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Banach spaces of continuous functions with few operators

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Abstract. We present two constructions of infinite, separable, compact Hausdorff spaces K for which the Banach space $C(K)$ of all continuous real-valued functions with the supremum norm has remarkable properties. In the first construction K is zero-dimensional and $C(K)$ is non-isomorphic to any of its proper subspaces nor any of its proper quotients. In particular, it is an example of a $C(K)$ space where the hyperplanes, one co-dimensional subspaces of $C(K)$, are not isomorphic to $C(K)$. In the second construction K is connected and $C(K)$ is indecomposable which implies that it is not isomorphic to any $C(K')$ for K' zero-dimensional. All these properties follow from the fact that there are few operators on our $C(K)$'s. If we assume the continuum hypothesis the spaces have few operators in the sense that every linear bounded operator $T : C(K) \to C(K)$ is of the form $gI + S$ where $g \in C(K)$ and S is weakly compact or equivalently (in C(K) spaces) strictly singular.

1. Introduction

Several fundamental questions concerning the structure of infinite dimensional Banach spaces (unless stated otherwise, all Banach spaces considered in this paper are meant to be infinite dimensional and over the field of the reals) remained unsolved for many decades. For example, whether every Banach space is decomposable ([Li]), i.e., whether it can be decomposed as $A \oplus B$ into two closed infinite dimensional subspaces, A and B; whether on every Banach space there is an operator other than those of the form $cI + C$ where c is a scalar and C is a compact operator; whether every Banach space has a proper closed subspace which is isomorphic to the entire space ([Ba]), or in particular whether the hyperplanes of a Banach space are isomorphic to the entire space. The first and the third question

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were answered in the negative only recently (see [G] and [GM]) and the second remains open. In this paper, we address similar questions in the context of the classical Banach space $C(K)$ of all continuous functions on a compact Hausdorff space K with the supremum norm. Our results are also negative.

The first construction concerns the question whether there is a Banach space X non-isomorphic to any of its proper subspaces whose particular case for one co-dimensional subspaces, became to be known as the hyperplane problem ([Ma]). For general Banach spaces the problem has been solved only recently by T. Gowers in [G] and W. Marciszewski constructed in [Mar] a compact K such that $C(K)$ with the topology of pointwise convergence is not linearly homeomorphic to $C(K) \times R$. Various authors (e.g., Semadeni [Se], Lacey [La], Arhangel'skii [Ar]) have posed the hyperplane problem for the particular case of classical Banach spaces of continuous functions on a compact Hausdorff space with the supremum norm. In this paper we solve this problem. Actually, our example addresses the original version of the hyperplane problem that is our $C(K)$ is non-isomorphic to any of its proper subspaces nor to any of its proper quotients. The main ingredient of the proof is showing that for any (linear, bounded) operator $T : C(K) \to C(K)$ its conjugate operator T^* has a decomposition $gI + S$ where g is a bounded Borel function on K and S is a weakly compact operator on the space of Radon measures on K. The space K is the Stone space of a Boolean algebra constructed by transfinite induction as a subalgebra of $\wp(N)$, thus the method is quite different than that of [G].

The second construction is motivated by the first, however K is connected and moreover $K - F$ is connected for any finite $F \subseteq K$. K is a subspace of the product $[0, 1]^{2^\omega}$ of the unit intervals. The key result again concerns the space of bounded linear operators on our $C(K)$, and shows that for any (linear, bounded) operator $T : C(K) \to C(K)$ its conjugate operator T^* has a decomposition $gI + S$ where g is a bounded Borel function on K and S is a weakly compact operator on the space of Radon measures on K. This result implies the indecomposability of $C(K)$ (see 2.5). However one cannot go as far as [GM]. Since every infinite-dimensional $C(K)$ contains a copy of c_0, there are no hereditarily indecomposable $C(K)$'s. Thus we obtain a natural example of an indecomposable space which is not hereditarily indecomposable.

The indecomposability of $C(K)$ implies (see 2.6) that it is non-isomorphic to any $C(K')$ for K' zero-dimensional, which solves, for example, a problem from [Se] (third problem on page 381). This question should be considered in the context of the classification of separable $C(K)$'s i.e., when K is metric, accomplished in the fifties by Miljutin [Mi], Bessaga and Pełczyński [BP]. According to this classification, all separable Banach spaces $C(K)$ are isomorphic to $C(K')$ zero-dimensional (see for example [Se], [Ro2], [Go]). The second $C(K)$ also has the property that it is non-isomorphic to any of its proper subspaces nor to any of its proper quotients. We present the first example since it is much simpler than the second and it is a motivation for the second.

If we assume the continuum hypothesis (abbreviated later CH) the results on the space of operators on $C(K)$ can be strengthened. We prove that any such operator is of the form $gI + S$ where $g \in C(K)$ and S is weakly compact or equivalently (in the context of $C(K)$ spaces) strictly singular. Let us see that this is the minimal possible space of operators in the context of $C(K)$'s: clearly multiplying by a continuous g is a legitimate operator, so let us see that for any $C(K)$ there is an operator $T : C(K) \to C(K)$ which is not of the form $gI + C$ where C is compact. If K is dispersed, then K has a convergent sequence and so, $C(K)$ has c_0 as a complemented subspace which leads to many non-compact operators. Otherwise, Lacey and Morris proved ([LM]) that any $C(K)$'s for K non-dispersed have l_2 as its quotient. In this case there is a continuous function from K into an uncountable set of reals which gives rise to an embedding of $C([0, 1])$ into $C(K)$, thus by Banach-Mazur theorem l_2 is also a subspace of the $C(K)$. Now, the composition of the quotient map onto l_2 with the isomorphic embedding with the shift on the copy of l_2 is an example of an operator of the desired form.

Using some ideas of this paper, it is possible (at least at the price of an additional set-theoretic assumption) to obtain a dispersed compact K such that all operators on $C(K)$ are of the form $cI + S$ where c is a real and S has the range included in c_0 (see [Ko2]).

However, there is no $C(K)$ space where all operators are of the form $cI + S$ where c is a real and S is weakly compact or in this context strictly singular (besides the original spaces of [GM], nonseparable Banach space with this property were constructed in [ALT]). This is because, if K has a convergent sequence, then there is a projection on an isomorphic copy of c_0 which can be composed with shifts which are not weakly compact; otherwise there are two non-first countable points in K, i.e., there are also two disjoint separable non-metrizable closed subsets $K_0, K_1 \subseteq K$. If $f : K \to R$ is in $C(K)$ such that $f|K_i = i$, then multiplying by f is not of the required form.

Recently, based on earlier versions of this paper, where CH was used in the connected construction, Plebanek showed (see [Pl]) that if analogous constructions are done in some nonseparable K's instead of the Stone space of $\wp(N)$ or the products of intervals, then one can guarantee, without assuming CH, that every operator on $C(K)$ is a multiplication by an element of $C(K)$ plus a weakly compact operator. However, this version is not the subspace of l_∞.

The origin of the method of the construction of K should be traced to [Fe]. This method is very flexible and leaves lots of room for custom-made variants (see [Ko1]). Fedorchuk's space besides Ostaszewski's space are two paradigmatic inverse limit constructions of compact spaces obtained under special set-theoretic assumptions, the latter already found many applications in the theory of $C(K)$'s (see [N] or [JM] where its version, the Kunen line, is used instead). However, the final version of our constructions has more similarities with [Ha] and [Ta]. The new crucial ingredient is the notion of a weak multiplier operator which links the topological properties of K with the geometric structure of $C(K)$.

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The notation is intended to be standard, some doubtful points are $f|A$ for the restriction of f to A and $f[A]$ for the image of A under f, N for the set of natural numbers. The terminology follows [Se], [E], [DS] and [Di]. All topological spaces are Hausdorff, measures are signed bounded measures, all linear operators or functionals are meant to be continuous, neighbourhood means an open neighbourhood, all norms on function spaces are supremum norms. By a convergent sequence we mean a nontrivial one, i.e., with all distinct terms.

We will consider a classical chain of inter-related structures: a Boolean algebra $\mathcal{A}$, its Stone space K, the Banach space $C(K)$ of continuous functions on K with the supremum norm, its dual, the Banach space $M(K)$ of Radon measures on K with the variation norm, even the dual to $M(K)$ does appear when we consider weakly compact operators on $M(K)$. Let us fix notation and terminology related to these structures.

Boolean algebras will be denoted by $\mathcal{A}$ and $\mathcal{A}_\alpha$, they will be identified with subalgebras of the algebra $\wp(N)$ of all subsets of the natural numbers with the usual set-theoretic operations. To avoid confusion, the supremum of an infinite family $\{A_n : n \in N\}$ of $\mathcal{A}$ will be denoted by $\bigvee\{A_n : n \in N\}$. The Stone space K of $\mathcal{A}$ consists of all ultrafilters of $\mathcal{A}$ where the basic sets are defined as $[A] = \{u \in K : A \in u\}$ for any $A \in \mathcal{A}$. If a Boolean algebra $\mathcal{A} \subseteq \wp(N)$ contains a singleton of n, then n^* denotes the principal ultrafilter (and, so an isolated point of the Stone space of $\mathcal{A}$) generated by $\{n\}$. K will denote a compact Hausdorff space. Recall that a Radon measure on K is a signed, Borel, scalar-valued, countably additive and regular measure. If K is the Stone space of a Boolean algebra $\mathcal{A}$, then there is a unique Radon measure which extends a finitely additive bounded measure on the algebra $\mathcal{A}$. The variation of a Radon measure μ will be denoted by $|\mu|$. The Banach space of Radon measures with the variation norm will be denoted by $M(K)$.

The structure of the paper is a follows. In the second section we introduce the concept of an operator which is a weak multiplier. A characterization of these operators will be essential in linking the topological properties of K with the properties of $C(K)$. The third section contains a construction of the zero-dimensional example K such that all operators on $C(K)$ are weak multipliers. The fourth section includes preparatory work for the fifth section where we present the connected example of a space $C(K)$ where all operators are weak multipliers. This preparatory work is mainly concerned with adding the suprema of infinite bounded pairwise disjoint sequences of functions in $C(K)$ where K is connected. We say that two real functions e, f on K are pairwise disjoint if and only if $e(x)f(x) = 0$ for any $x \in K$. Adding $\bigvee_{n\in N} A_n$ to a Boolean algebra which contains a pairwise disjoint sequence $(A_n)_{n\in N}$ is the main point of reference. The last sixth section is concerned with getting every operator of the form $gI + S$ where $g \in C(K)$ and S weakly compact under the continuum hypothesis.

2. Weak multipliers

Definition 2.1. *An operator $T : C(K) \to C(K)$ is called a weak multiplier if and only if for every bounded sequence $(e_n : n \in N)$ of pairwise disjoint elements of $C(K)$ (i.e., $e_n \cdot e_m = 0$ for $n \neq m$) and any sequence $(x_n : n \in N) \subseteq K$ such that $e_n(x_n) = 0$ we have*

$$\lim_{n\to\infty} T(e_n)(x_n) = 0.$$

Let us fix more terminology. For an operator $T : C(K) \to C(K)$ consider a function $g_T : K \to R$ defined by

$$g(x) = T^*(\delta_x)(\{x\}),$$

where δ_x is the Dirac measure concentrated at x and the functional $T^*(\delta_x)$ is considered as a Radon measure on K, i.e., we consider the measure of the singleton $\{x\}$. It is clear that $|g(x)| \leq ||T||$.

For a bounded function $f : K \to R$ and an open $U \subseteq K$ and $x \in K$ we define $osc(f, U) = \sup\{|f(x) - f(y)| : x, y \in U\}$ and $osc(f, x) = \inf\{osc(f, U) : x \in U\}$. It is clear that f is continuous at x if and only if $osc(f, x) = 0$.

If $g : K \to R$ is a bounded function integrable with respect to all Radon measures on K, we will consider an operator $gI : M(K) \to M(K)$ which sends the functional which is the integration of a function $f \in C(K)$ with respect to measure μ to the functional which is the integration of the product fg with respect to the measure μ.

Theorem 2.2. *Suppose that $T : C(K) \to C(K)$ is a bounded linear operator. Then the following are equivalent:*

a) T is a weak multiplier;

a') If K is zero-dimensional, for every sequence $(A_n : n \in N)$ of pairwise disjoint clopen sets of K we have

$$\lim_{n\to\infty} ||T(\chi_{A_n})|(K - A_n)|| = 0;$$

b) For every $\varepsilon > 0$ *the set* $\{x \in K : osc(g_T, x) > \varepsilon\}$ *is finite and* $T^* - g_T I : M(K) \to M(K)$ *is well defined and weakly compact;*
c) There is a bounded function $g : K \to R$ *which is integrable with respect to all Radon measures such that* $T^* - gI : M(K) \to M(K)$ *is weakly compact;*

Proof. First we prove that a) or a') in the zero-dimensional case imply b).

