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Global Solutions to the Two Dimensional Quasi-Geostrophic Equation with Critical or Super-Critical Dissipation

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Abstract. The two dimensional quasi-geostrophic (2D QG) equation with *critical* and *super-critical* dissipation is studied in Sobolev space $H^s(\mathbb{R}^2)$. For *critical* case $\left(\alpha = \frac{1}{2}\right)$, existence of global (large) solutions in H^s is proved for $s \geqslant \frac{1}{2}$ when $\|\theta_0\|_{L^\infty}$ is small. This generalizes and improves the results of Constantin, D. Cordoba and Wu [4] for $s = 1, 2$ and the result of A. Cordoba and D. Cordoba [8] for $s = \frac{3}{2}$. For $s \geq 1$, these solutions are also unique. The improvement for pushing s down from 1 to $\frac{1}{2}$ is somewhat surprising and unexpected. For *super-critical* case $\left(\alpha \in \left(0, \frac{1}{2}\right)\right)$, existence and uniqueness of global (large) solution in H^s is proved when the product $\|\theta_0\|_{H^s}^{\beta} \|\theta_0\|_{L^p}^{1-\beta}$ is small for suitable $s \geqslant 2 - 2\alpha$, $p \in [1, \infty]$ and $\beta \in (0, 1]$.

Key words. Two dimensional dissipative quasi-geostrophic equations, Existence, Uniqueness, Critical, Super-critical, Sobolev space.

Mathematics Subject Classification (2000): 76D, 35Q

1. Introduction

Consider the following dissipative two dimensional quasi-geostrophic (2D QG) equation:

$$\begin{aligned} \theta_t + u \cdot \nabla\theta + \kappa(-\Delta)^\alpha\theta = 0, \\ \theta(x, 0) = \theta_0(x), \end{aligned} \tag{1.1}$$

where $0 \leq \alpha \leq 1$, $\kappa \geq 0$, θ is temperature and u is fluid velocity. u and θ are related via a stream function ψ:

$$u = (u_1, u_2) = \left(-\frac{\partial\psi}{\partial x_2}, \frac{\partial\psi}{\partial x_1}\right), \qquad (-\Delta)^{\frac{1}{2}}\psi = -\theta. \tag{1.2}$$

The spacial domain Ω is either $\mathbb{R}^2$ or $\mathbb{T}^2 \equiv [0, 2\pi]^2$ with periodic boundary condition. In this article, we consider the case that $\Omega = \mathbb{R}^2$. The treatment for the case $\Omega = \mathbb{T}^2$ is similar and thus omitted.

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Equation (1.1) is an important model in geophysical fluid dynamics. See e.g. [5], [12], [22] and [23]. Mathematically, when $\kappa = 0$, it is an important example of 2D active scalar with a specific mathematical structure which remarkably resembles that of Euler equations for 3D incompressible fluid; when $\kappa > 0$ and $\alpha = \frac{1}{2}$, it is the dimensionally correct analogue of Navier-Stokes equations for viscous 3D incompressible fluid. The cases $\alpha > \frac{1}{2}$, $\alpha = \frac{1}{2}$ and $\alpha < \frac{1}{2}$ are called respectively *sub-critical*, *critical* and *super-critical*. In the following, we will completely ignore the case when $\alpha \geqslant 1$. Global well-posedness can be proved without much difficulty for this case. However, due to weak dissipation, well-posedness of 2D QG with $\alpha < 1$ is non-trivial, especially for critical and super-critical cases. Recently, these equations have been intensively studied because of their importance in mathematical and geophysical fluid dynamics. See among others, [1]-[2], [4]-[16], [21]-[25], [27] and references therein.

In this paper, we study global existence of the solutions to the 2D QG equation with critical and super-critical dissipation in Sobolev space H^s. For *critical* case, existence of global solutions in H^s is proved for $s \geqslant \frac{1}{2}$ when $\|\theta_0\|_{L^\infty}$ is small. This improves and generalizes the results of Constantin, D. Cordoba and Wu [4] for $s = 1, 2$ and the result of A. Cordoba and D. Cordoba [8] for $s = \frac{3}{2}$. Uniqueness of these solutions for $s \geqslant 1$ can be concluded by the result of [15]. Notice that the smallness condition on $\|\theta_0\|_{L^\infty}$ does *not* imply smallness of $\|\theta_0\|_{H^s}$. Both the improvement and the generalization discussed above are not trivial. In [4], the technique of integration by parts was crucial in obtaining some subtle estimates. If s is a non-integer, it seems impossible to play with integration by parts. Instead, in [8] for the special choice of $s = \frac{3}{2}$ and $\alpha = \frac{1}{2}$, very delicate commuting properties of Riesz transforms and their subtle estimates in Hardy space and BMO space were utilized in an elegant way. However, this method seems not suitable for general value of s, nor for super-critical case. It is interesting to point out that our result improves the result of [4] and [8] by pushing the allowed value of s from 1 down to $\frac{1}{2}$. This is somewhat surprising and unexpected for a few important reasons as explained in more details in the remarks following the statement of our main theorem.

