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Counting alternating knots by genus

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Abstract. It is shown that the number of alternating knots of given genus $g > 1$ grows as a polynomial of degree $6g - 4$ in the crossing number. The leading coefficient of the polynomial, which depends on the parity of the crossing number, is related to planar trivalent graphs with a Bieulerian path. The rate of growth of the number of such graphs is estimated.

1. Introduction

In the fundamental paper [14], Menasco and Thistlethwaite proved that two alternating diagrams of the same knot or link are related by a sequence of flypes, i.e. are *flype-equivalent*. This theorem has, among others, applications to enumeration problems of links. Apart from their knot-theoretic relevance, such problems are expected to have impact for, and so are of interest to, biologists, physicists and chemists studying knotting in their work (for example in DNA, or in statistical mechanics). The present paper deals with such a problem. Specifically, let $a_{n,g}$ denote the number of alternating knots of crossing number n and genus $g > 1$. Our effort will be to determine the behaviour of $a_{n,g}$ for g fixed and $n \to \infty$. Writing $a_n \asymp_n b_n$ for $\lim_{n\to\infty} a_n/b_n = 1$, our main result is stated as follows:

Theorem 1.1. *If $g > 1$, as $n \to \infty$ through the even/odd integers, we have*

$$a_{n,g} \asymp_n C_{g,e/o} n^{6g-4} + O(n^{6g-5}) \,, \tag{1}$$

with non-zero constants $C_{g,e}$ and $C_{g,o}$ (independent on n for fixed parity of n), and

$$400 \le \liminf_{g\to\infty} \sqrt[g]{(6g)!\, C_{g,e/o}} \le \limsup_{g\to\infty} \sqrt[g]{(6g)!\, C_{g,e/o}} \le \frac{2^{20}}{3^6} \approx 1438.38 \,.$$

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(In the following, a statement involving '$C_{g,e/o}$' is to be understood as a pair of statements, one involving '$C_{g,e}$' with n restricted to even integers, and one involving '$C_{g,o}$' with n restricted to odd integers.)

The main tool applied in the study of the numbers $a_{n,g}$ are certain group theoretic objects called Wicks forms.

A Wicks form is a canonical form of a product of commutators in a free group G [25]. The algebraic genus of a Wicks form w is the least positive integer g_a such that w is a product of g_a commutators in G. The topological genus of an oriented Wicks form $w = w_1 \dots w_{2l-1} w_{2l}$ is defined as the topological genus of the oriented compact connected surface obtained by labeling and orienting the edges of a $2l-$gon in the oriented plane by the letter of w and by identifying the edges labeled by a letter and its inverse. The algebraic and the topological genus coincide (cf. [10], [7]).

Wicks forms have been considered in [7], [25] to study products of commutators and products of squares in free groups. In [1] the exact formula for their number was computed, and it was proved that there is a bijection between Wicks forms of genus g and $1-$vertex triangulations of genus g orientable surfaces. L. Mosher [15] has constructed a complex whose fundamental group is the mapping class group of an orientable genus g surface. $1-$vertex triangulations appear as the vertices of this complex. Brenner and Lyndon considered such triangulations from a combinatorial point of view motivated by the study of non-parabolic subgroups in the modular group [6].

Wicks forms are also closely related to the structure of the class of alternating knots of given genus studied by the first author [21]. He showed that if $a_{n,g}$ denotes the number of alternating knots of crossing number n and genus g, then $a_{n,g}$ for g fixed and $n \to \infty$ grows polynomially in n. He also gave an estimate for the degree of this polynomial. In [22] the relation between alternating knot diagrams, Wicks forms, and the Gauß diagrams of [19] was established, and it was shown that the notions of genus for all of these objects coincide. Then the theory of Wicks forms [25, 1] was used to improve the estimate on the degree of the polynomial (in n) enumerating $a_{n,g}$ to $6g-4$ in genus $g > 1$.

In the present paper we deepen the relationship between Wicks forms and the genus of alternating knots. We will show that the above estimate $6g-4$ is exact for $g > 1$, thus determining the asymptotical behaviour of $a_{n,g}$ as $n \to \infty$ up to constants, depending on the parity of n, which we denote as $C_{g,e/o}$ (for even/odd n).

The main effort then will focus on identifying the constants $C_{g,e/o}$, i.e. the leading coefficient of the polynomial (for a given parity of n). We obtain a description of this coefficient in terms of the number of a special type of Wicks forms we call *planar maximal* Wicks forms. These are the forms whose graph is planar, trivalent (cubic), and 3-connected. As a Bieulerian path in the graph of such a form induces in each vertex a cyclic orientation, we arrive at another, quite unrelated, occurrence of at least a subclass of the 3-valent graphs well-known as Feynman diagrams [5], and then appearing in the theory of Vassiliev invariants [2,3]. Then we use the work of [1] and [24] to estimate the number of such graphs, and hence $C_{g,e/o}$, asymptotically for $g \to \infty$. More precisely, we show that the coefficients $C_{g,o}$ and $C_{g,e}$ are both non-zero for genus $g > 1$ and differ only at most by a linear term in g, so that the qualitative difference between even and odd crossing number for large g is minimal. We will also explain the reason for the degeneracy of the case $g = 1$ (where a difference between even and odd crossing number occurs) encountered in [21].

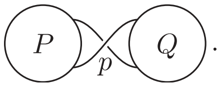
Since the alternating diagrams obtained from planar maximal Wicks forms are special alternating, an unexpected consequence of our investigation is that generically an alternating knot of any genus (higher than one) is a special alternating knot.

In a subsequent paper, the first author will extend the main result of this paper also to the number of positive knots of given genus and given crossing number. This requires an additional sharp estimate of this crossing number.

2. Preliminaries and statement of results

In order to state our results and introduce our tools, we start with some classical definitions and recall important properties of alternating knots and links.

Definition 2.1. A crossing p in a knot diagram D is called reducible (or nugatory) if D can be represented in the form



D is called reducible if it has a reducible crossing, else it is called reduced.

Definition 2.2. Denote by $c(D)$ the crossing number of a knot diagram D. The crossing number $c(K)$ of a knot K is the minimal crossing number $c(D)$ of all diagrams D of K.

Theorem 2.3. ([12,17,23]) *An alternating knot with a reduced alternating diagram of n crossings has crossing number n.*

Definition 2.4. For a diagram D of knot K, we define the genus $g(D)$ as the genus of the surface obtained by applying the Seifert algorithm to this diagram. It can be expressed as

$$g(D) = \frac{c(D) - s(D) + 1}{2},$$

with $s(D)$ being the number of Seifert circles of D.

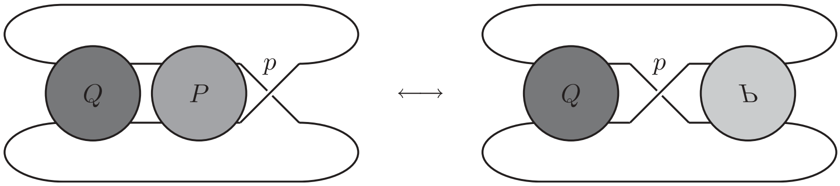


Fig. 1. A flype near the crossing p

For a knot K we call $g(K)$ the genus of K, and define it by the minimal genus of all Seifert surfaces (not only those obtained from the Seifert algorithm) of K.

The importance of this definition relies on the following classical fact:

Theorem 2.5. ([9,16]) *For an alternating knot K with an alternating diagram D we have $g(K) = g(D)$.*

By the work of Menasco and Thistlethwaite [14], alternating knots are intimately related to another diagrammatic move called flype.

Definition 2.6. A flype is a move on a diagram shown in figure 1.

Theorem 2.7. ([14]) *Two alternating diagrams of the same knot or link are flype-equivalent, that is, transformable into each other by a sequence of flypes.*

When we want to specify the distinguished crossing p, we say that it is a flype *near* the crossing p.

We call the tangle P of figure 1 *flypable*. We say that the crossing *p admits a flype* or that *the diagram admits a flype at (or near) p*.

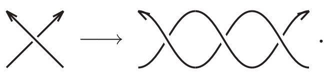
We call the flype *non-trivial*, if both tangles P and Q have at least two crossings.

