

# Normality of Marsden-Weinstein reductions for representations of quivers

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**Abstract.** We prove that the Marsden-Weinstein reductions for the moment map associated to representations of a quiver are normal varieties. We give an application to conjugacy classes of matrices.

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## 1. Introduction

Throughout, we work over an algebraically closed field  $K$  of characteristic zero. Kraft and Procesi have proved in [18] that the closure of any conjugacy class of  $n \times n$  matrices is a normal variety. They reduce to the case of a nilpotent conjugacy class  $C$ , realize the closure  $\overline{C}$  as an affine quotient variety  $Z//H$ , and then use Serre’s criterion to prove that  $Z$  is normal. Here  $Z$  is the variety of linear maps

$$K^{n_1} \begin{array}{c} \xrightarrow{A_1} \\ \xleftarrow{B_1} \end{array} K^{n_2} \begin{array}{c} \xrightarrow{A_2} \\ \xleftarrow{B_2} \end{array} \dots \begin{array}{c} \xrightarrow{A_{t-2}} \\ \xleftarrow{B_{t-2}} \end{array} K^{n_{t-1}} \begin{array}{c} \xrightarrow{A_{t-1}} \\ \xleftarrow{B_{t-1}} \end{array} K^n$$

satisfying  $B_1 A_1 = 0$  and  $B_i A_i = A_{i-1} B_{i-1}$  for  $2 \leq i \leq t - 1$ , and

$$H = \prod_{i=1}^{t-1} GL(n_i, K)$$

acting by conjugation (for suitable integers  $t$  and  $n_1 < n_2 < \dots < n_{t-1} < n$ ).

In this paper we take these ideas very much further, and prove the normality of quotients similar to  $Z//H$ , but associated to any quiver.

Let  $Q$  be a quiver with vertex set  $I$ , and let  $h(a)$  and  $t(a)$  denote the head and tail vertices of an arrow  $a \in Q$ . Let  $\overline{Q}$  be the double of  $Q$ , obtained from  $Q$  by adjoining a reverse arrow  $a^*$  for each arrow  $a$  in  $Q$ . If  $\alpha \in \mathbb{N}^I$ , then the space of representations of  $\overline{Q}$  of dimension vector  $\alpha$  is

$$Rep(\overline{Q}, \alpha) = \bigoplus_{a \in \overline{Q}} Mat(\alpha_{h(a)} \times \alpha_{t(a)}, K),$$

the group

$$GL(\alpha) = \prod_{i \in I} GL(\alpha_i, K)$$

acts on it by conjugation, its Lie algebra is

$$End(\alpha) = \prod_{i \in I} Mat(\alpha_i, K),$$

and there is a map

$$\mu_\alpha : Rep(\overline{Q}, \alpha) \rightarrow End(\alpha), \quad \mu_\alpha(x)_i = \sum_{\substack{a \in Q \\ h(a)=i}} x_a x_{a^*} - \sum_{\substack{a \in Q \\ t(a)=i}} x_{a^*} x_a,$$

which is in fact a moment map for the action of  $GL(\alpha)$  on  $Rep(\overline{Q}, \alpha)$  with respect to a natural symplectic form on  $Rep(\overline{Q}, \alpha)$ , and after identifying the Lie algebra with its dual.

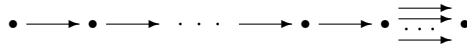
Identifying  $\lambda \in K^I$  with the element of  $End(\alpha)$  whose  $i$ th component is  $\lambda_i$  times the identity matrix of size  $\alpha_i$ , one can study the fibre  $\mu_\alpha^{-1}(\lambda)$  and the affine quotient variety  $N_Q(\lambda, \alpha) = \mu_\alpha^{-1}(\lambda) // GL(\alpha)$ . Such a quotient is an appropriate analogue in the category of affine varieties of a ‘symplectic quotient’, or ‘Marsden-Weinstein reduction’ [25]. Note that this variety is independent of the orientation of  $Q$ . (One might consider Marsden-Weinstein reductions associated to more general coadjoint orbits, but one obtains nothing new, see Remark 9.2.)

Constructions of this form were used by Kronheimer [19] to obtain the Kleinian singularities and their deformations and resolutions (see also [3, 9]), and they also play an important role in Nakajima’s geometric construction of integrable representations of Kac-Moody algebras [26]. For this reason we began a systematic study of them in [6]. (See the introduction to that paper for a detailed explanation of the connection with Nakajima’s quiver varieties.) Since then they have found applications back to Kleinian singularities [5], to the additive version of a problem of Deligne and Simpson [8], and to Kac’s conjectures on absolutely indecomposable representations of quivers [10]. For some connections with non-commutative geometry, see [1, 2].

Often the varieties are smooth, but this is not always the case, and it is an advantage of working in the algebraic geometric setting that one can deal with singular examples. In general we expect to obtain an interesting class of singularities, often having smooth deformations and resolutions with symplectic forms. (See the discussion in Section 8.) Our main result is as follows.

**Theorem 1.1.**  $N_Q(\lambda, \alpha)$  is a normal variety.

This result does not appear to apply to the situation discussed at the start of the introduction, since the group  $GL(n, K)$  is missing from  $H$ , and the relation  $A_{t-1}B_{t-1} = 0$  is also missing. However, these cancel out: by separating the rows of  $A_{t-1}$  into  $n$  matrices of size  $1 \times n_{t-1}$ , and the columns of  $B_{t-1}$  into  $n$  matrices of size  $n_{t-1} \times 1$ , one can see that  $Z \cong \mu_\alpha^{-1}(0)$  and  $Z//H \cong N_Q(0, \alpha)$  for a quiver  $Q$  of shape



with vertices  $1, \dots, t$ , with  $n$  arrows between the last two, and  $\alpha = (n_1, \dots, n_{t-1}, 1)$ . (This method of ‘deframing’ is discussed at the end of the introduction to [6].)

We remark that unlike Kraft and Procesi’s case, in the general case of Theorem 1.1 the fibre  $\mu_\alpha^{-1}(\lambda)$  itself need not be normal; see the discussion at the end of [4] for an example.

Using Kraft and Procesi’s construction of conjugacy class closures several times, or Lemma 9.1 in the non-nilpotent case, we have the following immediate consequence.

**Corollary 1.2.** If  $C_1, \dots, C_k$  are conjugacy classes in  $Mat(n, K)$ , then the quotient variety

$$\left\{ (M_i) \in \prod_{i=1}^k \overline{C_i} \mid \sum_{i=1}^k M_i = 0 \right\} // GL(n, K)$$

is normal.

Indeed the quotient is isomorphic to a variety of the form  $N_Q(\lambda, \alpha)$  where  $Q$  is a star with  $k$  arms and  $\alpha$  is a dimension vector which is  $n$  at the central vertex. The vanishing of the component of the moment map for the central vertex corresponds to the relation  $\sum_{i=1}^k M_i = 0$ .