For *super-critical* case, existence and uniqueness of the global solution in H^s is proved for $s \in \left[2 - 2\alpha, \frac{(1-\alpha)(1+2\alpha)}{2\alpha}\right]$ when the product $\|\theta_0\|_{H^s}^{\beta} \|\theta_0\|_{L^\infty}^{1-\beta}$ is small for some $\beta \in (0, 1)$ to be given in the following. Therefore, provided that $\|\theta_0\|_{L^\infty}$ is small enough, $\|\theta_0\|_{H^s}$ can still be very large and vice versa. Uniqueness of the solution follows from the result of [15] for any $s \geqslant 2 - 2\alpha$. In fact, more general results are obtained for $\theta_0 \in L^p$ with $p \in [1, +\infty]$. See Theorem 3.1 for more details. This generalizes the global existence results recently obtained in Ju [13] for $p = 2$.

The key ingredient of the proofs for these new improvements is an improved version of an important commutator estimate. The basic theory for commutator estimation was established in harmonic analysis via pseudo-differential calculus.

See Coifman and Meyer [3]. Using a theorem due to Coifman and Meyer, Kato and Ponce [19] proved an early version of the commutator estimate and used it in dealing with 3D Navier-Stokes and Euler Equations. Later, Kenig, Ponce and Vega [20] obtained an improved version of the commutator estimate and used it to study the well-posedness of KdV equation. In A. Cordoba and D. Cordoba [8], the commutator estimate of Kato and Ponce was used to study 2D QG. In Ju [13], the more general version of the commutator estimate of Kenig, Ponce and Vega was used to improve several results of [8]. In this paper, using the technique of generalized version of the commutator estimate of Kenig, Ponce and Vega, we improve global existence results of [4] and [8] for *critical* case. The strength of Kenig-Ponce-Vega commutator estimate is demonstrated in this paper by pushing s from 1 down to $\frac{1}{2}$ and by generalizing the result for all $s \geqslant \frac{1}{2}$; while methods in [4] and [8] fail for general value of s. Besides the improved result for *critical* case, the approach in this paper also provides new results for *super-critical* case, for which the previous two methods fail as well.

The rest of this article is organized as follows. In Section 2, we present some notations and recall some important preliminary results as preparation. In Section 3, we state our main results for the case when $\Omega = \mathbb{R}^2$. Similar results hold for $\Omega = \mathbb{T}^2$. We also give the proofs of these results in this section. In subsection 3.1, we prove global existence of the solution. In subsection 3.2, we prove the uniqueness of these solutions.

2. Notations and Preliminaries

We recall some notations and facts for $\Omega = \mathbb{R}^2$. Similar results are valid for $\Omega = \mathbb{T}^2$ as well. Fourier transform $\widehat{f}$ of a tempered distribution f is defined as

$$\widehat{f}(k) = \frac{1}{(2\pi)^2} \int_\Omega f(x) e^{-ik\cdot x} dx.$$

Denote $\Lambda \equiv (-\Delta)^{\frac{1}{2}}$. Obviously

$$\widehat{\Lambda f}(\xi) = |\xi| \widehat{f}(\xi).$$

More generally, $\Lambda^\beta f$ for $\beta \in \mathbb{R}$ can be identified with the Fourier transform

$$\widehat{\Lambda^\beta f}(\xi) = |\xi|^\beta \widehat{f}(\xi).$$

The following standard notations are used:

$$\|f\|_{L^p}^p = \int_\Omega |f|^p dx, \quad \|f\|_{L^\infty} = ess \sup_{x\in\Omega} |f(x)|,$$

$$\|f\|_{H^s} = \|\Lambda^s f\|_{L^2}, \quad \|f\|_{H^{s,p}} = \|\Lambda^s f\|_{L^p}.$$

By (1.2), we have

$$u = \left(\partial_{x_2}\Lambda^{-1}\theta,\ -\partial_{x_1}\Lambda^{-1}\theta\right) = (-\mathcal{R}_2\theta,\ \mathcal{R}_1\theta) \equiv \mathcal{R}^{\perp}\theta,$$

where $\mathcal{R}_j,\ j = 1, 2$ are the Riesz transforms defined by

$$\mathcal{R}_j f(x) = C P.V. \int_{\mathbb{R}^2} \frac{f(x-y)y_j}{|y|^3} dy, \qquad j = 1, 2.$$

Therefore, by Calderón-Zygmund Theorem, there is a constant $C_R(p) > 0$ such that

$$\|u\|_{L^p} \leq C_R(p)\|\theta\|_{L^p}, \quad \forall p \in (1, \infty). \tag{2.1}$$

Global existence of weak solution for 2D QG equation is obtained by Resnick [24].

Theorem 2.1. *Suppose $\theta_0 \in L^2$ and $k \geq 0$. Then, for any $T > 0$, there exists at least one weak solution to the 2D QGE in the following sense:*

$$\frac{d}{dt}\int_{\Omega} \theta\varphi dx - \int_{\Omega} \theta(u \cdot \nabla\varphi) dx + \kappa \int_{\Omega} \theta\Lambda^{2\alpha}\varphi dx = 0, \qquad \forall \varphi \in C_0^{\infty},$$

where $\theta \in L^{\infty}(0, T; L^2)$. Moreover, if $\kappa > 0$, $\theta \in L^2(0, T; H^{\alpha})$).

