,\nu}(\check{H}^1(S^1, \mathbb{Z})) = \mathbb{Z}w$. □

4.2. Cut & project sets: window independent cohomological information

We now consider cut & project sets which also come under the name projection point patterns or model sets. We restrict to the case in which the internal space is a real vector space. We will see that part of their cohomology is determined by a generalised Kronecker flow on a torus.

Consider a decomposition of $\mathbb{R}^M$ into two orthogonal subspaces E and $E^\perp$ and a regular lattice $\Gamma \subset \mathbb{R}^M$. Denoting π the orthogonal projection onto E and $\pi^\perp$ its orthogonal complement we suppose that the restriction of π to Γ is injective, i.e. $E^\perp \cap \Gamma = \{0\}$, and that $\pi^\perp(\Gamma)$ is dense in $E^\perp$. These data $(E, E^\perp, \Gamma)$ constitute a cut & project scheme as in [Mo97]. Identifying the dual of E and $E^\perp$ (as abelian groups) with E and $E^\perp$, respectively, the data $(E^\perp, E, \Gamma^*)$ define the dual cut & project scheme. Indeed, the restriction of π to Γ being injective implies denseness of $\pi^\perp(\Gamma^*)$, and denseness of $\pi^\perp(\Gamma)$ implies that the restriction of π to Γ^* is injective [Mo97].

To construct a so-called cut & project set in E one is also given subset $K \subset E^\perp$ which is compact and the closure of its interior. Then

$$P_K = \{\pi(\gamma) | \gamma \in \Gamma, \pi^\perp(\gamma) \in K\}$$

is called the cut & project set with window K. P_K and K are called non-singular if $\pi^\perp(\Gamma) \cap \partial K = \emptyset$.

As we will describe in the next section in more detail, P_K defines an $\mathbb{R}^n$-action on a topological space Ω_{P_K}, called the hull of P_K. For non-singular P_K, the hull has the form $\Omega_{P_K} = (E \oplus E_c^\perp)/\Gamma$ where $E_c^\perp$ is the cut up internal space. It is obtained from $E^\perp$ by disconnecting it along all points of $\partial K + \pi^\perp(\Lambda)$. Although the latter is a dense set, the resulting surjection $\eta : E_c^\perp \to E^\perp$ is almost one-to-one. The $\mathbb{R}^n$-action on Ω_{P_K} which is induced from the translation of P_K in $\mathbb{R}^n$ (see Section 5) is here simply given by $\psi_v(\tilde{q}(x, y)) = \tilde{q}(x + v, y)$ where $\tilde{q} : E \oplus E_c^\perp \to (E \oplus E_c^\perp)/\Gamma$ is the canonical projection. η induces an almost one-to-one surjection $\eta : \Omega_{P_K} \to X_\Gamma = \mathbb{R}^M/\Gamma$ interwining ψ with the flow associated to (A, Γ) where, in the notation above, $\mathbb{R}^n = E$ and A is the inclusion map $E \subset \mathbb{R}^M$. The generalised Kronecker flow is minimal and uniquely ergodic and the uniquely ergodic measure on Ω_{P_K} is the pull back of the measure on the Kronecker flow under η [FHK02].

Our aim is to study here the part of the cohomology which does not depend on the cuts but only on the cut & projection scheme. This is, more precisely, the image of $\eta^* : \check{H}(X_\Gamma, \mathbb{Z}) \to \check{H}(\Omega_{P_K}, \mathbb{Z})$.

Theorem 15. *Consider a cut & project set as above. Then the Ruelle-Sullivan map associated with its $\mathbb{R}^n$-action restricts to a graded ring isomorphism between $\eta^*(\check{H}(X_\Gamma, \mathbb{Z}))$ and $\Lambda\pi(\Gamma^*)$.*

Proof. Application of Proposition 11 reduces the system to the Kronecker flow associated to (A, Γ) where A is the inclusion $E \subset \mathbb{R}^M$. Now the dual of A is precisely the projection π whose restriction to Γ^* is injective. The statement follows therefore from Theorem 13. □

If also the restriction of $\pi^\perp$ to Γ is injective, i.e. $E \cap \Gamma = \{0\}$, then P_K has no periods, and vice versa. This is also equivalent to the denseness of $\pi(\Gamma^*)$ [Mo97]. We are thus led to the following conjecture. The definition of a Delone set of finite local complexity will be given in the next section.

