Mathematische Annalen

Ehrhart polynomials and stringy Betti numbers

Mircea Mustaţă · Sam Payne

Abstract. We study the connection between stringy Betti numbers of Gorenstein toric varieties and the generating functions of the Ehrhart polynomials of certain polyhedral regions. We use this point of view to give counterexamples to Hibi's conjecture on the unimodality of δ-vectors of reflexive polytopes.

1. Introduction

Let N be a lattice of rank d and let P be a d-dimensional lattice polytope in $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. For each nonnegative integer m, let $f_P(m)$ be the number of lattice points in mP. Then f_P is a polynomial in m of degree d, called the Ehrhart polynomial of P. The generating function $F_P(t) = \sum_{m \geq 0} f_P(m) t^m$ is a rational function in t and can be written as

$$F_P(t) = \frac{\delta_0 + \delta_1 t + \cdots + \delta_d t^d}{(1-t)^{d+1}},$$

for some nonnegative integers δ_i, with $\delta_0 = 1$. We put $\delta_P = (\delta_0, \ldots, \delta_d)$, and with a slight abuse of notation we denote by $\delta_P(t)$ the numerator of $F_P(t)$. If ℓ is the largest i such that δ_i is nonzero, then $\ell = d + 1 - r$, where r is the smallest positive integer such that rP contains a lattice point in its interior. Recall that a lattice polytope is reflexive if it contains 0 in its interior and its polar polytope has vertices in the dual lattice. Given the lattice polytope P, we have $\delta_i = \delta_{\ell-i}$ for all i if and only if rP is the translate of a reflexive polytope. Hibi conjectured in [Hi2, p. 111] that if this is the case, then δ_P is unimodal:

$$\delta_0 \leq \cdots \leq \delta_{[\ell/2]}. \tag{1}$$

M. MUSTAŢĂ, S. PAYNE
Department of Mathematics, University of Michigan
Ann Arbor, MI 48109, USA
(e-mail: mmustata@umich.edu; sdpayne@umich.edu)

In the particular case when P is the Birkhoff polytope of doubly stochastic $n \times n$ matrices, unimodality had been conjectured by Stanley [St] and was recently proved by Athanasiadis [At].

We assume now that P is reflexive, so $\ell = d$. Hibi showed that in this case

$$\delta_0 \leq \delta_1 \leq \delta_j \qquad (2)$$

for $2 \leq j \leq [d/2]$. If, in addition, the boundary of P admits a regular triangulation such that the vertices of each facet are a basis for the lattice N, then δ_P is the h-vector of the triangulation (see [Hi1]). In particular, if such a triangulation exists, then Stanley's theorem on the h-vectors of simplicial polytopes implies that δ_P is unimodal, so P satisfies Hibi's conjecture.

Note that if P is a reflexive polytope of dimension $d \leq 5$, then Hibi's conjecture follows from (2). The following reflexive polytope gives a counterexample to the conjecture for $d = 6$; for a more restricted and still open version of the conjecture, see [OH].

Example 1.1. Let $f = \frac{1}{3}(e_1 + \cdots + e_6)$ in $\mathbb{R}^6$, let N be the lattice $\mathbb{Z}^6 + \mathbb{Z} \cdot f$, and let P be the polyope with vertices $\{e_1, \ldots, e_6, e_1 - f, \ldots, e_6 - f\}$. It is straightforward to check that P is reflexive, and one computes that $2P$ and $3P$ contain 78 lattice points and 314 lattice points, respectively. It follows that $\delta_P = (1, 6, 8, 6, 8, 6, 1)$.

In this paper we give a combinatorial formula for δ_P when P is reflexive, as a positive linear combination of shifted h-vectors of simplicial polytopes, which we arrive at by using toric varieties to equate the combinatorial invariants δ_i of P with "stringy" invariants from complex algebraic geometry. This formula can also be proved directly, using elementary combinatorial arguments. We present proofs from both points of view. With this formula in hand, it is not difficult to construct examples, such as Example 1.1, where δ_P is not unimodal.