We say that the crossing *p admits* a (non-trivial) flype if the diagram can be represented as in figure 1 with p being the distinguished crossing (and both tangles having at least two crossings). A diagram admits a (non-trivial) flype if some crossing in it admits a (non-trivial) flype.

Since trivial flypes are of no interest we will assume from now on, unless otherwise noted, that all flypes are non-trivial, without mentioning this explicitly each time.

Let $a_{n,g}$ be the number of prime alternating knots K of genus $g(K) = g$ and crossing number $c(K) = n$. (We conform here to the notation of [22].) The set of such knots was shown to have special structure by a theorem of the first author.

Theorem 2.8. ([21, theorem 3.1]) *Reduced (that is, with no nugatory crossings) alternating knot diagrams of given genus decompose into finitely many equivalence classes under flypes and (reversed) applications of antiparallel twists at a crossing*



(2)

Henceforth we call the move in (2) a $\bar{t}_2$ move.

It was observed in [21] that in a sequence of flypes and $\bar{t}_2$ moves, all the flypes can be performed in the beginning. It follows then from [14] that there are only finitely many alternating knots with $\bar{t}_2$-irreducible diagrams of given genus g, and we call all such knots, and their alternating diagrams *generators* or *generating* knots/diagrams of genus g.

A *clasp* is a tangle made up of two crossings. According to the orientation of the strands we distinguish between reverse and parallel clasps.

reverse clasp parallel clasp

There is an obvious bijective correspondence between the crossings of the 2 diagrams in figure 1 before and after the flype, and under this correspondence we can speak of what is a specific crossing after the flype. In this sense, we make the following definition:

Definition 2.9. We call two crossings in a diagram $\sim$*-equivalent*, if they can be made to form a reverse clasp after some (sequence of) flypes.

Is is an easy exercise to check that $\sim$ is an equivalence relation.

Definition 2.10. We call an alternating diagram *generating*, if each $\sim$-equivalence class of its crossings has 1 or 2 elements. The set of diagrams which can be obtained by applying flypes and $\bar{t}_2$ moves on a generating diagram D we call *generating series* of D.

Thus theorem 2.8 says that alternating diagrams of given genus decompose into finitely many generating series.

Definition 2.11. Let c_g be the maximal crossing number of a generating diagram of genus g, and d_g the maximal number of $\sim$-equivalence classes of such a diagram.

A consequence of theorem 2.8 is

Corollary 2.12. *For any* $g \geq 1$

$$\sum_n a_{n,g} x^n = \frac{R_g(x)}{(x^{p_g} - 1)^{d_g}}$$

for some numbers $p_g, d_g \in \mathbb{N}$ *and* $R_g \in \mathbb{Q}[x]$.

This corollary can be written also in the following form (see corollary 3.2 of [21]): there are numbers $p_g, n_g \in \mathbb{N}$ and polynomials $P_{g,1}, \dots, P_{g,p_g} \in \mathbb{Q}[n]$ with $a_{n,g} = P_{g,n \bmod p_g}(n)$ for $n \geq n_g$.

Let us write $P_n = P_{g,n}$ when g is fixed. Using [21] we can say more on P_n (see §3). While the entire P_n depend on n mod p_g, the degree deg P_n and leading coefficient max cf P_n of P_n depend only on n mod 2. Let $d_{g,o} = \deg P_n + 1$ for n odd and $d_{g,e} = \deg P_n + 1$ for n even, and $d_g = \max(d_{g,o}, d_{g,e})$. The equivalence of this definition of d_g to the one in corollary 2.12 (where d_g is taken to be the smallest possible) follows from standard generating function theory, and to the one given in definition 2.11 follows from [14] (see [21], or §3 below for a better explanation). In [21], a rather rough estimate on the d_g was given, which was later improved in [22].

Theorem 2.13. ([22]) $d_g \leq 6g - 3$.

It was asked whether this is the best possible estimate. The first of the next 3 theorems, which summarize the results of this paper, answers partially this and some other questions of [22]. It is proved in §4.

For two sequences a_n and b_n we write $a_n \sim b_n$ if $\lim_{n\to\infty} |b_n/a_n|$ exists, and is not 0 or ∞.

Theorem 2.14. *The following holds:*

(1) $d_{g,o} = d_{g,e} = 6g - 3$ *for* $g > 1$. *That is,* $a_{n,g} \sim n^{6g-4}$.
(2) $c_g \geq 10g - 7$.

It will be convenient, from now on, to consider only genus $g > 1$. The case $g = 1$ is completely described in [21].

Definition 2.15. A *special* diagram is a diagram all of whose Seifert circles have either an empty interior or exterior. (Here interior and exterior denote the bounded and unbounded connected component of the complement of the Seifert circle in $\mathbb{R}^2$ and empty means not containing a crossing of the knot diagram.)

After we identify the degrees of the P_n, we relate their leading coefficients to 3-valent graphs. Note, that these coefficients can also be written as the limits

$$C_{g,o} = \lim_{n\to\infty} \frac{a_{2n+1,g}}{(2n+1)^{6g-4}} \qquad \text{and} \qquad C_{g,e} = \lim_{n\to\infty} \frac{a_{2n,g}}{(2n)^{6g-4}}\,.$$

We consider planar 3-connected 3-valent graphs (with no multiple edges and loops). When equipping such a graph with a Bieulerian path (whenever this is possible), we associate to it a special generating knot and prove that all generating knots with maximal degree contribution to the P_n correspond to such graphs.

As a Bieulerian path endows each vertex of such a graph with a cyclic orientation, we have yet another appearance of, at least some, 3-valent graphs from the theory of Vassiliev invariants [3] in a different context, after Bar-Natan's remarkable paper [2].

A consequence of our correspondence is that special alternating knots dominate among alternating knots of given genus (higher than 1), as the crossing number increases, as we prove in §4. (The conditions for an alternating knot to have a special alternating diagram and to be positive are equivalent, see [18].)

Theorem 2.16.

$$\frac{\#\{K\, alternating\ positive\ prime,\, c(K)=n,\, g(K)=g\,\}}{\#\{K\, alternating\ prime,\, c(K)=n,\, g(K)=g\,\}}\longrightarrow 1$$

as $n\to\infty$ *for any fixed* $g>1$.

(Here and below $\#S$ and $|S|$ both denote the cardinality of a finite set S.)

Later we tackle the asymptotic estimation of the number of 3-connected 3-valent planar graphs of genus g (that is, with $4g-2$ vertices and $6g-3$ edges) with Bieulerian paths up to cyclic permutation. Let B_g be this number. We have from our correspondence a relation to the numbers $C_{g,e} = \max \operatorname{cf} P_{g,2n}$ and $C_{g,o} = \max \operatorname{cf} P_{g,2n+1}$:

$$B_g \ge (6g-4)!\big(C_{g,o}+C_{g,e}\big) \ge \frac{B_g}{6}\,. \tag{3}$$

This inequality is explained in §3.

Moreover, we prove that the ratios of $C_{g,e}$ and $C_{g,o}$ can be double-sidedly estimated by polynomials in g (corollary 5.9). Thus the rate of growth of B_g becomes of interest. We obtain the following estimates on this rate. The proof uses the work of [24] and [25], and is given in §5.

Theorem 2.17.

$$400 \le \liminf_{g\to\infty} \sqrt[g]{B_g} \le \limsup_{g\to\infty} \sqrt[g]{B_g} \ \le\ \frac{2^{20}}{3^6} \approx 1438.37585\dots$$

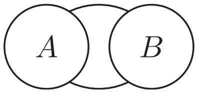
We conclude this section with a few more preliminary remarks.

In [19] the concept of Gauß diagrams was introduced as a tool for generating knot invariants. Given a knot diagram, one links by a chord on a circle the pre-images of the two passes of each crossing, orienting the chord from the underpass to the overpass. The resulting object is called a *Gauß diagram* (GD).

In general any circle with oriented chords is called a Gauß diagram. Not all Gauß diagrams come from knot diagrams; those that do are called *realizable* Gauß diagrams. We ignore in the sequel the sign of the crossings, that is, the direction of the arrows. Then realizable Gauß diagrams correspond bijectively to alternating knot diagrams up to mirroring.