Conjecture 16. *Let P be a Delone set of finite local complexity. Suppose that its associated $\mathbb{R}^n$-action is minimal and uniquely ergodic. Then $\mathbb{R}^{n*}/\overline{\tau_{\varphi,\mu}\check{H}^1(\Omega_P, \mathbb{Z})}$ is the dual group of the reciprocal lattice of the lattice of periods of P.*

5. Transversally locally constant forms and hulls of aperiodic systems

We now turn to the issue of a de Rham type isomorphism. We mentionned that the dynamical cohomology of (X, φ) coincides with the usual de Rham cohomology in the case that X consists only of one orbit and φ is locally free. In this case the de Rham theorem applies which states that when real coefficients are considered the homomorphism θ_φ of Theorem 6 becomes an isomorphism for good covers. This cannot be expected for the general case. Our main applications are to the study of aperiodic order and more specifically aperiodic tilings or point patterns of Euclidean space. In that case we will describe the appropriate refinement of dynamical cohomology which is needed to obtain an isomorphism.

We find it most convenient to work in the context of *Delone sets* but, as is well known, we could equally well talk about tilings. Recall that a set P in $\mathbb{R}^n$ is a Delone set if there are positive constants r_P, R_P such that

1. P is r_P-discrete, i.e. for all $x \neq y$ in P, $|x - y| \geq r_P$, and
2. for all x in $\mathbb{R}$, there is y in P with $|x - y| \leq R_P$.

We say that P is *aperiodic* if, for any x in $\mathbb{R}^n$, $P + x = P$ holds only if $x = 0$. Finally, P has *finite local complexity* if, for any fixed R, the number of sets $(P - x) \cap B(0, R)$, for x in P, is finite. Here $B(x, R)$ is the (open) ball of radius R around x. All Delone sets considered in this article are assumed to have finite local complexity. It is worth stating that the proper notion of finite local complexity is relative to a subgroup of the isometry group of $\mathbb{R}^n$. Here, we are implicitly using the subgroup of translations. When more general sub-Lie-groups of the isometry group are considered the generalisations indicated in Remarks 5,9 have to be taken into account.

The *hull* associated with P, denoted by Ω_P, is a compact metric space obtained by completion of the set of all translations of P, $\{P + x \mid x \in \mathbb{R}^n\}$, in the metric

$$D(P, P') = \inf\{\epsilon \mid \exists x, x' \in B(0, \frac{\epsilon}{2}) : (P - x) \cap B(0, \epsilon^{-1}) = (P' - x') \cap B(0, \epsilon^{-1})\}.$$

The elements of Ω_P can all be interpreted as Delone sets. The natural translation of Delone sets in $\mathbb{R}^n$ extends to an action of $\mathbb{R}^n$ on Ω_P. This is our action: $\varphi_v(P') = P' - v$, for P' in Ω_P and v in $\mathbb{R}^n$.

A fair amount is now known about the structure of such spaces. One particularly interesting feature is that the space possesses *canonical* local transversals to the action of $\mathbb{R}^n$. For each P' in Ω_P and $0 < \epsilon < R_P^{-1}$ we define

$$T_{P',\epsilon} = \{P'' \in \Omega_P \mid P'' \cap B(0, \epsilon^{-1}) = P' \cap B(0, \epsilon^{-1})\}.$$

These closed sets are transversal in the sense that $P'' \in T_{P',\epsilon}$ implies $\varphi_v(P'') \notin T_{P',\epsilon}$ if $0 \neq |v| < \frac{r_P}{2}$.

Definition 17. *A function f defined on Ω_P is transversally locally constant if, for every P' in Ω, there is $0 < \epsilon < R_P^{-1}$ such that f is constant on $T_{P',\epsilon}$. $C^\infty_{tlc}(\Omega_P, \Lambda\mathbb{R}^{n*})$ denotes the transversally locally constant functions in $C^\infty(\Omega_P, \Lambda\mathbb{R}^{n*})$. The Lie-algebra cohomology $H(\mathbb{R}^n, C^\infty_{tlc}(\Omega_P, \mathbb{R}))$ of $\mathbb{R}^n$ with coefficients in $C^\infty_{tlc}(\Omega_P, \Lambda\mathbb{R}^{n*})$ will also be called the transversally locally constant dynamical cohomology of Ω_P.*

For any open cover $\mathcal{U}$ of Ω_P and $j, k \geq 0$, we define $K^{j,k}_{tlc}(\mathcal{U})$ to be those elements of $K^{j,k}(\mathcal{U})$ which are transversally locally constant. It is clear from the definitions that $K_{tlc}(\mathcal{U})$ forms a sub-complex of the double complex $K(\mathcal{U})$.