Another important property of 2D QG equation obtained in [24] is that if $\theta_0 \in L^p$ for any $p \in [1, \infty]$, then

$$\|\theta(t)\|_{L^p} \leqslant \|\theta_0\|_{L^p}, \quad \forall t > 0.$$

We will use this property extensively in the later discussion.

For sub-critical case, global existence and uniqueness for strong solutions are obtained in [6] and [27]. For *critical* and *super-critical* case, see [4], [8], [13], [15] and [16]. See also [2] for the discussion in Besov space.

3. The Main Result and the Proof

First of all, we state our main result as the following theorem:

Theorem 3.1. *Assume $\alpha \in (0, \frac{1}{2}]$, $\kappa > 0$, $\Omega = \mathbb{R}^2$ and $\theta_0 \in H^s$.*

1. *Given $\alpha = \frac{1}{2}$ and $s \geqslant \frac{1}{2}$, there is a constant $C_1 > 0$ such that for any weak solution θ to equation (1.1), if*

$$\|\theta_0\|_{L^{\infty}} \leqslant \frac{\kappa}{C_1},$$

then

$$\|\theta(t)\|_{H^s} \leq \|\theta_0\|_{H^s}, \qquad \forall t > 0,$$

and the solution θ is unique for $s > 1$ and $\theta_0 \in L^2$.

Moreover, if $\|\theta_0\|_{L^\infty} < \frac{\kappa}{C_1}$*, then* $\theta \in L^2(0, +\infty; H^{s+\frac{1}{2}})$ *and the solution* θ *is unique if* $s \geqslant 1$ *and* $\theta_0 \in L^2$.

2. *Given* $\alpha \in \left(0, \frac{1}{2}\right]$, $p \in \left[1, \frac{1}{\alpha}\right]$, $s \geqslant 2 - 2\alpha$ *and*

$$\beta = \frac{\frac{2}{p} + 1 - 2\alpha}{\frac{2}{p} + s - 1},$$

which is within $\left[\frac{1}{s+\alpha}, \quad 1\right]$*, there is a constant* $C_2 > 0$ *such that for any weak solution* θ *to equation (1.1), if initially*

$$\|\theta_0\|_{H^s}^{\beta} \|\theta_0\|_{L^p}^{1-\beta} \leqslant \frac{\kappa}{C_2}, \tag{3.1}$$

then

$$\|\theta(t)\|_{H^s} \leq \|\theta_0\|_{H^s}, \qquad \forall t > 0.$$

The solution θ *is unique if* $s > 2 - 2\alpha$ *and* $\theta_0 \in L^2$*. Furthermore, if* $\|\theta_0\|_{H^s}^{\beta} \|\theta_0\|_{L^p}^{1-\beta} < \frac{\kappa}{C_2}$*, then* $\theta \in L^2(0, \infty; H^{s+\alpha})$ *and the solution* θ *is unique if* $s \geqslant 2 - 2\alpha$ *and* $\theta_0 \in L^2$.

3. *Given* $\alpha \in \left(0, \frac{1}{2}\right)$, $p \in \left(\frac{1}{\alpha}, \quad +\infty\right] \bigcap \left[\frac{2}{1-2\alpha}, \quad +\infty\right]$,

$$s \in \left[2 - 2\alpha, \quad \frac{1 - \frac{2}{p} + \alpha\left(1 - 2\alpha + \frac{2}{p}\right)}{2\alpha - \frac{2}{p}}\right],$$

and

$$\beta = \frac{\frac{2}{p} + 1 - 2\alpha}{\frac{2}{p} + s - 1}$$

which is within $\left[\frac{1}{s+\alpha}, \quad 1\right]$*, there is a constant* $C_3 > 0$ *such that for any weak solution* θ *to equation (1.1), if initially*

$$\|\theta_0\|_{H^s}^{\beta} \|\theta_0\|_{L^p}^{1-\beta} \leqslant \frac{\kappa}{C_3}, \tag{3.2}$$

then

$$\|\theta(t)\|_{H^s} \leq \|\theta_0\|_{H^s}, \qquad \forall t > 0.$$

The solution θ *is unique if* $s > 2 - 2\alpha$ *and* $\theta_0 \in L^2$*. Furthermore, if* $\|\theta_0\|_{H^s}^{\beta} \|\theta_0\|_{L^p}^{1-\beta} < \frac{\kappa}{C_3}$*, then* $\theta \in L^2(0, \infty; H^{s+\alpha})$ *and the solution* θ *is unique if* $s \geqslant 2 - 2\alpha$ *and* $\theta_0 \in L^2$.

Remark. 1. For the *critical* case, Constantin, D. Cordoba and Wu [4] proved as their main results that the solution $\theta \in H^s$ exists globally in time for $s = 2$ and

$s = 1$ if $\theta_0 \in H^s$ and $\|\theta_0\|_{L^\infty} \leqslant \frac{C}{\kappa}$ for some constant $C > 0$; while Theorem 3.3 of A. Cordoba and D. Cordoba [8] shows that the solution $\theta \in H^{\frac{3}{2}}$ exists globally in time if $\theta_0 \in H^{\frac{3}{2}}$ and $\|\theta_0\|_{L^\infty} \leqslant \frac{k}{C}$ for some constant $C > 0$. Therefore, part 1 of our main theorem improves these results. Notice that smallness of $\|\theta_0\|_{L^\infty}$ does *not* imply smallness of $\|\theta_0\|_{H^s}$.

