Math. Ann. 329, 161–196 (2004) **Mathematische Annalen**

DOI: 10.1007/s00208-004-0515-4

Non-uniqueness of solutions to the Cauchy problem for semilinear heat equations with singular initial data

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Received: 23 October 2002 / Revised version: 17 October 2003 / Published online: 17 February 2004 –

Abstract. The Cauchy problem for semilinear heat equations with singular initial data

$$w_t = \Delta w + w^p \quad \text{in } \mathbf{R}^N \times (0,\infty) \quad \text{and} \quad w(x,0) = \lambda a\,(x/|x|)\,|x|^{-2/(p-1)} \quad \text{in } R^N \setminus \{0\}$$

is studied, where $N \geq 2$, $\lambda > 0$ is a parameter, and $a \geq 0$, $a \not\equiv 0$. We show that when $p > (N+2)/N$ and $(N-2)p < N+2$, there exists a positive constant $\overline{\lambda}$ such that the problem has two positive self-similar solutions $\underline{w}_\lambda$ and $\overline{w}_\lambda$ with $\underline{w}_\lambda < \overline{w}_\lambda$ if $\lambda \in (0, \overline{\lambda})$ and no positive self-similar solutions if $\lambda > \overline{\lambda}$. Furthermore, for each fixed $t > 0$, $\underline{w}_\lambda(\cdot,t) \to 0$ and $\overline{w}_\lambda(\cdot,t) \to w_0(\cdot,t)$ in $L^\infty(\mathbf{R}^N)$ as $\lambda \to 0$, where w_0 is a non-unique solution to the problem with zero initial data, which is constructed by Haraux and Weissler.

Mathematics Subject Classification (2000): 35K55, 35J60

1. Introduction

We consider the Cauchy problem for semilinear heat equations with singular initial data:

$$w_t = \Delta w + w^p \qquad \text{in } \mathbf{R}^N \times (0,\infty), \tag{1.1}$$

$$w(x,0) = \lambda a\,(x/|x|)\,|x|^{-2/(p-1)} \qquad \text{in } \mathbf{R}^N \setminus \{0\}, \tag*{(1.2)$_\lambda$}$$

where $N \geq 2$, $p > (N+2)/N$, $a : S^{N-1} \to \mathbf{R}$, and $\lambda > 0$ is a parameter. We assume that $a \in L^\infty(S^{N-1})$ and $a \geq 0$, $a \not\equiv 0$. A typical case is $a \equiv 1$.

The equation (1.1) is invariant under the similarity transformation

$$w(x,t) \mapsto w_\mu(x,t) = \mu^{2/(p-1)} w(\mu x, \mu^2 t) \quad \text{for all } \mu > 0.$$

A solution w is said to be self-similar, when $w = w_\mu$ for all $\mu > 0$, that is,

$$w(x,t) = \mu^{2/(p-1)} w(\mu x, \mu^2 t) \quad \text{for all } \mu > 0. \tag{1.3}$$

Such self-similar solutions are global in time and often used to describe the large time behavior of global solutions to (1.1), see, e.g., [20,21,4,28,29].

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If $w(x,t)$ is a self-similar solution of (1.1) and has an initial value $A(x)$, then we easily see that A has the form $A(x) = A(x/|x|)|x|^{-2/(p-1)}$. Then the problem of existence of self-similar solutions is essentially depend on the solvablity of the Cauchy problem (1.1)-$(1.2)_\lambda$.

It is well known by [10,34,20] that if $1 < p \le (N+2)/N$ then (1.1) has no time global solution w such that $w \ge 0$ and $w \not\equiv 0$. Therefore, the condition $p > (N+2)/N$ is necessary for the existence of positive self-similar solutions of (1.1).

We briefly review some results concerning the Cauchy problem for (1.1) with initial date in $L^q(\mathbf{R}^N)$. Weissler [32,33] showed that the IVP (1.1) with $w(x,0) = A \in L^q(\mathbf{R}^N)$ admits a unique time-local solution if $q \ge N(p-1)/2$. He also showed in [34] that the solution exists time-globally if $q = N(p-1)/2$ and if $\|A\|_{L^q(\mathbf{R}^N)}$ is sufficiently small. Giga [12] has constructed a unique local regular solution in $L^\alpha(0, T : L^\beta)$, where α and β are chosen so that the norm of $L^\alpha(0, T : L^\beta)$ is invariant under scaling. On the other hand, for $1 \le q < N(p-1)/2$, Haraux and Weissler [16] constructed a solution $w_0 \in C([0,\infty); L^q(\mathbf{R}^N))$ of (1.1) satisfying $w_0(x,t) > 0$ for $t > 0$ and $\|w_0(\cdot,t)\|_{L^q(\mathbf{R}^N)} \to 0$ as $t \to 0$ when $p > (N+2)/N$ and $(N-2)p < N+2$ by seeking solutions of self-similar form. Therefore, if $p > (N+2)/N$ and $(N-2)p < N+2$, the Cauchy problem

$$w_t = \Delta w + w^p \quad \text{in } \mathbf{R}^N \times (0,\infty) \quad \text{and} \quad w(x,0) = 0 \quad \text{in } \mathbf{R}^N \tag{1.4}$$

admits a non-unique solution in $C([0,\infty); L^q(\mathbf{R}^N))$ for $1 \le q < N(p-1)/2$.

Kozono and Yamazaki [22] constructed Besov-type function spaces based on the Morrey spaces, and then obtained global existence results for the equation (1.1) and the Navier-Stokes system with small initial data in these spaces. By [22] the problem (1.1)-$(1.2)_\lambda$ admits a time-global solution for sufficiently small $\lambda > 0$. Cazenave and Weissler [4] proved the existence of global solutions, including self-similar solutions, to the nonlinear Schrödinger equations and the equations (1.1) with small initial data by using the weighted norms.

Galaktionov and Vazquez [11] have investigated the uniqueness of the solutions to the problem (1.1)-$(1.2)_\lambda$ with $a \equiv 1$. In [11, p. 41] they have conjectured that the problem (1.1)-$(1.2)_\lambda$ has exactly two solutions (the minimal and maximal) when $N \ge 3$ and $N/(N-2) < p \le (N+2)/(N-2)$.

Letting $\mu = t^{-1/2}$ in (1.3), we see that the self-similar solution w has the form

$$w(x,t) = t^{-1/(p-1)} u(x/\sqrt{t}), \tag{1.5}$$

where u satisfies the elliptic equation

$$\Delta u + \frac{1}{2} x \cdot \nabla u + \frac{1}{p-1} u + u^p = 0 \qquad \text{in } \mathbf{R}^N. \tag{1.6}$$

By Lemma B.1 in Appendix below we find that if w satisfies $(1.2)_\lambda$ in the sense of $L^1_{\rm loc}(\mathbf{R}^N)$, that is,

$$\int_K \left| w(x,t) - \lambda a(x/|x|)|x|^{-2/(p-1)} \right| dx \to 0 \quad \text{as } t \to 0$$

for any compact subset K of $\mathbf{R}^N$, then u satisfies

$$\lim_{r\to\infty} r^{2/(p-1)} u(r\omega) = \lambda a(\omega) \qquad \text{for a.e. } \omega \in S^{N-1}. \tag{1.7$_\lambda$}$$

Conversely, if $u \in C^2(\mathbf{R}^N)$ is a solution of (1.6) satisfying $(1.7)_\lambda$, then the function w defined by (1.5) satisfies (1.1) and $(1.2)_\lambda$ in the sense of $L^1_{\rm loc}(\mathbf{R}^N)$.

In this paper we investigate the problem (1.6)-$(1.7)_\lambda$ by making use of the methods for semilinear elliptic equations, and then derive the results for the Cauchy problem (1.1)-$(1.2)_\lambda$ to give a partially affirmative answer to the conjecture by [11]. First we will state the results concerning the problem (1.6)-$(1.7)_\lambda$, and then apply these results to the problem (1.1)-$(1.2)_\lambda$.

Before stating our results, we introduce some notations. Set $\rho(x) = e^{|x|^2/4}$. We define the weighted Sobolev space

$$H^1_\rho(\mathbf{R}^N) = \left\{ u \in H^1(\mathbf{R}^N) : \int_{\mathbf{R}^N} (|\nabla u|^2 + u^2)\rho dx < \infty \right\} \tag{1.8}$$

equipped with the norms

$$\|u\|_{H^1_\rho(\mathbf{R}^N)} = \left(\int_{\mathbf{R}^N} (|\nabla u|^2 + u^2)\rho dx \right)^{1/2}.$$

It has been shown by Weissler [36, Theorem 1] and Escobedo and Kavian [8, Theorem 0.14] independently that there exists a solution u_0 of the problem

$$\begin{cases} \Delta u + \dfrac{1}{2} x \cdot \nabla u + \dfrac{1}{p-1} u + u^p = 0 & \text{in } \mathbf{R}^N, \\ u \in H^1_\rho(\mathbf{R}^N) \quad \text{and} \quad u > 0 & \text{in } \mathbf{R}^N, \end{cases} \tag{1.9}$$

with $p > (N+2)/N$ and $(N-2)p < (N+2)$. The uniqueness of the solution to the problem (1.9) has been obtained by [25, Corollary 2].

We refer to u as a solution of (1.6) if $u \in C^2(\mathbf{R}^N)$ is a classical solution of (1.6). Our main results are stated in the following theorems:

Theorem 1. *Assume that $p > (N+2)/N$. Then there exists a constant $\overline{\lambda} > 0$ such that*

(i) for $0 < \lambda < \overline{\lambda}$, (1.6)-$(1.7)_\lambda$ *has a minimal positive solution $\underline{u}_\lambda$; $\underline{u}_\lambda$ is increase with respect to λ and satisfies $\|\underline{u}_\lambda\|_{L^\infty(\mathbf{R}^N)} = O(\lambda)$ as $\lambda \to 0$;*

(ii) for $\lambda > \overline{\lambda}$, there are no positive solutions of (1.6)–$(1.7)_\lambda$.

Theorem 2. *Assume that* $p > (N+2)/N$ *and* $(N-2)p < N+2$. *Let* $\overline{\lambda} > 0$ *be the constant in Theorem* 1. *Then, for* $0 < \lambda < \overline{\lambda}$, (1.6)-(1.7)$_\lambda$ *has a positive solution* $\overline{u}_\lambda$ *satisfying* $\overline{u}_\lambda > \underline{u}_\lambda$ *and*

$$\overline{u}_\lambda - \underline{u}_\lambda \in H^1_\rho(\mathbf{R}^N) \quad and \quad \overline{u}_\lambda(x) - \underline{u}_\lambda(x) = O(e^{-|x|^2/4}) \quad as\ |x| \to \infty.$$

Furthermore

$$\overline{u}_\lambda - \underline{u}_\lambda \to u_0 \quad in\ H^1_\rho(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N) \quad as\ \lambda \to 0,$$

where u_0 *is the solution of the problem* (1.9). *In particular, we have* $\overline{u}_\lambda \to u_0$ *in* $L^\infty(\mathbf{R}^N)$ *as* $\lambda \to 0$.

Remark 1. (i) We now restrict our attention to radial solutions of (1.6), i.e., a solution of the form $u = u(r)$, $r = |x|$. Then $u(r)$ must satisfies the initial value problem

$$\begin{cases} u_{rr} + \left(\dfrac{N-1}{r} + \dfrac{r}{2}\right) u_r + \dfrac{1}{p-1} u + |u|^{p-1}u = 0, \qquad r > 0, \\ u(0) = \alpha > 0 \quad \text{and} \quad u_r(0) = 0. \end{cases}$$

We denote by $u(r;\alpha)$ the unique solution of this problem. It has been shown by Haraux-Weissler [16, Proposition 3.4 and Theorem 5] that $u(r;\alpha)$ has the following properties: the limit $L(\alpha) = \lim_{r\to\infty} r^{2/(p-1)} u(r;\alpha)$ exists and is a locally Lipschitz function of $\alpha \in \mathbf{R}$; if $p > (N+2)/N$ then $u(r;\alpha) > 0$ for all $r > 0$ and $L(\alpha) > 0$ for sufficiently small $\alpha > 0$; if in addition $(N-2)p < (N+2)$ then there exists some $\alpha_0 > 0$ such that $L(\alpha_0) = 0$ and $u(r;\alpha_0) > 0$ for $r > 0$; $u(r;\alpha)$ is positive for $r > 0$ if $0 < \alpha < \alpha_0$. (See the proof of Proposition 3.7 in [16].) Moreover, it has been shown by Yanagida [38, Theorem 1] and Dohmen-Hirose [7, Theorem 1.2 and Corollary 1.3] that $u(r;\alpha)$ cannot be positive for all $r > 0$ if $\alpha > \alpha_0$. (In fact, they in [38,7] showed the uniqueness of the α_0 such that $L(\alpha_0) = 0$ and $u(r;\alpha_0) > 0$ for $r > 0$.) From these facts we find that, when $p > (N+2)/N$ and $(N-2)p < (N+2)$, $u(r;\alpha) > 0$ for $r > 0$ if and only if $\alpha \in (0,\alpha_0]$, and that L is a nonnegative continuous function on $[0,\alpha_0]$ with $L(0) = L(\alpha_0) = 0$ and $L \not\equiv 0$. Let $\overline{\lambda} = \max\{L(\alpha) : 0 \le \alpha \le \alpha_0\}$. Then $\overline{\lambda} > 0$, and it is clear that there are at least two different values of α satisfying $L(\alpha) = \lambda$ if $\lambda \in (0,\overline{\lambda})$, and there exists at least one value of α satisfying $L(\alpha) = \overline{\lambda}$. Thus, when $p > (N+2)/N$ and $(N-2)p < (N+2)$, the problem (1.6)–(1.7)$_\lambda$ with $a \equiv 1$ has at least two radial solutions if $0 < \lambda < \overline{\lambda}$, and at least one radial solution if $\lambda = \overline{\lambda}$. Moreover, there is no radial positive solution of (1.6)–(1.7)$_\lambda$ with $a \equiv 1$ if $\lambda > \overline{\lambda}$.

(ii) In the case $N = 1$, Weissler [35] has shown that $u(r;\alpha) > 0$ for $r > 0$ if and only if $\alpha \in (0,\alpha_0]$ for some $\alpha_0 > 0$, and that L is a positive concave function on $(0,\alpha_0)$ with $L(0) = L(\alpha_0) = 0$. Thus there are precisely two different value of α satisfying $L(\alpha) = \lambda$ if $0 < \lambda < \overline{\lambda}$, and so the problem has precisely two solutions if $0 < \lambda < \overline{\lambda}$.

(iii) At this time we do not know whether (1.6)–(1.7)$_\lambda$ with $\lambda = \overline{\lambda}$ has a positive solution when a is not constant. The exact multiplicity of positive solutions of (1.6)–(1.7)$_\lambda$ for $\lambda \in (0, \overline{\lambda}]$ is also an open question.