In our case, there are natural choices for the open covers $\mathcal{U}$. We let V be an open subset of $\mathbb{R}^n$ and

$$T_{P',\epsilon,V} = \{P'' + v : P'' \in T_{P',\epsilon}, v \in V\}.$$

Given any $R > 0$ the sets of the form $T_{P',\epsilon,B(x,r)}$ with $x \in \mathbb{R}^n$ and $r \leq R$ generate the topology of Ω_P.

Lemma 18. *$\check{C}^j(\mathcal{U}, \mathbb{R})$ is contained in $K^{j,0}_{tlc}(\mathcal{U})$ and coincides with the kernel of d on $K^{j,0}_{tlc}(\mathcal{U})$.*

Proof. Let $V = B(x, r)$ with $r < \frac{r_P}{2}$. Then $T_{P',\epsilon,V}$ is diffeomorphic to $T_{P',\epsilon} \times V$, the diffeomorphism's inverse being $(P', v) \mapsto P' + v$ (here the smoothness of the diffeomorphism should be understood w.r.t. φ-differentiability, in the transverse direction only continuity is required). Under this diffeomorphism a transversally locally constant function is constant in the second coordinate and the differential d becomes the identity times the exterior derivative on $\mathbb{R}^n$. Since the topology is generated by the above sets the lemma is thus a direct consequence of the definition of transversally locally constant. □

Lemma 19. *Let $\mathcal{U}$ be a covering by sets $T_{P',\epsilon,V}$ with convex V of diameter less than $\frac{r_P}{4}$. Then the complexes*

$$K_{tlc}^{j,0}(\mathcal{U}) \stackrel{d}{\to} K_{tlc}^{j,1}(\mathcal{U}) \stackrel{d}{\to} K_{tlc}^{j,2}(\mathcal{U}) \stackrel{d}{\to} \cdots$$

are acyclic. That is, their cohomology vanishes except in degree zero.

Proof. As usual, the result follows once we have found a contracting homotopy $\tilde{h} : K^{j,k} \to K^{j,k-1}$, $\tilde{h}d + d\tilde{h} = 1$, for $k \geq 1$.

For that we need to make sure that the covering is good in the sense that finite intersections of the type $\bigcap_i T_{P_i,\epsilon_i,V_i}$ have contractible leaves. Consider pairs (S, U) where S is a discrete subset of an open set $U \subset \mathbb{R}^n$. Let $T_{S,U} = \{P' \in \Omega_P : P' \cap U = S\}$ and $T_{S,U,V} = T_{S,U} + V$.Then

$$T_{S,U} \cap T_{S',U'} = \begin{cases} T_{S\cup S',U\cup U'} & \text{if } U \cap S' \subset S \text{ and } U' \cap S \subset S' \\ \emptyset & \text{otherwise} \end{cases}$$

and $T_{S,U} + v = T_{S+v,U+v}$. Using these rules one finds

$$T_{S,U,V} \cap T_{S',U',V'} = \bigcup_{v\in V} \bigcup_{w \in V'-v} T(w) + v$$

where

$$T(w) = \begin{cases} T_{S\cup(S'+w),U\cup(U'+w)} & \text{if } U \cap (S'+w) \subset S \\ & \text{and } (U'+w) \cap S \subset (S'+w), \\ \emptyset & \text{otherwise.} \end{cases}$$

If the diameter of V and V' is smaller than $\frac{r_P}{4}$ then $|w| < \frac{r_P}{2}$ so that, by the r_P-discreteness of P, $T(w) \neq \emptyset$ for at most one w. For such a w we thus have $T_{S,U,V} \cap T_{S',U',V'} = T_{S\cup(S'+w),U\cup(U'+w),V\cap(V'-w)}$. We see that $V \cap (V' - w)$ is convex and hence contractible. Finite intersections $\bigcap_i T_{P_i,\epsilon_i,V_i}$ fall in the above framework and are thus empty or of the form $T_{S,U,V}$ for some S, U, V with contractible V of diameter smaller than $\frac{r}{2}$.