2. In [4], the technique of integration by parts was crucial in obtaining several subtle estimates for the cases when $s = 1$ and $s = 2$. If s is not a integer, then it is impossible to play with integration by parts. Instead, in [8], for the special choice of $s = \frac{3}{2}$, very delicate commuting properties of Riesz transforms and their subtle estimates in Hardy space and BMO space were utilized elegantly. However, this method seems hard to be extended for general value of s.
3. Notice that the initial conditions (3.1) and (3.2), which yield the global solution in H^s for *super-critical* case, do *not* require either smallness of $\|\Lambda^s\theta_0\|_{L^2}$. Therefore, by part 2 and part 3 of our main theorem, for super-critical case, we can still obtain global existence of the solution in $H^s \cap L^2$ for arbitrarily large $\|\Lambda^s\theta_0\|_{L^2}$ provided that $\|\theta_0\|_{L^p}$ is small enough and vice versa. These results, which further generalize some of the recent results obtained in [13] for $p = 2 \in (1, \frac{1}{\alpha})$, would not be available via the methods of [8] or [4].
4. The method of proof provided in the following can also be used to study existence of local solutions without *any* smallness restrictions. Some new results have been obtained in [13] which improve those in [8]. More involved study will be explored in [17].
5. It is important to point out that the critical power $s = 2 - 2\alpha$ is very special from the point of view of scaling invariance as $H^{2-2\alpha}$ gives the important *scaling invariant* solution function space. Another important aspect of the special role for $2 - 2\alpha$ is that, so far, *uniqueness* result for the solutions in H^s can be obtained for $s \geqslant 2 - 2\alpha$ only. See more details in Subsection 3.2. See also [15]. On the other hand, existing best local *existence* results in Sobolev space H^s for *critical* and *super-critical* also require that $s \geqslant 2 - 2\alpha$. See [13]. The coincidence of these phenomena seems *not* accidental. This makes part 1 for our main theorem, especially the existence of H^s solution for $s \in [\frac{1}{2}, 1)$, particularly interesting from this point of view, as it gives the first family of such kind of solutions which break the barrier of the critical power $s = 2 - 2\alpha$ for *critical* case. So, this result is somewhat surprising and unexpected. It is not clear if such kind of solutions exist for *super-critical* case.
6. Finally, we point out that similar results are also valid for the case when $\Omega = \mathbb{T}^2$ with periodic boundary condition. For brevity, we omit the details.

We recall the following important commutator and product estimates:

Lemma 3.1 (Commutator and Product Estimates). *Suppose that $s > 0$ and $p \in (1, +\infty)$. If $f, g \in \mathcal{S}$, the Schwartz class, then*

$$\|\Lambda^s(fg) - f\Lambda^s g\|_{L^p} \leq C\left(\|\nabla f\|_{L^{p_1}}\|g\|_{H^{s-1,p_2}} + \|f\|_{H^{s,p_3}}\|g\|_{L^{p_4}}\right)$$

and

$$\|\Lambda^s(fg)\|_{L^p} \leq C\left(\|f\|_{L^{p_1}}\|g\|_{H^{s,p_2}} + \|f\|_{H^{s,p_3}}\|g\|_{L^{p_4}}\right)$$

with $p_2, p_3 \in (1, +\infty)$ *such that*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

Remark. The above lemma has been proved in Kenig, Ponce and Vega [20] with Λ being replaced by $J = (1-\Delta)^{\frac{1}{2}}$ and the homogeneous $H^{s,p}$ spaces being replaced by non-homogeneous ones. In the proof, the method of Kato and Ponce [19] is used which utilizes the results of Coifman and Meyer [3]. This lemma can be proved to be still valid for Λ by making use of a dilation argument of Kato as given in [18]. It is also clear that the lemma is valid whenever the corresponding right-hand sides terms are all finite.

For the rest of this section, we shall present the proof of our main theorem.

3.1. Existence

Now we start with some useful *a priori* estimates which will provide the formal proof of the regularity of the weak solutions in space H^s, i.e. the existence of solutions in H^s. First of all, multiplying (1.1) with θ and taking the inner product in L^2, we have

$$\frac{1}{2}\frac{d}{dt}\|\theta\|_{L^2}^2 + \kappa\|\Lambda^\alpha\theta\|_{L^2}^2 \leq 0.$$

Therefore, for any $t > 0$, we have

$$\|\theta(t)\|_{L^2}^2 + \kappa\int_0^t \|\Lambda^\alpha\theta(\tau)\|_{L^2}^2 d\tau \leq \|\theta_0\|_{L^2}^2,$$

which gives us the basic uniform boundedness of θ in L^2 and the property that $\theta \in L^2(0, +\infty, H^\alpha)$.

