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Tight, not semi–fillable contact circle bundles

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Abstract. Extending our earlier results, we prove that certain tight contact structures on circle bundles over surfaces are not symplectically semi–fillable, thus confirming a conjecture of Ko Honda.

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1. Introduction

Let Y be a closed, oriented three–manifold. A *positive, coorientable contact structure* on Y is the kernel

$$\xi = \ker\alpha \subset TY$$

of a one–form $\alpha \in \Omega^1(Y)$ such that $\alpha \wedge d\alpha$ is a positive volume form on Y. The pair (Y, ξ) is a *contact three–manifold*. In this paper we only consider positive, coorientable contact structures, so we call them simply 'contact structures'. For an introduction to contact structures the reader is referred to [1, Chapter 8] and [7].

There are two kinds of contact structures ξ on Y. If there exists an embedded disc $D \subset Y$ tangent to ξ along its boundary, ξ is called *overtwisted*, otherwise it is said to be *tight*. The isotopy classification of overtwisted contact structures coincides with their homotopy classification as tangent two–plane fields [4]. Tight contact structures are much more mysterious, and difficult to classify. A contact structure on Y is *virtually overtwisted* if its pull–back to some finite cover of Y becomes overtwisted, while it is called *universally tight* if its pull–back to the universal cover of Y is tight.

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A contact three–manifold (Y, ξ) is *symplectically fillable*, or simply *fillable*, if there exists a compact symplectic four–manifold (W, ω) such that (i) $\partial W = Y$ as oriented manifolds (here W is oriented by $\omega \wedge \omega$) and (ii) $\omega|_\xi \neq 0$ at every point of Y. (Y, ξ) is symplectically *semi*–fillable if there exists a fillable contact manifold (N, η) such that $Y \subset N$ and $\eta|_Y = \xi$. Semi–fillable contact structures are tight [6, 13]. The converse is known to be false by work of Etnyre and Honda, who recently found two examples of tight but not semi–fillable contact three–manifolds [8]. Nevertheless, all such examples known at present are virtually overtwisted, so it is natural to wonder whether every universally tight contact structure is symplectically semi–fillable.

In this paper we study certain virtually overtwisted tight contact structures discovered by Ko Honda. Denote by $Y_{g,n}$ the total space of an oriented S^1–bundle over Σ_g with Euler number n. Honda gave a complete classification of the tight contact structures on $Y_{g,n}$ [14]. The three–manifolds $Y_{g,n}$ carry infinitely many tight contact structures up to diffeomorphism. The hardest part of the classification involves two virtually overtwisted contact structures ξ_0 and ξ_1, which exist only when $n \geq 2g$. Honda conjectured that ξ_0 and ξ_1 are not symplectically semi–fillable [14]. The main theorem of the present paper extends our earlier results regarding these structures [19], establishing Honda's conjecture:

Theorem 1.1. *For $n \geq 2g > 0$, the tight contact structures ξ_0 and ξ_1 on $Y_{g,n}$ are not symplectically semi–fillable.*

The proof of Theorem 1.1 consists of two steps. In the first step, we derive a contact surgery presentation for ξ_0 and ξ_1 in the sense of [3], and we use it to determine the homotopy type of ξ_0 and ξ_1 considered as oriented two–plane fields. This is done in Sections 2 and 3.

In the second step, using specific properties of the $Spin^c$ structures $\mathbf{t}_{\xi_i}$ on $Y_{g,n}$ induced by ξ_i $(i = 0, 1)$ we generalize a result of the first author [17] so it applies to the situation at hand. Using this generalization together with an analytic computation of Nicolaescu's [20], we are able to determine the possible homotopy types of a semi–fillable contact structure inducing either $\mathbf{t}_{\xi_0}$ or $\mathbf{t}_{\xi_1}$. This is done in Section 4.

Theorem 1.1 follows immediately from the fact that the two sets of homotopy classes determined in the two steps above have empty intersection.

2. Contact surgery presentations for ξ_0 and ξ_1

A smooth knot K in a contact three–manifold (Y, ξ) which is everywhere tangent to ξ is called *Legendrian*. The contact structure ξ naturally induces a framing of K called the *contact framing*.

