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Propagation of polarization in elastodynamics with residual stress and travel times

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Abstract. We show that knowing the displacement-to-traction map associated to the equations of isotropic elastodynamics with residual stress we can determine the lens maps of compressional and shear waves. We derive several consequences of this for the inverse problem of determining the residual stress and the Lamé parameters from the displacement-to-traction map.

1. Introduction

Consider an elastic medium which occupies a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary $\partial\Omega$ and exterior normal ν. Displacement is a time-dependent vector field $u(t,\cdot)$ on $\overline{\Omega}$. Small displacements satisfy, in a source-free medium, the equations for (linearized) elastodynamics,

$$\rho\, \partial^2 u/\partial t^2 = \nabla \cdot S \quad \text{with} \quad S = R + \nabla u\, R + CE. \tag{1}$$

Here $0 < \rho \in C^\infty(\overline{\Omega})$ denotes the density, S is the Piola-Kirchhoff stress tensor which obeys the relation $SF^T = FS^T$ where $F = I + \nabla u$ is the deformation gradient. Divergence and transpose are taken with respect to the Euclidean metric $|\cdot|$. See [Gur72, Sect. 16] and [MH83, Sect. 4.2-3]. The elasticity tensor C maps infinitesimal strain tensors $E = (\nabla u + \nabla u^T)/2$ to symmetric stress tensors CE. C represents the elastic properties of the medium. $R(x)$, the residual stress tensor, is a symmetric 3×3-matrix, C^∞ on $\overline{\Omega}$. It satisfies $\nabla \cdot R = 0$. See [Gur72, Sect. 23],

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[Hog86]. There are no assumptions on the boundary values of R. We call

$$Pu = -\rho\, \partial^2 u/\partial t^2 + \nabla \cdot \big(R + \nabla u\, R + CE\big) \tag{2}$$

the operator for elastodynamics with residual stress. P is isotropic if the elasticity tensor is as follows,

$$CE = \lambda\, \mathrm{tr}(E) I + 2\mu E \quad \text{with } \lambda, \mu \in C^\infty(\overline{\Omega}), \quad 0 < \mu, \lambda + \mu. \tag{3}$$

λ and μ are the Lamé parameters of the elastic medium.

The initial displacement boundary value problem of elastodynamics is

$$Pu = 0 \text{ in } \mathbb{R} \times \Omega, \quad u = f \text{ on } \mathbb{R} \times \partial\Omega, \quad u = 0 \text{ initially.} \tag{4}$$

Given a boundary displacement $f \in \mathcal{E}'(\mathbb{R} \times \partial\Omega)^3$ a solution $u \in \mathcal{D}'(\mathbb{R} \times \Omega)^3$ is sought. By definition, $u = 0$ initially means that there exists $t_0 \in \mathbb{R}$ such that $u = 0$ in $\{t < t_0\}$. For existence and uniqueness see the discussion at the beginning of section 4.

The inverse problem for operators of elastodynamics is to recover as much as possible of the elasticity tensor C and of the residual stress tensor R from measurements performed at the space-time boundary $\mathbb{R} \times \partial\Omega$. See [ML87], [Rob97], and [Rac00c] for approaches to residual stress determination.

We deal with an inverse problem for subclasses $\mathcal{L}(L, \varepsilon)$, $L, \varepsilon > 0$, of operators for isotropic elastodynamics. By definition, $P \in \mathcal{L}(L, \varepsilon)$ if and only if

$$\lambda(x) + 2\mu(x), 1/\mu(x), 1/\rho(x) \le L \quad \text{when } x \in \overline{\Omega} \tag{5}$$

and

$$|R(x)| \le \varepsilon \mu(x) \quad \text{when } x \in \overline{\Omega}. \tag{6}$$

Here $|R(x)| = \sup_{|\xi|=1} |R(x)\xi \cdot \xi|$. Assumptions like (6) with ε small have been introduced before to ascertain well-posedness. See, e.g., [Rob97] for the static case. We will prove the well-posedness of (4) and the existence of microlocal parametrices if $P \in \mathcal{L}(L, \varepsilon)$, $\varepsilon > 0$ sufficiently small.

The (hyperbolic) displacement-to-traction map

$$\Lambda : u|_{\mathbb{R}\times\partial\Omega} \mapsto \nu \cdot S|_{\mathbb{R}\times\partial\Omega} \tag{7}$$

encodes boundary measurements. Here u solves (1) with zero initial data. We say that a property of an operator for elastodynamics from a given class is *determined by boundary measurements* if the property is the same for any two operators in the class with identical displacement-to-traction maps. It is also of interest to bound the time of measurement. We say that measurements of *duration* $T > 0$ suffice to determine a property if this property is determined by the restricted displacement-to-traction map, $u|_{[0,T]\times\partial\Omega} \mapsto \nu \cdot S|_{[0,T]\times\partial\Omega}$.

A useful approach to inverse problems consists in using high-frequency waves u generated by boundary data with singularities. From travel times of singularities of u recorded at $\partial\Omega$ one then aims to recover the requested properties. The latter problem is called an inverse kinematic problem. Obviously, an important step is to prove that travel times are in fact determined by boundary measurements. The main goal of this paper is provide a result of this kind which is applicable also when caustics may develop.

We study the propagation of polarization in the sense of Dencker [Den82] for the initial boundary problem of the operator for isotropic elastodynamics with residual stress, P. In Proposition 4.1 we show that P is a system of real principal type if the residual stress R satisfies

$$\mu(x)|\xi|^2 + R(x)\xi \cdot \xi > 0 \quad \text{when } (x, \xi) \in T^\star(\overline{\Omega}) \setminus 0. \tag{8}$$

If (8) holds then

$$\langle \xi, \xi \rangle_S = \big(\mu(x)|\xi|^2 + R(x)\xi \cdot \xi\big)/\rho(x), \tag{9}$$

$$\langle \xi, \xi \rangle_P = \big((\lambda(x) + 2\mu(x))|\xi|^2 + R(x)\xi \cdot \xi\big)/\rho(x), \tag{10}$$

are the duals $g_{S/P}^{-1}(x, \xi) = \langle \xi, \xi \rangle_{S/P}$, $(x, \xi) \in T^\star(\overline{\Omega})$, of Riemannian metrics $g_{S/P}$ on $\overline{\Omega}$. The characteristic variety of P is the union of the subvarieties $\tau^2 - \langle \xi, \xi \rangle_S = 0$ and $\tau^2 - \langle \xi, \xi \rangle_P = 0$ which correspond to shear and compressional waves, respectively.

The *lens map* or scattering relation $\mathcal{S} \subset \big(T^\star(\mathbb{R} \times \partial\Omega) \setminus 0\big)^2$ of a metric $g = (g^{ij})$ on $\overline{\Omega}$ with dual metric $g^{-1} = (g_{ij})$ is defined as follows. Consider bicharacteristic curves, $\gamma : [a, b] \to T^\star(\overline{\Omega} \times \mathbb{R})$, of the Hamilton function $H(t, x, \tau, \xi) = \tau^2 - g^{-1}(x, \xi)$ which satisfy the following: $\gamma(]a, b[)$ lies in the interior, γ intersects the boundary non-tangentially at $\gamma(a)$ and $\gamma(b)$, and time increases along γ. Then the the canonical projection from $(T^\star_{\mathbb{R}\times\partial\Omega}(\mathbb{R} \times \Omega) \setminus 0)^2$ onto $(T^\star(\mathbb{R} \times \partial\Omega) \setminus 0)^2$ maps the endpoint pair $(\gamma(b), \gamma(a))$ to a point in $\mathcal{S}$. Every point in $\mathcal{S}$ arises in this way. It is well-known that $\mathcal{S}$ is a homogeneous canonical relation on $T^\star(\mathbb{R} \times \partial\Omega) \setminus 0$. (See [Gui77] for the concept of a scattering relation.) $\mathcal{S}$ is, in fact, a diffeomorphism between open subsets (of hyperbolic regions) of $T^\star(\mathbb{R} \times \partial\Omega) \setminus 0$. We denote by $\mathcal{S}_S$ (resp. $\mathcal{S}_P$) the lens map of g_S (resp. g_P) and call it the shear (resp. compressional) lens map. The lens maps contain all travel time data.

Our main result is the following.

Theorem 1.1. *Given $L > 0$ there exists $\varepsilon > 0$ such that in the class $\mathcal{L}(L, \varepsilon)$ the shear and the compressional lens maps are determined by boundary measurements.*

Note that we are able to distinguish travel times of shear waves from travel times of compressional waves in boundary measurements. The proof implies, in addition, that observations of duration $T > 0$ suffice to recover all travel times $\leq T$. The proof of Theorem 1.1 is based on an analysis of the propagation of singularities for solutions to (4). Only singularities propagating through the interior of the medium occur. Solutions with singularities propagating in the boundary occur for traction boundary problems. These Rayleigh waves are not considered here.

Let g be Riemannian metric on $\overline{\Omega}$. Denote by D the open subset of $\partial\Omega \times \partial\Omega$ which consists of the pairs (x, y) of boundary points which can be joined by a geodesic which passes through the interior except for the endpoints x and y where it intersects $\partial\Omega$ transversally. By definition, the boundary distance function of $(\overline{\Omega}, g)$ is the function $d : D \to [0, \infty[$ which assigns to $(x, y) \in D$ the geodesic distance, i.e., the infimum of the lengths of such geodesics. If $(\overline{\Omega}, g)$ is strictly convex then D is the complement of the diagonal and d is smooth. Geodesics of g are projections of bicharacteristic curves of $\tau^2 - g^{-1}(x, \xi) = 0$. Geodesic distances equal travel times. When d is smooth it is a generating function of (a subset of) the lens maps $\mathcal{S}$ of g, i.e., $((t_1, x_1, \tau, \xi_1), (t_0, x_0, \tau, \xi_0)) \in \mathcal{S}$ if $t_1 - t_0 = d(x_1, x_0)$ and $\xi_j = -\tau\partial d(x_1, x_0)/\partial x_j$ for $j = 0, 1$. (See [Car35], [GS77].) Clearly, $\mathcal{S}$ determines d. The boundary distance functions of the metrics g_S and g_P are denoted d_S and d_P, the shear and the compressional boundary distance functions, respectively. Because of $\lambda + \mu > 0$ shear waves travel a slower speed than compressional waves. In particular, $d_P \leq d_S$ and the diameters satisfy $\mathrm{diam}_P(\overline{\Omega}) \leq \mathrm{diam}_S(\overline{\Omega})$. From Theorem 1.1 and the remarks following it we get a corollary on boundary distance (or travel time) functions.

Corollary 1.2. *Given $L > 0$ there exists $\varepsilon > 0$ such that in the class $\mathcal{L}(L, \varepsilon)$ the shear and the compressional boundary distance functions are determined by boundary measurements. Measurements of duration* $\mathrm{diam}_S(\overline{\Omega})$ *suffice.*

In the case $R = 0$ the metrics g_S and g_P are conformal to the Euclidean metric. Mukhometov [Muk82] solved the inverse kinematic problem for conformal classes of metrics under assumptions which exclude conjugate points. Rachele [Rac00b] proves Corollary 1.2 for $R = 0$ in the absence of conjugate points. That result, together with, e.g., Croke's theorem [Cro91, Theorem C], imply the following uniqueness result.

Corollary 1.3. [Rac00b, Theorem 1] *In the class of operators of isotropic elastodynamics with vanishing residual stresses, and with $(\overline{\Omega}, g_S)$, $(\overline{\Omega}, g_P)$ strictly convex, the compressional and shear wave speeds, $c_P = \sqrt{(\lambda + 2\mu)/\rho}$ and $c_S = \sqrt{\mu/\rho}$, are determined by boundary measurements. Measurements of duration* $\mathrm{diam}_S(\overline{\Omega})$ *suffice.*

If residual stresses do not vanish the metrics become anisotropic. From Corollary 1.2 and a result of Stefanov-Uhlmann on the anisotropic inverse kinematic problem [SU98, Theorem 1.1] we deduce the following result.