Noticing that $\nabla \cdot u = 0$, we have

$$(u \cdot \nabla(\Lambda^s\theta), \Lambda^s\theta) = 0.$$

Multiplying (1.1) with $\Lambda^{2s}\theta$ and taking the inner product in L^2, we have

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^s\theta\|_{L^2}^2 + \kappa\|\Lambda^{s+\alpha}\theta\|_{L^2}^2 = -(\Lambda^s(u\cdot\nabla\theta) - u\cdot\nabla(\Lambda^s\theta), \Lambda^s\theta).$$

Notice that Λ^s and ∇ are commutable, we have

$$\begin{aligned}|(\Lambda^s(u\cdot\nabla\theta) - u\cdot\nabla(\Lambda^s\theta), \Lambda^s\theta)| &= |(\Lambda^s(u\cdot\nabla\theta) - u\cdot(\Lambda^s\nabla\theta), \Lambda^s\theta)| \\ &\leq C\|\Lambda^s(u\cdot\nabla\theta) - u\cdot(\Lambda^s\nabla\theta)\|_{L^{p_2'}}\|\Lambda^s\theta\|_{L^{p_2}},\end{aligned}$$

where

$$\frac{1}{p_2} + \frac{1}{p_2'} = 1, \quad p_2 \in (2, \infty), \quad p_2' \in (1, 2)$$

and

$$\frac{1}{p_1} + \frac{2}{p_2} = 1, \quad p_1 \in (1, \infty), \quad p_2 \in (2, \infty).$$

Now we can make use of the commutator estimate. Since

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_2'},$$

we have

$$\begin{aligned} \|\Lambda^s(u \cdot \nabla\theta) - u \cdot (\Lambda^s \nabla\theta)\|_{L^{p_2'}} &\le C(\|\nabla u\|_{L^{p_1}} \|\Lambda^s\theta\|_{L^{p_2}} + \|\Lambda^s u\|_{L^{p_2}} \|\nabla\theta\|_{L^{p_1}}) \\ &\le C\|\Lambda\theta\|_{L^{p_1}} \|\Lambda^s\theta\|_{L^{p_2}} \end{aligned} \tag{3.3}$$

where, we have used (2.1) in the last inequality.

Therefore

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^s\theta\|_{L^2}^2 + \kappa\|\Lambda^{s+\alpha}\theta\|_{L^2}^2 \leqslant C\|\Lambda\theta\|_{L^{p_1}} \|\Lambda^s\theta\|_{L^{p_2}}^2.$$

Assume that $s + \alpha > 1$. Select

$$p_1 = 2(s + \alpha), \quad \beta = \frac{1}{s + \alpha}.$$

We have the following Gagliardo-Nirenberg inequality:

$$\|\Lambda\theta\|_{L^{p_1}} \leqslant C\|\Lambda^{s+\alpha}\theta\|_{L^2}^{\beta} \|\theta\|_{L^\infty}^{1-\beta}.$$

Since

$$\frac{2}{p_2} = 1 - \frac{1}{p_1} = 1 - \frac{1}{2(s+\alpha)},$$

for $\alpha = \frac{1}{2}$, we have the following Gagliardo-Nirenberg inequality:

$$\|\Lambda^s\theta\|_{L^{p_2}} \leqslant C\|\Lambda^{s+\alpha}\theta\|_{L^2}^{\delta} \|\theta\|_{L^\infty}^{1-\delta},$$

where

$$\delta = \frac{2s^2 - 2(1-\alpha)s + 1 - 2\alpha}{2(s+\alpha-1)(s+\alpha)} = \frac{s}{s+\alpha}.$$

Moreover, if $\alpha = \frac{1}{2}$, it is easy to that for $s > 1 - \alpha = \frac{1}{2}$, $\beta + 2\delta = 2$. Indeed,

$$\beta + 2\delta = \frac{2s^2 + (2\alpha - 1)s - \alpha}{(s+\alpha)(s+\alpha-1)} = \frac{2s^2 - \frac{1}{2}}{(s+\frac{1}{2})(s-\frac{1}{2})} = 2.$$

Therefore,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Lambda^s\theta\|_{L^2}^2+\kappa\|\Lambda^{s+\alpha}\theta\|_{L^2}^2 &\leqslant C\|\Lambda\theta\|_{L^{p_1}}\|\Lambda^s\theta\|_{L^{p_2}}^2\\ &\leqslant C\|\Lambda^{s+\alpha}\theta\|_{L^2}^{\beta+2\delta}\|\theta\|_{L^\infty}^{3-\beta-2\delta}\\ &= C\|\Lambda^{s+\alpha}\theta\|_{L^2}^2\|\theta\|_{L^\infty}.\end{aligned} \tag{3.4}$$

Hence, if $\alpha=\frac{1}{2}$ and

$$\|\theta_0\|_{L^\infty}\leqslant\frac{\kappa}{C},$$

then we have, for $s>\frac{1}{2}$,

$$\|\theta\|_{H^s}\leqslant\|\theta_0\|_{H^s},\quad \forall t\geqslant 0.$$

Furthermore, if

$$\|\theta_0\|_{L^\infty}<\frac{\kappa}{C},$$

then, for any $T>0$,

$$\int_0^T\|\theta(t)\|_{H^{s+\frac{1}{2}}}^2\,dt<\infty.$$

Now consider the special case that $\alpha=\frac{1}{2}$ and $s=\frac{1}{2}$.