Claim 1: *Suppose that* $x \in K$ *and* $\varepsilon > 0$*. Then for every neighbourhood* U *of* x *there is an* $f \in C(K)$ *such that* $0 \le f \le 1$*,* $f(x') = 1$ *for all* x' *in some neighbourhood of* x*,* $supp(f) \subseteq U$ *and* $|T(f)(x) - g_T(x)| < \varepsilon$*. If* K *is zerodimensional* f *could be assumed to be a characteristic function of a clopen set.*

Proof of Claim 1. The value of the Radon measure $T^*(\delta_x)$ at $\{x\}$ can be approximated (by the regularity) by $T^*(\delta_x)(\chi_A)$ for clopen A containing x in the zero-dimensional case or by $\int f dT^*(\delta_x)$ for f's as in the claim in the general case. But this integral is $T^*(\delta_x)(f)$ when $T^*(\delta_x)$ is interpreted as a functional, i.e., it is equal to $T(f)(x)$ as required.

Claim 2: $\{x \in K : osc(g_T, x) > \varepsilon\}$ *is finite for every* $\varepsilon > 0$*.*

Proof of Claim 2. Suppose the claim is false, then there are infinitely many points $(x_n : n \in N)$ and a fixed $\varepsilon > 0$ such that for every neighbourhood U_n of x_n there is $y_n \in U_n$ such that $|g_T(x_n) - g_T(y_n)| > \varepsilon$. As K is Hausdorff, we can assume without loss of generality that $(x_n : n \in N)$ is relatively discrete and so we can choose pairwise disjoint open neighbourhoods U_n of x_n and some e_n's with $0 \le e_n \le 1$, $supp(e_n) \subseteq U_n$ and $e_n|V_n = 1$ for some open neighbourhood $V_n \subseteq U_n$ of x_n such that
1) $|g_T(x_n) - T(e_n)(x_n)| < \varepsilon/4$.
This can be done by the discreteness of x_n's and claim 1. Now we choose certain y_n's as above i.e.,
2) $|g_T(x_n) - g_T(y_n)| > \varepsilon$,
By the continuity of the functions $T(e_n)$, the points y_n can be chosen to satisfy
3) $|T(e_n)(x_n) - T(e_n)(y_n)| < \varepsilon/4$. Of course we can also make sure that
4) $y_n \in V_n$.
Now apply claim 1 for g_T at y_n's i.e., find an open $W_n \subseteq U_n$ and $h_n \in C(K)$ with $||h_n|| \le 1$ such that $supp(h_n) \subseteq W_n$ for which there is a neighbourhood $Z_n \subseteq W_n$ of y_n such that $h_n|Z_n = 1$ and
5) $|g_1(y_n) - T(h_n)(y_n)| < \varepsilon/4$.
Thus by 1) - 3), 5) we have

$$|T(e_n)(y_n) - T(h_n)(y_n)| = |T(e_n - h_n)(y_n)| \ge \varepsilon/4.$$

To obtain a contradiction with the fact that T is a weak multiplier and complete the proof of claim 2, note that and $||e_n - h_n|| \leq 2$, $supp(e_n - h_n) \subseteq U_n$ (U_n's are pairwise disjoint) and $y_n \notin supp(e_n - h_n)$, since both of e_n and h_n are equal 1 on some neighbourhood of y_n by 4). To obtain a contradiction with a') note that, by claim 1, e_n's can be chosen to be of the form χ_{A_n} for some clopen $A_n \subseteq K$ and h_n can be chosen to be of the form χ_{B_n} for some clopen $B_n \subseteq A_n$. I.e., $e_n - h_n = \chi_{A_n - B_n}$ and $y_n \notin A_n - B_n$ as required in a').

Hence, we obtained the first part of b). It follows that the multiplication of a Radon measure by g_T is a well defined operator on $M(K)$ since g_T has a most countable set of discontinuities, i.e., is a Borel function and so, integrable with respect to any Radon measure. Now let us see that $T^* - gI : M(K) \to M(K)$ is weakly compact. Let $(\mu_n : n \in N)$ be a bounded sequence in $M(K)$, we need to show that $(T^* - gI)(\mu_n)$'s form a relatively weakly compact set. We will use the following version of the Dieudonné-Grothendieck theorem ([Di] VII. 14): If a bounded sequence $(\mu_n)_{n \in N}$ is not relatively weakly compact, then there are pairwise disjoint open $V_n \subseteq U_n \subseteq K$ and e_n's from $C(K)$ such that $0 \leq e_n \leq 1$, $e_n|V_n = 1$ and $e_n|(K - U_n) = 0$, $|\mu_n|(U_n - V_n) \to 0$ and the sequence $(\int e_n d\mu_k)_{k \in N}$ does not converges to zero uniformly for $k \in N$. This version can be easily obtained from the standard version using the Urysohn lemma and the regularity of Radon measures. So suppose that $(T^* - gI)(\mu_n)$'s do not form a relatively weakly compact set. Possibly renumerating the measures and going to a subsequence we can find an $\varepsilon > 0$ and sequences of pairwise disjoint open sets $(U_n)_{n \in N}$, $(V_n)_{n \in N}$ such that $V_n \subseteq U_n$ and a sequence of continuous functions $(e_n)_{n \in N}$ as above such that

$$|\int e_n d(T^* - g_T I)(\mu_n)| = |\int (T(e_n) - g_T e_n)\, d\mu_n| > \varepsilon.$$

for all $n \in N$. Note that

$$|\int_{U_n - V_n} (T(e_n) - g_T e_n)\, d\mu_n| \to 0$$

because $|\mu_n|(U_n - V_n) \to 0$. Also

$$|\int_{K - U_n} (T(e_n) - g_T e_n)\, d\mu_n| \to 0$$

because the support of e_n's are included in U_n's and Borel functions $T(e_n)|(K - U_n)$ converge uniformly to 0 since T is a weak multiplier. Hence

$$|\int_{V_n} (T(e_n) - g_T e_n)\, d\mu_n| = |\int_{V_n} (T(e_n) - g_T) d\mu_n| > \varepsilon,$$

for sufficiently large $n \in N$, since $e_n(x) = 1$ for $x \in V_n$.

Let M be such a positive real that $||\mu_n|| \leq M$ for all $n \in N$. We conclude that there are $x_n \in V_n$ such that $|T(e_n)(x_n) - g_T(x_n)| > \varepsilon/M$ for all $n \in N$. Thus, by claim 1, we can find continuous h_n's such that $0 \leq h_n \leq 1$, $supp(h_n) \subseteq V_n$, $h_n(x) = 1$ if $x \in W_n$ where W_n is some neighbourhood of x_n and

$$|T(h_n)(x_n) - g_T(x_n)| < \varepsilon/2M,$$

but we have

$$|T(e_n)(x_n) - g_T(x_n)| > \varepsilon/M$$

for sufficiently large $n \in N$. This implies that the sequence of $T(e_n - h_n)(x_n) = T(e_n)(x_n) - T(h_n)(x_n)$ does not tend to zero, but $(e_n - h_n)(x_n) = 1 - 1 = 0$ contradicting the fact that T is a weak multiplier. In the zero-dimensional case, as before, we can choose, by claim 1, e_n's and h_n's to be characteristic functions of clopen sets A_n and B_n, respectively, where $B_n \subseteq A_n$. In this case $\chi_{A_n} - \chi_{B_n} = \chi_{A_n - B_n}$ and $x_n \notin A_n - B_n$ contradicting a').

So we are left with showing that c) implies a), since a') trivially follows from a) and c) trivially follows from b). So, let g be as in c), let $(e_n)_{n\in N}$ be a bounded sequence of pairwise disjoint elements of $C(K)$ and let $(x_n)_{n\in N}$ be a sequence of points of K such that $e_n(x_n) = 0$. Consider the Radon measures $\mu_n = T^*(\delta_{x_n}) - g(x_n)\delta_{x_n}$ for $n \in N$. Clearly it is a bounded sequence of measures, since T and g are bounded. Suppose that

$$T(e_n)(x_n) = \int e_n dT^*(\delta_{x_n}) - 0 = \int e_n d\mu_n$$

do not converge to zero. It means that there are some open sets U_n included in the supports of e_n's, i.e., pairwise disjoint, such that $(|\mu_n(U_n)|)_{n\in N}$ does not converge to 0. This, by the Dieudonné-Grothendieck theorem, contradicts the fact that $T^* - gI$, as a weakly compact operator, sends bounded sequences to relatively weakly compact sequences, hence $(T(e_n)(x_n))_{n\in N}$ converges to 0 as required in a) which completes the proof of the theorem. □

Theorem 2.3. *Suppose that K is a compact space with no convergent sequences and that $T : C(K) \to C(K)$ is a weak multiplier. Then T is onto $C(K)$ if and only if it is an isomorphism onto its range.*

Proof. By theorem 2.2. $T^* = g_T I + S$ where S is weakly compact and $\{x \in K : osc(g_T, x) > \varepsilon\}$ is finite for any $\varepsilon > 0$, in particular the set of points where g_T is discontinuous is at most countable.

Claim 1: If $g_T(x) = 0$ for infinitely many x, then there is a nonmetrizable, separable $K_1 \subseteq K$ such that g_T is continuous and equal to zero at each point of K_1.

Proof of the Claim. Suppose that $g_T(x_n) = 0$ for infinitely many distinct $x_n \in K$. As K has no convergent sequences, the set of all limit points of $\{x_n : n \in N\}$ is a closed subset K' of K which has no isolated points. As g has at most countably many points of discontinuity, one can find an infinite closed, $K_1 \subseteq K'$ avoiding all these points, i.e., where g assumes only value 0. By taking the closure of a countable subset, one can assume that K_1 is separable, and using the fact that there are no convergent sequences in K, one concludes that K_1 is nonmetrizable as an infinite closed subset of K.

Claim 2: If $T : C(K) \to C(K)$ is an isomorphism onto its range, then T is onto $C(K)$.

Proof of the Claim. If T is an isomorphism onto its range, it means that T^* is onto M(K). If g is zero on an infinite set, use claim 1 to choose $\mu_n = \delta_{x_n}$, where $g(x_n) = 0$, and g is continuous at all points of relatively discrete $(x_n)_{n \in N}$. There are some Radon measures ν_n on K such that $T^*(\nu_n) = g\nu_n + S(\nu_n) = \mu_n$ and $||\nu_n|| \leq M$ for all $n \in N$. By the continuity of g at the points x_n one can find pairwise disjoint and open U_n's such that $|g(x)| \leq 1/n$ for $x \in U_n$. By the Dieudonné-Grothendieck theorem ([Di] VII. 14.) the fact that $S(\nu_n)$'s form a relatively weakly compact set means that the sequence $S(\nu_n)(U_n)$ converge to 0, also $|\int_{U_n} g d\nu_n| \leq M/n$ which contradicts the choice of μ_n's.

Claim 3: Suppose $K_1 \subseteq K$ is closed and separable and that $g_T(x) = 0$ for all $x \in K_1$. Then the composition $T_{K_1} : C(K) \to C(K_1)$ of T with the restriction on K_1 has separable range.

Proof of the Claim. First we will note that T_{K_1} is weakly compact. Note that a Radon measure μ on K_1 defines naturally a Radon measure on K concentrated on K_1, which we will denote by μ as well. Since $g_T I + S$ is the conjugate of T, we have that $\int T(f) d\mu = \int g_T f d\mu + \int f dS(\mu)$ for each $f \in C(K)$, so for μ concentrated on K_1 we get $\mu(T_{K_1}(f)) = S(\mu)(f)$ for each $f \in C(K)$, i.e., $(T_{K_1})^* = S$ and so $(T_{K_1})^*$ is weakly compact and we can use a Gantmacher's theorem which says that an operator is weakly compact if and only if its conjugate is ([DS] VI 4.8.).

Now we will note that weakly compact operators with ranges in $C(K_1)$ for K_1 separable have norm-separable ranges. As the range of an operator is a countable union of images of the balls, it is enough to note that convex weakly compact subsets of $C(K_1)$ are norm-separable. Let $D = \{x_n : n \in N\} \subseteq K_1$ be a countable dense set. Let $\phi : C(K_1) \to R^\omega$ be given by $\phi(f) = (f(x_n))_{n \in N}$. Note that ϕ is one-to-one and continuous with respect to the weak topology in $C(K_1)$ and the product topology in R^ω, thus it is a homeomorphism while restricted to compact sets of $C(K_1)$. Let $X \subseteq C(K_1)$ be convex and compact in the weak topology, R^ω is metrizable, so $\phi[X]$ is separable and so, its homeomorphic copy X is separable in the weak topology as well. Consider the set $Y \subseteq X$ of all convex combinations of elements of the countable weak-dense set $E \subseteq X$. Y is convex and separable in the norm topology (consider the convex combinations with rational coefficients). The weak closure of Y is X, but weak closures of convex sets coincide with closures in the norm topology ([Di] Theorem II.1.). So, the rational convex combinations of elements of E are norm dense in X.

Claim 4: If T is onto $C(K)$, then T is an isomorphism.