In order to explain our approach, we first reinterpret in algebro-geometric terms the proof of unimodality of δ_P in the special case mentioned above, due to Hibi. Here and throughout, P is assumed to be reflexive unless stated otherwise. Since P is reflexive, the polar polytope P° is reflexive, too. Note that the polytope P° corresponds to a toric variety X_{P° defined by the fan over the faces of P, and to an ample divisor D_{P° on X_{P°. The fact that P° is reflexive is equivalent with the fact that D_{P° is the canonical divisor on X_{P° (so in particular X_{P° is a Fano variety).

Consider a triangulation $\mathcal{P}$ of the boundary of P and let Δ be the fan whose maximal cones are the cones over the facets of $\mathcal{P}$. We have a proper birational morphism $f \colon X(\Delta) \to X = X_{P^\circ}$ induced by the identity on N. If $\mathcal{P}$ is a regular triangulation such that the vertices of each facet of $\mathcal{P}$ give a basis of N, then f is a resolution of singularities, $X(\Delta)$ is projective, and f is crepant, i.e. the pull-back of the canonical bundle on X is isomorphic to the canonical bundle on $X(\Delta)$. Conversely, every such resolution of singularities of X arises from a triangulation

as above. Given such a triangulation, δ_i is the $2i$-th Betti number of $X(\Delta)$, the dimension of the singular cohomology group $H^{2i}(X(\Delta); \mathbb{Q})$, and the unimodality of δ_P follows from the Hard Lefschetz Theorem on $X(\Delta)$.

In general, there may not exist any crepant resolution of X_{P°. However, using the theory of motivic integration, one can define "stringy Betti numbers" of X_{P° that agree with the Betti numbers of a crepant resolution whenever such a resolution exists [Bat]. A result of Batyrev and Dais shows that δ_i is the $2i$-th stringy Betti number of X_{P° [BD, Theorem 7.2]. We generalize this result as follows.

If $X = X(\Sigma)$ is a complete, d-dimensional Gorenstein toric variety, then there is a function Ψ_K on $N_\mathbb{R}$ that on each cone is given by an element of the dual lattice, and such that $\Psi_K(v_i) = 1$ for every primitive generator v_i of a ray of Σ. Consider the set

$$Q = \{v \in N_\mathbb{R} \mid \psi_K(v) \leq 1\}.$$

For every cone σ in Σ the intersection $Q \cap \sigma$ is a lattice polytope; it is the convex hull of the origin and of the primitive generators of the rays of σ. We see that Q, viewed as the union of the polytopes $Q \cap \sigma$, is naturally a polyhedral complex, and that Q is homeomorphic to a ball of dimension d.

Therefore we may define as in [Hi3] a polynomial of degree d (the Ehrhart polynomial) f_Q such that $f_Q(m)$ is the number of lattice points in mQ for every nonnegative integer m. Then we can write the generating function $F_Q(t) = \sum_{m\geq 0} f_Q(m)t^m$ in the form

$$F_Q(t) = \frac{\delta_0 + \delta_1 t + \cdots + \delta_d t^d}{(1-t)^{d+1}},$$

for some nonnegative integers δ_i.

Theorem 1.2. *For every complete Gorenstein toric variety X, δ_i is equal to the $2i$-th stringy Betti number of X.*

Although there may not exist any crepant resolution of singularities for X, we can always find a projective crepant morphism of toric varieties $f : X(\Delta) \to X$ such that $X(\Delta)$ has only Gorenstein orbifold singularities. Since f is crepant, the stringy Betti numbers of X are equal to the stringy Betti numbers of $X(\Delta)$. A theorem of Yasuda [Ya] then implies that the stringy Betti numbers of $X(\Delta)$ are equal to the dimensions of the graded pieces of the orbifold cohomology of $X(\Delta)$. We get a combinatorial formula for these dimensions using a toric formula due to Borisov, Chen, and Smith [BCS]. The resulting description of δ_Q is as follows. Fix a triangulation $\mathcal{T}$ of the boundary of Q whose vertices are in N, and let Δ be the fan over the faces of $\mathcal{T}$. For a face $F \in \mathcal{T}$ with vertices $v_1, \dots, v_r$, define