Select $p_1=2$ and $p_2=4$, then $\frac{1}{p_1}+\frac{2}{p_2}=1$. Notice that

$$\|\Lambda^s\theta\|_{L^{p_2}}=\|\Lambda^{\frac{1}{2}}\theta\|_{L^4}\leqslant C\|\Lambda\theta\|_{L^2}^{\frac{1}{2}}\|\theta\|_{L^\infty}^{\frac{1}{2}}.$$

Therefore

$$\|\Lambda\theta\|_{L^{p_1}}\|\Lambda^s\theta\|_{L^{p_2}}^2\leqslant C\|\Lambda\theta\|_{L^2}^2\|\theta\|_{L^\infty}.$$

Hence, we have

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^{\frac{1}{2}}\theta\|_{L^2}^2+\kappa\|\Lambda\theta\|_{L^2}^2\leqslant C\|\Lambda\theta\|_{L^2}^2\|\theta\|_{L^\infty}. \tag{3.5}$$

Therefore the above conclusion holds as well for $s=\frac{1}{2}$. This finish the *a priori* estimates for part 1 of Theorem 3.1.

In what follows, we consider the estimates in a more general way, which is valid for any $\alpha\in(0,1)$. However, since we are only interested in *critical* and *super-critical* cases. In the following we ignore detailed discussion when $\alpha>\frac{1}{2}$. Now, let us consider the following Gagliardo-Nirenberg inequalities:

$$\|\Lambda\theta\|_{L^{p_1}}\leqslant C\|\Lambda^{s+\alpha}\theta\|_{L^2}^{\beta}\|\theta\|_{L^p}^{1-\beta},\quad \|\Lambda^s\theta\|_{L^{p_2}}\leqslant C\|\Lambda^{s+\alpha}\theta\|_{L^2}^{\delta}\|\Lambda^s\theta\|_{L^2}^{1-\delta},$$

where p_1, p_2, β and δ satisfy the following equations:

$$\frac{1}{p_1} = \frac{1}{2} + \beta\left(\frac{1}{2} - \frac{s+\alpha}{2}\right) + \frac{1-\beta}{p}, \tag{3.6}$$

$$\frac{1}{p_2} = \delta\left(\frac{1}{2} - \frac{\alpha}{2}\right) + \frac{1-\delta}{2}, \tag{3.7}$$

$$\frac{1}{p_1} + \frac{2}{p_2} = 1, \tag{3.8}$$

$$\beta + 2\delta = 2, \tag{3.9}$$

and the following additional conditions:

$$\begin{gathered} p_1 \in (1,\infty), \quad p_2 \in (2,\infty), \quad p \geqslant 1, \\ \delta \in \left[\frac{1}{2}, \frac{2s+2\alpha-1}{2s+2\alpha}\right], \quad \beta \in \left[\frac{1}{s+\alpha}, 1\right]. \end{gathered} \tag{3.10}$$

If we can find suitable solutions of p_1, p_2, p, β and δ for the above equations satisfying the conditions given in (3.10), then we obtain the following:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\Lambda^s\theta\|_{L^2}^2 + \kappa\|\Lambda^{s+\alpha}\theta\|_{L^2}^2 &\leqslant C\|\Lambda^{s+\alpha}\theta\|_{L^2}^2\|\Lambda^s\theta\|_{L^2}^{\beta}\|\theta\|_{L^p}^{1-\beta} \\ &\leqslant C\|\Lambda^{s+\alpha}\theta\|_{L^2}^2\|\Lambda^s\theta\|_{L^2}^{\beta}\|\theta_0\|_{L^p}^{1-\beta}, \end{aligned} \tag{3.11}$$

which will then yield the global existence of θ in H^s if initially

$$\|\Lambda^s\theta_0\|_{L^2}^{\beta}\|\theta_0\|_{L^p}^{1-\beta} \leqslant \frac{\kappa}{C}. \tag{3.12}$$

Furthermore, if

$$\|\Lambda^s\theta_0\|_{L^2}^{\beta}\|\theta_0\|_{L^p}^{1-\beta} < \frac{\kappa}{C}, \tag{3.13}$$

then we have in addition that

$$\int_0^\infty \|\Lambda^{s+\alpha}\theta(t)\|_{L^2}^2 dt \leqslant \frac{C}{\kappa}\|\Lambda^s\theta_0\|_{L^2}^2. \tag{3.14}$$

As will be seen in subsection 3.2, this property is important for us to prove uniqueness of the solutions we obtained in the following.

In the following we try to find these suitable solutions when they exist. By (3.6), (3.7), (3.8) and (3.9), we have

$$p_1 = \frac{1}{\alpha\delta}, \quad p_2 = \frac{2}{1-\alpha\delta}.$$

We also have

$$\left(s-1+\frac{2}{p}\right)\beta=\frac{2}{p}+1-2\alpha$$

and

$$\left(s-1+\frac{2}{p}\right)\delta=\frac{1}{p}+s+\alpha-\frac{3}{2}.$$

Case 1: $s-1+\frac{2}{p}=0$.