In order to describe our earlier work on  $N_Q(\lambda, \alpha)$ , and for later use, we introduce some notation. Associated to a quiver  $Q$  with vertex set  $I$  there is a quadratic form

$$q(\alpha) = \sum_{i \in I} \alpha_i^2 - \sum_{a \in Q} \alpha_{h(a)} \alpha_{t(a)}$$

on  $\mathbb{Z}^I$ . We denote by  $(\alpha, \beta)$  the symmetric bilinear form with  $(\alpha, \alpha) = 2q(\alpha)$ , and we define  $p(\alpha) = 1 - q(\alpha)$ . (If we need to specify the quiver, we decorate  $q, (-, -)$  and  $p$  with subscripts.) Associated to  $Q$  there is a subset of  $\mathbb{Z}^I$  of ‘roots’ for  $Q$  (see for example [15]), and if  $\lambda \in K^I$  we denote by  $R_\lambda^+$  the set of positive roots  $\alpha$  with  $\lambda \cdot \alpha := \sum_{i \in I} \lambda_i \alpha_i = 0$ . We define  $\Sigma_\lambda$  to be the set of  $\alpha \in R_\lambda^+$  with the property that

$$p(\alpha) > p(\beta^{(1)}) + p(\beta^{(2)}) + \dots$$

whenever  $\alpha$  can be written as a sum  $\alpha = \beta^{(1)} + \beta^{(2)} + \dots$  of two or more elements of  $R_\lambda^+$ .

The elements of  $\mu_\alpha^{-1}(\lambda)$  correspond to representations of dimension vector  $\alpha$  of a certain algebra, the *deformed preprojective algebra*  $\Pi^\lambda$  of [9], and the quotient  $N_Q(\lambda, \alpha)$  classifies the isomorphism classes of semisimple representations. We showed in [6] that there is a simple representation of  $\Pi^\lambda$  of dimension vector  $\alpha$  if and only if  $\alpha \in \Sigma_\lambda$ . Moreover, in this case  $\mu_\alpha^{-1}(\lambda)$  and  $N_Q(\lambda, \alpha)$  are irreducible varieties. In [7] we studied  $N_Q(\lambda, \alpha)$  for arbitrary  $\alpha$ , and showed that  $N_Q(\lambda, \alpha)$  is either an irreducible variety or empty, according to whether or not  $\alpha$  can be written as a sum of elements of  $R_\lambda^+$ .

Note that, as in [7], we consider  $N_Q(\lambda, \alpha)$  as a variety, that is, we endow it with the reduced scheme structure. In [6] we worked with schemes, and showed that the natural scheme structures on  $\mu_\alpha^{-1}(\lambda)$  and  $N_Q(\lambda, \alpha)$  are reduced when  $\alpha \in \Sigma_\lambda$ . In general, however, this is not clear.

The contents of this paper are as follows. In Section 2 we study symplectic forms on bimodules over a semisimple algebra, and in Theorem 2.2 we prove an analogue of the (easy, linear) Darboux Theorem. In Section 3 we use this structure theory to give a coordinate-free construction of the varieties  $N_Q(\lambda, \alpha)$ . This is of interest in its own right, since this is how the varieties often arise, but our main application of it comes in Section 4, when we study the étale local structure of  $N_Q(\lambda, \alpha)$ . In Corollary 4.10 we show that any point in one of the varieties  $N_Q(\lambda, \alpha)$  has a neighbourhood which is isomorphic in the étale topology to a neighbourhood of the origin in  $N_{Q'}(0, \kappa)$  for some  $Q'$  and  $\kappa$ .

There is a stratification of  $N_Q(\lambda, \alpha)$  according to ‘representation type’ [6] with the stratum of type  $(k_1, \beta^{(1)}; \dots; k_r, \beta^{(r)})$  containing those semisimple representations of  $\Pi^\lambda$  which involve  $r$  non-isomorphic simples, with dimension vectors  $\beta^{(i)}$ , and multiplicities  $k_i$ . An important problem is to compute the dimension of the inverse image of this stratum in  $\mu_\alpha^{-1}(\lambda)$ . In Section 5 we prove some results which work for an arbitrary finitely generated algebra, and then in Section 6 we apply this to the preprojective algebra  $\Pi^0$  to get a bound on the dimensions of inverse images of strata in case  $\lambda = 0$ . This partially generalizes Lusztig’s result [24] that the nilpotent cone in  $\mu_\alpha^{-1}(0)$  is Lagrangian in  $\text{Rep}(\overline{Q}, \alpha)$  when  $Q$  has no oriented cycles.

In Section 7 we prove a normality condition for quotient schemes  $X//G$  with  $G$  a reductive group acting on a scheme  $X$ . If  $X$  is normal, then it is well-known that  $X//G$  is normal. Recall that Serre’s criterion says that a scheme is normal if and only if it has the regularity condition  $(R_1)$  and the depth condition  $(S_2)$ . In our situation,  $X$  has  $(S_2)$ , but may fail  $(R_1)$ . However  $X//G$  has  $(R_1)$ . Can one deduce that  $X//G$  is normal? In Corollary 7.2 we show that the answer is ‘yes’ if  $X//G$  has an open subscheme  $U$  which is normal, whose complement  $Z$  has codimension

at least two, and such that the inverse image of  $Z$  in  $X$  has codimension at least two.

In Section 8 we complete the proof of our main theorem. It is amusing that the cases that cannot be dealt with by the etale local structure result and an induction, are nearly Kleinian, and hence trivially normal. Finally we show that in many cases the varieties  $N_Q(0, \alpha)$  are genuinely singular.

For further developments based on the results in this paper we refer to [20, 21].

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## 2. Symplectic forms on bimodules

Let  $A$  be a finite-dimensional semisimple associative  $K$ -algebra. Since  $K$  is algebraically closed,  $A$  is isomorphic to a direct sum of full matrix algebras. It follows that any  $A$ - $A$ -bimodule is semisimple, so any sub-bimodule of an  $A$ - $A$ -bimodule has a complement which is a sub-bimodule.

If  $M$  is a finite-dimensional  $A$ - $A$ -bimodule, then a symplectic bilinear form  $\omega$  on  $M$  is said to be *balanced* if  $\omega(x, ay) = \omega(xa, y)$  for all  $a \in A$  and  $x, y \in M$ . Using the skew-symmetry of  $\omega$  one has  $\omega(ax, y) = -\omega(y, ax) = -\omega(ya, x) = \omega(x, ya)$ .

If  $U$  is a subspace of  $M$ , one defines  $U^\perp = \{m \in M \mid \omega(m, u) = 0 \text{ for all } u \in U\}$ . The non-degeneracy of  $\omega$  implies that  $\dim U^\perp = \dim M - \dim U$  and  $U^{\perp\perp} = U$ . If  $U$  is a sub-bimodule then so is  $U^\perp$ . Recall that  $U$  is *isotropic* (respectively *coisotropic*, respectively *Lagrangian*) if  $U \subseteq U^\perp$  (respectively  $U \supseteq U^\perp$ , respectively  $U = U^\perp$ ).