Let Σ_g be a closed, oriented surface of genus $g \geq 1$, and let

$$\pi : \ Y_{g,n} \to \Sigma_g$$

denote an oriented circle bundle over Σ_g with Euler number n. Let ξ be a contact structure on $Y_{g,n}$ such that a fiber

$$f = \pi^{-1}(s) \subset Y_{g,n} \quad (s \in \Sigma_g)$$

is Legendrian. We say that f has *twisting number* k if the contact framing of f is k with respect to the framing determined by the fibration π. A contact structure on $Y_{g,n}$ is called *horizontal* if it is isotopic to a contact structure transverse to the fibers of π.

Let ζ be a horizontal contact structure on $Y_{g,2g-2}$ such that a fiber f of the projection π is Legendrian with twisting number -1 (the existence of such a contact structure is well–known, cf. [10, §1.D]). Let $n \geq 2g$, and view the bundle $Y_{g,n} \to \Sigma_g$ as obtained by performing a $-\frac{1}{p+1}$–surgery, where

$$p = n - 2g + 1,$$

along the fiber f of $\pi : Y_{g,2g-2} \to \Sigma_g$ with respect to the trivialization induced by the fibration π. It was observed by Honda ([14, §5]) that when $n > 2g$ there are two possible ways of extending ζ from the complement of a standard neighborhood of f to a tight contact structure on $Y_{g,n}$, while when $n = 2g$ there is only one possible extension. This determines the contact structures ξ_0 and ξ_1, which coincide when $n = 2g$.

The construction of ξ_0 and ξ_1 can be viewed as a particular case of a more general construction. In fact, given a Legendrian knot K in a contact three–manifold (Y, ξ) and a nonzero rational number $r \in \mathbb{Q}$, it is possible to perform a *contact r–surgery* along K to obtain a new contact three–manifold (Y', ξ') [2,3]. Here Y' is the three-manifold obtained by a smooth r–surgery along K with respect to the contact framing, while ξ' is constructed by extending ξ from the complement of a standard neighborhood of K to a tight contact structure on the glued–up solid torus. Such extension exists once $r \neq 0$. In general there are several ways to extend ξ, but up to isotopy there is only one if

$$r = \frac{1}{k}, \quad k \in \mathbb{Z},$$

and two if

$$r = \frac{p}{p+1}, \quad p > 1,$$

as follows from [2, Propositions 3, 4 and 7]. When $r = -1$ the corresponding contact surgery coincides with Legendrian surgery [5,11,23]. A simple computation using the fact that the fiber f of $Y_{g,2g-2}$ has twisting number -1 with respect to the contact structure ζ shows that $\{\xi_0, \xi_1\}$ can be defined as the set of contact structures obtainable by contact $\frac{p}{p+1}$–surgery along f.

From now on, we shall indicate a contact r–surgery along a Legendrian knot K by writing the coefficient r next to it. Consider the result of performing contact

(-1)–surgery on the Legendrian knot in standard form in Figure 1 (here we are using the notation of [11], see especially Definition 2.1). Since contact (-1)–surgery is equivalent to Legendrian surgery, Figure 1 also represents a Stein four–manifold W with boundary [11]. As a smooth four–manifold, W is diffeomorphic to the two–disc bundle $D_{g,2g-2}$ with Euler number $2g-2$ over a surface of genus g. This can be checked by converting the contact surgery coefficient into the corresponding smooth surgery coefficient e.g. via the formulas found in [11] or [12]. Since by construction the boundaries of Stein four–manifolds come equipped with Stein fillable contact structures, we have a Stein fillable contact structure $\zeta(g)$ on $Y_{g,2g-2}$, which is tight by [6, 13].