Then

$$s=2-2\alpha,\quad \frac{1}{p}=\alpha-\frac{1}{2}.$$

Since $p\in[1,+\infty]$, we have $\alpha\geqslant\frac{1}{2}$ and that β and δ can be any numbers in their full ranges.

For $\alpha=\frac{1}{2}$, we have $p=\infty$ and $s=1$. Choosing $\beta=1$ yields global existence of $\theta\in H^1$ when $\|\theta_0\|_{H^1}\leqslant\frac{\kappa}{C}$. A more general result was previously obtained in [13] where it is proved that for any $\alpha\in(0,1)$, if $s=2-2\alpha$ and

$$\|\theta_0\|_{H^s}\leqslant\frac{\kappa}{C},$$

then θ exists globally in H^s.

For other choice of $\beta\in[2/3,1)$, we have global existence of $\theta\in H^1$ with

$$\|\theta_0\|_{H^1}^{\beta}\|\theta_0\|_{L^\infty}^{1-\beta}\leqslant\frac{\kappa}{C}.$$

Notice that a more general result than above for $\alpha=\frac{1}{2}$ can be obtained for any $\beta\in[0,1]$ by interpolating the previously obtained two global existence results with $\beta=0$ and $\beta=1$ respectively. Thus, it is not a new result. This ends the discussion of Case 1.

The left two cases are those when $s-1+\frac{2}{p}\neq 0$. Then we have

$$\beta=\frac{\frac{2}{p}+1-2\alpha}{\frac{2}{p}+s-1},\qquad \delta=\frac{\frac{1}{p}+s+\alpha-\frac{3}{2}}{\frac{2}{p}+s-1}. \tag{3.15}$$

Case 1. $s-1+\frac{2}{p}<0$.

Since we need $\beta>0$, that is,

$$\frac{1}{p}+\frac{1}{2}<\alpha,$$

this implies $\alpha>\frac{1}{2}$. Thus we omit discussion of this case completely since in this article we focus only on the critical and super-critical cases.

The final case is more interesting to us than the previous two. We will separate this case further into three subcases in the following discussion.

Case 2. $s - 1 + \frac{2}{p} > 0$.

Then $\beta > 0$ implies that

$$\frac{1}{p} + \frac{1}{2} > \alpha. \tag{3.16}$$

Consider the conditions that

$$\beta + 2\delta = 2, \quad \delta \in \left[\frac{1}{2}, \quad \frac{2s + 2\alpha - 1}{2s + 2\alpha}\right], \quad \beta \in \left[\frac{1}{s+\alpha}, \quad 1\right].$$

It is easy to see that the condition $\beta \leqslant 1$ is equivalent to

$$s \geqslant 2 - 2\alpha, \tag{3.17}$$

which is a stronger condition than $s - 1 + \frac{2}{p} > 0$ due to (3.16); while the condition that $\beta \geqslant \frac{1}{s+\alpha}$ is equivalent to

$$s\left(2\alpha - \frac{2}{p}\right) \leqslant 1 - \frac{2}{p} + \alpha\left(1 - 2\alpha + \frac{2}{p}\right). \tag{3.18}$$

Subcase 3a: $p = \frac{1}{\alpha}$. Then (3.18) is not a restriction for s. So, for any $\alpha \in (0, 1)$ and $s \geqslant 2 - 2\alpha$, we have

$$\beta = \frac{1}{2\alpha + s - 1} \in \left[\frac{1}{s+\alpha}, \quad 1\right].$$

Therefore, if initially

$$\|\theta_0\|_{H^s}^{\beta} \|\theta_0\|_{L^{\frac{1}{\alpha}}}^{1-\beta} \leqslant \frac{\kappa}{C},$$

then θ exists globally in H^s. Furthermore, if

$$\|\theta_0\|_{H^s}^{\beta} \|\theta_0\|_{L^{\frac{1}{\alpha}}}^{1-\beta} < \frac{\kappa}{C},$$

then

$$\int_0^\infty \|\Lambda^{s+\alpha}\theta(t)\|_{L^2}^2 dt \leqslant \frac{C}{\kappa} \|\Lambda^s \theta_0\|_{L^2}^2.$$

Subcase 3b: $p > \frac{1}{\alpha}$. Then (3.18) gives

$$s \leqslant \frac{1 - \frac{2}{p} + \alpha\left(1 - 2\alpha + \frac{2}{p}\right)}{2\alpha - \frac{2}{p}}.$$

So, the allowed range for s is

$$2 - 2\alpha \leqslant s \leqslant \frac{1 - \frac{2}{p} + \alpha\left(1 - 2\alpha + \frac{2}{p}\right)}{2\alpha - \frac{2}{p}}, \tag{3.19}$$

which means that, in order to have solution(s) for s, we need

$$\frac{1}{p} + \alpha \leqslant \frac{1}{2}.$$

Therefore, we have that only $\alpha \leqslant \frac{1}{2}$ is allowed for this subcase.

If $\alpha = \frac{1}{2}$, then $p = \infty$ and $s = 1$. This case is already covered in Case 1.