**Lemma 2.1.** *In an  $A$ - $A$ -bimodule with a balanced symplectic form, any simple sub-bimodule must be isotropic.*

*Proof.* Suppose that  $T$  is a simple sub-bimodule of  $M$ , and that  $M$  has a balanced symplectic form  $\omega$ . Then either  $T \cap T^\perp = T$ , or  $T \cap T^\perp = 0$ . In the former case  $T$  is isotropic, so suppose the latter holds, that is, that the restriction of  $\omega$  to  $T$  is non-degenerate, hence a symplectic form on  $T$ .

Since  $A$  is a product of full matrix algebras,  $T$  must be isomorphic to a bimodule of the form  $X \otimes_K Y^*$  where  $X$  and  $Y$  are simple left  $A$ -modules. The symplectic form gives a bimodule isomorphism  $T \rightarrow T^*$ , so  $X \cong Y$ . It follows that  $T \cong \text{End}_K(X)$ . Since  $X$  is simple, the homothety  $A \rightarrow \text{End}_K(X)$  is surjective, and since this is a bimodule map,  $T$  is a quotient of  $A$  as a bimodule.

Let  $t$  be the image in  $T$  of the identity element of  $A$ . Then every element of  $T$  is of the form  $at = ta$  for some  $a \in A$ . Now  $\omega(at, t) = \omega(ta, t) = \omega(t, at) = -\omega(at, t)$ . Thus  $\omega(at, t) = 0$ , and since this is true for all  $a$ , non-degeneracy implies that  $t = 0$ , a contradiction.  $\square$

We now have our analogue of the Darboux Theorem.

**Theorem 2.2.** *If  $M$  is an  $A$ - $A$ -bimodule with a balanced symplectic form  $\omega$ , then any maximal isotropic sub-bimodule  $S$  of  $M$  has an isotropic bimodule complement. Both  $S$  and the complement are Lagrangian. Moreover there is an isomorphism of bimodules  $M \cong S \oplus S^*$ , under which the symplectic form is given by  $\omega((s, f), (s', f')) = f(s') - f'(s)$ .*

*Proof.* Since bimodules over  $A$  are semisimple,  $S$  has a bimodule complement  $C$ . The fact that  $\omega$  is balanced implies that the map  $C \rightarrow S^*$  induced by  $\omega$  is a bimodule map. It is injective, for if  $T$  were a simple sub-bimodule of the kernel, then  $S \oplus T$  would be isotropic by Lemma 2.1, contradicting maximality.

Dualizing, the map  $f : S \rightarrow C^*$  induced by  $\omega$  is surjective. Semisimplicity for the category of bimodules then implies that the map  $C \rightarrow C^*$  induced by  $\omega$  factors through  $f$  by a bimodule map, say  $\theta : C \rightarrow S$ . Thus  $\omega(\theta(c), c') = \omega(c, c')$  for all  $c, c' \in C$ . By skew-symmetry, also  $\omega(c', \theta(c)) = \omega(c', c)$ .

Let  $D = \{c - \frac{1}{2}\theta(c) \mid c \in C\}$ . Clearly this is a bimodule complement to  $S$ . Moreover

$$\begin{aligned} \omega(c - \frac{1}{2}\theta(c), c' - \frac{1}{2}\theta(c')) &= \omega(c, c') - \omega(\frac{1}{2}\theta(c), c') - \omega(c, \frac{1}{2}\theta(c')) \\ &\quad + \omega(\frac{1}{2}\theta(c), \frac{1}{2}\theta(c')). \end{aligned}$$

Now the last term on the right hand side is zero since  $S$  is isotropic, and the second and third terms each give  $-\frac{1}{2}\omega(c, c')$ . Thus the right hand side vanishes.

Now  $S$  and  $D$  are isotropic, so both have dimension bounded by  $\frac{1}{2} \dim M$ . But this implies that they have dimension equal to this, and hence both are Lagrangian. Now the map  $D \rightarrow S^*$  induced by  $\omega$  is an isomorphism, and this gives the final assertion.  $\square$

**Corollary 2.3.** *If  $M$  is an  $A$ - $A$ -bimodule with a balanced symplectic form  $\omega$ , then any isotropic sub-bimodule  $S$  of  $M$  has a coisotropic bimodule complement.*

*Proof.* Extend  $S$  to a maximal isotropic sub-bimodule, which by semisimplicity we may write as  $S \oplus T$ , and choose a Lagrangian complement  $C$  to this. Then  $T \oplus C$  is a complement to  $S$ , and it is coisotropic since  $(T \oplus C)^\perp \subseteq C^\perp = C \subseteq T \oplus C$ .  $\square$

### 3. A coordinate-free description of $N_Q(\lambda, \alpha)$

We consider quadruples  $(A, M, \omega, \zeta)$  where  $A$  is a finite-dimensional semisimple associative  $K$ -algebra,  $M$  is a finite-dimensional  $A$ - $A$ -bimodule,  $\omega$  is a balanced symplectic bilinear form on  $M$ , and  $\zeta : A \rightarrow K$  is a linear map with  $\zeta(ab) = \zeta(ba)$  for all  $a, b \in A$  (a ‘trace function’).

Let  $G$  be the group of units of  $A$ . By Wedderburn’s Theorem it is a product of copies of the general linear group. It acts by conjugation on any  $A$ - $A$ -bimodule  $N$  via  $g \cdot n = gng^{-1}$  (where the multiplication on the right hand side of the equation is the bimodule action of  $A$  on  $N$ ). Recall that the *centre* of  $N$  is the set

$$\{n \in N \mid an = na \text{ for all } a \in A\}.$$

The fact that  $G$  is Zariski-dense in  $A$  implies that the centre of  $N$  coincides with the set of  $G$ -invariant elements of  $N$ . In particular, the condition imposed on  $\zeta$  is equivalent to insisting that it be a  $G$ -invariant element of  $A^*$ .

Let  $\mu : M \rightarrow A^*$  be the map defined by  $\mu(m)(a) = \omega(m, am)$ . Clearly we have  $\omega(m, am) = \omega(ma, m) = -\omega(m, ma)$ , so that  $\mu(m)(a) = \frac{1}{2}\omega(m, [a, m])$  where  $[a, m] = am - ma$ . Associated to the quadruple  $(A, M, \omega, \zeta)$  there is an affine variety

$$N(A, M, \omega, \zeta) = \mu^{-1}(\zeta) // G.$$

It is a Marsden-Weinstein reduction because of the following observation.

**Lemma 3.1.**  $\mu$  is a moment map for the action of  $G$  on  $M$ .

*Proof.* This means that  $d\mu_m(v)(a) = \omega_m(v, a_m)$  where  $m \in M$ ,  $v \in T_m M$  and  $a \in Lie(G)$ . Here  $\omega_m$  is the symplectic form induced by  $\omega$  on  $T_m M$  and  $m \mapsto a_m$  is the vector field on  $M$  induced by  $a$ .