Corollary 1.4. *There is a $C^{12}(\overline{\Omega})$ neighbourhood U of the euclidean metric such that the following holds. Let $P^{(1)}$ and $P^{(2)}$ be operators of isotropic elastodynamics. Assume $\Lambda^{(1)} = \Lambda^{(2)}$. Assume $\overline{\Omega}$ strictly convex with respect to the metrics $g_S^{(j)}$ and $g_P^{(j)}$. If $g_S^{(j)}, g_P^{(j)} \in U$ then $g_S^{(1)} = \Psi_S^\star g_S^{(2)}$, $g_P^{(1)} = \Psi_P^\star g_P^{(2)}$ with diffeomorphisms $\Psi_S, \Psi_P : \overline{\Omega} \to \overline{\Omega}$ which leave the boundary fixed, i.e., $\Psi_S(x) = \Psi_P(x) = x$ if $x \in \partial\Omega$.*

In [SU98, Theorem 1.1] an additional flatness assumption at the boundary of Ω is made. This assumption is superfluous in view of [LSU01, Theorem 2.1].

Rachele [Rac00c] proves Corollary 1.2 and Corollary 1.4 for a more general class of operators under additional assumptions which exclude conjugate points. Rachele's result also requires the determination [Rac00a] of the coefficients λ, μ, ρ to infinite order at the boundary, in order to extend the parameters smoothly to all of $\mathbb{R}^3$. Note that our result allows the presence of conjugate points and does not require the boundary determination or extension beyond the original manifold.

We prove Theorem 1.1 in section 6. The facts needed about propagation of singularities and polarizations in non-glancing boundary problems for systems of real principal type are proved in section 2 for first order systems. These are applied to second order systems and to elastodynamics in sections 3 and 4, respectively. In particular, section 4 contains an analysis of the displacement-to-traction map Λ and its pseudo-differential properties.

2. Singularities of first order boundary problems

We summarize some facts from the microlocal theory of boundary problems. The results are due to Dencker [Den82], Gérard [Gér85], Melrose [Mel81], and Taylor [Tay75].

Consider half-space $\overline{\mathbb{R}_+} \times \mathbb{R}^n$ as a manifold with boundary. Denote the natural coordinates $x \geq 0$, $y = (y_1, \ldots, y_n)$, and ξ, $\eta = (\eta_1, \ldots, \eta_n)$ the dual coordinates of cotangent space. Let $Z \subset \overline{\mathbb{R}_+} \times \mathbb{R}^n$ open with non-empty boundary $Y = Z \cap \{0\} \times \mathbb{R}^n$ and interior $Z^\circ = Z \cap \mathbb{R}_+ \times \mathbb{R}^n$. $\mathcal{D}'(Z)$ denotes the space of extendible distributions on Z°. Pseudo-differential operators of order at most m on Y and on Z acting along Y are written $A(y, D_y) \in \Psi^m(Y)$ and $B(x, y, D_y) \in \Psi_t^m(Z)$, respectively. $S^m = S^m(Y \times \mathbb{R}^n)$ and $S_t^m = S^m(Z \times \mathbb{R}^n)$ are the corresponding symbol spaces. Elements of $\Psi_t^m(Z)$ are called tangential pseudo-differential operators. Symbols are always assumed polyhomogeneous (classical). Pseudo-differential operators will always be chosen properly supported. We denote by $r_Y u = u|_Y$ the restriction of $u \in \mathcal{D}'(Z)$, when defined.

We consider $u \in \mathcal{D}'(Z)^K$ such that $Pu \in C^\infty(Z)^K$ where P is a $K \times K$ system of pseudo-differential operators which are differential with respect to x,

$$P = \sum_{j=0}^{m} P_j D_x^j \quad \text{with} \quad P_j = P_j(x, y, D_y) \in \Psi_t^j(Z), \quad P_0 = \mathrm{Id}_K . \tag{11}$$

The boundary wavefront set $\mathrm{WF}_b(u)$, defined in [Mel81], is a closed subset of the compressed cotangent bundle $\widetilde{T}^\star(Z)$. If the P_j are differential operators then $Pu \in C^\infty(Z)^K$ is a non-characteristic boundary problem and hence u is normally regular in the sense of Melrose [Mel81, II.9]. Recall from [Mel81] or [Hör85, 18.3] the following properties of a normally regular distribution u. $u \in C^\infty([0, \varepsilon[, \mathcal{D}'(\mathbb{R}^n))$ locally near Y. Au is normally regular if A is a tangential pseudo-differential operator. The boundary wavefront set $\mathrm{WF}_b(u) \subset T^\star(Y) \cup T^\star(Z^\circ) \subset \widetilde{T}^\star(Z)$. $(y, \eta) \in T^\star(Y) \setminus \mathrm{WF}_b(u)$ if and only if $Au \in C^\infty(Z)$ for some operator $A = A(x, y, D_y) \in \Psi_t^m(Z)$ which is non-characteristic at $(0, y, \eta)$. The polarization set $\mathrm{WF}_{\mathrm{pol}}^{(s)}(u)$ is, by definition the intersection of the sets

$$\mathcal{N}_A = \left\{(x, \xi; w) \in T^\star(Z^\circ) \times \mathbb{C}^K \ ; \ \sigma(A)(x, \xi) w = 0\right\}$$

where $A \in \Psi^0$ runs over all $1 \times K$ systems such that $Au \in H^{(s)}(Z^\circ)$. See [Den82] and [Gér85] for the precise definition and for results on the propagation of polarization along Hamilton orbits.

Let P as in (11) with principal symbol p. Following Dencker [Den82, Definition 3.1] we say that P is of real principal type if, microlocally near a given point, the characteristic variety is given by $q = 0$ with a scalar symbol q of real principal type and if there exists a matrix-valued symbol, $\tilde{p}$ such that $\tilde{p}p = q \,\mathrm{Id}_K$. If we assume P of real principal type then $H = H_q$ is a Hamilton field of the characteristic variety $V = q^{-1}(0)$ of P. A point $(y, \eta) \in T^\star Y \setminus 0$ is called glancing for P if, with respect to the natural projection from $T_Y^\star Z$ to $T^\star Z$, its preimage in $V \cap T_Y^\star Z$ contains a point where $Hx = 0$, else (y, η) is called non-glancing for P. Bicharacteristics intersect the boundary transversally at non-glancing points.

We now specialize to first order systems, $m = 1$. Let $G = G(x, y, D_y) \in \Psi_t^1(Z)$ be an $N \times N$ matrix of tangential pseudo-differential operators with homogeneous principal symbol g. We assume that $D_x \,\mathrm{Id}_N - G$ is of real principal type. We are interested in the singularities of normally regular solutions of

$$D_x w - G(x, y, D_y) w \equiv 0 \mod C^\infty(Z)^N. \tag{12}$$

Let $(y^{(0)}, \eta^{(0)}) \in T^\star(Y) \setminus 0$ non-glancing for $D_x \,\mathrm{Id}_N - G$.

The following decoupling lemma is due to Taylor [Tay75] in the case of simple real characteristics and to Gérard [Gér85] in the case of real principal type systems.

Lemma 2.1. *In a conic neighbourhood* Γ *of* $(0, y^{(0)}, \eta^{(0)})$*, the algebraic and geometric multiplicities of the real eigenvalues of* $g(x, y, \eta)$ *are equal and constant. There are homogeneous real-valued* $\mu_1, \ldots, \mu_J \in S_t^1$ *which enumerate, in* Γ*, the* *distinct real eigenvalues of g. Let N_j denote the multiplicity of μ_j. There is an elliptic $N \times N$ matrix $S \in \Psi_t^0$ such that microlocally near $(0, y^{(0)}, \eta^{(0)})$,*

$$\big(D_x \operatorname{Id}_N - G\big)S \equiv S\big(D_x \operatorname{Id}_N - H\big) \qquad \operatorname{mod} \Psi_t^{-\infty}. \tag{13}$$

$H \in \Psi_t^1$ is a block matrix with non-zero entries only on the diagonal,

$$H = \begin{pmatrix} \mu_1(x, y, D_y) \operatorname{Id}_{N_1} & & & & \\ & \ddots & & & \\ & & \mu_J(x, y, D_y) \operatorname{Id}_{N_J} & & \\ & & & E_+ & \\ & & & & E_- \end{pmatrix}. \tag{14}$$

The imaginary parts of the eigenvalues of the principal symbols of $E_+, E_- \in \Psi_t^1$ are positive and negative, respectively.

Proof. The following constructions hold in some conic neighbourhood Γ of $(0, y^{(0)}, \eta^{(0)})$. Γ may become smaller as the proof proceeds.

Since $A = D_x \operatorname{Id}_N - G$ is of real principal type its characterictic variety is $V = q^{-1}(0)$ with a scalar real principal type symbol q. The non-glancing assumption implies $\partial q/\partial\xi \neq 0$ at points $(0, y^{(0)}, \xi, \eta^{(0)}) \in V$. By the implicit function theorem, the real eigenvalues of $g(x, y, \eta)$ are smooth homogeneous functions $\mu_1(x, y, \eta) < \cdots < \mu_J(x, y, \eta)$ in Γ. We extend them as homogeneous real valued symbols $\mu_1, \ldots, \mu_J \in S_t^1(Z \times \mathbb{R}^n)$.

Let μ be a real eigenvalue of $g^{(0)} = g(0, y^{(0)}, \eta^{(0)})$. We show that the geometric multiplicity of μ equals its algebraic multiplicity,

$$\ker\big((\mu - g^{(0)})^r\big) = \ker\big(\mu - g^{(0)}\big), \quad \forall r \in \mathbb{N}. \tag{15}$$

Here and in the following, to ease notation, a scalar is identified with its multiple of the identity matrix, e.g., $\mu = \mu \operatorname{Id}_N$.

Let $a = \xi - g$ denote the principal symbol A. By the non-glancing hypothesis $\partial/\partial\xi$ is transversal to the characteristic variety $\det a = 0$ at $(y^{(0)}, \eta^{(0)})$. The intrinsic characterisation of real principal type [Den82, Prop. 3.2] shows that $\partial a/\partial\xi = \operatorname{Id}$ maps the kernel of a isomorphically onto the cokernel of a at $\xi = \mu$. Hence

$$\ker(\mu - g^{(0)}) \cap \operatorname{im}(\mu - g^{(0)}) = 0. \tag{16}$$

Equation (15) easily follows from (16).

Let $\gamma_1, \ldots, \gamma_J$ be non-intersecting closed positively oriented Jordan curves in the complex plane such that γ_j encloses $\mu_j(0, y^{(0)}, \eta^{(0)})$ but no other eigenvalue of $g^{(0)}$. (To enclose means that the winding number is non-zero.)

$$\pi_j(x, y, \eta) = \int_{\gamma_j} (\lambda - g(x, y, \eta)/|\eta|)^{-1} \frac{d\lambda}{2\pi i} \tag{17}$$

is the spectral projector onto the sum of generalized eigenspaces associated with the eigenvalues enclosed by γ_j of $g(x, y, \eta)/|\eta|$. Clearly, $\ker\left(\mu_j - g^{(0)}\right) \subset \operatorname{im}\pi_j$. By (15) equality holds at $(0, y^{(0)}, \eta^{(0)})$. By [Den82, Prop. 3.2] the dimension of $\ker\left(\mu_j - g^{(0)}\right)$ is constant. Also the rank of π_j is constant in Γ. Hence

$$\ker(\mu_j - g) = \operatorname{im}\pi_j \quad \text{in } \Gamma. \tag{18}$$

It follows from (18) that the geometric and the algebraic multiplicities of the real eigenvalues of g coincide everywhere in Γ. Therefore we can find an elliptic $N \times N$ matrix $s(x, y, \eta) \in S_t^0$ such that $s^{-1}gs \in S_t^1$ has, in Γ, the block structure of the principal symbol of the operator H claimed in (14).