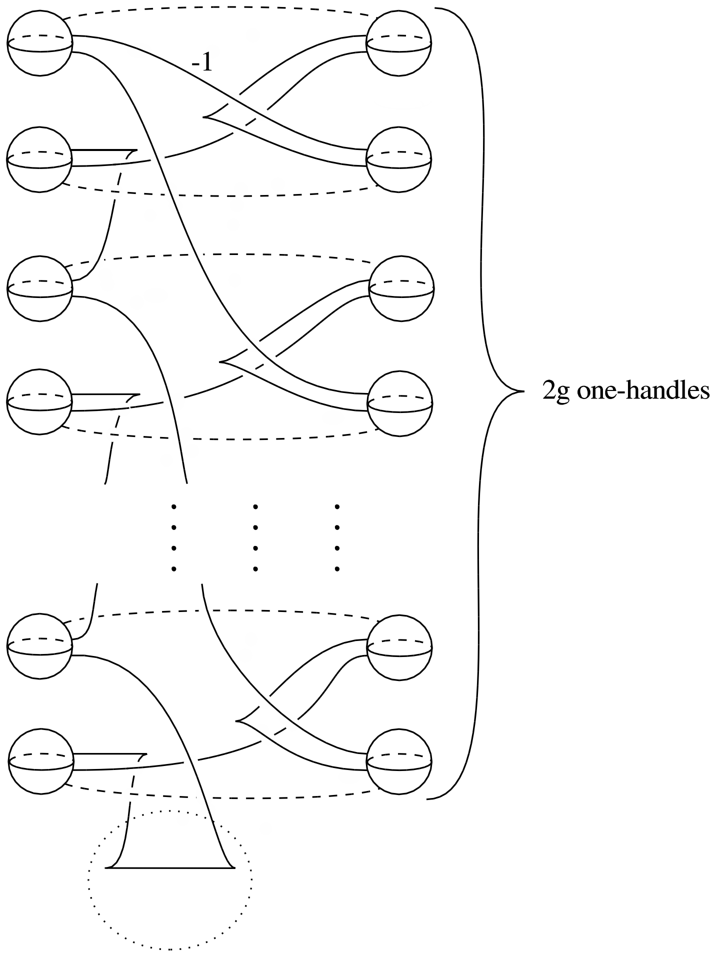


Fig. 1. The Stein four–manifold W (with boundary), diffeomorphic to $D_{g,2g-2}$

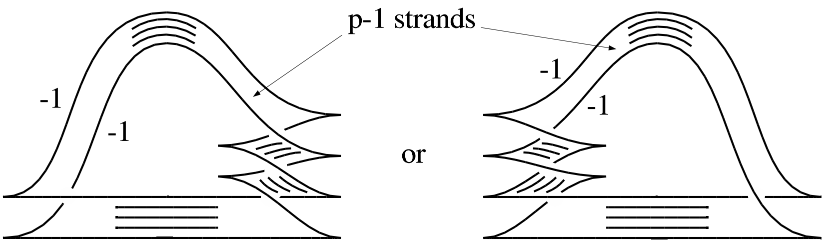


Fig. 2. The Legendrian surgeries equivalent to contact $-\frac{p}{p-1}$–surgery

Lemma 2.1. *The contact structure $\zeta(g)$ is horizontal.*

Proof. According to the classification [14, Theorem 2.11] of tight contact structures on $Y_{g,2g-2}$, the claim follows once we prove that the twisting number of any Legendrian knot isotopic to the fiber is negative. This latter negativity follows from the fact that contact (-1)–surgery on a Legendrian knot isotopic to a fiber and having twisting number ≥ 0 would result in a Stein four–manifold containing a sphere with self-intersection ≥ -1, contradicting the adjunction inequality for Stein four–manifolds [18]. □

By [3, Proposition 3], any contact r–surgery with $r < 0$ is equivalent to a Legendrian surgery along a Legendrian link. Moreover, the set of Legendrian links which correspond to some contact r–surgery is determined via a simple algorithm by the Legendrian knot and the continued fraction expansion of r. For example, let K be a Legendrian unknot in the standard contact three–sphere with Thurston–Bennequin invariant equal to -1. Then, a contact $-\frac{p}{p-1}$–surgery ($p > 1$) along K is equivalent to Legendrian surgery along one of the Legendrian links in Figure 2. According to [3, Proposition 7], a contact $\frac{p}{p+1}$–surgery on a Legendrian knot K is equivalent to a contact $\frac{1}{2}$–surgery on K followed by a contact $-\frac{p}{p-1}$–surgery on a Legendrian push–off of K. By [2, Proposition 9], a contact $\frac{1}{2}$–surgery on a Legendrian knot K can be replaced by two contact $(+1)$–surgeries, one on K and the other on a Legendrian push–off of K.