Each tangent space of  $M$  can be identified with  $M$  itself, and  $Lie(G)$  can be identified with  $A$ . The vector field induced by  $a$  is then the map  $M \rightarrow M$  sending  $m$  to  $[a, m] = am - ma$ . The equation thus becomes  $d\mu_m(v)(a) = \omega(v, [a, m])$  for  $m, v \in M$  and  $a \in A$ .

Now the formula for  $\mu$  immediately gives  $d\mu_m(v)(a) = \omega(v, am) + \omega(m, av)$ , and using the properties of  $\omega$  this becomes  $\omega(v, [a, m])$ , as required.  $\square$

A quiver  $Q$  with vertex set  $I$ , and elements  $\alpha \in \mathbb{N}^I$  and  $\lambda \in K^I$  give rise to a quadruple  $(End(\alpha), Rep(Q, \alpha), \omega_\alpha, \zeta_\lambda)$  where  $\omega_\alpha$  is the canonical symplectic form on  $Rep(Q, \alpha)$ ,

$$\omega_\alpha(x, y) = \sum_{a \in Q} (tr(x_{a^*} y_a) - tr(x_a y_{a^*})),$$

and  $\zeta_\lambda$  is defined by  $\zeta_\lambda(\theta) = \sum_{i \in I} \lambda_i tr(\theta_i)$ .

**Lemma 3.2.**  $N(\text{End}(\alpha), \text{Rep}(Q, \alpha), \omega_\alpha, \zeta_\lambda) \cong N_Q(\lambda, \alpha)$ .

*Proof.* Using the trace pairing  $(\theta, \phi) \mapsto \sum_{i \in I} \text{tr}(\theta_i \phi_i)$  to identify  $\text{End}(\alpha)$  with its dual, the map  $\mu$  coincides with the map  $\mu_\alpha$  and  $\zeta_\lambda$  coincides with  $\lambda$  as an element of  $\text{End}(\alpha)$ .  $\square$

We say that two quadruples  $(A, M, \omega, \zeta)$  and  $(A', M', \omega', \zeta')$  are *equivalent* if there is an algebra isomorphism  $f : A \rightarrow A'$  with  $\zeta'(f(a)) = \zeta(a)$  for all  $a \in A$ , and an  $A$ - $A$ -bimodule isomorphism  $g : M \rightarrow {}_f M'_f$  with  $\omega'(g(m_1), g(m_2)) = \omega(m_1, m_2)$  for all  $m_1, m_2 \in M$ . Clearly an equivalence of quadruples gives an isomorphism  $N(A, M, \omega, \zeta) \cong N(A', M', \omega', \zeta')$ . Our coordinate-free description of the spaces  $N_Q(\lambda, \alpha)$  is completed with the following result.

**Lemma 3.3.** *Every quadruple is equivalent to  $(\text{End}(\alpha), \text{Rep}(Q, \alpha), \omega_\alpha, \zeta_\lambda)$  for some  $Q, \alpha$  and  $\lambda$ .*

*Proof.* Let  $(A, M, \omega, \zeta)$  be a quadruple. Choose a maximal isotropic sub-bimodule  $S$  of  $M$  and identify  $M$  with  $S \oplus S^*$  as in Theorem 2.2.

The vertex set  $I$  of  $Q$  is chosen to index the simple left  $A$ -modules up to isomorphism. The dimension vector  $\alpha$  is defined by letting  $\alpha_i$  be the dimension of the  $i$ th simple module. Clearly  $A \cong \text{End}(\alpha)$ .

If  $X$  and  $Y$  are simple  $A$ -modules, then  $X \otimes_K Y^*$  is a simple  $A$ - $A$ -bimodule, and all simple modules are obtained in this way. We define the quiver  $Q$  by letting the number of arrows from  $X$  to  $Y$  be the multiplicity of this simple module in  $S$ .

Finally,  $\zeta$  is a  $G$ -invariant element of  $A^* \cong \text{End}(\alpha)^*$ . Using the trace pairing it corresponds to a  $G$ -invariant element of  $\text{End}(\alpha)$ . The  $G$ -invariance means that each of the matrices involved is a scalar multiple of the identity matrix, and combining the scalars gives the element  $\lambda \in K^I$  with  $\zeta_\lambda = \zeta$ .  $\square$

#### 4. Etale local structure

In this section we study the local structure of our Marsden-Weinstein reductions. In a differential geometric setting this has already been done in a special case by Kronheimer [19, Lemma 3.3]. In our situation we have the added complication of etale morphisms, so need Luna's Slice Theorem [23]. (See also [26, §6] and [22].) If  $G$  is a reductive group acting on an affine variety  $X$ , we use the notation  $X//G$  where Luna writes  $X/G$ , but otherwise we use Luna's notation. We denote by  $\pi_X$  the natural map  $X \rightarrow X//G$ .

Let  $(A, M, \omega, \zeta)$  be a quadruple as in the previous section. Let  $G$  be the group of units of  $A$ . Let  $\mu : M \rightarrow A^*$  be the moment map  $\mu(m)(a) = \omega(m, am)$ . Fix  $x \in \mu^{-1}(\zeta)$  with the property that its  $G$ -orbit is closed.

**Lemma 4.1.**  $[A, x]$  is an isotropic subspace of  $M$ . That is,  $[A, x] \subseteq [A, x]^\perp$ .



*Proof.* Since  $\omega$  is balanced and skew-symmetric, for  $a, b \in A$  we have

$$\begin{aligned} \omega([a, x], [b, x]) &= \omega(ax, bx) - \omega(ax, xb) - \omega(xa, bx) + \omega(xa, xb) \\ &= \omega(axb, x) + \omega(xb, ax) - \omega(x, abx) + \omega(x, axb) \\ &= \omega(x, bax) - \omega(x, abx). \end{aligned}$$

But this can be rewritten as  $\mu(x)(ba) - \mu(x)(ab) = \zeta(ba) - \zeta(ab) = 0$ .  $\square$

**Lemma 4.2.**  $A_x = \{a \in A \mid ax = xa\}$  is a semisimple subalgebra of  $A$ .

*Proof.* The stabilizer  $G_x$  of  $x$  is reductive by Matsushima's Theorem, see [23]. Now  $G_x$  is the group of units of  $A_x$ , and the elements  $1 + a$  with  $a$  in the radical of  $A_x$  form a closed connected unipotent normal subgroup. Thus the radical must be zero.  $\square$

Choose an  $A_x$ - $A_x$ -bimodule complement  $L$  to  $A_x$  in  $A$ , so that  $A = A_x \oplus L$ . Observe that  $[A, x]$  is an  $A_x$ - $A_x$ -sub-bimodule of  $M$ . By Corollary 2.3 we can choose a coisotropic  $A_x$ - $A_x$ -bimodule complement  $C$  to  $[A, x]$  in  $M$ , so  $M = [A, x] \oplus C$ . Let  $W = C \cap [A, x]^\perp$ . Let  $\mu_x : M \rightarrow A_x^*$  be the map obtained by composing  $\mu$  with the restriction map  $A^* \rightarrow A_x^*$ . Let  $\hat{\mu}$  be the restriction of  $\mu_x$  to  $W$ .