Proof of the Claim. Suppose that T is onto $C(K)$ and g is zero on an infinite set. Let $K_1 \subseteq K$ be as in claim 1. The space $C(K_1)$ is non-separable and its every member is in the range of the composition of T with the restriction on K_1. But the range of this composition is separable in $C(K_1)$ by claim 3. This contradicts the fact that T is onto $C(K)$. So we may assume that g can be 0 only at a finite subset of K.

Now, we will make use of Fredholm theory. An operator R on a Banach space is said to be Fredholm if and only if $dim(ker R)$ and $codim(Im(R))$ are finite. The index of such an operator is the number $i(R) = dim(ker R) - codim(Im(R))$. Proposition 2. c. 10. of [LT] implies that if $R : X \to Y$ is Fredholm (then its range is closed) and $S : X \to Y$ is strictly singular (i.e., not an isomorphism while restricted to any infinite dimensional subspace), then $R + S$ is Fredholm and $i(T + S) = i(T)$.

We will first observe that weakly compact operators from $M(K)$ into itself are strictly singular. It follows from the fact that such weakly compact operators send weakly Cauchy sequences to norm convergent sequences (i.e., $M(K)$ has the Dunford-Pettis property, see [DS], VI.8.10 for this property for $L_1(S, \Sigma, \mu)$ spaces and [DS] p. 394, Notes and Remarks for Kakutani's theorem which says that $M(K)$ is such a Banach space). Thus if a weakly compact operator on $M(K)$ were an isomorphism on an infinite dimensional subspace, any norm bounded sequence from such a subspace would have a weakly convergent subsequence (isomorphism preserve the fact that a sequence is weakly convergent, and restrictions of weakly compact operators are weakly compact by the Hahn-Banach theorem), which by the above property, would, in fact be a norm convergent sequence, hence the ball of our subspace would be compact.

Note that we may assume without loss of generality that $m < |g(x)| < M$ for some $m, M > 0$ and all $x \in K$. This is because a sequence of x_n's such that $|g(x_n)| < 1/n$ would give an uncountable set where g would be zero (the set of its accumulation points where g is continuous) and if a g' is a finite modification of g, then $(g - g')I$ is of finite rank, and so weakly compact, consequently it can replace g.

Multiplication by such a g is an isomorphism from the dual of $C(K)$ onto itself, i.e., a Fredholm operator of index 0, hence $gI + S$ has the same index 0. If T is onto, T^* is an isomorphism onto its image then, $dim(Ker(T^*)) = 0$, so since $i(T^*) = 0$, $codim(Im(T^*)) = 0$, i.e., we conclude that T^* is onto $M(K)$ or that T is an isomorphism. □

Theorem 2.4. *Suppose that $C(K)$ is such that all operators on it are weak multipliers. Then, finite co-dimensional subspaces of $C(K)$ are isomorphic if and only* *if they have the same co-dimension. In particular the hyperplanes of $C(K)$ are not isomorphic to the entire $C(K)$ and $C(K)$ is a Grothendieck space.*

Proof. It is well-known that all subspaces of the same finite co-dimension are isomorphic to each other (like in 21.5.7-8 of [Se]).

By the previous theorem none of the subspaces of a finite co-dimension is isomorphic to the entire $C(K)$. The remaining parts follow from the fact that $X \oplus R^n \sim C(K)$ if X is of co-dimension $n \in N$. It follows that c_0 cannot be a complemented subspace of $C(K)$, but this is sufficient for a $C(K)$ space to have the Grothendieck property (see [Sch]). □

Theorem 2.5. *Suppose that K is such that all operators on $C(K)$ are weak multipliers and that $K - F$ is connected for any finite $F \subseteq K$. Then, all projections in $C(K)$ are of the form $I + S$ or S where S is finite-dimensional. In other words, all complemented subspaces of $C(K)$ have finite dimension or finite co-dimension and $C(K)$ is an indecomposable Banach space.*

Proof. Let $P : C(K) \to C(K)$ be a projection (see [Se]). Let g_T and S, a weakly compact operator, be such that $P^* = g_T I + S$. We have $P^2 = P$, and so $(P^*)^2 = P^*$ i.e., $g_T^2 I + S^2 + g_T S + S g_T = g_T I + S$ and so $(g_T^2 - g_T)I$ is weakly compact. Suppose that x is a point where g_T is continuous. If $(g_T^2 - g_T)(x) \neq 0$, it would be separated from 0 over some open set, and so would not be a weakly compact operator. Thus $g_T(x)$ may assume value 0 or 1 if g_T is continuous at x. Let $F \subseteq K$ be the set of all points x of K where $osc(g_T, x) \geq 1/2$. By 2.2 b) F is finite. One easily notes that K has no non-trivial convergent sequences, because, it would produce a complemented copy of c_0 which would give many operators contradicting the assumption on K. Consequently as K is connected, all non-empty open sets must be uncountable, and so the points where g_T is continuous form a dense set in K. For any $x \in K$ such that $osc(g_T, x) < 1/2$, there must be an open neighbourhood U_x of x where $g_T(y) = 0$ or $g_T(y) = 1$ for all $y \in U_x$ such that g_T is continuous at y. This defines a continuous function $g : K - F \to \{0, 1\}$ such that $|g(x) - g_T(x)| \leq osc(g_T, x)$, in particular $g(x) = g_T(x)$ if $g_T(x)$ is continuous at x. By the assumption that $K - F$ is connected, we conclude that g is constantly 0 or 1. Denote by g the corresponding constant function defined on the entire K. Note that 2.2 b) implies that $\{x \in K : |g(x) - g_T(x)| > \varepsilon\}$ is finite for every $\varepsilon > 0$. To prove that $(g - g_T)I$ is weakly compact we will need the following:

Claim: Suppose $h : K \to R$. $hI : M(K) \to M(K)$ is weakly compact if and only if $\{x \in K : |h(x)| > \varepsilon\}$ is finite for every $\varepsilon > 0$.

Proof of the Claim. If there are $x_n \in K$ with $|h(x_n)| > \varepsilon$ for all n, we may assume that they form a discrete set and then the bounded sequence of measures $(\delta_{x_n} : n \in N)$ is send to $(h(x_n)\delta_{x_n} : n \in N)$ which is not relatively weakly compact by the Dieudonné-Grothendieck theorem. Conversely, if one wants to check that $h\mu_n$'s form a relatively weakly compact set in $M(K)$ for a bounded sequence in $M(K)$, again by the Dieudonné-Grothendieck theorem. it is enough to consider a family of pairwise disjoint open sets $(U_n : n \in N)$ and their $h\mu_k(U_n)$. They obviously tend to 0 uniformly in k, hence $(h\mu_n)_{n\in N}$ is weakly relatively compact which completes the proof of the claim.

Let $S' = (g_T - g)I + S$. The claim implies that S' is weakly compact. We also have that $P^* = I + S'$ or $P^* = S'$. Since $I - P^* = (I - P)^*$, by Gantmacher's theorem $I - P$ or P is a weakly compact projection. But using the fact that weakly compact operators on $C(K)$ are strictly singular ([Pe]) we conclude that P or $I - P$ has finite-dimensional range, as required. □

Theorem 2.6. *Suppose that a space $C(K)$ for some compact K is indecomposable, then $C(K)$ is non-isomorphic to any space $C(K')$ for any zero-dimensional compact K'.*

Proof. The space of continuous functions on one-point compactification of a discrete infinite set has, for example, many non-trivial projections, hence $C(K)$ cannot be isomorphic to such a space. Thus, if $C(K')$ is isomorphic to $C(K)$, then K' has at least two distinct non-isolated points, consequently, if K' is zero-dimensional, it has a clopen $A \subseteq K'$ such that both A and $K - A$ are infinite. The restriction of functions to A is an example of nontrivial projections. □

Recall that $Y \subseteq X$ is C^*-embedded in X if and only if every bounded continuous function on Y extends to a bounded continuous function on X.

Theorem 2.7. *The following are equivalent for a compact space K:*

a) All operators $T : C(K) \to C(K)$ are of the form $gI + S$ where $g \in C(K)$ and S is weakly compact.
b) All operators on $C(K)$ are weak multipliers and for every $x \in K$ the space $K - \{x\}$ is C^*-embedded in K.

Proof. To see that a) implies b) we may assume that K has no convergent sequences (otherwise a complemented copy of c_0 would give rise to many operators not as in a)). One only needs to prove that if there is a $y \in K$ and a continuous bounded function $h : (K - \{y\}) \to R$ which has no continuous extension to K, then $T(f) = h(f - f(y))$ is an operator which is not of the form as in a). Otherwise, suppose that $T = gI + S$ where $g \in C(K)$ and S is weakly compact. As h has no continuous extension to K, y must be non-isolated and so, there are distinct points $x_n \neq y$ and an $\varepsilon > 0$ such that $|h(x_n) - g(x_n)| \geq \varepsilon$ for all $n \in N$. As K has no convergent sequences, one of the accumulation points of the x_n's, say z, is distinct from y, it is also non-isolated and $|h(z) - g(z)| \geq \varepsilon$. This implies that there is an infinite open set U with the closure not containing y such that $|h(x) - g(x)| \geq \varepsilon/2$ for all $x \in U$. Note that $T - gI$ is an isomorphism on the subspace of all functions with supports in U, but on the other hand it is a weakly compact operator S which is impossible as weakly compact operators on $C(K)$'s are strictly singular ([Pe]) which completes the proof of b) assuming a).

Now let us see that b) implies a). Note that it would be enough to show that for every function $f : K \to [0, 1]$ such that $\{x : osc(f, x) > \varepsilon\}$ is finite for each $\varepsilon > 0$, $f|(K - X)$ can be extended to a continuous function f' on K where X is the set of all points of discontinuity of f. Indeed, theorem 2.2., b) implies that $T^* - g_T I : M(K) \to M(K)$ is weakly compact, and we would have that $(g_T - f')I : M(K) \to M(K)$ is weakly compact (see the claim of 2.5), and so $T^* - f'I$ is weakly compact. But f' is continuous so $T - f'I$ would be a well defined operator on $C(K)$ whose conjugate is $T^* - f'I$. By Gantmacher theorem, which says that an operator is weakly compact if and only if its conjugate is weakly compact, we would get that $T - f'I$ is weakly compact obtaining a).

Suppose that an f and X are as above and $f|(K - X)$ cannot be extended to a continuous function on K. This means that there is a point y of K and an $\varepsilon > 0$ such that every neighbourhood of y contains points $y', y'' \in K - X$ such that $|f(y') - f(y'')| > \varepsilon$. Making this assumption we will construct a g which is bounded and discontinuous only at y and cannot be extended to a continuous function on entire K which will contradict the second part of b). For this we need the following:

Claim: Suppose L is a compact space and $F, G \subseteq L$ are disjoint and closed and $\phi : L \to [0, 1]$ is a function such that $osc(f, x) < \varepsilon$ for all $x \in G$. Then there exists a function $\psi : L \to [0, 1]$ such that $\psi|F = \phi|F$, $\psi|G$ is continuous, $osc(\psi, x) \leq osc(\phi, x)$ for every $x \in L$ and $|\phi(x) - \psi(x)| \leq \varepsilon$ for all $x \in L$.

Proof of the Claim. Let U be an open set such that $F \subseteq U$ and $U \cap G = \emptyset$ and let $(f_i : i \in I)$ be a locally finite partition of unity subordinated to an open cover $\{U\} \cup \mathcal{V}$ where $osc(V, \phi) < \varepsilon$ and $V \cap F = \emptyset$ for all $V \in \mathcal{V}$. For each $i \in I$ pick an $r_i \in [0, 1]$ such that $\phi(x) = r_i$ for some $x \in X$ such that $f_i(x) \neq 0$. Define $I_1 \cup I_2 = I$ so that $i \in I_1$ if there is $V \in \mathcal{V}$ such that $\{x \in X : f_i(x) \neq 0\} \subseteq V$ and otherwise $i \in I_2$. Define

$$\psi(x) = \sum\{r_i f_i(x) : i \in I_1\} + \sum\{\phi(x) f_i(x) : i \in I_2\}.$$

It is clear that $\psi|F = \phi|F$ because $f_i(x) = 0$ if $x \in F$ and $i \in I_1$, that $\psi|G$ is continuous because $f_i(x) = 0$ if $x \in G$ and $i \in I_2$ and that the oscillation does not grow. Also note that if $i \in I_1$ and $f_i(x) \neq 0$, then $\phi(x) - \varepsilon \leq r_i \leq \phi(x) + \varepsilon$, i.e., $|\psi(x) - \sum_{i \in I} \phi(x) f_i(x)| \leq \varepsilon$, as required in the claim.

Now using the claim we can construct a sequence of functions $f_n : K \to [0, 1]$ such that:

1) $f_0 = f$, $osc(f_n, K - \{y\}) \leq 1/2^n$, $osc(f_n, x) \leq osc(f, x)$ for all $x \in X$,
2) $|f_n(x) - f_{n+1}(x)| \leq 1/2^n$,
3) for each $n \in N$ there is an open neighbourhood U_n of y such that $f_n|U_n = f|U$.