Choose S with principal symbol equal to s. We obtain (13) with the error class $\Psi_t^{-\infty}$ replaced by Ψ_t^0, however. We use the uncoupling technique of [Tay75] to obtain $K \in \Psi_t^{-1}$ such that the error is $\Psi_t^{-\infty}$ if we replace S by $S(\mathrm{Id} + K)$. After doing this, however, H will only satisfy a weaker form than (14) with $\mu_j(x, y, D_y)\,\mathrm{Id}_{N_j}$ is replaced by $\mu_j(x, y, D_y)\,\mathrm{Id}_{N_j} + M_j$ with some $M_j \in \Psi_t^0$. By [Gér85, Lemme 2.1.] there exist elliptic $N_j \times N_j$ matrices $E_j \in \Psi_t^0$ such that

$$\left((D_x - \mu_j(x, y, D_y))\,\mathrm{Id}_{N_j} - M_j\right)E_j \equiv E_j\left((D_x - \mu_j(x, y, D_y))\,\mathrm{Id}_{N_j}\right)$$

holds modulo operators in $\Psi^{-\infty}$. Let $E \in \Psi_t^0$, $N \times N$, denote the diagonal block matrix with blocks $E_1, \ldots, E_J$ and, in the lower right corner, Id. Finally, to remove the M_j's, we replace S by SE. □

Let B be a $K \times N$ matrix in Ψ_t^0 with homogeneous principal symbol b. Given $h \in \mathcal{D}'(Y)^K$ we wish to solve equation (12) under the following boundary condition specified by B and h,

$$\begin{aligned} D_x w - Gw &\equiv 0 \mod C^\infty(Z)^N, \\ Bw|_Y &\equiv h \mod C^\infty(Y)^K. \end{aligned} \tag{19}$$

Let $M_+ \cup M_- = \{1, \ldots, J\}$ be a disjoint union decomposing the set of real eigenvalues of $g(0, y^{(0)}, \eta^{(0)})$ into two parts. We call the eigenvalue μ_j forward (resp. backward) if $j \in M_+$ (resp. $j \in M_-$). Correspondingly, we call characteristics and bicharacteristic curves forward or backward. In case $D_x\,\mathrm{Id}_N - G$ is hyperbolic with respect to a time variable $t(x, y)$ such a decomposition arises as follows. A bicharacteristic γ issuing from the boundary into the interior is forward (resp. backward) if t increases (resp. decreases) along γ.

We shall find a microlocal parametrix of the boundary problem (19) if a condition of Lopatinski type holds. Define, for (y, η) sufficiently close to $(y^{(0)}, \eta^{(0)})$, the forward Lopatinski space as the following linear subspace of $\mathbb{C}^N$,

$$L_g^+(y, \eta) = \operatorname{im} \int_{\gamma^+} (\lambda - g(0, y, \eta))^{-1}\, d\lambda. \tag{20}$$

γ^+ is a closed positively oriented Jordan curve in the complex plane which encloses the eigenvalues of $g^{(0)} = g(0, y^{(0)}, \eta^{(0)})$ which are real and forward or which have positive imaginary part. γ^+ encloses no other eigenvalues of $g^{(0)}$.

Proposition 2.2. *Assume that $b(0, y^{(0)}, \eta^{(0)})$ maps $L_g^+(y^{(0)}, \eta^{(0)})$ onto $\mathbb{C}^K$. Then there exists a conic neighbourhood $\Gamma \subset T^\star(Y)$ of $(y^{(0)}, \eta^{(0)})$ and an operator $W : \mathcal{D}'(Y)^K \to \mathcal{D}'(Z)^N$ such that the following holds. For every $h \in \mathcal{D}'(Y)^K$ with* $\mathrm{WF}(h) \subset \Gamma$ *the distribution $w = Wh \in \mathcal{D}'(Z)^N$ is normally regular and solves* (19). $\mathrm{WF}(w|_{Z^\circ})$ *is contained in the union of the forward bicharacteristics which issue from* $\mathrm{WF}(h)$. *$W_0 := r_Y W$, W followed by restriction to Y, is a $N \times K$ pseudo-differential operator of order* 0 *on Y. The principal symbol $w_0(y, \eta)$ of W_0 maps $\mathbb{C}^K$ into $L_g^+(y, \eta)$ and satisfies $bw_0 = \mathrm{Id}$ in Γ.*

Proof. Let S and H as in Lemma 2.1, and denote their principal symbols by s and h, respectively. Clearly, $gs = sh$. Hence $L_g^+ = sL_h^+$. The block structure of H and the partitioning into forward and backward eigenvalues defines a projector Π on $\mathbb{C}^N$. Π projects onto the subspace corresponding to the blocks $\mu_j(x, y, D_y)\,\mathrm{Id}_{N_j}$, $j \in M_+$, and E_+ of H along the subspace corresponding to the blocks $\mu_j(x, y, D_y)\,\mathrm{Id}_{N_j}$, $j \in M_-$, and E_- of H. Notice that $L_h^+ = \Pi\mathbb{C}^N$. By assumption

$$\mathbb{C}^K = bL_g^+ = bsL_h^+ = bs\Pi\mathbb{C}^N. \tag{21}$$

The Cauchy problems

$$(D_x - \mu_j(x, y, D_y))v \in C^\infty(Z), \quad v|_Y \equiv f \mod C^\infty(Y),$$

are solved using scalar Fourier integral operators V_j, [Dui73]. The wavefront set of the solution $v = V_j f$ is contained in the image of the bicharacteristics associated with $\xi - \mu_j(x, y, \eta) = 0$ which issue from $\mathrm{WF}(f)$. The parabolic system

$$D_x v - E_+ v \in C^\infty(Z), \quad v|_Y \equiv f \mod C^\infty(Y),$$

is solved using a Poisson operator V_+, [Tay75]. The solution $v = V_+ f$ has no singularities in Z°. Therefore we may construct an operator $V : \mathcal{D}'(Y)^N \to \mathcal{D}'(Z)^N$ such that the following holds for any $f \in \mathcal{D}'(Y)^N$. $v = Vf$ is normally regular, $D_x v - Hv \in C^\infty(Z)^N$, and $\mathrm{WF}(v|_{Z^\circ})$ is contained in the union of the forward bicharacteristics which issue from $\mathrm{WF}(f)$. Furthermore, modulo $C^\infty(Y)^N$, $v|_Y \equiv \Pi f$.

Recall the operator of restriction to Y, r_Y. $r_Y BSV$ is a $K \times N$ system of pseudo-differential operators on Y. Its principal symbol $bs\Pi$ is, close to $(y^{(0)}, \eta^{(0)})$, surjective by (21). Choose a $N \times K$ operator $C \in \Psi^0(Y)$ which is a right inverse, $r_Y BSVC \equiv Br_Y SVC \equiv \mathrm{Id}$. $W = SVC$ satisfies the claims. $\square$

Remark 1. If the boundary data f is a Lagrangian distribution then the solution $w = Wf$ is Lagrangian with respect to the forward characteristics. Röhrig [Röh] derives the transport equations for the principal symbol of w along the bicharacteristics.

To prepare waves with specified polarization we need the following result about propagation of polarization at the boundary. Essentially this is a corollary of [Gér85, Théorème 6.1].

Proposition 2.3. *Let $w \in \mathcal{D}'(Z)^N$ normally regular such that $(y^{(0)}, \eta^{(0)}) \notin \mathrm{WF}_b(D_x w - Gw)$. Assume $w|_Y \in H^{(s-1)}(Y)$, $s > 1$. Let $\mu \in S^1(Y \times \mathbb{R}^n)$ be a real eigenvalue of $g(0,\cdot)$ in a conic neighbourhood of $(y^{(0)}, \eta^{(0)})$. Let $Q \in \Psi^0(Y)$ have principal symbol equal to, in a neighbourhood of $(y^{(0)}, \eta^{(0)})$, the spectral projector on the eigenspace of the eigenvalue μ. Then $(y^{(0)}, \eta^{(0)}) \in \mathrm{WF}^{(s)}(Qw|_Y)$ if and only if $\mathrm{WF}^{(s)}_{\mathrm{pol}}(w)$ contains a Hamilton orbit above the μ-bicharacteristic which issues from $(y^{(0)}, \eta^{(0)})$.*

Proof. Choose a parametrix S^{-1} of S in Lemma 2.1 and put $w' = S^{-1}w$. The hypotheses of the Proposition still hold with w replaced by w' and with G replaced by H of (14). Let Q_μ denote the projection to the components of the block which corresponds to μ in the block decomposition (14). Then $Q - Q_\mu \in \Psi^{-1}(Y)$ and, using the assumption on $w|_Y$, $\mathrm{WF}^{(s)}(Q_\mu w|_Y) = \mathrm{WF}^{(s)}(Qw|_Y)$. $v = Q_\mu w$ solves the diagonal system $(y^{(0)}, \eta^{(0)}) \notin \mathrm{WF}_b(D_x v - \mu(x, y, D_y)v)$. The assertion follows from well-known results on propagation of singularities in the Cauchy problem for scalar strictly hyperbolic equations and from [Den82, Theorem 4.2]. □

3. Second order boundary problems

Here we reduce the Dirichlet problem for second order real principal systems to a boundary problem for a first order real principal type system.

Let $P = D_x^2 \,\mathrm{Id}_K + P_1(x, y, D_y)D_x + P_2(x, y, D_y)$ be a $K \times K$ matrix of differential operators of second order. We are interested in the Dirichlet problem

$$\begin{aligned} Pu &\equiv 0 \mod C^\infty(Z)^K, \\ u|_Y &\equiv f \mod C^\infty(Y)^K. \end{aligned} \tag{22}$$

Any solution u is normally regular.

We associate with (22) an equivalent first order boundary problem (19) as follows. Set $N = 2K$, $G \in \Psi_t^1$ the $N \times N$ matrix

$$G = \begin{pmatrix} 0 & \langle D_y \rangle \,\mathrm{Id}_K \\ -P_2 \langle D_y \rangle^{-1} & -P_1 \end{pmatrix}, \tag{23}$$

and $B \in \Psi_t^0$ the $K \times N$ matrix with $Bw = w_1$, $w = (w_1, w_2)$. Here $\langle D_y \rangle \in \Psi_t^1$ denotes the operator with full symbol $\langle \eta \rangle = (1 + |\eta|^2)^{1/2} \in S_t^1$.

Lemma 3.1. *Let $f \in \mathcal{D}'(Y)^K$ and $h = \langle D_y \rangle f$. Solutions u of* (22) *and $w = (w_1, w_2)$ of* (19) *are related as follows. If u solves* (22) *then $w = (w_1, w_2) = (\langle D_y \rangle u, D_x u)$ solves* (19). *Conversely, if w solves* (19) *then $u = \langle D_y \rangle^{-1} w_1$ solves* (22).