Now we consider the Cauchy problem (1.1)–(1.2)$_\lambda$. We refer to w as a solution of (1.1) if $w \in C^2(\mathbf{R}^N \times (0,\infty))$ is a classical solution of (1.1). If u is a solution of (1.6)–(1.7)$_\lambda$, then the function w defined by (1.5) is a solution of (1.1) satisfying (1.2)$_\lambda$ in the sense of $L^1_{\text{loc}}(\mathbf{R}^N)$ by Lemma B.1 below. Put

$$w_0(x,t) = t^{-1/(p-1)} u_0(x/\sqrt{t}), \tag{1.10}$$

where u_0 is the solution of the problem (1.9). It has been shown by [8, Proposition 3.5] that $u_0 \in C^2(\mathbf{R}^N)$ and $u_0(x) = O(e^{-|x|^2/8})$ as $|x| \to \infty$. (See, also, [26, Theorem 1].) Then we have $u_0 \in L^q(\mathbf{R}^N)$ for all $q \ge 1$ and

$$\|w_0(\cdot,t)\|_{L^q(\mathbf{R}^N)} = t^{-1/(p-1)+N/2q} \|u_0\|_{L^q(\mathbf{R}^N)}.$$

Consequently, w_0 solves the the Cauchy problem (1.4) in $C([0,\infty); L^q(\mathbf{R}^N))$ for $1 \le q < N(p-1)/2$. We note that the positive solution u of (1.6) satisfying

$$u(x) = o(|x|^{-2/(p-1)}) \quad \text{as } |x| \to \infty$$

is unique and radially symmetric by [25, Corollary 1]. Therefore, w_0 defined by (1.10) coincides with the non-unique solution constructed by Haraux and Weissler [16].

As a consequence of Theorems 1 and 2, we obtain the following results.

Corollary 1. *Assume that $p > (N+2)/N$. Then there exists a constant $\overline{\lambda} > 0$ such that*

(i) for $0 < \lambda < \overline{\lambda}$, (1.1)-(1.2)$_\lambda$ has a positive self-similar solution $\underline{w}_\lambda$; for each fixed $t > 0$, the solution $\underline{w}_\lambda(\cdot,t)$ is increasing with respect to λ and satisfies

$$\|\underline{w}_\lambda(\cdot,t)\|_{L^\infty(\mathbf{R}^N)} = O(\lambda) \quad \textit{as } \lambda \to 0;$$

(ii) for $\lambda > \overline{\lambda}$, (1.1)-(1.2)$_\lambda$ has no positive self-similar solutions.

Corollary 2. *Assume that $p > (N+2)/N$ and $(N-2)p < N+2$. Let $\overline{\lambda} > 0$ be the constant in Corollary 1. Then, for $0 < \lambda < \overline{\lambda}$, (1.1)-(1.2)$_\lambda$ has a positive self-similar solution $\overline{w}_\lambda$ satisfying $\overline{w}_\lambda(x,t) > \underline{w}_\lambda(x,t)$ for $(x,t) \in (\mathbf{R}^N \times (0,\infty))$; the solution $\overline{w}_\lambda$ satisfies, for each fixed $t > 0$,*

$$\|\overline{w}_\lambda(\cdot,t) - w_0(\cdot,t)\|_{L^\infty(\mathbf{R}^N)} \to 0 \quad \textit{as } \lambda \to 0,$$

where w_0 is the non-unique solution of (1.4) in $C([0,\infty); L^q(\mathbf{R}^N))$ for $1 \le q < N(p-1)/2$, which is constructed by [16].

Remark 2. (i) The existence of a positive self-similar solution of (1.1)–$(1.2)_\lambda$ has been shown by [4] under a weaker condition on a.

(ii) It is already known that there is no solutions of (1.1)–$(1.2)_\lambda$ if λ is large enough, see, e.g., [33, Corollary 5.1], [37, Corollary 1.1], and [24, Remark 3.7]. These results, however, do not quite apply to self-similar solutions in stated, we easily see that the proofs easily apply to self-similar solutions, or any positive measurable solutions.

(iii) From (i) of Remark 1 we see that, when $p > (N+2)/N$ and $(N-2)p < (N+2)$, the problem (1.1)–$(1.2)_\lambda$ with $\lambda = \overline{\lambda}$ has a radial positive self-similar solution if $a \equiv 1$. It is an open question whether (1.1)–$(1.2)_\lambda$ with $\lambda = \overline{\lambda}$ has a positive self-similar solution if a is not constant. The exact multiplicity of positive self-similar solutions of (1.1)–$(1.2)_\lambda$ for $\lambda \in (0, \overline{\lambda}]$ is still an open question.

We prove Theorem 1 by using of the explicit supersolution and comparison arguments based on the maximum principle. We prove Theorem 2 by variational approach essentially due to Ambrosetti-Rabinowitz [1] and Crandall-Rabinowitz [5].

As far as we are aware, the idea of constructing self-similar solutions by solving the initial value problem for homogeneous initial data was first used by Giga and Miyakawa [13], for the Navier-Stokes equation in vorticity form. The idea of [13] was used later by several authors for various problems. Concerning the equation

$$u_t - \Delta u + u^p = 0 \qquad \text{in } R^N, \tag{1.11}$$

we refer to Kwak [23] and Cazenave et al. [3]. They also obtained the asymptotically self-similar behavior for a class of general solutions. See, also [15, 19, 9, 18].

After the paper was completed, we learned the work by Souplet and Weissler [30] where the existence of radial self-similar solutions of (1.6) were studied precisely in the subcritical, supercritical, and critical cases by using a shooting argument.

This paper is organized as follows: in Section 2 we show the maximum principle and comparison results for the operator related to the equation (1.6). In Section 3 we consider the linearized eigenvalue problems. Sections 4 and 5 devoted to the proofs of Theorems 1 and 2, respectively. For completeness, we show the regularity and some properties of the solutions in the appendixes.

In the remaining part of the paper, we assume that $p > (N+2)/N$.

2. Preliminaries

In this section we show the following two propositions which are crucial for the proofs of the theorems. For simplicity, we define Lu by

$$Lu = -\Delta u - \frac{1}{2} x \cdot \nabla u - \frac{1}{p-1} u$$

for $u \in C^2(\mathbf{R}^N)$.

Proposition 2.1. *Assume that $Lu \geq 0$ in $\mathbf{R}^N$, and that*

$$\liminf_{|x|\to\infty} |x|^{2/(p-1)} u(x) \geq 0.$$

Then $u > 0$ or $u \equiv 0$ in $\mathbf{R}^N$. In particular, if $Lu \geq 0$ and $u \geq 0$ in $\mathbf{R}^N$ then $u > 0$ or $u \equiv 0$ in $\mathbf{R}^N$.

Proposition 2.2. *Assume that $\alpha, \beta : S^{N-1} \to \mathbf{R}$ satisfy $\alpha, \beta \in L^\infty(S^{N-1})$ and*

$$0 \leq \alpha(\omega) \leq \beta(\omega) \quad \text{for a.e. } \omega \in S^{N-1}.$$

Suppose that there exists a positive function v satisfying $Lv \geq v^p$ in $\mathbf{R}^N$ and

$$\lim_{r\to\infty} r^{2/(p-1)} v(r\omega) = \beta(\omega) \quad \text{for a.e. } \omega \in S^{N-1}.$$

Then there exists a positive solution u of $Lu = u^p$ in $\mathbf{R}^N$ satisfying $u \leq v$ in $\mathbf{R}^N$ and

$$\lim_{r\to\infty} r^{2/(p-1)} u(r\omega) = \alpha(\omega) \quad \text{for a.e. } \omega \in S^{N-1}. \tag{2.1}$$

Moreover, for any positive function w satisfying $Lw \geq w^p$ in $\mathbf{R}^N$ and

$$\liminf_{r\to\infty} r^{2/(p-1)} w(r\omega) \geq \alpha(\omega) \quad \text{for a.e. } \omega \in S^{N-1},$$

we have $u \leq w$ in $\mathbf{R}^N$.

First we show the following lemma.

Lemma 2.1. *Assume that $\alpha : S^{N-1} \to \mathbf{R}$ satisfies $\alpha \in L^\infty(S^{N-1})$ and $\alpha \geq 0$, $\alpha \not\equiv 0$ for a.e. $\omega \in S^{N-1}$. Then there exists a positive function $\phi_\alpha \in C^2(\mathbf{R}^N)$ satisfying $L\phi_\alpha = 0$ in $\mathbf{R}^N$ and*

$$\lim_{r\to\infty} r^{2/(p-1)} \phi_\alpha(r\omega) = \alpha(\omega) \qquad \text{for a.e. } \omega \in S^{N-1}. \tag{2.2}$$

Proof. Put

$$w(x,t) = \frac{1}{(4\pi t)^{N/2}} \int_{\mathbf{R}^N} e^{-|x-y|^2/(4t)} \alpha(y/|y|) |y|^{-2/(p-1)} dy. \tag{2.3}$$

We note that $\alpha(y/|y|)|y|^{-2/(p-1)} \in L^1_{\text{loc}}(\mathbf{R}^N)$ from $p > (N+2)/N$. By [6, Chapter 5, Theorem 6.1], w defined by (2.3) satisfies $w_t = \Delta w$ in $\mathbf{R}^N \times (0,\infty)$ and

$$w(x,t) \to \alpha(x/|x|)|x|^{-2/(p-1)} \quad \text{in } L^1_{\text{loc}}(\mathbf{R}^N) \quad \text{as } t \to 0.$$

Define the rescaled functions w_μ by $w_\mu(x,t) = \mu^{2/(p-1)} w(\mu x, \mu^2 t)$ for $\mu > 0$. From (2.3) we obtain $w(x,t) = w_\mu(x,t)$ for all $\mu > 0$. Putting $\mu = 1/\sqrt{t}$, we find that

$$w(x,t) = t^{-1/(p-1)} \phi_\alpha(x/\sqrt{t}), \tag{2.5}$$

where $\phi_\alpha(x) = w(x, 1)$. It can be easily checked that ϕ_α satisfies $L\phi_\alpha = 0$ in $\mathbf{R}^N$. By Lemma B.1 in Appendix B below, we obtain (2.2). □

Lemma 2.2. *Let $\Omega \subset \mathbf{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$.*

(i) Assume that $Lu \geq 0$ in Ω and $u \geq 0$ on $\partial\Omega$. Then $u > 0$ or $u \equiv 0$ in Ω.
(ii) Assume that $f \in C^\theta(\Omega)$ for some $\theta \in (0, 1)$ and $g \in C(\partial\Omega)$. Then there exists a solution u of $Lu = f$ in Ω and $u = g$ on $\partial\Omega$.

Proof. (i) From Lemma 2.1 there exists a positive function ϕ satisfying

$$L\phi = 0 \quad \text{in } \mathbf{R}^N \quad \text{and} \quad \lim_{r\to\infty} r^{2/(p-1)}\phi(r\omega) = 1 \quad \text{for a.e. } \omega \in S^{N-1}. \tag{2.6}$$

Let $v(x) = u(x)/\phi(x)$. Then v satisfies

$$-\Delta v - \left(\frac{2}{\phi}\nabla\phi + \frac{1}{2}x\right)\cdot\nabla v \geq 0 \quad \text{in } \Omega \quad \text{and} \quad v \geq 0 \text{ on } \partial\Omega.$$

By the maximum principle [27] we have $v > 0$ or $v \equiv 0$ in Ω, which implies that $u > 0$ or $u \equiv 0$ in Ω.

(ii) Let ϕ be a positive function satisfying (2.6). We have a solution $v \in C^{2,\theta}(\Omega)$ of

$$\begin{cases} -\Delta v - \left(\dfrac{2}{\phi}\nabla\phi + \dfrac{1}{2}x\right)\cdot\nabla v = \dfrac{f}{\phi} & \text{in } \Omega, \text{ and} \\ \hfill v = \dfrac{g}{\phi} & \text{on } \partial\Omega. \end{cases}$$

(See, e.g., [14].) Then $u(x) = v(x)\phi(x)$ satisfies $Lu = f$ in Ω and $u = g$ on $\partial\Omega$. □

Proof of Proposition 2.1. Let $v(x) = u(x)/\phi(x)$, where ϕ is a positive function satisfying (2.6). Then v satisfies

$$-\Delta v - \left(\frac{2}{\phi}\nabla\phi + \frac{1}{2}x\right)\cdot\nabla v \geq 0 \quad \text{in } \mathbf{R}^N \quad \text{and} \quad \liminf_{|x|\to\infty} v(x) \geq 0.$$

First we show $v \geq 0$ in $\mathbf{R}^N$. Assume to the contrary that $v(x_0) < 0$ for some $x_0 \in \mathbf{R}^N$. Choose $\varepsilon > 0$ so small that $\varepsilon < -v(x_0)$, and take $R > 0$ so large that $R > |x_0|$ and $v(x) \geq -\varepsilon$ on $|x| = R$. By the maximum principle [27] we have $v \geq -\varepsilon$ in $|x| \leq R$. This contradicts $v(x_0) < -\varepsilon$. Hence, $v \geq 0$ in $\mathbf{R}^N$. As a consequence of (i) of Lemma 2.2 we have $v > 0$ or $v \equiv 0$ in $\mathbf{R}^N$, which implies that $u > 0$ or $u \equiv 0$ in $\mathbf{R}^N$. □

Lemma 2.3. *Let $f \in C^{\theta}_{\mathrm{loc}}(\mathbf{R}^N)$ for some $\theta \in (0,1)$, and let $f \geq 0$. Assume that there exists a positive function v such that $Lv \geq f$ in $\mathbf{R}^N$. Then there exists a solution u of $Lu = f$ in $\mathbf{R}^N$ such that $0 \leq u \leq v$ in $\mathbf{R}^N$.*

Proof. Define $B_r = \{x \in \mathbf{R}^N : |x| < r\}$ for $r > 0$. From (ii) of Lemma 2.2, there exists a solution u_k of

$$Lu_k = f \quad \text{in } B_k \quad \text{and} \quad u_k = v \quad \text{on } \partial B_k$$

for each $k = 1, 2, \ldots$. From (i) of Lemma 2.2 we have $u_k > 0$ in B_k. Put $w_k(x) = v(x) - u_k(x)$. Then w_k satisfies $Lw_k \geq 0$ in B_k and $w_k = 0$ on ∂B_k. From (i) of Lemma 2.2 again we have $w_k \geq 0$. Thus we have $0 < u_k \leq v$ in B_k.

Take $R > 0$. Since u_k satisfies $Lu_k = f$ in B_R for $k \geq R$, by the Schauder estimates $\{u_k\}$ is bounded in $C^{2,\theta}_{\mathrm{loc}}(B_R)$ for some $0 < \theta < 1$. Then, by the Ascolli-Arzela, a subsequence in $\{u_k\}$ converges in $C^2_{\mathrm{loc}}(B_R)$. We may do the same arguments for a sequence $\{R_n\}$ such that $R_n \to \infty$ as $n \to \infty$. By the diagonal method there exists a function $u \in C^2(\mathbf{R}^N)$ such that a subsequence converges to u in $C^2_{\mathrm{loc}}(\mathbf{R}^N)$. Thus u satisfies $Lu = f$ in $\mathbf{R}^N$ with $0 \leq u \leq v$ in $\mathbf{R}^N$. This concludes the proof. □

Lemma 2.4. *Let ϕ_α be a positive function satisfying $L\phi_\alpha = 0$ in $\mathbf{R}^N$ and (2.2). Assume that there exists a positive function $\hat{v}$ satisfying*

$$L\hat{v} \geq (\hat{v} + \phi_\alpha)^p \quad \textit{in } \mathbf{R}^N \quad \textit{and} \quad \lim_{|x|\to\infty} |x|^{2/(p-1)} \hat{v}(x) = 0.$$

Then there exists a solution $\hat{u}$ of $L\hat{u} = (\hat{u} + \phi_\alpha)^p$ in $\mathbf{R}^N$ satisfying $0 \leq \hat{u} \leq \hat{v}$. Moreover, for any positive function $\hat{w}$ satisfying

$$L\hat{w} \geq (\hat{w} + \phi_\alpha)^p \quad \textit{in } \mathbf{R}^N \quad \textit{and} \quad \liminf_{|x|\to\infty} |x|^{2/(p-1)} \hat{w}(x) \geq 0, \tag{2.7}$$

we have $\hat{u} \leq \hat{w}$ in $\mathbf{R}^N$.