In [22] we remarked that the Gauß diagram of a generating diagram has no triple of chords, not intersecting each other, and intersecting the same subset of the remaining chords.

Definition 2.18. The diagram



is called *connected sum* $A\#B$ of the diagrams A and B. If a diagram D can be represented as the connected sum of diagrams A and B, such that both A and B have at least one crossing, then D is called *disconnected* (or composite), else it is called *connected* (or prime).

By the work of Menasco, diagrammatic primeness and topological primeness coincide for alternating knots.

Theorem 2.19. ([13]) *A prime alternating diagram depicts a prime alternating knot.*

We should remark that the Gauß diagrams of prime diagrams are those for which any two chords a and b can be connected by a sequence of chords $c_1, \dots, c_n$ with $a = c_1$, $b = c_n$, such that c_i and c_{i+1} intersect. We call such Gauß diagrams *prime*.

For the following discussion, it will be most convenient to consider prime alternating knots up to mirroring, but with orientation.

The reason for considering prime diagrams is that, once we prove theorem 2.14.(1), the contribution to $a_{n,g}$ from composite diagrams is negligible. Genus and number of $\sim$-equivalence classes is additive under connected sum, and by theorem 2.13 the number of $\sim$-equivalence classes of composite genus g diagrams can be at most $6g - 6$.

We now turn again to flypes and introduce a distinction according to the orientation near the crossing p at which the flype is performed. See figure 2.

An important observation is that each crossing admits at most one of the types A and B of flypes, and this remains so after applying any sequence on flypes on the diagram.

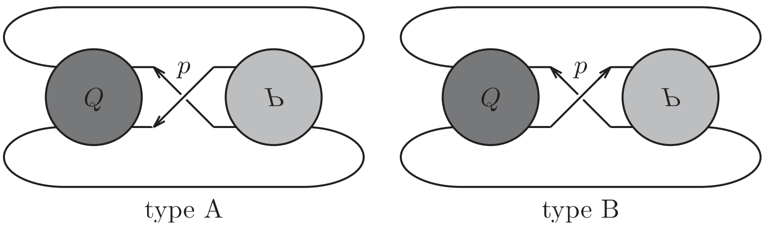
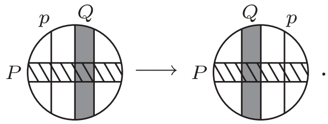


Fig. 2. A flype of type A and B

We remark that on the Gauß diagram, a flype of type B looks like



3. Enumerating alternating knots of given genus

The Flyping theorem 2.7 is the central ingredient of the method to take account of duplications of the same alternating knot occurring among diagrams of given genus. This has been carried out in [21], but we repeat most of it and describe some of the parts in more detail, since it is decisive and provides a guideline for what follows. Our objective will be to identify the leading coefficient of the polynomials P_n.

A first observation is that if two Gauß diagrams G and H can be transformed by flypes and cyclic rotations, so can their generating diagrams. Thus, to remove duplications of the same alternating knot in different generating series is the same as to remove duplications of the knots represented by the generating diagrams, thereby reducing the list of such diagrams. Once this is done, duplications of the same knot can occur only within each generating series separately.

A symmetry of the Gauß diagram can be described as follows. Assign to the chords of G numbers $1, \dots, n$. Then a symmetry is a permutation $\sigma \in S_n$ such that, when replacing the labels of G from i to $\sigma(i)$ and calling G' this new labeled diagram, G' can be transformed into G by a sequence of flypes and a cyclic rotation of the circle.

It is clear that any symmetry descends to a permutation of $\sim$-equivalence classes, and each such permutation of $\sim$-equivalence classes comes from exactly one symmetry up to flypes (flypes permute the crossings of a $\sim$-equivalence class arbitrarily). Thus the symmetry group S_G of a Gauß diagram G can be considered as a subgroup of the permutation group of G's $\sim$-equivalence classes.

If G is a generating diagram, each one of G's symmetries can be also considered to permute the $\sim$-equivalence classes of a diagram in the generating series of G (because they correspond tautologically to $\sim$-equivalence classes of G).

Flypes and cyclic rotations carry over under passing to the generating diagram (removing pairs of chords from each $\sim$-equivalence class, until one or two are left). Thus two diagrams of a(n alternating) knot in the same generating series are transformable into each other by the action of a symmetry of the generating diagram.

Conversely, a symmetry of the generating diagram G transforms a diagram D in G's generating series into a diagram of the same knot, unless in the symmetry of G type A flypes have been performed at crossings to which $\bar{t}_2$ moves have been applied in passing from G to D.

This can be accounted for by considering instead of G the diagram $\tilde{G}$, obtained from G by applying a $\bar{t}_2$ move at each crossing which is the single crossing in its $\sim$-equivalence class in G. (It will follow from (5) that the contribution of diagrams with a $\sim$-equivalence class of one single crossing is negligible.)

However, if we know (and we will prove that in theorem 4.7; see also remark 4.9) that the G we need to consider do not admit non-trivial flypes and have no parallel clasps (that is, don't admit even trivial flypes of type A with one of the tangles having a single crossing), then this subtlety does not come about, and we can still work with G (rather than $\tilde{G}$). We really need to ensure the lack of parallel clasps, because in the definition of symmetry we worked with *marked* chords (or crossings), and a flype at a crossing in a parallel clasp, although not altering the diagram, does alter the markings, and thus permutes $\sim$-equivalence classes.

Thus, to enumerate the alternating knots of n crossings in the generating series of a diagram G by $a_{G,n}$ (which are the same as equivalence classes of Gauß diagrams in this series modulo symmetries), we apply Burnside's lemma [11, lemma 14.3 on p. 1058]. Let $a_{G,s,n}$ be the number of n crossing diagrams in the (fixed) generating series of G fixed by some symmetry $s \in S_G$. Then

$$a_{G,n} = \frac{1}{|S_G|} \sum_{s \in S_G} a_{G,s,n} \,. \tag{4}$$

Let G have d_G $\sim$-equivalence classes. Then $a_{G,s,n}$ counts compositions of n of d_G numbers, some of which are equal, unless $s = Id$. This is polynomial in n of degree

$$d_G - 1 - \#\text{identifications} = \#\text{cycles of } s - 1 \,.$$

Thus the maximal contribution is exactly this for $s = Id$, which is

$$\binom{n + d_G - 1}{d_G - 1} = \frac{1}{(d_G - 1)!} n^{d_G - 1} + O(n^{d_G - 2})$$

and

$$a_{G,n} = \frac{1}{|S_G|\,(d_G - 1)!} n^{d_G - 1} + O(n^{d_G - 2}) \,. \tag{5}$$

Consequently, we have shown (modulo the proof of theorem 4.7)

$$C_{g,o} = \frac{1}{(d_{g,o} - 1)!} \sum_{\substack{G \text{ generating,} \\ c(G) \text{ odd,} \\ d_G = d_{g,o}}} \frac{1}{|S_G|} \,. \tag{6}$$

We will show in the next section that the G occurring in the sum are exactly those coming (in a way defined there) from planar 3-valent 3-connected graphs with a Bieulerian path.

As a cyclic rotation of the Gauß diagram, under the identification of the Gauß diagram with a 3-valent graphs with Bieulerian path, corresponds to a cyclic permutation of the sequence of edge passes described by the path, we will thus consider graphs with Bieulerian path up to such cyclic permutations of the path.

Another important knot diagrammatic step will be to get disposed of the flypes, showing that for the planar 3-valent graphs with Bieulerian path the knot diagrams do not admit flypes, and thus the reduction of generating diagrams is necessary only by cyclic rotations.

The case of considering only cyclic rotations as symmetries has been studied, as for such graphs G it is known that $|S_G| \in \{1, 2, 3, 6\}$ [4]. Thus we obtain from (6) and its analogue for $c(G)$ even the relation (3). Therefore, the basic problem is to estimate the number of such graphs B_g. This is done in §5.