Proof. The first statement follows immediately from the definition of G and B. For the proof of the converse statement let w be a solution of (19). The first row of $D_x w \equiv Gw$ and the ellipticity of $\langle D_y \rangle$ imply $D_x u \equiv w_2$. Hence the second row implies $Pu \equiv 0$. By our choice of B and h the boundary conditions are equivalent: $\langle D_y \rangle u = Bw \equiv h = \langle D_y \rangle f$. □

Let $(x, y, \xi, \eta) \in T^\star(Z)$, $\eta \neq 0$. Let $p = \xi^2 + p_1 \xi + p_2$ denote the principal symbol of P. Then the principal symbol of G is

$$g = \begin{pmatrix} 0 & |\eta| \\ -p_2/|\eta| & -p_1 \end{pmatrix}.$$

Lemma 3.2. *Let $\eta \neq 0$. Then*

$$(\xi - g')(\xi - g) = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} \quad \textit{where} \quad g' = \begin{pmatrix} -p_1 & -|\eta| \\ p_2/|\eta| & 0 \end{pmatrix} \tag{24}$$

and

$$\ker(\xi - g) = \begin{pmatrix} |\eta| \operatorname{Id}_K \\ \xi \operatorname{Id}_K \end{pmatrix} \ker(p). \tag{25}$$

The characteristic varieties of P and $D_x \operatorname{Id}_N - G$ are equal. If P is of real principal type then so is $D_x \operatorname{Id}_N - G$.

Proof. Equation (24) is verified by direct computation. Clearly, $(\xi - g)w = 0$ with $w = (w_1, w_2)^T$, holds if and only if $\xi w_1 = |\eta| w_2$ and $p w_2 = 0$. To prove the last assertion assume there is a $K \times K$ matrix of symbols, $\tilde{p}$, such that $\tilde{p} p = q \operatorname{Id}_K$ holds with a scalar real principal type smbol q. Then, using (24), we obtain a $N \times N$ matrix of symbols, $\tilde{a}$, such that $\tilde{a}(\xi - g) = q \operatorname{Id}_N$. □

Remark 2. Assume P of real principal type. Let $C_0 u = (\langle D_y \rangle u|_Y, D_x u|_Y)$ denote the Cauchy data of a solution of $Pu \equiv 0$. It follows from Proposition 2.3 and Lemma 3.2 that $\mathrm{WF}_{\mathrm{pol}}^{(s+1)}(u)$ contains a Hamilton orbit above a given bicharacteristic issuing from $\gamma = (y, \eta) \in T^\star(Y) \setminus 0$ if and only if $\gamma \in \mathrm{WF}^{(s)}(QC_0 u)$ where $Q \in \Psi^0(Y)$ with principal symbol equal to the spectral projector onto the eigenspace $\{(\langle \eta \rangle a, \xi a);\ p(0, y, \xi, \eta) a = 0\}$ which corresponds to the given characteristic.

We give sufficient conditions for the existence of a microlocal parametrix for the boundary problem (22).

Proposition 3.3. *Assume P of real principal type. Let $(y^{(0)}, \eta^{(0)}) \in T^\star Y \setminus 0$ be non-glancing for P. Let γ^+ be a closed positively oriented Jordan curve which does not meet the poles of $\lambda \mapsto p(0, y^{(0)}, \lambda, \eta^{(0)})^{-1}$ and which has winding number* 1 *(resp.* 0*) with respect to the poles with positive (resp. negative) imaginary part. Assume that*

$$K \geq \operatorname{rank} \int_{\gamma^+} \left(\lambda - g(0, y^{(0)}, \eta^{(0)})\right)^{-1} d\lambda, \tag{26}$$

$$K \leq \operatorname{rank} \int_{\gamma^+} p(0, y^{(0)}, \lambda, \eta^{(0)})^{-1} d\lambda. \tag{27}$$

Then there exists a conic neighbourhood $\Gamma \subset T^\star(Y)$ of $(y^{(0)}, \eta^{(0)})$ and an operator $U : \mathcal{D}'(Y)^K \to \mathcal{D}'(Z)^K$ such that for any $f \in \mathcal{D}'(Y)^K$ with $\mathrm{WF}(f) \subset \Gamma$ *the distribution $u = Uf \in \mathcal{D}'(Z)^K$ is normally regular and solves* (22). $\mathrm{WF}(u|_{Z^\circ})$ *is contained in the union of the forward bicharacteristics which issue from* $\mathrm{WF}(f)$. *The restriction of the normal derivative to Y, $U' := r_Y D_x U \in \Psi^1(Y)$, is a $K \times K$ pseudo-differential operator with principal symbol u' which satisfies*

$$u'(y^{(0)}, \eta^{(0)}) \int_{\gamma^+} p(0, y^{(0)}, \lambda, \eta^{(0)})^{-1} d\lambda = \int_{\gamma^+} \lambda\, p(0, y^{(0)}, \lambda, \eta^{(0)})^{-1} d\lambda. \tag{28}$$

Proof. We use the equivalence, stated in Lemma 3.1, of (22) with the first order boundary problem (19).

A real eigenvalue of $g(0, y^{(0)}, \eta^{(0)})$ is, by definition, forward if it is enclosed by γ^+. First we show that our assumptions imply the following formula for the Lopatinski space,

$$L_g^+(y^{(0)}, \eta^{(0)}) = \operatorname{im} \begin{pmatrix} \int_{\gamma^+} |\eta^{(0)}|\, p(0, y^{(0)}, \lambda, \eta^{(0)})^{-1} d\lambda \\ \int_{\gamma^+} \lambda\, p(0, y^{(0)}, \lambda, \eta^{(0)})^{-1} d\lambda \end{pmatrix}. \tag{29}$$

From (24) we infer that the resolvent of g is

$$(\lambda - g)^{-1} = \begin{pmatrix} * & |\eta| p(\lambda)^{-1} \\ * & \lambda p(\lambda)^{-1} \end{pmatrix} \quad \text{where } p(\lambda) = \lambda^2 + p_1 \lambda + p_2, \lambda \in \mathbb{C}. \tag{30}$$

Here $*$ indicates unspecified expressions. Hence the right hand side in (29) is contained in the left hand side. Equality follows from the dimension assumptions (26) and (27).

The principal symbol of B is $b = (\mathrm{Id}_K, 0)$. Therefore (29), (26), and (27) imply $bL_g^+ = \mathbb{C}^K$ at $(0, y^{(0)}, \eta^{(0)})$. Proposition 2.2 applies to give a solution operator of (19), $W = (W_1, W_2)^T$ with $W_1 = BW$. Define $U = \langle D_y \rangle^{-1} W_1 \langle D_y \rangle$. It follows from Lemma 3.1 that $PU \equiv 0$, and $r_Y U \equiv \mathrm{Id}$, and $W_2 \langle D_y \rangle \equiv$

$D_x U$. Hence $(\langle D_y \rangle, U') \equiv r_Y W \langle D_y \rangle \in \Psi^1(Y)$. The principal symbol of $r_Y W$, $(\mathrm{Id}_K, |\eta|^{-1} u'(y,\eta))^T$, maps $\mathbb{C}^K$ into the Lopatinski space $L_g^+(y,\eta)$. Now we can read the formula (28) off the equation (29).

The bound on $\mathrm{WF}(u|_{Z^\circ})$ follows from the bound on $\mathrm{WF}(w|_{Z^\circ})$ in Proposition 2.2. □

4. Isotropic elastodynamic equations

In the following P denotes an operator for isotropic elastodynamics introduced in (2) and (3) such that (8) holds.

The boundary problem $Pu = g$ in $\mathbb{R} \times \Omega$, and $u = 0$ on $\mathbb{R} \times \partial\Omega$, has the variational formulation: $(\rho \ddot{u}, v) + a(u, v) + (g, v) = 0$, $\forall v \in \big(\overset{\circ}{H}{}^{(1)}(\Omega)\big)^3$. Here $a = a_0 + a_R$, $a_0(u, v) = \int_\Omega \mathrm{tr}\left(C E(u)\, E(v)^T\right) dx$, and $a_R(u, v) = \int_\Omega \mathrm{tr}\left((\nabla u)\, R\, (\nabla v)^T\right) dx$. It follows from Korn's inequality that a_0 satisfies a coerciveness estimate $|a_0(u, u)| \geq c\|u\|_1^2$ with a positive constant depending only on Ω and on a lower bound on the Lamé coefficient μ, [DL76]. Given $L > 0$ there exists $\varepsilon > 0$ such that a is coercive if $P \in \mathcal{L}(L, \varepsilon)$. In fact, a_R is absorbed into the coerciveness estimate if (6) is assumed with $0 < \varepsilon = \varepsilon(L)$ sufficiently small. We use [DL76, Thm. III.4.1.] to conclude that the initial boundary value problem (4) for displacement boundary data $f \in H_c^{(s)}(\mathbb{R} \times \partial\Omega)^3$, $s \geq 3$, is well-posed. In particular, the displacement-to-traction map (7) is defined,

$$\Lambda : H_c^{(s+1)}(\mathbb{R} \times \partial\Omega)^3 \to H^{(s)}(\mathbb{R} \times \partial\Omega)^3 \quad \text{if } s \geq 2. \tag{31}$$

Let (t, x, τ, ξ) denote a generic point in $T^\star(\mathbb{R} \times \overline{\Omega}) \setminus 0$. The Euclidean metric $\xi^2 = \xi \cdot \xi$ is used to identify tangent and cotangent vectors of $\overline{\Omega}$. For $\eta, \zeta \in \mathbb{C}^3$ the dot product is the analytic (non-Hermitian) extension, $\eta \cdot \zeta = \eta_1\zeta_1 + \eta_2\zeta_2 + \eta_3\zeta_3$. $\pi = \pi(\xi) = (\xi \otimes \xi)/(\xi \cdot \xi)$ denotes the orthogonal projection onto a nonzero direction if $\xi \in \mathbb{R}^3 \setminus 0$.

As a consequence of (8) and $\lambda + \mu > 0$ the metrics defined in (9) and (10) satisfy

$$0 < \langle \xi, \xi \rangle_S < \langle \xi, \xi \rangle_P \quad \text{if } \xi \neq 0. \tag{32}$$

The norms associated with these metrics are denoted $|\xi|_{S/P} = \sqrt{\langle \xi, \xi \rangle_{S/P}}$.

Proposition 4.1. *The scalar symbols $q_{S/P}(t, x, \tau, \xi) = \rho(x)\big(\tau^2 - \langle \xi, \xi \rangle_{S/P}\big)$ and their product $q_S q_P$ are of real principal type. P is a system of real principal type with principal symbol*

$$p = q_S(\mathrm{Id}_3 - \pi) + q_P \pi. \tag{33}$$

Proof. A straightforward computation gives the principal symbol p of P at $(t, x, \tau, \xi) \in T^\star(\mathbb{R} \times \overline{\Omega})$ as follows:

$$p = \rho\tau^2 \operatorname{Id}_3 -(\lambda + \mu)(\xi \otimes \xi) - \mu\xi^2 \operatorname{Id}_3 -(\xi \cdot R\xi) \operatorname{Id}_3 .$$

Hence p has the asserted form. It follows from (32) that q_S and q_P are of real principal type. Furthermore

$$q_P(t, x, \tau, \xi) < q_S(t, x, \tau, \xi) \quad \text{if } \xi \neq 0. \tag{34}$$

Hence also $q = q_S q_P$ is a scalar symbol of real principal type. Now $q^{-1}(0) = (\det p)^{-1}(0)$ and $\tilde{p}p = q \operatorname{Id}$ for $\tilde{p} = q_P(\operatorname{Id}_3 -\pi) + q_S\pi$. According to [Den82, Definition 3.1] P is a system of real principal type with characteristic variety $q = 0$. □

Remark 3. Man [Man98] proposes for elastodynamics with residual stress R a more general constituitive law $S = R + \nabla u \cdot R + CE$ where the elasticity tensor C also depends linearly on R. In the isotropic case CE consists of the right-hand side in (3) plus the R dependent terms

$$\beta_1 \operatorname{tr}(E) \operatorname{tr}(R)I + \beta_2 \operatorname{tr}(R)E + \beta_3\big(\operatorname{tr}(E)R + \operatorname{tr}(ER)I\big) + \beta_4\big(ER + RE\big). \tag{35}$$

In the inverse problem for real media the additional terms should not be neglected since typically R is much larger than the stress CE. See [Man98, Sect. 2]. A straightforward calculation shows that the elastodynamic operator P with this isotropic stress-strain relation is still of real principal type in case $\beta_3 = \beta_4 = 0$, $\lambda + 2\mu + \beta_1 \operatorname{tr}(R) > \mu + \beta_2 \operatorname{tr}(R)/2$, and $(\mu + \beta_2 \operatorname{tr}(R)/2)|\xi|^2 + R\xi \cdot \xi > 0$ when $\xi \in T^\star_x(\overline{\Omega}) \setminus 0$.