$$\mathrm{Box}(F) = \{a_1 v_1 + \cdots + a_r v_r \in N_\mathbb{R} : 0 < a_i < 1\},$$

and let Δ_F be the fan in $N/(N \cap \mathrm{span}\, F)$ whose cones are the projections of the cones in Δ containing F. For a positive integer m, let $h_{\Delta_F}[m]$ denote the h-vector of Δ_F shifted by m, defined by

$$
h_{\Delta_F}[m]_i = \begin{cases} 0 \text{ for } i < m. \\ (h_{\Delta_F})_{i-m} \text{ for } i \geq m. \end{cases}
$$

Note that Δ_F is the simplicial fan corresponding to the T-invariant subvariety of $X(\Delta)$ determined by the cone over F, and $(h_{\Delta_F})_i$ is the $2i$-th Betti number of $X(\Delta_F)$. In particular, if $X(\Delta)$ is projective, then the Hard Lefschetz Theorem on $X(\Delta_F)$ implies that h_{Δ_F} is unimodal.

Theorem 1.3. *If $\mathcal{T}$ is any triangulation of the boundary of Q whose vertices are in N, then*

$$
\delta_Q = h_{\mathcal{T}} + \sum_{F \in \mathcal{T},\, v \in \mathrm{Box}(F) \cap N} h_{\Delta_F}[\Psi_K(v)].
$$

In particular, the sum of shifted h-vectors in Theorem 1.3 is independent of the choice of triangulation.

2. δ-vectors and stringy Betti numbers

A d-dimensional Gorenstein variety X with canonical singularities has a stringy E-function

$$
E_{\mathrm{st}}(X; w, z) \in \mathbb{Z}[[w, z]] \cap \mathbb{Q}(w, z)
$$

defined using Hodge theory and motivic integration on a resolution of singularities of X. If $E_{\mathrm{st}}(X; w, z) = \sum_{p,q} a_{pq} w^p z^q$ is a polynomial, then the j-th stringy Betti number of X is defined to be $(-1)^j \sum_{p+q=j} a_{pq}$.

Suppose now that $X = X(\Sigma)$ is a complete Gorenstein toric variety (see [Fu] for basic facts on toric varieties). In this case $E_{\mathrm{st}}(X; w, z)$ is a polynomial in wz, so the odd stringy Betti numbers vanish and the $2i$-th stringy Betti number of X is the coefficient of $(wz)^i$ [Bat, Section 3]. Our proof of Theorem 1.2 is based on the following formula for $E_{\mathrm{st}}(X; w, z)$ as a rational function [Bat, Theorem 4.3]. Since X is Gorenstein, we have a function ψ_K on $N_{\mathbb{R}}$ that on each cone is given by an element in the dual lattice, and such that $\Psi_K(v_i) = 1$ for every primitive generator v_i of a ray of Σ. For each cone $\sigma \in \Sigma$, let σ° denote the relative interior of σ. Recall that $\sum_{v \in \sigma^\circ} (wz)^{-\Psi_K(v)}$ is a rational function in wz (see, for example, [Bar, VIII.1]). Batyrev has shown that we have the following equality of rational functions,

$$
E_{\mathrm{st}}(X; w, z) = (wz - 1)^d \sum_{\sigma \in \Delta} \sum_{v \in \sigma^\circ \cap N} (wz)^{-\Psi_K(v)}. \tag{3}
$$

As in the Introduction, we define

$$Q = \{v \in N_{\mathbb{R}} \mid \Psi_K(v) \leq 1\}.$$

There is an Ehrhart polynomial f_Q such that, for positive integers m, $f_Q(m)$ is the number of lattice points in mQ, and f_Q satisfies Ehrhart reciprocity: $f_Q(-m)$ is the number of lattice points in the interior of mQ. The proofs of these assertions follow as in [Hi2], using the fact that Q is homeomorphic to a ball of dimension d. The generating function $F_Q(t) = \sum_{m \geq 0} f_Q(m)t^m$ can then be written

$$F_Q(t) = \frac{\delta_0 + \delta_1 t + \cdots + \delta_d t^d}{(1-t)^{d+1}},$$

for some nonnegative integers δ_i.

