

Moduli spaces of holomorphic triples over compact Riemann surfaces

Steven B. Bradlow^{1,2} · Oscar García-Prada^{1,3,5,6} · Peter B. Gothen^{1,4,5}

Received: 9 December 2002 / Revised version: 11 March 2003 /

Published online: 24 November 2003 – © Springer-Verlag 2003

Abstract. A holomorphic triple over a compact Riemann surface consists of two holomorphic vector bundles and a holomorphic map between them. After fixing the topological types of the bundles and a real parameter, there exist moduli spaces of stable holomorphic triples. In this paper we study non-emptiness, irreducibility, smoothness, and birational descriptions of these moduli spaces for a certain range of the parameter. Our results have important applications to the study of the moduli space of representations of the fundamental group of the surface into unitary Lie groups of indefinite signature ([5, 7]). Another application, that we study in this paper, is to the existence of stable bundles on the product of the surface by the complex projective line.

1. Introduction

Let X be a closed Riemann surface of genus $g \geq 2$. The theory of holomorphic triples has its origins [13, 4] in the search for solutions to certain gauge theoretic equations on X , obtained by dimensional reduction of the Hermitian–Einstein

S.B. BRADLOW

Department of Mathematics, University of Illinois, Urbana, IL 61801, USA
(e-mail: bradlow@math.uiuc.edu)

O. GARCÍA-PRADA

Instituto de Matemáticas y Física Fundamental, Consejo Superior de Investigaciones Científicas, Serrano 113 bis, 28006, Madrid, Spain
(e-mail: oscar.garcia-prada@uam.es)

P.B. GOTHEN

Departamento de Matemática Pura, Faculdade de Ciências, Universidade do Porto, Rua do Campo Alegre 687, 4169-007, Porto, Portugal (e-mail: pbgothen@fc.up.pt)

¹ Members of VBAC (Vector Bundles on Algebraic Curves), which is partially supported by EAGER (EC FP5 Contract no. HPRN-CT-2000-00099) and by EDGE (EC FP5 Contract no. HPRN-CT-2000-00101).

² Partially supported by the National Science Foundation under grant DMS-0072073.

³ Partially supported by the Ministerio de Ciencia y Tecnología (Spain) under grant BFM2000-0024.

⁴ Partially supported by the Fundação para a Ciência e a Tecnologia (Portugal) through the Centro de Matemática da Universidade do Porto and through grant no. SFRH/BPD/1606/2000.

⁵ Partially supported by the Portugal/Spain bilateral Programme Acciones Integradas, grant nos. HP2000-0015 and AI-01/24.

⁶ Partially supported by a British EPSRC grant (October–December 2001).

equation in 4 dimensions. More precisely, solutions to the Hermitian–Einstein equation on $X \times \mathbb{P}^1$ which are invariant under the standard action of $SU(2)$ on \mathbb{P}^1 correspond to solutions to the so-called *vortex equations* on X . The Hitchin–Kobayashi correspondence states that a solution to the Hermitian–Einstein equation on $X \times \mathbb{P}^1$ gives rise to a polystable holomorphic bundle and that, conversely, any stable holomorphic bundle admits a Hermitian–Einstein metric. The counterpart on X states that there is a Hitchin–Kobayashi correspondence between solutions to the coupled vortex equations and *stable holomorphic triples*. A holomorphic triple consists of a pair of holomorphic vector bundles, E_1 and E_2 , over X and a holomorphic map $\phi: E_2 \rightarrow E_1$ between them. An important feature of the stability condition for triples is that it depends on a real parameter α , corresponding to the fact that there is a real parameter in the vortex equations; thus one is led to the concept of α -stability of a holomorphic triple. This parallels the fact that when studying Hermitian–Einstein metrics and stable bundles on $X \times \mathbb{P}^1$ it is necessary to choose a polarization on this complex surface. We note that, as usual, there are corresponding concepts of α -polystable and α -semistable triples (see Section 2 below for precise definitions).

It was shown in [4] (see also [13]) that projective moduli spaces for holomorphic triples exist. (Later a direct construction was given by Schmitt using geometric invariant theory [25].) Since the stability condition depends on the real parameter α , so do the moduli spaces. Fixing the topological invariants $n_i = \text{rk}(E_i)$ and $d_i = \text{deg}(E_i)$, we denote the moduli space of α -polystable triples with the given invariants by

$$\mathcal{N}_\alpha = \mathcal{N}_\alpha(n_1, n_2, d_1, d_2),$$

and the moduli space of α -stable triples by $\mathcal{N}_\alpha^s \subseteq \mathcal{N}_\alpha$. In this paper we address the questions of smoothness, non-emptiness and irreducibility of these moduli spaces.

Before describing our results in more detail, we explain our motivation, which comes from the problem of determining the connected components of the moduli space of representations of the fundamental group of X in $PU(p, q)$. A detailed study of this moduli space appears in a companion paper [7]; in the following we briefly outline the main ideas. The first point to notice is that we may as well study the connected components of the moduli space of projectively flat $U(p, q)$ bundles on X . This moduli space can be divided into disjoint closed subspaces $\mathcal{M}(a, b)$ indexed by a pair of integers (a, b) , the Chern classes obtained from a reduction of structure group to the maximal compact subgroup $U(p) \times U(q)$. The values of (a, b) are bounded by the Milnor–Wood type inequality

$$\left| \frac{aq - bp}{p + q} \right| \leq \min\{p, q\}(g - 1).$$

For each allowed value of (a, b) one expects the space $\mathcal{M}(a, b)$ to be non-empty and connected, thus forming a connected component of the moduli space.

By the work of Hitchin [18, 19], Donaldson [12], Simpson [26–29] and Corlette [9], the moduli spaces $\mathcal{M}(a, b)$ are homeomorphic to moduli spaces of so-called $U(p, q)$ -Higgs bundles on X : these are pairs (E, Φ) , where E is a holomorphic vector bundle which decomposes as a direct sum $E = V \oplus W$ and the Higgs field $\Phi: E \rightarrow E \otimes K$ is of the form

$$\phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$$

with respect to the direct sum decomposition of E . Here K is the canonical line bundle of X and the invariants a and b appear as the degrees of V and W respectively. The L^2 -norm of the Higgs field gives us a Bott-Morse function on the moduli space (cf. Hitchin [18, 19]). Thus, connectedness of the spaces $\mathcal{M}(a, b)$ will be a consequence of connectedness of the corresponding subspaces of local minima. In the case of flat $U(2, 2)$ -bundles, it was shown in [15] that the local minima are represented by Higgs bundles for which either β or γ vanishes. One of the main results in [7] is that this is true in general. The crucial observation is now that there is a bijective correspondence between $U(p, q)$ -Higgs bundles (E, Φ) with $\beta = 0$ or $\gamma = 0$ and holomorphic triples: if, say, $\gamma = 0$, we obtain a holomorphic triple $T = (E_1, E_2, \phi)$ by setting $E_1 = V \otimes K$, $E_2 = W$ and $\phi = \beta$. It turns out that (E, Φ) is (poly)stable as a $U(p, q)$ -Higgs bundle if and only if the corresponding holomorphic triple T is α -(poly)stable for $\alpha = 2g - 2$. It follows that the subspace of local minima on $\mathcal{M}(a, b)$ is isomorphic to a moduli space of $(2g - 2)$ -polystable holomorphic triples. Thus the results of the present paper imply results on non-emptiness and connectedness of the moduli spaces $\mathcal{M}(a, b)$. We refer the reader to [7] for the precise statements.

We now return to our main subject of study, the holomorphic triples. In order for \mathcal{N}_α to be non-empty, one must have $\alpha \geq \alpha_m$ with $\alpha_m = d_1/n_1 - d_2/n_2 \geq 0$. In the case $n_1 \neq n_2$ there is also a finite upper bound α_M . When the parameter α varies, the nature of the α -stability condition only changes for a discrete number of so-called *critical values* of α (see Section 2 for the precise statements). We can now state our main results.

Theorem A. (1) *A triple $T = (E_1, E_2, \phi)$ of type (n_1, n_2, d_1, d_2) is α_m -polystable if and only if $\phi = 0$ and E_1 and E_2 are polystable. We thus have*

$$\mathcal{N}_{\alpha_m}(n_1, n_2, d_1, d_2) \cong M(n_1, d_1) \times M(n_2, d_2).$$

where $M(n, d)$ denotes the moduli space of polystable bundles of rank n and degree d . In particular, $\mathcal{N}_{\alpha_m}(n_1, n_2, d_1, d_2)$ is non-empty and irreducible.

(2) *If $\alpha > \alpha_m$ is any value such that $2g - 2 \leq \alpha$ (and $\alpha < \alpha_M$ if $n_1 \neq n_2$) then the moduli space $\mathcal{N}_\alpha^s(n_1, n_2, d_1, d_2)$ is non-empty, irreducible, and smooth of dimension $(g - 1)(n_1^2 + n_2^2 - n_1n_2) - n_1d_2 + n_2d_1 + 1$. Moreover:*

- If $n_1 = n_2 = n$ then the moduli space $\mathcal{N}_\alpha^s(n, n, d_1, d_2)$ is birationally equivalent to a \mathbb{P}^N -fibration over $M^s(n, d_2) \times \text{Sym}^{d_1-d_2}(X)$, where $M^s(n, d_2)$ denotes the subspace of stable bundles of type (n, d_2) , $\text{Sym}^{d_1-d_2}(X)$ is the symmetric product, and the fiber dimension is $N = n(d_1 - d_2) - 1$.
 - If $n_1 > n_2$ then the moduli space $\mathcal{N}_\alpha^s(n_1, n_2, d_1, d_2)$ is birationally equivalent to a \mathbb{P}^N -fibration over $M^s(n_1 - n_2, d_1 - d_2) \times M^s(n_2, d_2)$, where the fiber dimension is $N = n_2d_1 - n_1d_2 + n_2(n_1 - n_2)(g - 1) - 1$.
 - If $n_1 < n_2$ then the moduli space $\mathcal{N}_\alpha^s(n_1, n_2, d_1, d_2)$ is birationally equivalent to a \mathbb{P}^N -fibration over $M^s(n_2 - n_1, d_2 - d_1) \times M^s(n_1, d_1)$, where the fiber dimension is $N = n_2d_1 - n_1d_2 + n_1(n_2 - n_1)(g - 1) - 1$.
- (3) If $n_1 \neq n_2$ then the moduli space $\mathcal{N}_{\alpha_M}(n_1, n_2, d_1, d_2)$ is non-empty and irreducible. Moreover

$$\mathcal{N}_{\alpha_M}(n_1, n_2, d_1, d_2) \cong \begin{cases} M(n_2, d_2) \times M(n_1 - n_2, d_1 - d_2) & \text{if } n_1 > n_2 \\ M(n_1, d_1) \times M(n_2 - n_1, d_2 - d_1) & \text{if } n_1 < n_2. \end{cases}$$

Our strategy for studying the moduli spaces is similar in spirit to the one used by Thaddeus [31]: basically it consists in obtaining a good understanding of the moduli space for a particular (large) value of α and then keeping track of how the moduli space changes as α varies. In the following we explain this in more detail and outline the contents of the paper.

After recalling the basic facts about holomorphic triples in Section 2, we go on to study extensions and deformations of triples in Section 3. Here we show that the quasi-projective variety $\mathcal{N}_\alpha^s \subseteq \mathcal{N}_\alpha$ corresponding to α -stable triples is smooth for all values of α greater than or equal to $2g - 2$ (Theorem 3.8).

In Sections 4 and 5 we examine how the moduli spaces differ for values of α on opposite sides of a critical value. If $\mathcal{N}_{\alpha_c^\pm}$ denote the moduli spaces for values of α above and below a critical value α_c , we denote the loci along which they differ by $\mathcal{S}_{\alpha_c^\pm}$ respectively. Our main result (Theorem 5.12) is that for all $\alpha \geq 2g - 2$ the codimension of $\mathcal{S}_{\alpha_c^\pm}$ is strictly positive. It follows that the number of irreducible components of the spaces \mathcal{N}_α^s are the same for all α satisfying $\alpha \geq 2g - 2$ and $\alpha_m < \alpha < \alpha_M$. In order to estimate the codimension of the $\mathcal{S}_{\alpha_c^\pm}$ we need to estimate the dimension of certain spaces of extensions of triples. It is notable that this requires us to consider objects more general than triples, namely the *holomorphic chains* studied in [1]. The rather technical details are in Section 4: the main result is Proposition 4.3 which is then used to deduce the key Proposition 4.7.

Next we turn to the question of understanding the moduli spaces \mathcal{N}_α for large values of the parameter α . After obtaining some preliminary results in Section 6, we consider the case of triples with $n_1 \neq n_2$ in Section 7. Let \mathcal{N}_L denote the moduli space of α -polystable triples for α between α_M and the largest critical value smaller than α_M . We show that this ‘large α ’ moduli space is birationally equivalent to a \mathbb{P}^N -fibration over a product of moduli spaces of stable bundles

(Theorem 7.7). Combining this fact with our codimension estimates we obtain our main results on non-emptiness and irreducibility of the moduli spaces \mathcal{N}_α and \mathcal{N}_α^s ; these appear as Theorem 7.9 and Corollary 7.10.

In Section 8 we obtain analogous results in the case when $n_1 = n_2$. Even though there is no upper limit to α in this case, the moduli spaces do stabilize for α sufficiently large (Theorem 8.6) and hence it makes sense to consider the large α moduli space \mathcal{N}_L also in this case. The birational description of \mathcal{N}_L is given in Theorem 8.15, while the main results on non-emptiness and irreducibility are in Theorem 8.16.

Finally, in Section 9, we go back to the origins of the theory of holomorphic triples and apply our results on moduli of triples to deduce the existence of $SU(2)$ -invariant Hermitian–Einstein metrics on complex vector bundles on $X \times \mathbb{P}^1$; equivalently, our results imply the existence of stable vector bundles on $X \times \mathbb{P}^1$.

This paper and its companion [7] form a substantially revised version of the preprint [6]. The main results proved in this paper were announced in the note [5]. In that note we claim (without proof) that for $\alpha \geq 2g - 2$, the moduli spaces \mathcal{N}_α are irreducible without imposing the conditions in (2) or (3) of Theorem A. This is a reasonable conjecture, which we hope to come back to in a future publication.

Acknowledgements. We thank the Mathematics Departments of the University of Illinois at Urbana-Champaign, the University Autónoma of Madrid and the University of Aarhus, the Department of Pure Mathematics of the University of Porto, the Mathematical Sciences Research Institute of Berkeley, the Mathematical Institute of the University of Oxford, and the Erwin Schrödinger International Institute for Mathematical Physics in Vienna for their hospitality during various stages of this research. We thank Ron Donagi, Tomás Gómez, Rafael Hernández, Nigel Hitchin, Alastair King, Vicente Muñoz, Peter Newstead, and S. Ramanan for many insights and patient explanations.

2. Definitions and basic facts

2.1. Holomorphic triples and their moduli spaces

Let X be a compact Riemann surface (some of what follows is also true also for a compact Kähler manifold [13, 1]). Recall ([4] and [13]) that a *holomorphic triple* $T = (E_1, E_2, \phi)$ on X consists of two holomorphic vector bundles E_1 and E_2 on X and a holomorphic map $\phi: E_2 \rightarrow E_1$. A homomorphism from $T' = (E'_1, E'_2, \phi')$ to $T = (E_1, E_2, \phi)$ is a commutative diagram

$$\begin{array}{ccc} E'_2 & \xrightarrow{\phi'} & E'_1 \\ \downarrow & & \downarrow \\ E_2 & \xrightarrow{\phi} & E_1, \end{array}$$

where the vertical arrows are holomorphic maps. A triple $T' = (E'_1, E'_2, \phi')$ is a subtriple of $T = (E_1, E_2, \phi)$ if the sheaf homomorphisms $E'_1 \rightarrow E_1$ and $E'_2 \rightarrow E_2$ are injective. A subtriple $T' \subset T$ is called *proper* if $T' \neq 0$ and $T' \neq T$.

Definition 2.1. For any $\alpha \in \mathbb{R}$ the α -degree and α -slope of T are defined to be

$$\begin{aligned} \text{deg}_\alpha(T) &= \text{deg}(E_1) + \text{deg}(E_2) + \alpha \text{rk}(E_2), \\ \mu_\alpha(T) &= \frac{\text{deg}_\alpha(T)}{\text{rk}(E_1) + \text{rk}(E_2)} \\ &= \mu(E_1 \oplus E_2) + \alpha \frac{\text{rk}(E_2)}{\text{rk}(E_1) + \text{rk}(E_2)}, \end{aligned}$$

where $\text{deg}(E)$, $\text{rk}(E)$ and $\mu(E) = \text{deg}(E)/\text{rk}(E)$ are the degree, rank and slope of E , respectively.

We say $T = (E_1, E_2, \phi)$ is α -stable if

$$\mu_\alpha(T') < \mu_\alpha(T)$$

for any proper subtriple $T' = (E'_1, E'_2, \phi')$. Sometimes it is convenient to use

$$\Delta_\alpha(T') = \mu_\alpha(T') - \mu_\alpha(T), \tag{2.1}$$

in terms of which the α -stability of T is equivalent to $\Delta_\alpha(T') < 0$ for any proper subtriple T' . We define α -semistability by replacing the above strict inequality with a weak inequality. A triple is called α -polystable if it is the direct sum of α -stable triples of the same α -slope.

Write $\mathbf{n} = (n_1, n_2)$ and $\mathbf{d} = (d_1, d_2)$. We denote by

$$\mathcal{N}_\alpha = \mathcal{N}_\alpha(\mathbf{n}, \mathbf{d}) = \mathcal{N}_\alpha(n_1, n_2, d_1, d_2)$$

the moduli space of α -polystable triples $T = (E_1, E_2, \phi)$ which have $\text{rk}(E_i) = n_i$ and $\text{deg}(E_i) = d_i$ for $i = 1, 2$. The subspace of α -stable triples is denoted by \mathcal{N}_α^s . We refer to $(\mathbf{n}, \mathbf{d}) = (n_1, n_2, d_1, d_2)$ as the *type* of the triple.

There are certain necessary conditions in order for α -semistable triples to exist. Let $\mu_i = d_i/n_i$ for $i = 1, 2$. We define

$$\alpha_m = \mu_1 - \mu_2, \tag{2.2}$$

$$\alpha_M = \left(1 + \frac{n_1 + n_2}{|n_1 - n_2|}\right)(\mu_1 - \mu_2), \quad n_1 \neq n_2. \tag{2.3}$$

Proposition 2.2. [4, Theorem 6.1] *The moduli space $\mathcal{N}_\alpha(n_1, n_2, d_1, d_2)$ is a complex analytic variety, which is projective when α is rational. A necessary condition for $\mathcal{N}_\alpha(n_1, n_2, d_1, d_2)$ to be non-empty is*

$$0 \leq \alpha_m \leq \alpha \leq \alpha_M \quad \text{if } n_1 \neq n_2,$$

$$0 \leq \alpha_m \leq \alpha \quad \text{if } n_1 = n_2.$$

Remark 2.3. If $\alpha_m = 0$ and $n_1 \neq n_2$ then $\alpha_m = \alpha_M = 0$ and the moduli space of α stable triples is empty unless $\alpha = 0$.

A direct construction of these moduli spaces has been given by Schmitt [25] using geometric invariant theory.

Given a triple $T = (E_1, E_2, \phi)$ one has the dual triple $T^* = (E_2^*, E_1^*, \phi^*)$, where E_i^* is the dual of E_i and ϕ^* is the transpose of ϕ . The following is not difficult to prove ([4, Proposition 3.16]).

Proposition 2.4. *The α -(semi)stability of T is equivalent to the α -(semi)stability of T^* . The map $T \mapsto T^*$ defines a bijection*

$$\mathcal{N}_\alpha(n_1, n_2, d_1, d_2) = \mathcal{N}_\alpha(n_2, n_1, -d_2, -d_1),$$

which is moreover an isomorphism.

This can be used to restrict our study to $n_1 \geq n_2$ and appeal to duality to deal with the case $n_1 < n_2$.

2.2. Critical values

A holomorphic triple $T = (E_1, E_2, \phi)$ of type (n_1, n_2, d_1, d_2) is strictly α -semi-stable if and only if it has a proper subtriple $T' = (E'_1, E'_2, \phi')$ such that $\mu_\alpha(T') = \mu_\alpha(T)$, i.e.

$$\mu(E'_1 \oplus E'_2) + \alpha \frac{n'_2}{n'_1 + n'_2} = \mu(E_1 \oplus E_2) + \alpha \frac{n_2}{n_1 + n_2}. \tag{2.4}$$

There are two ways in which this can happen: The first one is if there exists a subtriple T' such that

$$\frac{n'_2}{n'_1 + n'_2} = \frac{n_2}{n_1 + n_2}, \text{ and} \\ \mu(E'_1 \oplus E'_2) = \mu(E_1 \oplus E_2).$$

In this case the terms containing α drop from (2.4) and T is strictly α -semistable for all values of α . We refer to this phenomenon as α -independent semistability. This cannot happen if $\text{GCD}(n_2, n_1 + n_2, d_1 + d_2) = 1$. The other way in which strict α -semistability can happen is if equality holds in (2.4) but

$$\frac{n'_2}{n'_1 + n'_2} \neq \frac{n_2}{n_1 + n_2}. \tag{2.5}$$

The values of α for which this happens are called critical values.