Proof. For each $u \in C^2(\mathbf{R}^N)$, we define the mapping Tu as follows: $v = Tu$ if

$$Lv = (u + \phi_\alpha)^p \quad \text{in } \mathbf{R}^N \quad \text{and} \quad 0 \leq v \leq \hat{v} \quad \text{in } \mathbf{R}^N. \tag{2.8}$$

Assume that $0 \leq u \leq \hat{v}$. Since $\hat{v}$ satisfies $L\hat{v} \geq (u + \phi_\alpha)^p$, from Lemma 2.3 there exists a function v satisfying (2.8). Then the mapping T is well defined for each $u \in C^2(\mathbf{R}^N)$ satisfying $0 \leq u \leq \hat{v}$. We also find that

$$\lim_{|x|\to\infty} |x|^{2/(p-1)} Tu(x) = 0 \tag{2.9}$$

from $0 \leq Tu \leq \hat{v}$ and $\lim_{|x|\to\infty} |x|^{2/(p-1)} \hat{v}(x) = 0$.

Assume that $u_1, u_2 \in C^2(\mathbf{R}^N)$. We show that $0 \le u_1 < u_2 \le \hat{v}$ implies $Tu_1 < Tu_2$. In fact, if $u_1 < u_2$ then $L(Tu_2 - Tu_1) > 0$ in $\mathbf{R}^N$. From (2.9) we have

$$\lim_{|x|\to\infty} |x|^{2/(p-1)}(Tu_2(x) - Tu_1(x)) = 0.$$

Hence, from Proposition 2.1 we have $Tu_1 < Tu_2$.

Define $\{\hat{u}_k\}$ inductively by

$$\hat{u}_0 \equiv 0 \quad \text{and} \quad \hat{u}_k = T\hat{u}_{k-1} \quad \text{for } k = 1, 2, \dots . \tag{2.10}$$

Since we have $L(T\hat{u}_0) = \phi_\alpha^p > 0$ in $\mathbf{R}^N$ and $\lim_{|x|\to\infty} |x|^{2/(p-1)}T\hat{u}_0(x) = 0$, we obtain $T\hat{u}_0 > 0$ in $\mathbf{R}^N$ by Proposition 2.1. Then, by induction, $\hat{u}_k$ is well defined and satisfies

$$0 \equiv \hat{u}_0 < \hat{u}_1 < \cdots < \hat{u}_k < \hat{u}_{k+1} < \cdots < \hat{v} \quad \text{in } \mathbf{R}^N.$$

Define $\hat{u}(x) = \lim_{k\to\infty} \hat{u}_k(x)$. Take $R > 0$ and define $B_R = \{x \in \mathbf{R}^N : |x| < R\}$. Since $\{\hat{u}_k\}$ satisfies

$$L\hat{u}_k = (\hat{u}_{k-1} + \phi_\alpha)^p \le (\hat{v} + \phi_\alpha)^p \quad \text{in } B_R,$$

it follows from elliptic interior estimates that $\{\hat{u}_k\}$ is bounded in $W^{2,p}_{\text{loc}}(B_R)$ for every $p > 1$. By the Sobolev embedding theorem and the Schauder estimates, $\{\hat{u}_k\}$ is bounded in $C^{2,\theta}_{\text{loc}}(B_R)$ for some $\theta \in (0, 1)$. Therefore, $\{\hat{u}_k\}$ converges to $\hat{u}$ in $C^2_{\text{loc}}(B_R)$. We may do the same arguments for a sequence $\{R_n\}$ such that $R_n \to \infty$ as $n \to \infty$. By the diagonal method $\{\hat{u}_k\}$ converges to $\hat{u}$ in $C^2_{\text{loc}}(\mathbf{R}^N)$, and thus we have $L\hat{u} = (\hat{u} + \phi_\alpha)^p$ and $0 < \hat{u} \le \hat{v}$ in $\mathbf{R}^N$.

Let $\hat{w}$ be a positive function satisfying (2.7). We claim that $\hat{w} > u$ implies $\hat{w} > Tu$ for $u \in C^2(\mathbf{R}^N)$ satisfying $0 \le u \le \hat{v}$. In fact, if $\hat{w} > u$ we have $L(\hat{w} - Tu) > 0$ in $\mathbf{R}^N$ and

$$\liminf_{|x|\to\infty} |x|^{2/(p-1)}(\hat{w} - Tu(x)) = \liminf_{|x|\to\infty} |x|^{2/(p-1)}\hat{w} \ge 0.$$

From Proposition 2.1 we obtain $\hat{w} > Tu$.

Let $\{\hat{u}_n\}$ be the sequence defined by (2.10). Then we have $\hat{w} > \hat{u}_0 \equiv 0$ and $\hat{w} > \hat{u}_k$ for $k = 1, 2, \dots$, by induction. Therefore, we have $\hat{w} \ge \hat{u}$. □

Proof of Proposition 2.2. Let ϕ_β be a positive function satisfying $L\phi_\beta = 0$ in $\mathbf{R}^N$ and

$$\lim_{r\to\infty} r^{2/(p-1)}\phi(r\omega) = \beta(\omega) \quad \text{for a.e. } \omega \in S^{N-1}.$$

Then, from Proposition 2.1 we have $\phi_\alpha \le \phi_\beta$ in $\mathbf{R}^N$. Define $\hat{v}(x) = v(x) - \phi_\beta(x)$. From $L\hat{v} = v^p > 0$ in $\mathbf{R}^N$ and $\lim_{|x|\to\infty} |x|^{2/(p-1)}\hat{v}(x) = 0$ we have $\hat{v} > 0$ by Proposition 2.1. We also find that $\hat{v}$ satisfies $L\hat{v} \ge (\hat{v} + \phi_\beta)^p \ge (\hat{v} + \phi_\alpha)^p$ in $\mathbf{R}^N$. Then it follows from Lemma 2.4 that there exists a solution $\hat{u}$ of $L\hat{u} = (\hat{u}+\phi_\alpha)^p$ in

$\mathbf{R}^N$ satisfying $0 \le \hat{u} \le \hat{v}$ in $\mathbf{R}^N$. In particular, we have $\lim_{|x|\to\infty} |x|^{2/(p-1)}\hat{u}(x) = 0$. Put $u = \hat{u} + \phi_\alpha$. Then u satisfies $Lu = u^p$ in $\mathbf{R}^N$ with (2.1).

Define $\hat{w} = w - \phi_\alpha$. Then $\hat{w}$ satisfies

$$L\hat{w} \ge w^p = (\hat{w} + \phi_\alpha)^p > 0 \quad \text{in } \mathbf{R}^N \quad \text{and} \quad \liminf_{|x|\to\infty} |x|^{2/(p-1)}\hat{w}(x) \ge 0.$$

Proposition 2.1 implies that $\hat{w} > 0$ in $\mathbf{R}^N$. From Lemma 2.4 we have $\hat{u} \le \hat{w}$, which implies $u \le w$ in $\mathbf{R}^N$. This completes the proof of Proposition 2.2. □

3. Eigenvalue problems

We recall here some results about the weighted Sobolev space $H^1_\rho(\mathbf{R}^N)$ defined by (1.8). For $1 \le p < \infty$, we define

$$L^p_\rho(\mathbf{R}^N) = \left\{ u \in L^p(\mathbf{R}^N) : \int_{\mathbf{R}^N} u^p \rho dx < \infty \right\} \quad \text{and} \quad \|u\|_{L^p_\rho} = \left(\int_{\mathbf{R}^N} u^p \rho dx \right)^{1/p},$$

where $\rho(x) = e^{|x|^2/4}$.

Lemma 3.1. *(i) For every $u \in H^1_\rho(\mathbf{R}^N)$, we have*

$$\frac{N}{2} \int_{\mathbf{R}^N} u^2 \rho dx \le \int_{\mathbf{R}^N} |\nabla u|^2 \rho dx.$$

(ii) The embedding $H^1_\rho(\mathbf{R}^N) \subset L^2_\rho(\mathbf{R}^N)$ is compact.

(iii) If $N \ge 3$, then the embedding $H^1_\rho(\mathbf{R}^N) \subset L^{p+1}_\rho(\mathbf{R}^N)$ is continuous for $1 \le p \le (N+2)/(N-2)$, and is compact for $1 < p < (N+2)/(N-2)$. If $N = 2$ then the embedding $H^1_\rho(\mathbf{R}^2) \subset L^{p+1}_\rho(\mathbf{R}^2)$ is continuous and compact for $p > 1$.

For the proof, see Escobedo and Kavian [8] and Kavian [20]. From (i) of Lemma 3.1, for $u \in H^1_\rho(\mathbf{R}^N)$ we have

$$\int_{\mathbf{R}^N} \left(|\nabla u|^2 - \frac{1}{p-1} u^2 \right) \rho dx \ge \left(\frac{N}{2} - \frac{1}{p-1} \right) \int_{\mathbf{R}^N} u^2 \rho dx \tag{3.1}$$

and

$$\int_{\mathbf{R}^N} \left(|\nabla u|^2 - \frac{1}{p-1} u^2 \right) \rho dx \ge \left(1 - \frac{2}{N(p-1)} \right) \int_{\mathbf{R}^N} |\nabla u|^2 \rho dx. \tag{3.2}$$

Let us consider the eigenvalue problem

$$\begin{cases} -\Delta u - \dfrac{1}{2} x \cdot \nabla u - \dfrac{1}{p-1} u = \mu m(x) u \quad \text{in } \mathbf{R}^N, \\ u \in H^1_\rho(\mathbf{R}^N), \end{cases} \tag{3.3}$$

where $m \in L^\infty(\mathbf{R}^N) \cap C^\theta(\mathbf{R}^N)$ for some $\theta \in (0, 1)$ and $m > 0$ in $\mathbf{R}^N$. First, we show the following:

Lemma 3.2. *The problem* (3.3) *has the first eigenvalue* $\mu_0 > 0$ *and the corresponding eigenfunction* $u_0 > 0$ *in* $\mathbf{R}^N$. *Furthermore, we have*

$$\mu_0 = \inf \left\{ \int_{\mathbf{R}^N} \left(|\nabla u|^2 - \frac{1}{p-1} u^2 \right) \rho dx : u \in H^1_\rho(\mathbf{R}^N),\ \int_{\mathbf{R}^N} m u^2 \rho dx = 1 \right\}. \tag{3.4}$$

Proof. We claim that $\mu_0 > 0$ and the minimization problem (3.4) is achieved by some function $u_0 > 0$. First we show $\mu_0 > 0$. Indeed, we see that

$$1 = \int_{\mathbf{R}^N} m u^2 \rho dx \le \|m\|_{L^\infty} \int_{\mathbf{R}^N} u^2 \rho dx.$$

Then it follows from (3.1) that

$$\int_{\mathbf{R}^N} \left(|\nabla u|^2 - \frac{1}{p-1} u^2 \right) \rho dx \ge \left(\frac{N}{2} - \frac{1}{p-1} \right) \frac{1}{\|m\|_{L^\infty}}$$

for $u \in H^1_\rho(\mathbf{R}^N)$, which implies $\mu_0 > 0$.

Let $\{u_k\} \subset H^1_\rho(\mathbf{R}^N)$ be a minimizing sequence of μ_0, that is,

$$\int_{\mathbf{R}^N} m u_k^2 \rho dx = 1 \quad \text{and} \quad \int_{\mathbf{R}^N} \left(|\nabla u_k|^2 - \frac{1}{p-1} u_k^2 \right) \rho dx \to \mu_0 \quad \text{as } k \to \infty.$$

From (3.2) and (i) of Lemma 3.1 we find that $\{u_k\}$ is bounded in $H^1_\rho(\mathbf{R}^N)$. Then, from (ii) of Lemma 3.1, there exist a subsequence that we still denoted $\{u_k\}$ and a function $u_0 \in H^1_\rho(\mathbf{R}^N)$ such that

$$u_k \rightharpoonup u_0 \quad \text{weakly in } H^1_\rho(\mathbf{R}^N) \quad \text{as } k \to \infty,$$

$$u_k \to u_0 \quad \text{strongly in } L^2_\rho(\mathbf{R}^N) \quad \text{as } k \to \infty.$$

Then we obtain

$$\int_{\mathbf{R}^N} \left(|\nabla u_0|^2 - \frac{1}{p-1} u_0^2 \right) \rho dx \le \liminf_{k \to \infty} \int_{\mathbf{R}^N} \left(|\nabla u_k|^2 - \frac{1}{p-1} u_k^2 \right) \rho dx = \mu_0$$

and

$$1 = \lim_{k \to \infty} \int_{\mathbf{R}^N} m u_k^2 \rho dx = \int_{\mathbf{R}^N} m u_0^2 \rho dx.$$

Hence, we find that u_0 achieves μ_0. Clearly, $|u_0|$ also achieves μ_0. By the elliptic regularity theory and Proposition 2.1, we have $u_0 \in C^2(\mathbf{R}^N)$ and $u_0 > 0$ in $\mathbf{R}^N$. □

In this section we show the following two propositions.

Proposition 3.1. *Assume that there is a positive function $w \in C^2(\mathbf{R}^N)$ satisfying*

$$\Delta w + \frac{1}{2}x \cdot \nabla w + \frac{1}{p-1}w + \mu m(x)w \leq 0, \qquad x \in \mathbf{R}^N, \tag{3.5}$$

for some $\mu \in \mathbf{R}$. Then $\mu \leq \mu_0$, where μ_0 is the first eigenvalue of the problem (3.3).

Proposition 3.2. *Assume that $m_1,\ m_2 \in L^\infty(\mathbf{R}^N)$ satisfy $0 < m_1(x) \leq m_2(x)$, $m_1(x) \not\equiv m_2(x)$. Let μ_i be the first eigenvalue of the problem*

$$\begin{cases} -\Delta u - \frac{1}{2}x \cdot \nabla u - \frac{1}{p-1}u = \mu m_i(x)u \quad \text{in } \mathbf{R}^N, \\ \qquad\qquad u \in H^1_\rho(\mathbf{R}^N), \end{cases} \tag{3.6$_i$}$$

for each $i = 1, 2$. Then $\mu_1 > \mu_2$.

To prove Proposition 3.1, we consider the eigenvalue problems

$$\begin{cases} -\Delta v - \frac{1}{2}x \cdot \nabla v - \frac{1}{p-1}v = \mu m(x)v \quad \text{in } B_k, \\ \qquad\qquad v \in H^1_0(B_k), \end{cases} \tag{3.7$_k$}$$

where $B_k = \{x \in \mathbf{R}^N : |x| < k\}$, for $k = 1, 2, \ldots$. We can prove that the problem $(3.7)_k$ has the first eigenvalue $\mu_k > 0$ and the corresponding eigenfunction $v_k > 0$ in B_k. Furthermore, we find that

$$\mu_k = \inf\left\{\int_{B_k}\left(|\nabla v|^2 - \frac{1}{p-1}v^2\right)\rho dx : v \in H^1_0(B_k),\ \int_{B_k} mv^2\rho dx = 1\right\}, \tag{3.8$_k$}$$

and that $v_k \in C^2(\overline{B_k})$ achieves the minimization $(3.8)_k$.