For the proof of Theorem 1.2 we will need the following two lemmas. A proof of the first lemma in the case when Q is a polytope can be found in [Hi2] and the general case is similar, but we include the proof for the reader's convenience.

Lemma 2.1. *With the above notation, we have $\delta_i = \delta_{d-i}$ for every i.*

Proof. Note first that if m is a positive integer, then a lattice point v is in the interior of mQ if and only if v is in $(m-1)Q$. Indeed, v is in the interior of mQ if and only if $\psi_K(v) < m$, and since $\psi_K(v)$ is an integer this is the case if and only if $\psi_K(v) \leq m-1$, which happens if and only if v is in $(m-1)Q$. Ehrhart reciprocity implies that

$$f_Q(m-1) = (-1)^d f_Q(-m) \tag{4}$$

for every positive integer m, and therefore for all m.

If we write $f_Q(m) = \sum_{i=0}^d a_i \binom{i+m}{i}$, then we deduce

$$F_Q(t) = \sum_{m \in \mathbb{N}} \sum_{i=0}^d a_i \binom{i+m}{i} t^m = \sum_{i=0}^d a_i \cdot \sum_{m \in \mathbb{N}} \binom{i+m}{i} t^m = \sum_{i=0}^d \frac{a_i}{(1-t)^{i+1}}.$$

If we put $\widetilde{F}_Q(t) = \sum_{m \geq 1} f_Q(-m)t^m$, then

$$\widetilde{F}_Q(t) = \sum_{i=0}^d a_i \cdot \sum_{m \geq i+1} (-1)^i \binom{m-1}{i} t^m = \sum_{i=0}^d (-1)^i \frac{a_i t^{i+1}}{(1-t)^{i+1}},$$

so we have the equality of rational functions $\widetilde{F}_Q(t) = -F_Q(t^{-1})$.

On the other hand, (4) gives $\widetilde{F}_Q(t) = (-1)^d t F_Q(t)$, hence $F_Q(t^{-1}) = (-1)^{d+1} t F_Q(t)$. Since $(1-t)^{d+1} F_Q(t) = \sum_{i=0}^d \delta_i t^i$, this equality gives $\delta_i = \delta_{d-i}$ for every i. $\square$

Lemma 2.2. *With the above notation, we have*

$$(1 - t)F_Q(t) = \sum_{v \in N} t^{\psi_K(v)}.$$

Proof. We can write

$$F_Q(t) = \sum_{m \in \mathbb{N}} \sum_{v \in mQ \cap N} t^m = \sum_{v \in N} \sum_{m \geq \psi_K(v)} t^m,$$

using the fact that v is in mQ if and only if $m \geq \psi_K(v)$. The assertion in the lemma follows. $\square$

Proof of Theorem 1.2. It is enough to show that $E_{\mathrm{st}}(X; t, 1) = \delta_Q(t)$. Combining Lemma 2.2 with (3), we have

$$E_{\mathrm{st}}(X; t, 1) = (t - 1)^d (1 - t^{-1}) F_Q(t^{-1}).$$

Now

$$F_Q(t^{-1}) = \frac{\delta_Q(t^{-1})}{(1 - t^{-1})^{d+1}}.$$

By Lemma 2.1 we have $\delta_i = \delta_{d-i}$, so $\delta_Q(t^{-1}) = t^{-d}\delta_Q(t)$. Hence

$$E_{\mathrm{st}}(X; t, 1) = (t - 1)^d \frac{\delta_Q(t)}{t^d (1 - t^{-1})^d} = \delta_Q(t).$$