If $\alpha \in \left(0, \frac{1}{2}\right)$, then for any p such that

$$p \in \left(\frac{1}{\alpha}, \quad +\infty\right] \bigcap \left[\frac{2}{1 - 2\alpha}, \quad +\infty\right], \tag{3.20}$$

and for any s such that

$$2 - 2\alpha \leqslant s \leqslant \frac{1 - \frac{2}{p} + \alpha\left(1 - 2\alpha + \frac{2}{p}\right)}{2\alpha - \frac{2}{p}},$$

we have

$$\beta = \frac{\frac{2}{p} + 1 - 2\alpha}{\frac{2}{p} + s - 1} \in \left[\frac{1}{s + \alpha}, \quad 1\right].$$

Therefore, if initially

$$\|\theta_0\|_{H^s}^{\beta} \|\theta_0\|_{L^p}^{1-\beta} \leqslant \frac{\kappa}{C},$$

then θ exists globally in H^s. Furthermore, if

$$\|\theta_0\|_{H^s}^{\beta} \|\theta_0\|_{L^{\frac{1}{\alpha}}}^{1-\beta} < \frac{\kappa}{C},$$

then

$$\int_0^\infty \|\Lambda^{s+\alpha}\theta(t)\|_{L^2}^2 dt \leqslant \frac{C}{\kappa} \|\Lambda^s \theta_0\|_{L^2}^2.$$

Notice that especially if $p = \infty$, then the range for s is

$$2 - 2\alpha \leqslant s \leqslant \frac{1 + \alpha(1 - 2\alpha)}{2\alpha} = \frac{(1 + 2\alpha)(1 - \alpha)}{2\alpha}.$$

Subcase 3c: $1 \leqslant p < \frac{1}{\alpha}$. Then (3.18) gives

$$s \geqslant \frac{1 - \frac{2}{p} + \alpha\left(1 - 2\alpha + \frac{2}{p}\right)}{2\alpha - \frac{2}{p}}. \tag{3.21}$$

It can be shown by elementary calculations that for $\alpha \in (0, 1)$

$$\frac{1 - \frac{2}{p} + \alpha\left(1 - 2\alpha + \frac{2}{p}\right)}{2\alpha - \frac{2}{p}} > 2 - 2\alpha$$

if and only if

$$\alpha > \frac{1}{2} + \frac{1}{p}.$$

Now consider only the interested $\alpha \in \left(0, \frac{1}{2}\right]$. Then

$$\frac{1 - \frac{2}{p} + \alpha\left(1 - 2\alpha + \frac{2}{p}\right)}{2\alpha - \frac{2}{p}} \leqslant 2 - 2\alpha.$$

So, for $\alpha \in \left(0, \frac{1}{2}\right]$, $p \in \left[1, \frac{1}{\alpha}\right)$ and $s \geqslant 2 - 2\alpha$, we can let

$$\beta = \frac{\frac{2}{p} + 1 - 2\alpha}{\frac{2}{p} + s - 1} \in \left[\frac{1}{s + \alpha}, \quad 1\right].$$

Therefore, if initially

$$\|\theta_0\|_{H^s}^{\beta} \|\theta_0\|_{L^p}^{1-\beta} \leqslant \frac{\kappa}{C},$$

then θ exists globally in H^s. Furthermore, if

$$\|\theta_0\|_{H^s}^{\beta} \|\theta_0\|_{L^{\frac{1}{\alpha}}}^{1-\beta} < \frac{\kappa}{C},$$

then

$$\int_0^\infty \|\Lambda^{s+\alpha}\theta(t)\|_{L^2}^2 dt \leqslant \frac{C}{\kappa} \|\Lambda^s \theta_0\|_{L^2}^2.$$

This ends the discussion of Case 3.

Now we see that Subcases 3a and 3c yield the needed *a priori* estimates for part 2 of Theorem 3.1; while Subcase 3b yields the needed *a priori* estimates for part 3 of Theorem 3.1.

By now, we have formally proved the existence results by the above corresponding *a priori* estimates. To finish the proof rigorously, we can make use of the standard method of retarded mollification to first obtain as above the uniform *a priori* bounds for the mollified solutions, and then use Theorem 2.1 and pass to the limit to obtain the same bounds for the weak solution θ. Since this is a standard procedure, it is therefore omitted for simplicity of presentation.

The above discussion finishes the proof of the existence part of Theorem 3.1.

3.2. Uniqueness

The solutions we have obtained in above subsection are all unique. Indeed, we have the following uniqueness results proved in Ju [15].

Theorem 3.2. *Suppose that $\kappa > 0$, $\alpha \in (0, 1)$ and that θ is a weak solutions of the 2D dissipative QGE (1.1) with the initial data $\theta_0 \in L^2$.*

1. *If $s \geq 2(1 - \alpha)$ and $\theta \in L^2(0, T; H^{s+\alpha})$, then the weak solution to (1.1) is unique.*
2. *If $s > 2(1 - \alpha)$ and $\theta \in L^\infty(0, T; H^s)$, then the weak solution to (1.1) is unique.*

Using Theorem 3.2, it is easy to check that the uniqueness results stated in Theorem 3.1 are all valid.

This finishes the proof of the uniqueness part of Theorem 3.1.

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