Suppose that $v \in H^1_0(B_k)$, and extend v to be zero outside B_k. Then $v \in H^1_0(B_{k+1})$. From $(3.8)_k$ we have $\mu_k \geq \mu_{k+1}$ for $k = 1, 2, \ldots$.

Lemma 3.3. *We have* $\lim_{k\to\infty}\mu_k = \mu_0$*, where μ_0 is the first eigenvalue of the problem* (3.3).

Proof. Suppose that $v_k \in H^1_0(B_k)$ is the first eigenfunction of the problem $(3.7)_k$, and extend v_k to be zero outside B_k. Then $v_k \in H^1_\rho(\mathbf{R}^N)$ and satisfies

$$\int_{\mathbf{R}^N}\left(\nabla v_k \cdot \nabla\phi - \frac{1}{p-1}v_k\phi - \mu_k m v_k\phi\right)\rho dx = 0$$

for any $\phi \in C^\infty_0(B_k)$. Since v_k achieves the the minimization $(3.8)_k$, we have

$$\int_{\mathbf{R}^N} mv_k^2\rho dx = 1 \quad \text{and} \quad \int_{\mathbf{R}^N}\left(|\nabla v_k|^2 - \frac{1}{p-1}v_k^2\right)\rho dx = \mu_k.$$

From (3.4) we have $\mu_k \geq \mu_0$. From (3.2) and $\mu_k \geq \mu_{k+1}$, $k = 1, 2, \dots$, it follows that

$$\int_{\mathbf{R}^N} |\nabla v_k|^2 \rho dx \leq \left(1 - \frac{2}{N(p-1)}\right)^{-1} \mu_k \leq \left(1 - \frac{2}{N(p-1)}\right)^{-1} \mu_1.$$

Therefore, from (i) of Lemma 3.1, $\{v_k\}$ is bounded in $H^1_\rho(\mathbf{R}^N)$. Then, from (ii) of Lemma 3.1, there exist a subsequence that we still denote $\{v_k\}$ and a function $v_0 \in H^1_\rho(\mathbf{R}^N)$ such that

$$v_k \rightharpoonup v_0 \quad \text{weakly in } H^1_\rho(\mathbf{R}^N) \quad \text{as } k \to \infty,$$

$$v_k \to v_0 \quad \text{strongly in } L^2_\rho(\mathbf{R}^N) \quad \text{as } k \to \infty.$$

Then we obtain $\int_{\mathbf{R}^N} m v_0^2 \rho dx = 1$, which implies that $v_0 \not\equiv 0$. We also obtain

$$\int_{\mathbf{R}^N} \left(\nabla v_0 \cdot \nabla \phi - \frac{1}{p-1} v_0 \phi - \mu_\infty m v_0 \phi\right) \rho dx = 0 \tag{3.9}$$

for any $\phi \in C_0^\infty(\mathbf{R}^N)$, where $\mu_\infty = \lim_{k\to\infty} \mu_k$. Since $C_0^\infty(\mathbf{R}^N)$ is dense in $H^1_\rho(\mathbf{R}^N)$, we obtain (3.9) for any $\phi \in H^1_\rho(\mathbf{R}^N)$. Putting $\phi = u_0 > 0$ in (3.9), where u_0 is the eigenfunction of the problem (3.3), we obtain

$$\begin{aligned}\mu_\infty \int_{\mathbf{R}^N} m v_0 u_0 \rho dx &= \int_{\mathbf{R}^N} \left(\nabla v_0 \cdot \nabla u_0 - \frac{1}{p-1} v_0 u_0\right) \rho dx \\ &= \mu_0 \int_{\mathbf{R}^N} m v_0 u_0 \rho dx,\end{aligned}$$

which implies $\mu_\infty = \mu_0$. □

Proof of Proposition 3.1. We claim that $\mu < \mu_k$ for each $k = 1, 2, \dots$, where μ_k is the first eigenvalue of the problem $(3.7)_k$. Assume that v_k is the corresponding eigenfunction. We note that $v_k \in C^2(\overline{B_k})$ and satisfies

$$\int_{B_k} m v_k^2 \rho dx = 1 \quad \text{and} \quad \int_{B_k} \left(|\nabla v_k|^2 - \frac{1}{p-1} v_k^2\right) \rho dx = \mu_k. \tag{3.10}$$

Let $w \in C^2(\mathbf{R}^N)$ be a positive function satisfying (3.5). Then, by the straight forward calculation we have the following Picone's identity (cf. [17,31]):

$$\rho w^2 \left|\nabla \left(\frac{v_k}{w}\right)\right|^2 + \nabla \cdot \left(\frac{v_k^2}{w} (\rho \nabla w)\right) = \rho |\nabla v_k|^2 + \frac{v_k^2}{w} \nabla \cdot (\rho \nabla w) \quad \text{in } B_k.$$

Since w satisfies

$$\nabla \cdot (\rho \nabla w) + \rho \left(\frac{1}{p-1} w + \mu m w\right) \leq 0,$$

we obtain

$$\rho w^2 \left|\nabla\left(\frac{v_k}{w}\right)\right|^2 + \nabla\cdot\left(\frac{v_k^2}{w}(\rho\nabla w)\right) \le \left(|\nabla v_k|^2 - \left(\frac{1}{p-1}+\mu m\right)v_k^2\right)\rho \tag{3.11}$$

in B_k. Note that $v_k = 0$ on ∂B_k. Then, by using Green's formula, we have

$$\int_{B_k} \nabla\cdot\left(\frac{v_k^2}{w}(\rho\nabla w)\right)dx = 0.$$

Therefore, integrating (3.11) on B_k we obtain

$$0 < \int_{B_k} \rho w^2\left|\nabla\left(\frac{v_k}{w}\right)\right|^2 dx \le \int_{B_k}\left(|\nabla v_k|^2 - \frac{1}{p-1}v_k^2\right)\rho dx - \mu\int_{B_k} m v_k^2 \rho dx.$$

From (3.10) we have $0 < \mu_k - \mu$. Then $\mu_k > \mu$ for each $k = 1, 2, \ldots$. From Lemma 3.3 we obtain $\mu \le \mu_0$. □

Proof of Proposition 3.2. Let $u_i > 0$ be the first eigenfunction of the problem $(3.6)_i$ for each $i = 1, 2$. Then u_i, $i = 1, 2$, satisfies

$$\int_{\mathbf{R}^N}\left(\nabla u_i\cdot\nabla\phi - \frac{1}{p-1}u_i\phi\right)\rho dx = \mu_i\int_{\mathbf{R}^N} m_i u_i\phi\rho dx$$

for any $\phi \in H^1_\rho(\mathbf{R}^N)$. Therefore, we have

$$\begin{aligned}\mu_1\int_{\mathbf{R}^N} m_1u_1u_2\rho dx &= \int_{\mathbf{R}^N}\left(\nabla u_1\cdot\nabla u_2 - \frac{1}{p-1}u_1u_2\right)\rho dx\\ &= \mu_2\int_{\mathbf{R}^N} m_2u_1u_2\rho dx.\end{aligned}$$

Since $m_1 \le m_2$, $m_1 \not\equiv m_2$, we obtain $\mu_1 > \mu_2$. □

4. Existence of the minimal solution: Proof of Theorem 1

For each $\lambda > 0$ we introduce the solution set

$$S_\lambda = \{u \in C^2(\mathbf{R}^N) : u \text{ is a positive solution of } (1.6)\text{-}(1.7)_\lambda\}.$$

We call a minimal solution $\underline{u}_\lambda \in S_\lambda$, if $\underline{u}_\lambda$ satisfies $\underline{u}_\lambda \le u$ for all $u \in S_\lambda$.

First we show the following results.

Lemma 4.1. *(i) We have $S_\lambda \neq \emptyset$ for some $\lambda > 0$. Moreover, if $S_{\lambda_0} \neq \emptyset$ for some $\lambda_0 > 0$, then $S_\lambda \neq \emptyset$ for all $\lambda \in (0, \lambda_0)$.*

(ii) If $S_\lambda \neq \emptyset$ then there exists a minimal solution $\underline{u}_\lambda \in S_\lambda$. Moreover, for any positive function w satisfying

$$\begin{cases} -\Delta w - \dfrac{1}{2}x \cdot \nabla w - \dfrac{1}{p-1}w \geq w^p & \text{in } \mathbf{R}^N \quad \text{and} \\ \liminf\limits_{r\to\infty} r^{2/(p-1)} w(r\omega) \geq \lambda a(\omega) & \text{for a.e. } \omega \in S^{N-1}, \end{cases} \tag{4.1}$$

we have $\underline{u}_\lambda \leq w$.

Proof. (i) Let $v = v(r)$, $r = |x|$, be a positive solution of (1.6) satisfying

$$\lim_{r\to\infty} r^{2/(p-1)} v(r) = \ell$$

for some $\ell > 0$. The existence of such v is obtained by [16, Theorem 5]. Take $\lambda_* > 0$ so small that $\lambda_* \leq \ell/\|a\|_{L^\infty(S^{N-1})}$. By applying Proposition 2.2 with $\alpha(\omega) = \lambda_* a(\omega)$ and $\beta(\omega) \equiv \ell$, we obtain a positive solution u of (1.6)–$(1.7)_\lambda$ with $\lambda = \lambda_*$, that is, $S_{\lambda_*} \neq \emptyset$.

Assume that $S_{\lambda_0} \neq \emptyset$ for some $\lambda_0 > 0$. Let $\lambda \in (0, \lambda_0)$. Then, by applying Proposition 2.2 with $\alpha(\omega) = \lambda a(\omega)$ and $\beta(\omega) = \lambda_0 a(\omega)$, we have a positive solution u of (1.6)–$(1.7)_\lambda$. Therefore, $S_\lambda \neq \emptyset$ for all $\lambda \in (0, \lambda_0)$.

(ii) Assume that $u_\lambda \in S_\lambda$. Applying Proposition 2.2 with $v = u_\lambda$ and $\alpha(\omega) = \beta(\omega) = \lambda a(\omega)$, we have a positive solution $\underline{u}_\lambda$ of (1.6)-$(1.7)_\lambda$ such that $\underline{u}_\lambda \leq w$ for any $w > 0$ satisfying (4.1). In particular, we obtain $\underline{u}_\lambda \leq u$ for all $u \in S_\lambda$. This implies that $\underline{u}_\lambda$ is the minimal solution of S_λ. □

Lemma 4.2. *(i) Assume that $\underline{u}_{\lambda_1} \in S_{\lambda_1}$ and $\underline{u}_{\lambda_2} \in S_{\lambda_2}$ are minimal solutions with $0 < \lambda_1 < \lambda_2$. Then*

$$\frac{\underline{u}_{\lambda_1}}{\lambda_1} \leq \frac{\underline{u}_{\lambda_2}}{\lambda_2} \quad \text{in } \mathbf{R}^N. \tag{4.2}$$

In particular, $\underline{u}_{\lambda_1} < \underline{u}_{\lambda_2}$ in $\mathbf{R}^N$.

(ii) Let $\underline{u}_\lambda \in S_\lambda$ be the minimal solution. Then $\|\underline{u}_\lambda\|_{L^\infty(\mathbf{R}^N)} = O(\lambda)$ as $\lambda \to 0$.

(iii) Let $\overline{\lambda} = \sup\{\lambda > 0 : S_\lambda \neq \emptyset\}$. Then $\overline{\lambda} < \infty$.

Remark 4.1. As already mentioned in (ii) of Remark 2, the result (iii) of this lemma is essentially obtained by [33,37,24]. However, we give here a slight simple proof for convenience.

Proof. (i) Define $v = \underline{u}_{\lambda_2}/\lambda_2$. Then v satisfies

$$\begin{cases} -\Delta v - \dfrac{1}{2}x \cdot \nabla v - \dfrac{1}{p-1}v = \lambda_2^{p-1} v^p \geq \lambda_1^{p-1} v^p & \text{in } \mathbf{R}^N \quad \text{and} \\ \lim\limits_{r\to\infty} r^{2/(p-1)} v(r\omega) = a(\omega) & \text{for a.e. } \omega \in S^{N-1}. \end{cases}$$

Put $w = \lambda_1 v$. Then w satisfies

$$\begin{cases} -\Delta w - \dfrac{1}{2}x\cdot\nabla w - \dfrac{1}{p-1}w \geq w^p & \text{in } \mathbf{R}^N \quad \text{and} \\ \lim\limits_{r\to\infty} r^{2/(p-1)} w(r\omega) = \lambda_1 a(\omega) & \text{for a.e. } \omega \in S^{N-1}. \end{cases}$$

From (ii) of Lemma 4.1 we have $\underline{u}_{\lambda_1} \leq w$, which implies that (4.2) holds. In particular, we have $\underline{u}_{\lambda_1} < \underline{u}_{\lambda_2}$ in $\mathbf{R}^N$.

(ii) Take $\lambda_0 > 0$ so that $S_{\lambda_0} \neq \emptyset$. Let $\lambda \in (0, \lambda_0)$. From (i) of this lemma, we have

$$\frac{\underline{u}_\lambda}{\lambda} \leq \frac{\underline{u}_{\lambda_0}}{\lambda_0} \quad \text{in } \mathbf{R}^N.$$

Then we obtain $\|\underline{u}_\lambda\|_{L^\infty(\mathbf{R}^N)} \leq (\lambda/\lambda_0)\|\underline{u}_{\lambda_0}\|_{L^\infty(\mathbf{R}^N)}$ for $\lambda \in (0, \lambda_0)$. This implies that (ii) holds.

(iii) Assume that $S_\lambda \neq \emptyset$ for some $\lambda > 0$. Let $\underline{u}_\lambda \in S_\lambda$ be the minimal solution. Then $v = \underline{u}_\lambda/\lambda$ satisfies

$$\Delta v + \frac{1}{2}x\cdot\nabla v + \frac{1}{p-1}v + \underline{u}_\lambda^{p-1} v = 0 \qquad \text{in } \mathbf{R}^N. \tag{4.3}$$

Take $\lambda_0 \in (0, \lambda)$, and let $\underline{u}_{\lambda_0} \in S_{\lambda_0}$ be the minimal solution. Then, from (i) of this lemma, we have $\underline{u}_\lambda/\lambda \geq \underline{u}_{\lambda_0}/\lambda_0$. Hence, from (4.3) we have

$$\Delta v + \frac{1}{2}x\cdot\nabla v + \frac{1}{p-1}v + \lambda^{p-1}\left(\frac{\underline{u}_{\lambda_0}}{\lambda_0}\right)^{p-1} v \leq 0 \qquad \text{in } \mathbf{R}^N.$$

On the other hand, from Lemma 3.2 the eigenvalue problem

$$\begin{cases} -\Delta w - \dfrac{1}{2}x\cdot\nabla w - \dfrac{1}{p-1}w = \mu\left(\dfrac{\underline{u}_{\lambda_0}}{\lambda_0}\right)^{p-1} w \quad \text{in } \mathbf{R}^N, \\ w \in H^1_\rho(\mathbf{R}^N), \end{cases}$$

has the first eigenvalue $\mu_0 > 0$. By Proposition 3.1 we have $\lambda^{p-1} \leq \mu_0$. This implies that $\sup\{\lambda > 0 : S_\lambda \neq \emptyset\} \leq \mu_0^{1/(p-1)}$. □

Proof of Theorem 1. (i) Let $\overline{\lambda} = \sup\{\lambda > 0 : S_\lambda \neq \emptyset\}$. Then, from (i) of Lemma 4.1 and (iii) of Lemma 4.2, we have $0 < \overline{\lambda} < \infty$. By Lemma 4.1, for $\lambda \in (0, \overline{\lambda})$, $S_\lambda \neq \emptyset$ and there exists a minimal solution $\underline{u}_\lambda \in S_\lambda$. From (i) and (ii) of Lemma 4.2, $\underline{u}_\lambda$ is increasing in λ and satisfies $\|\underline{u}_\lambda\|_{L^\infty(\mathbf{R}^N)} = O(\lambda)$ as $\lambda \to 0$.