In particular, as in the proof of Lemma 18, $\bigcap_i T_{P_i,\epsilon_i,V_i} = T_{S,U,V}$ is diffeomorphic to the Cartesian product $T_{S,U} \times V$. Taking therefore in these product coordinates $\tilde{h} = 1 \times H$, where H is the standard contracting homotopy on $\mathbb{R}^n$ [AM85], we obtain a map which preserves transversally locally constant functions and satisfies $\tilde{h}d + d\tilde{h} = 1 \times (H\tilde{d} + \tilde{d}H) = 1$ (here $\tilde{d}$ is the exterior derivative on $\mathbb{R}^n$). □

The following theorem follows now as in the classical situation of the de Rham theorem from the last two lemmata together with the observation that any cover of Ω_P has a refinement by covers of the type used in the last lemma.

Theorem 20. *The ring homomorphism θ_φ of Theorem 6 provides an isomorphism between $\check{H}(\Omega_P, \mathbb{R})$, the Cech cohomology of Ω_P with real coefficients, and $H(\mathbb{R}^n, C^\infty_{tlc}(\Omega_P, \mathbb{R}))$, the transversally locally constant dynamical cohomology.*

6. *P*-equivariant cohomology

The notion introduced in the last section of transversally locally constant forms on the hull Ω_P (and hence of the transversally locally constant dynamical cohomology) can be reduced to something simpler and much less technical. This is the aim of the current section.

We begin by introducing the notion of a strongly P-equivariant function on $\mathbb{R}^n$ [Ke03] where P is as in the last section a Delone set of $\mathbb{R}^n$ of finite local complexity. Roughly speaking, a function f on $\mathbb{R}^n$ is strongly P-equivariant if there is some constant R such that, if the patterns in P surrounding two points x and y of radius R are equal (after translating by $-x$ and $-y$, respectively), then f must take the same values at x and y. More precisely, we have the following.

Definition 21. *Let f be a function defined on $\mathbb{R}^n$. We say that f is strongly P-equivariant if there is a constant $R > 0$ such that, if x, y in $\mathbb{R}^n$ satisfy*

$$(P - x) \cap B(0, R) = (P - y) \cap B(0, R),$$

then $f(x) = f(y)$.

For a finite dimensional vector space W, we let $\check{C}^\infty_P(\mathbb{R}^n, W)$ denote the smooth strongly P-equivariant functions from $\mathbb{R}^n$ to W. The cohomology of the sub-complex of strongly P-equivariant forms on $\mathbb{R}^n$ is called the *strongly P-equivariant cohomology of* $\mathbb{R}^n$ and denoted by $\check{H}_P(\mathbb{R}^n)$.

The functions of the closure $C^\infty_P(\mathbb{R}^n, W)$ of $\check{C}^\infty_P(\mathbb{R}^n, W)$ in $C^\infty(\mathbb{R}^n, W)$ (w.r.t. the standard Fréchet topology) are called weakly P-equivariant. The cohomology of the sub-complex of weakly P-equivariant forms on $\mathbb{R}^n$ is called the *weakly P-equivariant cohomology of* $\mathbb{R}^n$ and denoted by $H_P(\mathbb{R}^n)$. We note that P-equivariant forms are always bounded.

Proposition 22. *1. Let $f : \Omega_P \to W$ be transversally locally constant. The function f_P defined on $\mathbb{R}^n$ by*

$$f_P(x) = f(P - x)$$

is strongly P-equivariant. In addition, if f is φ-smooth, then f_P is a smooth function.

2. If $g : \mathbb{R}^n \to W$ is a strongly P-equivariant function, then the function $\tilde{g}(P - x) = g(x)$, defined apriori on the orbit of P, extends to a continuous transversally locally constant function on Ω_P. If g is smooth, then $\tilde{g}$ is φ-smooth.

3. *The map* $f \mapsto f_P$ *is a graded ring isomorphism between* $C^\infty_{tlc}(\Omega_P, \Lambda\mathbb{R}^{n*})$ *and* $\check{C}^\infty_P(\mathbb{R}^n, \Lambda\mathbb{R}^{n*})$ *intertwining the differential d on the former with the usual exterior derivative on forms on* $\mathbb{R}^n$. *It extends by continuity to a graded ring isomorphism between* $C^\infty(\Omega_P, \Lambda\mathbb{R}^{n*})$ *and* $C^\infty_P(\mathbb{R}^n, \Lambda\mathbb{R}^{n*})$.