**Lemma 4.3.** *The restriction of the symplectic form  $\omega$  to  $W$  is non-degenerate, and hence a symplectic form. It is balanced for the  $A_x$ - $A_x$ -bimodule structure, and the corresponding moment map is  $\hat{\mu}$ .*

*Proof.* Let  $0 \neq h \in W$ . We show that there is an element  $h' \in W$  with  $\omega(h, h') \neq 0$ . Since  $\omega$  is non-degenerate,  $C^\perp \cap [A, x]^\perp = 0$ , so by dimensions  $M = C^\perp \oplus [A, x]^\perp$ . Non-degeneracy now implies that there are elements  $p \in C^\perp$  and  $q \in [A, x]^\perp$  with  $\omega(h, p + q) \neq 0$ . Thus  $\omega(h, q) \neq 0$ . Now since  $M = [A, x] \oplus C$  and  $[A, x] \subseteq [A, x]^\perp$  we have  $[A, x]^\perp = [A, x] \oplus W$ . The claim follows on writing  $q = q' + h'$  corresponding to this decomposition, and observing that  $\omega(h, q') = 0$ . The rest is clear.  $\square$

Define  $v : C \rightarrow L^*$  by  $v(c)(\ell) = \omega(c, \ell x) + \omega(c, \ell c) + \omega(x, \ell c)$ . Thus

$$\mu(x + c)(a + \ell) = \zeta(a + \ell) + \mu_x(c)(a) + v(c)(\ell)$$

for  $c \in C$ ,  $a \in A_x$  and  $\ell \in L$ .

**Lemma 4.4.** *The assignment  $c \mapsto x + c$  induces a  $G_x$ -equivariant map  $\mu_x^{-1}(0) \cap v^{-1}(0) \rightarrow \mu^{-1}(\zeta)$ , and the induced map  $(\mu_x^{-1}(0) \cap v^{-1}(0)) // G_x \rightarrow \mu^{-1}(\zeta) // G$  is etale at 0.*

*Proof.* We consider the map  $C \rightarrow M$  sending  $c$  to  $x + c$ . The formula for  $\mu(x + c)$  shows that  $x + c \in \mu^{-1}(\zeta)$  if and only if  $c \in \mu_x^{-1}(0) \cap v^{-1}(0)$ , from which the first assertion follows. The conjugation action of  $G$  on  $M$  gives a map  $G \times C \rightarrow M$ ,

and the fact that  $M = [A, x] \oplus C$  implies that the differential of this map at  $(1, 0)$  is surjective. Let  $X = G \times_{G_x} C$ . (See [23, p86].) By dimensions it follows that the induced map  $\phi : X \rightarrow M$  is etale at  $(1, 0)$ .

Thus Luna's Fundamental Lemma [23, p94] applies to  $\phi$ . There is an affine open subset  $U$  of  $X$  which contains  $(1, 0)$ , is saturated for  $\pi_X$ , the restriction of  $\phi$  to  $U$  is etale, the image  $V = \phi(U)$  is an affine open subset of  $M$ , saturated for  $\pi_M$ , the morphism  $\phi/G : U//G \rightarrow V//G$  is etale, and the morphisms  $\phi : U \rightarrow V$  and  $\pi_U$  induce a  $G$ -isomorphism  $U \rightarrow V \times_{V//G} U//G$ .

Let  $V' = V \cap \mu^{-1}(\zeta)$ . Since  $V'$  is a closed subvariety of  $V$ , one can identify  $U' = V' \times_V U$  with a closed subvariety of  $U$ , and clearly  $U' = U \cap X'$  where  $X' = G \times_{G_x} C'$  and  $C' = \mu_x^{-1}(0) \cap v^{-1}(0)$ .

Now [23, Lemma, p95], applied to the map  $\phi : U \rightarrow V$  and the subvariety  $V'$ , implies that the morphism

$$\phi'/G : (U \cap X')//G \rightarrow (V \cap \mu^{-1}(\zeta))//G$$

is etale.

The fact that  $U$  is saturated for  $\pi_X$  implies that  $U \cap X'$  is saturated for  $\pi_{X'}$ , so the domain of  $\phi'/G$  is an open subset of  $X'//G \cong C'//G_x$ . Similarly, the fact that  $V$  is saturated for  $\pi_M$  implies that  $V \cap \mu^{-1}(\zeta)$  is saturated for  $\pi_{\mu^{-1}(\zeta)}$ , so the codomain of  $\phi'/G$  is an open subset of  $\mu^{-1}(\zeta)//G$ . The claim follows.  $\square$

**Lemma 4.5.**  $\dim W = \dim C - \dim L$ .

*Proof.* Since  $[A, x] \subseteq [A, x]^\perp$  we have  $C + [A, x]^\perp = M$ . Thus

$$\dim W = \dim C + \dim[A, x]^\perp - \dim M = \dim C - \dim L,$$

where the last equality holds because  $\dim[A, x]^\perp = \dim M - \dim[A, x] = \dim M - \dim A + \dim A_x = \dim M - \dim L$ .  $\square$

**Lemma 4.6.** *The map  $v : C \rightarrow L^*$  is smooth at 0, so  $v^{-1}(0)$  is smooth at 0. Its tangent space at 0 is identified with  $W$ .*

*Proof.* Identifying  $C$  and  $L^*$  with their tangent spaces at the origin, the differential of  $v$  at the origin is given by

$$dv_0(c)(\ell) = \omega(c, \ell x) + \omega(x, \ell c) = \omega(c, [\ell, x])$$

so  $\text{Ker } dv_0 = C \cap [L, x]^\perp = W$ . By the dimension formula  $dv_0$  must be surjective, so  $v$  is smooth at 0.  $\square$

**Lemma 4.7.**  $C = C^\perp \oplus W$  and  $C^\perp = C \cap W^\perp$ .

*Proof.*  $W = C \cap [A, x]^\perp = C^{\perp\perp} \cap [A, x]^\perp = (C^\perp + [A, x])^\perp$ , so  $W^\perp = C^\perp + [A, x]$ . Since  $C$  is coisotropic, this implies that  $C \cap W^\perp = C^\perp + (C \cap [A, x]) = C^\perp$ . The left hand side can be rewritten as  $C^{\perp\perp} \cap W^\perp = (C^\perp + W)^\perp$ , and hence  $C^\perp + W = C$ . Finally,  $C^\perp \cap W = C^\perp \cap [A, x]^\perp = (C + [A, x])^\perp = M^\perp = 0$ .  $\square$

**Lemma 4.8.** *There is a morphism  $(\mu_x^{-1}(0) \cap \nu^{-1}(0))//G_x \rightarrow \hat{\mu}^{-1}(0)//G_x$ , sending 0 to 0, which is etale at 0.*

*Proof.* Let  $p : C \rightarrow W$  be the projection coming from the decomposition of Lemma 4.7. Since  $C^\perp$  is isotropic, it follows that  $\hat{\mu}p = \mu_x$ . Let  $\phi : \nu^{-1}(0) \rightarrow W$  be the restriction of  $p$ . By Lemma 4.6 it is etale at 0.