Definition 2.5. We say that $\alpha \in [\alpha_m, \infty)$ is a critical value if there exist integers n'_1, n'_2, d'_1 and d'_2 such that

$$\frac{d'_1 + d'_2}{n'_1 + n'_2} + \alpha \frac{n'_2}{n'_1 + n'_2} = \frac{d_1 + d_2}{n_1 + n_2} + \alpha \frac{n_2}{n_1 + n_2},$$

that is,

$$\alpha = \frac{(n_1 + n_2)(d'_1 + d'_2) - (n'_1 + n'_2)(d_1 + d_2)}{n'_1 n_2 - n_1 n'_2},$$

with $0 \leq n'_i \leq n_i$, $(n'_1, n'_2, d'_1, d'_2) \neq (n_1, n_2, d_1, d_2)$, $(n'_1, n'_2) \neq (0, 0)$ and $n'_1 n_2 \neq n_1 n'_2$. We say that α is generic if it is not critical.

Proposition 2.6. [4] Fix (n_1, n_2, d_1, d_2) .

- (1) The critical values of α form a discrete subset of $\alpha \in [\alpha_m, \infty)$, where α_m is as in (2.2).
- (2) If $n_1 \neq n_2$ the number of critical values is finite and lies in the interval $[\alpha_m, \alpha_M]$, where α_M is as in (2.3).
- (3) The stability criteria for two values of α lying between two consecutive critical values are equivalent; thus the corresponding moduli spaces are isomorphic.
- (4) If α is generic and $\text{GCD}(n_2, n_1 + n_2, d_1 + d_2) = 1$, then α -semistability is equivalent to α -stability.

For the application of triples to $U(p, q)$ -Higgs bundles ([7]; see also the Introduction), it is important to have criteria to rule out strict α -semistability when $\alpha = 2g - 2$, where g is the genus of the surface. One such criterion, dealing actually with any integral values of α , is given by the following.

Lemma 2.7. Let m be an integer such that $\text{GCD}(n_1 + n_2, d_1 + d_2 - mn_1) = 1$. Then

- (1) $\alpha = m$ is not a critical value,
- (2) there are no α -independent semistable triples.

Proof. To prove (1), suppose that $\alpha = m$ is a critical value. There exist then a triple T and a proper subtriple T' so that

$$(d'_1 + d'_2 + mn'_2)(n_1 + n_2) = (d_1 + d_2 + mn_2)(n'_1 + n'_2).$$

Thus $n_1 + n_2$ divides $(d_1 + d_2 + mn_2)(n'_1 + n'_2)$. But $n_1 + n_2 > n'_1 + n'_2$, so we get that $\text{GCD}(n_1 + n_2, d_1 + d_2 + mn_2) > 1$. Writing $d_1 + d_2 + mn_2 = d_1 + d_2 - mn_1 + m(n_1 + n_2)$, we see that $\text{GCD}(n_1 + n_2, d_1 + d_2 - mn_1) > 1$, in contradiction with the hypothesis. To prove (2), we show that $\text{GCD}(n_2, n_1 +$

$n_2, d_1 + d_2) = 1$, from which the result follows by (4) in Proposition 2.6. Suppose that $\text{GCD}(n_2, n_1 + n_2, d_1 + d_2) \neq 1$. Then there is (n'_1, n'_2, d'_1, d'_2) such that $\frac{n'_2}{n'_1+n'_2} = \frac{n_2}{n_1+n_2}$ and $\frac{d'_1+d'_2}{n'_1+n'_2} = \frac{d_1+d_2}{n_1+n_2}$. It follows that

$$\frac{d'_1 + d'_2 - mn'_1}{n'_1 + n'_2} = \frac{d_1 + d_2 - mn_1}{n_1 + n_2},$$

and hence $\text{GCD}(n_1 + n_2, d_1 + d_2 - mn_1) \neq 1$, in contradiction with the hypothesis. □

2.3. Vortex equations

There is a correspondence between stability and the existence of solutions to certain gauge-theoretic equations on a triple $T = (E_1, E_2, \phi)$, known as the *vortex equations* ([4] and [13]). The vortex equations

$$\begin{aligned} \sqrt{-1}\Lambda F(E_1) + \phi\phi^* &= \tau_1 \text{Id}_{E_1}, \\ \sqrt{-1}\Lambda F(E_2) - \phi^*\phi &= \tau_2 \text{Id}_{E_2}, \end{aligned} \tag{2.6}$$

are equations for Hermitian metrics on E_1 and E_2 . Here Λ is contraction by the Kähler form of a metric on X (normalized so that $\text{vol}(X) = 2\pi$), $F(E_i)$ is the curvature of the unique connection on E_i compatible with the Hermitian metric and the holomorphic structure of E_i , and τ_1 and τ_2 are real parameters satisfying $d_1 + d_2 = n_1\tau_1 + n_2\tau_2$. Here ϕ^* is the adjoint of ϕ with respect to the Hermitian metrics. One has the following.

Theorem 2.8. [4, Theorem 5.1] *A solution to (2.6) exists if and only if T is α -polystable for $\alpha = \tau_1 - \tau_2$.*

Using the vortex interpretation of the moduli space of triples one can easily identify the moduli space of triples for $\alpha = \alpha_m$.

Proposition 2.9. *A triple $T = (E_1, E_2, \phi)$ is α_m -polystable if and only if $\phi = 0$ and E_1 and E_2 are polystable. We thus have*

$$\mathcal{N}_{\alpha_m}(n_1, n_2, d_1, d_2) \cong M(n_1, d_1) \times M(n_2, d_2),$$

where $M(n_i, d_i)$ is the moduli space of semistable bundles of rank n_i and degree d_i .

Proof. Consider equations (2.6) on T . If $\alpha = \alpha_m$ then $\tau_1 = \mu_1$ and $\tau_2 = \mu_2$ and hence in order to have solutions of (2.6) we must have $\phi = 0$. In this case, (2.6) say that the Hermitian metrics on E_1 and E_2 have constant central curvature. But this is equivalent to the polystability of E_1 and E_2 by the theorem of Narasimhan and Seshadri [24]. □

3. Extensions and deformations of triples

In order to analyse the differences between the moduli spaces \mathcal{N}_α as α changes, as well as the smoothness properties of the moduli space for a given value of α , we need to study the homological algebra of triples. This is done by considering the hypercohomology of a certain complex of sheaves, in a similar way to what is done in the study of infinitesimal deformations by Biswas and Ramanan [3]. In fact, it is a special case of the more general situation considered in [16].

3.1. Extensions

Let $T' = (E'_1, E'_2, \phi')$ and $T'' = (E''_1, E''_2, \phi'')$ be two triples and, as usual, let

$$\begin{aligned} (\mathbf{n}', \mathbf{d}') &= (n'_1, n'_2, d'_1, d'_2), \\ (\mathbf{n}'', \mathbf{d}'') &= (n''_1, n''_2, d''_1, d''_2), \end{aligned}$$

where $n'_i = \text{rk}(E'_i)$, $n''_i = \text{rk}(E''_i)$, $d'_i = \text{deg}(E'_i)$ and $d''_i = \text{deg}(E''_i)$. Let $\text{Hom}(T'', T')$ denote the linear space of homomorphisms from T'' to T' , and let $\text{Ext}^1(T'', T')$ denote the linear space of equivalence classes of extensions of the form

$$0 \longrightarrow T' \longrightarrow T \longrightarrow T'' \longrightarrow 0,$$

where by this we mean a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E'_1 & \longrightarrow & E_1 & \longrightarrow & E''_1 & \longrightarrow & 0 \\ & & \phi' \uparrow & & \phi \uparrow & & \phi'' \uparrow & & \\ 0 & \longrightarrow & E'_2 & \longrightarrow & E_2 & \longrightarrow & E''_2 & \longrightarrow & 0. \end{array}$$

Hence, to analyse $\text{Ext}^1(T'', T')$ one considers the complex of sheaves

$$C^\bullet(T'', T'): E''_1{}^* \otimes E'_1 \oplus E''_2{}^* \otimes E'_2 \xrightarrow{c} E''_2{}^* \otimes E'_1, \tag{3.1}$$

where the map c is defined by

$$c(\psi_1, \psi_2) = \phi' \psi_2 - \psi_1 \phi''.$$

Proposition 3.1. *There are natural isomorphisms*

$$\begin{aligned} \text{Hom}(T'', T') &\cong \mathbb{H}^0(C^\bullet(T'', T')), \\ \text{Ext}^1(T'', T') &\cong \mathbb{H}^1(C^\bullet(T'', T')), \end{aligned}$$

and a long exact sequence associated to the complex $C^\bullet(T'', T')$:

$$\begin{aligned} 0 &\longrightarrow \mathbb{H}^0(C^\bullet(T'', T')) \longrightarrow H^0(E''_1{}^* \otimes E'_1 \oplus E''_2{}^* \otimes E'_2) \longrightarrow H^0(E''_2{}^* \otimes E'_1) \\ &\longrightarrow \mathbb{H}^1(C^\bullet(T'', T')) \longrightarrow H^1(E''_1{}^* \otimes E'_1 \oplus E''_2{}^* \otimes E'_2) \longrightarrow H^1(E''_2{}^* \otimes E'_1) \\ &\longrightarrow \mathbb{H}^2(C^\bullet(T'', T')) \longrightarrow 0. \end{aligned} \tag{3.2}$$

Proof. The proof is omitted since it is very similar to that given in [3] in the study of deformations, and it is a special case of a much more general result proved in [16]. □

We introduce the following notation:

$$\begin{aligned} h^i(T'', T') &= \dim \mathbb{H}^i(C^\bullet(T'', T')), \\ \chi(T'', T') &= h^0(T'', T') - h^1(T'', T') + h^2(T'', T'). \end{aligned} \tag{3.3}$$

Proposition 3.2. *For any holomorphic triples T' and T'' we have*

$$\begin{aligned} \chi(T'', T') &= \chi(E_1''^* \otimes E_1') + \chi(E_2''^* \otimes E_2') - \chi(E_2''^* \otimes E_1') \\ &= (1 - g)(n_1''n_1' + n_2''n_2' - n_2''n_1') \\ &\quad + n_1''d_1' - n_1'd_1'' + n_2''d_2' - n_2'd_2'' - n_2''d_1' + n_1'd_2'', \end{aligned}$$

where $\chi(E) = \dim H^0(E) - \dim H^1(E)$ is the Euler characteristic of E .

Proof. Immediate from the long exact sequence (3.2) and the Riemann–Roch formula. □

Corollary 3.3. *For any extension $0 \rightarrow T' \rightarrow T \rightarrow T'' \rightarrow 0$ of triples,*

$$\chi(T, T) = \chi(T', T') + \chi(T'', T'') + \chi(T'', T') + \chi(T', T''). \tag{□}$$

Remark 3.4. Proposition 3.2 shows that $\chi(T'', T')$ depends only on the topological invariants $(\mathbf{n}', \mathbf{d}')$ and $(\mathbf{n}'', \mathbf{d}'')$ of T' and T'' . Whenever convenient we shall therefore use the notation

$$\chi(\mathbf{n}'', \mathbf{d}'', \mathbf{n}', \mathbf{d}') = \chi(T'', T').$$

3.2. Vanishing of \mathbb{H}^0 and \mathbb{H}^2

The following vanishing results play a central role in our study.

Proposition 3.5. *Suppose that T' and T'' are α -semistable.*

- (1) *If $\mu_\alpha(T') < \mu_\alpha(T'')$ then $\mathbb{H}^0(C^\bullet(T'', T')) \cong 0$.*
- (2) *If $\mu_\alpha(T') = \mu_\alpha(T'')$ and T'' is α -stable, then*

$$\mathbb{H}^0(C^\bullet(T'', T')) \cong \begin{cases} \mathbb{C} & \text{if } T' \cong T'' \\ 0 & \text{if } T' \not\cong T''. \end{cases}$$

Proof. By Proposition 3.1 we can identify $\mathbb{H}^0(C^\bullet(T'', T'))$ with $\text{Hom}(T'', T')$. The statements (1) and (2) are thus the direct analogs for triples of the same results for semistable bundles. The proof is identical. Suppose that $h : T'' \rightarrow T'$ is a non-trivial homomorphism of triples. If $T' = (E'_1, E'_2, \Phi')$ and $T'' = (E''_1, E''_2, \Phi'')$ then h is given by a pair of holomorphic maps $u_i : E''_i \rightarrow E'_i$ for $i = 1, 2$ such that $\Phi' \circ u_2 = u_1 \circ \Phi''$. We can thus define subtriples of T'' and T' respectively by $T_N = (\ker(u_1), \ker(u_2), \Phi')$ and $T_I = (\text{im}(u_1), \text{im}(u_2), \Phi')$, where in T_I , it is in general necessary to take the saturations of the image $\text{im}(u_1)$ and $\text{im}(u_2)$. By the semistability conditions, we get

$$\mu_\alpha(T_N) \leq \mu_\alpha(T'') \leq \mu_\alpha(T_I) \leq \mu_\alpha(T').$$

The conclusions follow directly from this. □

Proposition 3.6. *Suppose that the triples T' and T'' are α -semistable and satisfy $\mu_\alpha(T') = \mu_\alpha(T'')$. Then*

- (1) $\mathbb{H}^2(C^\bullet(T'', T')) = 0$ whenever $\alpha > 2g - 2$.
- (2) If one of T', T'' is $\alpha + \epsilon$ -stable for some $\epsilon \geq 0$, then $\mathbb{H}^2(C^\bullet(T'', T')) = 0$ whenever $\alpha \geq 2g - 2$.

Proof. From (3.2) it is clear that the vanishing of $\mathbb{H}^2(C^\bullet(T'', T'))$ is equivalent to the surjectivity of the map

$$H^1(E''_1^* \otimes E'_1 \oplus E''_2^* \otimes E'_2) \longrightarrow H^1(E''_2^* \otimes E'_1).$$

By Serre duality this is equivalent to the injectivity of the map

$$H^0(E''_1^* \otimes E''_2 \otimes K) \xrightarrow{P} H^0(E''_1^* \otimes E''_1 \otimes K) \oplus H^0(E''_2^* \otimes E''_2 \otimes K) \quad (3.4)$$

$$\psi \qquad \longmapsto \qquad ((\phi'' \otimes \text{Id}) \circ \psi, \psi \circ \phi').$$

Proof of (1). Suppose that P is not injective. Then there is a non-trivial homomorphism $\psi : E''_1 \rightarrow E''_2 \otimes K$ in $\ker P$. Let $I = \text{im } \psi$ and $N = \ker \psi$. Since $(\phi'' \otimes \text{Id}_K) \circ \psi = 0$, $I \subset \ker \phi''$ and hence $T''_I = (0, I \otimes K^*, 0)$ is a proper subtriple of T'' . Similarly, the fact that $\psi \circ \phi' = 0$ implies that $\text{im } \phi' \subset N$ and thus $T'_N = (\ker \psi, E'_2, \phi')$ is a proper subtriple of T' . Let $k = \text{rk}(N)$ and $l = \text{deg}(N)$. Then, from the exact sequence

$$0 \longrightarrow N \longrightarrow E'_1 \longrightarrow I \longrightarrow 0$$

we see that $\text{rk}(I) = n'_1 - k$ and $\text{deg}(I) = d'_1 - l$. Hence

$$\mu_\alpha(T'_N) = \frac{l + d'_2}{k + n'_2} + \alpha \frac{n'_2}{k + n'_2},$$

$$\mu_\alpha(T''_I) = \frac{d'_1 - l}{n'_1 - k} + 2 - 2g + \alpha.$$

Adding these two expressions, and clearing denominators we see that

$$\begin{aligned} d'_1 + d'_2 + (n'_1 - k)(2 - 2g) + \alpha(n'_1 + n'_2 - k) \\ = (k + n'_2)\mu_\alpha(T'_N) + (n'_1 - k)\mu_\alpha(T''_I). \end{aligned}$$

But $\mu_\alpha(T'_N) \leq \mu_\alpha(T')$, $\mu_\alpha(T''_I) \leq \mu_\alpha(T'')$ and $\mu_\alpha(T') = \mu_\alpha(T'')$. From this we obtain that

$$d'_1 + d'_2 + (n'_1 - k)(2 - 2g) + \alpha(n'_1 + n'_2 - k) \leq d'_1 + d'_2 + \alpha n'_2, \quad (3.5)$$

and hence

$$\alpha(n'_1 - k) \leq (n'_1 - k)(2g - 2).$$

Since $n'_1 - k > 0$ we get that $\alpha \leq 2g - 2$. Hence P must be injective if the hypotheses of the part (1) of the proposition are satisfied.

Proof of (2). Suppose that T'' is $\alpha + \epsilon$ -stable for some $\epsilon \geq 0$. It follows that $\mu_{\alpha+\epsilon}(T''_I) < \mu_{\alpha+\epsilon}(T'')$, i.e

$$\mu_\alpha(T''_I) - \mu_\alpha(T'') < \epsilon \left(\frac{n''_2}{n''_1 + n''_2} - 1 \right) \leq 0.$$

Thus, following exactly the same argument as in the proof of (1), we get a strict inequality in (3.5). We conclude that that if P is not injective then $\alpha < 2g - 2$, i.e. if $\alpha \geq 2g - 2$ then P must be injective. If T' is $\alpha + \epsilon$ -stable for some $\epsilon \geq 0$ then we get that

$$\mu_\alpha(T'_N) - \mu_\alpha(T') < \epsilon \left(\frac{n'_2}{n'_1 + n'_2} - \frac{n'_2}{k + n'_2} \right) \leq 0.$$

The rest of the argument is the same as in the case that T'' is $\alpha + \epsilon$ -stable. \square

Corollary 3.7. *Let T' and T'' be α -semistable triples with $\mu_\alpha(T') = \mu_\alpha(T'')$, and $\alpha > 2g - 2$. Then*

$$\dim \text{Ext}^1(T'', T') = h^0(T'', T') - \chi(T'', T').$$

The same holds for $\alpha \geq 2g - 2$ if in addition T' or T'' is $\alpha + \epsilon$ -stable for some $\epsilon \geq 0$.

Proof. It follows from Proposition 3.1 and (3.3) that

$$\dim \text{Ext}^1(T'', T') = h^0(T'', T') + h^2(T'', T') - \chi(T'', T'). \quad (3.6)$$

The result follows immediately from this and the vanishing of $h^2(T'', T')$ given by Proposition 3.6. \square

3.3. Deformation theory for triples

Since the space of infinitesimal deformations of T is isomorphic to $\mathbb{H}^1(C^\bullet(T, T))$, the considerations of the previous sections also apply to studying deformations of a holomorphic triple T . To be precise, one has the following.

Theorem 3.8. *Let $T = (E_1, E_2, \phi)$ be an α -stable triple of type (n_1, n_2, d_1, d_2) .*

- (1) *The Zariski tangent space at the point defined by T in the moduli space of stable triples is isomorphic to $\mathbb{H}^1(C^\bullet(T, T))$.*
- (2) *If $\mathbb{H}^2(C^\bullet(T, T)) = 0$, then the moduli space of α -stable triples is smooth in a neighbourhood of the point defined by T .*
- (3) *$\mathbb{H}^2(C^\bullet(T, T)) = 0$ if and only if the homomorphism*

$$H^1(E_1^* \otimes E_1 \oplus E_2^* \otimes E_2) \longrightarrow H^1(E_2^* \otimes E_1)$$

in the corresponding long exact sequence is surjective.

- (4) *At a smooth point $T \in \mathcal{N}_\alpha^s(n_1, n_2, d_1, d_2)$ the dimension of the moduli space of α -stable triples is*

$$\begin{aligned} \dim \mathcal{N}_\alpha^s(n_1, n_2, d_1, d_2) &= h^1(T, T) = 1 - \chi(T, T) \\ &= (g - 1)(n_1^2 + n_2^2 - n_1 n_2) - n_1 d_2 + n_2 d_1 + 1. \end{aligned} \tag{3.7}$$

- (5) *If ϕ is injective or surjective then $T = (E_1, E_2, \phi)$ defines a smooth point in the moduli space.*
- (6) *If $\alpha \geq 2g - 2$, then T defines a smooth point in the moduli space, and hence $\mathcal{N}_\alpha^s(n_1, n_2, d_1, d_2)$ is smooth.*

Proof. Statements (1) and (2) follow from Theorems 2.3 and 3.1 in [3], respectively. An indirect proof of (1) and (2), exploiting the correspondence between triples on X and stable bundles on $X \times \mathbb{P}^1$ (see Section 9) also follows from [4]. Statement (3) follows from the long exact sequence (3.2) with $T = T' = T''$. (4) follows from (1), (2) and Propositions 3.2 and 3.7. (5) is proved in [4, Proposition 6.3]. (6) is a consequence of Proposition 3.6. □

4. Bounds for χ

In our approach to the study of how the moduli spaces of triples vary with the parameter, it is of crucial importance to be able to estimate the Euler characteristics $\chi(T'', T') = \chi(\mathbf{n}'', \mathbf{d}'', \mathbf{n}', \mathbf{d}')$ when T' and T'' are polystable triples with the same α -slope. The basic idea is to identify $\chi(T'', T')$ as a hypercohomology Euler characteristic for the complex $C^\bullet(T'', T')$ defined in (3.1) and to notice that the complex is itself a holomorphic triple. As such it ought to satisfy a stability

condition induced from the stability condition of T' and T'' . In principle, a way to obtain the stability condition for $C^\bullet(T'', T')$ should be provided by the correspondence between the stability of the holomorphic triples and the existence of solutions to the vortex equations given by Theorem 2.8. However, there seem to be no simple way to construct a solution to the vortex equations for $C^\bullet(T'', T')$ from solutions on T' and T'' . Instead we consider slightly more general objects than triples, known as *holomorphic chains*. These are studied in [1].