Consider the Legendrian knot $L \subset \partial W$ whose diagram is obtained by replacing the "dotted ellipse" in Figure 1 with Figure 3. If we perform contact $\frac{p}{p+1}$–surgery on L, the resulting contact structures will have contact surgery presentations obtained by replacing the "dotted ellipse" in Figure 1 with either Figure 4(a) or 4(b). The Legendrian knot

$$L \subset \partial W = Y_{g,2g-2}$$

is isotopic to a fiber of

$$\pi:\ Y_{g,2g-2} \to \Sigma_g$$

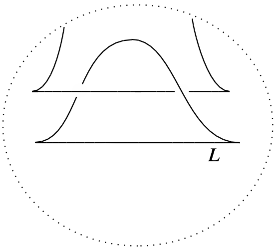
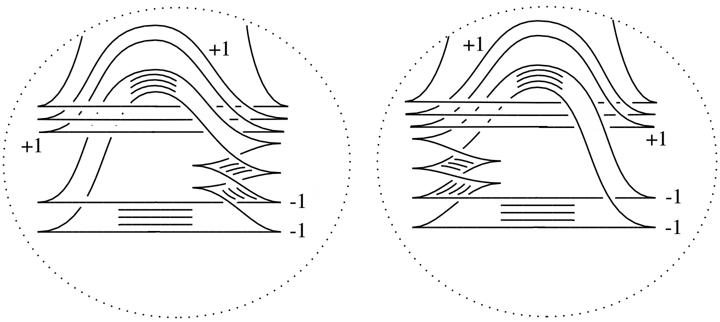


Fig. 3. The Legendrian knot L



(a) (b)

Fig. 4. Pictures to be pasted in Figure 1 to obtain ξ_0 and ξ_1

and it has contact framing -1 with respect to the framing induced by the fibration. Since $\zeta(g)$ is horizontal, we can define ξ_0, respectively ξ_1, as the contact structure obtained by using the surgery diagram of Figure 4(a), respectively Figure 4(b).

3. Homotopy classes of ξ_0 and ξ_1

Homotopy theory of oriented two–plane fields on three–manifolds

Let Ξ_Y denote the space of oriented two–plane fields on the closed, oriented three–manifold Y. Since a $Spin^c$ structure on a three–manifold can be interpreted as an equivalence class of nowhere vanishing vector fields [22], by taking the oriented normal a coorientable two–plane field $\xi \in \Xi_Y$ naturally induces a $Spin^c$ structure $\mathbf{t}_\xi$, which depends only on the homotopy class $[\xi]$. Therefore there is a map

$$p\colon \pi_0(\Xi_Y) \to Spin^c(Y)$$

defined as $p([\xi]) = \mathbf{t}_\xi$. It is not difficult to show that, if Y is connected, there is a non–canonical identification of each fiber $p^{-1}(\mathbf{t}_\xi)$ with $\mathbb{Z}/d(\mathbf{t}_\xi)\mathbb{Z}$, where $d(\mathbf{t}_\xi) \in \mathbb{Z}$ is the divisibility of

$$c_1(\xi) \in H^2(Y;\mathbb{Z}),$$

and it is zero if $c_1(\xi)$ is a torsion element (see, e.g. [11, Proposition 4.1]).

When $c_1(\xi)$ is torsion the two–plane fields inducing the same $Spin^c$ structure $\mathbf{t}_\xi$ can be distinguished by a numerical invariant. Suppose that X is a compact 4-manifold with $\partial X = Y$, with X carrying an almost–complex structure J whose complex tangents at the boundary form an oriented two–plane field homotopic to ξ on Y. Observe that the fact that $c_1(\xi)$ is torsion implies that $c_1^2(X, J) \in \mathbb{Q}$ makes sense.

Theorem 3.1 ([11]). *The rational number*

$$d_3(\xi) = \frac{1}{4}(c_1^2(X, J) - 3\sigma(X) - 2\chi(X)) \in \mathbb{Q}$$

depends only on $[\xi]$*, not on the almost–complex four–manifold* (X, J)*. Moreover, two two–plane fields* ξ_1 *and* ξ_2 *inducing the same* $Spin^c$ *structure with torsion first Chern class are homotopic if and only if*

$$d_3(\xi_1) = d_3(\xi_2).$$ □

In the following, the invariant d_3 will be called the *three–dimensional invariant.*

Attaching two–handles and homotopy invariants

Recall that contact (-1)–surgery, i.e. Legendrian surgery, can be viewed as the result of attaching a symplectic two–handle [23]. In fact, attaching the two–handle to a contact three–manifold (Y_1, ξ_1) gives rise to a cobordism W between Y_1 and the three–manifold underlying the contact three–manifold (Y_2, ξ_2) resulting from the three–dimensional contact surgery. Furthermore, W carries an almost–complex structure whose complex tangent lines at the boundary coincide with ξ_1 and ξ_2 (see e.g. [9]).