4. Identifying maximal generating diagrams

Let G be a connected 3-valent graph. Fix some arbitrary orientation (direction) of the edges in G. A *Bieulerian path* in G is a closed path that traverses each edge of G exactly twice, only once in each direction, and does not traverse any edge followed immediately by its inverse (itself in the opposite direction).

To a Bieulerian path one can associate a word in some alphabet (called *Wicks form* and considered in more detail later), obtained by labeling each edge by a letter, and putting this letter (resp. its inverse) when the edge is traversed in (resp. oppositely to) its orientation.

In [22] we described the bijection between a graph with Bieulerian path G and a Gauß diagram G'.

To obtain G' from G one just writes the letters of its word (Wicks form) w along a circle and links by a chord each letter and its inverse. To obtain G from G', we consider the circle of G' as a $2n$-gon (each side corresponding to a basepoint of a chord) and identify sides corresponding to the basepoints of the same chord, obtaining G lying on a surface S. (The circle G' bounds a disk that yields S under the identifications.) To indicate the origin of G and S, we write $G = G(w)$ and $S = S(w)$. The dual of G forms a 1-vertex triangulation of S.

We call a graph with a Bieulerian path *realizable* if and only if its associated Gauß diagram is realizable (as a knot diagram). In this case each Seifert circle of the knot diagram corresponds to a vertex of the graph, and each crossing of the knot diagram attached to a pair of Seifert circles corresponds to an edge joining the vertices of these Seifert circles. In this sense we call the number of crossings attached a Seifert circle its *valence* (the valence of its corresponding vertex in the graph).

Then in [22] we defined the genus of Gauß diagrams and of graphs in different ways and showed that they coincide. Also the genus of a knot diagram (which is equal for alternating diagrams to the genus of the knot; see Theorem 2.5 was showed to be equal to the genus of its Gauß diagram.

We do not repeat these definitions, but recall that in the case of a trivalent graph,

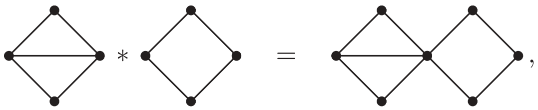
$$\text{genus of graph} \;=\; \frac{3+\#\text{ edges}}{6} \;=\; \frac{2+\#\text{ vertices}}{4}\,.$$

If the graph is additionally planar, then

$$\text{genus of graph} \;=\; \frac{\#\text{ faces}-1}{2},$$

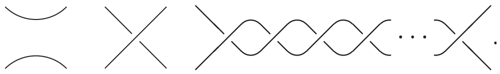
where the infinite face is counted.

It is easy to see that composite knot diagrams give composite Gauß diagrams, which in turn correspond to graphs with a cut vertex. Since genus is additive under the join of graphs



as mentioned, a composite genus g knot diagram can have at most $6g-6$ $\sim$-equivalence classes. Thus the contribution of such diagrams is negligible, once we have shown that there are diagrams with more $\sim$-equivalence classes (see the proof of theorem 2.14).

Definition 4.1. A primitive Conway tangle [8] is a tangle of the form



We call two crossings a and b in a diagram D *neighbored*, if they belong to a reversely oriented primitive Conway tangle in D, that is, there are crossings $c_1,\dots,c_n$ with $a=c_1$ and $b=c_n$, such that c_i and c_{i+1} form a reverse clasp in D. (Equivalently, a and b correspond in the graph to edges which can be connected by a path passing only through vertices of valence 2.)

This is a similar definition to $\sim$-equivalence, but with no flypes allowed. Thus the number of $\sim$-equivalence classes of a diagram is not more than the number of neighbored equivalence classes of the same diagram, or of any flyped version of it.

The following was proved in [22] in a slightly implicit way, so we will recapture the proof in more detail.

Lemma 4.2. *A knot diagram of genus g has at most* $6g-3$ *neighbored equivalence classes (and hence at most* $6g-3$ $\sim$*-equivalence classes).*

Moreover, knot diagrams of genus g having exactly $6g-3$ *neighbored equivalence classes come exactly from graphs with Bieulerian path, all whose vertices have valence 2 or 3.*

Proof. Note that in the Gauß diagram neighbored equivalent crossings correspond to chords we called in [22] parallel. Then note that in fact we needed in the proof of theorem 3.6 in [22] (second paragraph) only the lack of parallel pairs of chords to ensure that $\hat{G}$ has no vertices of valence 1 and 2. Then we showed that $\hat{G}$ has at most $6g-3$ edges, and exactly $6g-3$ edges if and only if all its vertices have valence 2 or 3. □

The lemma means in particular, that if G' is realizable and its knot diagram D has $6g-3$ $\sim$-equivalence (or just neighbored equivalence) classes, then all vertices of G' have valence 2 or 3, and thus the Seifert circles of D have 2 or 3 adjacent crossings. Hence the knot diagram is special.

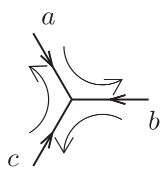
In general the condition of being realizable is difficult to test for G', but in the trivalent case it is surprisingly simple.

Theorem 4.3. *A trivalent graph with Bieulerian path is realizable if and only if it is planar(ly embeddable). In this case the knot diagram is special.*

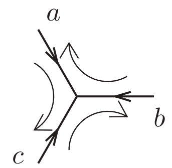
We should remark that a planar graph is in fact a graph equipped with a concrete planar embedding, while the realizability of the graph does not depend on the planar embedding. However, we will shortly show that for the cases we need to consider the planar embedding is unique (see remark 4.8).

For the proof, and later, we will need the following additional structure on a trivalent graph with Bieulerian path.

Definition 4.4. A Bieulerian path in a trivalent graph induces an orientation on each 3-valent vertex v given by a cyclic order of the 3 adjacent edges. To define it, orient the 3 adjacent edges a, b and c towards v. Then if the word of the Bieulerian path contains the subwords ab^{-1}, bc^{-1} and ca^{-1} (in whatever order), then the orientation at v is given by (a,b,c).



If the Bieulerian path contains the subwords ac^{-1}, cb^{-1} and ba^{-1} (in whatever order), then the orientation at v is (c,b,a).



The proof of theorem 4.3 we give now establishes a natural correspondence between a plane 3-valent graph with Bieulerian path and a special knot diagram.

Proof of theorem 4.3. Let G be a 3-valent graph with Bieulerian path. The path induces the orientation of vertices. If two ends of the edge have the same orientation, put on the edge an additional vertex of degree two. We have a graph G' with vertices of degree two and three. Every edge x of G, which was divided in two parts, will be replaced in the Bieulerian path by x_1x_2. We can change the orientations of the edges of G' such that in the Bieulerian path the orientations of edges alternate. Now we have an oriented graph such that for every vertex all edges incident to it either all are incoming or all are outgoing.

If all edges incident to a vertex all are outgoing (incoming) we say, that the vertex is of the first (second) type.

In the middle of any edge of G' we put a small cross, it will be a future crossing of the knot diagram. Now we draw a circle with the center in each vertex, such that the circles with centers in the ends of the same edge are tangent at the small cross. We equip each circle with the orientation induced by the orientation of the vertex. These circles will be the Seifert circles for our knot diagram.

Now we form the knot diagram from the Seifert circles by an algorithm, which is inverse to the Seifert algorithm. Overcrossings and undercrossings are defined as follows: if the knot strand goes from a vertex of the first type to a vertex of the second type, we have an overcrossing; if the strand goes from a vertex of the second type to a vertex of the first type, we have an undercrossing.

Note, that even after inserting vertices of valence 2, the graph has no edge connecting different vertices of valence 2, and thus the resulting knot diagram has not more than two neighbored crossings in each neighbored equivalence class. □

Not all vertex orientations come from Bieulerian paths. However, those that do, come in a unique way.

Lemma 4.5. *Any vertex orientation coming from a Bieulerian path determines the Bieulerian path uniquely (& vice versa).*

Proof. This is rather obvious, since the the Bieulerian path is determined uniquely by its local pieces around each vertex. □

Proof of theorem 2.14. We give an example of a planar 3-valent graph of genus g and $6g-3$ $\sim$-equivalence classes and even and odd crossing number of the corresponding diagram. In the case of odd crossing number we have $10g-7$ crossings. (This graph has $6g-3$ $\sim$-equivalence classes since it is 3-connected and 3-valent.)