4.1. Holomorphic chains

A *holomorphic chain* is a diagram

$$C: E_m \xrightarrow{\phi_m} E_{m-1} \xrightarrow{\phi_{m-1}} \dots \xrightarrow{\phi_1} E_0,$$

where each E_i is a holomorphic vector bundle and $\phi_i: E_i \rightarrow E_{i-1}$ is a holomorphic map. Let

$$\begin{aligned} \mu(C) &= \mu(E_0 \oplus \dots \oplus E_m), \\ \lambda_i(C) &= \frac{\text{rk}(E_i)}{\sum_{i=0}^m \text{rk}(E_i)}, \quad i = 0, \dots, m. \end{aligned}$$

For $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m$, the α -slope of C is defined to be

$$\mu_\alpha(C) = \mu(C) + \sum_{i=1}^m \alpha_i \lambda_i(C).$$

The notion of α -stability is defined via the standard α -slope condition on subchains, that is, for any holomorphic subchain $C' \subset C$ we must have $\mu_\alpha(C') < \mu_\alpha(C)$. Semistability and polystability are defined as usual. A holomorphic triple is a holomorphic chain of length 2, and the stability notions coincide, taking $\alpha = (\alpha)$. As for triples, there are natural gauge-theoretic equations for holomorphic chains, which we now describe. Define $\tau = (\tau_0, \dots, \tau_m) \in \mathbb{R}^{m+1}$ by

$$\tau_i = \mu_\alpha(C) - \alpha_i, \quad i = 0, \dots, m, \tag{4.1}$$

where we make the convention $\alpha_0 = 0$. Then α can be recovered from τ by

$$\alpha_i = \tau_0 - \tau_i, \quad i = 0, \dots, m. \tag{4.2}$$

The τ -vortex equations

$$\sqrt{-1}\Lambda F(E_i) + \phi_{i+1}\phi_{i+1}^* - \phi_i^*\phi_i = \tau_i \text{Id}_{E_i}, \quad i = 0, \dots, m,$$

are equations for Hermitian metrics on E_0, \dots, E_m . Here, as in (2.6), $F(E_i)$ is the curvature of the Hermitian connection on E_i , Λ is contraction with the Kähler form and $\text{vol}(X) = 2\pi$. By convention $\phi_0 = \phi_{m+1} = 0$. One has the generalization of Theorem 2.8 to the case of holomorphic chains.

Theorem 4.1. [1, Theorem 3.4] *A holomorphic chain C is α -polystable if and only if the τ -vortex equations have a solution, where α and τ are related by (4.1).*

4.2. A length 3 holomorphic chain

Let $T' = (E'_1, E'_2, \phi')$ and $T'' = (E''_1, E''_2, \phi'')$ be two triples. Let us consider the length 3 holomorphic chain

$$\widetilde{C}^\bullet(T'', T'): E_1''^* \otimes E_2' \xrightarrow{a_2} E_1''^* \otimes E_1' \oplus E_2''^* \otimes E_2' \xrightarrow{a_1} E_2''^* \otimes E_1', \quad (4.3)$$

where

$$\begin{aligned} a_2(\psi) &= (\phi' \psi, -\psi \phi''), \\ a_1(\psi_1, \psi_2) &= \phi' \psi_2 - \psi_1 \phi''. \end{aligned}$$

We shall sometimes write this chain briefly as

$$\widetilde{C}^\bullet(T'', T'): C_2 \xrightarrow{a_2} C_1 \xrightarrow{a_1} C_0.$$

Note that the last two terms of $\widetilde{C}^\bullet(T'', T')$ coincide with the complex $C^\bullet(T'', T')$. Note also that $\widetilde{C}^\bullet(T'', T')$ is not in general a complex. Our goal in this section is to prove, using Theorem 4.1, that if T' and T'' are α -polystable then $\widetilde{C}^\bullet(T'', T')$ is α -polystable for a suitable choice of α .

Lemma 4.2. *Let T' and T'' be holomorphic triples and suppose we have solutions to the (τ'_1, τ'_2) -vortex equations on T' and the (τ''_1, τ''_2) -vortex equations on T'' , such that $\tau'_1 - \tau''_1 = \tau'_2 - \tau''_2$. Then the induced Hermitian metric on $\widetilde{C}^\bullet(T'', T')$ satisfies the chain vortex equations*

$$\sqrt{-1} \Lambda F(C_0) + a_1 a_1^* = \tilde{\tau}_0 \text{Id}_{C_0}, \quad (4.4)$$

$$\sqrt{-1} \Lambda F(C_1) + a_2 a_2^* - a_1^* a_1 = \tilde{\tau}_1 \text{Id}_{C_1}, \quad (4.5)$$

$$\sqrt{-1} \Lambda F(C_2) - a_2^* a_2 = \tilde{\tau}_2 \text{Id}_{C_2}, \quad (4.6)$$

for $\tau = (\tilde{\tau}_0, \tilde{\tau}_1, \tilde{\tau}_2)$ given by

$$\begin{aligned} \tilde{\tau}_0 &= \tau'_1 - \tau''_2, \\ \tilde{\tau}_1 &= \tau'_1 - \tau''_1 = \tau'_2 - \tau''_2, \\ \tilde{\tau}_2 &= \tau'_2 - \tau''_1. \end{aligned}$$

Proof. We shall only show that the induced Hermitian metric satisfies (4.5), since the proofs that it satisfies the two remaining equations are similar (but simpler).

The vortex equations for T' and T'' are

$$\begin{aligned} \sqrt{-1} \Lambda F(E'_1) + \phi' \phi'^* &= \tau'_1 \text{Id}_{E'_1}, & \sqrt{-1} \Lambda F(E''_1) + \phi'' \phi''^* &= \tau''_1 \text{Id}_{E''_1}, \\ \sqrt{-1} \Lambda F(E'_2) - \phi'^* \phi' &= \tau'_2 \text{Id}_{E'_2}, & \sqrt{-1} \Lambda F(E''_2) - \phi''^* \phi'' &= \tau''_2 \text{Id}_{E''_2}. \end{aligned}$$

We shall write the left hand side of (4.5) in terms of these known data of the triples T' and T'' . First, we note that

$$F(E_i'^*) = -F(E_i')^t, \quad i = 1, 2,$$

and similarly for $F(E_i''^*)$. Hence

$$\begin{aligned} F(C_1) &= F(E_1''^* \otimes E_1' \oplus E_2''^* \otimes E_2') \\ &= (F(E_1''^*) \otimes \text{Id} + \text{Id} \otimes F(E_1'), F(E_2''^*) \otimes \text{Id} + \text{Id} \otimes F(E_2')) \\ &= (-F(E_1')^t \otimes \text{Id} + \text{Id} \otimes F(E_1'), -F(E_2')^t \otimes \text{Id} + \text{Id} \otimes F(E_2')). \end{aligned} \tag{4.7}$$

Next we calculate a_1^* : note that for $\xi \otimes x \in C_0$ and $(\eta_1 \otimes y_1, \eta_2 \otimes y_2) \in C_1$ we have

$$\begin{aligned} &\langle a_1^*(\xi \otimes x), (\eta_1 \otimes y_1, \eta_2 \otimes y_2) \rangle_{C_1} \\ &= \langle \xi \otimes x, a_1(\eta_1 \otimes y_1, \eta_2 \otimes y_2) \rangle_{C_0} \\ &= \langle \xi \otimes x, -\eta_1 \phi'' \otimes y_1 + \eta_2 \otimes \phi'(y_2) \rangle_{C_0} \\ &= \langle \xi \otimes x, -\phi'''(\eta_1) \otimes y_1 + \eta_2 \otimes \phi'(y_2) \rangle_{C_0} \\ &= -\langle \xi, \phi'''(\eta_1) \rangle_{E_2''^*} \langle x, y_1 \rangle_{E_1'} + \langle \xi, \eta_2 \rangle_{E_2''^*} \langle x, \phi'(y_2) \rangle_{E_1'} \\ &= -\langle \phi'''^*(\xi), \eta_1 \rangle_{E_1''^*} \langle x, y_1 \rangle_{E_1'} + \langle \xi, \eta_2 \rangle_{E_2''^*} \langle \phi'^*(x), y_2 \rangle_{E_2'} \\ &= \langle (-\phi'''^*(\xi) \otimes x, \xi \otimes \phi'^*(x)), (\eta_1 \otimes y_1, \eta_2 \otimes y_2) \rangle_{C_1}. \end{aligned}$$

Hence,

$$a_1^*(\xi \otimes x) = (-\phi'''^*(\xi) \otimes x, \xi \otimes \phi'^*(x)). \tag{4.8}$$

Similarly, to calculate a_2^* consider $\xi \otimes x \in C_2$ and $(\eta_1 \otimes y_1, \eta_2 \otimes y_2) \in C_1$. Then

$$a_2^*(\eta_1 \otimes y_1, \eta_2 \otimes y_2) = \eta_1 \otimes \phi'^*(y_1) - \phi'''^*(\eta_2) \otimes y_2. \tag{4.9}$$

Using (4.9) and (4.8) we can now calculate for $(\eta_1 \otimes y_1, \eta_2 \otimes y_2) \in C_1$:

$$\begin{aligned} a_2 a_2^*(\eta_1 \otimes y_1, \eta_2 \otimes y_2) &= (\eta_1 \otimes \phi' \phi'^*(y_1) - \phi'''^*(\eta_2) \otimes \phi'(y_2), \\ &\quad -\phi'''(\eta_1) \otimes \phi'^*(y_1) + \phi''' \phi'''^*(\eta_2) \otimes y_2), \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} a_1^* a_1(\eta_1 \otimes y_1, \eta_2 \otimes y_2) &= (\phi'''^* \phi'''(\eta_1) \otimes y_1 - \phi'''^*(\eta_2) \otimes \phi'(y_2), \\ &\quad -\phi'''(\eta_1) \otimes \phi'^*(y_1) + \eta_2 \otimes \phi'^* \phi'(y_2)). \end{aligned} \tag{4.11}$$

Putting together (4.7), (4.10) and (4.11) we finally obtain

$$\begin{aligned}
 & (\sqrt{-1}\Lambda F(C_1) + a_2a_2^* - a_1^*a_1)(\eta_1 \otimes y_1, \eta_2 \otimes y_2) \\
 &= \left(\eta_1 \otimes (\sqrt{-1}\Lambda F(E'_1) + \phi'\phi'^*)(y_1) + (-\sqrt{-1}\Lambda F(E''_1))^t \right. \\
 & \quad \left. + \phi''^*\phi''^t\right)(\eta_1) \otimes y_1, \eta_2 \otimes (\sqrt{-1}\Lambda F(E'_2) - \phi'^*\phi')(y_2) \\
 & \quad - (\sqrt{-1}\Lambda F(E''_2))^t - \phi''^t\phi''^*) (\eta_2) \otimes y_2 \Big). \tag{4.12}
 \end{aligned}$$

Notice that the unpleasant mixed term $(-\phi''^*(\eta_2) \otimes \phi'(y_2), -\phi''^t(\eta_1) \otimes \phi'^*(y_1))$ appears both in $a_1^*a_1$ and $a_2a_2^*$ and therefore cancels. This would not have been the case if we had considered the vortex equations on the triple $C^\bullet(T'', T')$ and is the reason why we must consider the chain $\widetilde{C}^\bullet(T'', T')$. Combining (4.12) with the vortex equations (or their transposes) for the triples T' and T'' we get

$$\begin{aligned}
 & (\sqrt{-1}\Lambda F(C_1) + a_2a_2^* - a_1^*a_1)(\eta_1 \otimes y_1, \eta_2 \otimes y_2) \\
 &= ((\tau'_1 - \tau''_1)\eta_1 \otimes y_1, (\tau'_2 - \tau''_2)\eta_2 \otimes y_2). \tag{4.13}
 \end{aligned}$$

Since $\tau'_1 - \tau''_1 = \tau'_2 - \tau''_2$ this concludes the proof. □

Proposition 4.3. *Let T' and T'' be α -polystable triples. Then the holomorphic chain $\widetilde{C}^\bullet(T'', T')$ is α -polystable for $\alpha = (\alpha_1, \alpha_2) = (\alpha, 2\alpha)$.*

Proof. Since the triples T' and T'' are α -polystable, it follows from Theorem 2.8 that they support solutions to the (τ'_1, τ'_2) - and (τ''_1, τ''_2) -vortex equations, respectively, where $\alpha = \tau'_1 - \tau'_2 = \tau''_1 - \tau''_2$. Notice that $\tau'_1 - \tau''_1 = \tau'_2 - \tau''_2$. Thus it follows from Lemma 4.2 that the holomorphic chain $\widetilde{C}^\bullet(T'', T')$ supports a solutions to the chain vortex equations for $\tau = (\tau'_1 - \tau''_2, \tau'_1 - \tau''_1, \tau'_2 - \tau''_1)$. Now Theorem 4.1 and (4.2) imply that $\widetilde{C}^\bullet(T'', T')$ is α -polystable for

$$\begin{aligned}
 \alpha_1 &= \tau'_1 - \tau''_2 - \tau'_2 + \tau''_2 = \alpha, \\
 \alpha_2 &= \tau'_1 - \tau''_2 - \tau'_2 + \tau''_1 = 2\alpha. \tag{□}
 \end{aligned}$$

4.3. Bounds for $\chi(T'', T')$

We start with some technical lemmas needed to estimate the Euler characteristic $\chi(T'', T')$.

Lemma 4.4. *Let $T' = (E'_1, E'_2, \phi')$ and $T'' = (E''_1, E''_2, \phi'')$ be triples for which the chain $\widetilde{C}^\bullet(T'', T')$ is $\alpha = (\alpha, 2\alpha)$ -polystable. Let*

$$\begin{aligned}
 C_1 &= E''^*_1 \otimes E'_1 \oplus E''^*_2 \otimes E'_2, \\
 C_0 &= E''^*_2 \otimes E'_1,
 \end{aligned}$$

and $a_1: C_1 \rightarrow C_0$ be defined as in (4.3). Then the following inequalities hold.

$$\text{deg}(\ker(a_1)) \leq \text{rk}(\ker(a_1))(\mu_\alpha(T') - \mu_\alpha(T'')), \tag{4.14}$$

$$\begin{aligned}
 \text{deg}(\text{im}(a_1)) &\leq (\text{rk}(C_0) - \text{rk}(\text{im}(a_1)))(\mu_\alpha(T'') \\
 &\quad - \mu_\alpha(T') - \alpha) + \text{deg}(C_0). \tag{4.15}
 \end{aligned}$$

Proof. If $\text{rk}(\ker(a_1)) = 0$ then (4.14) is obvious. Assume therefore that $\text{rk}(\ker(a_1)) > 0$. Using $\ker(a_1)$, we can then define a quotient of the chain $\widetilde{C}^\bullet(T'', T')$ by

$$\mathcal{K}: 0 \rightarrow \ker(a_1) \rightarrow 0.$$

Thus, since $\mu_\alpha(\mathcal{K}) = \mu(\ker(a_1)) + \alpha$, it follows from the definition of α -polystability that

$$\mu(\ker(a_1)) + \alpha \leq \mu_\alpha(\widetilde{C}^\bullet(T'', T')) = \mu_\alpha(T') - \mu_\alpha(T'') + \alpha.$$

We therefore have

$$\mu(\ker(a_1)) \leq \mu_\alpha(T') - \mu_\alpha(T''),$$

which is equivalent to (4.14). The second inequality, i.e. (4.15), is obvious when $\text{rk}(\text{im}(a_1)) = \text{rk}(C_0)$. We thus assume $\text{rk}(\text{im}(a_1)) < \text{rk}(C_0)$. Using the cokernel $\text{coker}(a_1)$ (or its saturation if it is not torsion free), we can define a subchain of the chain $\widetilde{C}^\bullet(T'', T')$ by

$$\mathcal{Q}: 0 \rightarrow 0 \rightarrow \text{coker}(a_1).$$

By the α -polystability of $\widetilde{C}^\bullet(T'', T')$ we have $\mu_\alpha(\mathcal{Q}) \geq \mu_\alpha(\widetilde{C}^\bullet(T'', T'))$. This, together with the fact that

$$\mu(\text{coker}(a_1)) \leq \frac{\text{deg}(C_0) - \text{deg}(\text{im}(a_1))}{\text{rk}(C_0) - \text{rk}(\text{im}(a_1))},$$

leads directly to 4.15. □

Lemma 4.5. *Let $c': V_2' \rightarrow V_1'$ and $c'': V_2'' \rightarrow V_1''$ be linear maps between finite dimensional vector spaces. Assume that $V_1' \oplus V_2' \neq 0$ and $V_1'' \oplus V_2'' \neq 0$. Define*

$$\begin{aligned} f: \text{Hom}(V_1'', V_1') \oplus \text{Hom}(V_2'', V_2') &\longrightarrow \text{Hom}(V_2'', V_1') \\ (\psi_1, \psi_2) &\longmapsto c'\psi_2 - \psi_1c''. \end{aligned}$$

If f is an isomorphism, then exactly one of the following alternatives must occur:

- (1) $V_1' = V_2'' = 0$ and $c' = c'' = 0$.
- (2) $V_1'' = 0$, $V_1', V_2', V_2'' \neq 0$ and $c': V_2' \xrightarrow{\cong} V_1'$.
- (3) $V_2' = 0$, $V_1', V_1'', V_2'' \neq 0$ and $c'': V_2'' \xrightarrow{\cong} V_1''$.

In particular, if V_1', V_2', V_1'' and V_2'' are all non-zero then f cannot be an isomorphism.

Proof. If $(c', c'') = (0, 0)$ then $f = 0$ and therefore

$$\text{Hom}(V_2'', V_1') = \text{Hom}(V_1'', V_1') = \text{Hom}(V_2'', V_2') = 0.$$

If $V_1' \neq 0$ then $V_1'' = V_2'' = 0$ i.e. $V_1'' \oplus V_2'' = 0$. Hence $V_1' = 0$. Similarly one sees that $V_2'' = 0$ and thus alternative (1) occurs. Henceforth assume that $(c', c'') \neq (0, 0)$. Let $r_i' = \dim V_i'$ and $r_i'' = \dim V_i''$ for $i = 1, 2$. If f is an isomorphism then $r_1''r_1' + r_2''r_2' = r_2''r_1'$ from which it follows that $r_2''(r_1' - r_2') = r_1''r_1'$ and $r_1'(r_2'' - r_1'') = r_2''r_2'$. Hence

$$r_1' \geq r_2', \tag{4.16}$$

$$r_2'' \geq r_1''. \tag{4.17}$$

Assume that we have strict inequality in (4.16) and (4.17). Then, in particular, $\text{coker}(c')$ and $\text{ker}(c'')$ must both be non-zero. Choose a complement to $\text{im}(c')$ in V_1' so that

$$V_1' = \text{im}(c') \oplus \text{im}(c')^\perp.$$

We then have an inclusion

$$\text{Hom}(\text{ker}(c''), \text{im}(c')^\perp) \hookrightarrow \text{Hom}(V_2'', V_1').$$

Let $\psi = (\psi_1, \psi_2) \in \text{Hom}(V_1'', V_1') \oplus \text{Hom}(V_2'', V_2')$ and $x \in \text{ker}(c'')$, then

$$f(\psi)(x) = c'\psi_2(x) - \psi_1c'(x) = c'\psi_2(x),$$

which belongs to $\text{im}(c')$. Hence $\text{im}(f)$ and $\text{Hom}(\text{ker}(c''), \text{im}(c')^\perp)$ have trivial intersection and, therefore, f cannot be an isomorphism, which is absurd. It follows that equality must hold in at least one of the inequalities (4.16) and (4.17). Suppose that equality holds in (4.16), i.e. $r_1' = r_2' = 0$. Then $r_1''r_1' = 0$, i.e. $r_1'' = 0$ or $r_1' = 0$. Suppose first that $r_1' = 0$, then $r_2' = r_1' = 0$, which contradicts our assumption that $V_1' \oplus V_2' \neq 0$. Thus we must have $r_1'' = 0$ and $r_1' \neq 0$. We thus also have $V_2' \neq 0$ (since $r_2' = r_1'$) and $V_2'' \neq 0$ (since $r_2'' + r_1'' \neq 0$). Furthermore, since $(c', c'') \neq (0, 0)$ we can assume that $c'' \neq 0$. In this case $f(\psi_1, \psi_2) = f(\psi_1, 0) = -\psi_1c''$. In particular, if f is an isomorphism then so is c'' . Thus alternative (2) occurs. In a similar manner one sees that if equality holds in (4.17) then alternative (3) occurs. Obviously the three alternatives are mutually exclusive. \square

Lemma 4.6. *Suppose that T' and T'' are non-zero triples of types (n_1', n_2', d_1', d_2') and $(n_1'', n_2'', d_1'', d_2'')$ respectively. Let $n_1 = n_1' + n_1''$, $n_2 = n_2' + n_2''$, $d_1 = d_1' + d_1''$, $d_2 = d_2' + d_2''$, $\mu_1 = d_1/n_1$, and $\mu_2 = d_2/n_2$. Let α_m and α_M be the extreme α values for the triples of type (n_1, n_2, d_1, d_2) , as defined in (2.2) and (2.3), with the convention that $\alpha_M = \infty$ if $n_1 = n_2$. Let $\alpha_m < \alpha < \alpha_M$ and suppose that $\mu_\alpha(T') = \mu_\alpha(T'')$, then the map*

$$a_1: E_1''^* \otimes E_1' \oplus E_2''^* \otimes E_2' \rightarrow E_2''^* \otimes E_1'$$

cannot be an isomorphism.

Proof. Let us consider the triple $T = T' \oplus T''$. It is clear that $\mu_\alpha(T) = \mu_\alpha(T') = \mu_\alpha(T'')$.