In the case of contact $(+1)$-surgery, there is still a smooth cobordism W between Y_1 and Y_2. One can easily check the existence of an almost–complex structure J on the complement of a ball B in the interior of W, with J inducing ξ_1 and ξ_2 as tangent complex lines. We define q to be the three–dimensional invariant of the two–plane field induced by J on ∂B. Observe that, although J may not extend to the whole cobordism, J induces a $Spin^c$ structure $\mathbf{s}_J$ which does extend – uniquely – to W. *A priori* q depends on (Y_1, ξ_1) and the Legendrian knot K along which we perform the contact surgery. In fact, it can be shown that q is independent of these data; we will not use this fact here.

Lemma 3.1. *With the above notation, if* (Y_1, ξ_1) *is the standard contact three–sphere and* K *is a Legendrian unknot with Thurston–Bennequin invariant equal to* -1*, then* $q = \frac{1}{2}$.

Proof. Consider K in the standard contact three–sphere as above. We view the standard contact three–sphere as the contact boundary of the unit ball $B_1(0) \subset \mathbb{C}^2$. Attach a smooth two–handle H_1 to $B_1(0)$ along K with framing $+1$ with respect to the contact framing. The result is a smooth four–manifold X diffeomorphic to $S^2 \times D^2$. The unique $Spin^c$ structure on $B_1(0)$ extends to a $Spin^c$ structure $\mathbf{s}$ on X, restricting to H_1 as the $Spin^c$ structure defined above. Denote by k the value of $c_1(\mathbf{s})$ on a generator of the second homology group of X.

Let K' be a Legendrian push–off of K, which we may assume disjoint from H_1, and attach a symplectic two–handle H_2 to K' realizing Legendrian surgery on K'. The $Spin^c$ structure $\mathbf{s}$ extends over H_2, and the value of its first Chern class on the homology generator corresponding to K' is 0, because K' has vanishing rotation number (see [11], especially the proof of Proposition 2.3). By [2, Proposition 8] the resulting contact three–manifold is just the standard contact three–sphere. Its three–dimensional invariant d_3 is $-\frac{1}{2}$, but when viewed as the result of the above construction, d_3 can also be expressed as

$$\frac{1}{4}(2k^2 - 4) + q.$$

We can generalize this argument using Legendrian push–offs

$$K_1, K_1', \dots, K_n, K_n'$$

of K by performing contact $(+1)$–surgeries on

$$K_1, \dots, K_n$$

and contact (-1)–surgeries on

$$K_1', \dots, K_n'.$$

The resulting contact three–manifold is the standard contact three–sphere again. A homological computation as before gives the identity

$$\frac{1}{4}(n+1)(k^2 n - 2) + nq = -\frac{1}{2},$$

which must hold for all $n \in \mathbb{N}$. This implies that $k = 0$ and $q = \frac{1}{2}$. □

Spin^c structures on disc and circle bundles

Let $D_{g,n}$ be the oriented disc bundle with Euler number n over a closed oriented surface of genus g. By e.g. fixing a metric on $D_{g,n}$ one sees that the tangent bundle of $D_{g,n}$ is isomorphic to the direct sum of the pull–back of $T\Sigma_g$ and the vertical tangent bundle, which is isomorphic to the pull–back of the real oriented two–plane bundle

$$E_{g,n} \to \Sigma_g$$

with Euler number n. In short, we have

$$TD_{g,n} \cong \pi^*(T\Sigma_g \oplus E_{g,n}).$$

This splitting of $TD_{g,n}$ naturally endows $D_{g,n}$ with an almost–complex structure which induces a $Spin^c$ structure $\mathbf{s}_0$ on $D_{g,n}$. The orientation on $D_{g,n}$ determines an isomorphism