We recall some notions of the microlocal theory of boundary problems and apply them to the system of elastodynamics. Let $\gamma = (t, x, \tau, \xi_|) \in T^\star(\mathbb{R} \times \partial\Omega) \setminus 0$. By this we mean that there is given $(t, x, \tau, \xi) \in T^\star(\mathbb{R} \times \overline{\Omega}) \setminus 0$ with $x \in \partial\Omega$, and that we define $\xi_| = \xi|_{T_x(\partial\Omega)} \in T^\star_x(\partial\Omega)$. γ is called an elliptic, a hyperbolic, or a glancing point of S/P mode if the following quadratic equation in z,

$$q_{S/P}(t, x, \tau, \xi - z\nu(x)) = 0,$$

has no real roots, two distinct real roots, or a double real root, respectively. $T^\star(\mathbb{R} \times \partial\Omega) \setminus 0$ decomposes into the disjoint union of the elliptic region $\mathcal{E}_{S/P}$, the hyperbolic region $\mathcal{H}_{S/P}$, and the glancing hypersurface $\mathcal{G}_{S/P}$ of the S/P mode. Because of (34) we have $\mathcal{E}_S \subset \mathcal{E}_P$ and $\mathcal{H}_P \subset \mathcal{H}_S$. $T^\star(\mathbb{R} \times \partial\Omega) \setminus 0$ is the disjoint union of the hyperbolic region $\mathcal{H}_P$, the mixed region $\mathcal{E}_P \cap \mathcal{H}_S$, the elliptic region $\mathcal{E}_S$, and the glancing set $\mathcal{G} = \mathcal{G}_S \cup \mathcal{G}_P$. The lens maps satisfy $\mathcal{S}_S \subset \mathcal{H}_S \times \mathcal{H}_S$ and $\mathcal{S}_P \subset \mathcal{H}_P \times \mathcal{H}_P$.

A simple real root z is called forward (resp. backward) if the bicharacteristic curve starting in direction $\xi - z\nu$ enters $\mathbb{R} \times \Omega$ when time increases (resp. decreases). Characteristics and bicharacteristics are called forward or backward correspondingly. Observe from Hamiltons equations that a characteristic $\xi - z\nu$, z real, of $q_{S/P}$ is forward (resp. backward) if $\tau\langle \xi - z\nu, \nu\rangle_{S/P}$ is positive (resp. negative). We denote by $z_{S/P} = z_{S/P}(t, x, \tau, \xi, \nu)$ the forward real root z or the complex root z with positive imaginary part of $q_{S/P}(t, x, \tau, \xi - z\nu) = 0$. We shall use the abbreviation $\xi_{S/P} = \xi - z_{S/P}\nu(x)$.

Given $\delta > 0$ we define

$$\Gamma_\delta = \{(t, x, \tau, \xi_|) \in T^\star(\mathbb{R} \times \partial\Omega) \setminus 0;\ |\tau| \geq \delta|\xi_||\}. \tag{36}$$

Here $|\xi_|| = |\xi|$ if and only if $\xi \cdot \nu = 0$. If $P \in \mathcal{L}(L, 1/2)$ then a straightforward estimate shows that we can choose $\delta = \delta(L) > 0$ such that $T^\star(\mathbb{R}\times\partial\Omega)\setminus(\Gamma_\delta \cup 0) \subset \mathcal{E}_S$ holds. Therefore the following result implies, in particular, that the displacement-to-traction map Λ is pseudo-differential microlocally in the hyperbolic and in the mixed regions if $P \in \mathcal{L}(L, \varepsilon)$, $\varepsilon > 0$ sufficiently small. Moreover, Λ is pseudo-differential at every non-glancing point if $R = 0$.

Proposition 4.2. *Let $L, \delta > 0$. Assume* (5). *There exists $0 < \varepsilon = \varepsilon(L, \delta)$ such that under the assumption* (6) *the following holds. Given $\gamma \in \Gamma_\delta \setminus \mathcal{G}$ then Λ equals, in a microlocal neighbourhood of (γ, γ), a first order pseudo-differential operator with principal symbol given as follows.*

$$\sigma(\Lambda) : a \mapsto \lambda(a \cdot \xi')\nu + \mu(a \cdot \nu)\xi' + \mu(\xi' \cdot \nu)a + (R\xi' \cdot \nu)a \tag{37}$$

when $a \in \mathbb{C}\xi'$, resp. $a \cdot \xi' = 0$, with $\xi' = \xi_P$, resp. $\xi' = \xi_S$.

We need the following fact about the characteristics of P.

Lemma 4.3. *Let $L, \delta > 0$. Assume* (5). *There exists $0 < \varepsilon = \varepsilon(L, \delta)$ such that the following is true if* (6) *holds. For $\gamma = (t, x, \tau, \xi_|) \in \Gamma_\delta \setminus \mathcal{G}$ we have $z_S \neq z_P$, ${\xi_S}^2 \neq 0$, ${\xi_P}^2 \neq 0$, and*

$$\xi_S \cdot \xi_P \neq 0. \tag{38}$$

We prove Lemma 4.3 in section 7.

Proof (of Proposition 4.2). Choose ε as in Lemma 4.3. Decreasing $\varepsilon > 0$ if necessary we assume that the initial boundary value problem (4) is well-posed and that, therefore, Λ is defined. Let $\gamma = (t, x, \tau, \xi_|) \in \Gamma_\delta \setminus \mathcal{G}$, $\xi \cdot \nu(x) = 0$. Flatten the boundary $\partial\Omega$ near x by a change of coordinates such that the differential at x is orthogonal.

We consider the symbol $p = p(t, x, \cdot)$. Its inverse is $p^{-1} = q_S^{-1}(\mathrm{Id}_3 - \pi) + q_P^{-1}\pi$. Let γ^+ be a closed Jordan curve enclosing z_S and z_P but no other roots of

$q(\tau, \xi - z\nu) = 0$. Observe $r_{S/P} := (d/dz)q_{S/P}(\tau, \xi - z\nu)|_{z=z_{S/P}} \neq 0$. By the residue theorem

$$\begin{aligned} A_j &:= \int_{\gamma^+} z^j\, p(\tau, \xi - z\nu)^{-1} \frac{dz}{2\pi i} \\ &= (z_S^j/r_S)\,(\mathrm{Id}_3 - \pi(\xi_S)) + (z_P^j/r_P)\,\pi(\xi_P) \end{aligned} \tag{39}$$

for every non-negative integer j. We show that A_0 is non-singular. Assume $A_0 w = 0$. Then $0 = \xi_S \cdot A_0 w = (\xi_S \cdot \xi_P)(\xi_P \cdot w)/(r_P \xi_P^2)$. Applying Lemma 4.3 we infer $\xi_P \cdot w = 0$. Therefore $w = \pi(\xi_S)w$ and $0 = (\xi_P \cdot \xi_S)(\xi_S \cdot w)$. Applying Lemma 4.3 again we get $\xi_S \cdot w = 0$. Hence $w = \pi(\xi_S)w = 0$.

The invertibility of A_0 implies that (27) holds with $K = 3$. Inequality (26) holds because the rank is bounded by $N = 6$ minus the dimension of the eigenspaces corresponding to eigenvalues not enclosed by γ^+ which is 3.

We apply Proposition 3.3. We find, microlocally near γ, a parametrix $U : f \mapsto u$ of the boundary problem (4). For a vector field $V = (V_1, V_2, V_3)$ on the closure of Ω define $D_V = \sum_{j=1}^3 V_j D_j$ and B_V as $(D_V\, \mathrm{Id}_3)U$ followed by restriction to $\mathbb{R} \times \partial\Omega$. B_V is a 3×3 system of pseudo-differential operators on $\mathbb{R} \times \partial\Omega$ of order 1. We compute its principal symbol. Formula (28) translates into $\sigma(B_{-\nu})(\gamma)A_0 = A_1$. Using (39) we obtain

$$\sigma(B_{-\nu})(\gamma) : a \mapsto \begin{cases} z_S a & \text{if } a \in A_0 \xi_P^\perp, \\ z_P a & \text{if } a \in A_0 \mathbb{C}\xi_S. \end{cases}$$

Formula (39) and the invertibility of A_0 imply $A_0 \xi_P^\perp = \xi_S^\perp$ and $A_0 \mathbb{C}\xi_S = \mathbb{C}\xi_P$. Hence, if we recall $\xi \cdot \nu(x) = 0$, we rewrite the above as

$$\sigma(B_V)(\gamma) : a \mapsto \begin{cases} (V(x) \cdot \xi_S)a & \text{if } a \in \xi_S^\perp, \\ (V(x) \cdot \xi_P)a & \text{if } a \in \mathbb{C}\xi_P, \end{cases} \tag{40}$$

with $V = \nu$. For vector fields V tangent to $\partial\Omega$ we compute $\sigma(B_V)(\gamma) = (V(x) \cdot \xi)\,\mathrm{Id}_3$. Thus (40) holds for any vector field V.

Observe that $\gamma \notin \mathrm{WF}\left(\Lambda f - (\nu \cdot S(Uf))|_{\mathbb{R}\times\partial\Omega}\right)$ if WF f is contained in a small conic neighbourhood of γ. Here $S(u)$ denotes the stress tensor which corresponds to the displacement u. The displacement-to-traction map $u \mapsto \nu \cdot S(u)$ is a first order differential operator with principal symbol

$$s(x, \eta) := \lambda(\nu \otimes \eta) + \mu(\eta \otimes \nu) + \mu(\eta \cdot \nu)\,\mathrm{Id}_3 + (R\eta \cdot \nu)\,\mathrm{Id}_3, \tag{41}$$

$\eta \in T_x^\star(\overline{\Omega})$. We can write Λ as a sum of terms CB_V, C a 3×3 matrix of smooth functions. It follows that that $\Lambda \in \Psi^1$ in a conic neigbourhood of γ. Formula (37) for the principal symbol of Λ follows from (41) and (40). □

Remark 4. Given $L > 0$ we can choose $\delta > 0$ and $0 < \varepsilon \leq 1/2$ such that the following holds if $P \in \mathcal{L}(L, \varepsilon)$: $T^\star(\mathbb{R} \times \partial\Omega) \setminus (\Gamma_\delta \cup 0) \subset \mathcal{E}_S$; the initial boundary value problem (4) for $f \in H_c^{(s)}(\mathbb{R} \times \partial\Omega)^3$, $s \geq 3$, is well-posed; the assertions in Proposition 4.2 hold. It is clear from the proof of Proposition 4.2 that, under such hypotheses, microlocal forward and backward parametrices exist in the hyperbolic, in the mixed, and in part of the elliptic region.

5. Propagation of polarization

Let $L > 0$. Choose $0 < \delta, \varepsilon$ as in Remark 4 at the end of section 4. In the following we assume $P \in \mathcal{L}(L, \varepsilon)$.