If a_1 is an isomorphism then, applying Lemma 4.5 fibrewise, it follows that one of the following alternatives must occur:

- (a) $E'_1 = E''_2 = 0$ and $\phi' = \phi'' = 0$.
- (b) $E''_1 = 0, E'_1, E'_2, E''_2 \neq 0$ and $\phi' : E'_2 \xrightarrow{\cong} E'_1$.
- (c) $E'_2 = 0, E'_1, E''_1, E''_2 \neq 0$ and $\phi'' : E''_2 \xrightarrow{\cong} E''_1$.

We shall consider each case in turn. *Case (a).* In this case we have $T' = (0, E_2, 0)$, $T'' = (E_1, 0, 0)$ and $T = (E_1, E_2, 0)$. It follows from $\mu_\alpha(T') = \mu_\alpha(T)$ that $\alpha = \mu(E_1) - \mu(E_2) = \alpha_m$. *Case (b).* In this case we have $n_1 = n'_1$ and $n_2 = n'_2 + n''_2 = n'_1 + n''_2$. Hence $n_2 > n_1$. Furthermore, from $\mu_\alpha(T') = \mu_\alpha(T)$ we get $\mu(E_1) + \frac{\alpha}{2} = \mu(E_1 \oplus E_2) + \frac{n_2}{n_1 + n_2}$, i.e. $\alpha = \frac{2n_2}{n_2 - n_1} \alpha_m = \alpha_M$. *Case (c).* In this case we have $n_2 = n''_2$ and $n_1 = n'_1 + n''_1 = n'_1 + n''_2$. Hence $n_1 > n_2$. Furthermore, from $\mu_\alpha(T'') = \mu_\alpha(T)$ we get $\alpha = \frac{2n_1}{n_1 - n_2} \alpha_m = \alpha_M$. If $n_1 = n_2$ then case (a) is the only possibility, so $\alpha = \alpha_m$. If $n_1 \neq n_2$, then (a) or exactly one of (b) and (c) are the only possibilities, depending on whether $n_1 < n_2$ or $n_1 > n_2$. In both cases we see that $\alpha = \alpha_m$ or $\alpha = \alpha_M$. □

Proposition 4.7. *Suppose that T' and T'' are non-zero triples of types (n'_1, n'_2, d'_1, d'_2) and $(n''_1, n''_2, d''_1, d''_2)$ respectively. Let $n_1 = n'_1 + n''_1, n_2 = n'_2 + n''_2, d_1 = d'_1 + d''_1, d_2 = d'_2 + d''_2, \mu_1 = d_1/n_1$, and $\mu_2 = d_2/n_2$. Let α_m and α_M be the extreme α values for the triples of type (n_1, n_2, d_1, d_2) , as defined in (2.2) and (2.3), with the convention that $\alpha_M = \infty$ if $n_1 = n_2$. Let $\alpha_m < \alpha < \alpha_M$. Suppose that $\mu_\alpha(T') = \mu_\alpha(T'')$ and that the chain $\widetilde{C}^\bullet(T'', T')$, as defined in (4.3), is $(\alpha, 2\alpha)$ -stable. Then*

$$\chi(T'', T') \leq 1 - g$$

if $\alpha \geq 2g - 2$. In particular, if $g \geq 2$ then $\chi(T'', T') \leq 0$.

Proof. From the long exact sequence (3.2) and the Riemann-Roch formula we obtain

$$\chi(T'', T') = (1 - g)(\text{rk}(C_1) - \text{rk}(C_0)) + \text{deg}(C_1) - \text{deg}(C_0), \tag{4.18}$$

where C_1 and C_0 are as in (4.3). We can apply Lemma 4.4, and then use the estimates (4.14) and (4.15). Together with

$$\text{deg}(C_1) = \text{deg}(\ker(a_1)) + \text{deg}(\text{im}(a_1)), \tag{4.19}$$

$$\text{rk}(C_1) = \text{rk}(\ker(a_1)) + \text{rk}(\text{im}(a_1)), \tag{4.20}$$

these yield

$$\begin{aligned} \deg(C_1) \leq & (\mu_\alpha(T') - \mu_\alpha(T''))(\text{rk}(C_1) - \text{rk}(C_0)) \\ & - \alpha(\text{rk}(C_0) - \text{rk}(\text{im}(a_1))) + \deg(C_0). \end{aligned}$$

Using that $\mu_\alpha(T') = \mu_\alpha(T'')$, we can then deduce that

$$\deg(C_1) - \deg(C_0) \leq -\alpha(\text{rk}(C_0) - \text{rk}(\text{im}(a_1))).$$

Combining this with (4.18) we get

$$\chi(T'', T') \leq (1 - g)(\text{rk}(C_1) - \text{rk}(C_0)) - \alpha(\text{rk}(C_0) - \text{rk}(\text{im}(a_1))). \tag{4.21}$$

If $\alpha \geq 2g - 2$ then we get

$$\chi(T'', T') \leq (1 - g)(\text{rk}(C_0) + \text{rk}(C_1) - 2\text{rk}(\text{im}(a_1))),$$

with equality if and only if $\alpha = 2g - 2$. Furthermore $\text{rk}(\text{im}(a_1)) \leq \text{rk}(C_0)$ and $\text{rk}(a_1) \leq \text{rk}(C_1)$, with equality in both if and only if a_1 is an isomorphism. Thus in all cases we get $\chi(T'', T') \leq 0$, with equality if and only if $\alpha = 2g - 2$ and a_1 is an isomorphism. But by Lemma 4.6, since $\alpha_m < \alpha < \alpha_M$, then a_1 cannot be an isomorphism. Thus in all cases we get $\text{rk}(C_0) + \text{rk}(C_1) - 2\text{rk}(\text{im}(a_1)) \geq 1$ and hence $\chi(T'', T') \leq 1 - g$. \square

Remark 4.8. Since the roles of T' and T'' in Proposition 4.7 are symmetric, we obtain the same bound for $\chi(T', T'')$.

5. Crossing critical values

In this section we study the differences between the stable loci $\mathcal{N}_\alpha^s(\mathbf{n}, \mathbf{d})$ in the moduli spaces $\mathcal{N}_\alpha(\mathbf{n}, \mathbf{d})$, for fixed values of $\mathbf{n} = (n_1, n_2)$ and $\mathbf{d} = (d_1, d_2)$ but different values of α . Since in this section \mathbf{n} and \mathbf{d} are fixed, we use the abbreviated notation

$$\mathcal{N}_\alpha^s = \mathcal{N}_\alpha^s(\mathbf{n}, \mathbf{d}) \quad \text{and} \quad \mathcal{N}_\alpha = \mathcal{N}_\alpha(\mathbf{n}, \mathbf{d}).$$

Our main result is that for all $\alpha \geq 2g - 2$ any differences between the \mathcal{N}_α^s are confined to subvarieties of positive codimension. In particular, the number of irreducible components of the spaces \mathcal{N}_α^s are the same for all α satisfying $\alpha \geq 2g - 2$ and $\alpha_m < \alpha < \alpha_M^1$. If the coprimality condition $\text{GCD}(n_2, n_1 + n_2, d_1 + d_2) = 1$ is satisfied, then $\mathcal{N}_\alpha^s = \mathcal{N}_\alpha$ at all non-critical values of α , so the results apply to \mathcal{N}_α for all non-critical $\alpha \geq 2g - 2$.

We begin with a set theoretic description of the differences between two spaces $\mathcal{N}_{\alpha_1}^s$ and $\mathcal{N}_{\alpha_2}^s$ when α_1 and α_2 are separated by a critical value (as defined in section 2.2). For the rest of this section we adopt the following notation: Let α_c be a critical value such that

$$\alpha_m < \alpha_c < \alpha_M. \tag{5.1}$$

¹ When $n_1 \neq n_2$ the bounds α_m and α_M are as in (2.2) and (2.3). When $n_1 = n_2$ we adopt the convention that $\alpha_M = \infty$

Set

$$\alpha_c^+ = \alpha_c + \epsilon, \quad \alpha_c^- = \alpha_c - \epsilon, \tag{5.2}$$

where $\epsilon > 0$ is small enough so that α_c is the only critical value in the interval (α_c^-, α_c^+) .

5.1. Flip Loci

Definition 5.1. Let $\alpha_c \in (\alpha_m, \alpha_M)$ be a critical value for triples of type (\mathbf{n}, \mathbf{d}) . We define flip loci $\mathcal{S}_{\alpha_c^\pm} \subset \mathcal{N}_{\alpha_c^\pm}^s$ by the conditions that the points in $\mathcal{S}_{\alpha_c^+}$ represent triples which are α_c^+ -stable but α_c^- -unstable, while the points in $\mathcal{S}_{\alpha_c^-}$ represent triples which are α_c^- -stable but α_c^+ -unstable.

Remark 5.2. The definition of $\mathcal{S}_{\alpha_c^+}$ can be extended to the extreme case $\alpha_c = \alpha_m$. However, since all α_m^+ -stable triples must be α_m^- -unstable, we see that $\mathcal{S}_{\alpha_m^+} = \mathcal{N}_{\alpha_m^+}^s$. Similarly, when $n_1 \neq n_2$ we get $\mathcal{S}_{\alpha_M^-} = \mathcal{N}_{\alpha_M^-}^s$. The only interesting cases are thus those for which $\alpha_m < \alpha_c < \alpha_M$.

Lemma 5.3. In the above notation:

$$\mathcal{N}_{\alpha_c^+}^s - \mathcal{S}_{\alpha_c^+} = \mathcal{N}_{\alpha_c}^s = \mathcal{N}_{\alpha_c^-}^s - \mathcal{S}_{\alpha_c^-}. \tag{5.3}$$

Proof. By definition we can identify $\mathcal{N}_{\alpha_c^+}^s - \mathcal{S}_{\alpha_c^+} = \mathcal{N}_{\alpha_c^-}^s - \mathcal{S}_{\alpha_c^-}$.

Suppose now that t is a point in $\mathcal{N}_{\alpha_c^+}^s - \mathcal{S}_{\alpha_c^+} = \mathcal{N}_{\alpha_c^-}^s - \mathcal{S}_{\alpha_c^-}$, but that t is not in $\mathcal{N}_{\alpha_c}^s$. Let T be a triple representing t . Then T has a subtriple $T' \subseteq T$ for which $\mu_{\alpha_c}(T') \geq \mu_{\alpha_c}(T)$, and also $\mu_{\alpha_c^\pm}(T') < \mu_{\alpha_c^\pm}(T)$. This is not possible, and hence $t \in \mathcal{N}_{\alpha_c}^s$.

Finally, suppose that $t \in \mathcal{N}_{\alpha_c}^s$ and let T be a triple representing t . Then $\mu_{\alpha_c}(T') < \mu_{\alpha_c}(T)$ for all subtriples $T' \subset T$. But since the set of possible values for $\mu_{\alpha_c}(T')$ is a discrete subset of \mathbb{R} , we can find a $\delta > 0$ such that $\mu_{\alpha_c}(T') - \mu_{\alpha_c}(T) \leq -\delta$ for all subtriples $T' \subset T$. Thus $\mu_{\alpha_c^\pm}(T') - \mu_{\alpha_c^\pm}(T) < 0$. That is, t is in $\mathcal{N}_{\alpha_c^\pm}^s$, and hence $\mathcal{N}_{\alpha_c}^s \subseteq \mathcal{N}_{\alpha_c^\pm}^s - \mathcal{S}_{\alpha_c^\pm}$. \square

Our goal is to show that the flip loci $\mathcal{S}_{\alpha_c^\pm}$ are contained in subvarieties of positive codimension in $\mathcal{N}_{\alpha_c^\pm}^s$ respectively.

Proposition 5.4. Let $\alpha_c \in (\alpha_m, \alpha_M)$ be a critical value for triples of type $(\mathbf{n}, \mathbf{d}) = (n_1, n_2, d_1, d_2)$. Let $T = (E_1, E_2, \phi)$ be a triple of this type.

(1) Suppose that T represents a point in $\mathcal{S}_{\alpha_c^+}$, i.e. suppose that T is α_c^+ -stable but α_c^- -unstable. Then T has a description as the middle term in an extension

$$0 \rightarrow T' \rightarrow T \rightarrow T'' \rightarrow 0 \tag{5.4}$$

in which

- (a) T' and T'' are both α_c^+ -stable, with $\mu_{\alpha_c^+}(T') < \mu_{\alpha_c^+}(T)$,
 - (b) T' and T'' are both α_c -semistable with $\mu_{\alpha_c}(T') = \mu_{\alpha_c}(T)$.
- (2) Similarly, if T represents a point in $\mathcal{S}_{\alpha_c^-}$, i.e. if T is α_c^- -stable but α_c^+ -unstable, then T has a description as the middle term in an extension (5.4) in which
- (a) T' and T'' are both α_c^- -stable with $\mu_{\alpha_c^-}(T') < \mu_{\alpha_c^-}(T)$,
 - (b) T' and T'' are both α_c -semistable with $\mu_{\alpha_c}(T') = \mu_{\alpha_c}(T)$.

Proof. In both cases (i.e. (1) and (2)), since its stability property changes at α_c , the triple T must be strictly α_c -semistable, i.e. it must have a proper subtriple T' with $\mu_{\alpha_c}(T') = \mu_{\alpha_c}(T)$. We can thus consider the (non-empty) set

$$\mathcal{F}_1 = \{T' \subsetneq T \mid \mu_{\alpha_c}(T') = \mu_{\alpha_c}(T)\}.$$

Proof of (1). Suppose first that T is α_c^+ -stable but α_c^- -unstable. We observe that if $T' \in \mathcal{F}_1$, then $\frac{n'_2}{n'_1+n'_2} < \frac{n_2}{n_1+n_2}$, since otherwise T could not be α_c^+ -stable. But the allowed values for $\frac{n'_2}{n'_1+n'_2}$ are limited by the constraints $0 \leq n'_1 \leq n_1, 0 \leq n'_2 \leq n_2$ and $n'_1 + n'_2 \neq 0$. We can thus define

$$\lambda_0 = \max \left\{ \frac{n'_2}{n'_1 + n'_2} \mid T' \in \mathcal{F}_1 \right\}$$

and set

$$\mathcal{F}_2 = \left\{ T_1 \subset \mathcal{F}_1 \mid \frac{n'_2}{n'_1 + n'_2} = \lambda_0 \right\}.$$

Now let T' be any triple in \mathcal{F}_2 . Since T' has maximal α_c -slope, we can assume that $T'' = T/T'$ is a locally free triple, i.e. if $T'' = (E''_2, E''_1, \Phi)$ then E''_2 and E''_1 are both locally free. Furthermore, since T is α_c -semistable and $\mu_{\alpha_c}(T') = \mu_{\alpha_c}(T) = \mu_{\alpha_c}(T'')$, it follows that both T' and T'' are α_c -semistable and of the same α_c -slope. We now show that T'' is α_c^+ -stable. Suppose not. Then there is a proper subtriple $\tilde{T}'' \subset T''$ with $\mu_{\alpha_c^+}(\tilde{T}'') \geq \mu_{\alpha_c^+}(T'')$. However, since we can assume that α_c^+ is not a critical value for triples of type (\tilde{T}'') , we must have

$$\mu_{\alpha_c^+}(\tilde{T}'') > \mu_{\alpha_c^+}(T'').$$

Thus, since (T'') is α_c -semistable, we must have $\mu_{\alpha_c}(\tilde{T}'') \leq \mu_{\alpha_c}(T'')$ and also

$$\frac{\tilde{n}''_2}{\tilde{n}''_1 + \tilde{n}''_2} > \frac{n''_2}{n''_1 + n''_2}.$$

If $\mu_{\alpha_c}(\tilde{T}'') < \mu_{\alpha_c}(T'')$, say $\mu_{\alpha_c}(\tilde{T}'') = \mu_{\alpha_c}(T'') - \delta$, then in order to have $\mu_{\alpha_c^+}(\tilde{T}'') > \mu_{\alpha_c^+}(T'')$ we must have

$$\frac{\tilde{n}''_2}{\tilde{n}''_1 + \tilde{n}''_2} > \frac{n''_2}{n''_1 + n''_2} + \frac{\delta}{\epsilon}.$$

Letting ϵ approach zero, we see that $\frac{\tilde{n}_2''}{\tilde{n}_1'' + \tilde{n}_2''}$ must be arbitrarily large. This cannot be if $0 \leq \tilde{n}_1'' \leq n_1''$ and $0 \leq \tilde{n}_2'' \leq n_2''$ (and $\tilde{n}_1'' + \tilde{n}_2'' > 0$). We may thus assume that $\mu_{\alpha_c}(\tilde{T}'') = \mu_{\alpha_c}(T'')$. Consider now the subtriple $\tilde{T}' \subset T$ defined by the pull-back diagram

$$0 \rightarrow T' \rightarrow \tilde{T}' \rightarrow \tilde{T}'' \rightarrow 0.$$

This has $\mu_{\alpha_c}(\tilde{T}') = \mu_{\alpha_c}(T'') = \mu_{\alpha_c}(T)$ and thus

$$\frac{\tilde{n}_2'}{\tilde{n}_1' + \tilde{n}_2'} \leq \lambda_0 = \frac{n_2'}{n_1' + n_2'}.$$

It follows from this and the above extension that

$$\frac{\tilde{n}_2''}{\tilde{n}_1'' + \tilde{n}_2''} \leq \lambda_0 = \frac{n_2'}{n_1' + n_2'}.$$

However, since $\mu_{\alpha_c}(T') = \mu_{\alpha_c}(T)$ but $\mu_{\alpha_c^+}(T') < \mu_{\alpha_c^+}(T)$, we have that

$$\frac{n_2'}{n_1' + n_2'} < \frac{n_2''}{n_1'' + n_2''}.$$

Combining the previous two inequalities we get

$$\frac{\tilde{n}_2''}{\tilde{n}_1'' + \tilde{n}_2''} < \frac{n_2''}{n_1'' + n_2''}$$

which is a contradiction.

Now take $T' \in \mathcal{F}_2$ with minimum rank (i.e. minimum $n_1' + n_2'$) in \mathcal{F}_2 . We claim that T' is α_c^+ -stable. If not, then as before it has a proper subtriple \tilde{T}' with $\mu_{\alpha_c}(\tilde{T}') \leq \mu_{\alpha_c}(T')$ and $\frac{\tilde{n}_2'}{\tilde{n}_1' + \tilde{n}_2'} > \frac{n_2'}{n_1' + n_2'}$. Then $\tilde{n}_1' + \tilde{n}_2' < n_1' + n_2'$, which contradicts the minimality of $n_1' + n_2'$. Thus T' is α_c^+ -stable. Moreover, since T is α_c^+ -stable it follows that $\mu_{\alpha_c^+}(T') < \mu_{\alpha_c^+}(T)$. Thus taking $T' \in \mathcal{F}_2$ with minimum rank, and $T'' = T/T'$, we get a description of T as an extension in which (a)-(b) are satisfied.

Proof of (2). If T is α_c^- -stable but α_c^+ -unstable, then $\frac{n_2'}{n_1' + n_2'} > \frac{n_2}{n_1 + n_2}$ for all $T' \in \mathcal{F}_1$. The proof of (a) must thus be modified as follows. With

$$\lambda_0 = \min \left\{ \frac{n_2'}{n_1' + n_2'} \mid T' \in \mathcal{F}_1 \right\}$$

we can define

$$\mathcal{F}_2 = \left\{ T' \subset \mathcal{F}_1 \mid \frac{n_2'}{n_1' + n_2'} = \lambda_0 \right\}$$

and select $T' \in \mathcal{F}_2$ such that T' has minimal rank in \mathcal{F}_2 . It follows in a similar fashion to that above that T has a description as

$$0 \rightarrow T' \rightarrow T \rightarrow T'' \rightarrow 0$$

in which all the requirements of the proposition are satisfied. □

Remark 5.5. Unlike for Jordan-Hölder filtrations for semistable objects, the filtrations produced by the above proposition are always of length two, i.e. always yield a description of the semistable object as an extension of stable objects. This is achieved by exploiting the extra ‘degree of freedom’ provided by the parameter α_c . The true advantage of never having to consider extensions of length greater than two is that it removes the need for inductive procedures in the analysis of the flip loci.

Definition 5.6. Let $\alpha_c \in (\alpha_m, \alpha_M)$ be a critical value for triples of type (\mathbf{n}, \mathbf{d}) . Let $(\mathbf{n}', \mathbf{d}') = (n'_1, n'_2, d'_1, d'_2)$ and $(\mathbf{n}'', \mathbf{d}'') = (n''_1, n''_2, d''_1, d''_2)$ be such that

$$(\mathbf{n}, \mathbf{d}) = (\mathbf{n}', \mathbf{d}') + (\mathbf{n}'', \mathbf{d}''), \tag{5.5}$$

(i.e. $n_1 = n'_1 + n''_1, n_2 = n'_2 + n''_2, d_1 = d'_1 + d''_1$, and $d_2 = d'_2 + d''_2$), and also

$$\frac{d'_1 + d'_2}{n'_1 + n'_2} + \alpha_c \frac{n'_2}{n'_1 + n'_2} = \frac{d''_1 + d''_2}{n''_1 + n''_2} + \alpha_c \frac{n''_2}{n''_1 + n''_2}. \tag{5.6}$$

(1) Define $\tilde{\mathcal{S}}_{\alpha_c^+}(\mathbf{n}'', \mathbf{d}'', \mathbf{n}', \mathbf{d}')$ to be the set of all isomorphism classes of extensions

$$0 \longrightarrow T' \longrightarrow T \longrightarrow T'' \longrightarrow 0,$$

where T' and T'' are α_c^+ -stable triples with topological invariants $(\mathbf{n}', \mathbf{d}')$ and $(\mathbf{n}'', \mathbf{d}'')$ respectively, and the isomorphism is on the triple T .