$$H^2(D_{g,n};\mathbb{Z}) \cong \mathbb{Z},$$

so the set

$$Spin^c(D_{g,n}) = \mathbf{s}_0 + H^2(D_{g,n};\mathbb{Z})$$

can be canonically identified with the integers. We denote by

$$\mathbf{s}_e = \mathbf{s}_0 + e \in Spin^c(D_{g,n})$$

the element corresponding to the integer

$$e \in \mathbb{Z} \cong H^2(D_{g,n};\mathbb{Z}).$$

Consider $Y_{g,n} = \partial D_{g,n}$. We have

$$H_1(Y_{g,n};\mathbb{Z}) \cong H^2(Y_{g,n};\mathbb{Z}) \cong \mathbb{Z}^{2g} \oplus \mathbb{Z}/n\mathbb{Z},$$

where the summand $\mathbb{Z}/n\mathbb{Z}$ is generated by the Poincaré dual F of the class of a fiber of the projection

$$\pi\colon\ Y_{g,n} \to \Sigma_g.$$

Each $Spin^c$ structure

$$\mathbf{s}_e \in Spin^c(D_{g,n})$$

determines by restriction a $Spin^c$ structure

$$\mathbf{t}_e \in Spin^c(Y_{g,n}),$$

with

$$\mathbf{t}_e = \mathbf{t}_0 + eF, \quad e \in \mathbb{Z}.$$

Since $nF = 0$, we see that $\mathbf{t}_{e+n} = \mathbf{t}_e$ for every e. Therefore,

$$\mathbf{t}_0, \ldots, \mathbf{t}_{n-1}$$

is a complete list of *torsion $Spin^c$ structures* on $Y_{g,n}$, i.e. $Spin^c$ structures on $Y_{g,n}$ with torsion first Chern class. In short, the $Spin^c$ structures on $Y_{g,n}$ which extend to the disc bundle are precisely the torsion ones.

Homotopy invariants of the contact structures ξ_i

Let W be the Stein four–manifold with boundary diffeomorphic to $D_{g,2g-2}$ given by Figure 1. Consider the smooth four–dimensional handlebody X obtained by attaching to W the two–handles realizing the contact surgeries described in Figure 4(a) or 4(b). Converting the contact framing coefficients into the usual ones, we see that a framed link presentation of X is obtained by pasting Figure 5(a) in place of the "dotted ellipse" in Figure 1.

By the discussion above on attaching two–handles we know that, corresponding to each of Figure 4(a) and 4(b), there is an almost–complex structure on X minus two balls lying in the interior of the two–handles realizing the $(+1)$–surgeries. Moreover, the two almost–complex structures determine the two–plane fields ξ_0 and ξ_1 on ∂X and two $Spin^c$ structures $\mathbf{s}_0$ and $\mathbf{s}_1$ on X. Observe that, since the rotation number of the Legendrian knot in Figure 1 vanishes, it follows from [11, Theorem 4.12] that $c_1(W) = 0$. In the same way, it follows that we can choose an orientation of the $n - 2g$ linking knots with framing -3 in Figure 5(a) so that $c_1(\mathbf{s}_i)$ evaluates as $(-1)^i$ on all the corresponding homology classes. Finally, by the argument given in the proof of Lemma 3.1, $c_1(\mathbf{s}_i)$ evaluates trivially on the generators of $H_2(X;\mathbb{Z})$ determined by the two–handles corresponding to the $(+1)$–surgeries.

The four–manifold X is diffeomorphic to

$$D_{g,n}\#S^2 \times S^2\#(n-2g)\overline{\mathbb{CP}}^2.$$

One can see this by performing a sequence of handleslides on the Kirby diagram as shown in Figure 5. In fact, start by sliding over the knot K_1 in Figure 5(a) the remaining $(n-2g-1)$ (-3)–framed circles. Then, slide K_1 over K_2 and finally K_2 over K_3, obtaining 5(b). Sliding the long $(2g-2)$–framed arc over the 2–framed knot and using the 0–framed normal circle to separate the 2–framed circle from the rest of the diagram, we get 5(c). Blowing down the (-1)–circle results in 5(d), and $(n-2g-1)$ further blow downs give 5(e). Following the handle slides of Figure 5 on the homological level we see that $c_1(\mathbf{s}_i)$ evaluates on the generator of the second homology of $D_{g,n}$ as $(-1)^i(n-2g)$. Moreover, it evaluates as $(-1)^i$ on generators of the $\overline{\mathbb{CP}}^2$ summands, and vanishes when restricted to the $S^2 \times S^2$ summand. This immediately implies that the $Spin^c$ structure $\mathbf{t}_{\xi_i}$ is equal to the restriction of the unique $Spin^c$ structure