(ii) By the definition of $\overline{\lambda}$, we can conclude that (1.6)–(1.7)$_\lambda$ has no positive solution for $\lambda > \overline{\lambda}$. □

5. Existence of the second solution: Proof of Theorem 2

Let $\underline{u}_\lambda$ be the minimal positive solution of (1.6)-(1.7)$_\lambda$ for $\lambda \in (0, \overline{\lambda})$ obtained in Theorem 1. In order to find a second solution of (1.6)-(1.7)$_\lambda$ we introduce the following problem:

$$\begin{cases} \Delta u + \dfrac{1}{2}x \cdot \nabla u + \dfrac{1}{p-1}u + (u + \underline{u}_\lambda)^p - \underline{u}_\lambda^p = 0 \quad \text{in } \mathbf{R}^N, \\ \qquad u \in H^1_\rho(\mathbf{R}^N) \quad \text{and} \quad u > 0 \quad \text{in } \mathbf{R}^N. \end{cases} \tag{5.1$_\lambda$}$$

Clearly, we can get another positive solution $\overline{u}_\lambda = \underline{u}_\lambda + u_\lambda$ of (1.6)–(1.7)$_\lambda$, if (5.1)$_\lambda$ possesses a solution u_λ satisfying (5.2) below. In this section we show the following two propositions.

Proposition 5.1. *Let $p > (N+2)/N$ and $(N-2)p < N+2$. For $\lambda \in (0, \overline{\lambda})$, there exists a solution $u_\lambda \in C^2(\mathbf{R}^N)$ of* (5.1)$_\lambda$ *satisfying*

$$u_\lambda(x) = O(e^{-|x|^2/4}) \quad \text{as } |x| \to \infty. \tag{5.2}$$

Proposition 5.2. *Assume that $p > (N+2)/N$ and $(N-2)p < N+2$. Let u_λ be the solution of* (5.1)$_\lambda$ *obtained in Proposition* 5.1. *Then $u_\lambda \to u_0$ in $H^1_\rho(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$ as $\lambda \to 0$, where u_0 is the solution of the problem* (1.9).

As a consequence of Propositions 5.1 and 5.2 we obtain Theorem 2.

We show the existence of the solution of (5.1)$_\lambda$ by using a variational method. To this end we define the corresponding variational functional of (5.1)$_\lambda$ by

$$I_\lambda(u) = \frac{1}{2}\int_{\mathbf{R}^N} \left(|\nabla u|^2 - \frac{1}{p-1}u^2 \right) \rho dx - \int_{\mathbf{R}^N} G(u, \underline{u}_\lambda) \rho dx$$

with $u \in H^1_\rho(\mathbf{R}^N)$, where

$$G(t,s) = \frac{1}{p+1}(t+s)^{p+1} - \frac{1}{p+1}s^{p+1} - s^p t.$$

We know that the nontrivial critical point $u \in H^1_\rho(\mathbf{R}^N)$ of the functional I_λ is a weak solution of the equation in (5.1)$_\lambda$, that is, u satisfies

$$\int_{\mathbf{R}^N} \left(\nabla u \cdot \nabla \phi - \frac{1}{p-1}u\phi \right) \rho dx - \int_{\mathbf{R}^N} g(u, \underline{u}_\lambda)\phi \rho dx = 0$$

for any $\phi \in H^1_\rho(\mathbf{R}^N)$, where

$$g(t,s) = (t+s)^p - s^p.$$

We easily see that $u_\lambda \in C^2(\mathbf{R}^N)$ and $u_\lambda > 0$ in $\mathbf{R}^N$ from Proposition A.1 in Appendix A and Proposition 2.1.

First we investigate the properties of the functions $g(t,s)$ and $G(t,s)$.

Lemma 5.1. *(i) For $s_0>0$, there is a constant $C=C(s_0)>0$ such that*

$$0\le g(t,s)\le C(t+t^p),\qquad t\ge 0,\ 0\le s\le s_0.$$

(ii) For $\delta>0$, there is a constant $C=C(\delta)>0$ such that

$$0\le g(t,s)\le Ct,\qquad 0\le s,t\le\delta.$$

Furthermore, $C(\delta)\to 0$ as $\delta\to 0$.
(iii) We have

$$G(t,s)\ge\frac{1}{p+1}t^{p+1},\qquad s,t\ge 0.$$

(iv) For any $\varepsilon>0$ and $s_0>0$, there is a constant $C=C(\varepsilon,s_0)>0$ such that

$$G(t,s)-\frac{p}{2}s^{p-1}t^2\le\varepsilon t^2+Ct^{p+1},\qquad t\ge 0,\ 0\le s\le s_0.$$

(v) Put $c_p=\min\{1,p-1\}$. Then

$$g(t,s)t-(2+c_p)G(t,s)\ge-\frac{c_p p}{2}s^{p-1}t^2,\qquad s,t\ge 0.$$

Proof. (i) For $0\le s\le s_0$ we have

$$\lim_{t\to\infty}\frac{g(t,s)}{t^p}=1\quad\text{and}\quad\lim_{t\to 0}\frac{g(t,s)}{t}=ps^{p-1}$$

by using l'Hospital's rule. Hence we obtain (i).

(ii) For $0\le s,t\le\delta$ we have

$$g_t(t,s)=p(t+s)^{p-1}\le p(2\delta)^{p-1}.$$

Integrating the above on $[0,t]$ with respect t, we obtain $g(t,s)\le C(\delta)t$, where $C(\delta)=p(2\delta)^{p-1}$. Thus, $C(\delta)\to 0$ as $\delta\to 0$.

(iii) We have $G(0,s)=G_t(0,s)=0$ and $G_{tt}(t,s)=p(t+s)^{p-1}\ge pt^{p-1}$ for $t,s\ge 0$. By integrating on $[0,t]$ twice with respect t, we obtain (iii).

(iv) Put $h(t,s)=G(t,s)-(p/2)s^{p-1}t^2$. We have $h(0,s)=h_t(0,s)=h_{tt}(0,s)=0$. Then, by using l'Hospital's rule, we obtain

$$\lim_{t\to 0}\frac{h(t,s)}{t^2}=0.$$

By virtue of

$$\lim_{t\to\infty}\frac{h(t,s)}{t^{p+1}}=\frac{1}{p+1},$$

we obtain (iv).

(v) Define

$$H(t,s) = g(t,s)t - (2+c_p)G(t,s) + \frac{c_p p}{2} s^{p-1} t^2.$$

Then we have $H(0,s) = H_t(0,s) = H_{tt}(0,s) = 0$ and

$$H_{ttt}(t,s) = \begin{cases} p(p-1)(2-p)(t+s)^{p-3}s & \text{if } 1 < p < 2, \\ p(p-1)(p-2)(t+s)^{p-3}t & \text{if } \quad p \geq 2. \end{cases}$$

Thus $H_{ttt}(t,s) \geq 0$ for $s, t \geq 0$. By integrating on [0, t] three times with respect t, we obtain $H(t,s) \geq 0$ for $s, t \geq 0$. Thus (v) holds. □

Let $\underline{u}_\lambda$ be the minimal positive solution of (1.6)-$(1.7)_\lambda$ for $\lambda \in (0, \overline{\lambda})$. By Lemma 3.2 the corresponding eigenvalue problem

$$\begin{cases} -\Delta w - \frac{1}{2} x \cdot \nabla w - \frac{1}{p-1} w = \mu p \underline{u}_\lambda^{p-1} w \quad \text{in } \mathbf{R}^N, \\ w \in H^1_\rho(\mathbf{R}^N), \end{cases}$$

has the first eigenvalue $\mu(\lambda) > 0$. Furthermore, we have

$$\mu(\lambda) = \inf \left\{ \int_{\mathbf{R}^N} \left(|\nabla w|^2 - \frac{1}{p-1} w^2 \right) \rho dx : w \in H^1_\rho(\mathbf{R}^N),\ p \int_{\mathbf{R}^N} \underline{u}_\lambda^{p-1} w^2 \rho dx = 1 \right\}.$$

Then it follows that

$$\int_{\mathbf{R}^N} \left(|\nabla w|^2 - \frac{1}{p-1} w^2 \right) \rho dx \geq \mu(\lambda) p \int_{\mathbf{R}^N} \underline{u}_\lambda^{p-1} w^2 \rho dx \tag{5.3}$$

for any $w \in H^1_\rho(\mathbf{R}^N)$.

Lemma 5.2. *For $0 < \lambda < \overline{\lambda}$, we have $\mu(\lambda) > 1$. Moreover, $\mu(\lambda)$ is strictly decreasing in $\lambda \in (0, \overline{\lambda})$.*

Proof. Take $\lambda_1, \lambda_2 \in (0, \overline{\lambda})$ with $\lambda_1 < \lambda_2$. From (i) of Theorem 1 we have $\underline{u}_{\lambda_2} > \underline{u}_{\lambda_1}$ in $\mathbf{R}^N$, and hence $\underline{u}_{\lambda_2}^{p-1} > \underline{u}_{\lambda_1}^{p-1}$. By Proposition 3.2, we have $\mu(\lambda_2) < \mu(\lambda_1)$. Therefore, $\mu(\lambda)$ is strictly decreasing in λ.

Let $\lambda \in (0, \overline{\lambda})$, and let $\lambda_0 \in (\lambda, \overline{\lambda})$. Put $w = \underline{u}_{\lambda_0} - \underline{u}_\lambda$. Then $w > 0$ and w satisfies

$$\Delta w + \frac{1}{2} x \cdot \nabla w + \frac{1}{p-1} w + p \underline{u}_\lambda^{p-1} w \leq 0, \qquad x \in \mathbf{R}^N.$$

By Proposition 3.1 we have $\mu(\lambda) \geq 1$. Then $\mu(\lambda) \geq 1$ for $\lambda \in (0, \overline{\lambda})$. Since $\mu(\lambda)$ is strictly decreasing, we have $\mu(\lambda) > 1$ for all $\lambda \in (0, \overline{\lambda})$. □

In the following we verify the existence of nontrivial solution of $(5.1)_\lambda$ by means of the Mountain Pass lemma.

Lemma 5.3. *Assume that $\{u_k\}$ is the Palais-Smale sequence for $I_\lambda(u)$, that is,*

$$u_k \in H^1_\rho(\mathbf{R}^N), \quad \{I_\lambda(u_k)\} \text{ is bounded}, \quad \text{and} \quad I'_\lambda(u_k) \to 0 \text{ as } k \to \infty \tag{5.4}$$

in the dual space of $H^1_\rho(\mathbf{R}^N)$. Then $\{u_k\}$ is bounded in $H^1_\rho(\mathbf{R}^N)$.

Proof. Since $\{I_\lambda(u_k)\}$ is bounded, we have

$$\frac{1}{2}\int_{\mathbf{R}^N}\left(|\nabla u_k|^2 - \frac{1}{p-1}u_k^2\right)\rho dx - \int_{\mathbf{R}^N} G(u_k, \underline{u}_\lambda)\rho dx \le M \tag{5.5}$$

for some $M > 0$. Let $\varepsilon > 0$. From $I'_\lambda(u_k) \to 0$ as $k \to \infty$, we have, for sufficient large k,

$$\left|\int_{\mathbf{R}^N}\left(\nabla u_k \cdot \nabla\phi - \frac{1}{p-1}u_k\phi\right)\rho dx - \int_{\mathbf{R}^N} g(u_k, \underline{u}_\lambda)\phi\rho dx\right| \le \varepsilon\|\phi\|_{H^1_\rho}$$

for any $\phi \in H^1_\rho(\mathbf{R}^N)$. Putting $\phi = u_k/\|u_k\|_{H^1_\rho}$, we have

$$\left|\int_{\mathbf{R}^N}\left(|\nabla u_k|^2 - \frac{1}{p-1}u_k^2\right)\rho dx - \int_{\mathbf{R}^N} g(u_k, \underline{u}_\lambda)u_k\rho dx\right| \le \varepsilon\|u_k\|_{H^1_\rho}.$$

Then we obtain

$$\int_{\mathbf{R}^N}\left(|\nabla u_k|^2 - \frac{1}{p-1}u_k^2\right)\rho dx \ge \int_{\mathbf{R}^N} g(u_k, \underline{u}_\lambda)u_k\rho dx - \varepsilon\|u_k\|_{H^1_\rho}. \tag{5.6}$$

Put $c_p = \min\{1, p-1\}$. From (5.5) and (5.6) we have

$$\begin{aligned}(2+c_p)M &\ge \left(1 + \frac{c_p}{2}\right)\int_{\mathbf{R}^N}\left(|\nabla u_k|^2 - \frac{1}{p-1}u_k^2\right)\rho dx \\ &\quad -(2+c_p)\int_{\mathbf{R}^N} G(u_k, \underline{u}_\lambda)\rho dx \\ &\ge \frac{c_p}{2}\int_{\mathbf{R}^N}\left(|\nabla u_k|^2 - \frac{1}{p-1}u_k^2\right)\rho dx \\ &\quad + \int_{\mathbf{R}^N}\left(g(u_k, \underline{u}_\lambda)u_k - (2+c_p)G(u_k, \underline{u}_\lambda)\right)\rho dx - \varepsilon\|u_k\|_{H^1_\rho}.\end{aligned}$$

From (v) of Lemma 5.1, (5.3), and (3.2), it follows that

$$\begin{aligned}(2+c_p)M &\ge \frac{c_p}{2}\left(\int_{\mathbf{R}^N}\left(|\nabla u_k|^2 - \frac{1}{p-1}u_k^2\right)\rho dx - p\int_{\mathbf{R}^N} \underline{u}_\lambda^{p-1}u_k^2\rho dx\right) \\ &\quad -\varepsilon\|u_k\|_{H^1_\rho} \\ &\ge \frac{c_p}{2}\left(1 - \frac{1}{\mu(\lambda)}\right)\int_{\mathbf{R}^N}\left(|\nabla u_k|^2 - \frac{1}{p-1}u_k^2\right)\rho dx - \varepsilon\|u_k\|_{H^1_\rho} \\ &\ge \frac{c_p}{2}\left(1 - \frac{1}{\mu(\lambda)}\right)\left(1 - \frac{2}{N(p-1)}\right)\|\nabla u_k\|^2_{L^2_\rho} - \varepsilon\|u_k\|_{H^1_\rho}.\end{aligned}$$