$\square$

3. δ-vectors via orbifold cohomology

The orbifold cohomology of a Gorenstein variety Y with quotient singularities was defined by Chen and Ruan [CR] and Yasuda [Ya], as follows. There is a canonically associated orbifold (smooth Deligne-Mumford stack) $\mathcal{Y}$ whose coarse moduli space is Y. Let $I(\mathcal{Y})$ be the inertia stack of $\mathcal{Y}$. We denote by $\mathcal{Y}_i \subset I(\mathcal{Y})$ the connected components of $I(\mathcal{Y})$ and let $\overline{\mathcal{Y}_i}$ be the coarse moduli space of $\mathcal{Y}_i$. The "age" s_i of $\mathcal{Y}_i$ is a positive integer determined by the action of the inertia group. As a graded vector space, the orbifold cohomology of Y is given by

$$H^*_{\mathrm{orb}}(Y, \mathbb{Q}) = \bigoplus_{\mathcal{Y}_i \subset I(\mathcal{Y})} H^*(\overline{\mathcal{Y}_i}, \mathbb{Q})[s_i],$$

where $[s_i]$ denotes a grading shift by s_i, so $H^j(\overline{\mathcal{Y}_i}, \mathbb{Q})[s_i] = H^{j-s_i}(\overline{\mathcal{Y}_i}, \mathbb{Q})$.

It is a theorem of Yasuda [Ya] that the j-th stringy Betti number of Y is equal to the dimension of $H^j_{\mathrm{orb}}(Y, \mathbb{Q})$. See also [Po] for a proof of this result in the case of toric varieties.

We mention that Chen and Ruan have constructed a ring structure on orbifold cohomology in [CR]. J. Fernandez gave in [Fe] a necessary and sufficient condition for when the Chen-Ruan cohomology satisfies the Hard Lefschetz Theorem. His condition inspired us in looking for the counterexamples to Hibi's Conjecture.

There is an algebraic version of orbifold cohomology, due to Abramovich, Graber and Vistoli [AGV], the so-called orbifold Chow ring. Note that when Y is a simplicial toric variety, each $\overline{\mathcal{Y}_i}$ is also a simplicial toric variety, so the odd cohomology of $\overline{\mathcal{Y}_i}$ vanishes and $H^{2*}(\overline{\mathcal{Y}_i}, \mathbb{Q})$ is isomorphic to the Chow ring $A^*(\overline{\mathcal{Y}_i}, \mathbb{Q})$. It follows that at least as vector spaces, $H^{2*}_{\text{orb}}(Y, \mathbb{Q})$ agrees in this case with the [AGV] version $A^*_{\text{orb}}(Y, \mathbb{Q})$ as used by Borisov, Chen and Smith [BCS]. We mention that while there seems to be agreement among experts that the ring structures are also the same in this case, there is no available reference. We stress however that we do not need this, as we are interested only in the vector space structure of the orbifold cohomology.

Proof of Theorem 1.3. Let Y be the toric variety corresponding to the fan Δ whose maximal cones are the cones over the facets of the triangulation $\mathcal{T}$. For a face $F \in \mathcal{T}$ and a lattice point $v \in \text{Box}(F)$, Δ_F is the fan associated to the stacky fan $\Delta/\sigma(\overline{v})$ defined in [BCS], and hence h_{Δ_F} is the vector whose i-th entry is the dimension of $A^i(X(\Delta_F))$. Furthermore, the integer $\Psi_K(v)$ is equal to $\deg y^v$ as defined in [BCS]. Hence the theorem follows from [BCS, Proposition 5.2]. □

Although we arrived at Theorem 1.3 through the connection with orbifold cohomology and the results of [BCS], it is also possible to prove this result directly using elementary combinatorial methods, as follows. For a fan Δ with h-vector $h_\Delta = (h_0, \ldots, h_r)$, we write $h_\Delta(t)$ for the polynomial $h_\Delta(t) = h_0 + h_1 t + \cdots + h_r t^r$.