$$\mathbf{s}_e \in Spin^c(D_{g,n})$$

such that $c_1(\mathbf{s}_e)$ evaluates on the generator of $H_2(D_{g,n};\mathbb{Z})$ as $(-1)^i(n-2g)$. Since the value of $c_1(\mathbf{s}_0)$ on the generator is $2-2g+n$, e satisfies the equation:

$$2-2g+n+2e = (-1)^i(n-2g).$$

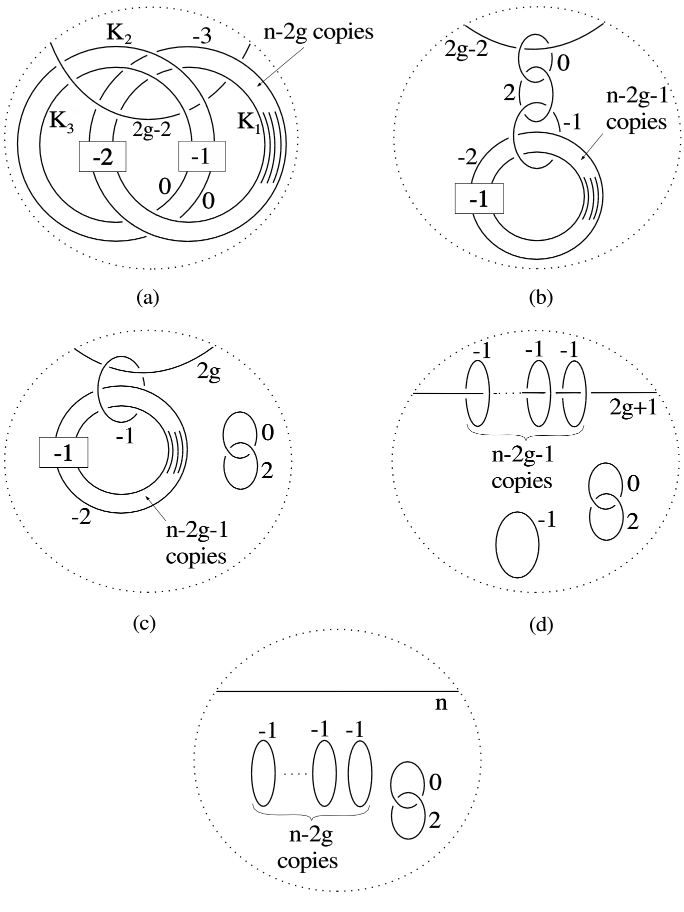


Fig. 5. The diffeomorphism between X and $D_{g,n}\#S^2 \times S^2\#(n-2g)\overline{\mathbb{CP}}^2$

Therefore we get $e = -1$ or $e = 2g - 1 + n$ respectively for $i = 0$ or $i = 1$. Since $\mathbf{s}_e|_{Y_{g,n}} = \mathbf{t}_e$, we conclude that

$$\mathbf{t}_{\xi_i} = \mathbf{t}_{2ig-1}$$

for $i = 0, 1$. Observe that this result is consistent with the independent calculation made in [19].

Lemma 3.2. *The value of the three–dimensional invariant of ξ_i is*

$$d_3(\xi_i) = \frac{n^2 - 3n + 4g^2}{4n}.$$

Proof. We have

$$\chi(X) = n - 4g + 4, \quad \sigma(X) = 1 - n + 2g.$$

From what we know about $c_1(\mathbf{s}_i)$ it is easy to deduce that

$$c_1^2(\mathbf{s}_i) = -2\frac{g(n-2g)}{n}.$$

In order to compute the three–dimensional invariant we need to take into account the correction term q for each of the two contact $(+1)$–surgeries. Using Lemma 3.1 we conclude

$$d_3(\xi_i) = \frac{1}{4}(c_1^2(\mathbf{s}_i) - 2\chi(X) - 3\sigma(X)) + 2 = \frac{n^2 - 3n + 4g^2}{4n}. \qquad \square$$