Indeed, given f_n, find an open neighbourhood U_n of y such that $F = \overline{U_n}$ is disjoint from the finite set G of points $x \neq y$ where $osc(f_n, x) \geq 1/2^{n+1}$, we may apply the lemma obtaining the f_{n+1}.

It is easy to see that since f_n's form a uniformly Cauchy sequence of functions on $X - \{y\}$ of decreasing oscillations, they converge to a continuous function on $K - \{y\}$. By 3) the function cannot be extended to a continuous function g on K, since in any neighbourhood of y there are y', y'' satisfying $|g(y'') - g(y'')| > \varepsilon$ because f had this property. This contradicts the fact that $K - \{y\}$ is C^*-embedded in K. □

Lemma 2.8. *Suppose that K is a compact space such that whenever U_1, U_2 are open subsets of K satisfying $\overline{U}_1 \cap \overline{U}_2 \neq \emptyset$, then $\overline{U}_1 \cap \overline{U}_2$ contains more than one point, then for every $x \in K$ the space $K - \{x\}$ is C^*-embedded in K.*

Proof. Suppose that $f : (K - \{x\}) \to R$ is bounded and continuous. Without loss of generality we may assume that it is into $[0, 1]$ and that x is non-isolated. The family

$$\{\overline{f[U - \{x\}]} : x \in U,\ U \text{ is open in } K\}$$

is a centered family of closed sets in a compact space, hence its intersection contains a point $t \in [0, 1]$. If it is the unique point of the intersection, one can easily show that putting $f(x) = t$ defines a continuous extension of f.

But otherwise, if the intersection contains two points $t_1 < t_2$, the closures in K of open sets $U_1 = f^{-1}[[0, t_1+\varepsilon)]$ and $U_1 = f^{-1}[(t_1-\varepsilon, 1]]$ where $\varepsilon = (t_2-t_1)/3$ contain x. By the hypothesis on K, they must contain some other point $x' \in K$ as well, which would contradict the continuity of f at x'. □

3. Construction of the zero-dimensional compact space

Theorem 3.1. *There is a Boolean algebra $\mathcal{A} \subseteq \wp(N)$ containing all finite sets such that given*

a) a sequence $(A_n : n \in N)$ of pairwise disjoint elements of $\mathcal{A}$,
b) a sequence $(l_n : n \in N)$ of distinct natural numbers such that $l_n \notin A_m$ for $n, m \in N$, there is an infinite $b \subseteq N$ such that
c) $\{A_m : m \in b\}$ has its supremum A in $\mathcal{A}$ and
d) the intersection of the sets $\overline{\{l_n{}^* : n \in b\}}$ and $\overline{\{l_n{}^* : n \notin b\}}$ in the Stone space K of $\mathcal{A}$ is nonempty.

The rest of this section is devoted to the proof of the above theorem. We construct by transfinite induction Boolean algebras $\mathcal{A}_\alpha$ for $\alpha \leq 2^\omega$ which can be considered as subalgebras of the algebra $\wp(N)$, taking unions at limit ordinals, i.e., $\mathcal{A}_\lambda = \bigcup_{\alpha<\lambda} \mathcal{A}_\alpha$ for $\lambda \leq 2^\omega$ limit. Along this construction we consider the Stone spaces K_α of $\mathcal{A}_\alpha$ and the dense discrete subset of isolated points $N_\alpha =$

$\{n^*|\alpha : n \in N\} \subseteq K_\alpha$, where $n^*|\alpha$ is the principal ultrafilter of $\mathcal{A}_\alpha$ generated by $\{n\}$. We require that $\mathcal{A}_1$ is the algebra of finite and cofinite subsets of N, i.e., K_1 is a convergent sequence with its limit. The construction is very similar to a construction of R. Haydon ([Ha]).

At a successor stage, $\mathcal{A}_{\alpha+1}$ is obtained from $\mathcal{A}_\alpha$ so that a case of c) of theorem 2.1. is taken care of. This extension is obtained by adding one element A_α, to $\mathcal{A}_\alpha$ which works as the A of 2.1. c). At each stage we will have to preserve some cases of d).

At the beginning of the construction we fix an enumeration:

$$(A_n(\alpha), l_n(\alpha))_{\alpha<2^\omega, n\in N}$$

such that for every $\alpha < 2^\omega$ and for every pair $((A_n)_{n\in N}, (l_n)_{n\in N})$ consisting of a countable pairwise disjoint sequence of sets of integers (remember we work in $\wp(N)$) and of an increasing sequences of integers satisfying

$$*) \qquad l_n \notin A_m \ \text{ for } n, m \in N,$$

there is $\alpha < \alpha' < 2^\omega$ such that $A_n(\alpha') = A_n$ and $l_n = l_n(\alpha')$ for all $n \in N$. This enumeration exists since, $(2^\omega)^\omega = 2^\omega$ implies that there are 2^ω such pairs and $2^\omega \times 2^\omega = 2^\omega$ implies that 2^ω can be divided into 2^ω pairwise disjoint sets, each cofinal in 2^ω.

At stage $\alpha < 2^\omega$ we are given $\mathcal{A}_\alpha$ and its Stone space K_α and families $(b_\beta)_{\beta<\alpha}$, $(a_\beta)_{\beta<\alpha}$ of subsets of N such that for every $\beta < \alpha$ we have $b_\beta \subseteq a_\beta$ and

$$**) \qquad \overline{\{n^*|\alpha : n \in b_\beta\}} \cap \overline{\{n^*|\alpha : n \in a_\beta - b_\beta\}} \neq \emptyset,$$

where the closures are taken in K_α.

At successor stages $\alpha + 1$ consider the Stone space K_α of the algebra $\mathcal{A}_\alpha$. We will choose $a \subseteq N$ so that for any choice of an infinite $b \subseteq a$ after adding $A = \bigvee_{n\in b} A_n$ to $\mathcal{A}_\alpha$ the conditions **) are preserved in $K_{\alpha+1}$ for all $\beta < \alpha$.

There is no doubt that some choices of $b \subseteq a \subseteq N$ destroy **), for example, if one of the sets in **) is included in such an A and the other is disjoint from it. The point here is that since we have 2^ω choices and less than 2^ω commitments in **), there is one choice which works for all the commitments. This idea is based on lemma 1D of [Ha].

Let $(a^\theta : \theta < 2^\omega)$ denote an almost disjoint family of subsets of N, i.e., a^θ's are infinite while $a^\theta \cap a^{\theta'}$ is finite for $\theta < \theta' < 2^\omega$. We claim that there is a $\theta < 2^\omega$ such that **) is satisfied in any extension $\mathcal{A}_{\alpha+1}$ of $\mathcal{A}_\alpha$ obtained by adding $A = \bigvee\{A_n : n \in b\}$ for $b \subseteq a^\theta \subseteq N$.

Assume, to the contrary, that for each choice of $\theta < 2^\omega$ there is a $b^\theta \subseteq a^\theta$ such that there is $\beta < \alpha$ so that the closures of the sets $R' = \{n^*|\alpha+1 : n \in b_\beta\} \subseteq K_{\alpha+1}$ and $S' = \{n^*|\alpha + 1 : n \in a_\beta - b_\beta\} \subseteq K_{\alpha+1}$ get separated in $K_{\alpha+1}$ obtained using b^θ. Using general form of an element of a Boolean algebra generated by one element A over a subalgebra, we conclude that there are pairwise disjoint elements $B, C, D \in \mathcal{A}_\alpha$ such that $R' \subseteq ([B] \cap [A]) \cup ([C] - [A]) \cup [D]$ and $S' \cap \{([B] \cap [A]) \cup ([C] - [A]) \cup [D]\} = \emptyset$ (Recall that $[X]$ stands for the clopen set of the Stone space of an algebra corresponding to the element X of the algebra). Since the ultrafilters $n^*|\alpha + 1$ are generated from $\mathcal{A}_\alpha$, we conclude in $\wp(N)$ that

$$R \subseteq (B \cap A^\theta) \cup (C - A^\theta) \cup D$$

$$S \cap \{(B \cap A^\theta) \cup (C - A^\theta) \cup D\} = \emptyset,$$

where $R = \{n : n^* \in b_\beta\}$ and $S = \{n : n^* \in a_\beta - b_\beta\}$ and $A^\theta = \bigcup\{A_n : n \in b^\theta\}$. As we have 2^ω choices for θ and less than 2^ω sets B's, C's and D's in $\mathcal{A}_\alpha$, there are distinct θ and θ' for which we have the same R, S and the same B's, C's and D's work. Taking intersections with B or C and intersections or unions, we get

$$R \cap B \subseteq (B \cap A^\theta \cap A^{\theta'})$$

$$S \cap (B \cap (A^\theta \cap A^{\theta'})) = \emptyset,$$

$$R \cap C \subseteq C \cap (-A^\theta \cup -A^{\theta'}) = C - (A^\theta \cap A^{\theta'})$$

$$S \cap C \cap (C \cap (-A^\theta \cup -A^{\theta'})) = S \cap C \cap (C - (A^\theta \cap A^{\theta'})) = \emptyset.$$

As $A^\theta \cap A^{\theta'}$ is a finite union of A_n's (since $b^\theta \cap b^{\theta'}$ is finite), it corresponds to a clopen set in K_α and this implies that R' and S' can be separated in K_α, a contradiction with the inductive assumption **). Thus one of a^θ's works for all choices of $b \subseteq a^\theta$, call it a.

Now we can choose $b \subseteq a$. For every infinite $a \subseteq N$ there is $b \subseteq a$ such that $\overline{\{l_n^* : n \in b\}}$ intersects $\overline{\{l_n^* : n \in a - b\}}$ where the closures are taken in K_α. This follows from the fact that the topological weight of K_α is less than 2^ω and in a compact space the separations of closed sets can be done by finitely many basic open sets. As $l_n^*|\alpha + 1$ are generated from $\mathcal{A}_\alpha$ and by *), $l_n{}^*|\alpha$'s do not belong to A_m's for $n, m \in N$, one can conclude that all $l_n{}^*|\alpha + 1$'s do not belong to $[\bigcup\{A_n : n \in b\}]$ that is the above sets are not separated in $K_{\alpha+1}$. So we can define $b_\alpha = \{l_n : n \in b\}$ and $a_\alpha = \{l_n : n \in a\}$ and add these sets to our list from **) of pairs of sets whose closures will never be disjoint.

We define $A_\alpha = \bigcup\{A_n : n \in b\} = \bigvee\{A_n : n \in b\}$ which exists in the complete Boolean algebra $\wp(N)$. This completes the description of the construction

Now to prove that $\mathcal{A}$ satisfies conditions c) and d) of theorem 2.1, let $(A_n : n \in N) \subseteq \mathcal{A}$ and $(l_n : n \in N)$ satisfy conditions a), b) of 2.1. There are $\alpha_n \in 2^\omega$ such that $A_n \in \mathcal{A}_{\alpha_n}$. Since the cofinality of 2^ω is uncountable (see [Ku] or [Je]), there is an $\alpha < 2^\omega$ such that $\alpha_n < \alpha$ for each $n \in N$. Applying the properties of our enumeration from *), we conclude that the pair $((A_n)_{n \in N}, (l_n)_{n \in N})$ is considered at some $\alpha < 2^\omega$ such that $\{A_n : n \in N\} \subseteq \mathcal{A}_\alpha$. The construction guarantees e) and that the corresponding case of d) holds in $\mathcal{A}_{\alpha+1}$. But this case of d) enters our list **), which is preserved, by the construction at successor stages.

The requirements **) cannot be destroyed first time at limit stages, in particular at 2^ω, by the fact that separated closed sets in compact zero-dimensional spaces are separated by clopen sets i.e., belonging to the clopen algebra which at the limit stage is the union of the previous algebras. This completes the proof of theorem 3.1. □

Lemma 3.2. *Every operator $T : C(K) \to C(K)$ is a weak multiplier.*

Proof. Suppose that a bounded, linear operator $T : C(K) \to C(K)$ is not a weak multiplier, i.e., by 2.2 that there exist a sequence of pairwise disjoint clopen sets $(A_n : n \in N)$ such that there is $\varepsilon > 0$ and points $x_n \in K$ such that $x_n \notin A_n$ and $|T(\chi_{A_n})(x_n)| > \varepsilon$ for infinitely many n's. Since finite sums of the characteristic functions of A_n's are of norm one, if x_n were constant for infinitely many n's, we would get a contradiction with the fact that T is bounded. Thus, we may assume without loss of generality that $|T(\chi_{A_n})(x_n)| > \varepsilon$ holds for all $n \in N$ and that $x_n = {l_n}^*$ for some increasing sequence $(l_n : n \in N)$ of positive integers, since $\{n^* : n \in N\}$ is dense in K.