Proof. 1. By a partition of unity argument it suffices to show the statement for a function f which has support in some $T_{P',\epsilon,B(0,\frac{r_P}{2})}$ and on which it is transversally constant. In other words, under identification $T_{P',\epsilon,B(0,\frac{r_P}{2})} \cong T_{P',\epsilon} \times B(0, \frac{r_P}{2})$, f is of the form $f = 1 \times g$ where $g : \mathbb{R}^n \to W$ has support in $B(0, \frac{r_P}{2})$. Then $f_P(x) = g(y)$ for each $y \in \mathbb{R}^n$ such that $P - x \in T_{P'-y,\epsilon}$. This is well-defined since there is at most one such y in $B(0, \frac{r_P}{2})$. Now it is clear that f_P is strongly P-equivariant with constant $R = \epsilon^{-1}$. The preservation of smoothness follows directly using the local product structure.

2. If g is strongly P-equivariant and R the constant from Definition 21 then $\tilde{g}$ is transversally locally constant with $\epsilon = R^{-1}$. The preservation of smoothness follows as in 1.

3. The first statement is a direct consequence of 1 and 2 above together with the straightforward observation that $\tilde{f_P} = f$ and $\tilde{g}_P = g$. The second statement follows from the fact that the completion of $C^\infty_{tlc}(\Omega_P, \varphi, \mathbb{R})$ in the Fréchet topology defined by the semi-norms $\|\omega\|_\infty$ (supremum norm) and $\|\omega\|_{i_1\cdots i_k} = \|\partial_{e_{i_1}} \cdots \partial_{e_{i_k}} \omega\|_\infty$ is $C^\infty(\Omega_P, \varphi, \mathbb{R})$. This is directly seen using the local product structure.

□

As a consequence of this proposition we have

Theorem 23. $f \mapsto f_P$ *induces graded ring isomorphisms between the transversally locally constant dynamical cohomology* $H(\mathbb{R}^n, C^\infty_{tlc}(\Omega_P, \mathbb{R}))$ *and the strongly* P*-equivariant cohomology* $\check{H}_P(\mathbb{R}^n)$*, and between the dynamical cohomology* $H(\mathbb{R}^n, C^\infty(\Omega_P, \mathbb{R}))$ *and the weakly* P*-equivariant cohomology* $H_P(\mathbb{R}^n)$.

Remark 24. Let us call a complex continuous function over $\mathbb{R}^n$ P-equivariant if it lies in the closure of $\check{C}_P(\mathbb{R}^n, \mathbb{C})$ w.r.t the supremum norm. Then $f \mapsto f_P$ induces an algebra isomorphism between $C(\Omega_P)$ and this closure. Hence the topological space Ω_P is described by the complex continuous P-equivariant functions. The Cech cohomology of Ω_P is, however, the cohomology of the sub-complex of *strongly* P-equivariant forms. The latter was therefore simply called the P-equivariant cohomology (of $\mathbb{R}^n$) in [Ke03].

6.1. Multiplicativity of the Ruelle-Sullivan map

The advantage of Theorem 23 is that elements of $\check{H}_P(\mathbb{R}^n)$ have representatives which are forms over $\mathbb{R}^n$. These are not only more easily accessible but also allow to use techniques from differential topology, as is done in the proposition below. It is used to show that the Ruelle-Sullivan map for *ergodic* invariant measures of $\mathbb{R}^n$-actions on hulls of Delone sets with finite local complexity is multiplicative.

We denote for $\omega \in C^\infty(X, \Lambda\mathbb{R}^{n*})$

$$\|\omega\|_1 = \int_X |\omega(x)| d\mu(x)$$

where $|\cdot|$ is a norm on $\Lambda\mathbb{R}^{n*}$.

Proposition 25. *Let P be a Delone set with finite local complexity and μ be an ergodic invariant probability measure for the action of $\mathbb{R}^n$ on Ω_P. Let ω be any closed strongly P-equivariant k-form on $\mathbb{R}^n$, $k \geq 1$. For any $\epsilon > 0$ there exists a closed strongly P-equivariant form ω_ϵ which defines the same element as ω in $\check{H}_P(\mathbb{R}^n)$ and such that*

$$\|\omega_\epsilon - \tilde{\tau}_{\varphi,\mu}(\omega)\|_1 \leq \epsilon.$$

Here $\tilde{\tau}_{\varphi,\mu}$ is the expression of the Ruelle-Sullivan map on $\check{H}_P(\mathbb{R}^n)$ and we view $\tilde{\tau}_{\varphi,\mu}(\omega)$ as constant form on $\mathbb{R}^n$.