We give the examples in the form of Bieulerian path, parametrized by $t = 2(g-2)$.

The case of odd crossing number (Figure 3) is

$$\begin{gathered} ab^{-1}cd^{-1}x_1x_2^{-1}y_1z_3^{-1}z_4y_2^{-1}x_5 \dots x_{2t-2}^{-1}y_{t-1} \\ z_{2t-1}^{-1}z_{2t}y_t^{-1}ef^{-1}gh^{-1}ia^{-1}fe^{-1}x_{2t}x_{2t-1}^{-1} \\ x_{2t-2}\dots x_2x_1^{-1}kl^{-1}bi^{-1}hn^{-1}mz_{2t}^{-1}z_{2t-1}\dots \\ z_2^{-1}z_1c^{-1}lk^{-1}dz_1^{-1}z_2y_1^{-1}x_3x_4^{-1}y_2z_5^{-1}\dots \\ z_{2t-2}y_{t-1}^{-1}x_{2t-1}x_{2t}^{-1}y_tm^{-1}ng^{-1}\,. \end{gathered}$$

(Note that this is one single word, split because of its length, and the dots, also at the end of a line, indicate only a finite number of letters to be inserted suggestively according to the figure.)

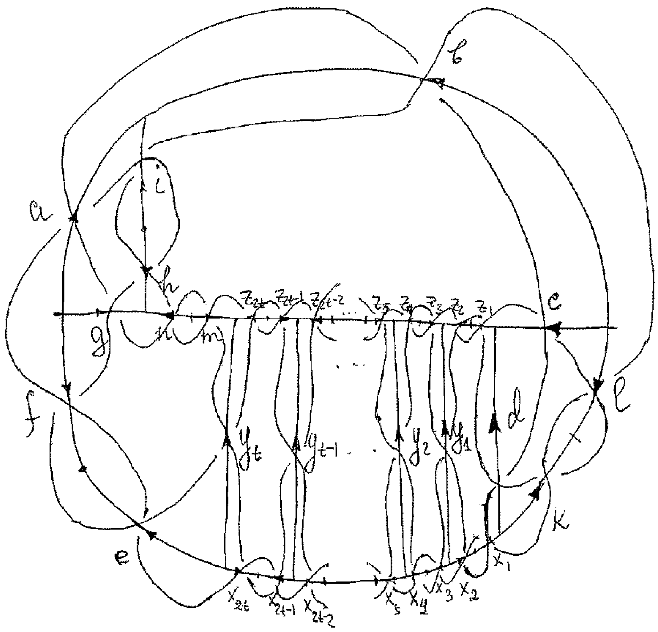


Fig. 3.

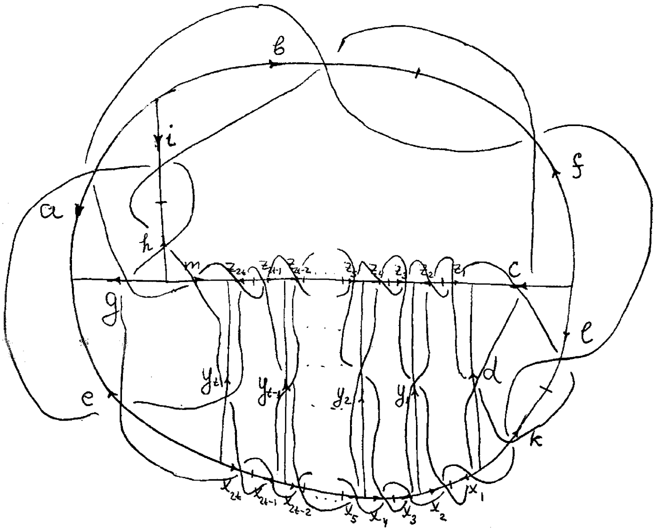


Fig. 4.

The case of even crossing number (Figure 4) is

$$
\begin{gathered}
a^{-1}bf^{-1}lk^{-1}x_1x_2^{-1}x_3x_4^{-1}x_5\dots \\
x_{2t-2}^{-1}x_{2t-1}x_{2t}^{-1}eg^{-1}hi^{-1}ae^{-1} \\
y_tz_{2t}^{-1}z_{2t-1}y_{t-1}^{-1}x_{2t-2}\dots x_5^{-1} \\
y_2z_4^{-1}z_3y_1^{-1}x_2x_1^{-1}dc^{-1}fb^{-1}ih^{-1} \\
my_t^{-1}x_{2t}x_{2t-1}^{-1}y_{t-1}z_{2t-2}^{-1}\dots \\
z_5y_2^{-1}x_4x_3^{-1}y_1z_2^{-1}z_1d^{-1}kl^{-1}c \\
z_1^{-1}z_2z_3^{-1}z_4z_5^{-1}\dots z_{2t-2}z_{2t-1}^{-1}z_{2t}m^{-1}g\,.
\end{gathered}
$$

□

Remark 4.6. The lower bound $10g-7$ is almost optimal at least for special alternating generators. If we take a planar cubic graph of genus g, and put two crossings on each of its $6g-3$ edges (that is, add a vertex of valence two on each edge), then we obtain by the above described construction an alternating link diagram with as many components as regions we have, namely $2g+1$. Since the change of 2 crossings to 1 along each edge changes the number of components by ± 1, we need at least $2g$ such replacements to obtain a knot diagram. Thus the maximal number of crossings we can have is

$$2\,(6g-3)\;-\;2g\;=\;10g-6\,.$$

If we have a $\bar{t}_2$-irreducible special diagram with a Seifert circle of valence ≥ 4, then one can see that one can always perform a Reidemeister II move on an appropriate pair of non-neighbored edges in this Seifert circle so as to obtain a special diagram of the same genus and two crossings more, which is still $\bar{t}_2$-irreducible (although not always any pair of such edges will do).

Not for all planar 3-valent graphs of genus g with Bieulerian path the corresponding diagram has $6g-3$ $\sim$-equivalence classes. Such graphs are described in the following theorem.

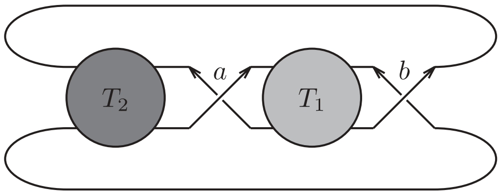
Theorem 4.7. *Let G be a planar 3-valent graph (with Bieulerian path) and D its knot diagram (as constructed in the proof of theorem 4.3). Then the following conditions are equivalent:*

(1) *G is 3-connected (i.e., removing any pair of edges does not disconnect it),*
(2) *D has $6g-3$ $\sim$-equivalence classes,*
(3) *D admits no (non-trivial) flypes.*

Remark 4.8. By a theorem of Whitney each 3-valent 3-connected graph has, if any, a unique planar embedding up to moves in S^2 (see [2]). Thus for the cases that are of interest to us we do not need to care about ambiguities of the planar embedding, and can consider the graph also abstractly.

Proof. We prove by a ring conclusion the equivalence of the negations of the 3 conditions stated.

$\neg 2 \Rightarrow \neg 1$. If 2. does not hold, then D has $< 6g-3$ $\sim$-equivalence classes, but $6g-3$ neighbored equivalence classes. Thus there are 2 crossings a and b in D which can be made neighbored only after flypes, (As crossings which can be made neighbored after trivial flypes are already neighbored, these flypes must be non-trivial.) Since neighbored crossings cannot admit a type A flype (and the type of flype a crossing admits is not changed under flypes), these flypes must be of type B. Thus we have the following picture:



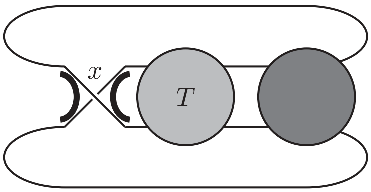
(7)

As each of the T_i contains at least 2 crossings, they both must contain Seifert circles of valence 3 (otherwise one of them would be a reversely oriented primitive Conway tangle, and a and b would be neighbored), and passing to the graph, this graph is disconnected by the removing of the edges corresponding to a and b.