(2) Define $\tilde{\mathcal{S}}_{\alpha_c^+}^0(\mathbf{n}'', \mathbf{d}'', \mathbf{n}', \mathbf{d}')$ $\subset \tilde{\mathcal{S}}_{\alpha_c^+}(\mathbf{n}'', \mathbf{d}'', \mathbf{n}', \mathbf{d}')$ to be the set of all extensions for which moreover T is α_c^+ -stable. In an analogous manner, define $\tilde{\mathcal{S}}_{\alpha_c^-}(\mathbf{n}'', \mathbf{d}'', \mathbf{n}', \mathbf{d}')$ and $\tilde{\mathcal{S}}_{\alpha_c^-}^0(\mathbf{n}'', \mathbf{d}'', \mathbf{n}', \mathbf{d}')$ $\subset \tilde{\mathcal{S}}_{\alpha_c^-}(\mathbf{n}'', \mathbf{d}'', \mathbf{n}', \mathbf{d}')$.

(3) Define

$$\tilde{\mathcal{S}}_{\alpha_c^+} = \bigcup \tilde{\mathcal{S}}_{\alpha_c^+}(\mathbf{n}'', \mathbf{d}'', \mathbf{n}', \mathbf{d}'), \quad \tilde{\mathcal{S}}_{\alpha_c^+}^0 = \bigcup \tilde{\mathcal{S}}_{\alpha_c^+}^0(\mathbf{n}'', \mathbf{d}'', \mathbf{n}', \mathbf{d}')$$

where the union is over all (n'_1, n'_2, d'_1, d'_2) and $(n''_1, n''_2, d''_1, d''_2)$ such that the above conditions apply, and also $\frac{n'_2}{n'_1+n'_2} < \frac{n''_2}{n''_1+n''_2}$.

(4) Similarly, define

$$\tilde{\mathcal{S}}_{\alpha_c^-} = \bigcup \tilde{\mathcal{S}}_{\alpha_c^-}(\mathbf{n}'', \mathbf{d}'', \mathbf{n}', \mathbf{d}'), \quad \tilde{\mathcal{S}}_{\alpha_c^-}^0 = \bigcup \tilde{\mathcal{S}}_{\alpha_c^-}^0(\mathbf{n}'', \mathbf{d}'', \mathbf{n}', \mathbf{d}')$$

where the union is over all (n'_1, n'_2, d'_1, d'_2) and $(n''_1, n''_2, d''_1, d''_2)$ such that the above conditions apply, and also $\frac{n'_2}{n'_1+n'_2} > \frac{n''_2}{n''_1+n''_2}$.

Remark 5.7. It can happen that $\tilde{\mathcal{S}}_{\alpha_c^-}^0$ or $\tilde{\mathcal{S}}_{\alpha_c^-}$ is empty. For instance there may be no possible choices of (n'_1, n'_2, d'_1, d'_2) and $(n''_1, n''_2, d''_1, d''_2)$ which satisfy all the required conditions. In this case, the implication of the next lemma is that one or both of the flip loci $\mathcal{S}_{\alpha_c^\pm}$ is empty.

Lemma 5.8. *There are maps, say $v^\pm : \tilde{\mathcal{S}}_{\alpha_c^\pm}^0 \longrightarrow \mathcal{N}_{\alpha_c^\pm}^s$, which map triples to their equivalence classes. The images contain the flip loci $\mathcal{S}_{\alpha_c^\pm}$.*

Proof. The existence of the maps is clear. The second statement, about the images of the maps, follows by Proposition 5.4. Indeed, suppose that T represents a point in $\mathcal{S}_{\alpha_c^+}$ and that

$$0 \rightarrow T' \rightarrow T \rightarrow T'' \rightarrow 0$$

is an extension of the type described in proposition 5.4, with T' a triple of type $(\mathbf{n}', \mathbf{d}')$ and T'' a triple of type $(\mathbf{n}'', \mathbf{d}'')$. Then $(\mathbf{n}', \mathbf{d}')$ and $(\mathbf{n}'', \mathbf{d}'')$ satisfy conditions (5.5) and (5.6). Furthermore, since $\mu_{\alpha_c^+}(T') < \mu_{\alpha_c^+}(T'')$, we must have $\frac{n'_2}{n'_1+n'_2} < \frac{n''_2}{n''_1+n''_2}$. Thus T is contained in $v^+(\tilde{\mathcal{S}}_{\alpha_c^+}^0)$. A similar argument shows that $\mathcal{S}_{\alpha_c^+}$ is contained in $v^-(\tilde{\mathcal{S}}_{\alpha_c^-}^0)$. □

5.2. Codimension estimates and comparison of moduli spaces

Consider a critical value $\alpha_c \in (\alpha_m, \alpha_M)$ for triples of type (\mathbf{n}, \mathbf{d}) . Fix $(\mathbf{n}', \mathbf{d}') = (n'_1, n'_2, d'_1, d'_2)$ and $(\mathbf{n}'', \mathbf{d}'') = (n''_1, n''_2, d''_1, d''_2)$ as in Definition 5.6. For simplicity we shall denote the moduli spaces of α_c^\pm -semistable triples of type $(\mathbf{n}', \mathbf{d}')$, respectively $(\mathbf{n}'', \mathbf{d}'')$, by

$$\mathcal{N}'_{\alpha_c^\pm} = \mathcal{N}_{\alpha_c^\pm}(\mathbf{n}', \mathbf{d}') \quad \text{and} \quad \mathcal{N}''_{\alpha_c^\pm} = \mathcal{N}_{\alpha_c^\pm}(\mathbf{n}'', \mathbf{d}'').$$

Proposition 5.9. *If $\alpha_c > 2g - 2$ then $\tilde{\mathcal{S}}_{\alpha_c^\pm}(\mathbf{n}'', \mathbf{d}'', \mathbf{n}', \mathbf{d}')$ is a locally trivial fibration over $\mathcal{N}'_{\alpha_c^\pm} \times \mathcal{N}''_{\alpha_c^\pm}$, with projective fibers of dimension*

$$-\chi(\mathbf{n}'', \mathbf{d}'', \mathbf{n}', \mathbf{d}') - 1.$$

In particular, $\tilde{\mathcal{S}}_{\alpha_c^\pm}(\mathbf{n}'', \mathbf{d}'', \mathbf{n}', \mathbf{d}')$ has dimension

$$1 - \chi(\mathbf{n}', \mathbf{d}', \mathbf{n}', \mathbf{d}') - \chi(\mathbf{n}'', \mathbf{d}'', \mathbf{n}'', \mathbf{d}'') - \chi(\mathbf{n}'', \mathbf{d}'', \mathbf{n}', \mathbf{d}'),$$

where $\chi(\mathbf{n}', \mathbf{d}', \mathbf{n}', \mathbf{d}')$ etc. are as in section 3. The same is true for $\tilde{\mathcal{S}}_{\alpha_c^+}(\mathbf{n}'', \mathbf{d}'', \mathbf{n}', \mathbf{d}')$ when $\alpha_c = 2g - 2$.

Proof. From the defining properties of $\tilde{\mathcal{S}}_{\alpha_c^\pm}(\mathbf{n}'', \mathbf{d}'', \mathbf{n}', \mathbf{d}')$ there is map

$$\tilde{\mathcal{S}}_{\alpha_c^\pm}(\mathbf{n}'', \mathbf{d}'', \mathbf{n}', \mathbf{d}') \longrightarrow \mathcal{N}'_{\alpha_c^\pm} \times \mathcal{N}''_{\alpha_c^\pm} \tag{5.7}$$

which sends an extension

$$0 \rightarrow T' \rightarrow T \rightarrow T'' \rightarrow 0$$

to the pair $([T'], [T''])$, where $[T']$ denotes the class represented by T' and similarly for $[T'']$. We first examine the fibers of this map.

Notice that T' and T'' satisfy the hypothesis of Proposition 3.6 and therefore of Corollary 3.7. Notice moreover that, since $\mu_{\alpha_c^\pm}(T') < \mu_{\alpha_c^\pm}(T'')$, it is not possible to have $T' \cong T''$. Thus (cf. Corollary 3.7 and Proposition 3.5(2)) we have

$$\begin{aligned} \dim \mathbb{P}(\text{Ext}^1(T'', T')) &= \dim \text{Ext}^1(T'', T') - 1 \\ &= -\chi(T'', T') - 1 \\ &= -\chi(\mathbf{n}'', \mathbf{d}'', \mathbf{n}', \mathbf{d}') - 1, \end{aligned} \tag{5.8}$$

which is independent of T' and T'' . Note that if $\alpha_c = 2g - 2$, T' and T'' satisfy the hypothesis of Proposition 3.5(2) for α_c^+ , but not for α_c^- .

It remains to establish that the fibration (5.7) is locally trivial. If the coprimality conditions $\text{GCD}(n'_1, n'_2, d'_1 + d'_2) = 1 = \text{GCD}(n''_1, n''_2, d''_1 + d''_2)$ hold then the moduli spaces $\mathcal{N}'_{\alpha_c^\pm}$ and $\mathcal{N}''_{\alpha_c^\pm}$ are fine moduli spaces (cf. [25]). That is, there are universal objects, say \mathcal{U}' and \mathcal{U}'' , defined over $\mathcal{N}'_{\alpha_c^\pm} \times X$ and $\mathcal{N}''_{\alpha_c^\pm} \times X$. These can be viewed as coherent sheaves of algebras (cf. [2]), or more precisely as examples of the \mathcal{Q} -bundles considered in [16]. Pulling these back to $\mathcal{N}'_{\alpha_c^\pm} \times \mathcal{N}''_{\alpha_c^\pm} \times X$ we can construct $\text{Hom}(\mathcal{U}'', \mathcal{U}')$ (where we have abused notation for the sake of clarity). Taking the projection from $\mathcal{N}'_{\alpha_c^\pm} \times \mathcal{N}''_{\alpha_c^\pm} \times X$ onto $\mathcal{N}'_{\alpha_c^\pm} \times \mathcal{N}''_{\alpha_c^\pm}$, we can then construct the first direct image sheaf. By the results in [16], we can identify the fibers as hypercohomology groups which, in this case, parameterize extensions of triples. We thus obtain $\tilde{\mathcal{S}}_{\alpha_c^\pm}$ as the projectivization of the first direct image of $\text{Hom}(\mathcal{U}'', \mathcal{U}')$. If the coprimality conditions fail, then the universal objects do not exist globally. However they still exist locally over (analytic) open sets in the stable locus in the base $\mathcal{N}'_{\alpha_c^\pm} \times \mathcal{N}''_{\alpha_c^\pm}$. This is sufficient for our purpose since by construction the image of the map in (5.7) lies in the stable locus. The result now follows from (5.8) and formula (3.7) (in Theorem 3.8) as applied to $\mathcal{N}'_{\alpha_c^\pm}$ and $\mathcal{N}''_{\alpha_c^\pm}$. \square

Proposition 5.10. *If $\alpha_c > 2g - 2$ then the loci $\mathcal{S}_{\alpha_c^\pm} \subset \mathcal{N}_{\alpha_c^\pm}^s$ are locally contained in subvarieties of codimension bounded below by*

$$\min\{-\chi(\mathbf{n}'', \mathbf{d}'', \mathbf{n}', \mathbf{d}')\},$$

where the minimum is over all $(\mathbf{n}', \mathbf{d}')$ and $(\mathbf{n}'', \mathbf{d}'')$ which satisfy (5.5) and (5.6) and also $\frac{n'_2}{n'_1+n'_2} < \frac{n''_2}{n''_1+n''_2}$ (in the case of $\mathcal{S}_{\alpha_c^+}$) or $\frac{n'_2}{n'_1+n'_2} > \frac{n''_2}{n''_1+n''_2}$ (in the case of $\mathcal{S}_{\alpha_c^-}$). The same is true for $\mathcal{S}_{\alpha_c^+}$ when $\alpha_c = 2g - 2$.

Proof. If $\alpha_c > 2g - 2$ then we can assume $\alpha_c^\pm \geq 2g - 2$. Clearly also, $\alpha_c^+ \geq 2g - 2$ when $\alpha_c = 2g - 2$. Thus by Theorem 3.8 the moduli spaces $\mathcal{N}_{\alpha_c^\pm}^s$ are smooth and have dimension $1 - \chi(\mathbf{n}, \mathbf{d}, \mathbf{n}, \mathbf{d})$. By Corollary 3.3 and Proposition 5.9 we obtain

$$\begin{aligned} \dim \mathcal{N}_{\alpha_c^\pm}^s &= 1 - \chi(\mathbf{n}', \mathbf{d}', \mathbf{n}', \mathbf{d}') - \chi(\mathbf{n}'', \mathbf{d}'', \mathbf{n}'', \mathbf{d}'') \\ &\quad - \chi(\mathbf{n}'', \mathbf{d}'', \mathbf{n}', \mathbf{d}') - \chi(\mathbf{n}', \mathbf{d}', \mathbf{n}'', \mathbf{d}'') \\ &= \dim \tilde{\mathcal{S}}_{\alpha_c^\pm}(\mathbf{n}'', \mathbf{d}'', \mathbf{n}', \mathbf{d}') - \chi(\mathbf{n}', \mathbf{d}', \mathbf{n}'', \mathbf{d}''). \end{aligned} \quad \square$$

Proposition 5.11. *Let T' and T'' be α_c^\pm -polystable triples. Then the holomorphic chain $\widetilde{C}^\bullet(T'', T')$ (as defined in (4.3)) is $(\alpha_c, 2\alpha_c)$ -polystable.*

Proof. From Proposition 4.3, we have that the α_c^\pm -polystability of T' and T'' implies the $(\alpha_c^\pm, 2\alpha_c^\pm)$ -polystability of $\widetilde{C}^\bullet(T'', T')$.

Now, the critical values for the chain form a discrete set of points in the (α_1, α_2) plane. We can thus pick $\epsilon > 0$ so that, with $\alpha_c^\pm = \alpha_c \pm \epsilon$, the point $(\alpha_c^\pm, 2\alpha_c^\pm)$ is not a critical point. We can in fact assume that there are no critical points in $B_\epsilon^\circ(\alpha_c, 2\alpha_c)$, i.e. in the punctured ball of radius ϵ centered at $(\alpha_c, 2\alpha_c)$. Thus $(\alpha_c^\pm, 2\alpha_c^\pm)$ -polystability is equivalent to $(\alpha_c, 2\alpha_c)$ -polystability. \square

Theorem 5.12. *Let $\alpha_c \in (\alpha_m, \alpha_M)$ be a critical value for triples of type (\mathbf{n}, \mathbf{d}) . If $\alpha_c > 2g - 2$ then the loci $\mathcal{S}_{\alpha_c^\pm} \subset \mathcal{N}_{\alpha_c^\pm}^s$ are contained in subvarieties of codimension at least $g - 1$. In particular, they are contained in subvarieties of strictly positive codimension if $g \geq 2$. If $\alpha_c = 2g - 2$ then the same is true for $\mathcal{S}_{\alpha_c^+}$.*

Proof. Combining Propositions 4.7 and 5.11 we have that

$$-\chi(\mathbf{n}', \mathbf{d}', \mathbf{n}'', \mathbf{d}'') = -\chi(T', T'') \geq g - 1$$

(notice that the order of T' and T'' in these Propositions is irrelevant). The result follows now from Proposition 5.10. \square

Theorem 5.13. *Let α_1 and α_2 be any two values in (α_m, α_M) such that $2g - 2 \leq \alpha_1$ and $\alpha_m < \alpha_1 < \alpha_2 < \alpha_M$. Then the moduli spaces $\mathcal{N}_{\alpha_1}^s$ and $\mathcal{N}_{\alpha_2}^s$ have the same number of irreducible components, in particular, $\mathcal{N}_{\alpha_1}^s$ is irreducible if and only if $\mathcal{N}_{\alpha_2}^s$ is.*

Proof. This follows immediately from Theorem 5.12 if α_1 and α_2 are non-critical, and from Theorem 5.12 together with Lemma 5.3 if either of them is critical. \square

6. Special values of α

In this section we identify some critical values and special subintervals in the range (α_m, α_M) . We describe their significance for the structure of α -stable triples.

6.1. Small α

Let $\alpha_m^+ = \alpha_m + \epsilon$, with ϵ such that the interval $(\alpha_m, \alpha_m^+]$ does not contain any critical value (sometimes we refer to this value of α as *small*). The following is important in the construction of the moduli space for small α ([4]).

Proposition 6.1. *If a triple $T = (E_1, E_2, \phi)$ is α_m^+ -semistable triple, E_1 and E_2 are semistable. In the converse direction, if one of E_1 and E_2 is stable and the other is semistable, $T = (E_1, E_2, \phi)$ is α_m^+ -stable.*

Corollary 6.2. *If $\text{GCD}(n_1, d_1) = 1$ and $\text{GCD}(n_2, d_2) = 1$, then $\mathcal{N}_{\alpha_m^+}^s(n_1, n_2, d_1, d_2)$ is isomorphic to the projectivization of a Picard sheaf over $M(n_1, d_1) \times M(n_2, d_2)$, where $M(n, d)$ is the moduli space of stable bundles of rank n and degree d .*

Proof. Let \mathbb{E}_1 and \mathbb{E}_2 the universal bundles over $X \times M(n_1, d_1)$ and $X \times M(n_2, d_2)$, respectively. Consider the canonical projections

$$\begin{aligned} \pi &: X \times M(n_1, d_1) \times M(n_2, d_2) \rightarrow M(n_1, d_1) \times M(n_2, d_2) ; \\ \hat{\pi} &: X \times M(n_1, d_1) \times M(n_2, d_2) \rightarrow X ; \\ \pi_1 &: X \times M(n_1, d_1) \times M(n_2, d_2) \rightarrow X \times M(n_1, d_1) ; \\ \pi_2 &: X \times M(n_1, d_1) \times M(n_2, d_2) \rightarrow X \times M(n_2, d_2) . \end{aligned}$$

From Proposition 6.1 we deduce that

$$\mathcal{N}_{\alpha_m^+}^s(n_1, n_2, d_1, d_2) = \mathbb{P}(R^1\pi_*(\pi_1^*\mathbb{E}_1 \otimes \pi_2^*\mathbb{E}_2^* \otimes \hat{\pi}^*K)^*). \quad \square$$

6.2. Critical values determined by the kernel

Throughout this section we assume that the triple (E_1, E_2, ϕ) has type (n_1, n_2, d_1, d_2) , with $n_1 \geq n_2$. The case $n_1 < n_2$ can be dealt with via duality of triples.

Definition 6.3. *For each integer $0 \leq j < n_2$ set*

$$\alpha_j = \frac{2n_1n_2}{n_2(n_1 - n_2) + (j + 1)(n_1 + n_2)}(\mu_1 - \mu_2). \quad (6.1)$$

Proposition 6.4. *Let $T = (E_1, E_2, \phi)$ be a triple in which $n_1 \geq n_2$. Let $N \subset E_2$ be the kernel of $\phi : E_2 \rightarrow E_1$. Suppose that T is α -semistable for some $\alpha > \alpha_j$. Then N has rank at most j . In particular, if T is α -semistable for some $\alpha > \alpha_0$ then $N = 0$, i.e. ϕ is injective.*

Proof. Suppose that

$$\text{rk}(N) = k > 0. \quad (6.2)$$

We consider the subtriples $T_N = (0, N, 0)$ and $T_I = (I, E_2, \phi)$, where I denotes the image sheaf $\text{im}(\phi)$. If $N \neq 0$, then the triple T_N is a proper subtriple, and so is T_I since $n_1 \geq n_2$. The α -semistability condition applied to T_N yields

$$\alpha n_1 \leq (n_1 + n_2)(\mu - \mu_N), \quad (6.3)$$

where μ_N denotes the slope of N and μ is the slope of $E_1 \oplus E_2$.

The α -semistability condition applied to T_I yields

$$\mu(E_2 \oplus I) + \alpha \frac{n_2}{i + n_2} \leq \mu + \alpha \frac{n_2}{n_1 + n_2}, \quad (6.4)$$

where $i = \text{rk}(I)$. Furthermore, from the exact sequence

$$0 \longrightarrow N \longrightarrow E_2 \longrightarrow I \longrightarrow 0, \tag{6.5}$$

we get

$$k + i = n_2, \tag{6.6}$$

$$k\mu_N + i\mu_I = n_2\mu_2. \tag{6.7}$$

Using (6.7) we can write

$$\mu(E_2 \oplus I) = \frac{2n_2\mu_2 - k\mu_N}{2n_2 - k} \tag{6.8}$$

and hence (6.4) yields

$$-k(n_1 + n_2)\mu_N \leq (n_1 + n_2)((2n_2 - k)\mu - 2n_2\mu_2) + \alpha n_2(n_2 - k - n_1). \tag{6.9}$$

Combining k times (6.3) and (6.9) yields

$$\alpha \leq \frac{2n_1n_2}{n_2(n_1 - n_2) + k(n_1 + n_2)}(\mu_1 - \mu_2). \tag{6.10}$$

We have thus shown that if $\text{rk}(N) = k$ and the triple is α -stable, then

$$\alpha \leq \alpha_{k-1}$$

where α_{k-1} is given by (6.1) with $j = k - 1$. Since

$$\alpha_{k-1} > \alpha_k > \dots > \alpha_{n_2-1},$$

we can conclude that if the triple is α -semistable with $\alpha > \alpha_{k-1}$, then the rank of N is strictly less than k . In particular, if $\alpha > \alpha_0$, where

$$\alpha_0 = \frac{2n_1n_2}{n_2(n_1 - n_2) + (n_1 + n_2)}(\mu_1 - \mu_2), \tag{6.11}$$

then T is injective. □

As an immediate consequence we obtain the following.