We analyze the polarization of solutions of $Pu = 0$ by applying polarization filters, i.e., certain approximate projection operators, to the Cauchy data of u. The Cauchy data of a solution of $Pu \in C^\infty(\mathbb{R} \times \overline{\Omega})$ are, by definition, $Cu = (Eu|_{\mathbb{R}\times\partial\Omega},\ \nu \cdot S|_{\mathbb{R}\times\partial\Omega})$. Here, in order to have C of order 1, we have chosen a fixed scalar elliptic operator $E \in \Psi^1(\mathbb{R} \times \partial\Omega)$. Denote the principal symbol of E by e. Notice that the Cauchy data of a solution u of the initial boundary value problem $Pu = 0$, $u = f$ at $\mathbb{R} \times \partial\Omega$, and $u = 0$ initially, are represented using the displacement-to-traction map as follows: $Cf = (Ef, \Lambda f) = Cu$. Here, abusing notation, we also defined Cf.

We now describe, on the principal symbol level, the spaces onto which polarization filters project. Let $\gamma = (t, x; \tau, \xi_|) \in T^\star(\mathbb{R} \times \partial\Omega) \setminus 0$. Set, with s defined in (41),

$$B^\pm_{S/P}(\gamma) = \begin{pmatrix} e(\gamma)\,\mathrm{Id}_3 \\ s(x, \xi - z^\pm_{S/P}\nu) \end{pmatrix} \ker p(t, x, \tau, \xi - z^\pm_{S/P}\nu) \subset \mathbb{C}^6$$

if $\gamma \in \mathcal{H}_{S/P}$. Here $z^+_{S/P}$ and $z^-_{S/P}$ are the forward and backward roots of $q_{S/P}(t, x, \tau, \xi - z\nu) = 0$, respectively. Also define the linear subspaces

$$B_{S/P}(\gamma) = \sum \begin{pmatrix} e(\gamma)\,\mathrm{Id}_3 \\ s(x, \xi - z\nu) \end{pmatrix} \ker p(t, x, \tau, \xi - z\nu)$$

where the sum ranges over the roots $z^\pm_{S/P}$ of $q_{S/P}(t, x, \tau, \xi - z\nu) = 0$. Clearly, $B_{S/P}(\gamma) = B^+_{S/P}(\gamma) + B^-_{S/P}(\gamma)$ if $\gamma \in \mathcal{H}_{S/P}$. The disjoint unions

$$B^\pm_{S/P} = \dot\cup_{\gamma\in\mathcal{H}_{S/P}} B^\pm_{S/P}(\gamma) \quad \text{and} \quad B_{S/P} = \dot\cup_{\gamma\notin\mathcal{G}_{S/P}} B_{S/P}(\gamma)$$

are subsets of the trivial bundles, $\mathbb{C}^6$.

Lemma 5.1. *$B^\pm_S$, resp. $B^\pm_P$, are vector subbundles of $\mathbb{C}^6$ over $\mathcal{H}_S$, resp. $\mathcal{H}_P$, of ranks 2, resp. 1. B_P is a vector subbundle of $\mathbb{C}^6$ over $\mathcal{H}_P \cup \mathcal{E}_P$ of rank 2. Furthermore*

$$\begin{aligned} \mathbb{C}^6 &= B^+_S \oplus B^-_S \oplus B^+_P \oplus B^-_P \quad \text{over } \mathcal{H}_P, \\ \mathbb{C}^6 &= B^+_S \oplus B^-_S \oplus B_P \quad \text{over } \mathcal{H}_S \cap \mathcal{E}_P. \end{aligned} \tag{42}$$

Proof. Given a point in $\partial\Omega$ we introduce coordinates $x = (x_1, \dots, x_n) = (x', x_n)$, $n = 3$, such that Ω and $\partial\Omega$ correspond to $x_n > 0$ and $x_n = 0$, respectively. Let $\xi = (\xi', \xi_n)$ denote the dual variables. We also arrange that, at the given point in the coordinates, formula (41) with $\eta = \xi$ still holds. At $x_n = 0$ we define h_1 by

$$\begin{pmatrix} e\,\mathrm{Id}_3 \\ s \end{pmatrix} = \begin{pmatrix} h_1\,\mathrm{Id}_3 & 0 \\ * & \partial s/\partial\xi_n \end{pmatrix} \begin{pmatrix} (\tau^2 + \xi'^2)^{1/2}\,\mathrm{Id}_3 \\ \xi_n\,\mathrm{Id}_3 \end{pmatrix}.$$

Here $*$ indicates an unspecified expression. h_1 is elliptic because e is. μ, $\lambda+\mu \geq 0$ and (8) imply the ellipticity of $\partial s/\partial\xi_n$. Hence

$$h(t, x', \tau, \xi') = \begin{pmatrix} h_1\,\mathrm{Id}_3 & 0 \\ * & \partial s/\partial\xi_n \end{pmatrix} \tag{43}$$

defines an elliptic 6×6 symbol of order 0. Therefore, it suffices to prove the Lemma when, in the definitions of the vector spaces $B^*_{S/P}(\gamma)$, the symbols e and $s(x, \xi)$ are replaced by $(\tau^2+\xi'^2)^{1/2}\,\mathrm{Id}_3$ and $\xi_n\,\mathrm{Id}_3$, respectively. Having made this replacement the assertions follow from spectral decomposition of the first order symbol g associated with p in Lemma 3.2 and from (25) together with the known dimensions of ker p. □

Let $\pi^{\pm}_{S/P}$ and π_P denote the projectors associated with the decompositions (42). In the following $\Pi^{\pm}_{S/P}$ and Π_P denote 6×6 systems of pseudo-differential operators of order 0 having principal symbols $\pi^{\pm}_{S/P}$ and π_P, respectively. The operators $\Pi^{\pm}_S$, $\Pi^{\pm}_P$, and Π_P are defined microlocally in $\mathcal{H}_S$, $\mathcal{H}_P$, and $\mathcal{H}_P \cup \mathcal{E}_P$, respectively. These operators serve to define wave fronts sets which allow to distinguish the mode, S/P, and the forward or backward propagation direction, $\pm$, of Cauchy data Cu. Given a regularity $s \in \mathbb{R}$, a mode $m = S/P$, and a direction $\sigma = \pm$ the corresponding wave front set of Cu consists of those elements in $\mathcal{H}_m$ which belong to $\mathrm{WF}^{(s)}(\Pi^{\sigma}_m Cu)$. Abusing notation we denote this set $\mathrm{WF}^{(s)}(\Pi^{\sigma}_m Cu)$, i.e., by convention, we always have $\mathrm{WF}^{(s)}(\Pi^{\pm}_{S/P} Cu) \subset \mathcal{H}_{S/P}$.

We now state how the wave front set of the Cauchy data can be used to test the polarization of the displacement u.

Proposition 5.2. *Let $u \in \mathcal{D}'(\mathbb{R} \times \overline{\Omega})^3$ and $s \geq 3$ such that $Pu \in C^\infty(\mathbb{R} \times \overline{\Omega})^3$ and $Cu \in H^{(s-1)}(\mathbb{R} \times \partial\Omega)^6$. Let $\gamma \in T^\star(\mathbb{R} \times \partial\Omega) \setminus 0$, $\gamma \notin \mathcal{G}_P$. Then $\gamma \in \mathrm{WF}^{(s)}(\Pi^{\pm}_{S/P} Cu)$ if and only if $\gamma \in \mathcal{H}_{S/P}$ and $\mathrm{WF}^{(s+1)}_{\mathrm{pol}}(u)$ contains a Hamilton orbit above the shear/compressional wave bicharacteristic which issues from γ into the interior. The bicharacteristic is forward for the plus sign and backward for the minus sign.*

We point out that the assumption $s \geq 3$ is introduced only because of (31).

Proof. Introduce coordinates $x = (x_1, \dots, x_n)$, $n = 3$, as in the proof of Lemma 5.1. Abbreviate $D_x = (D', D_n)$. The simplified Cauchy data

$$C_0 u = (\langle D_t, D'\rangle u|_{x_n=0}, D_n u|_{x_n=0})$$

are related to the Cauchy data Cu as follows: $Cu \equiv HC_0u$ where $H \in \Psi^0(\mathbb{R} \times \partial\Omega)^{6\times 6}$ with principal symbol equal to h of equation (43). It now suffices to prove the Proposition with Cu replaced by C_0u and the operators $\Pi^{\pm}_{S/P}$ replaced by $H^{-1}\Pi^{\pm}_{S/P}H$. The assertions now follow from Proposition 2.3 if we recall the argument in Remark 2 of section 3. □

Curves which are bicharacteristics over the interior and reflected at non-glancing boundary points, with or without conversion between shear and compressional mode, are called broken bicharateristics. The propagation of singularities in the Cauchy data is stated recursively as follows.

Proposition 5.3. *Let* $f \in H^{(s)}_c(\mathbb{R} \times \partial\Omega)$, $s \geq 3$, *with* $\mathrm{WF}^{(s+1)}(f) \subset \Gamma_\delta$. *Let* $T \in \mathbb{R}$ *such that no forward broken bicharacteristic which issues from*

$$\mathrm{WF}^{(s+1)}(f) \cap \big(\mathrm{WF}^{(s)}(\Pi^+_S Cf) \cup \mathrm{WF}^{(s)}(\Pi^+_P Cf)\big)$$

intersects $\mathcal{G} \cap \{t \leq T\}$. *Then*

$$\begin{aligned} \mathrm{WF}^{(s)}(Cf) \cap \{t \leq T\} = \big(&\mathrm{WF}^{(s+1)}(f) \cup \mathcal{S}_S(\mathrm{WF}^{(s)}(\Pi^+_S Cf)) \\ &\cup \mathcal{S}_P(\mathrm{WF}^{(s)}(\Pi^+_P Cf))\big) \cap \{t \leq T\}. \end{aligned} \tag{44}$$

The functional notation for $\mathcal{S}_{S/P}$ in (44) is justified because these relations are in fact maps.

Proof. Let u denote the solution of $Pu = 0$ with Dirichlet boundary value f and zero initial data. By [Den82, Theorem 4.2] and [Gér85] the polarization set $\mathrm{WF}^{(s+1)}_{\mathrm{pol}}(u) \cap \{t \leq T\}$ is contained in the union of Hamilton orbits which lie above the broken bicharacteristics which issue from $\mathrm{WF}^{(s+1)}(f)$. We apply Proposition 5.2 at both endpoints of bicharacteristics which connect boundary points. We obtain $\mathcal{S}_{S/P}\big(\mathrm{WF}^{(s)}(\Pi^+_{S/P}Cf)\big) \subset \mathrm{WF}^{(s)}(\Pi^-_{S/P}Cf)$. Therefore the right hand side of (44) is contained in the left hand side. To prove the opposite inclusion let $\gamma \in \mathrm{WF}^{(s)}(Cf)\setminus\mathrm{WF}^{(s+1)}(f)$. Then γ is the endpoint of a forward bicharacteristic contained in $\mathrm{WF}^{(s+1)}(u)$ and, by Proposition 5.2, issued from $\mathrm{WF}^{(s)}(\Pi^+_S Cf)$ or $\mathrm{WF}^{(s)}(\Pi^+_P Cf)$. □

Proposition 5.2 combined with the following result permits us to specify, without having to know the coefficients of P, sources f for which compressional singularities are muted.