$\neg 1 \Rightarrow \neg 3$. Assume that G is not 3-connected (it is 2-connected, as an edge disconnecting it would give a nugatory crossing in D). Then D looks like in (7) with both T_i containing Seifert circles of valence 3 (coming from a 3-valent vertex in each of the two disconnected components remaining from G after deleting the edges corresponding to a and b). If some T_i contained only one crossing, then there would be only one Seifert circle S in T_i of valence 3, to which a and b are attached. The third crossing attached to S would be reducible. Therefore, both T_i have at least 2 crossings, and D admits a non-trivial flype.

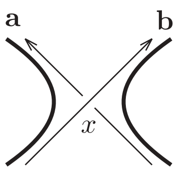
$\neg 3 \Rightarrow \neg 2$. We can assume that each Seifert circle in D bounds only 2 or 3 crossings. Otherwise by lemma 4.2, D will have $< 6g-3$ neighbored equivalence classes, and so $< 6g-3$ $\sim$-equivalence classes, and we would be done.

First, assume that D admits a type B flype at a crossing x. Then the picture is like this:

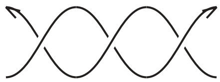


(the thickened lines should depict parts of the Seifert circles).

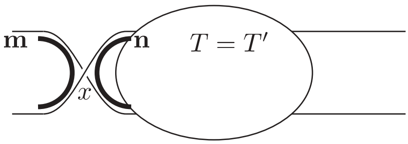
The crossing x joins 2 Seifert circles **a** and **b**:



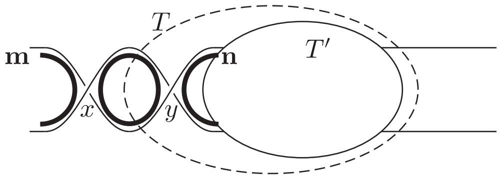
One of **a** and **b** must have 3 crossings at it, otherwise x will be the middle crossing in

 , (8)

contradicting the fact that D by construction has no neighbored equivalence class of more than 2 crossings. If both **a** and **b** have 3 crossings, we have

 ,

with $\mathbf{m} = \mathbf{a}$ and $\mathbf{n} = \mathbf{b}$. Else let without loss of generality $\mathbf{b}$ be the Seifert circle with 2 crossings. Then we have



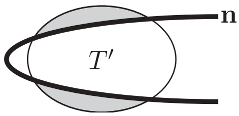
(9)

for $\mathbf{m} = \mathbf{a}$ and $\mathbf{n}$ being a new Seifert circle (so far possibly equal to $\mathbf{a}$).

In both cases we have that $\mathbf{m}$ and $\mathbf{n}$ bound 3 crossings each (in latter case because D by construction has no neighbored equivalence class of more than 2 crossings).

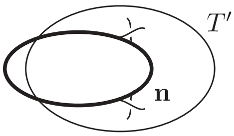
The Seifert circle $\mathbf{n}$ does not leave T' through its other 2 ends, i.e., does not look like

Otherwise, because of the specialty of D, all crossings attached to $\mathbf{n}$ in T' (if any) would be on the outside of $\mathbf{n}$, i.e., in the shaded regions



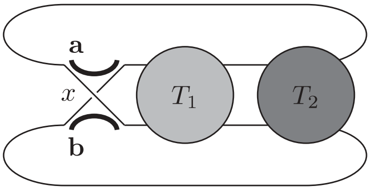
Then T' would have no crossing (and hence T at most one) or would contain reducible crossings, or D would be composite, in all cases giving a contradiction. (Therefore, in particular $\mathbf{n} \neq \mathbf{a}$.)

Thus $\mathbf{n}$ remains within T', and T' contains the two other crossings attached to $\mathbf{n}$.



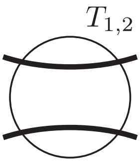
Then a flype at x (and y if in case (9)), would join $\mathbf{m}$ and $\mathbf{n}$ into a Seifert circle of 4 crossings. This flyped version of D has by lemma 4.2 less than $6g-3$ neighbored equivalence classes, and so D will have $< 6g-3$ $\sim$-equivalence classes.

If now D admits a type A flype



,

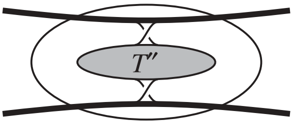
in both $T_{1,2}$ the Seifert circles entering and leaving the tangle must look like



,

because they must respect the orientation at the in- and outputs and must not cross.

As both Seifert circles must have ≥ 1 crossing attached to them in each of the T_i (else some of the tangles is empty or the diagram is composite), and **a** and **b** have at most 3 crossings attached to them, each of **a** and **b** has exactly one crossing attached to it in each of the T_i. If this crossing is the same for both Seifert circles (i.e., connects them) for both T_i, then the T_i are both 1 crossing tangles, and D is the trefoil diagram (of genus 1, which we don't consider). Otherwise, in one of the T_i the crossings y at **a** and z at **b** are different, and then (because the diagram is special) T_i must look like



.

If the shaded tangle T'' has no crossing, then for orientation reasons the two crossings at **a** and **b** are reducible. If it has one crossing, then we have a fragment like (8). Therefore, T'' has at least two crossings, and T'' (and hence D) admits a non-trivial type B flype (near y or z), which leads this case back to the previous one. □

Remark 4.9. It follows that in fact D admits not even trivial type A flypes with one of the tangles having a single crossing, that is, D has no parallel clasp (unless it is the trefoil diagram of genus 1 which we don't consider). This is because a 3-connected trivalent graph G has no multiple edges (in genus $g > 1$).

Theorems 4.3 and 4.7 prove

Corollary 4.10. *There is a bijective correspondence between genus g diagrams with $6g-3$ $\sim$-equivalence classes and planar 3-connected 3-valent graphs with Bieulerian paths (considered up to moves in S^2 on the graph and cyclic permutations of the path).* □

Proof of theorem 2.16. This is now putting together the previous results. Clearly, we need to consider only genus g generators D of the maximal number of $\sim$-equivalence classes. By lemma 4.2, this maximal number is $6g-3$, and generators with that many $\sim$-equivalence classes have graphs with vertices of valence 2 and 3. By theorem 4.3 the diagrams of such graphs are special, and we know from theorem 2.14 that for any crossing number parity, at least one such example exists. Finally, from Part 3 of theorem 4.7 we know that diagrams in the series of D have only symmetries coming from the Bieulerian path, and the order of such a symmetry is at most 6. □

5. Asymptotical estimates

To study the behaviour of $C_{g,e/o}$ for $g \to \infty$, we need to recall the notion of Wicks forms. We will concentrate only on properties relevant to our context (and thus do not discuss all of those mentioned in the introduction).

An *oriented Wicks form* is a cyclic word $w = w_1 w_2 \dots w_{2l}$ (a cyclic word is the orbit of a linear word under cyclic permutations) in some alphabet $a_1^{\pm 1}, a_2^{\pm 1}, \dots$ of letters $a_1, a_2, \dots$ and their inverses $a_1^{-1}, a_2^{-1}, \dots$, such that

(i) if a_i^{ϵ} appears in w (for $\epsilon \in \{\pm 1\}$) then $a_i^{-\epsilon}$ appears exactly once in w,
(ii) the word w contains no cyclic factor (subword of cyclically consecutive letters in w) of the form $a_i a_i^{-1}$ or $a_i^{-1} a_i$ (no cancellation),
(iii) if $a_i^{\epsilon} a_j^{\delta}$ is a cyclic factor of w then $a_j^{-\delta} a_i^{-\epsilon}$ is not a cyclic factor of w (substitutions of the form $a_i^{\epsilon} a_j^{\delta} \longmapsto x,\ \ a_j^{-\delta} a_i^{-\epsilon} \longmapsto x^{-1}$ are impossible).

An oriented Wicks form $w = w_1 w_2 \dots$ in an alphabet A is *isomorphic* to $w' = w_1' w_2' \dots$ in an alphabet A' if there exists a bijection $\varphi : A \longrightarrow A'$ with $\varphi(a^{-1}) = \varphi(a)^{-1}$ such that w' and $\varphi(w) = \varphi(w_1)\varphi(w_2)\dots$ define the same cyclic word.