Proposition 6.5. *Let $\alpha > \alpha_0$, where α_0 is given by (6.11).*

(1) *An α -semistable triple (E_1, E_2, ϕ) defines a sequence of the form*

$$0 \longrightarrow E_2 \xrightarrow{\phi} E_1 \longrightarrow F \oplus S \longrightarrow 0, \tag{6.12}$$

where F is locally free and S is a torsion sheaf.

(2) If $n_1 = n_2$ then an α -semistable triple (E_1, E_2, ϕ) defines a sequence of the form

$$0 \longrightarrow E_2 \xrightarrow{\phi} E_1 \longrightarrow S \longrightarrow 0, \tag{6.13}$$

where S is a torsion sheaf of degree $d_1 - d_2$.

Lemma 6.6. *Let α_0 be given by (6.11).*

(1) If $n_1 > n_2$ then

$$\alpha_0 = \frac{n_2(n_1 - n_2)}{n_2(n_1 - n_2) + n_1 + n_2} \alpha_M = \frac{2n_1n_2}{n_2(n_1 - n_2) + n_1 + n_2} \alpha_m, \tag{6.14}$$

where α_m and α_M are given by (2.2) and (2.3), respectively.

(2) If $n_1 = n_2 = n$ then

$$\alpha_0 = n\alpha_m = n(\mu_1 - \mu_2) = d_1 - d_2. \tag{6.15}$$

(3) If $n_1 \geq n_2$ then $\alpha_0 \geq \alpha_m$, with equality if and only if $\alpha_m = 0$ or $n_2 = 1$.

Proof. Parts (1) and (2) are immediate. Using (1) we compute

$$\alpha_0 - \alpha_m = \frac{n_1 + n_2}{n_2(n_1 - n_2) + n_1 + n_2} (n_2 - 1)\alpha_m,$$

from which (3) follows. □

6.3. Critical values determined by the cokernel

In this section we assume that $n_1 > n_2$. The range for α is then $[\alpha_m, \alpha_M]$, where α_m and α_M are given by (2.2) and (2.3). Let us define

$$\alpha_t := \alpha_M - \frac{n_1 + n_2}{n_2(n_1 - n_2)}. \tag{6.16}$$

Proposition 6.7. *Suppose that a triple $T = (E_1, E_2, \phi)$ of the form (6.12) with $n_1 > n_2$ is α -semistable for some $\alpha > \alpha_m$. Then*

$$s \leq \frac{n_2(n_1 - n_2)}{(n_1 + n_2)} (\alpha_M - \alpha),$$

where s is the degree of S . In particular, if $\alpha > \alpha_t$, then $S = 0$, i.e. the quotient sheaf E_1/E_2 is locally free.

Proof. If $T = (E_1, E_2, \phi)$ is of the form in (6.12), with $S \neq 0$, then we can find a proper subtriple $T' = (E'_1, E_2, \phi)$ of the form

$$0 \longrightarrow E_2 \xrightarrow{\phi} E'_1 \longrightarrow S \longrightarrow 0. \tag{6.17}$$

Indeed, E'_1 is the kernel of the sheaf map $E_1 \longrightarrow F \oplus S \longrightarrow S$. Notice that $n'_1 = n_2$ and $d'_1 = d_2 + s$, where n'_1, d'_1 denote the rank and degree of E'_1 , etc. We compute

$$\Delta_\alpha(T') = \frac{n_1}{n_1 + n_2}(\mu_2 - \mu_1) + \frac{\alpha}{2} \left(\frac{n_1 - n_2}{n_1 + n_2} \right) + \frac{s}{2n_2}, \tag{6.18}$$

where, as in (2.1), $\Delta_\alpha(T') = \mu_\alpha(T') - \mu_\alpha(T)$. But

$$\frac{n_1}{n_1 + n_2}(\mu_1 - \mu_2) = \frac{\alpha_M}{2} \left(\frac{n_1 - n_2}{n_1 + n_2} \right)$$

and hence

$$\Delta_\alpha(T') = \frac{n_1 - n_2}{2(n_1 + n_2)} \left(\alpha - \alpha_M + \frac{n_1 + n_2}{n_2(n_1 - n_2)}s \right). \tag{6.19}$$

If the triple is α -semistable then $\Delta_\alpha(T') \leq 0$ and the result follows. □

Let us define

$$\alpha_e = \max\{\alpha_m, \alpha_0, \alpha_t\}. \tag{6.20}$$

The following is an immediate consequence of Proposition 6.7.

Proposition 6.8. *Let $\alpha > \alpha_e$. An α -semistable triple (E_1, E_2, ϕ) defines an extension*

$$0 \longrightarrow E_2 \xrightarrow{\phi} E_1 \longrightarrow F \longrightarrow 0, \tag{6.21}$$

with F locally free.

It turns out that for extension like (6.21), arising from semistable triples the dimension of $H^1(E_2 \otimes F^*)$ does not depend on the given triple. More precisely:

Proposition 6.9. *Let (E_1, E_2, ϕ) be an α -semistable triple in which $\ker \phi = 0$ and $\text{coker } \phi$ is locally free, defining an extension like (6.21). Then $H^0(E_2 \otimes F^*) = 0$ and hence*

$$\dim H^1(E_2 \otimes F^*) = n_2d_1 - n_1d_2 + n_1(n_1 - n_2)(g - 1). \tag{6.22}$$

Proof. From [4, Lemma 4.5] we have that the α -semistability of (E_1, E_2, ϕ) for arbitrary α implies that $H^0(E_1 \otimes E_2^*) = 0$. From (6.21), we have an injective homomorphism $F^* \rightarrow E_1^*$, which after tensoring with E_2 gives that $H^0(E_2 \otimes F^*)$ injects in $H^0(E_1 \otimes E_2^*)$, and hence the desired vanishing. By Riemann–Roch we obtain (6.22). □

7. Moduli space of triples with $n_1 \neq n_2$

Throughout this section we assume that $n_1 > n_2$. The case $n_1 < n_2$ can be dealt with by triples duality. Recall that the allowed range for the stability parameter is $\alpha_m \leq \alpha \leq \alpha_M$, where $\alpha_m = \mu_1 - \mu_2$ and $\alpha_M = \frac{2n_1}{n_1 - n_2} \alpha_m$, and we assume that $\mu_1 - \mu_2 > 0$. We describe the moduli space \mathcal{N}_α for $2g - 2 \leq \alpha \leq \alpha_M$, beginning with $\alpha = \alpha_M$.

7.1. Moduli space for $\alpha = \alpha_M$

Proposition 7.1. *Let $T = (E_1, E_2, \phi)$ be an α_M -polystable triple. Then $E_1 = \text{im } \phi \oplus F$, where $F = \text{coker } \phi$, and T decomposes as the direct sum of two α_M -polystable triples of the same α_M -slope, T' and T'' , where $T' = (\text{im } \phi, E_2, \phi)$, and $T'' = (F, 0, 0)$. In particular, T is never α_M -stable. Moreover, $E_2 \cong \text{im } \phi$ and E_2 and F are polystable.*

Proof. By Proposition, 6.8, T defines an extension

$$0 \longrightarrow E_2 \xrightarrow{\phi} E_1 \longrightarrow F \longrightarrow 0, \tag{7.1}$$

with F locally free. Let $T' = (\text{im } \phi, E_2, \phi)$. Of course $\phi : E_2 \rightarrow \text{im } \phi$ is an isomorphism, and

$$\mu_{\alpha_M}(T') = \mu(E_2) + \frac{\alpha_M}{2},$$

but this is equal to $\mu_{\alpha_M}(T)$ and hence T cannot be α_M -stable. Since we assume that T is α_M -polystable, it must decompose as $T' \oplus T''$, where $T'' = (F, 0, 0)$. It is clear from the polystability of T that T' and T'' are α_M -polystable with the same α_M -slope. Applying the α_M -semistability condition to the subtriples $(E'_2, \phi(E'_2), \phi) \subset T'$ and $(F', 0, 0) \subset T''$, we obtain that $\mu(E'_2) \leq \mu(E_2)$ and $\mu(F') \leq \mu(F)$, and hence E_2 and F are semistable. In fact the polystability of T' and T'' imply the polystability of E_2 and F , respectively. \square

As a consequence of Proposition 7.1, we obtain the following.

Corollary 7.2. *Suppose that $n_1 > n_2$ and $\mu_1 - \mu_2 > 0$. Then*

$$\mathcal{N}_{\alpha_M}(n_1, n_2, d_1, d_2) \cong \mathcal{N}_{\alpha_M}(n_2, n_2, d_2, d_2) \times M(n_1 - n_2, d_1 - d_2), \tag{7.2}$$

where $M(n_1 - n_2, d_1 - d_2)$ is the moduli space of polystable bundles of rank $n_1 - n_2$ and degree $d_1 - d_2$.

7.2. Moduli space for large α

Let α_L be the largest critical value in (α_m, α_M) , and let \mathcal{N}_L (respectively \mathcal{N}_L^s) denote the moduli space of α -polystable (respectively α -stable) triples for $\alpha_L < \alpha < \alpha_M$. We refer to \mathcal{N}_L as the ‘large α ’ moduli space. By definition, α_L is at least as big as α_e (where α_e is as in (6.20)). Thus if T is α -stable for $\alpha > \alpha_L$, then we can assume it is of the form (6.21), i.e. it gives rise to an extension

$$0 \longrightarrow E_2 \xrightarrow{\phi} E_1 \longrightarrow F \longrightarrow 0.$$

In particular, $I = \text{im } \phi$ is a subbundle with torsion free quotient in E_1 , and $\phi : E_2 \longrightarrow I$ is an isomorphism. Thus we get a subtriple $T_I = (I, E_2, \phi)$ in which the bundles have the same rank and degree, and ϕ is an isomorphism.

Proposition 7.3. *Let $T = (E_1, E_2, \phi)$ represent a point in \mathcal{N}_L , i.e. suppose that the triple is α -semistable for some α in the range $\alpha_L < \alpha < \alpha_M$. Then*

- (1) *the triple $T_I = (I, E_2, \phi)$ is α_M -semistable,*
- (2) *the bundle E_2 is semistable.*

Proof. (1). Let $T' = (E'_1, E'_2, \phi')$ be any subtriple of T_I . Since T' is also a subtriple of T , we get

$$\mu_\alpha(T') \leq \mu_\alpha(T). \tag{7.3}$$

A direct computation shows that

$$\mu_\alpha(T) = \mu_\alpha(T_I) + \frac{n_1 - n_2}{2(n_1 + n_2)}(\alpha_M - \alpha), \tag{7.4}$$

where we have used the fact that $n_1 > n_2$ in T and hence $\alpha_M = \frac{2n_1}{n_1 - n_2}(\mu_1 - \mu_2) = 2(\mu(F) - \mu_2)$. Thus for all $\alpha_L < \alpha < \alpha_M$ we have

$$\mu_\alpha(T') - \mu_\alpha(T_I) \leq \frac{n_1 - n_2}{2(n_1 + n_2)}(\alpha_M - \alpha).$$

Taking the limit $\alpha \rightarrow \alpha_M$, we get

$$\mu_{\alpha_M}(T') - \mu_{\alpha_M}(T_I) \leq 0,$$

i.e. T_I is α_M -semistable.

- (2). Let $E'_2 \subset E_2$ be any proper subsheaf. Then $(\phi(E'_2), E'_2, \phi)$ is a subtriple of T_I . Since $\phi : E_2 \longrightarrow \phi(E_2)$ is an isomorphism, this subtriple has $\mu(\phi(E'_2)) = \mu(E'_2)$ and $n'_2 = n'_1$. The α_M -semistability condition of T_I thus gives

$$\mu(E'_2) + \frac{\alpha_M}{2} \leq \mu_2 + \frac{\alpha_M}{2},$$

(where we have made use of the fact that $\mu(\phi(E_2)) = \mu(E_2) = \mu_2$). It follows from this that $\mu(E'_2) \leq \mu_2$, i.e. that E_2 is semistable. \square

Proposition 7.4. *Suppose that the triple $T = (E_1, E_2, \phi)$ is of the form in (6.21), i.e.*

$$0 \longrightarrow E_2 \xrightarrow{\phi} E_1 \longrightarrow F \longrightarrow 0,$$

with F locally free. Then there is an $\epsilon > 0$ such that F is semistable if the triple is α -semistable for any $\alpha > \alpha_M - \epsilon$. Indeed the conclusion holds for any

$$0 < \epsilon < \frac{2}{m(m-1)^2}, \tag{7.5}$$

where $m = n_1 - n_2 = \text{rk}(F)$.

Proof. Let $F' \subset F$ be any proper subsheaf. Denote the rank and slope of F (resp. F') by m and μ_F (resp. m' and $\mu_{F'}$). We can always find $E'_1 \subset E_1$ such that $F' = E'_1/E_2$, i.e. such that we have

$$0 \longrightarrow E_2 \xrightarrow{\phi} E'_1 \longrightarrow F' \longrightarrow 0.$$

Let $T' = (E'_1, E_2, \phi)$. For convenience, define

$$\Delta_\alpha \equiv \Delta_\alpha(T') = \mu_\alpha(T') - \mu_\alpha(T). \tag{7.6}$$

Using

$$\begin{aligned} n_1 &= n_2 + m, \\ n'_1 &= n_2 + m', \\ n_1\mu_1 &= n_2\mu_2 + m\mu_F, \\ n'_1\mu'_1 &= n_2\mu_2 + m'\mu_{F'}, \end{aligned} \tag{7.7}$$

we get

$$\mu_{F'} - \mu_F = \frac{(2n_2 + m)(2n_2 + m')}{2n_2m'} \Delta_\alpha - \left(\frac{m - m'}{2m'} \right) (\alpha - 2(\mu_F - \mu_2)). \tag{7.8}$$

But $2(\mu_F - \mu_2) = \alpha_M$. Thus, setting

$$\alpha = \alpha_M - \epsilon, \tag{7.9}$$

we get

$$\mu_{F'} - \mu_F = \frac{(2n_2 + m)(2n_2 + m')}{2n_2m'} \Delta_\alpha + \left(\frac{m - m'}{m'} \right) \frac{\epsilon}{2}. \tag{7.10}$$

If now we take

$$\frac{\epsilon}{2} < \frac{1}{m(m-1)^2},$$

then for all $0 < m' < m$ we get

$$\left(\frac{m - m'}{m'}\right) \frac{\epsilon}{2} < \frac{1}{m(m - 1)}. \tag{7.11}$$

Hence, if the triple is α -semistable, so that $\Delta_\alpha \leq 0$, then we get

$$\mu_{F'} - \mu_F < \frac{1}{m(m - 1)}. \tag{7.12}$$

Since μ_F and $\mu_{F'}$ are rational numbers, the first with denominator m , and the second with denominator $m' \leq (m - 1)$, equation (7.12) equivalent to the condition $\mu_{F'} - \mu_F \leq 0$. □

We can combine Propositions 6.8, 7.4 and 7.3 to obtain the following.

Proposition 7.5. *Let $T = (E_1, E_2, \phi)$ be an α -semistable triple for some α in the range $\alpha_L < \alpha < \alpha_M$. Then T is of the form*

$$0 \longrightarrow E_2 \xrightarrow{\phi} E_1 \longrightarrow F \longrightarrow 0,$$

with F locally free, and E_2 and F are semistable.

In the converse direction we have:

Proposition 7.6. *Let $T = (E_1, E_2, \phi)$ be a triple of the form*

$$0 \longrightarrow E_2 \xrightarrow{\phi} E_1 \longrightarrow F \longrightarrow 0,$$

with F locally free. If E_2 is semistable and F is stable then T is α -stable for $\alpha = \alpha_M - \epsilon$ in the range $\alpha_L < \alpha < \alpha_M$.

Proof. Any subtriple $T' = (E'_1, E'_2, \phi')$ defines a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_2 & \xrightarrow{\phi} & E_1 & \longrightarrow & F & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & E'_2 & \xrightarrow{\phi'} & E'_1 & \longrightarrow & F' & \longrightarrow & 0, \end{array}$$

where $F' \subset F$. Then

$$\begin{aligned} \Delta_\alpha &\equiv \Delta_\alpha(T') = \mu_\alpha(T') - \mu_\alpha(T) \\ &= \mu(E'_1 \oplus E'_2) - \mu(E_1 \oplus E_2) + \alpha\left(\frac{n'_2}{n'_1 + n'_2} - \frac{n_2}{n_1 + n_2}\right). \end{aligned} \tag{7.13}$$

Denote the rank and slope of F (resp. F') by m and μ_F (resp. m' and $\mu_{F'}$). Using

$$\begin{aligned} n_1 &= n_2 + m, \\ n'_1 &= n'_2 + m', \\ n_1\mu_1 &= n_2\mu_2 + m\mu_F, \\ n'_1\mu'_1 &= n'_2\mu'_2 + m'\mu_{F'}, \end{aligned}$$

and the fact that $\alpha_M = 2(\mu_F - \mu_2)$, and setting $\alpha = \alpha_M - \epsilon$, we obtain

$$\begin{aligned} \Delta_\alpha &= \frac{2n'_2 + \mu'_2 + m\mu_{F'}}{2n'_2 + m'} - \frac{2n_2 + \mu_2 + m\mu_F}{2n_2 + m} \\ &\quad + 2(\mu_F - \mu_2)\left(\frac{n'_2}{2n'_2 + m'} - \frac{n_2}{2n_2 + m}\right) - \epsilon\left(\frac{n'_2}{2n'_2 + m'} - \frac{n_2}{2n_2 + m}\right) \\ &= \frac{2n'_2}{2n'_2 + m'}(\mu'_2 - \mu_2) + \frac{m'}{2n'_2 + m'}(\mu_{F'} - \mu_F) - \epsilon\left(\frac{n'_2m - n_2m'}{(2n'_2 + m')(2n_2 + m)}\right). \end{aligned} \tag{7.14}$$

Clearing denominators in (7.14) we obtain

$$(2n'_2 + m')\Delta_\alpha = n'_2(2\Delta_2 - \frac{m}{2n_2 + m}\epsilon) + m'(\Delta_F + \frac{n_2}{2n_2 + m}\epsilon),$$

where

$$\Delta_2 = \mu'_2 - \mu_2, \quad \text{and} \quad \Delta_F = \mu_{F'} - \mu_F. \tag{7.15}$$

Now suppose that E_2 is semistable and F is stable. The semistability of E_2 implies that

$$2\Delta_2 - \frac{m}{2n_2 + m}\epsilon < 0,$$

while the stability of F implies there exists $\delta > 0$ such that $\Delta_F \leq -\delta < 0$. Thus by taking $\epsilon < \frac{2n_2+m}{n_2}\delta$, we have $\Delta_F + \frac{n_2}{2n_2+m}\epsilon < 0$, and hence $\Delta_\alpha < 0$. \square

Theorem 7.7. *Assume that $n_1 > n_2$ and $d_1/n_1 > d_2/n_2$. Then the moduli space $\mathcal{N}_L^s = \mathcal{N}_L^s(n_1, n_2, d_1, d_2)$ is smooth of dimension*

$$(g - 1)(n_1^2 + n_2^2 - n_1n_2) - n_1d_2 + n_2d_1 + 1,$$

and is birationally equivalent to a \mathbb{P}^N -fibration over $M^s(n_1 - n_2, d_1 - d_2) \times M^s(n_2, d_2)$, where $M^s(n, d)$ is the moduli space of stable bundles of rank n and degree d , and

$$N = n_2d_1 - n_1d_2 + n_1(n_1 - n_2)(g - 1) - 1.$$

In particular, $\mathcal{N}_L^s(n_1, n_2, d_1, d_2)$ is non-empty and irreducible.

If $\text{GCD}(n_1 - n_2, d_1 - d_2) = 1$ and $\text{GCD}(n_2, d_2) = 1$, the birational equivalence is an isomorphism.

Moreover, in all cases, $\mathcal{N}_L = \mathcal{N}_L(n_1, n_2, d_1, d_2)$ is irreducible and hence birationally equivalent to \mathcal{N}_L^s .

Proof. For every triple $T = (E_1, E_2, \phi)$ in \mathcal{N}_L^s , the homomorphism ϕ is injective and hence, by (5) in Proposition 3.8, T defines a smooth point in the moduli space, whose dimension is then given by (4) in Proposition 2.6.

Given $F \in M^s(n_1 - n_2, d_1 - d_2)$ and $E_2 \in M^s(n_2, d_2)$, we know from Proposition 2.6 that every extension

$$0 \longrightarrow E_2 \xrightarrow{\phi} E_1 \longrightarrow F \longrightarrow 0,$$

defines a triple $T = (E_1, E_2, \phi)$ in \mathcal{N}_L^s . These extensions are classified by $H^1(E_2 \otimes F^*)$. In fact two classes defining the same element in the projectivization $\mathbb{P}H^1(E_2 \otimes F^*)$ define equivalent extensions and therefore equivalent triples. Now,

$$\text{deg}(E_2 \otimes F^*) = (n_1 - n_2)d_2 - n_2(d_1 - d_2) = n_1n_2(\mu_2 - \mu_1) < 0$$

and, since $E_2 \otimes F^*$ is semistable, then $H^0(E_2 \otimes F^*) = 0$. Hence, by the Riemann–Roch theorem

$$h^1(E_2 \otimes F^*) = n_2d_1 - n_1d_2 + n_1(n_1 - n_2)(g - 1).$$

In particular this dimension is constant as F and E_2 vary in their corresponding moduli spaces.