We may also assume without loss of generality that the points ${l_n}^*$ are not in the sets A_m for $n, m \in N$: if there is one n_0 such that ${l_n}^* \in A_{n_0}$ for n's from an infinite set $M \subseteq N$, we may consider $M - \{n_0\}$ and use the disjointedness of A_n's. Otherwise, one can construct by induction an infinite set of indices as required. Thus $(A_n : n \in N)$ and $(l_n : n \in N)$ satisfy a) and b) of theorem 3.1.
Let μ_n be the Radon measures on K which correspond (see [Se], §18.), by the Riesz representation theorem, to the linear bounded functionals ϕ_n on $C(K)$ given by the relation

$$\phi_n(f) = T(f)({l_n}^*) = \int_K f d\mu_n$$

which holds for all $f \in C(K)$, i.e., $\mu_n = T^*(\delta_{{l_n}^*})$. In particular, we have $|\mu_n(A_n)| > \varepsilon$. Now we may find an infinite $N' \subseteq N$ such that for $n \in N'$ we have

$$***) \qquad \sum\{|\mu_n|(A_m) : \; n \neq m, \; m \in N'\} < \varepsilon/3.$$

This is a lemma of Rosenthal (see [Ro] lemma 1.1. also in the stronger version that we are using see [Di] page 82.) applied to the measures μ_n as above.
For $M \subseteq N'$ define

$$\partial_M = \overline{\bigcup_{n\in M} A_n} - \bigcup_{n\in M} A_n.$$

Note that ∂_M is exactly the set of all points of K whose all neighbourhoods intersect infinitely many sets A_n for $n \in M$. It follows that $\partial_{M'} \subseteq \partial_M$, if $M' \subseteq M$ and moreover that $\partial_{M'} \subseteq \partial_M$, if $M' - M$ is finite.
Let $\{N_\xi : \xi < \omega_1\}$ be a family of infinite subsets of N' such that $N_\xi \cap N_{\xi'}$ is finite whenever $\xi \neq \xi'$ for all $\xi \in \omega_1$. We may apply theorem 2.1. to $\{A_n : n \in N_\xi\}$

and $(l_n : n \in N_\xi)$ for each $\xi < \omega_1$, obtaining $b_\xi \subseteq N_\xi$, and a supremum A_ξ like in theorem 3.1. Note that, as a clopen set, A_ξ includes ∂_{b_ξ} and is disjoint from $\partial_{b_\eta - b_\xi}$ for any $\eta \neq \xi$, hence the family $(\partial_{b_\xi})_{\xi<\omega_1}$ is pairwise disjoint, and so, one of its sets, for instance $\partial_{b_{\xi_0}}$ must be null with respect to all the measures μ_n. Since $A_{\xi_0} = \bigcup\{A_n : n \in b_{\xi_0}\} \cup \partial_{b_{\xi_0}}$ we have

$$****) \qquad \mu_n(A_{\xi_0}) = \mu_n(\bigcup_{m\in b_{\xi_0}} A_m)$$

$A_{\xi_0} = A$ is clopen and so χ_A is continuous, let us analyze $T(\chi_A)$. By ***) and ****) we conclude that if $n \in b = b_{\xi_0}$ we have

$$\begin{aligned}|T(\chi_A)(l_n{}^*)| = |\mu_n(A)| &= |\mu_n(A_n) + \sum\{\mu_n(A_m) : m \neq n,\ m \in b\}| \\ &\geq \varepsilon - \varepsilon/3 = 2\varepsilon/3.\end{aligned}$$

But if $n \in N_{\xi_0} - b$, where $a = a_{\xi_0}$, we have

$$|T(\chi_A)(l_n{}^*)| = |\mu_n(A)| = |\sum\{\mu_n(A_m) :\ m \in b\}| \leq \varepsilon/3.$$

Since $T(\chi_A)$ is continuous we have that the closures of the sets $\{l_n{}^* : n \in b\}$ and $\{l_n{}^* : n \in N_\xi - b\}$ are disjoint. This contradicts d) of theorem 2.1. and completes the proof of lemma 3.2. □

Theorem 3.3. *There is a separable compact zero-dimensional space K such that no proper subspace and no proper quotient of $C(K)$ is isomorphic to $C(K)$, in particular the hyperplanes of $C(K)$ are not isomorphic to the entire $C(K)$.*

Proof. Apply theorems 2.4 and lemma 3.2. □

4. Adding suprema of pairwise disjoint families of functions on connected spaces

First, let us make elementary observations and introduce some notation about bounded pairwise disjoint sequences of continuous functions. Infinite sums of functions are always considered pointwise. We consider the lattice $C(K)$ with the usual pointwise order.

Lemma 4.1. *Let K be a compact, Hausdorff space and let $(f_n)_{n\in N}$ be a bounded pairwise disjoint sequence of continuous functions from K into* [0, 1].

a) $f \in C(K)$ is $\sup\{f_n : n \in N\}$ in the lattice $C(K)$ if and only if

$$\Delta(f, (f_n)_{n\in N}) = \{x \in K : \sum_{n\in N} f_n(x) \neq f(x)\}$$

is nowhere dense in K.

b) $\sum_{n\in N} f_n$ *is well-defined and continuous in the dense open set*

$$D((f_n)_{n\in N}) = \bigcup\{U : U \text{ is open and } \{n : supp(f_n) \cap U \neq \emptyset\} \text{ is finite}\}.$$

Proof. For a), first suppose that $f = \sup\{f_n : n \in N\}$, in particular $f \geq f_n \geq 0$ for all $n \in N$. It is clear that there cannot be any $x \in K$ where $0 < f_n(x) < f(x)$ for some n, i.e., $\Delta(f, (f_n)_{n\in N}) \subseteq \{x \in K : \forall n \in N \ f_n(x) = 0\}$. Now, note that if $\Delta(f, (f_n)_{n\in N})$ were not nowhere dense, then, there would be an open set U such that $f_n(x) = 0$ for all $n \in U$ and $f(x) > 0$ for some $x \in U$. A slight modification of f would contradict the fact that f is the supremum. Conversely if $g \geq f_n$ for all $n \in N$ but $f \not\leq g$, then there is an $\varepsilon > 0$ and an open U such that $g(x) \leq f(x) - \varepsilon$ for $x \in U$, then $U \subseteq \Delta(f, (f_n)_{n\in N})$.

In b) the density of $D((f_n)_{n\in N})$ follows from the fact that if an open U intersects some $supp(f_n)$, then there is $x \in U$ such that $f_n(x) \neq 0$ and consequently there is an open $V \subseteq U \cap D((f_n)_{n\in N})$, since $(f_n)_{n\in N}$ is pairwise disjoint.

The continuity follows from the fact that $\sum_{n\in N} f_n$ is locally a finite sum of continuous functions in $D((f_n)_{n\in N})$. □

In this section we discuss an operation on connected compact spaces that is analogous to adding $\bigvee_{n\in N} A_n$ for a pairwise disjoint sequence $(A_n)_{n\in N}$ to Boolean algebras. The latter played an important role in the construction leading to theorem 3.1. and will be used here as the motivation. To understand topologically the process of adding $\bigvee_{n\in N} A_n$ to a Boolean algebra, we need to use the Stone duality to translate the natural notion like $\bigvee_{n\in N} A_n$ into, at the first sight, less natural topological concepts.

Recall that if X is an element of a Boolean algebra, then $[X]$ denotes the clopen basic set of the Stone space of the algebra consisting of all ultrafilters which contain X. We may use $[X]_K$ if the Boolean algebra and its Stone space K is to be specified. If K is the Stone space of a Boolean algebra $\mathcal{A}$ then the Stone space L of the algebra $\mathcal{B}$ generated by a new element A over $\mathcal{A}$ can be interpreted as a subset of $K \times \{0, 1\}$ defined by

$$L = \{(x, 0) : x \in K : x \cup \{-A\} \text{ extends to an ultrafilter of } \mathcal{B}\} \cup$$

$$\cup \{(x, 1) : x \in K : x \cup \{A\} \text{ extends to an ultrafilter of } \mathcal{B}\}.$$

That is, if π stands for the projection on the first coordinate, i.e., from L onto K, and if $x \in K$, then $\pi^{-1}(\{x\})$ is sometimes only the singleton $\{(x, 1)\}$, sometimes only the singleton $\{(x, 0)\}$ and sometimes $\{(x, 0), (x, 1)\}$, depending on the relation between x and A. In fact, we are in the first case if and only if there is $A' \in x$ such that $A' \leq A$ in the Boolean order in $\mathcal{B}$, we are in the second case if and only if there is $A' \in x$ such that $A' \leq -A$ and we are in the third case if none of the above holds.

Suppose that $(A_n)_{n\in N}$ is a pairwise disjoint sequence of elements of $\mathcal{A}$ and $\bigvee_{n\in N} A_n$ does not exist in $\mathcal{A}$, however $\bigvee_{n\in N} A_n = A$ appears in $\mathcal{B}$ as above.

Simple applications of the Stone duality can be summarized in the following list which uses the notation K, L and π as above:

1) $(\chi_{[A_n]_K})_{n\in N}$ has no supremum in $C(K)$.
2) $\chi_{[\bigvee_{n\in N} A_n]_L}$ is the supremum of $(\chi_{[A_n]_L})_{n\in N}$ in $C(L)$,
3) $\chi_{[A_n]_K} \circ \pi = \chi_{[A_n]_L}$,
4)

$$\overline{\bigcup\{[A_n]_K : n \in N\}} - \bigcup\{[A_n]_K : n \in N\}$$

is the set of all points in K whose preimages under π have two elements, it is closed and nowhere dense in K and its preimage under π is nowhere dense in L.

5) L is the closure in $K \times \{0, 1\}$ of the graph of a continuous function $f : dom(f) \to \{0, 1\}$ which is given by $f(x) = 1$ for $x \in \bigcup\{[A_n]_K : n \in N\}$ and $f(x) = 0$ for $x \notin \overline{\bigcup\{[A_n]_K : n \in N\}}$; the domain of f is the complement of the set in 4).

This motivates the following definition which should be traced back to [Fe].

Definition 4.2. *Suppose that K is a compact space, $L \subseteq K \times [0, 1]$ and $(f_n)_{n\in N}$ is a pairwise disjoint sequence of continuous functions from K into $[0, 1]$. We say that L is an extension of K by $(f_n)_{n\in N}$ if and only if L is the closure of the graph of the restriction $(\sum_{n\in N} f_n)|D((f_n)_{n\in N})$. Moreover we say that L as above is a strong extension of K by $(f_n)_{n\in N}$ if and only if the graph of $\sum_{n\in N} f_n$ is a subset of L.*

Lemma 4.3. *Suppose that K is a compact space, that $(f_n)_{n\in N}$ is a pairwise disjoint sequence of continuous functions from K into $[0, 1]$, that L is an extension of K by $(f_n)_{n\in N}$ and π is the projection from L onto K, then the following hold:*

a) If $M \subseteq K$ is nowhere dense in K, then $\pi^{-1}[M]$ is nowhere dense in L.
b) There is $sup\{f_n \circ \pi : n \in N\}$ in $C(L)$.

Proof. a) Let $(x, t) \in L$ and let $U \times V$ be an open neighbourhood of (x, t) in $K \times [0, 1]$. Let (x', t') be in the graph of $\sum_{n\in N} f_n|D((f_n)_{n\in N})$ and in $U \times V$. Since $D((f_n)_{n\in N})$ is open dense where $\sum_{n\in N} f_n$ is continuous by 4.1. b), there is a non-empty open $U' \subseteq U \cap D((f_n)_{n\in N})$ such that $(\sum_{n\in N} f_n)[U'] \subseteq V$ and $M \cap U' = \emptyset$. So $(U' \times V) \cap L$ is nonempty open subset of $U \times V$ disjoint from $\pi^{-1}[M]$ as required.

b) Let $f(x, t) = t$ for $(x, t) \in L$. As the restriction of a continuous function, f is continuous. It is clear, by the definition of L, that

$$f(x, t) = \sum_{n\in N} f_n(x) = \sum_{n\in N} f_n(\pi(x, t))$$

if $x \in D((f_n)_{n\in N})$ and $(x,t) \in L$. By 4.3 a) and 4.1. b) the remaining points of L, i.e., $\pi^{-1}[K - D((f_n)_{n\in N})]$ form a nowhere dense set. Hence f is the supremum of $(f_n \circ \pi : n \in N)$ in $C(L)$ by 4.1. a). □

In higher dimensions there appear extensions as above which are not strong. The reason we consider strong extensions is the following:

Lemma 4.4. *Suppose that K is compact and connected and $(f_n : n \in N)$ is a sequence of pairwise disjoint continuous functions from K into* $[0, 1]$. *Then the closure of the graph of $\sum_{n\in N} f_n$ is connected subspace of $K \times [0, 1]$. In particular, if L is a strong extension of a connected K by $(f_n)_{n\in N}$, then L is connected.*

Proof. Let M be the closure of the graph of $\sum_{n\in N} f_n$. Suppose that it is not connected. By the compactness, there are two disjoint open subsets U_1, U_2 of $K \times [0, 1]$ such that $M \subseteq U_1 \cup U_2$ and $M \cap U_1 \neq \emptyset \neq M \cap U_2$. Let M_n be the graph of $\sum_{i<n} f_i$. It is clear that M_n intersect both U_1 and U_2 for n large enough, so we may assume without loosing generality that it happens for all $n \in N$. Note that M_n's are compact and connected, as graphs of continuous functions on K, since the projection from the graph is a homeomorphism ([E], 2.3). Hence $F_n = M_n - (U_1 \cup U_2) \neq \emptyset$. Note that F_n's are compact and decreasing and included in $\{x : \sum_{i<n} f(x) = 0\} \times \{0\}$, since if $f_i(x) > 0$, then $(x, f_i(x)) \in M$. Hence the F_n's have nonempty intersection which is included in M which contradicts the fact that $M \subseteq U_1 \cup U_2$.