The *genus* $g_t(w)$ of an oriented Wicks form $w = w_1 \dots w_{2l-1} w_{2l}$ is defined as the topological genus of the oriented compact connected surface $S(w)$ obtained as described in §4.

The *automorphism group* $\mathrm{Aut}(w)$ of an oriented Wicks form $w = w_1 w_2 \dots w_{2l}$ of length $2l$ is the group of all cyclic permutations μ of the linear word $w_1 w_2 \dots w_{2l}$ such that w and $\mu(w)$ are isomorphic linear words (i.e. $\mu(w)$ is obtained from w by permuting the letters of the alphabet). The group $\mathrm{Aut}(w)$ is a subgroup of the cyclic group $\mathbb{Z}/2l\mathbb{Z}$ acting by cyclic permutations on linear words representing w.

The automorphism group $\mathrm{Aut}(w)$ of an oriented Wicks form w can of course also be described in terms of permutations on the oriented edge set of $G = G(w)$ induced by orientation-preserving homeomorphisms of $S = S(w)$ leaving G invariant. In particular an oriented maximal Wicks form and the associated dual 1-vertex triangulation have isomorphic automorphism groups. Thus $\mathrm{Aut}(w)$ is the same as S_G in (6).

Let G be a cubic (3-valent) connected graph on $4g-2$ vertices and the word U be one of its Bieulerian paths. Note that a Bieulerian path can be presented as a word, which is called an oriented Wicks form of genus g. We will consider only Wicks forms, which came from Bieulerian paths of 3-connected planar cubic graphs on $4g-2$ vertices and we will call them *planar Wicks forms*. (These forms are also maximal in the sense of [1], but we will drop this term to simplify language, since all Wicks forms we deal with are in fact maximal.) Note that these are graphs without multiple edges in genus $g > 1$.

Let us call a planar Wicks form $\hat{w}$ *based* if one interval (*basepoint*) between a pair of (cyclically) consecutive letters is distinguished. Let the genus of $\hat{w}$ be that of w, where w is obtained from $\hat{w}$ by forgetting the basepoint. Based planar Wicks forms are considered equivalent only up to bijections of their letters.

We define the *mass* $m_g = |W^g|$ to be the cardinality of the (finite) set W^g of based planar Wicks forms of given genus g.

Any planar Wicks form w of genus g can give rise (by adding a basepoint) to at most $6g-3$ different based planar Wicks forms. Thus it suffices to consider the rate of growth of m_g. Our first goal is to prove that asymptotically as $g \to \infty$,

$$\liminf_{g\to\infty} \frac{m_{g+1}}{m_g} \geq 400\,. \tag{10}$$

Let H_3 be a subgraph which consists of three edges a, b, c leaving one common vertex. Let us choose three different points of the graph G, which are not vertices and identify them with the 1-valent vertices of the subgraph H_3.

Definition 5.1. A vertex V (with oriented edges a, b, c pointing toward V) in a planar Wicks form w is *positive* if

$$w = ab^{-1}\dots bc^{-1}\dots ca^{-1}\dots \quad \text{or} \quad w = ac^{-1}\dots cb^{-1}\dots ba^{-1}\dots$$

and V is *negative* if

$$w = ab^{-1}\dots ca^{-1}\dots bc^{-1}\dots \quad \text{or} \quad w = ac^{-1}\dots ba^{-1}\dots ab^{-1}\dots \quad .$$

If $\hat{w}$ is a based planar Wicks form obtained by adding a basepoint to w, then a vertex V is called positive (or negative) in $\hat{w}$ if and only if it is positive (or negative) in w.

Let V be a negative vertex of a planar Wicks form of genus $g > 1$. Since we noted that the graph has no multiple edges, the vertex V has three distinct neighbors. We have then

$$w = x_1 a b^{-1} y_2 u_1 z_1 c a^{-1} x_2 u_2 y_1 b c^{-1} z_2 u_3$$

(some identifications among $x_i,\ y_j$ and z_k may occur, see [25] for all the details) and the word w is obtained by a so-called *γ−construction* from the word $w' = x\tilde{u}_2 y\tilde{u}_1 z\tilde{u}_3$. The subwords u_i of w are obtained from the subwords $\tilde{u}_i$ of w' by replacing $x,\ y,\ z$ by x_1x_2, y_1y_2, z_1z_2 respectively.

By γ−construction we obtain a Bieulerian path w on a plane cubic graph H with $4g+2$ vertices from a Bieulerian path w' on a plane cubic graph G with $4g-2$ vertices. Here H is obtained from G by adding the vertex V with three edges $a,\ b,\ c$ incident to it. We attach the edges $a,\ b,\ c$ to (not necessarily distinct) edges $x,\ y,\ z$ of G. Since the label of each edge $x,\ y,\ z$ appears exactly twice in w (i.e., we can interchange it with its inverse), we have eight possibilities to do the γ−construction by adding the negative vertex V in a given face of G with three specified edges $x,\ y,\ z$ in its boundary. Alternatively speaking, these eight possibilities are given by the orientations of the three vertices adjacent to V in H.

For based Wicks forms the same construction applies, and the basepoint is inherited naturally.

Definition 5.2. We call the application which associates to a planar Wicks form w of genus $g > 2$ with a chosen negative vertex V the planar Wicks form w' of genus $g-1$ defined as above the *reduction* of w with respect to the negative vertex V. This notion extends naturally to based planar Wicks forms.

Lemma 5.3. *An oriented Wicks form (or based planar Wicks form) of genus g has exactly* $2g$ *negative and* $2g-2$ *positive vertices.*

Proof. See [1, proposition 2.1]. □

Proof of theorem 2.17. Let $g > 1$. We compare W^{g+1} and W^g by estimating the number of possible γ−constructions that lead from the one set to the other and backward.

Forgetting for a moment the basepoint, each element of W^{g+1} can be obtained by applying γ−construction to an element in W^g. The Lemma 5.3 shows that we can construct $2(g+1)$ planar Wicks forms in W^g by applying reduction with respect to a negative vertex to a given element in W^{g+1}. When taking basepoints back into account, though, not any such reduction is admissible, because the basepoint must not separate the letters removed or identified. So, the number of such reductions is at most $(2g+2)m_{g+1}$.

Let us estimate the number of "augmentations". Since the resulting and the initial graphs are both planar, we can apply the γ−construction only inside of each face of the embedding of the planar graph G into the plane. Also, to be sure, that the resulting graph will be 3-connected, we will apply γ-construction only to two or three different edges. (That is, not all of x, y and z in the above description are equal.) In this case for an element of genus g a γ-construction gives eight elements of genus $g+1$: we can choose the orientations of the three adjacent vertices to the new negative one. There are $2g+1$ faces, denote the number of edges of one face by n_i. Then, starting with a given element in W^g, there are not less than $8\sum_{i=1}^{2g+1}(\binom{n_i}{3}+2\binom{n_i}{2})$ possibilities for $\gamma-$construction, with $\sum_{i=1}^{2g+1} n_i = 12g-6$. In the sum of the n_i every edge of the graph is counted twice. (The method to estimate the number of possibilities for $\gamma-$construction is similar to one in the proof of Theorem 1.1 in [1].)

Consider a function $f(n_i) = \binom{n_i}{3} + 2\binom{n_i}{2}$, where $\sum_{i=1}^{2g+1} n_i = 12g-6$. We need to know when the function $F(g) = 8\sum_{i=1}^{2g+1}(\binom{n_i}{3} + 2\binom{n_i}{2})$ is minimal. The following inequalities are true for the function $f(n_i)$:

1) $f(n_i+2) + f(n_i) > 2f(n_i+1)$
2) if $n_i < n_j - 1$, then $f(n_i) + f(n_j) > f(n_j - 1) + f(n_i + 1)$.

So, our function $F(g)$ is minimal for $g \geq 6$ if and only if $2g-11$ of n_i are equal to 6 and twelve of n_i are equal to 5.

Since $f(5) = 30$ and $f(6) = 50$, we have that the lower bound for the number of "augmentations" is $8(100g-190)m_g$.