Second proof of Theorem 1.3. By Lemma 2.2, it will suffice to show that

$$(1 - t)^d \cdot \sum_{v \in N} t^{\Psi_K(v)} = \sum_{F \in \mathcal{T}, v \in \text{Box}(F)} t^{\Psi_K(v)} \cdot h_{\Delta_F}(t).$$

Now each lattice point in the cone over a face $G \in \mathcal{T}$ can be written uniquely as a nonnegative integer linear combination of the vertices of G plus a fractional part. Hence any lattice point v_0 in the relative interior of this cone can be written uniquely as

$$v_0 = v + v_{G|F} + v',$$

where v is in $\text{Box}(F)$ for some face $F \prec G$, $v_{G|F}$ is the sum of the vertices of G that are not in F, and v' is a nonnegative integer linear combination vertices of G. Since each lattice point $v \in N$ is in the relative interior of exactly one cone, it

follows that

$$(1-t)^d \sum_{v \in N} t^{\Psi_K(v)} = \sum_{F \in \mathcal{T}, v \in \mathrm{Box}(F)} t^{\Psi_K(v)} \cdot \sum_{G \succ F} t^{\dim G - \dim F}(1-t)^{\mathrm{codim}\, G}$$

$$= \sum_{F \in \mathcal{T}, v \in \mathrm{Box}(F)} t^{\Psi_K(v)} \cdot h_{\Delta_F}(t),$$

as required.   □

Example 3.1. Let m be a positive integer. Let $f \in \mathbb{R}^{2m}$ be the vector $f = (\frac{1}{m}, \ldots, \frac{1}{m})$, and let N be the lattice $N = \mathbb{Z}^{2m} + \mathbb{Z} \cdot f$. We take $P \subset \mathbb{R}^{2m}$ to be the polytope with vertices $e_1, \ldots, e_{2m}, e_1 - f, \ldots, e_{2m} - f$. It is straightforward to check that P is reflexive. We will show that

$$\delta_P = (1, 2m, 2m+2, 2m, 2m+2, \ldots, 2m, 2m+2, 2m, 1).$$

This generalizes Example 1.1, and shows that for $m > 0$ there are $2m$-dimensional reflexive polytopes with $[\frac{m-1}{2}]$ descents in $(\delta_0, \delta_1, \ldots, \delta_m)$.

We compute δ_P by applying Theorem 1.3 to the triangulation $\mathcal{P}$ of the boundary of P whose facets are $\langle e_1, \ldots, e_{2m} \rangle$, $\langle e_1 - f, \ldots, e_{2m} - f \rangle$, $\langle e_1, \ldots, \widehat{e_j}, \ldots, e_k,$ $e_k - f, \ldots, e_{2m-f} \rangle$ and $\langle e_1, \ldots, e_j, e_j - f, \ldots, \widehat{e_k - f}, \ldots, e_{2m} - f \rangle$ for $1 \leq j < k \leq 2m$. This triangulation is obtained by "pulling" the sequence of points $e_1, \ldots, e_{2m-1}$. In particular, $\mathcal{P}$ is a regular triangulation, and hence $h_{\mathcal{P}}$ is unimodal. Now $\mathcal{P}$ has $4m$ vertices, so $(h_{\mathcal{P}})_1 = 2m$, and $\mathcal{P}$ has $4m^2 - 2m + 2$ facets, so $(h_{\mathcal{P}})_0 + \cdots + (h_{\mathcal{P}})_{2m} = 4m^2 - 2m + 2$. It then follows from unimodality and the fact that $(h_{\mathcal{P}})_0 = (h_{\mathcal{P}})_{2m} = 1$ that

$$h_{\mathcal{P}} = (1, 2m, 2m, \ldots, 2m, 2m, 1).$$

To compute δ_P, it remains to compute the contributions of the points in $\mathrm{Box}(F)$ for the faces $F \in \mathcal{P}$. The only faces of $\mathcal{P}$ whose Box is nonempty are $F = \langle e_1, \ldots, e_{2m} \rangle$ and $F' = \langle e_1 - f, \ldots, e_{2m} - f \rangle$, which contain $\{f, \ldots, (m-1)f\}$ and $\{-f, \ldots, (1-m)f\}$, respectively. Since F and F' are facets, $\Delta_F = \Delta_{F'} = 0$ and $h_{\Delta_F} = h_{\Delta_{F'}} = 1$. Since $m_v = 2k$ for $v = \pm k \cdot f$, it follows that

$$\delta_P = h_{\mathcal{P}} + (0, 0, 2, 0, 2, \ldots, 2, 0, 2, 0, 0),$$

as required.

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