Proposition 5.4. *Choose $M \in \Psi^0(\mathbb{R} \times \partial\Omega)^{3\times 3}$ such that its principal symbol m equals at every $(t, x, \tau, \xi_|) \in T^\star(\mathbb{R} \times \partial\Omega)$ the orthogonal projector onto the one-dimensional subspace of $\mathbb{R}^3$ which is orthogonal to ξ and $\nu(x)$. Let $\gamma \in \mathcal{H}_P$. There is a conic neighbourhood $\Gamma \subset \mathcal{H}_P$ of γ such that the following inclusion holds for every $f \in H_c^{(s)}(\mathbb{R} \times \partial\Omega)^3$, $s \geq 3$, with* $\mathrm{WF}(f) \subset \Gamma$*:*

$$\mathrm{WF}^{(s)}\left(\Pi_P^+ Cf\right) \subset \mathrm{WF}^{(s+1)}(f - Mf). \tag{45}$$

Proof. First we show

$$\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \mathbb{C}^6 \subset B_S^+ + B_S^-. \tag{46}$$

To see this let $\gamma = (t, x, \tau, \xi_|)$ and $a \in \mathbb{C}^3$ with $\xi \cdot a = \nu \cdot a = 0$, $\nu = \nu(x)$. In view of (33) a belongs to the kernel of $p(t, x, \tau, \xi_S^\pm)$. In view of the definition (41) and (9) we have $s(x, \xi_S^\pm)a = \langle \xi_S^\pm, \nu\rangle_S a$. Hence $(e(\gamma)a, \langle \xi_S^\pm, \nu\rangle_S a) \in B_S^\pm(\gamma)$. $\langle \xi_S^+ - \xi_S^-, \nu\rangle_S \neq 0$ because $z_S^+ \neq z_S^-$. Therefore we obtain $(0, a), (a, 0) \in B_S^+(\gamma) + B_S^-(\gamma)$ proving (46).

$\pi_P = \pi_P^+ + \pi_P^-$ vanishes on $B_S^+ + B_S^-$. Therefore (46) and the symbol calculus imply

$$\Pi_P \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix} \in \Psi^{-1}. \tag{47}$$

Choose the conic neighbourhood Γ of γ in such a way that the displacement-to-traction map Λ is a pseudo-differential operator in $\Gamma \times \Gamma$. Shrinking Γ if necessary we may assume that every solution of $Pu = 0$ which has zero initial data and Dirichlet data f with $\mathrm{WF}(f) \subset \Gamma$ does not contain backward bicharacteristics issuing from Γ in its wavefront set.

Observe from formula (37) that the principal symbol of Λ maps the space onto which m projects into itself. Hence $\Lambda M - M\Lambda M \in \Psi^0$ and therefore

$$\begin{pmatrix} EM \\ \Lambda M \end{pmatrix} \equiv \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix} \begin{pmatrix} E\,\mathrm{Id}_3 \\ \Lambda M \end{pmatrix} \qquad \mathrm{mod}\ \Psi^0. \tag{48}$$

Let $f \in H_c^{(s)}(\mathbb{R} \times \partial\Omega)^3$, $s \geq 3$, with $\mathrm{WF}(f) \subset \Gamma$. Equations (47) and (48) imply

$$\Pi_P \begin{pmatrix} EMf \\ \Lambda Mf \end{pmatrix} \in H^{(s)}(\mathbb{R} \times \partial\Omega).$$

Assume $\gamma \notin \mathrm{WF}^{(s+1)}(f - Mf)$. Then, recalling $Cf = (Ef, \Lambda f)$, we obtain

$$\gamma \notin \mathrm{WF}^{(s)}(\Pi_P Cf). \tag{49}$$

Proposition 5.2 and our choice of Γ imply that (49) holds with Π_P replaced by Π_P^-. Hence (49) also holds with Π_P replaced by Π_P^+. $\square$

6. Proof of Theorem 1.1

The idea is to recover the lens maps from the elements of $\mathrm{WF}(\Lambda f)\setminus\mathrm{WF}(f)$ with least time where f ranges over microlocal point sources.

Let $L>0$. Choose $0<\delta,\varepsilon$ as in Remark 4 at the end of section 4. Let $P^{(1)},P^{(2)}\in\mathcal{L}(L,\varepsilon)$. Assume $\Lambda^{(1)}=\Lambda^{(2)}$. We show that the shear and compressional lens maps are equal: $\mathcal{S}_S^{(1)}=\mathcal{S}_S^{(2)}$ and $\mathcal{S}_P^{(1)}=\mathcal{S}_P^{(2)}$. Let $\tilde{\mathcal{S}}$ denote the union of the sets $\mathcal{S}_{S/P}^{(j)}\cap\big((\mathcal{G}^{(k)}\times\Gamma_\delta)\cup(\Gamma_\delta\times\mathcal{G}^{(k)})\big)$. Observe that $\mathcal{S}_{S/P}^{(j)}\setminus\tilde{\mathcal{S}}$ is open and dense in $\mathcal{S}_{S/P}^{(j)}$. We first prove that the lens maps agree outside $\tilde{\mathcal{S}}$.

Fix $s\geq 3$. Below we choose, given $\gamma^{\mathrm{in}}\in\Gamma_\delta$ nonglancing (for $j=1,2$ and S/P), sources f with the properties

$$f\in H_c^{(s)}(\mathbb{R}\times\partial\Omega)^3,\quad \mathrm{WF}^{(s+1)}(f)=\mathbb{R}_+\gamma^{\mathrm{in}}. \tag{50}$$

$\mathbb{R}_+\gamma^{\mathrm{in}}$ is the orbit of γ^{in} under the natural action of $\mathbb{R}_+$ on the cotangent bundle. A construction of distributions f as in (50) is given in, e.g., [Den82, Example 2.6]. We study the singularities of the Cauchy data $Cf=(Ef,\Lambda^{(1)}f)=(Ef,\Lambda^{(2)}f)$. Because of the zero initial condition in (4) the backward bicharacteristics issuing from γ^{in} are disjoint from $\mathrm{WF}^{(s+1)}(u)$. By Proposition 5.2 $\gamma^{\mathrm{in}}\notin\mathrm{WF}^{(s)}(\Pi_S^{-(j)}Cf)\cup\mathrm{WF}^{(s)}(\Pi_P^{-(j)}Cf)$, $j=1,2$. $\gamma^{\mathrm{in}}\in\mathrm{WF}^{(s)}(Cf)$ by assumption on f since E is elliptic. Therefore $\gamma^{\mathrm{in}}\in\mathrm{WF}^{(s)}(\Pi_S^{+(j)}Cf)\cup\mathrm{WF}^{(s)}(\Pi_P^{+(j)}Cf)$ or $\gamma^{\mathrm{in}}\in\mathcal{E}_S^{(j)}$.

Let $(\gamma^{\mathrm{out}},\gamma^{\mathrm{in}})\in\mathcal{S}_S^{(1)}\setminus\tilde{\mathcal{S}}$. Choose f with (50) and $\gamma^{\mathrm{in}}\notin\mathrm{WF}^{(s+1)}(f-Mf)$. Then we have $\gamma^{\mathrm{in}}\notin\mathrm{WF}^{(s)}(\Pi_P^{+(j)}Cf)$ for $j=1$ and $j=2$. In fact, this follows from Proposition 5.4 if $\gamma^{\mathrm{in}}\in\mathcal{H}_P^{(j)}$ and from our convention $\mathrm{WF}^{(s)}(\Pi_P^{+(j)}Cf)\subset\mathcal{H}_P^{(j)}$ if $\gamma^{\mathrm{in}}\notin\mathcal{H}_P^{(j)}$. Consequently,

$$\gamma^{\mathrm{in}}\in\mathrm{WF}^{(s)}(\Pi_S^{+(1)}Cf)\quad\text{and}\quad\gamma^{\mathrm{in}}\in\mathrm{WF}^{(s)}(\Pi_S^{+(2)}Cf)\cup\mathcal{E}_S^{(2)}.$$

Consider the set J of all $T\in\mathbb{R}$ with $\mathrm{WF}^{(s)}(Cf)\cap\{t\leq T\}=\mathbb{R}_+\gamma^{\mathrm{in}}$. J is a non-empty interval with $t(\gamma^{\mathrm{in}})=\min J$. We prove $t(\gamma^{\mathrm{out}})=\sup J$ using Proposition 5.3 with $\mathcal{S}_{S/P}^{(1)}$. For $T\in J$ we have $\mathrm{WF}^{(s)}(\Pi_S^{+(1)}Cf)\cap\{t\leq T\}=\mathbb{R}_+\gamma^{\mathrm{in}}$ and $\mathrm{WF}^{(s)}(\Pi_P^{+(1)}Cf)\cap\{t\leq T\}=\emptyset$. Therefore equation (44) becomes

$$\mathrm{WF}^{(s)}(Cf)\cap\{t\leq T\}=\big(\mathbb{R}_+\gamma^{\mathrm{in}}\cup\mathbb{R}_+\gamma^{\mathrm{out}}\big)\cap\{t\leq T\}.$$

Since lens maps strictly increase time t this equation remains valid when $T=\sup J$. It follows that $t(\gamma^{\mathrm{out}})=\sup J$ and

$$\mathrm{WF}^{(s)}(Cf)\cap\{t\leq t(\gamma^{\mathrm{out}})\}=\mathbb{R}_+\gamma^{\mathrm{in}}\cup\mathbb{R}_+\gamma^{\mathrm{out}}. \tag{51}$$

By the same reasoning, now using Proposition 5.3 with $\mathcal{S}_{S/P}^{(2)}$, we prove that

$$\mathrm{WF}^{(s)}(Cf)\cap\{t\leq T\}\subset\big(\mathbb{R}_+\gamma^{\mathrm{in}}\cup\mathcal{S}_S^{(2)}(\mathbb{R}_+\gamma^{\mathrm{in}})\big)\cap\{t\leq T\}$$

holds for $T \leq \sup J = t(\gamma^{\text{out}})$. Comparing this with (51) implies $(a\gamma^{\text{out}}, \gamma^{\text{in}}) \in \mathcal{S}_S^{(2)}$ for some $a > 0$. The covariable τ is constant along bicharacteristics. Therefore $a = 1$. Thus we have shown $\mathcal{S}_S^{(1)} \setminus \tilde{\mathcal{S}} \subset \mathcal{S}_S^{(2)}$. Interchanging $P^{(1)}$ with $P^{(2)}$ we obtain

$$\mathcal{S}_S^{(1)} \setminus \tilde{\mathcal{S}} = \mathcal{S}_S^{(2)} \setminus \tilde{\mathcal{S}}. \tag{52}$$

Let $(\gamma^{\text{out}}, \gamma^{\text{in}}) \in \mathcal{S}_P^{(1)} \setminus \tilde{\mathcal{S}}$. Choose f with (50) and

$$\gamma^{\text{in}} \in \text{WF}^{(s)}(\Pi_P^{+(1)} Cf) \setminus \text{WF}^{(s)}(\Pi_S^{+(1)} Cf). \tag{53}$$

Arguing as in the previous case, using Proposition 5.3 with $\mathcal{S}_{S/P}^{(1)}$, we obtain (51). Using Proposition 5.3 again, now with $\mathcal{S}_{S/P}^{(2)}$, we infer

$$\text{WF}^{(s)}(Cf) \cap \{t \leq t(\gamma^{\text{out}})\} \subset \mathbb{R}_+\gamma^{\text{in}} \cup \mathcal{S}_S^{(2)}(\mathbb{R}_+\gamma^{\text{in}}) \cup \mathcal{S}_P^{(2)}(\mathbb{R}_+\gamma^{\text{in}}).$$