We apply [23, Lemma 3, p93] to the  $G_x$ -morphism  $\phi$ . Since the codomain is a vector space it is smooth, so certainly normal. Thus there is an affine open subvariety  $U'$  of  $\nu^{-1}(0)//G_x$  containing 0 such that the restriction of

$$\phi/G_x : \nu^{-1}(0)//G_x \rightarrow W//G_x$$

to  $U'$  is etale. In particular  $\phi/G_x$  is etale at 0.

Now since  $\hat{\mu}p = \mu_x$  we have  $\phi^{-1}(\hat{\mu}^{-1}(0)) = \mu_x^{-1}(0) \cap \nu^{-1}(0)$ . Thus the pull-back of  $\phi/G_x$  along the closed immersion  $\hat{\mu}^{-1}(0)//G_x \rightarrow W//G_x$  is a morphism as in the statement of the lemma. The assertion follows.  $\square$

Given points  $x_i$  in varieties  $X_i$  ( $i = 1, 2$ ), we say that the  $x_i$  have neighbourhoods which are *isomorphic in the etale topology* if there is a variety  $Z$ , a point  $z \in Z$ , and morphisms from  $Z$  to the  $X_i$ , etale at  $z$  and sending  $z$  to  $x_i$ . For later use, note that  $X_1$  is normal at  $x_1$  if and only if  $X_2$  is normal at  $x_2$  by [12, I, Theorem 9.5].

**Theorem 4.9.** *The points  $x \in \mu^{-1}(\zeta)//G$  and  $0 \in \hat{\mu}^{-1}(0)//G_x$  have neighbourhoods which are isomorphic in the etale topology.*

*Proof.* Combine Lemmas 4.4 and 4.8.  $\square$

Combining this result with the coordinate-free description of  $N_Q(\lambda, \alpha)$  we obtain the following.

**Corollary 4.10.** *Suppose  $Q$  is a quiver with vertex set  $I$ ,  $\lambda \in K^I$  and  $\alpha \in \mathbb{N}^I$ . If  $x \in N_Q(\lambda, \alpha)$ , then there is a quiver  $Q'$  and dimension vector  $\kappa$  such that the points  $x \in N_Q(\lambda, \alpha)$  and  $0 \in N_{Q'}(0, \kappa)$  have neighbourhoods which are isomorphic in the etale topology.*

Here  $A = \bigoplus_{i \in I} \text{Mat}(\alpha_i, K)$ . If  $x$  has representation type  $(k_1, \beta^{(1)}; \dots; k_r, \beta^{(r)})$ , then

$$A_x \cong \text{Mat}(k_1, K) \oplus \dots \oplus \text{Mat}(k_r, K).$$

Thus  $Q'$  has vertex set  $\{1, \dots, r\}$  and the dimension vector is  $\kappa = (k_1, \dots, k_r)$ . To determine the arrows in  $Q'$ , or rather  $\overline{Q'}$ , one needs to determine the multiplicities of the simple  $A_x$ - $A_x$ -bimodules  $K^{k_s} \otimes_K (K^{k_t})^*$  in  $W$ . Now the natural map  $W \rightarrow [A, x]^\perp/[A, x]$  is an isomorphism of  $A_x$ - $A_x$ -bimodules, and we have  $[A, x] \cong A/A_x$  and  $M/[A, x]^\perp \cong (A/A_x)^*$ . It thus suffices to determine how the

simple  $A$ - $A$ -bimodules  $K^{\alpha_i} \otimes_K (K^{\alpha_j})^*$  decompose into simple  $A_x$ - $A_x$ -bimodules. This follows from the decomposition

$$\begin{aligned} K^{\alpha_i} &\cong (K^{\beta_i^{(1)}})^{k_1} \oplus \dots \oplus (K^{\beta_i^{(r)}})^{k_r} \\ &\cong (K^{k_1})^{\beta_i^{(1)}} \oplus \dots \oplus (K^{k_r})^{\beta_i^{(r)}} \end{aligned}$$

It follows that  $\overline{Q}$  has  $2p_Q(\beta^{(i)})$  loops at vertex  $i$ , and  $-(\beta^{(i)}, \beta^{(j)})_Q$  arrows from  $i$  to  $j$ , if  $i \neq j$ .

### 5. Varieties of module extensions

We prove a result about varieties of modules which is used in the next section.

Let  $A$  be a finitely generated associative  $K$ -algebra, let  $I$  be a finite set, and let  $e_i$  ( $i \in I$ ) be a complete set of orthogonal idempotents in  $A$ , meaning that  $e_i e_j = \delta_{ij} e_i$  and  $\sum_{i \in I} e_i = 1$ . If  $M$  is a finite dimensional  $A$ -module, its *dimension vector* is the element of  $\mathbb{N}^I$  whose  $i$ th component is  $\dim e_i M$ . For  $\alpha \in \mathbb{N}^I$  we define  $Rep(A, \alpha)$  to be the variety of  $A$ -module structures on  $K^\alpha = \bigoplus_{i \in I} K^{\alpha_i}$  for which  $e_i$  acts as the projection onto the  $i$ th summand. (Any such  $A$  can be realized as a quotient of a path algebra  $KQ$ , with the  $e_i$  corresponding to the trivial paths, and then  $Rep(A, \alpha)$  is a closed subvariety of  $Rep(Q, \alpha)$ .) If  $x \in Rep(A, \alpha)$  we denote the corresponding  $A$ -module by  $K_x$ . The group

$$GL(\alpha) = \prod_{i \in I} GL(\alpha_i, K)$$

acts by conjugation on  $Rep(A, \alpha)$  and its orbits correspond to isomorphism classes of  $A$ -modules  $M$  of dimension vector  $\alpha$ .

**Lemma 5.1.** *Suppose that  $M$  and  $L$  are finite-dimensional  $A$ -modules and that  $M \oplus L$  has dimension vector  $\alpha$ . The set  $\mathcal{E}$  of triples*

$$(x, \theta, \phi) \in Rep(A, \alpha) \times Hom_K(M, K^\alpha) \times Hom_K(K^\alpha, L)$$

*such that  $\theta$  and  $\phi$  are  $A$ -module maps for the module structure on  $K^\alpha$  given by  $x$ , and the sequence*

$$0 \rightarrow M \xrightarrow{\theta} K_x \xrightarrow{\phi} L \rightarrow 0$$

*is exact, is locally closed, and  $\dim \mathcal{E} = \alpha \cdot \alpha + \dim Ext^1(L, M) - \dim Hom(L, M)$ .*

*Proof.*  $\mathcal{E}$  is locally closed since the conditions that  $\theta$  and  $\phi$  be  $A$ -module maps and that  $\phi\theta = 0$  are closed, while the conditions that  $\theta$  be injective and  $\phi$  surjective are open.