4. The proof of Theorem 1.1

Theorem 4.1. *Let* $n \geq 2g > 0$*, and let* ξ *be an oriented two–plane field on* $Y_{g,n}$ *such that* $\mathbf{t}_\xi \in \{\mathbf{t}_{\xi_0}, \mathbf{t}_{\xi_1}\}$*. If* ξ *is homotopic to a semi–fillable contact structure, then*

$$d_3(\xi) = \frac{n^2 + n + 4g^2}{4n} - 2g - 2.$$

Proof. In the proof of [17, Theorem 2.1] it is shown that if Y is a closed three–manifold and $\mathbf{t} \in Spin^c(Y)$ is torsion and satisfies:

- all Seiberg-Witten solutions in $\mathbf{t}$ are reducible and
- the moduli space of the Seiberg-Witten solutions in $\mathbf{t}$ is a smooth manifold and the corresponding Dirac operators have trivial kernels,

then the expected dimension d_1 of the Seiberg-Witten moduli space of solutions over a symplectic semi–filling of $(Y_{g,n}, \xi_i)$ equipped with a cylindrical end metric and fixed asymptotic limit is equal to

$$d_1 = -1 - b_1(Y_{g,n}). \tag{1}$$

The moduli space of Seiberg-Witten solutions on $Y_{g,n}$ has been determined in [15] (see also [21]). These results show that the assumptions listed above hold for the moduli spaces associated to the $Spin^c$ structures $\mathbf{t}_{\xi_i}$. Therefore, Equation (1) holds. This implies that for each $i = 0, 1$, $Y_{g,n}$ carries only one homotopy type of two–plane fields potentially containing semi–fillable contact structures inducing $\mathbf{t}_{\xi_i}$, because d_1 is equal to the three–dimensional invariant plus an expression involving some topological terms and an η–invariant which depends only on $\mathbf{t}_{\xi_i}$ [16, Formula 3.1]. In fact, such an expression has been explicitely calculated in [20], in the formula preceding (3.29), so our proof reduces to translating that formula into our notations.

In Nicolaescu's notations the integer κ corresponds to $\mathbf{t}_{g-1+\kappa}$. This is because his "base" $Spin^c$ structure is induced by a $Spin$ structure on $Y_{g,n}$ with associated bundle of spinors

$$\mathbb{S} = \pi^* K_{\Sigma_g}^{-\frac{1}{2}} \oplus \pi^* K_{\Sigma_g}^{\frac{1}{2}} \to Y_{g,n}$$

(see text following Formula (2.6) in [20]), and $\mathbb{S}$ is the restriction of

$$T D_{g,n} \otimes \pi^* K_{\Sigma_g}^{\frac{1}{2}} \to D_{g,n}$$

to the boundary.

The result we need is obtained by substituting n for ℓ and g or $n-g$ in place of κ into the formula preceding (3.29) in [20]. (The formula we are using here differs from Formula (3.29) by the additive term $2g-1$ because (3.29) computes the dimension of the whole moduli space rather than the dimension of the moduli space of solutions with a fixed asymptotic limit, i.e. d_1.) Explicitely, in our notation we have:

$$-1 - b_1(Y_{g,n}) = d_1 = d_3(\xi) - \frac{1}{2}(2g-1) - \frac{1}{4}(n-1) - \frac{\kappa^2}{n} + \kappa$$

where $b_1(Y_{g,n}) = 2g$ and the value of κ to be substituted is either g or $n-g$ according to whether $\mathbf{t}_\xi = \mathbf{t}_{\xi_1}$ or $\mathbf{t}_\xi = \mathbf{t}_{\xi_0}$, respectively. In both cases we obtain for $d_3(\xi)$ the value given in the statement. □

Proof of Theorem 1.1. Let ξ be a two–plane field representing a homotopy class inducing $\mathbf{t}_{\xi_i}$ which might be represented by a semi–fillable contact structure. Then, by Lemma 3.2 and Theorem 4.1 we have

$$d_3(\xi_i) - d_3(\xi) = 2g + 1 > 0.$$

Therefore, the homotopy classes $[\xi_i]$ cannot be represented by semi–fillable contact structures. □

Remarks. (1) For $n < 2g$ the circle bundle $Y_{g,n}$ admits no $Spin^c$ structure for which the Seiberg-Witten moduli space has the properties required by the proof of Theorem 4.1.

(2) The assumption $g > 0$ in Theorem 4.1 is necessary, since $Y_{0,n}$ is a lens space on which all tight contact structures are Stein fillable. The proof of Theorem 4.1 breaks down since the formula from [20] used in the proof holds only for $g \geq 1$.

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