We can describe this globally in terms of Picard sheaves. To do that we consider first the case in which $\text{GCD}(n_1 - n_2, d_1 - d_2) = 1$ and $\text{GCD}(n_2, d_2) = 1$. In this situation there exist universal bundles \mathbb{F} and \mathbb{E}_2 over $X \times M(n_1 - n_2, d_1 - d_2)$ and $X \times M(n_2, d_2)$, respectively. Consider the canonical projections

$$\begin{aligned} \pi &: X \times M(n_1 - n_2, d_1 - d_2) \times M(n_2, d_2) \\ &\quad \rightarrow M(n_1 - n_2, d_1 - d_2) \times M(n_2, d_2), \\ \nu &: X \times M(n_1 - n_2, d_1 - d_2) \times M(n_2, d_2) \rightarrow X \times M(n_1 - n_2, d_1 - d_2), \end{aligned}$$

and

$$\pi_2: X \times M(n_1 - n_2, d_1 - d_2) \times M(n_2, d_2) \rightarrow X \times M(n_2, d_2).$$

The Picard sheaf

$$\mathcal{S} := R^1\pi_*(\pi_2^*\mathbb{E}_2 \otimes \nu^*\mathbb{F}^*),$$

is then locally free and we can identify $\mathcal{N}_L = \mathcal{N}_L^s$ with $\mathcal{P} = \mathbb{P}(\mathcal{S})$. This is indeed a \mathbb{P}^N fibration with $N = n_2d_1 - n_1d_2 + n_1(n_1 - n_2)(g - 1) - 1$, which in particular is non-empty since $M(n_1 - n_2, d_1 - d_2)$ and $M(n_2, d_2)$ are non-empty and $N > 0$.

If $\text{GCD}(n_1 - n_2, d_1 - d_2) \neq 1$ and $\text{GCD}(n_2, d_2) \neq 1$, there are no universal bundles and hence the Picard bundle does not exist. However, its projectivization over

$$M^s(n_1 - n_2, d_1 - d_2) \times M^s(n_2, d_2)$$

does exist. This can be constructed by working in the open set R of the Quot scheme corresponding to stable bundles. The point is that an appropriate linear group GL acts on R , with the centre acting trivially and such that PGL acts freely with the quotient being $M^s(n_1 - n_2, d_1 - d_2) \times M^s(n_2, d_2)$. For the action on the projective bundle associated to the universal bundle over R , the centre of GL still acts trivially, and standard descent arguments produce the required \mathbb{P}^N fibration \mathcal{P} over $M^s(n_1 - n_2, d_1 - d_2) \times M^s(n_2, d_2)$.

We now show that the complement of \mathcal{P} has strictly positive codimension in \mathcal{N}_L^s . This follows from two facts. The first one is that any family of strictly semistable bundles of rank $n_1 - n_2$ and degree $d_1 - d_2$ depends on a number of parameters strictly less than the dimension of $M^s(n_1 - n_2, d_1 - d_2)$ (cf. e.g. [8]). The same argument applies to any family of strictly semistable bundles of rank n_2 and degree d_2 . The second fact is that the dimension of $H^1(E_2 \otimes F^*)$ is fixed by the Riemann–Roch theorem (we use here that E_2 and F are semistable).

To prove the last statement, i.e. to extend the results to \mathcal{N}_L , we consider the family $\tilde{\mathcal{P}}$ of equivalence classes of extensions

$$0 \longrightarrow E_2 \xrightarrow{\phi} E_1 \longrightarrow F \longrightarrow 0,$$

where F and E_2 are semistable. Clearly, $\tilde{\mathcal{P}}$ contains the family \mathcal{P} . The family $\tilde{\mathcal{P}}$ is irreducible. This is because since F and E_2 are semistable they vary (for fixed ranks and degrees) in irreducible families \mathcal{F} and \mathcal{E}_2 , respectively, and as shown above $H^0(E_2 \otimes F^*) = 0$. Hence $\tilde{\mathcal{P}}$ is a projective bundle over $\mathcal{F} \times \mathcal{E}_2$. From Proposition 7.5, we know that $\mathcal{N}_L \subset \tilde{\mathcal{P}}$, and since α -semistability is an open condition (which follows from the construction of the moduli space given in [4] and [25]), we have that \mathcal{N}_L is irreducible. □

Remark 7.8. If $n_1 < n_2$, we have an analogous theorem for $\mathcal{N}_L^s(n_1, n_2, d_1, d_2)$ via the isomorphism

$$\mathcal{N}_\alpha^s(n_1, n_2, d_1, d_2) = \mathcal{N}_\alpha^s(n_2, n_1, -d_2, -d_1)$$

given by duality (Proposition 2.4).

7.3. Moduli space for $\alpha_m < 2g - 2 \leq \alpha < \alpha_M$

Theorem 7.9. *Let α be any value in the range $\alpha_m < 2g - 2 \leq \alpha < \alpha_M$. Then \mathcal{N}_α^s is birationally equivalent to \mathcal{N}_L^s . In particular it is non-empty and irreducible.*

Proof. This follows from Theorem 5.13 and Theorem 7.7. □

Corollary 7.10. *Let (n_1, n_2, d_1, d_2) be such that $\text{GCD}(n_2, n_1 + n_2, d_1 + d_2) = 1$. If α is a generic value satisfying $\alpha_m < 2g - 2 \leq \alpha < \alpha_M$, then \mathcal{N}_α is birationally equivalent to \mathcal{N}_L , and in particular it is irreducible.*

Proof. From (4) in Proposition 2.6 one has that $\mathcal{N}_\alpha = \mathcal{N}_\alpha^s$ if $\text{GCD}(n_2, n_1 + n_2, d_1 + d_2) = 1$ and α is generic. In particular, $\mathcal{N}_L = \mathcal{N}_L^s$, and hence the result follows from Theorem 7.9. \square

8. Moduli space of triples with $n_1 = n_2$

Throughout this section we assume that $n_1 = n_2 = n$ and $d_1 \geq d_2$.

8.1. Moduli space for $d_1 = d_2$

Proposition 8.1. *Suppose that $n_1 = n_2 = n$ and $d_1 = d_2 = d$. Let $T = (E_1, E_2, \phi)$ be a triple of type (n_1, n_2, d_1, d_2) , and let $\alpha > 0$. Then T is α -(poly)stable if and only if E_1 and E_2 are (poly)stable and ϕ is an isomorphism.*

Proof. In this case the injectivity bound α_0 given by (6.11) is $\alpha_0 = \alpha_m = 0$. Hence for every α -semistable triple $T = (E_1, E_2, \phi)$ with $\alpha > 0$, ϕ must be injective and therefore an isomorphism. The polystability of E_1 and E_2 is now straightforward to see. To show the converse, suppose that E_1 and E_2 are both polystable and let $T' = (E'_1, E'_2, \phi')$ be any subtriple of T .

$$\begin{aligned} \mu_\alpha(T') &= \mu(E'_1 \oplus E'_2) + \alpha \frac{n'_2}{n'_1 + n'_2} \\ &\leq \mu(E_1 \oplus E_2) + \alpha \frac{n'_2}{n'_1 + n'_2} \\ &\leq \mu_\alpha(T) + \alpha \left(\frac{n'_2}{n'_1 + n'_2} - \frac{1}{2} \right) \\ &\leq \mu_\alpha(T), \end{aligned}$$

since $n'_1 \geq n'_2$ if ϕ is injective. \square

Corollary 8.2. *The moduli space $\mathcal{N}_\alpha(n, n, d, d)$ and the moduli space $M(n, d)$ of polystable bundles of rank n and degree d are isomorphic. In particular $\mathcal{N}_\alpha(n, n, d, d)$ is non-empty and irreducible.*

Proof. From Proposition 8.1 it is clear that we have a surjective map, say

$$\pi : \mathcal{N}_\alpha(n, n, d, d) \rightarrow M(n, d).$$

Suppose that $\pi([T]) = \pi([T'])$, where $[T]$ and $[T']$ are points in $\mathcal{N}_\alpha(n, n, d, d)$ represented by triples $T = (E, E, \phi)$ and $T' = (E', E', \phi')$ respectively. We may assume that T and T' are polystable triples, and hence that E and E' are polystable vector bundles. Thus, since $\pi([T]) = \pi([T'])$, we can find an isomorphism $h_1 : E \rightarrow E'$. Set $h_2 = \phi' \circ h_1 \circ \phi^{-1}$ (remember that ϕ and ϕ' are bundle isomorphisms). Then (h_1, h_2) defines an isomorphism from T to T' . Thus π is injective. \square

Combining Proposition 7.1 and Corollaries 7.2 and 8.2, we obtain the following.

Corollary 8.3. *Suppose that $n_1 > n_2$ and $\mu_1 - \mu_2 > 0$. Then*

$$\mathcal{N}_{\alpha_M}(n_1, n_2, d_1, d_2) \cong M(n_2, d_2) \times M(n_1 - n_2, d_1 - d_2).$$

In particular, $\mathcal{N}_{\alpha_M}(n_1, n_2, d_1, d_2)$ is non-empty and irreducible.

8.2. *Bounds on E_1 and E_2 for $\alpha > \alpha_0$*

Lemma 8.4. *Let (E_1, E_2, ϕ) be a triple with $\ker \phi = 0$. Let (E'_1, E'_2, ϕ') be a subtriple with $n'_1 = n'_2 = n'$. Thus we get the following diagram, in which S and S' are torsion sheaves:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_2 & \xrightarrow{\phi} & E_1 & \longrightarrow & S & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & E'_2 & \xrightarrow{\phi'} & E'_1 & \longrightarrow & S' & \longrightarrow & 0. \end{array}$$

Then

$$\begin{aligned} \Delta_\alpha(T') \equiv \mu_\alpha(T') - \mu_\alpha(T) &= (\mu(E'_2) - \mu_2) + \frac{1}{2} \left(\frac{s'}{n'} - \frac{s}{n} \right) \\ &= (\mu(E'_1) - \mu_1) - \frac{1}{2} \left(\frac{s'}{n'} - \frac{s}{n} \right). \end{aligned}$$

Here s and s' are the degrees of S and S' respectively.

Proof. From the above diagram we get

$$\begin{aligned} n\mu_2 + s &= n\mu_1, \\ n\mu(E'_2) + s' &= n\mu(E'_1). \end{aligned}$$

Thus

$$\begin{aligned} \mu_\alpha(T') &= \frac{1}{2}(\mu(E'_1) + \mu(E'_2)) + \frac{\alpha}{2} = \frac{1}{2} \left(2\mu(E'_2) + \frac{s'}{n'} \right) + \frac{\alpha}{2} \\ &= \frac{1}{2} \left(2\mu(E'_1) - \frac{s'}{n'} \right) + \frac{\alpha}{2}, \end{aligned}$$

and similarly for $\mu_\alpha(T)$.

□

Proposition 8.5. *Let (E_1, E_2, ϕ) be an α -semistable triple with $\ker \phi = 0$. Then*

(1) *For any subsheaf $E'_1 \subset E_1$*

$$\mu(E'_1) \leq \mu_1 + \frac{1}{2}(n-1)(\mu_1 - \mu_2).$$

(2) *For any subsheaf $E'_2 \subset E_2$*

$$\mu(E'_2) \leq \mu_2 + \frac{1}{2}(\mu_1 - \mu_2).$$

Proof. Since $\ker \phi = 0$ the results of Lemma 8.4 apply. Furthermore, any subsheaf $E'_1 \subset E_1$ is part of a subtriple (E'_1, E'_2, ϕ') with $n'_1 = n'_2 = n'$. Likewise, given any subsheaf $E'_2 \subset E_2$, we can take $E'_1 = \phi(E'_2)$. Thus we can use the results of Lemma 8.4, plus the fact that α -stability implies $\Delta_\alpha(T') < 0$ for all subtriples, to conclude

$$\mu(E'_1) - \mu_1 - \frac{1}{2} \left(\frac{s'}{n'} - \frac{s}{n} \right) < 0$$

for all $E'_1 \subset E_1$. Similarly

$$\mu(E'_2) - \mu_2 + \frac{1}{2} \left(\frac{s'}{n'} - \frac{s}{n} \right) < 0$$

for all $E'_2 \subset E_2$. The results now follow using the fact that $0 \leq s' \leq s$ and $1 \leq n' < n$. □

8.3. Stabilization of moduli

Theorem 8.6 (Stabilization Theorem). *Let α_0 be as in (6.15).*

(1) *Let α_1, α_2 be any real numbers such that $\alpha_0 < \alpha_1 \leq \alpha_2$, then*

$$\mathcal{N}_{\alpha_1}(n, n, d_1, d_2) \subseteq \mathcal{N}_{\alpha_2}(n, n, d_1, d_2).$$

(2) *There is a real number $\alpha_L \geq \alpha_0$ such that*

$$\mathcal{N}_{\alpha_1}(n, n, d_1, d_2) = \mathcal{N}_{\alpha_2}(n, n, d_1, d_2)$$

for all $\alpha_1 \geq \alpha_2 > \alpha_L$.

Proof. (1). Recall from Proposition 6.5 that if $\alpha > \alpha_0$ then any triple, $T = (E_1, E_2, \phi)$, in $\mathcal{N}_\alpha(n, n, d_1, d_2)$ has $\text{rk}(\phi) = n$. It follows that in any subtriple, say $T' = (E'_1, E'_2, \phi')$, the rank of E'_1 is at least as big as the rank of E'_2 , i.e. $n'_1 \geq n'_2$. We treat the cases $n'_1 > n'_2$ and $n'_1 = n'_2$ separately. In both cases we must show that

$$\Delta_{\alpha_1}(T') \leq 0 \Rightarrow \Delta_{\alpha_2}(T') \leq 0$$

if $\alpha_1 \leq \alpha_2$. If $n'_1 = n'_2$ then for any α

$$\Delta_\alpha(T') = \mu(E'_1 \oplus E'_2) - \mu(E_1 \oplus E_2). \tag{8.1}$$

In particular, $\Delta_\alpha(T')$ is independent of α and hence $\Delta_{\alpha_1}(T') = \Delta_{\alpha_2}(T')$. If $n'_1 > n'_2$, then for any α

$$\Delta_\alpha(T') = \mu(E'_1 \oplus E'_2) - \mu(E_1 \oplus E_2) + \left(\frac{n'_2}{n'_1 + n'_2} - \frac{1}{2} \right) \alpha. \tag{8.2}$$

For each subtriple, $\Delta_\alpha(T')$ is thus a linear function of α , with slope

$$\lambda(T') = \left(\frac{n'_2}{n'_1 + n'_2} - \frac{1}{2} \right) = \frac{n'_2 - n'_1}{2(n'_1 + n'_2)} \tag{8.3}$$

and constant term

$$M(T') = \mu(E'_1 \oplus E'_2) - \mu(E_1 \oplus E_2). \tag{8.4}$$

We see that if $n'_1 > n'_2$ then $\lambda(T') < 0$. It follows from this that

$$\Delta_{\alpha_1}(T') \leq 0 \implies \Delta_{\alpha_2}(T') \leq 0$$

if $\alpha_1 \leq \alpha_2$.

(2). Consider any α_1, α_2 such that $\alpha_0 < \alpha_1 \leq \alpha_2$. By Part (1), the difference (if any) between \mathcal{N}_{α_1} and \mathcal{N}_{α_2} is due entirely to triples which are α_2 -stable but not α_1 -stable. Any such triple must have a subobject, say $T' = (E'_1, E'_2, \phi')$, for which

$$\Delta_{\alpha_2}(T') \leq 0 < \Delta_{\alpha_1}(T'). \tag{8.5}$$

As in (1), we need only consider subobjects for which the rank of E'_1 is at least as big as the rank of E'_2 , i.e. $n'_1 \geq n'_2$. Clearly (8.5) is not possible for a subobject with $n'_1 = n'_2$ (since in that case $\Delta_{\alpha_1}(T') = \Delta_{\alpha_2}(T')$). Suppose then that $n'_1 > n'_2$. By (8.2) and the fact that for such a subobject $\lambda(T') < 0$, we get that

$$\Delta_\alpha(T') \geq 0 \iff \alpha \leq \frac{M(T')}{-\lambda(T')}. \tag{8.6}$$

We claim that there is a bound, α_L , depending only on the degrees and ranks of E_1 and E_2 , such that

$$\frac{M(T')}{-\lambda(T')} \leq \alpha_L \tag{8.7}$$

for all possible subtriples with $n'_1 > n'_2$. For a triple $T = (E_1, E_2, \phi)$ in \mathcal{N}_{α_2} Proposition 8.5 applies, giving upper bounds on slopes of subsheaves of both E_1 and E_2 . Using these bounds we compute

$$M(T') \leq \frac{nn'_1}{2(n'_1 + n'_2)}(\mu_1 - \mu_2). \tag{8.8}$$

Combined with (8.3), this gives

$$\begin{aligned} \frac{M(T')}{-\lambda(T')} &\leq \frac{nn'}{(n'_1 - n'_2)}(\mu_1 - \mu_2) \\ &\leq n(n - 1)(\mu_1 - \mu_2). \end{aligned}$$

We can thus take

$$\alpha_L = n(n - 1)(\mu_1 - \mu_2). \tag{8.9}$$

We can now complete the proof of Part (2): if $\alpha_1 > \alpha_L$ then no triple in \mathcal{N}_{α_2} can have a subtriple satisfying (8.6). Hence $\mathcal{N}_{\alpha_2} = \mathcal{N}_{\alpha_1}$. \square

Remark 8.7. If $n = 2$ then $\alpha_L = \alpha_0 = d_1 - d_2$, i.e. the stabilization parameter coincides with the injectivity parameter.

It is clear from (8.9) that $\alpha_L = 0$ correspond to the following especial cases.

Proposition 8.8. *If $n = 1$ or $\alpha_m = 0$ then $\alpha_L = 0$. Hence if ϵ is any positive real number,*

- (1) *if $n = 1$, then \mathcal{N}_α is isomorphic to $\mathcal{N}_{\alpha_m + \epsilon}(1, 1, d_1, d_2)$ for every $\alpha \in (\alpha_m, \infty)$;*
- (2) *if $\alpha_m = 0$, then \mathcal{N}_α is isomorphic to $\mathcal{N}_\epsilon(n, n, d_1, d_2)$ for every $\alpha \in (0, \infty)$.*

8.4. Moduli for large α and $\alpha \geq 2g - 2$

Let $\alpha > \alpha_0$, where α_0 is as in (6.15). By Proposition 6.5, we know that all triples in $\mathcal{N}_\alpha(n, n, d_1, d_2)$ are of the form

$$0 \longrightarrow E_2 \xrightarrow{\phi} E_1 \longrightarrow S \longrightarrow 0, \tag{8.10}$$

where S is a torsion sheaf of degree $d = d_1 - d_2$. By analogy with the $n_1 \neq n_2$, let us denote by $\mathcal{N}_L(n, n, d_1, d_2)$ the ‘large α ’ moduli space, i.e. the moduli space of α -semistable triples for any $\alpha \in (\alpha_L, \infty)$. Since $\alpha_L \geq \alpha_0$ we have that all triples in $\mathcal{N}_L(n, d_1, d_2)$ are of the form in (8.10) and that E_1 and E_2 are bounded by the constraints in Proposition 8.5.

In the converse direction we have the following.

Proposition 8.9. *Let $T = (E_1, E_2, \phi)$ be a triple such that $\ker \phi = 0$. If E_1 and E_2 are semistable, then T is α -semistable for large enough α , i.e. $T \in \mathcal{N}_L(n, n, d_1, d_2)$. If either E_1 or E_2 is stable, then T is α -stable.*

Proof. Since $\ker \phi = 0$, it follows (as in the proof of Theorem 8.6) that in any subtriple, say $T' = (E'_1, E'_2, \phi')$, the rank of E'_1 is at least as big as the rank of E'_2 , i.e. $n'_1 \geq n'_2$. If $n'_1 > n'_2$, then (8.2), (8.3) and (8.4) apply, with $\lambda(T') < 0$ and

$\frac{M(T')}{-\lambda(T')} \leq \alpha_L$. Thus $\mu_\alpha(T') - \mu_\alpha(T) < 0$ for $\alpha > \alpha_L$. For subtriples with $n'_1 = n'_2$, equation (8.1) says that

$$\Delta_\alpha(T') = \mu(E'_1 \oplus E'_2) - \mu(E_1 \oplus E_2)$$

for any α . For such subtriples, and for any α , it thus follows that

- (1) $\Delta_\alpha(T') \leq 0$ if both E_1 and E_2 are semistable, and
- (2) $\Delta_\alpha(T') < 0$ if at least one of the bundles is stable. □

Theorem 8.10. *The moduli space $\mathcal{N}_L^S(n, n, d_1, d_2)$ is non-empty.*

Proof. Our strategy is to show that there exist rank n stable bundles E_1 and E_2 of degree d_1 and d_2 , respectively, and a torsion sheaf S of degree $d_1 - d_2$, fitting in an exact sequence

$$0 \longrightarrow E_2 \longrightarrow E_1 \longrightarrow S \longrightarrow 0.$$

The result will then follow from Proposition 8.9.

To prove this, let E be a vector bundle, and let $\text{Quot}^d(E)$ be the Quot scheme of quotients $E \longrightarrow S$ where S is a torsion sheaf of degree d . The basic fact we need is the following.