We note here that $\mu(\lambda) > 1$ from Lemma 5.2. Therefore, $\{\|\nabla u_k\|_{L^2_\rho}\}$ is bounded, and hence, from (i) of Lemma 3.1, $\{u_k\}$ is bounded in $H^1_\rho(\mathbf{R}^N)$. □

Lemma 5.4. *The functional I_λ satisfies the Palais-Smale condition, that is, any Palais-Smale sequence contains a subsequence which converges in $H^1_\rho(\mathbf{R}^N)$.*

Proof. We show the case where $N \geq 3$. We can verify the case where $N = 2$ with a slight modification. Let $\{u_k\}$ be a Palais-Smale sequence, that is, (5.4) holds. By Lemma 5.3 we have $\{u_k\}$ is bounded in $H^1_\rho(\mathbf{R}^N)$. Then, from (ii) and (iii) of Lemma 3.1, there exist a subsequence that we still denote $\{u_k\}$ and a function $u \in H^1_\rho(\mathbf{R}^N)$ such that

$$u_k \rightharpoonup u \quad \text{weakly in } H^1_\rho(\mathbf{R}^N) \text{ as } k \to \infty, \tag{5.7}$$

$$u_k \to u \quad \text{strongly in } L^2_\rho(\mathbf{R}^N) \cap L^{p+1}_\rho(\mathbf{R}^N) \text{ as } k \to \infty. \tag{5.8}$$

We claim that $\|\nabla(u_k - u)\|_{L^2_\rho} \to 0$ as $k \to \infty$. We see that

$$\|\nabla(u_k - u)\|_{L^2_\rho} = \int_{\mathbf{R}^N} \nabla u_k \cdot (\nabla u_k - \nabla u)\rho dx - \int_{\mathbf{R}^N} \nabla u \cdot (\nabla u_k - \nabla u)\rho dx.$$

It follows from (5.7) that

$$\int_{\mathbf{R}^N} \nabla u \cdot (\nabla u_k - \nabla u)\rho dx \to 0 \quad \text{as } k \to \infty.$$

We observe that

$$\int_{\mathbf{R}^N} \nabla u_k \cdot (\nabla u_k - \nabla u)\rho dx = I'_\lambda(u_k)(u_k - u) + \frac{1}{p-1}\int_{\mathbf{R}^N} u_k(u_k - u)\rho dx + \int_{\mathbf{R}^N} g(u_k, \underline{u}_\lambda)(u_k - u)\rho dx.$$

Since $I'_\lambda(u_k) \to 0$ as $k \to \infty$, we have

$$|I'_\lambda(u_k)(u_k - u)| \leq |I'_\lambda(u_k)| \|u_k - u\|_{H^1_\rho} \to 0 \quad \text{as } k \to \infty.$$

From (i) of Lemma 5.1 we obtain

$$\left|\int_{\mathbf{R}^N} g(u_k, \underline{u}_\lambda)(u_k - u)\rho dx\right| \leq C\left(\int_{\mathbf{R}^N} u_k(u_k - u)\rho dx + \int_{\mathbf{R}^N} u_k^p(u_k - u)\rho dx\right)$$

for some constant $C > 0$. By using Hölder inequality and (5.8), we obtain

$$\left|\int_{\mathbf{R}^N} u_k(u_k - u)\rho dx\right| \leq \|u_k\|^2_{L^2_\rho} \|u_k - u\|^2_{L^2_\rho} \to 0 \quad \text{as } k \to \infty$$

and

$$\left|\int_{\mathbf{R}^N} u_k^p(u_k - u)\rho dx\right| \leq \|u_k\|^{p+1}_{L^{p+1}_\rho} \|u_k - u\|_{L^{p+1}_\rho} \to 0 \quad \text{as } k \to \infty.$$

Therefore, we have

$$\int_{\mathbf{R}^N} \nabla u_k \cdot (\nabla u_k - \nabla u)\rho dx \to 0 \quad \text{as } k \to \infty,$$

and conclude that $\|\nabla(u_k - u)\|_{L^2_\rho} \to 0$ as $k \to \infty$. From (i) of Lemma 3.1 we have $u_k \to u$ in $H^1_\rho(\mathbf{R}^N)$. □

Lemma 5.5. *There exist some constants* $\delta = \delta(\lambda) > 0$ *and* $\eta = \eta(\lambda) > 0$ *such that*

$$I_\lambda(u) \geq \eta > 0 \tag{5.9}$$

for all $u \in H^1_\rho(\mathbf{R}^N)$ *satisfying* $\|\nabla u\|_{L^2_\rho} = \delta$.

Proof. For any $u \in H^1_\rho(\mathbf{R}^N)$ we have

$$\begin{aligned} I_\lambda(u) = \frac{1}{2}\int_{\mathbf{R}^N} \left(|\nabla u|^2 - \frac{1}{p-1}u^2 - p\underline{u}_\lambda^{p-1}u^2\right)\rho dx \\ - \int_{\mathbf{R}^N} \left(G(u, \underline{u}_\lambda) - \frac{p}{2}\underline{u}_\lambda^{p-1}u^2\right)\rho dx \equiv J_1 - J_2. \end{aligned}$$

From (5.3) and Lemma 5.2 we obtain

$$J_1 \geq \frac{1}{2}\left(1 - \frac{1}{\mu(\lambda)}\right)\int_{\mathbf{R}^N} \left(|\nabla u|^2 - \frac{1}{p-1}u^2\right)\rho dx$$

with $\mu(\lambda) > 1$. Then, from (3.2), we have

$$J_1 \geq C_0\|\nabla u\|^2_{L^2_\rho}, \quad \text{where } C_0 = \frac{1}{2}\left(1 - \frac{1}{\mu(\lambda)}\right)\left(1 - \frac{2}{N(p-1)}\right) > 0.$$

From (iv) of Lemma 5.1, for any $\varepsilon > 0$ there is a constant $C_1 = C_1(\varepsilon, \|\underline{u}_\lambda\|_{L^\infty}) > 0$ such that

$$J_2 \leq \varepsilon \int_{\mathbf{R}^N} u^2 \rho dx + C_1 \int_{\mathbf{R}^N} u^{p+1}\rho dx.$$

From (i) and (iii) of Lemma 3.1 we have

$$J_2 \leq \frac{2}{N}\varepsilon\|\nabla u\|^2_{L^2_\rho} + C_1C_2\|\nabla u\|^{p+1}_{L^2_\rho}$$

for some constant $C_2 > 0$. Take $\varepsilon > 0$ so small that $\varepsilon < NC_0/2$. Then we have

$$I_\lambda(u) \geq C_3\|\nabla u\|^2_{L^2_\rho} - C_1C_2\|\nabla u\|^{p+1}_{L^2_\rho}, \quad \text{where } C_3 = C_0 - \frac{2}{N}\varepsilon > 0,$$

which implies that (5.9) holds for some $\delta > 0$ and $\eta > 0$. □

Define the corresponding functional of (1.9) by

$$I_0(u) = \frac{1}{2}\int_{\mathbf{R}^N}\left(|\nabla u|^2 - \frac{1}{p-1}u^2\right)\rho dx - \frac{1}{p+1}\int_{\mathbf{R}^N} u^{p+1}\rho dx$$

with $u \in H^1_\rho(\mathbf{R}^N)$. Let u_0 be the solution of the problem (1.9). Then u_0 satisfies

$$\int_{\mathbf{R}^N}\left(|\nabla u_0|^2 - \frac{1}{p-1}u_0^2\right)\rho dx = \int_{\mathbf{R}^N} u_0^{p+1}\rho dx. \tag{5.10}$$

Therefore, we have

$$I_0(u_0) = \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\mathbf{R}^N} u_0^{p+1}\rho dx. \tag{5.11}$$

Lemma 5.6. *Let u_0 be the solution of the problem* (1.9)*, and let* $0 < \lambda < \overline{\lambda}$*. Then*

(i) $I_\lambda(tu_0) < 0$ *for* $t > ((p+1)/2)^{1/(p-1)}$;
(ii) $\sup_{t>0} I_\lambda(tu_0) \le I_0(u_0)$.

Proof. From (5.10) we have

$$\begin{aligned} I_\lambda(tu_0) &= \frac{t^2}{2}\int_{\mathbf{R}^N}\left(|\nabla u_0|^2 - \frac{1}{p-1}u_0^2\right)\rho dx - \int_{\mathbf{R}^N} G(tu_0, \underline{u}_\lambda)\rho dx \\ &= \frac{t^2}{2}\int_{\mathbf{R}^N} u_0^{p+1}\rho dx - \int_{\mathbf{R}^N} G(tu_0, \underline{u}_\lambda)\rho dx. \end{aligned}$$

From (iii) of Lemma 5.1 we have

$$G(tu_0, \underline{u}_\lambda) \ge \frac{t^{p+1}}{p+1}u_0^{p+1}.$$

Then it follows that

$$I_\lambda(tu_0) \le \left(\frac{t^2}{2} - \frac{t^{p+1}}{p+1}\right)\int_{\mathbf{R}^N} u_0^{p+1}\rho dx. \tag{5.12}$$

Since $(t^2/2 - t^{p+1}/(p+1)) < 0$ for $t > ((p+1)/2)^{1/(p-1)}$, we obtain (i). From (5.11) and (5.12) we obtain

$$\begin{aligned} \sup_{t>0} I_\lambda(tu_0) &\le \sup_{t>0}\left(\frac{t^2}{2} - \frac{t^{p+1}}{p+1}\right)\int_{\mathbf{R}^N} u_0^{p+1}\rho dx \\ &= \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\mathbf{R}^N} u_0^{p+1}\rho dx = I_0(u_0), \end{aligned}$$

which implies that (ii) holds. □

Lemma 5.7. *For* $0 < \lambda < \overline{\lambda}$*, there exists a critical point* $u_\lambda \in H^1_\rho(\mathbf{R}^N)$ *of* $I_\lambda(u)$ *such that* $I_\lambda(u_\lambda) \le I_0(u_0)$*. Moreover,* $u_\lambda \in C^2(\mathbf{R}^N)$ *and* $u_\lambda(x) \to 0$ *as* $|x| \to \infty$.

Proof. From Lemma 5.4, $I_\lambda(u)$ satisfies the Palais-Smale condition. From (i) of Lemma 5.6, there exists a constant $T_1 > 0$ such that $e = T_1 u_0$ satisfies $\|\nabla e\|_{L^2_\rho} > \delta$ and $I_\lambda(e) \le 0$, where δ is the constant appearing in Lemma 5.5. Denote

$$c = \inf_{v\in\Gamma} \max_{s\in[0,1]} I_\lambda(v(s)),$$

where $\Gamma = \{v \in C([0,1]; H^1_\rho(\mathbf{R}^N)) : v(0) = 0,\ v(1) = e\}$. Then, from Lemma 5.5 and (ii) of Lemma 5.6, it follows that

$$0 < \eta \le c \le I_0(u_0).$$

The Mountain Pass Lemma [1, 5] enables us to find a critical point $u_\lambda \in H^1_\rho(\mathbf{R}^N)$ of $I_\lambda(u)$. Hence, u_λ is a weak solution of the equation in $(5.1)_\lambda$ and satisfies $I_\lambda(u_\lambda) \le I_0(u_0)$. By Proposition A.1 in Appendix A, we have $u_\lambda \in C^2(\mathbf{R}^N)$ and $u_\lambda(x) \to 0$ as $|x| \to \infty$. □

Proof of Proposition 5.1. The existence of solution u_λ of the problem $(5.1)_\lambda$ has been obtained by Lemma 5.7. Therefore it suffices to show (5.2). Take a constant c_0 so that $0 < c_0 < (N/2) - 1/(p-1)$. Recall that both $\underline{u}_\lambda(x)$ and $u_\lambda(x)$ tend to 0 as $|x| \to \infty$. Then, from (ii) of Lemma 5.1, there is a constant $R > 0$ such that

$$0 \le g(u_\lambda(x), \underline{u}_\lambda(x)) \le c_0 u_\lambda(x), \quad |x| \ge R. \tag{5.13}$$

Put $w(x) = C_1 e^{-|x|^2/4}$, where $C_1 = \max_{|x|\le R} u_\lambda(x) e^{|x|^2/4}$. Clearly, $w(x) \ge u_\lambda(x)$ for $|x| \le R$. Since $w \in H^1_\rho(\mathbf{R}^N)$ and w satisfies $-\nabla\cdot(\rho\nabla w) = (N/2)\rho w$ in $\mathbf{R}^N$, we have

$$\int_{\mathbf{R}^N} \nabla w \cdot \nabla\phi \rho dx = \frac{N}{2}\int_{\mathbf{R}^N} w\phi\rho dx \tag{5.14}$$

for any $\phi \in H^1_\rho(\mathbf{R}^N)$. Let $\phi(x) = (u_\lambda(x) - w(x))^+$, where $a^+ = \max\{a, 0\}$. Then $\phi \in H^1_\rho(\mathbf{R}^N)$, $\phi \equiv 0$ for $|x| \le R$, and

$$\nabla\phi = \begin{cases} \nabla u_\lambda - \nabla w & \text{if } u_\lambda \ge w, \\ 0 & \text{if } u_\lambda < w. \end{cases} \tag{5.15}$$

Now we claim that $\phi \equiv 0$ in $\mathbf{R}^N$. We observe that

$$\int_{\mathbf{R}^N} \left(\nabla u_\lambda \cdot \nabla\phi - \left(\frac{1}{p-1} + c_0\right) u_\lambda\phi\right)\rho dx = \int_{\mathbf{R}^N} \left(g(u_\lambda, \underline{u}_\lambda) - c_0 u_\lambda\right)\phi\rho dx.$$

From (5.13) and $\phi \equiv 0$ for $|x| \le R$ it follows that

$$\begin{aligned} &\int_{\mathbf{R}^N} \left(\nabla u_\lambda \cdot \nabla\phi - \left(\frac{1}{p-1} + c_0\right) u_\lambda\phi\right)\rho dx \\ &= \int_{|x|\ge R} \left(g(u_\lambda, \underline{u}_\lambda) - c_0 u_\lambda\right)\phi\rho dx \le 0. \end{aligned} \tag{5.16}$$

From (5.14) and $c_0 < (N/2) - 1/(p-1)$, we obtain

$$\begin{aligned}\int_{\mathbf{R}^N} \left(\nabla w \cdot \nabla \phi - \left(\frac{1}{p-1} + c_0 \right) w\phi \right) \rho dx \\ = \left(\frac{N}{2} - \frac{1}{p-1} - c_0 \right) \int_{\mathbf{R}^N} w\phi\rho dx \geq 0.\end{aligned} \tag{5.17}$$

Then, from (5.16) and (5.17) we obtain

$$\int_{\mathbf{R}^N} \left((\nabla u_\lambda - \nabla w) \cdot \nabla \phi - \left(\frac{1}{p-1} + c_0 \right) (u_\lambda - w)\phi \right) \rho dx \leq 0.$$

By virtue of (5.15) it follows that

$$\int_{\mathbf{R}^N} \left(|\nabla \phi|^2 - \left(\frac{1}{p-1} + c_0 \right) \phi^2 \right) \rho dx \leq 0.$$

From (i) of Lemma 3.1 we have

$$\left(\frac{N}{2} - \frac{1}{p-1} - c_0 \right) \int_{\mathbf{R}^N} \phi^2 \rho dx \leq 0.$$

This implies that $\phi \equiv 0$ in $\mathbf{R}^N$, and hence, $u_\lambda(x) \leq w(x) = C_1 e^{-|x|^2/4}$ for $x \in \mathbf{R}^N$. □

The next result is fundamental to the proof of Proposition 5.2.