Proof. Let ω be a closed strongly P-equivariant k-form. We let ω_r be the form obtained by averaging over r-cubes $I_r = [-\frac{r}{2}, \frac{r}{2}]^n$:

$$\omega_r(x) = \frac{1}{r^n} \int_{I_r} \omega(x+y) dy.$$

By the Birkhoff ergodic theorem ω_r converges to $\tilde{\tau}_{\varphi,\mu}(\omega)$ for $r \to \infty$ in the norm $\|\cdot\|_1$ [Li01]. Moreover, each ω_r is a closed strongly P-equivariant k-form. The proposition follows therefore if we can verify that $\omega_r - \omega = dG_r$ for a strongly P-equivariant $k-1$-form G_r.

For any $e_1, \ldots, e_{k-1} \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$, we let $g_y : \mathbb{R}^n \to \Lambda^{k-1}\mathbb{R}^{n*}$ be given by

$$\langle e_1 \wedge \cdots \wedge e_{k-1}, g_y(x)\rangle = (-1)^{k-1} \int_0^1 dt \langle e_1 \wedge \cdots \wedge e_{k-1} \wedge y, \omega(x+ty)\rangle.$$

g_y is strongly P-equivariant, as the integral depends only on the values of ω between x and $x+y$. Furthermore,

$$\begin{aligned}\langle e_0 \wedge \cdots \wedge e_{k-1}, dg_y(x)\rangle &= \sum_{i=0}^{k-1} (-1)^i e_i \cdot \nabla \langle e_0 \wedge \cdots \hat{e_i} \cdots \wedge e_{k-1}, g_y(x)\rangle \\ &= (-1)^{k-1} \int_{t=0}^1 dt\, \big(\langle e_1 \wedge \cdots \wedge e_{k-1} \wedge y, d\omega(x+ty)\rangle \\ &\quad -(-1)^k y \cdot \nabla \langle e_0 \wedge \cdots \wedge e_{k-1}, \omega(x+ty)\rangle\big)\end{aligned}$$

where $y \cdot \nabla f$ is the derivative of the function f in the direction y. Since ω is closed the first term vanishes, and using $y \cdot \nabla f(x+ty) = \frac{d}{dt} f(x+ty)$ the second integral gives $\langle e_0 \wedge \cdots \wedge e_{k-1}, \omega(x+y) - \omega(x) \rangle$. Hence $\omega(x+y) - \omega(x) = dg_y(x)$ and

$$G_r = \frac{1}{r^n} \int_{I_r} g_y dy.$$

G_r is thus a strongly P-equivariant form satisfying $\omega_r - \omega = dG_r$. □

Theorem 26. *Let P be a Delone set with finite local complexity and μ be an ergodic invariant probability measure for the action of $\mathbb{R}^n$ on Ω_P. The Ruelle-Sullivan map*

$$\tau_{\varphi,\mu} : \check{H}(\Omega_P, \mathbb{Z}) \to \Lambda \mathbb{R}^{n*}$$

is a homomorphism of graded rings.

Proof. We have seen that the map θ_φ is a ring homomorphism. We show that also $\tilde{\tau}_{\varphi,\mu}$ is multiplicative. Let ω, ρ be two closed strongly P-equivariant forms over $\mathbb{R}^n$. Given $\epsilon > 0$ let ω_ϵ be as in the statement of Proposition 25. Then

$$\tilde{\tau}_{\varphi,\mu}(\omega\rho) = \tilde{\tau}_{\varphi,\mu}(\omega_\epsilon \rho)$$

since $(\omega - \omega_\epsilon)\rho$ is exact. Furthermore

$$\tilde{\tau}_{\varphi,\mu}(\omega_\epsilon \rho) = \tilde{\tau}_{\varphi,\mu}(\tilde{\tau}_{\varphi,\mu}(\omega_\epsilon)\rho) + \int_{\Omega_P} (\omega_\epsilon - \tilde{\tau}_{\varphi,\mu}(\omega_\epsilon))\rho \, d\mu$$

and hence

$$|\tilde{\tau}_{\varphi,\mu}(\omega\rho) - \tilde{\tau}_{\varphi,\mu}(\omega)\tilde{\tau}_{\varphi,\mu}(\rho)| \le \|\omega_\epsilon - \tilde{\tau}_{\varphi,\mu}(\omega)\|_1 \|\rho\|_\infty \le \epsilon \|\rho\|_\infty$$

where $\|\rho\|_\infty = \sup_x |\rho(x)|$. The result follows since ϵ was arbitrary. □

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