Comparing with (51) we deduce $(a\gamma^{\text{out}}, \gamma^{\text{in}}) \in \mathcal{S}_S^{(2)} \cup \mathcal{S}_P^{(2)}$ for some $a > 0$. Again $a = 1$ follows. If $(\gamma^{\text{out}}, \gamma^{\text{in}}) \in \mathcal{S}_S^{(2)}$ then also $(\gamma^{\text{out}}, \gamma^{\text{in}}) \in \mathcal{S}_S^{(1)}$ by (52). Hence $(\gamma^{\text{out}}, \gamma^{\text{in}}) \in \mathcal{S}_P^{(2)}$ if $(\gamma^{\text{out}}, \gamma^{\text{in}}) \notin \mathcal{S}_S^{(1)}$. Suppose $(\gamma^{\text{out}}, \gamma^{\text{in}}) \in \mathcal{S}_S^{(1)}$. Choose f as above but now with projectors $\Pi_{S/P}^{+(1)}$ in (53) replaced by $\Pi_{S/P}^{+(2)}$. Recall $(\gamma^{\text{out}}, \gamma^{\text{in}}) \in \mathcal{S}_S^{(1)} \cap \mathcal{S}_P^{(1)}$. In particular, $\gamma^{\text{in}} \notin \mathcal{E}_S^{(1)}$. Therefore, $\gamma^{\text{in}} \in \text{WF}^{(s)}(\Pi_P^{+(1)} Cf) \cup \text{WF}^{(s)}(\Pi_S^{+(1)} Cf)$. We argue as before and obtain (51) using Proposition 5.3 with $\mathcal{S}_{S/P}^{(1)}$. We now apply Proposition 5.3 with $\mathcal{S}_{S/P}^{(2)}$ and, using $\gamma^{\text{in}} \notin \text{WF}^{(s)}(\Pi_S^{+(2)} Cf)$, we obtain

$$\text{WF}^{(s)}(Cf) \cap \{t \leq t(\gamma^{\text{out}})\} \subset \mathbb{R}_+\gamma^{\text{in}} \cup \mathcal{S}_P^{(2)}(\mathbb{R}_+\gamma^{\text{in}}).$$

Comparing with (51) we deduce $(\gamma^{\text{out}}, \gamma^{\text{in}}) \in \mathcal{S}_P^{(2)}$. Thus we have shown $\mathcal{S}_P^{(1)} \setminus \tilde{\mathcal{S}} \subset \mathcal{S}_P^{(2)}$. Interchanging $P^{(1)}$ with $P^{(2)}$ we obtain

$$\mathcal{S}_P^{(1)} \setminus \tilde{\mathcal{S}} = \mathcal{S}_P^{(2)} \setminus \tilde{\mathcal{S}}. \tag{54}$$

It remains to show that (52) and (54) hold with $\tilde{\mathcal{S}}$ replaced by the empty set. Assume $(\gamma^{\text{out}}, \gamma^{\text{in}}) \in \mathcal{S}_S^{(1)}$. We use a limit argument to prove $(\gamma^{\text{out}}, \gamma^{\text{in}}) \in \mathcal{S}_S^{(2)}$. First we observe that $\gamma^{\text{in}}, \gamma^{\text{out}} \notin \mathcal{G}_S^{(2)}$. Suppose not. Then a neighbourhood of $(\gamma^{\text{out}}, \gamma^{\text{in}})$ in $\mathcal{S}_S^{(1)}$ contains a point which is in $(\Gamma_\delta \times \mathcal{E}_S^{(2)}) \setminus \tilde{\mathcal{S}}$ or in $(\mathcal{E}_S^{(2)} \times \Gamma_\delta) \setminus \tilde{\mathcal{S}}$. This point cannot be in $\mathcal{S}_S^{(2)}$. This contradicts (52). Choose a sequence $(\gamma_k^{\text{out}}, \gamma_k^{\text{in}}) \in \mathcal{S}_S^{(1)} \cap \mathcal{S}_S^{(2)}$, $k \in \mathbb{N}$, which converges to $(\gamma^{\text{out}}, \gamma^{\text{in}})$. Let γ_k denote the shear wave bicharacteristic for $P^{(2)}$ with $\gamma_k(0) = \gamma_k^{\text{in}}$ and $\gamma_k(t_k) = \gamma_k^{\text{out}}$, $t_k > 0$. The length sequence of the corresponding sequence of geodesics is bounded. By compactness there is a limit geodesic and thus a bicharacteristic $\gamma : [0, T] \to T^\star(\mathbb{R} \times \overline{\Omega})$ of $P^{(2)}$ with $\gamma(0) = \gamma^{\text{in}}$ and $\gamma(T) = \gamma^{\text{out}}$. It suffices to show $\gamma(t)$ lies over the interior when $0 < t < T$. Suppose we had $\gamma(t^*)$ above the boundary for some $0 < t^* < T$. Then every neighbourhood of $(\gamma(t^*), \gamma^{\text{in}})$ has non-empty intersection with $\mathcal{S}_S^{(2)}$ hence, by (52), also with $\mathcal{S}_S^{(1)}$. This contradicts the continuity of the map $\mathcal{S}_S^{(1)}$ at γ^{in}. Hence we have shown $\mathcal{S}_S^{(1)} \subset \mathcal{S}_S^{(2)}$. The other inclusions are proved in the same way.

7. Proof of Lemma 4.3

Let $\gamma = (t, x, \tau, \xi_|) \in \Gamma_\delta \setminus \mathcal{G}$. To ease notation we drop the coordinates (t, x).

Recall the definitions of the symbols $q_{S/P}(\tau, \xi)/\rho = \tau^2 - |\xi|^2_{S/P}$, the metrics $\rho|\xi|_P^2 = \rho|\xi|_S^2 + (\lambda + \mu)|\xi|^2$, $\rho|\xi|_S^2 = \mu|\xi|^2 + R\xi \cdot \xi$, and the characteristics $\xi_{S/P} = \xi - z_{S/P}\nu$, $q_{S/P}(\tau, \xi_{S/P}) = 0$. We have $|\nu| = 1$.

The equation $q_S(\tau, \xi - z\nu) - q_P(\tau, \xi - z\nu) = (\lambda + \mu)(\xi - z\nu)^2$ holds for $z \in \mathbb{C}$. It implies $\xi_S \cdot \xi_P = \xi^2_{S/P} = 0$ if $z_S = z_P$. Also it implies $q_P(\tau, \xi_S) = 0$ if $\xi_S^2 = 0$. So if we had $\xi_S^2 = 0$ then $z_S \notin \mathbb{R}$ because of the real principal type property of $q_S q_P$. Since $\operatorname{Im} z_{S/P} \geq 0$ this can only happen if $z_P = z_S$. Therefore $\xi_S^2 = 0$ implies $\xi_S \cdot \xi_P = 0$. In the same way $\xi_P^2 = 0$ implies $\xi_S \cdot \xi_P = 0$. Therefore, it suffices to prove the inequality (38).

Choose $0 < \varepsilon \leq 1/2$. $\varepsilon > 0$ will be decreased further depending on L and δ only. The smallness assumption (6) on the residual stress tensor implies

$$1 - \varepsilon \leq \frac{\rho|\eta|_S^2}{\mu|\eta|^2}, \ \frac{\rho|\eta|_P^2}{(\lambda + 2\mu)|\eta|^2} \leq 1 + \varepsilon \quad \text{if } \eta \neq 0. \tag{55}$$

Consider the elliptic case, $\gamma \in \mathcal{E}_S \subset \mathcal{E}_P$. Without loss of generality we assume $\xi \cdot \nu = 0$. Then $\xi_S \cdot \xi_P = |\xi|^2 + z_S z_P$ and $|\xi| = |\xi_||\leq |\tau|/\delta$. $(\lambda + 2\mu)/\rho \leq L^2$ by (5). Therefore, decreasing ε we assume

$$\rho\tau^2 > 2\varepsilon(\lambda + 2\mu)|\xi|^2 \geq 2\varepsilon\mu|\xi|^2. \tag{56}$$

$z = z_{S/P}$ is the solution with positive imaginary part of the quadratic equation

$$|\nu|^2_{S/P} z^2 - 2bz + (|\xi|^2_{S/P} - \tau^2) = 0 \quad \text{where } b = R\xi \cdot \nu.$$

Notice that the signs of the real parts of z_S and z_P are equal to the sign of b. Hence $z_S z_P \notin \mathbb{R}$ and thus $\xi_S \cdot \xi_P \neq 0$ if $b \neq 0$. Assume $b = 0$. We solve the quadratic equations and then estimate using (55) and (56):

$$\begin{aligned}
|z_S z_P|^2 &= \frac{\rho(|\xi|_S^2 - \tau^2)}{\rho|\nu|_S^2} \cdot \frac{\rho(|\xi|_P^2 - \tau^2)}{\rho|\nu|_P^2} \\
&< \frac{\big(\mu(1+\varepsilon) - 2\varepsilon\mu\big)|\xi|^2}{(1-\varepsilon)\mu|\nu|^2} \cdot \frac{\big((\lambda + 2\mu)(1+\varepsilon) - 2\varepsilon(\lambda + 2\mu)\big)|\xi|^2}{(1-\varepsilon)(\lambda + 2\mu)|\nu|^2} \\
&\leq |\xi|^4.
\end{aligned}$$

Hence $|\xi|^2 + z_S z_P \neq 0$, i.e., (38) holds.

In the mixed case, $\gamma \in \mathcal{E}_P \cap \mathcal{H}_S$, we have $z_S \in \mathbb{R}$ and $\operatorname{Im} z_P > 0$. This implies (38).

Consider the hyperbolic case, $\gamma \in \mathcal{H}_P \subset \mathcal{H}_S$. Without loss of generality we assume

$$\langle \xi, \nu \rangle_P = 0. \tag{57}$$

Then the roots z of $0 = \tau^2 - |\xi - z\nu|_P^2 = -|\nu|_P^2 z^2 - |\xi|_P^2 + \tau^2$ have opposite signs. Since $q_P < q_S$ this is also true for the roots of $0 = \tau^2 - |\xi - z\nu|_S^2$. Furthermore $0 < |z_P| < |z_S|$. Since z_S and z_P are both forward they have the same sign. For simplicity we assume $0 < z_P < z_S$. We now derive the estimate

$$\xi_S \cdot \xi_P \geq |\xi|^2 - \varepsilon z_S |\xi| + z_S z_P. \tag{58}$$

$(\lambda + 2\mu)\xi \cdot \nu + R\xi \cdot \nu = 0$ is equation (57) restated. From this and (6) we deduce $2|\xi \cdot \nu| \leq \varepsilon |\xi|$. Hence $|(z_S + z_P)(\xi \cdot \nu)| \leq 2 z_S |\xi \cdot \nu| \leq \varepsilon z_S |\xi|$ (58) follows.

The equation $q_S(\tau, \xi - z_S \nu) - q_P(\tau, \xi - z_P \nu) = 0$ is equivalent to

$$z_S^2(|\nu|_S^2 - t^2|\nu|_P^2) - 2z_S\langle \xi, \nu\rangle_S - ((\lambda + \mu)/\rho)|\xi|^2 = 0 \quad \text{where } t = z_P/z_S. \tag{59}$$

We estimate the root z_S of this quadratic equation. Using the Cauchy-Schwarz inequality and (55) to estimate $|\nu|_S$ from below we deduce from (59)

$$|z_S|/4 \leq |\xi|_S + |\xi|\sqrt{(\lambda + \mu)/\mu} \quad \text{if } 2t|\nu|_P \leq |\nu|_S.$$

Decreasing $\varepsilon > 0$ if necessary, we assume $\varepsilon z_S \leq |\xi|$ if $2t|\nu|_P \leq |\nu|_S$. Inserting this estimate into (58) we get $\xi_S \cdot \xi_P \geq z_S z_P > 0$. It remains to prove (38) when $2t|\nu|_P > |\nu|_S$. From (55) and (5) we get $|\nu|_P^2/|\nu|_S^2 \leq 3(\lambda + 2\mu)/\mu \leq 3L^2$. Decreasing $\varepsilon > 0$ if necessary, we assume $\varepsilon^2|\nu|_P \leq 2|\nu|_S$. Hence $4t > \varepsilon^2$, $\varepsilon^2 z_S^2/4 < z_S z_P$. We estimate the right hand side of (58) from below and get $\xi_S \cdot \xi_P > (|\xi| - \varepsilon z_S/2)^2 \geq 0$.

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