Since  $A$  is finitely generated as a  $K$ -algebra, it follows that  $L$  is a finitely-presented  $A$ -module, so that  $Ext^1(L, M)$  is finite dimensional. Choose an exact sequence

$$\xi : 0 \rightarrow U \xrightarrow{p} V \xrightarrow{q} L \rightarrow 0$$

with the property that the connecting map  $Hom(U, M) \rightarrow Ext^1(L, M)$  is surjective. For example one can take  $\xi$  to be the universal exact sequence, with  $U = M^k$  where  $k = \dim Ext^1(L, M)$ .

Let  $\mathcal{D}$  be the set of tuples

$$((x, \theta, \phi), f, g) \in \mathcal{E} \times Hom(U, M) \times Hom_K(V, K^\alpha)$$

where  $g$  is an  $A$ -module map for the module structure given by  $x$ , and the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \xrightarrow{p} & V & \xrightarrow{q} & L & \longrightarrow & 0 \\ & & f \downarrow & & g \downarrow & & \parallel & & \\ 0 & \longrightarrow & M & \xrightarrow{\theta} & K_x & \xrightarrow{\phi} & L & \longrightarrow & 0. \end{array}$$

commutes. Clearly  $\mathcal{D}$  is a closed subset.

We relate the dimensions of  $\mathcal{E}$  and  $\mathcal{D}$  by computing the fibres of the projection  $\pi : \mathcal{D} \rightarrow \mathcal{E}$ . Suppose  $(x, \theta, \phi) \in \mathcal{E}$ . By the surjectivity of the connecting map, there is some  $((x, \theta, \phi), f, g) \in \mathcal{D}$ . Moreover, given homomorphisms  $f' \in Hom(U, M)$  and  $g' \in Hom(V, K_x)$ , we have  $((x, \theta, \phi), f + f', g + g') \in \mathcal{D}$  if and only if  $\phi g' = 0$  and  $\theta f' = g' p$ , so if and only if  $g' = \theta h$  and  $f' = hp$  for some  $h \in Hom(V, M)$ . Thus  $\pi^{-1}(x, \theta, \phi)$  is isomorphic to  $Hom(V, M)$ . It follows that  $\dim \mathcal{D} = \dim \mathcal{E} + \dim Hom(V, M)$ .

There is also a projection  $\mathcal{D} \rightarrow Hom(U, M)$ . To compute the fibre over  $f \in Hom(U, M)$ , we construct the pushout

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \xrightarrow{p} & V & \xrightarrow{q} & L & \longrightarrow & 0 \\ & & f \downarrow & & a \downarrow & & \parallel & & \\ 0 & \longrightarrow & M & \xrightarrow{r} & N & \xrightarrow{s} & L & \longrightarrow & 0. \end{array}$$

Any invertible linear map  $b : N \rightarrow K^\alpha$  induces an  $A$ -module structure on  $K^\alpha$  such that  $b$  is a module homomorphism. If  $b$  is given by invertible linear maps  $e_i N \rightarrow K^{\alpha_i}$  ( $i \in I$ ), this structure corresponds to an element  $x \in Rep(A, \alpha)$  with  $b \in Hom(N, K_x)$  and  $((x, br, sb^{-1}), f, ba) \in \mathcal{D}$ . Conversely, if  $((x, \theta, \phi), f, g)$  is in  $\mathcal{D}$ , then by the universal property of the pushout we have constructed, there is a map  $b \in Hom(N, K_x)$  giving a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{r} & N & \xrightarrow{s} & L & \longrightarrow & 0 \\ & & \parallel & & b \downarrow & & \parallel & & \\ 0 & \longrightarrow & M & \xrightarrow{\theta} & K_x & \xrightarrow{\phi} & L & \longrightarrow & 0. \end{array}$$

and clearly  $b$  is an isomorphism. Thus the fibre over  $f$  is isomorphic to the product over  $i \in I$  of the set of invertible linear maps  $e_i N \rightarrow K^{\alpha_i}$ . Thus each fibre is irreducible of dimension  $\alpha \cdot \alpha$ , so

$$\dim \mathcal{D} = \dim \operatorname{Hom}(U, M) + \alpha \cdot \alpha.$$

The assertion follows.  $\square$

**Lemma 5.2.** *Let  $\mathcal{C}$  be a  $GL(\gamma)$ -stable constructible subset of  $\operatorname{Rep}(A, \gamma)$ , let  $T$  be a simple  $A$ -module of dimension vector  $\beta$ , let  $m$  be a positive integer, and let  $\alpha = \gamma + m\beta$ . Let  $\mathcal{F}$  be the set of  $x \in \operatorname{Rep}(A, \alpha)$  such that  $\dim \operatorname{Hom}(K_x, T) = m$  and there is an exact sequence*

$$0 \rightarrow K_c \xrightarrow{\theta} K_x \xrightarrow{\phi} T^m \rightarrow 0$$

with  $c \in \mathcal{C}$ . Then  $\mathcal{F}$  is constructible and

$$\dim \mathcal{F} \leq \dim \mathcal{C} + \alpha \cdot \alpha - \gamma \cdot \gamma - m^2 + md$$

where

$$d = \max_{c \in \mathcal{C}} (\dim \operatorname{Ext}^1(T, K_c) - \dim \operatorname{Hom}(T, K_c)).$$

*Proof.* Let  $\mathcal{G}$  be the set of quadruples

$$(x, c, \theta, \phi) \in \operatorname{Rep}(A, \alpha) \times \operatorname{Rep}(A, \gamma) \times \operatorname{Hom}_K(K^\gamma, K^\alpha) \times \operatorname{Hom}_K(K^\alpha, T^m)$$

forming an exact sequence as above, with  $c \in \mathcal{C}$ , and with  $\dim \operatorname{Hom}(K_x, T) = m$ . Clearly  $\mathcal{G}$  is constructible. The first projection  $\mathcal{G} \rightarrow \operatorname{Rep}(A, \alpha)$  has image  $\mathcal{F}$ , so that  $\mathcal{F}$  is constructible. If  $(x, c, \theta, \phi) \in \mathcal{G}$ , since the components of  $\phi$  must form a basis of  $\operatorname{Hom}(K_x, T)$ , any other surjective map  $K_x \rightarrow T^m$  must be obtained by composing  $\phi$  with an automorphism of  $T^m$ . It follows that  $GL(\gamma) \times \operatorname{Aut}_A(T^m)$  acts freely and transitively on the fibres of the projection, so

$$\dim \mathcal{F} = \dim \mathcal{G} - \gamma \cdot \gamma - m^2.$$

The second projection  $\mathcal{G} \rightarrow \operatorname{Rep}(A, \gamma)$  has image contained in  $\mathcal{C}$ , and the fibre over any point  $c$  is contained in the variety  $\mathcal{E}$  of Lemma 5.1 for  $L = T^m$  and  $M = K_c$ . It follows that the fibre has dimension at most  $\alpha \cdot \alpha + md$ . Thus  $\dim \mathcal{G} \leq \dim \mathcal{C} + \alpha \cdot \alpha + md$ , giving the result.  $\square$

### 6. Dimensions of strata

Let  $\Pi = \Pi^0$  be the (undeformed) preprojective algebra for  $\mathcal{Q}$ . The trivial paths in  $K\overline{\mathcal{Q}}$  give a complete set of orthogonal idempotents  $e_i$  in  $\Pi$ , and for  $\alpha \in \mathbb{N}^I$  we can identify  $Rep(\Pi, \alpha)$  with  $\mu_\alpha^{-1}(0)$ .