Lemma 8.11. *Let L be a line bundle and let $\psi : L^{\oplus n} \longrightarrow S$ be an element in $\text{Quot}^d(L^{\oplus n})$. Then, if L has big enough degree (depending on n and d), for a generic S , the vector bundle $E = \ker \psi$ is stable.*

Proof. The proof is implicit in the papers by Hernández [17] and Maruyama [22], where they deal with the case $L = \mathcal{O}$. There, one needs an extra condition on n and d , which is not required in the twisted case when the degree of L is big enough.

Let L be a line bundle of degree m and $d'' > 0$ such that $d_1 = nm - d''$. By Lemma 8.11, if $\psi : L^n \longrightarrow S'' \in \text{Quot}^{d''}(L^n)$ is generic, then $E_1 = \ker \psi$ is a stable bundle of rank n and degree d_1 . Let $d = d_1 - d_2$ and consider a generic element $\eta : E_1 \longrightarrow S \in \text{Quot}^d(E_1)$. Let $E_2 = \ker \eta$, and let S' the cokernel of the natural inclusion $E_2 \longrightarrow L^n$. We have the following commutative diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & E_2 & \rightarrow & E_1 & \rightarrow & S \rightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \rightarrow & E_2 & \rightarrow & L^n & \rightarrow & S' \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & \rightarrow & S'' = S'' & \rightarrow & 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0.
 \end{array}$$

We see from the diagram that E_2 coincides with the kernel of $L^n \rightarrow S'$. If S' is general enough we can again apply Lemma 8.11 and conclude that E_2 is stable. To show that this is indeed the case, we observe that the diagram defines a map

$$\text{Quot}_0^d(E_1) \times \text{Quot}_0^{d''}(L^n) \rightarrow \text{Quot}_0^{d+d''}(L^n),$$

where Quot_0 denotes an open non-empty subscheme of Quot , which is surjective and finite. □

We deal now with the irreducibility of the moduli spaces. Although not formulated in precisely this form, the following result is implicit in [21], in the proof of Proposition 5.1.

Theorem 8.12 (Markman-Xia [21]). *There is an irreducible family \mathcal{S} parameterizing quotients $E_1 \rightarrow S \rightarrow 0$, where E_1 is a rank n and degree d_1 vector bundle such that all its subbundles have their slope bounded above by a given universal constant, and S is a torsion sheaf of degree $d > 0$.*

Theorem 8.13. *If $\alpha > \alpha_0$, then $\mathcal{N}_\alpha(n, n, d_1, d_2)$ is irreducible.*

Proof. Since $\alpha > \alpha_0$, an α -semistable triple $T = (E_1, E_2, \phi)$ defines a sequence as in (8.10) and hence a quotient $E_1 \rightarrow S \rightarrow 0$ in \mathcal{S} . By (1) in Proposition 8.5, the slopes of subbundles of E_1 are bounded above by the universal constant

$$\mu_1 + \frac{1}{2}(n - 1)(\mu_1 - \mu_2).$$

Let $\mathcal{S}^0 \subset \mathcal{S}$ consist of elements $E_1 \rightarrow S \rightarrow 0$ in \mathcal{S} that come from an α -semistable triple $0 \rightarrow E_2 \rightarrow E_1 \rightarrow \mathcal{S} \rightarrow 0$. Since α -semistability is an open condition \mathcal{S}^0 is a Zariski open set of \mathcal{S} , which is non-empty by Theorem 8.10, and hence irreducible by Theorem 8.12. The irreducibility of $\mathcal{N}_\alpha(n, n, d_1, d_2)$ follows now from that of \mathcal{S}^0 . □

Proposition 8.14. *The moduli space $\mathcal{N}_L(n, n, d_1, d_2)$ is birationally equivalent to a \mathbb{P}^N -fibration \mathcal{P} over $M^s(n, d_2) \times \text{Sym}^d(X)$, where $N = n(d_1 - d_2) - 1$, $\text{Sym}^d(X)$ is the d -symmetric product of X , and $M^s(n_2, d_2)$ is the moduli space of stable bundles of rank n_2 and degree d_2 .*

Proof. Let E_2 be a rank n and degree d_2 vector bundle and let S be a torsion sheaf of degree $d > 0$. We construct E_1 as an extension

$$0 \rightarrow E_2 \rightarrow E_1 \rightarrow S \rightarrow 0. \tag{8.11}$$

Such extensions are parameterized by $\text{Ext}^1(S, E_2)$. Suppose that S is of the form $S = \mathcal{O}_D$, where D is a divisor in $\text{Sym}^d(X)$. Let L be a line bundle. Consider the short exact sequence

$$0 \rightarrow L^*(-D) \rightarrow L^* \rightarrow \mathcal{O}_D \rightarrow 0,$$

and apply to it the functor $\text{Hom}(\cdot, E_2)$, to obtain the long exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(E_2 \otimes L) & \longrightarrow & H^0(E_2 \otimes L(D)) & \longrightarrow & \\ \text{Ext}^1(\mathcal{O}_D, E_2) & \longrightarrow & H^1(E_2 \otimes L) & \longrightarrow & H^1(E_2 \otimes L(D)) & \longrightarrow & 0. \end{array} \tag{8.12}$$

We thus have

$$\dim \text{Ext}^1(\mathcal{O}_D, E_2) = \chi(E_2 \otimes L) - \chi(E_2 \otimes L(D)) = nd,$$

where $\chi(E) = \dim H^0(E) - \dim H^1(E)$. Taking L so that $\text{deg}(L) \gg 0$, we have that $H^1(E_2 \otimes L) = 0$. If E_2 is semistable (or more generally, if it moves in a bounded family) we can take the same L for every E_2 . Then

$$\text{Ext}^1(\mathcal{O}_D, E_2) = H^0(E_2 \otimes L(D))/H^0(E_2 \otimes L).$$

Let \mathcal{P} be the set of equivalence classes of extensions (8.11), where E_2 is stable then \mathcal{P} is a \mathbb{P}^N -fibration over $M^s(n, d_2) \times \text{Sym}^d(X)$, where $N = nd - 1 = \dim \mathbb{P}(\text{Ext}^1(\mathcal{O}_D, E_2))$. Since we assume that d is positive, N is non-negative and positive if $n > 1$. Setting $d = d_1 - d_2$, a simple computation shows that

$$\dim \mathcal{P} = (g - 1)(n_1^2 + n_2^2 - n_1n_2) - n_1d_2 + n_2d_1 + 1.$$

Clearly \mathcal{P} is irreducible of the same dimension as \mathcal{N}_L , and since it is contained in \mathcal{S} (like \mathcal{N}_L) it must be birationally equivalent to \mathcal{N}_L . Notice that If $\text{GCD}(n, d_2) = 1$, then \mathcal{P} is the projectivization of a Picard bundle. □

Combining the results of this section, we arrive at the following theorems.

Theorem 8.15. *The moduli space $\mathcal{N}_L(n, n, d_1, d_2)$ is non-empty and irreducible. Furthermore, it is birationally equivalent to a \mathbb{P}^N -fibration over $M^s(n, d_2) \times \text{Sym}^d(X)$, where the fiber dimension is $N = n(d_1 - d_2) - 1$.*

Proof. It follows from Theorems 8.13 and 8.10 and Proposition 8.14. □

Theorem 8.16. *Let $\alpha \geq 2g - 2 > \alpha_m$. Then*

(1) *The moduli space \mathcal{N}_α^s is birationally equivalent to \mathcal{N}_L and it is hence non-empty and irreducible.*

(2) *If in addition either*

- $\text{GCD}(n, 2n, d_1 + d_2) = 1$ and $\alpha \geq 2g - 2 > \alpha_m$ is generic, or
- $d_1 - d_2 < \alpha$,

then $\mathcal{N}_\alpha(n, n, d_1, d_2)$ is birationally equivalent to $\mathcal{N}_L(n, n, d_1, d_2)$ and hence irreducible.

Proof. If $2g - 2 > \alpha_L$, the result follows from Theorems 8.6 and 8.15. Assume then that $2g - 2 \leq \alpha_L$.

(1) From Theorem 8.13 we know that \mathcal{N}_L is birationally equivalent to \mathcal{N}_L^s . The result follows now from Theorem 5.13 and Theorem 8.15.

(2) For the first part, we observe that from (4) in Proposition 2.6 one has that $\mathcal{N}_\alpha = \mathcal{N}_\alpha^s$ if $\text{GCD}(n, 2n, d_1 + d_2) = 1$ and α is generic, and hence the result follows from (1). The second part is a consequence of Theorem 8.13. □

9. Triples and dimensional reduction

Let \mathbb{P}^1 be the complex projective line. The Lie group $SL(2, \mathbb{C})$ acts on $X \times \mathbb{P}^1$ via the trivial action on X and the identification $\mathbb{P}^1 = SL(2, \mathbb{C})/P$, where P is the subgroup of lower triangular matrices.

The theory of holomorphic triples and vortex equations on X is related with the study of stable $SL(2, \mathbb{C})$ -equivariant bundles on $X \times \mathbb{P}^1$ and the existence of invariant solutions to the Hermitian–Einstein equations. In fact, it is in this way (known as dimensional reduction) that the theory originated (see [4] and [13] for details).

In this section we recall the basics of this correspondence and apply our main results on triples to the theory of vector bundles on $X \times \mathbb{P}^1$.

9.1. Existence of stable bundles on $X \times \mathbb{P}^1$ and triples

Proposition 9.1. [4, Proposition 2.3] *There is a one-to-one correspondence between holomorphic triples (E_1, E_2, ϕ) on X and holomorphic extensions on $X \times \mathbb{P}^1$ of the form*

$$0 \longrightarrow p^*E_1 \longrightarrow E \longrightarrow p^*E_2 \otimes q^*\mathcal{O}(2) \longrightarrow 0, \tag{9.1}$$

where p and q are the canonical projections from $X \times \mathbb{P}^1$ to X and \mathbb{P}^1 , respectively, and $\mathcal{O}(2)$ is the degree 2 line bundle of \mathbb{P}^1 (the tangent bundle).

Proof. The proof given in [13] is simply that extensions over $X \times \mathbb{P}^1$ of the form (9.1) are parametrized by

$$H^1(X \times \mathbb{P}^1, p^*(E_1 \otimes E_2^*) \otimes q^*\mathcal{O}(-2)).$$

By the Künneth formula, this is isomorphic to

$$H^0(X, E_1 \otimes E_2^*) \otimes H^1(\mathbb{P}^1, \mathcal{O}(-2)) \cong H^0(X, E_1 \otimes E_2^*).$$

After fixing an element in $H^1(\mathbb{P}^1, \mathcal{O}(-2))$, the homomorphism ϕ can thus be identified with the extension class defining E . □

Notice that the vector bundles E of the form in (9.1) come equipped with an action of $SL(2, \mathbb{C})$ which lifts the action on $X \times \mathbb{P}^1$. The action on E is trivial on p^*E_1 and p^*E_2 , is the standard one on $\mathcal{O}(2)$, and leaves invariant the extension class.

To talk about the stability of E one needs a Kähler metric on $X \times \mathbb{P}^1$. Let us fix a metric on X and the Fubini-Study metric on \mathbb{P}^1 , both normalized to have volume 2π . Let $\alpha > 0$ be a real number. We consider on $X \times \mathbb{P}^1$ the one-parameter family of $SU(2)$ -invariant Kähler metrics with Kähler form

$$\omega_\alpha = \alpha p^*\omega_X \oplus q^*\omega_{\mathbb{P}^1}.$$

Here ω_X and $\omega_{\mathbb{P}^1}$ are the Kahler forms on X and \mathbb{P}^1 , respectively. The degree of a complex vector bundle E over $X \times \mathbb{P}^1$ with respect to ω_α is given by

$$\text{deg}(E) = \int_{X \times \mathbb{P}^1} c_1(E) \wedge \omega_\alpha,$$

where $c_1(E)$ is the first Chern class of E . Recall that E is said to be *stable* with respect to ω_α if for every non-trivial coherent reflexive subsheaf $E' \subset E$,

$$\mu(E') < \mu(E),$$

where $\mu(E) = \text{deg}(E)/\text{rk}(E)$ is the *slope* of E . Since we are in complex dimension 2, E' is locally free.

Theorem 9.2. [4, Theorem 4.1] *Let $T = (E_1, E_2, \phi)$ be a holomorphic triple over X and let E be the holomorphic bundle over $X \times \mathbb{P}^1$ defined by T as in Proposition 9.1. Then, if E_1 and E_2 are not isomorphic, T is α -stable if and only if E is stable with respect to ω_α . If $E_1 \cong E_2$, the triple T is α -stable if and only if E decomposes as a direct sum*

$$E = p^*E_1 \otimes q^*\mathcal{O}(1) \oplus p^*E_2 \otimes q^*\mathcal{O}(1),$$

and $p^*E_i \otimes q^*\mathcal{O}(1)$ is stable with respect to ω_α .

Remark 9.3. The stability of $p^*E_i \otimes q^*\mathcal{O}(1)$ is equivalent to the stability of E_i .

Let (n_1, n_2, d_1, d_2) be the type of the triple $T = (E_1, E_2, \phi)$. Let \mathcal{M}_α be the moduli space of stable bundles on $X \times \mathbb{P}^1$ with respect to ω_α , whose topological type is that of E in (9.1). Combining Theorems 9.2, 7.9 and 8.16 we can prove existence of stable bundles on $X \times \mathbb{P}^1$. More precisely.

Theorem 9.4. \mathcal{M}_α is non-empty if

- (1) $2g - 2 \leq \alpha \leq \alpha_M$ if $n_1 \neq n_2$, where α_M is given by (2.3);
- (2) $2g - 2 \leq \alpha$ if $n_1 = n_2$.

Remark 9.5. The moduli space \mathcal{N}_α can be identified with the $\text{SL}(2, \mathbb{C})$ -invariant part of \mathcal{M}_α ([4]). Hence from Theorems 7.9 and 8.16 we can say that within the range for α in Theorem 9.4 the invariant loci for different values of α are birationally equivalent. Whether this is true or not for the whole moduli spaces \mathcal{M}_α for different values of α is something that deserves further study (see [23] for a discussion on this in the rank two case).

9.2. Existence of $SU(2)$ -invariant Hermitian–Einstein metrics

By the Hitchin–Kobayashi correspondence proved by Donaldson, Uhlenbeck and Yau [10, 11, 32], the stability of the bundle E on $X \times \mathbb{P}^1$ is equivalent to the existence of an irreducible solution to the Hermitian–Einstein equation. Recall that this is a Hermitian metric on E such that

$$\sqrt{-1}\Lambda F(E) = \mu \text{Id}_E, \tag{9.2}$$

where, as usual, Λ is contraction with the Kähler form of $X \times \mathbb{P}^1$, $F(E)$ is the curvature of the unique connection determined by the Hermitian metric and the holomorphic structure of E , Id_E is the identity endomorphism of E and μ is the slope of E .

The action of $SL(2, \mathbb{C})$ on E restricts to an action of the compact subgroup $SU(2) \subset SL(2, \mathbb{C})$, and, since the metric ω_α on $X \times \mathbb{P}^1$ is $SU(2)$ -invariant, one can consider $SU(2)$ -invariant solutions to (9.2). The relevant fact is the following.

Proposition 9.6. [13, Proposition 3.11] *Let $T = (E_1, E_2, \phi)$ be a holomorphic triple of type (n_1, n_2, d_1, d_2) over X and let E be the holomorphic bundle over $X \times \mathbb{P}^1$ associated to T by Proposition 9.1. Let τ_1 and τ_2 be real numbers such that $d_1 + d_2 = n_1\tau_1 + n_2\tau_2$, and $\tau_1 - \tau_2 > 0$. Then T admits a solution to (2.6) if and only if E admits an $SU(2)$ -invariant Hermitian–Einstein metric with respect to ω_α .*

Combining the previous results we have the following.

Corollary 9.7. *The vector bundle E associated to a triple T of type (n_1, n_2, d_1, d_2) has a Hermitian–Einstein metric, with respect to ω_α if*

- (1) $2g - 2 \leq \alpha \leq \alpha_M$ if $n_1 \neq n_2$, where α_M is given by (2.3);
- (2) $2g - 2 \leq \alpha$ if $n_1 = n_2$.

In fact this metric is $SU(2)$ -invariant and it is given by a vortex solution on T .

Remark 9.8. This is similar in spirit to the instanton solutions of vortex type on \mathbb{R}^4 studied by Witten [33] and Taubes [30].

References

1. Álvarez-Cónsul, L., García-Prada, O.: Dimensional reduction, $SL(2, \mathbb{C})$ -equivariant bundles and stable holomorphic chains. *Int. J. Math.* **12**, 159–201 (2001)
2. Álvarez-Cónsul, L., García-Prada, O.: Hitchin–Kobayashi correspondence, quivers and vortices. *Comm. Math. Phys.* **238**, 1–31 (2003)
3. Biswas, I., Ramanan, S.: An infinitesimal study of the moduli of Hitchin pairs. *J. London Math. Soc.* (2) **49**, 219–231 (1994)
4. Bradlow, S.B., García-Prada, O.: Stable triples, equivariant bundles and dimensional reduction. *Math. Ann.* **304**, 225–252 (1996)

5. Bradlow, S.B., García-Prada, O., Gothen P.B.: Representations of the fundamental group of a surface in $PU(p,q)$ and holomorphic triples. *C.R. Acad. Sci. Paris* **333**, 347–352 (2001)
6. Bradlow, S.B., García-Prada, O., Gothen P.B.: Surface group representations, Higgs bundles and holomorphic triples. Preprint, 2002, [arXiv:math.AG/0206012](https://arxiv.org/abs/math/0206012)
7. Bradlow, S.B., García-Prada, O., Gothen P.B.: Surface group representations and $U(p, q)$ -Higgs bundles. *J. Differential Geometry* **64**, 111–170 (2003)
8. Brambila-Paz, L., Grzegorzczak, I., Newstead, P.E.: Geography of Brill–Noether loci for small slopes. *J. Algebraic Geom.* **6**, 645–669 (1997)
9. Corlette, K.: Flat G -bundles with canonical metrics. *J. Diff. Geom.* **28**, 361–382 (1988)
10. Donaldson, S.K.: Anti-self-dual Yang–Mills connections on a complex algebraic surface and stable vector bundles. *Proc. Lond. Math. Soc.* **3**, 1–26 (1985)
11. Donaldson, S.K.: Infinite determinants, stable bundles and curvature. *Duke Math. J.* **54**, 231–247 (1987)
12. Donaldson, S.K.: Twisted harmonic maps and the self-duality equations. *Proc. London Math. Soc.* (3) **55**, 127–131 (1987)
13. García-Prada, O.: Dimensional reduction of stable bundles, vortices and stable pairs. *Int. J. Math.* **5**, 1–52 (1994)
14. Gieseker D.: On moduli of vector bundles on an algebraic surface. *Ann. Math.* **106**, 45–60 (1977)
15. Gothen, P.B.: Components of spaces of representations and stable triples. *Topology* **40**, 823–850 (2001)
16. Gothen, P.B., King, A.D.: Homological algebra of quiver bundles. Preprint, 2002, [arXiv:math.AG/0202033](https://arxiv.org/abs/math/0202033)
17. Hernández, R.: On Harder-Narasimhan stratification over Quot schemes. *J. Reine Angew. Math.* **371**, 115–124 (1986)
18. Hitchin, N.J.: The self-duality equations on a Riemann surface. *Proc. London Math. Soc.* **55**, 59–126 (1987)
19. Hitchin, N.J.: Lie groups and Teichmüller space. *Topology* **31**, 449–473 (1992)
20. Kobayashi S.: *Differential Geometry of Complex Vector Bundles*. Princeton University Press, New Jersey, 1987
21. Markman, E., Xia, E.Z.: The moduli of flat $PU(p, p)$ structures with large Toledo invariants. *Math. Z.* **240**, 95–109 (2002)
22. Maruyama M.: Elementary transformations in the theory of algebraic vector bundles. *LNM* **961**, 241–266 (1982)
23. Qin, Z.B.: Equivalence classes of polarizations and moduli spaces of sheaves. *J. Diff. Geom.* **37**, 397–415 (1993)
24. Narasimhan, M.S., Seshadri, C.S.: Stable and unitary bundles on a compact Riemann surface. *Ann. Math.* **82**, 540–564 (1965)
25. Schmitt, A.: Moduli problems of sheaves associated with oriented trees. *Algebras and Representation theory* **6**, 1–32 (2003)
26. Simpson, C.T.: Constructing variations of Hodge structure using Yang-Mills theory and applications to uniformization. *J. Am. Math. Soc.* **1**, 867–918 (1988)
27. Simpson, C.T.: Higgs bundles and local systems. *Inst. Hautes Études Sci. Publ. Math.* **75**, 5–95 (1992)
28. Simpson, C.T.: Moduli of representations of the fundamental group of a smooth projective variety I. *Inst. Hautes Études Sci. Publ. Math.* **79**, 867–918 (1994)
29. Simpson, C.T.: Moduli of representations of the fundamental group of a smooth projective variety II. *Inst. Hautes Études Sci. Publ. Math.* **80**, 5–79 (1994)
30. Taubes, C.H.: On the equivalence of the first and second order equations for gauge theories. *Commun. Math. Phys.* **75**, 207–227 (1980)
31. Thaddeus, M.: Stable pairs, linear systems and the Verlinde formula. *Invent. Math.* **117**, 317–353 (1994)

-
32. Uhlenbeck, K.K., Yau S.T.: On the existence of Hermitian–Yang–Mills connections on stable bundles over compact Kähler manifolds. *Comm. Pure and Appl. Math.* **39–S**, 257–293 (1986)
 33. Witten E.: Some exact multipseudoparticle solutions of classical Yang–Mills theory. *Phys. Rev. Lett.* **38**, 121 (1977)