The second part of the lemma is a consequence of the fact that if the graph of $\sum_{n\in N} f_n$ is included in L, then it its closure is L. □

Note that we consider extensions like in definition 4.2. and not the closures of the graph of $\sum_{n\in N} f_n$, because the latter in general may not satisfy 4.3. a) The following lemma is a tool for obtaining strong extensions.

Lemma 4.5. *Suppose that K is compact, and of topological weight $\kappa < 2^\omega$ and $X_1, X_2 \subseteq K$ be two disjoint relatively discrete subsets of K such that $\overline{X_1} \cap \overline{X_2} \neq \emptyset$. Suppose that $(f_n)_{n\in N}$ is pairwise disjoint sequence of continuous functions from K into* $[0, 1]$ *and $(N_\xi : \xi < 2^\omega)$ is a family of infinite subsets of N such that $N_\xi \cap N_{\xi'}$ is finite for any $\xi \neq \xi'$. For an infinite $b \subseteq N$ let $K(b)$ be the extension of K by $(f_n)_{n\in b}$ and let $\pi(b)$ denote the projection from $K(b)$ onto K. Let*

$$X_i' = \{(x, 0) : x \in X_i,\ x \notin D((f_n)_{n\in b})\} \cup$$
$$\cup \{(x, t) : x \in X_i,\ x \in D((f_n)_{n\in b}),\ (\sum_{n\in b} f_n)(x) = t, \}.$$

There is $A \subseteq 2^\omega$ of cardinality not bigger than κ such that the following hold for all $\xi \in 2^\omega - A$ and all infinite $b \subseteq N_\xi$:

a) $K(b)$ is a strong extension of K by $(f_n)_{n\in b}$.
b) $\overline{X_1'} \cap \overline{X_2'} \neq \emptyset$, where the closures are taken in $K(b)$.

Proof. (a) suppose that a) does not hold, i.e., there is a subset $A_1 \subseteq 2^\omega$ of cardinality bigger than κ such that a) fails for some $b_\xi \subseteq N_\xi$ for all $\xi \in A_1$. It is clear that if (x_ξ, t) witnesses the fact that the difference between the graph of $\sum_{n\in b_\xi} f_n$ and the closure of the graph of $(\sum_{n\in b_\xi} f_n)|D(\sum_{n\in b_\xi} f_n) = K(b_\xi)$ is non-empty, then $t = 0$. Note also that x_ξ is not in the closure of $\bigcup\{supp(f_n) : n \notin b_\xi\}$ since $(\bigcup\{supp(f_n) : n \notin b_\xi\}) \times \{0\}$ is included in the closure of the graph of $(\sum_{n\in b_\xi} f_n)|D(\sum_{n\in b_\xi} f_n)$. Let $U_\xi \subseteq X$ be an open neighbourhood of x_ξ disjoint from $\bigcup\{supp(f_n) : n \notin b_\xi\}$. But x_ξ cannot be in $D(\sum_{n\in b_\xi} f_n)$, hence x_ξ is in the closure of $\bigcup\{supp(f_n) : n \in b_\xi - F\}$ for every finite $F \subseteq N$. Thus we can conclude that $x_\xi \in U_\eta$ if and only if $\xi = \eta$, and so $\{x_\xi : \xi \in A_1\}$ is a discrete subspace of X of cardinality bigger than κ which is impossible since the weight of X is κ. This completes the proof of a).

(b) It is clear that if $x \in D(\sum_{n\in b} f_n) \cap \overline{X_1} \cap \overline{X_2}$ then $(x, t) \in \overline{X_1}' \cap \overline{X_2}'$ where $(\sum_{n\in b} f_n)(x) = t$.

So consider $x \in \overline{X_1} \cap \overline{X_2}$ such $f_n(x) = 0$ for all $n \in N$. In this case by a), if $\xi \notin A_1$ then $(x, 0) \in K(b_\xi)$, so it is enough to prove that $(x, 0) \in \overline{X_1'} \cap \overline{X_2'}$ where the closures are taken in $K(b_\xi)$ for $\xi \in 2^\omega - A$ for some $A \supseteq A_1$ of cardinality κ.

If there is a finite $F \subseteq N$ such that x is in the closure of $\{x \in X_i : \sum_{n\in F} f_n(x_i) > 0\}$, then $(x, 0)$ is in the closure of X_i' relative to any $K(b_\xi)$. So we may assume that for any open neighbourhood U of x, for every finite set $F \subseteq N$ there are points $x_i \in X_i \cap U$ such that $\sum_{n\in F} f_n(x_i) = 0$. Thus for any open neighbourhood U of x for any two $\xi_1, \xi_2 \in A$ there are points $y \in U \cap X_1$ such that $\sum_{n\in b_{\xi_1}\cap b_{\xi_2}} f_n(y) = 0$. So for all but one $\xi \in \kappa - A_1$ we have $\sum_{n\in b_\xi} f_n(y) = 0$ for some $y \in U \cap X_1$. Applying the same argument to X_2 and all κ basic neighbourhoods of x we conclude that for all but κ many ξ's in $\kappa - A_1$ and each neighbourhood of x there are points y of X_1 and X_2 such that $\sum_{n\in b_\xi} f_n(y) = 0$. But this means that $(x, 0)$ is in $\overline{X_1'} \cap \overline{X_2'}$ which completes the proof of b). □

This completes the description of the techniques used at successor stages of the transfinite inductive construction. At limit stages, we again use the analogy from the zero-dimensional construction. Taking unions of subalgebras at limit ordinals corresponds to taking topological inverse limits (see [E]), i.e.,

$$K_\lambda = \{x \in [0, 1]^\lambda : \forall\alpha < \lambda\ x|\alpha \in K_\alpha\},$$

if λ is a limit ordinal. As the goal of the successor step was to add the supremum of a pairwise disjoint bounded sequence of positive functions, it would be desirable not too loose this supremum in the following stages. In general it is possible when passing from $(f_n)_{n\in N}$ to $(f_n \circ \pi)_{n\in N}$ however it cannot happen in our case. It also turns out that the property of $[0, 1] \times [0, 1]$ that it is connected after removing any finite subset is preserved by strong extensions. This is useful in 2.5. $\pi_{\alpha,\beta}$ will denote the projection from K_β onto K_α where K_α, K_β are as below. We will omit β if it is clear from the context.

Lemma 4.6. *Suppose that β is an ordinal and $(K_\alpha)_{\alpha\leq\beta}$ is such that $K_1 = [0,1]^2$, $K_\alpha \subseteq [0,1]^\alpha$ is compact, $\pi_\alpha[K_\alpha] = K_{\alpha'}$ for $\alpha' \leq \alpha \leq \beta$, the extensions at limits are inverse limits and the extensions from K_α to $K_{\alpha+1}$ are strong extensions by pairwise disjoint sequences of continuous functions into* $[0,1]$. *Then*

a) If $f, f_n \in C(K_\alpha)$ for $n \in N$ and $\alpha \leq \beta$ are such that

$$f = \sup\{f_n : n \in N\},$$

then

$$f \circ \pi_{\alpha,\beta} = \sup\{f_n \circ \pi_{\alpha,\beta} : n \in N\}.$$

b) $K_\beta - F$ is connected whenever $F \subseteq K_\beta$ is finite.

Proof. (a) The proof is by induction in β. If β is a limit ordinal, we note that if $\Delta(f, \{f_n : n \in N\})$ of 4.1. a) were dense in some basic open set, the lemma would fail at some ordinal less than β. Now suppose that $\beta = \beta' + 1$ and the lemma holds for β', i.e., we may assume that $\alpha = \beta'$. It is enough to apply lemma 4.3 a) to $\Delta(f, \{f_n : n \in N\})$ which is nowhere dense in K_α by lemma 4.1. b). Applying 4.1. b) again in $K_{\alpha+1}$ implies the desired conclusion.

(b) It is enough to prove that any finite subset $F \subseteq K_\beta$ is included in a nowhere dense subset of K whose complement includes a dense connected space. Indeed then the closure of this set is dense in $K_\beta - F$, and hence $K_\beta - F$ is connected. We will work with nowhere dense subset of K_α's of the form $N_\alpha = \pi^{-1}_{1,\alpha}[K_1 - \pi_{1,\beta}[F]]$. One proves that they are nowhere dense by induction on α. For successor α it follows from lemma 4.3. a) and for a limit α one proves it using the fact that basic open sets are determined by finitely many coordinates. So, we are left with proving for $1 \leq \alpha \leq \beta$ that $K_\alpha - N_\alpha$'s have dense connected subset.

By induction on α we prove that there are $M^n_\alpha \subseteq K_\alpha$ such that:

1) $\pi_{\alpha',\alpha}[M^n_\alpha] = M^n_{\alpha'}$ for $\alpha' \leq \alpha \leq \beta$,
2) M^n_α's are compact and connected,
3) $M^n_\alpha \cap N_\alpha = \emptyset$, $M^n_\alpha \subseteq M^{n+1}_\alpha$,
4) $\bigcup_{n\in N} M^n_\alpha$ is dense in $K_\alpha - N_\alpha$.

We start by choosing M^n_1 to satisfy 2) - 4) and such that $[0,1]^2 - \bigcup_{n\in N} M^n_\alpha$ is the finite set $\pi_{1,\beta}[F] = N_1 \subseteq [0,1]^2$.

Suppose we are given M^n_α's as above. The extension from K_α to $K_{\alpha+1}$ is a strong extension by some pairwise disjoint bounded sequence of continuous functions $(f_i)_{i\in N}$ from K_α into $[0,1]$. Define $M^n_{\alpha+1}$ to be the closure of the graph of $\sum_{i\in N} f_i|M^n_\alpha$. Since the extensions are strong, we have that $M^n_{\alpha+1} \subseteq K_{\alpha+1}$. By lemma 4.4., we conclude that $M^n_{\alpha+1}$ is compact and connected, and $\pi_{\alpha+1,\alpha}[M^n_{\alpha+1}] = M^n_\alpha$. It is also clear that $M^n_\alpha \cap N_\alpha = \emptyset$ and $M^n_{\alpha+1} \subseteq M^{n+1}_{\alpha+1}$. To prove that $\bigcup_{n\in N} M^n_{\alpha+1}$ is dense in $K_{\alpha+1} - N_{\alpha+1}$, note that $D((f_i)_{i\in N}) \cap \bigcup_{n\in N} M^n_\alpha$ is a dense subset of $D((f_i)_{i\in N})$ by 4) of the inductive assumption, hence the graph of $\sum_{i\in N} f_i$ restricted to it is dense in $K_{\alpha+1}$, but the graph of this restriction is included in $\bigcup_{n\in N} M^n_{\alpha+1}$.

To preserve 1) - 4) at a limit α, just consider M^n_α to be the inverse limit of $M^n_{\alpha'}$'s for $\alpha' < \alpha$. Now, 1) and 3) follow from the definitions of the sets. 2) follows from the fact that inverse limits of compact connected spaces are compact and connected (see [E]). 4) follows from the inductive assumption and from the fact that the failure of the density is witnessed by some basic open set., i.e., determined by finitely many coordinates. □

5. Construction of a connected compact space

In this section we construct a connected compact space which has some analogous properties to the zero-dimensional construction of section 3, the latter can serve as a motivating example for the present construction. Theorem 3.1. corresponds to the following:

Theorem 5.1. *There is an infinite, compact, connected separable space K with a countable dense set $Q = \{q_n : n \in N\}$ having the following properties:*

i) Given:

a) a bounded sequence $(f_n : n \in N)$ of pairwise disjoint continuous functions from K into $[0, 1]$,

b) a relatively discrete (i.e., no point is in the closure of the remaining points) sequence $(q_{l_n} : n \in N)$ of distinct points of Q such that $f_m(q_{l_n}) = 0$ for $n, m \in N$, there is an infinite $b \subseteq N$ such that

c) $\{f_n : n \in b\}$ has its supremum f in the lattice $C(K)$,

d) the intersection of the sets $\overline{\{q_{l_n} : n \in N - b\}}$ and $\overline{\{q_{l_n} : n \in b\}}$ is nonempty.

ii) For every finite $F \subseteq K$ the subspace $K - F$ is connected.

Proof. We construct by transfinite induction compact connected spaces $K_\alpha \subseteq [0, 1]\times[0, 1]^\alpha$ for $1 \le \alpha \le 2^\omega$ taking inverse limits (see section 4) at limit ordinals. Along this construction we will also construct dense sets $Q_\alpha = \{q_n|\alpha : n \in N\}$ of K_α. We require that $K_1 = [0, 1] \times [0, 1]$ and Q_1 are the points of the square $[0, 1] \times [0, 1]$ with both rational coordinates.