Lemma 5.8. *Let* $M_\lambda = \sup_{x \in \mathbf{R}^N} u_\lambda(x)$ *for* $0 < \lambda < \overline{\lambda}$. *Then* $\liminf_{\lambda \to 0+} M_\lambda > 0$.

Proof. Assume to the contrary that $\liminf_{\lambda \to 0+} M_\lambda = 0$. Take a constant c_0 so that $0 < c_0 < (N/2) - 1/(p-1)$. Recall that $\|\underline{u}_\lambda\|_{L^\infty} \to 0$ as $\lambda \to 0$. From (ii) of Lemma 5.1, we can take a $\lambda > 0$ so that $g(u_\lambda(x), \underline{u}_\lambda(x)) \leq c_0 u_\lambda(x)$ for $x \in \mathbf{R}^N$. Then we have

$$\int_{\mathbf{R}^N} \left(|\nabla u_\lambda|^2 - \frac{1}{p-1} u_\lambda^2 \right) \rho dx = \int_{\mathbf{R}^N} g(u_\lambda, \underline{u}_\lambda) u_\lambda \rho dx \leq c_0 \int_{\mathbf{R}^N} u_\lambda^2 \rho dx.$$

It follows that

$$\int_{\mathbf{R}^N} |\nabla u_\lambda|^2 \rho dx \leq \left(c_0 + \frac{1}{p-1} \right) \int_{\mathbf{R}^N} u_\lambda^2 \rho dx < \frac{N}{2} \int_{\mathbf{R}^N} u_\lambda^2 \rho dx$$

with $u_\lambda \in H^1_\rho(\mathbf{R}^N)$. This contradicts (i) of Lemma 3.1. Hence, we obtain $\liminf_{\lambda \to 0+} M_\lambda > 0$. □

Proof of Proposition 5.2. Let $\{\lambda_k\}$ be a sequence such that $\lambda_k > \lambda_{k+1}$ and $\lambda_k \to 0$ as $k \to \infty$. For simplicity, one sets $v_k = u_{\lambda_k}$ and $\underline{v}_k = \underline{u}_{\lambda_k}$. The proof is divided into several steps.

Step 1. We claim that $\{v_k\}$ is bounded in $H^1_\rho(\mathbf{R}^N)$.

From Lemma 5.7 we have $I_{\lambda_k}(v_k) \le I_0(u_0)$, that is,

$$\frac{1}{2}\int_{\mathbf{R}^N}\left(|\nabla v_k|^2 - \frac{1}{p-1}v_k^2\right)\rho dx - \int_{\mathbf{R}^N} G(v_k, \underline{v}_k)\rho dx \le I_0(u_0).$$

Since v_k satisfies

$$\int_{\mathbf{R}^N}\left(|\nabla v_k|^2 - \frac{1}{p-1}v_k^2\right)\rho dx = \int_{\mathbf{R}^N} g(v_k, \underline{v}_k)v_k\rho dx,$$

we obtain

$$\begin{aligned}(2+c_p)I_0(u_0) &\ge \left(1+\frac{c_p}{2}\right)\int_{\mathbf{R}^N}\left(|\nabla v_k|^2 - \frac{1}{p-1}v_k^2\right)\rho dx \\ &\quad -(2+c_p)\int_{\mathbf{R}^N} G(v_k, \underline{v}_k)\rho dx \\ &\ge \frac{c_p}{2}\int_{\mathbf{R}^N}\left(|\nabla v_k|^2 - \frac{1}{p-1}v_k^2\right)\rho dx \\ &\quad +\int_{\mathbf{R}^N}\left(g(v_k, \underline{v}_k)v_k - (2+c_p)G(v_k, \underline{v}_k)\right)\rho dx,\end{aligned}$$

where $c_p = \min\{1, p-1\}$. From (v) of Lemma 5.1 and (5.3), it follows that

$$\begin{aligned}(2+c_p)I_0(u_0) &\ge \frac{c_p}{2}\left(\int_{\mathbf{R}^N}\left(|\nabla v_k|^2 - \frac{1}{p-1}v_k^2\right)\rho dx - p\int_{\mathbf{R}^N}\underline{v}_k^{p-1}v_k^2\rho dx\right) \\ &\ge \frac{c_p}{2}\left(1-\frac{1}{\mu(\lambda_k)}\right)\int_{\mathbf{R}^N}\left(|\nabla v_k|^2 - \frac{1}{p-1}v_k^2\right)\rho dx.\end{aligned}$$

Since $\mu(\lambda)$ is strictly decreasing and $\mu(\lambda) > 1$ by Lemma 5.2, we have $\mu(\lambda_k) > \mu(\lambda_1) > 1$. From (3.2) we obtain

$$(2+c_p)I_0(u_0) \ge \frac{c_p}{2}\left(1-\frac{1}{\mu(\lambda_1)}\right)\left(1-\frac{2}{N(p-1)}\right)\|\nabla v_k\|^2_{L^2_\rho},$$

which implies that $\{\|\nabla v_k\|_{L^2_\rho}\}$ is bounded. Hence, $\{v_k\}$ is bounded in $H^1_\rho(\mathbf{R}^N)$.

Step 2. We show that there exist a subsequence that we still denote $\{v_k\}$ and a function $v_0 \in H^1_\rho(\mathbf{R}^N)$ such that $v_k \to v_0$ in $H^1_\rho(\mathbf{R}^N)$ as $k \to \infty$.

Since $\{v_k\}$ is bounded in $H^1_\rho(\mathbf{R}^N)$, from (ii) and (iii) of Lemma 3.1, there exist a subsequence (still denoted by $\{v_k\}$) and some $v_0 \in H^1_\rho(\mathbf{R}^N)$ such that

$$v_k \rightharpoonup v_0 \quad \text{weakly in } H^1_\rho(\mathbf{R}^N) \text{ as } k \to \infty,$$

$$v_k \to v_0 \quad \text{strongly in } L^2_\rho(\mathbf{R}^N) \cap L^{p+1}_\rho(\mathbf{R}^N) \text{ as } k \to \infty.$$

We claim that $\|\nabla(v_k - v_0)\|_{L^2_\rho} \to 0$ as $k \to \infty$. We observe that

$$\|\nabla(v_k - v_0)\|_{L^2_\rho} = \int_{\mathbf{R}^N} \nabla v_k \cdot (\nabla v_k - \nabla v_0)\rho dx - \int_{\mathbf{R}^N} \nabla v_0 \cdot (\nabla v_k - \nabla v_0)\rho dx$$

and

$$\int_{\mathbf{R}^N} \nabla v_k \cdot (\nabla v_k - \nabla v_0)\rho dx = \frac{1}{p-1}\int_{\mathbf{R}^N} v_k(v_k - u)\rho dx + \int_{\mathbf{R}^N} g(v_k, \underline{v}_k)(v_k - v_0)\rho dx.$$

By the similar argument as in the proof of Lemma 5.4, we obtain $\|\nabla(v_k - v_0)\|_{L^2_\rho} \to 0$ as $k \to \infty$, and hence, $v_k \to v_0$ in $H^1_\rho(\mathbf{R}^N)$ as $k \to \infty$.

Step 3. We show that $v_0 = u_0$, where u_0 is the solution of the problem (1.9). Furthermore, we have $v_k \to u_0$ in $H^1_\rho(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$ as $k \to \infty$.

First we show that v_0 satisfies the equation in (1.9). Since $v_k \to v_0$ in $H^1_\rho(\mathbf{R}^N)$ by Step 2, it suffices to prove that

$$\int_{\mathbf{R}^N} g(v_k, \underline{v}_k)\phi\rho dx \to \int_{\mathbf{R}^N} v_0^p \phi\rho dx \quad \text{as } k \to \infty \tag{5.18}$$

for any $\phi \in H^1_\rho(\mathbf{R}^N)$. From $v_k \to v_0$ in $L^2_\rho(\mathbf{R}^N) \cap L^{p+1}_\rho(\mathbf{R}^N)$, there exist a subsequence (still denoted by $\{v_k\}$) and a function $h \in L^2_\rho(\mathbf{R}^N) \cap L^{p+1}_\rho(\mathbf{R}^N)$ such that

$$v_k(x) \le h(x) \quad \text{a.e. } x \in \mathbf{R}^N \tag{5.19}$$

for $k = 1, 2, \ldots$, and $v_k \to v_0$ a.e. $x \in \mathbf{R}^N$. (See, e.g., [2].) By virtue of $\|\underline{v}_k\|_{L^\infty} \to 0$ as $k \to \infty$, we have

$$g(v_k, \underline{v}_k) = (v_k - \underline{v}_k)^p - \underline{v}_k^p \to v_0^p \quad \text{a.e. } x \in \mathbf{R}^N.$$

From (i) of Lemma 5.1 and (5.19) it follows that

$$g(v_k, \underline{v}_k) \le C(v_k + v_k^p) \le C(h + h^p) \quad \text{a.e. } x \in \mathbf{R}^N.$$

By the Hölder inequality we have

$$\int_{\mathbf{R}^N} (h + h^p)\phi\rho dx \le \|h\|_{L^2_\rho}\|\phi\|_{L^2_\rho} + \|h\|^p_{L^{p+1}_\rho}\|\phi\|_{L^{p+1}_\rho} < \infty.$$

Therefore, by the Lebesgue convergence theorem, we obtain (5.18). Hence, v_0 satisfies the equation in (1.9).

Next we show $v_0 > 0$. From Proposition A.1 in Appendix A, $v_0 \in C^2(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$ and $|\nabla v_0| \in L^\infty(\mathbf{R}^N)$. By (i) of Proposition A.2, $\{v_k\}$ is bounded in $C^1(\mathbf{R}^N)$. Thus $\{v_k - v_0\}$ is bounded in $C^1(\mathbf{R}^N)$. Recall that $v_k - v_0 \to 0$ in

$H^1_\rho(\mathbf{R}^N)$ by Step 2. Then, by (ii) of Proposition A.2 we have $v_k \to v_0$ in $L^\infty(\mathbf{R}^N)$, and hence $v_0 \geq 0$. Lemma 5.8 yields $v_0 \not\equiv 0$. Thus $v_0 > 0$ by Proposition 2.1. Therefore, v_0 solves the problem (1.9). Since the solution of the problem (1.9) is unique by [25, Corollary 2], we conclude that $v_0 = u_0$. In particular, we obtain $v_k \to u_0$ in $H^1_\rho(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$ as $k \to \infty$.

Let λ_k be a sequence satisfying $\lambda_k \to 0$ as $k \to 0$. Then, by Steps 1-3, there exists a subsequence (still denoted by $\{\lambda_k\}$) such that $u_{\lambda_k} \to u_0$ in $H^1_\rho(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$ as $k \to 0$, which implies that $u_\lambda \to u_0$ in $H^1_\rho(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$ as $\lambda \to 0$. This completes the proof of Proposition 5.2. □

Appendix A.

Proposition A.1. *Let $u \in H^1_\rho(\mathbf{R}^N)$ be a solution of*

$$\Delta u + \frac{1}{2} x \cdot \nabla u + f(x, u) = 0 \qquad \text{in } \mathbf{R}^N, \tag{A.1}$$

where f is Hölder continuous and satisfies

$$|f(x, u)| \leq C(u + u^p), \quad x \in \mathbf{R}^N,\ u \in [0, \infty), \tag{A.2}$$

for some constants $C > 0$ and $p > 1$, $(N-2)p < N + 2$. Then $u \in C^2(\mathbf{R}^N)$, and both $u(x)$ and $|\nabla u(x)|$ tend to 0 as $|x| \to \infty$.

We prove Proposition A.1 by following the idea of Escobedo and Kavian [8]. First we prepare the following lemma.

Lemma A.1. *(i) Let $u \in H^1_\rho(\mathbf{R}^N)$. Then $u \in L^r(\mathbf{R}^N)$ and $|x||\nabla u| \in L^r(\mathbf{R}^N)$ for all $r \in [1, 2]$.*

(ii) Assume that $u \in L^2_\rho(\mathbf{R}^N) \cap L^q(\mathbf{R}^N)$ for some $q > 2$. Then $u \in L^r(\mathbf{R}^N)$ for all $r \in [2, q]$.

(iii) Assume that $|\nabla u| \in L^2_\rho(\mathbf{R}^N) \cap L^q(\mathbf{R}^N)$ for some $q > 2$. Then $|x||\nabla u| \in L^r(\mathbf{R}^N)$ for all $r \in [2, q)$.

(iv) Let $u \in L^2_\rho(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$. Then $u \in L^q(\mathbf{R}^N)$ for all $q > 2$.

(v) Let $|\nabla u| \in L^2_\rho(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$. Then $|x||\nabla u| \in L^q(\mathbf{R}^N)$ for all $q > 2$.

Proof. (i) It is clear if $r = 2$. For $1 \leq r < 2$ we have

$$\begin{aligned} \int_{\mathbf{R}^N} u^r dx &= \int_{\mathbf{R}^N} u^r \rho^{r/2} \rho^{-r/2} dx \\ &\leq \left(\int_{\mathbf{R}^N} u^2 \rho dx \right)^{r/2} \left(\int_{\mathbf{R}^N} \rho^{-r/(2-r)} dx \right)^{(2-r)/2} < \infty \end{aligned}$$

and

$$\int_{\mathbf{R}^N} |x|^r |\nabla u|^r dx = \int_{\mathbf{R}^N} |x|^r |\nabla u|^r \rho^{r/2} \rho^{-r/2} dx$$
$$\leq \left(\int_{\mathbf{R}^N} |\nabla u|^2 \rho dx \right)^{r/2}$$
$$\times \left(\int_{\mathbf{R}^N} |x|^{2r/(2-r)} \rho^{-r/(2-r)} dx \right)^{(2-r)/2} < \infty.$$

(ii) Let $r \in (2, q)$. Put $s = (q-2)/(r-2) > 1$. Then we have

$$\int_{\mathbf{R}^N} |u|^r dx \leq \int_{\mathbf{R}^N} u^{q/s} u^{(2s-2)/s} \rho^{(s-1)/s} dx$$
$$\leq \left(\int_{\mathbf{R}^N} u^q dx \right)^{1/s} \left(\int_{\mathbf{R}^N} u^2 \rho dx \right)^{(s-1)/s} < \infty.$$

(iii) Let $r \in (2, q)$. Put $s = (q-2)/(r-2) > 1$. Then we have

$$\int_{\mathbf{R}^N} |x|^r |\nabla u|^r dx \leq \sup_{x \in \mathbf{R}^N} \left(|x|^r \rho^{-(s-1)/s} \right) \int_{\mathbf{R}^N} |\nabla u|^{q/s} |\nabla u|^{(2s-2)/s} \rho^{(s-1)/s} dx$$
$$\leq \sup_{x \in \mathbf{R}^N} \left(|x|^r \rho^{-(s-1)/s} \right) \left(\int_{\mathbf{R}^N} |\nabla u|^q dx \right)^{1/s}$$
$$\times \left(\int_{\mathbf{R}^N} |\nabla u|^2 \rho dx \right)^{(s-1)/s} < \infty.$$

(iv) For $q > 2$, we have

$$\int_{\mathbf{R}^N} |u|^q dx \leq \int_{\mathbf{R}^N} |u|^q \rho dx \leq \|u\|_{L^\infty}^{q-2} \int_{\mathbf{R}^N} u^2 \rho dx < \infty.$$

(v) For $q > 2$, we have

$$\int_{\mathbf{R}^N} |x|^q |\nabla u|^q dx \leq \sup_{x \in \mathbf{R}^N} (|x|^q \rho^{-1}) \int_{\mathbf{R}^N} |\nabla u|^q \rho dx$$
$$\leq \sup_{x \in \mathbf{R}^N} (|x|^q \rho^{-1}) \|\nabla u\|_{L^\infty}^{q-2} \int_{\mathbf{R}^N} |\nabla u|^2 \rho dx < \infty.$$

□

Set

$$h(x, u) = \frac{1}{2} x \cdot \nabla u + u + f(x, u). \tag{A.3}$$

Then the solution u of (A.1) satisfies

$$-\Delta u + u = h(x, u) \qquad \text{in } \mathbf{R}^N. \tag{A.4}$$

We show the following:

Lemma A.2. *Let* $u \in H^1_\rho(\mathbf{R}^N)$ *be a solution of* (A.1) *such that* $u \in L^q(\mathbf{R}^N)$ *for some* $q > 2$. *Then* $h(x,u)$ *defined by* (A.3) *satisfies* $h \in L^{q/p}(\mathbf{R}^N)$.