Finally, we obtain for $g \geq 6$

$$m_{g+1} \geq \frac{8(100g-190)m_g}{2g+2} .$$

This shows (10), and hence the stated lower bound.

To obtain the upper bound, we use the work of Tutte [24]. (In the following an equation reference of the type $(x.y)$ always refers to that paper.)

Let T_g be the number of plane (that is, concretely planarly embedded) 3-valent 3-connected graphs G of genus g, that is, with $2g+1$ regions. By Whitney's theorem the planar embedding of each G is unique up to change of the infinite region, and the number of such choices is linearly bounded in g, and thus we can fix a favorable choice of the infinite region without loss of generality. By an easy argument, any planar 3-valent graph has a region R whose boundary ∂R has at most 5 vertices, so fix the infinite region to be a k-gon with $k \leq 5$. Then G turns into what is called in [24] a triangulation of ∂R. (The 3-connectedness implies the condition stated below (1.2).)

It follows then from (5.11)-(5.13) that the number of triangulations for $k = 3, 4, 5$ differ only by a polynomial in n. Here n is the number given in (1.4), and it is, up to a constant, equal to $1/3$ of the number of edges of G, and thus (up to a constant) to $2g$. Then (8.1) gives

$$\lim_{g\to\infty} \sqrt[g]{T_g} = \left(\frac{256}{27}\right)^2 = \frac{2^{16}}{3^6} .$$

The remaining factor 2^4 comes from the possible cyclic orientations of the $4g-2$ vertices of G and Lemma 4.5. □

Remark 5.4. Obviously, the interval between lower and upper bound remains wide open. On the lower side, many more sequences of the transformations of [25] than we could handle lead to 3-connected planar graphs. On the upper side, it is clear that many choices of cyclic orientations of vertices will not give connected paths. However, we do not know how to (substantially) benefit from these circumstances to narrow the gap.

The next lemma is needed to establish the correspondence between the even and odd crossing number case.

Lemma 5.5. *The orientation of a positive vertex can be reversed without altering the orientation of the other vertices.*

Proof. Let v be a positive vertex, such that edges a,b,c are leaving v. By the definition of the positive vertex

$$w = ab^{-1}U_1bc^{-1}U_2ca^{-1}U_3 \quad \text{or} \quad w = ac^{-1}W_1cb^{-1}W_2ba^{-1}W_3\,.$$

Reversing of the orientation will give

$$w = ac^{-1}U_2cb^{-1}U_1ba^{-1}U_3 \quad \text{or} \quad w = ab^{-1}W_2bc^{-1}W_1ca^{-1}W_3\,,$$

which does not change the orientation of other vertices. □

Definition 5.6. Call a 3-valent planar graph G with Bieulerian path *even* resp. *odd*, if its associated knot diagram has even resp. odd number of crossings. This is equivalent to saying that the number of edges of G connecting oppositely oriented vertices is even resp. odd.

Corollary 5.7. *There is a bijection between even and odd 3-valent planar graphs of genus $g>1$ with Bieulerian paths and distinguished positive vertex.* □

Remark 5.8. Of course, for genus 1 there are no positive vertices and the corollary does not hold. This explains the degeneracy of this case. (We have from [21] that $d_{1,e} = 2 \neq 3 = d_{1,o}$, and there are no special even crossing number genus 1 alternating diagrams.)

Corollary 5.9.

$$\frac{1}{6(2g-2)} \le \frac{C_{g,o}}{C_{g,e}} \le 6(2g-2)\,.$$

Proof. Use the previous corollary, (6) and $|S_G| \le 6$. □

From this corollary and (3) we obtain

$$\liminf_{g\to\infty} \sqrt[g]{(6g-4)!C_{g,o}} = \liminf_{g\to\infty} \sqrt[g]{B_g},$$

and the analogous statement for 'lim sup' and/or '$C_{g,e}$'. Thus we have

Corollary 5.10.

$$\begin{aligned} 400 \le \liminf_{g\to\infty} \sqrt[g]{(6g-4)!C_{g,o}} &\le \limsup_{g\to\infty} \sqrt[g]{(6g-4)!C_{g,o}} \\ &\le \frac{2^{20}}{3^6} \approx 1438.37585\dots, \end{aligned}$$

with the same inequality for $C_{g,e}$. □

Accumulating the previous results, we obtain our main theorem.

Proof of theorem 1.1. Theorem 2.14, with the explanation of §3, and Theorem 4.7 establish (1) with $C_{g,o}$ as in (6) (and $C_{g,e}$ similarly given), where $a_n \asymp_n b_n + O(c_n)$ means $a_n \asymp_n b_n$ and $a_n - b_n = O(c_n)$. The estimates on $C_{g,e/o}$ were given in Corollary 5.10. □

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References

1. Bacher, R., Vdovina, A.: Counting 1-vertex triangulations of oriented surfaces. Discrete Mathematics **246**(1–3), 13–27 (2002)
2. Bar-Natan, D.: Lie algebras and the four color theorem. Combinatorica **17**(1), 43–52 (1997)
3. Bar-Natan, D.: On the Vassiliev knot invariants. Topology **34**(2), 423–472 (1995)
4. Bianchi, G., Cori, R.: Colorings of hypermaps and a conjecture of Brenner and Lyndon. Pacific J. Math. **110**(1), 41–48 (1984)
5. Bott, R., Taubes, C.: On the self-linking of knots. Jour. Math. Phys. **35**(10), 5247–5287 (1994)
6. Brenner, J.L., Lyndon, R.C.: Permutations and cubic graphs. Pacific Journal of Math. **104**, 285–315 (1983)
7. Comerford, L., Edmunds, C.: Products of commutators and products of squares in a free group. Int. J. of Algebra and Comput. **4**(3) (1994), 469–480.
8. Conway, J.H.: On enumeration of knots and links. In: Computational Problems in abstract algebra J. Leech (ed.), Pergamon Press, 1969, pp. 329–358
9. Crowell, R.: Genus of alternating link types. Ann. of Math. **69**(2), 258–275 (1959)
10. Culler, M.: Using surfaces to solve equations in free groups. Topology **20**(2), 133–145 (1981)
11. Handbook of Combinatorics Vol. **II** P.L. Graham, M. Grötschel and L. Lovácz (eds.), North-Holland, 1995

12. Kauffman, L.H.: State models and the Jones polynomial. Topology **26**, 395–407 (1987)
13. Menasco, W.W.: Closed incompressible surfaces in alternating knot and link complements. Topology **23**(1), 37–44 (1986)
14. Menasco, W.W., Thistlethwaite, M.B.: The classification of alternating links. Ann. of Math. **138**(1), 113–171 (1993)
15. Mosher, L.: A User's Guide to the Mapping Class Group: Once Punctured Surfaces. DIMACS Series **25**, (1994)
16. Murasugi, K.: On the genus of the alternating knot. I, II, J. Math. Soc. Japan **10**, 94–105, 235–248 (1958)
17. Murasugi, K.: Jones polynomial and classical conjectures in knot theory. Topology **26**, 187–194 (1987)
18. Nakamura, T.: Positive alternating links are positively alternating. J. Knot Theory Ramifications **9**(1), 107–112 (2000)
19. Polyak, M., Viro, O.: Gauss diagram formulas for Vassiliev invariants. Int. Math. Res. Notes **11**, 445–454 (1994)
20. Rolfsen, D.: Knots and links. Publish or Perish Inc., Mathematical Lecture Series **7**, Berkeley CA 1976
21. Stoimenow, A.: Knots of genus one. Proc. Amer. Math. Soc. **129**(7), 2141–2156 (2001)
22. Stoimenow, A., Tchernov, V., Vdovina, A.: The canonical genus of a classical and virtual knot. Geom. Dedicata **95**, 215–225 (2002)
23. Thistlethwaite, M.B.: A spanning tree expansion for the Jones polynomial. Topology **26**, 297–309 (1987)
24. Tutte, W.T.: A census of planar triangulations. Canad. J. Math. **14**, 21–38 (1962)
25. Vdovina, A.A.: Constructing Orientable Wicks Forms and Estimation of Their Number. Communications in Algebra **23**(9), 3205–3222 (1995)