At a successor stage, $K_{\alpha+1}$ is obtained from K_α using a strong extension by an appropriate pairwise disjoint sequence $(g_n)_{n\in N}$ of positive continuous functions (see section 4) so that c) of theorem 5.1. is taken care of. At each stage we will have to preserve some cases of d).

At the beginning of the construction we fix an enumeration:

$$(f_n(\alpha), l_n(\alpha))_{\alpha<2^\omega, n\in N}$$

such that for every $\alpha' < 2^\omega$ and for every pair $((f_n)_{n\in N}, (l_n)_{n\in N})$ consisting of a countable bounded sequence of positive continuous functions $f_n : [0, 1]^{2^\omega} \to$

$[0, 1]$ and of an increasing sequence of integers $(l_n)_{n\in N}$ there is $\alpha' < \alpha < 2^\omega$ such that $f_n(\alpha) = f_n$ and $l_n(\alpha) = l_n$ for all $n \in N$. This enumeration exists since, the separability of $[0, 1]^{2^\omega}$ implies that there are $2^\omega \times 2^\omega$ such pairs and $2^\omega \times 2^\omega = 2^\omega$ implies that 2^ω can be divided into 2^ω pairwise disjoint sets, each cofinal in 2^ω.

At stage $\alpha < 2^\omega$ we are given K_α and sequences $(b_\beta)_{\beta<\alpha}$, $(a_\beta)_{\beta<\alpha}$ of subsets of N such that for every $\beta < \alpha$ we have $b_\beta \subseteq a_\beta$, $\{q_n|\alpha : n \in a_\beta\}$ is relatively discrete in K_α and

$$*)\qquad \overline{\{q_n|\alpha : n \in b_\beta\}} \cap \overline{\{q_n|\alpha : n \in a_\beta - b_\beta\}} \neq \emptyset,$$

where the closures are taken in K_α.
The successor stage from α to $\alpha + 1$ is non-trivial, if $(f_n(\alpha) : n \in N)$ well-defines functions on K_α which satisfy a) and b) of 3.1. together with $\{q_{l_n(\alpha)} : n \in N)$. That is, we require that $\{q_{l_n}|\alpha : n \in N\}$ is relatively discrete and that there is a pairwise disjoint sequence $(g_n : n \in N)$ of continuous functions from K_α into $[0, 1]$ such that $g_n(q_{l_m}|\alpha) = 0$ for every $n, m \in N$ and $f_n(\alpha)(y) = g_n(x)$, whenever $x \in K_\alpha$ and $y|\alpha = x$. Otherwise we call the case trivial and define $K_{\alpha+1} = K_\alpha \times \{0\}$ which is homeomorphic to K_α and $q_n|\alpha + 1 = q_n|\alpha^\frown 0$.

So assume that we are in a non-trivial successor stage from α to $\alpha + 1$. We will choose $a_\alpha \subseteq N$ so that for any choice of an infinite $b \subseteq a_\alpha$ after extending K_α by $(g_n : n \in b)$ the conditions *) are preserved in $K_\alpha(b)$ for all $\beta < \alpha$ and the extension is strong (see section 4). Let $(N_\xi : \xi < 2^\omega)$ be a family of infinite subsets of N such that $N_\xi \cap N_{\xi'}$ is finite for $\xi \neq \xi'$. Consider $X_1(\beta) = \{q_n|\alpha : n \in a_\beta - b_\beta\}$ and $X_2(\beta) = \{q_n|\alpha : n \in b_\beta\}$. By lemma 4.5 there are sets $A(\beta) \subseteq 2^\omega$ of cardinality not bigger than $|\alpha|$ which satisfies the lemma 4.5. for $X_1 = X_1(\beta)$ and $X_2 = X_2(\beta)$ for all $\beta < \alpha$. Since $|\alpha| \times |\alpha| = |\alpha| < 2^\omega$, we may conclude that there is $\xi \in 2^\omega$ such that $\xi \notin A(\beta)$ for all $\beta < \alpha$.

First, it means that the extension from K_α to $K_\alpha(b)$ for any infinite $b \subseteq N_\xi$ is a strong extension. Secondly, it means that if we define

$$q_k|\alpha + 1 = q_k|\alpha^\frown x \text{ where } x = \sum_{n\in b} f_n(q_k|\alpha)$$

if $x \in D((f_n)_{n\in b})$ and $x = 0$ otherwise, then *) is satisfied for every $\beta < \alpha$ in the extension $K_\alpha(b)$ (it also serves as the definition of $Q_{\alpha+1}$). So we are left with defining $b_\alpha \subseteq a_\alpha \subseteq N_\xi$ so that *) is satisfied for $\beta = \alpha$ in $K_\alpha(b_\alpha)$. Choose $a_\alpha = \{l_n : n \in N_\xi\}$, since our case is nontrivial, we know that $\{q_n|\alpha : n \in a_\alpha\}$ is discrete in K_α. Now using the fact that K_α has weight less than 2^ω and the fact that in a compact space, disjoint closed sets can be separated by finite unions of basic open sets, one can find b_α such that

$$\overline{\{q_n|\alpha : n \in b_\alpha\}} \cap \overline{\{q_n|\alpha : n \in a_\alpha - b_\alpha\}} \neq \emptyset$$

Since $g_n(q_{l_m}) = 0$ for all $n, m \in N$, we conclude that if $x \in K_\alpha$ is in the above intersection, then $\sum_{n\in b_\alpha} g_n(x) = 0$ and hence

$$(x,0) \in \overline{\{(q_n|\alpha)^\frown 0 : n \in b_\alpha\}} \cap \overline{\{(q_n|\alpha)^\frown 0 : n \in a_\alpha - b_\alpha\}}$$

that is *) is satisfied for $\beta = \alpha$ as well in $K_{\alpha+1} = K_\alpha(b)$.
As we mentioned before, at limit stages we take inverse limits. This completes the description of the construction.

Now, let us verify that the theorem 5.1. holds for $K = K_{2^\omega}$. Suppose $(f_n : n \in N)$ and $(l_n : n \in N)$ are as in the theorem. By the Tietze extension theorem, there is a bounded sequence of continuous $(f'_n : n \in N)$ such that $f'_n|K = f_n$ for each $n \in N$. By a theorem of Mibu (see [Mi] or [CN]), there are countable $X_n \subseteq 2^\omega$ such that if $x|X_n = y|X_n$, then $f'_n(x) = f'_n(y)$. Using the fact that there is no countable set cofinal in 2^ω (see Konig's lemma in [Ku] or [Je]), let $\alpha' = \sup(\bigcup_{n\in N} X_n)$. We have that $\alpha' < 2^\omega$ an that $f_n(x) = g_n \circ \pi_\alpha(x)$ for any $n \in N$ and any $\alpha' < \alpha \leq 2^\omega$ and any $x \in [0,1]^{2^\omega}$ such that $\pi_\alpha(x) \in K_\alpha$ and some pairwise disjoint continuous $g_n : K_\alpha \to [0,1]$. This means that there is an α as above such that $f'_n = f_n(\alpha)$ and $l_n = l_n(\alpha)$. By the construction the extension from K_α to $K_{\alpha+1}$ is strong and $(g_n \circ \pi_\alpha)_{n\in b_\alpha}$ has the supremum in $K_{\alpha+1}$ by lemma 4.5. Lemma 4.6 implies that $(g_n \circ \pi_\alpha)_{n\in b_\alpha}$ has the supremum in K_{2^ω}. On the other hand the α-case of *) which is preserved at each successor stage of the construction, and hence at all stages (again, separations of closed disjoint sets can be done by finitely many basic open sets, i.e., which use finitely many coordinates) implies d) of theorem 5.1.

The fact that ii) is satisfied follows from lemma 4.6 b) and the construction. One easily proves by induction on $\alpha \leq 2^\omega$ that Q_α is dense in K_α. □

Note that K as in 5.1 is topologically quite rigid, i.e., if $F : K \to K$ is continuous, then F is constant or the identity. To see this, if F is not constant, nor the identity, by the connectedness, there must be a relatively discrete infinite sequence of distinct points $y_n \in K$ and $x_n \neq y_n$ such that $F(x_n) = y_n$. Take $f_n : K \to [0,1]$ such that $f(x_n) = 0$, $f(y_n) = 1$ and f_n's are pairwise disjoint and continuous. Following the argument as at the begining of the lemma below, one can assume that $f_m(x_n) = 0$ for all $n, m \in N$ and that $x_n = q_{l_n}$'s are relatively discrete. Now an application of 5.1. gives that the closures of $\{y_n : n \in b\}$ and $\{y_n : n \in N - b\}$ (by taking $f^{-1}[[2/3,1]]$ and $f^{-1}[[0,1/3]]$) are disjoint which contradicts the fact that the closures of $\{q_{l_n} : n \in b\}$ and $\{q_{l_n} : n \in N - b\}$ are not disjoint. However, in the following lemma, using the notion of a weak multiplier we can transfer this rigidity to the space $C(K)$.

Lemma 5.2. *Every operator $T : C(K) \to C(K)$ is a weak multipliplier.*

Proof. Suppose that a bounded, linear operator $T : C(K) \to C(K)$ is not a weak multiplier, i.e., that there exist a sequence of pairwise disjoint elements $f_n \in C(K)$ with ranges in $[-1,1]$ and there is $\varepsilon > 0$ and points $x_n \in K$ such that $f_n(x_n) = 0$ for all $n \in N$ and $|T(f_n)(x_n)| > \varepsilon$ for infinitely many n's. We may assume without loss of generality that it happens for all $n \in N$ and, by the density of Q, that $x_n = q_{l_n}$ for some $l_n \in N$.

Since finite sums of f_n's are uniformly bounded, if q_{l_n} were constant for infinitely many n's, we would get a contradiction with the fact that T is bounded. Thus, we may assume without loss of generality that $(l_n : n \in N)$ is a strictly increasing sequence of positive integers. As K is compact, any sequence of distinct points contains a relatively discrete subsequence, hence we may assume without loss of generality that $\{q_{l_n} : n \in N\}$ is relatively discrete.

We may assume without loss of generality that $f_m(q_{l_n}) = 0$ for $n, m \in N$: if there is one k_0 such that $f_{k_0}(q_{l_n}) \neq 0$ for n's from an infinite set $N' \subseteq N$, thin-out N to $N' - \{k_0\}$ and use the disjointness of f_n's. Otherwise, one can construct an infinite subsequence as required by induction. Finally, by considering multiples of $-\min(f_n, 0)$ and $\max(f_n, 0)$ one may assume without loss of generality that all functions f_n have ranges included in $[0, 1]$.

Let μ_n be the Radon measures on K which correspond (see [Se], §18.), by the Riesz representation theorem, to the linear bounded functionals ϕ_n on $C(K)$ given by the relation

$$\phi_n(f) = T(f)(q_{l_n}) = \int_K f d\mu_n$$

which holds for all $f \in C(K)$, i.e., $\mu_n = T^*(\delta_{q_{l_n}})$ and so, $(\mu_n)_{n\in N}$ is a bounded sequence in $M(K)$. Hence, we have $|\int f_n d\mu_n| > \varepsilon$. Now we may find an infinite $N' \subseteq N$ such that for $n \in N'$ we have

$$**) \qquad \sum\{|\int f_m d\mu_n| : n \neq m, m \in N'\} < \varepsilon/3.$$

This is a lemma of Rosenthal (see [Ro] lemma 1.1. also in the stronger version that we are using see [Di] page 82.) applied to countably additive atomic measures ν_n on subsets of N determined by $\nu_n(\{m\}) = \int f_m d\mu_n$. They are uniformly bounded since both $(\mu_n)_{n\in N}$ and $(f_n)_{n\in N}$ are bounded.
We need to do one more thinning-out to obtain an infinite subset $N'' \subseteq N'$ such that for any infinite $b \subseteq N''$ the following holds:

$$***) \qquad \int \sup\{f_m : m \in b\} d\mu_n = \int \sum_{m\in b} f_m d\mu_n.$$

whenever $\sup\{f_m : m \in b\}$ exists in $C(K)$. For this it will be enough to prove that there is an infinite $N'' \subseteq N'$ such that for every infinite $b \subseteq N''$ such that $\sup\{f_m : m \in b\}$ exists, we have

$$\int [\sup\{f_m : m \in b\} - \sum_{m\in b} f_m(x)] d\mu_n = 0$$

for all $n \in N$. Define $f_b = \sup\{f_m : m \in b\} - \sum_{m\in b} f_m(x)$, if the supremum exists. Suppose that there is no infinite $N'' \subseteq N'$ satisfying ***). Let $\{N_\xi : \xi < \omega_1\}$ be a family of infinite subsets of N' such that $N_\xi \cap N_{\xi'}$ is finite whenever $\xi \neq \xi'$ for all $\xi, \xi' \in \omega_1$. By the assumption, there are infinite $b_\xi \subseteq N_\xi$ with $\int f_{b_\xi} d\mu_n \neq 0$ for some $n \in N$ (the supremum exists for some b_ξ by the properties of K). One $n \in N$ works for uncountably many ξ's. Since uncountably many disjoint Borel sets cannot be non-null with respect to a Radon measure, to get a contradiction, it is enough to prove the following:

Claim. The family $\{f_{b_\xi} : \xi \in \omega_1\}$ is pairwise disjoint family of Borel functions on K.