Proof. From (i) and (ii) of Lemma A.1, $u \in L^r(\mathbf{R}^N)$ for all $r \in [1,q]$. Since f satisfies (A.2), we have $f(x,u) \in L^r(\mathbf{R}^N)$ for $1 \le r \le q/p$. Then it suffices to show that $|x||\nabla u| \in L^{q/p}(\mathbf{R}^N)$. If $q/p \le 2$, from (i) of Lemma A.1, the result is established. So we assume that $q/p > 2$. From $u \in H^1_\rho(\mathbf{R}^N)$, we have $h \in L^2(\mathbf{R}^N)$. Then, by using the equation (A.4) we obtain $u \in W^{2,2}(\mathbf{R}^N)$. By the Sobolev embeddings, we have

$$|\nabla u| \in L^{r_1}(\mathbf{R}^N), \quad \frac{1}{r_1} = \frac{1}{2} - \frac{1}{N} \quad \text{if } N > 2,$$

$$|\nabla u| \in L^r(\mathbf{R}^N) \quad \text{for all } r > 2 \quad \text{if } N = 2.$$

In the cases where $N = 2$ or $r_1 > q/p$, from (iii) of Lemma A.1, we have $|x||\nabla u| \in L^{q/p}(\mathbf{R}^N)$, and the result is established. In the cases where $N > 2$ and $r_1 \le q/p$, from (iii) of Lemma A.1, we have $|x||\nabla u| \in L^r(\mathbf{R}^N)$ for all $r \in [1, r_1)$. Then $h \in L^r(\mathbf{R}^N)$ for $r \in [1, r_1)$, and so $u \in W^{2,r}(\mathbf{R}^N)$ for $r \in [1, r_1)$. The Sobolev embeddings now yield

$$|\nabla u| \in L^r(\mathbf{R}^N) \quad \text{for all } r \in [1, r_2), \quad \frac{1}{r_2} = \frac{1}{r_1} - \frac{1}{N}, \quad \text{if } r_1 < N,$$

$$|\nabla u| \in L^\infty(\mathbf{R}^N), \quad \text{if } r_1 > N.$$

In the cases where $r_1 > N$ or $r_2 > q/p$, we have $|x||\nabla u| \in L^{q/p}(\mathbf{R}^N)$. In the cases where $r_2 \le p/q$, we have $|x||\nabla u| \in L^r(\mathbf{R}^N)$ for all $r \in [1, r_2)$. Repeating the arguments in finite times, we obtain $|x||\nabla u| \in L^{q/p}(\mathbf{R}^N)$. □

Proof of Proposition A.1. We show the case where $N \ge 3$. We can verify the case where $N = 2$ with a slight modification.

First we show $u \in L^\infty(\mathbf{R}^N)$. From (iii) of Lemma 3.1, $u \in L^{q_0}(\mathbf{R}^N)$, where $q_0 = 2N/(N-2)$. Then, from Lemma A.2, we have $h \in L^{q_0/p}(\mathbf{R}^N)$. By using the equation (A.4) we obtain $u \in W^{2,q_0/p}(\mathbf{R}^N)$. Then the Sobolev embedding implies that

$$u \in L^{q_1}(\mathbf{R}^N), \quad \frac{1}{q_1} = \frac{p}{q_0} - \frac{2}{N} \quad \text{if } q_0 < \frac{pN}{2},$$

$$u \in L^\infty(\mathbf{R}^N), \quad \text{if } q_0 > \frac{pN}{2}.$$

We note that $q_1 > q_0$ from the assumption $p < (N+2)/(N-2)$. If $q_0 < pN/2$, from Lemma A.2, we have $h \in L^{q_1/p}(\mathbf{R}^N)$, and hence $u \in W^{2,q_1/p}(\mathbf{R}^N)$. Then the Sobolev embedding implies that

$$u \in L^{q_2}(\mathbf{R}^N), \quad \frac{1}{q_2} = \frac{p}{q_1} - \frac{2}{N} \quad \text{if } q_1 < \frac{pN}{2},$$

$$u \in L^\infty(\mathbf{R}^N), \quad \text{if } q_1 > \frac{pN}{2}.$$

Repeating above arguments in finite times, we obtain $u \in L^\infty(\mathbf{R}^N)$.

From (iv) of Lemma A.1 we have $u \in L^{pq}(\mathbf{R}^N)$ for all $q > N$. Then form Lemma A.2 we have $h \in L^q(\mathbf{R}^N)$, and hence $u \in W^{2,q}(\mathbf{R}^N)$ for all $q > N$. By the Sobolev embedding theorem, $u \in C^{1,\theta}(\mathbf{R}^N)$ for some $\theta \in (0, 1)$. Then, since f is Hölder continuous, we obtain $u \in C^2(\mathbf{R}^N)$. We note that $C_0^\infty(\mathbf{R}^N)$ is dense in $W^{2,q}(\mathbf{R}^N)$. Then, by using the Sobolev embedding theorem again, we obtain $u(x) \to 0$ and $|\nabla u(x)| \to 0$ as $|x| \to \infty$. □

Proposition A.2. *(i) Assume that $\{u_k\}$ is bounded in $H^1_\rho(\mathbf{R}^N)$, and that u_k satisfies*

$$\Delta u_k + \frac{1}{2} x \cdot \nabla u_k + f_k(x, u_k) = 0 \qquad \text{in } \mathbf{R}^N \tag{A.5$_k$}$$

for $k = 1, 2, \ldots$. We assume in (A.5)$_k$ that f_k satisfies

$$|f_k(x, u)| \le C(u + u^p), \quad x \in \mathbf{R}^N, \ u \in [0, \infty),$$

for some constants $C > 0$ and $p > 1$, $(N-2)p < N+2$, where C and p are independent of k. Then $\{u_k\}$ is bounded in $C^1(\mathbf{R}^N)$.

(ii) Assume that $\{u_k\}$ is bounded in $C^1(\mathbf{R}^N)$, and that $u_k \to 0$ in $H^1_\rho(\mathbf{R}^N)$. Then $u_k \to 0$ in $L^\infty(\mathbf{R}^N)$ as $k \to \infty$.

From the proof of Lemma A.1 we obtain the following results.

Lemma A.3. *(i) Assume that $\{u_k\}$ is bounded in $H^1_\rho(\mathbf{R}^N)$. Then $\{u_k\}$ and $\{|x||\nabla u_k|\}$ are bounded in $L^r(\mathbf{R}^N)$ for all $r \in [1, 2]$.*

(ii) Assume that $\{u_k\}$ is bounded in $L^2_\rho(\mathbf{R}^N) \cap L^q(\mathbf{R}^N)$ for some $q > 2$. Then $\{u_k\}$ is bounded in $L^r(\mathbf{R}^N)$ for all $r \in [2, q]$.

(iii) Assume that $\{|\nabla u_k|\}$ is bounded in $L^2_\rho(\mathbf{R}^N) \cap L^q(\mathbf{R}^N)$ for some $q > 2$. Then $\{|x||\nabla u_k|\}$ is bounded in $L^r(\mathbf{R}^N)$ for all $r \in [2, q)$.

(iv) Let $\{u_k\}$ is bounded in $L^2_\rho(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$. Then $\{u_k\}$ is bounded in $L^q(\mathbf{R}^N)$ for all $q > 2$.

(v) Let $\{|\nabla u_k|\}$ is bounded in $L^2_\rho(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$. Then $\{|x||\nabla u_k|\}$ is bounded in $L^q(\mathbf{R}^N)$ for all $q > 2$.

Set

$$h_k(x, u) = \frac{1}{2} x \cdot \nabla u + u + f_k(x, u).$$

Then the solution u_k of (A.5)$_k$ satisfies

$$-\Delta u_k + u_k = h_k(x, u_k) \qquad \text{in } \mathbf{R}^N.$$

By the similar arguments in the proof of Lemma A.2, we obtain the following results.

Lemma A.4. *Assume that u_k is a solution of* $(A.5)_k$ *such that* $\{u_k\}$ *is bounded in* $L^q(\mathbf{R}^N)$ *for some* $q > 2$. *Then* $\{h_k(\cdot, u_k)\}$ *is bounded in* $L^{q/p}(\mathbf{R}^N)$.

Proof of Proposition A.2. (i) Following the arguments in the proof of Proposition A.1, we obtain $\{u_k\}$ is bounded in $L^\infty(\mathbf{R}^N)$. From (iv) of Lemma A.3, $\{u_k\}$ is bounded in $L^{pq}(\mathbf{R}^N)$ for all $q > N$. Then $\{h_k\}$ is bounded in $L^q(\mathbf{R}^N)$, and hence, $\{u_k\}$ is bounded in $W^{2,q}(\mathbf{R}^N)$ for all $q > N$. By the Sobolev embedding theorem, $\{u_k\}$ is bounded in $C^1(\mathbf{R}^N)$.

(ii) Let $q > N$. Since $\{u_k\}$ is bounded in $C^1(\mathbf{R}^N)$ and $u_k \to 0$ in $H^1_\rho(\mathbf{R}^N)$, we have

$$\|u_k\|^q_{L^q} \le \int_{\mathbf{R}^N} u_k^q \rho dx \le \|u_k\|^{q-2}_{L^\infty} \int_{\mathbf{R}^N} u_k^2 \rho dx \to 0 \quad \text{as } k \to \infty$$

and

$$\|\nabla u_k\|^q_{L^q} \le \int_{\mathbf{R}^N} |\nabla u_k|^q \rho dx \le \|\nabla u_k\|^{q-2}_{L^\infty} \int_{\mathbf{R}^N} |\nabla u_k|^2 \rho dx \to 0 \quad \text{as } k \to \infty.$$

Hence, $u_k \to 0$ in $W^{1,q}(\mathbf{R}^N)$ as $k \to \infty$ for $q > N$. Then by the Sobolev embedding theorem, we have $u_k \to 0$ in $L^\infty(\mathbf{R}^N)$. □

Appendix B.

Lemma B.1. *Let u be a positive function on $\mathbf{R}^N$, and let w be a function defined by* (1.5) *on* $\mathbf{R}^N \times (0, \infty)$. *Then w satisfies* $(1.2)_\lambda$ *in the sense of* $L^1_{\mathrm{loc}}(\mathbf{R}^N)$, *if and only if u satisfies* $(1.7)_\lambda$.

In order to prove Lemma B.1 we need the following

Lemma B.2. *Let w be the function in Lemma* B.1. *Put* $B_R = \{x \in \mathbf{R}^N : |x| < R\}$, *where* $R > 0$. *Then*

$$|x|^{2/(p-1)} w(x,t) \to \lambda a(x/|x|) \quad \textit{as } t \to 0 \quad \textit{for a.e. } x \in B_R \tag{B.1}$$

if and only if

$$w(\omega, t) \to \lambda a(\omega) \quad \textit{as } t \to 0 \quad \textit{for a.e. } \omega \in S^{N-1}. \tag{B.2}$$

Proof. Define $E \subset \mathbf{R}^N$ such that if $x \in \mathbf{R}^N \setminus E$ then

$$|x|^{2/(p-1)} w(x,t) \to \lambda a(x/|x|) \quad \text{as } t \to 0.$$

It follows from (1.5) that if $x \in \mathbf{R}^N \setminus E$ and $\mu > 0$ then

$$|\mu x|^{2/(p-1)} w(\mu x, t) = |x|^{2/(p-1)} w(x, t/\mu^2) \to \lambda a(x/|x|) \quad \text{as } t \to 0.$$

This implies that $x \in \mathbf{R}^N \setminus E$ if and only if $\mu x \in \mathbf{R}^N \setminus E$ for all $\mu > 0$. Thus we have

$$x \in E \quad \text{if and only if} \quad \mu x \in E \quad \text{for all } \mu > 0. \tag{B.3}$$

Put $E_S = E \cap S^{N-1}$ and $E_B = E \cap B_R$. Then it follows from (B.3) that

$$\int_{E_B} dx = \int_0^R N\omega_N r^{N-1} \left(\int_{E_S} dS \right) dr = \omega_N R^N \int_{E_S} dS,$$

where ω_N is the volume of unit ball in $\mathbf{R}^N$ and dS denotes the surface measure on E_S. This implies that (B.1) holds if and only if (B.2) holds. □

Proof of Lemma B.1. From (1.5) we see that $|x|^{2/(p-1)} w(x,t) = |y|^{2/(p-1)} u(y)$, where $y = x/\sqrt{t}$. In particular, we have

$$w(\omega, t) = r^{2/(p-1)} u(r\omega), \quad \text{where } r = 1/\sqrt{t}. \tag{B.4}$$

Assume that u satisfies $(1.7)_\lambda$. Then, from (B.4), we obtain (B.2). Lemma B.2 implies that (B.1) holds for any $R > 0$. Now, fix a compact set $K \subset \mathbf{R}^N$. Then, by the Lebesgue dominated convergence theorem, we have

$$\begin{aligned} &\int_K |w(x,t) - \lambda a(x/|x|)|\, dx \\ &\quad = \int_K |x|^{-2/(p-1)} \left| |x|^{2/(p-1)} w(x,t) - \lambda a(x/|x|) \right| dx \to 0 \end{aligned}$$

as $t \to 0$. Therefore, w satisfies $(1.2)_\lambda$ in the sense of $L^1_{\mathrm{loc}}(\mathbf{R}^N)$.

Conversely, assume that w satisfies $(1.2)_\lambda$ in the sense of $L^1_{\mathrm{loc}}(\mathbf{R}^N)$. Then (B.1) holds for $R > 0$, which implies (B.2) by Lemma B.2. From (B.4) we find that u satisfies $(1.7)_\lambda$. This completes the proof of Lemma B.1. □

Acknowledgements. The author is grateful to the referee for his carefully reading of the manuscript and for many giving invaluable comments and suggestions. He also gives thanks to Professor Ryuji Kajikiya of Nagasaki Institute of Applied Science for helpful comments.

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