Proof of the Claim. Note that since $(f_n : n \in b_\xi)$ is a bounded sequence of pairwise disjoint positive continuous functions which possess its supremum,

$$\sup(f_m : m \in b_\xi) = \sup(f_m : m \in b_\xi - F) + \sum_{m\in F\cap b_\xi} f_m$$

for any finite $F \subseteq b_\xi$. This implies that $f_{b_\xi} = f_{b_\xi - F}$ for any finite $F \subseteq N$. On the other hand $f_{b_\xi - b_{\xi'}} \leq \sup(f_m : m \in b_\xi - b_{\xi'})$, $f_{b_{\xi'} - b_\xi} \leq \sup(f_m : m \in b_{\xi'} - b_\xi)$ and the functions $\sup(f_m : m \in b_\xi - b_{\xi'})$ and $\sup(f_m : m \in b_{\xi'} - b_\xi)$ are disjoint for distinct $\xi, \xi' < \omega_1$. This completes the proof of the claim.
Now we may apply theorem 5.1. to $\{f_n : n \in N''\}$, $(l_n : n \in N'')$, $\{\mu_n : n \in N''\}$ and ε where N'' is as in ***), obtaining b and f as in theorem 5.1. Let us analyze the function $T(f)$. By **) and ***) we conclude that if $n \in b$ we have

$$|T(f)(q_{l_n})| = |\int f d\mu_n| = |\int f_n d\mu_n + \int \sum\{f_m : m \neq n,\ m \in b\} d\mu_n| \geq$$
$$\geq \varepsilon - \varepsilon/3 = 2\varepsilon/3.$$

But if $n \in N - b$, we have

$$|T(f)(q_{l_n})| = |\int \sum_{n\in b} f_n d\mu_n| \leq \varepsilon/3.$$

Since $T(f)$ is continuous we have that the closures of the sets $\{q_{l_n} : n \in b\}$ and $\{q_{l_n} : n \in N - b\}$ are disjoint. This contradicts d) of theorem 5.1. and completes the proof of lemma 5.2. □

Lemma 5.3. *There is a compact, connected separable space K, such that $C(K)$ is an indecomposable Banach space, whose hyperplanes are non-isomorphic to the entire space and which is non-isomorphic with any $C(K')$ for K' zero-dimensional.*

Proof. Apply 5.2. and 2.5, 2.6 and 2.7. □

6. $C(K)$'s where all operators are of the form $gI + S$

In this section we assume the continuum hypothesis and describe how to construct K zero-dimensional or connected where besides all the properties obtained in section 3 or 5 all operators on $C(K)$ are of the form $gI+S$, where $g \in C(K)$ and S is weakly compact or equivalently strictly singular. As explained in the introduction, this is in a sense the minimal possible space of operators for a Banach spaces $C(K)$. Also this property implies that the hyperplanes of the $C(K)$ are non-isomorphic to the entire $C(K)$, and in the connected case that $C(K)$ is indecomposable and non-isomorphic to $C(K')$ for K' zero-dimensional. The hyperplane problem form $C(K)$ spaces and the problem whether all spaces $C(K)$ are isomorphic to a $C(L)$ for an L zero-dimensional can be solved using the constructions presented in the previous sections, however assuming CH, the proofs of these results are simpler. CH can be removed from these simpler arguments by working, unlike we below, with certain nonseparable K's as shown in [Pl]. We do not know if obtaining a subspace of l_∞ of the form $C(K)$ with few operators in the sense as above can be done without any special set-theoretic assumption.

Theorem 6.1. *Assume the continuum hypothesis. There is a compact connected (resp. zero-dimensional) K such $C(K)$ can be isometrically embedded into l_∞ and every bounded operator $T : C(K) \to C(K)$ is of the form $gI + S$ where $g \in C(K)$ and S is weakly compact or equivalently strictly singular.*

The rest of this section is devoted to the sketch of the proof of this theorem. In the light of theorem 2.7 and lemma 2.8 the crucial observation is the following:

Lemma 6.2. *Suppose that K is metric and compact, and has a dense countable subset $Q = \{q_n : n \in N\}$. Suppose that U_1, U_2 are open subsets of K with $\overline{U_1}\cap\overline{U}_2$ nonempty. Then there exist a sequence $(f_n)_{n\in N}$ of pairwise disjoint continuous functions $f_n : K \to [0, 1]$ and strictly increasing sequences $(l_n(i) : n \in N)$ and infinite, co-infinite sets $b(0), b(1) \subseteq N$ for $i = 0, 1$ such that $(q_{l_n(i)} : n \in N)$ is relatively discrete for each $i = 1, 2$ such that for every infinite $b \subseteq N$ in the extension $K(b)$ of K by $(f_n)_{n\in b}$ the following holds: there are distinct $x(0), x(1)$ in $K(b)$ and disjoint closed sets $F_i \subseteq K(b)$ such that*

$$x(i) \in \overline{\pi^{-1}[U_1] \cap \{q'_{l_n(i)} : n \in b(i)\}} \cap \overline{\pi^{-1}[U_2] \cap \{q'_{l_n(i)} : n \in N - b(i)\}},$$

$$\{q'_{l_n(i)} : n \in b(i)\} \subseteq F_i,$$

where $q_j{}' = (q_j, t)$ and $\sum_{n\in b} f_n(q_j) = t$ if $q_j \in D((f_n)_{n\in N})$ or otherwise $t = 0$.

Proof. Let $x \in \overline{U_1} \cap \overline{U}_2$. Using the fact that K is metrizable we can find two sequences of non-empty open sets $(W_n)_{n\in N}$ and $(V_n)_{n\in N}$ and a two strictly increasing sequences of integers $(k_n(i))_{n\in N}$ for $i = 0, 1$ such that:

1) every neighbourhood of x includes all but finitely many W_n's and all but finitely many V_n's.

2) $W_n \subseteq U_1$ and $V_n \subseteq U_2$ for all $n \in N$.
 Let $f_n : K \to [0, 1]$ be such a continuous function that there are x_n, y_n satisfying:
3) $q_{k_n(0)} \in W_n, q_{k_n(1)} \in V_n$,
4) $f_n(q_{k_{2n}(0)}) = f_n(q_{k_{2n}(1)}) = 1$.
5) $supp(f_n) \subseteq W_{2n} \cup V_{2n}$.

Let $b \subseteq N$ be infinite, and let us verify that $(f_n)_{n\in b}$ satisfies the lemma for $l_{2n}(1) = k_{2n}(0)$ and $l_{2n+1}(1) = k_{2n}(1)$ where $b(1)$ is the set of even integers; and for $l_{2n}(0) = k_{2n+1}(0)$ $l_{2n+1}(0) = k_{2n+1}(1)$ where $b(0)$ is the set of even integers. Note that $(q_{k_{2n}(0)}, 1) \in \pi^{-1}[U_1]$, $(q_{k_{2n}(1)}, 1) \in \pi^{-1}[U_2]$ are on the graph on f_n's for $n \in b$, hence $x(1) = (x, 1)$ satisfies the lemma. On the other hand the sets $W_{2n+1} \times \{0\}$ and $V_{2n+1} \times \{0\}$ are included in $\pi^{-1}[U_1] \cap K(b)$ and $\pi^{-1}[U_1] \cap K(b)$ respectivly as well. So $x(0) = (x, 0)$ works as well. To complete the lemma note that $F_0 = K \times [0, 1/3]$ and $F_1 = K \times [2/3, 1]$ work. □

Now we will describe how to modify the constructions from section 3 or 5 to obtain K's of 6.1. As we noted, by theorem 2.7 and lemma 2.8 it is enough to get K's as in theorems 3.1 or 5.1 such that additionally whenever $\overline{U}_1 \cap \overline{U}_2$ is non-empty for two open sets U_1 and U_2, then it has at least two points. Let us concentrate on the connected case K which is more complicated. We assume that the reader is familiar with the details of sections 4 and 5 which will not be repeated.

The construction is as in section 5, by induction on $\alpha < 2^\omega = \omega_1$ we construct an inverse system of K_α's using strong extensions of pairwise disjoint sequences of continuous functions from K_α into $[0, 1]$ where K_α's have countable dense sets Q_α. This time the enumeration

$$(f_n(\alpha), l_n(\alpha))_{n\in N,\alpha\in Even}$$

has the domain $Even$, which is the set of even countable ordinals. At even countable ordinals we proceed as in section 5. We are also given an enumeration

$$(U_1(\alpha), U_2(\alpha))_{\alpha\in Odd}$$

where Odd is the set of odd countable ordinals and such that for every pair U_1, U_2 of open subsets of $[0, 1]^{\omega_1}$ which are countable unions of basic open sets, there are cofinaly many α's such that $U_1(\alpha) = U_1$ and $U_2(\alpha) = U_2$. It is clear that there is such an enumeration.

For $\beta \in Even$ we are also given conditions like in *) of section 5, which have to be preserved at all stages $\alpha < \omega_1$, even or odd. For $\beta < \omega_1$ and odd, we have a couple of conditions like in *), namely we have infinite $b_\beta(i) \subseteq a_\beta(i) \subseteq N$ for $i = 1, 2$ such that

+) $$\overline{\{q_n : n \in b_\beta(i)\}} \cap \overline{\{q_n : n \in a_\beta(i) - b_\beta(i)\}} \neq \emptyset$$

for each $i = 0, 1$, where the closures are taken in K_α and these conditions must be valid at each $\alpha \geq \beta$.

At odd stages $\alpha < \omega_1$, if the closures in K_α of $\pi_\alpha[U_1(\alpha)]$ and $\pi_\alpha[U_2(\alpha)]$ have nonempty intersection, we use lemma 6.2 to find appropriate $(f_n)_{n\in N}$, $(l_n(i))_{n\in N}$ and $b(i)$ for $i = 0, 1$. Then, as in the even case of section 5, we follow lemma 4.5 to find an appropriate infinite $b \subseteq N$ which implies that $K(b)$ is a strong extension and all the conditions from *) and +) can be preserved. For $i = 0, 1$ we define

$$a_\alpha(i) = \{l_n(i) : n \in N\},\ b_\alpha(i) = \{l_n(i) : n \in b(i)\}.$$

Thus we have +) at $\alpha + 1$-th stage. This completes the description of the successor stage of the modification of the construction. Again we take inverse limits at limit ordinals.

The proof of the fact that $K = K_{\omega_1}$ satisfies 5.1. is the same as in section 5. To finish the proof of 6.1., we only need to prove that in K if the closures of two open set have nonempty intersection, then the intersection has at least two points. Note that, as K is separable, for every open set U there exist a countable family $(V_n)_{n\in N}$ of basic open sets included in V such that the closure of V is the same as the closure of the union of $(V_n)_{n\in N}$. So we may focus only on closures of such unions of sequences $(V_n)_{n\in N}$. For any such two sets U_1, U_2, there are cofinally in ω_1 countable α's such that $\overline{U_i} = \overline{U_i(\alpha)}$ for $i = 1, 2$. So, if α is above all the coordinates which determine the basic open sets whose unions are the U_i's we also have

$$++) \qquad \pi_\alpha^{-1}[\pi_\alpha[U_i]] = U_i.$$

If the closures of U_1 and U_2 have non-empty intersection, it also holds for the projections onto K_α. By the construction at such α's we applied lemma 6.2. obtaining two distinct $x(i)$ for $i = 0, 1$ in $K_{\alpha+1}$ which are in the intersections of the closures of the sets $\{q_n : n \in b_{\alpha+1}(i)\}$ and $\{q_n : n \in a_{\alpha+1}(i) - b_{\alpha+1}(i)\}$ respectively for $i = 0, 1$. But these are conditions +) which hold in all K_α's and consequently in K. As the sets $\{q_n|\alpha+1 : n \in a_{\alpha+1}(0)\}$, $\{q_n|\alpha+1 : n \in a_{\alpha+1}(1)\}$ have disjoint closures in $K_{\alpha+1}$ by 6.2., they must have disjoint closures in K, hence ++) guarantees that the intersections of the closures of U_1 and U_2 in K has at least two distinct points as required.

The modification of the zero-dimensional construction is similar, since one can assume that f_n's of lemma 6.2. are characteristic functions of clopen sets. The spaces $C(K)$ can be isometrically embedded into l_∞ since K is separable. □

One can also consider other versions of the space constructed in this paper. One can strengthen the properties of the space, following the method of [Ta], making some of them hereditary with respect to a large class of quotients. This way one can obtain a separable K as in this paper such that for every infinite closed $K' \subseteq K$ the space $C(K')$ has few operators. These results will be published elsewhere.

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