Let  $T_1, \dots, T_r$  be a collection of non-isomorphic simple  $\Pi$ -modules, of dimensions  $\beta^{(1)}, \dots, \beta^{(r)}$ .

If  $h \geq 0$ ,  $j_1, \dots, j_h$  are integers in the range  $1, \dots, r$ , and  $m_1, \dots, m_h$  are positive integers, we say that a  $\Pi$ -module  $M$  has *top-type*  $(j_1, m_1; \dots; j_h, m_h)$  if it has a filtration by submodules

$$0 = M_0 \subset M_1 \subset \dots \subset M_h = M$$

such that  $M_i/M_{i-1} \cong T_{j_i}^{m_i}$  and  $\dim Hom(M_i, T_{j_i}) = m_i$  for each  $i$ . This means that having chosen the sequence  $j_1, \dots, j_h$ , the submodules  $M_i$  are determined by decreasing induction on  $i$  by the fact that  $M_{i-1}$  is the intersection of the kernels of all maps from  $M_i$  to  $T_{j_i}$  (the ‘top’ of  $M_i$  with respect to the simple  $T_{j_i}$ ).

If  $(j_1, m_1; \dots; j_h, m_h)$  is a top-type and  $1 \leq i \leq h$ , define  $z_i$  to be zero if  $p(\beta^{(j_i)}) = 0$  or there is no  $k < i$  with  $j_k = j_i$ , and otherwise to be equal to  $m_k$  for the largest  $k < i$  with  $j_k = j_i$ .

**Lemma 6.1.** *If  $M$  has top-type  $(j_1, m_1; \dots; j_h, m_h)$  then  $\dim Hom(M_{i-1}, T_{j_i}) \leq z_i$ .*

*Proof.* For simplicity of notation, let  $T = T_{j_i}$ . First suppose that  $Ext^1(T, T) = 0$ . Applying  $Hom(-, T)$  to the exact sequence  $0 \rightarrow M_{i-1} \rightarrow M_i \rightarrow T^{m_i} \rightarrow 0$  gives an exact sequence

$$0 \rightarrow Hom(T^{m_i}, T) \rightarrow Hom(M_i, T) \rightarrow Hom(M_{i-1}, T) \rightarrow Ext^1(T^{m_i}, T) = 0.$$

Thus  $Hom(M_{i-1}, T) = 0$  by dimensions.

Now suppose that  $Ext^1(T, T) \neq 0$ . If there is no  $k < i$  with  $j_k = j_i$  then  $M_{i-1}$  has no composition factors isomorphic to  $T$ , so the Hom space is zero. Thus suppose otherwise, and choose  $k < i$  maximal with  $j_k = j_i$ . By the definition of top-type we have  $\dim Hom(M_k, T) = m_k$ . Now applying  $Hom(-, T)$  to the exact sequence

$$0 \rightarrow M_k \rightarrow M_{i-1} \rightarrow M_{i-1}/M_k \rightarrow 0$$

and using the fact that  $M_{i-1}/M_k$  has no composition factor isomorphic to  $T$ , we deduce that  $Hom(M_{i-1}, T)$  embeds in  $Hom(M_k, T)$ . □

**Lemma 6.2.** *The subset  $\mathcal{D}$  of  $\mu_\alpha^{-1}(0)$  consisting of those elements with top-type  $(j_1, m_1; \dots; j_h, m_h)$  is constructible and has dimension at most*

$$\alpha \cdot \alpha - 1 + p(\alpha) + \sum_{i=1}^h m_i z_i - \sum_{i=1}^h m_i^2 p(\beta^{(j_i)}).$$

*Proof.* Let  $\gamma = \alpha - m_h \beta^{(j_h)}$ . Let  $\mathcal{C}$  be the subset of  $\mu_\gamma^{-1}(0) = \text{Rep}(\Pi, \gamma)$  consisting of those elements  $c$  such that  $K_c$  has top-type  $(j_1, m_1; \dots; j_{h-1}, m_{h-1})$  and with  $\dim \text{Hom}(K_c, T_{j_h}) \leq z_h$ . By induction on  $h$  it is constructible and

$$\dim \mathcal{C} \leq \gamma \cdot \gamma - 1 + p(\gamma) + \sum_{i=1}^{h-1} m_i z_i - \sum_{i=1}^{h-1} m_i^2 p(\beta^{(j_i)}).$$

By Lemma 6.1, for any element  $x \in \mathcal{D}$  there is an exact sequence

$$0 \rightarrow K_c \rightarrow K_x \rightarrow T_{j_h}^{m_h} \rightarrow 0$$

with  $c \in \mathcal{C}$ . Thus by Lemma 5.2 we have

$$\dim \mathcal{D} \leq \dim \mathcal{C} + \alpha \cdot \alpha - \gamma \cdot \gamma - m_h^2 + m_h d$$

where

$$d = \max_{c \in \mathcal{C}} (\dim \text{Ext}^1(T_{j_h}, K_c) - \dim \text{Hom}(T_{j_h}, K_c)).$$

Now by [5] we have

$$\dim \text{Ext}^1(T_{j_h}, K_c) - \dim \text{Hom}(T_{j_h}, K_c) = \dim \text{Hom}(K_c, T_{j_h}) - (\gamma, \beta^{(j_h)}),$$

so  $d \leq z_h - (\gamma, \beta^{(j_h)})$ . The result follows.  $\square$

Let  $\pi : \mu_\alpha^{-1}(0) \rightarrow N_Q(0, \alpha)$  be the quotient map. Fix a representation type  $(k_1, \beta^{(1)}; \dots; k_r, \beta^{(r)})$  for representations of  $\Pi$  of dimension  $\alpha$ .

**Theorem 6.3.** *If  $x \in N_Q(0, \alpha)$  has representation type  $(k_1, \beta^{(1)}; \dots; k_r, \beta^{(r)})$ , then*

$$\dim \pi^{-1}(x) \leq \alpha \cdot \alpha - 1 + p(\alpha) - \sum_t p(\beta^{(t)}).$$

*Proof.* Any element of  $\pi^{-1}(x)$  has top-type  $(j_1, m_1; \dots; j_h, m_h)$  for suitable  $j_i$  and  $m_i$ . It thus suffices to prove that

$$\sum_{i=1}^h (m_i z_i - m_i^2 p(\beta^{(j_i)})) \leq - \sum_{t=1}^r p(\beta^{(t)}).$$

for each possible top-type. We group the first sum according to the value of  $j_i$ . Fix  $t$ , and let the values of  $i$  with  $j_i = t$  be  $i_1, \dots, i_q$ . It clearly suffices to prove that

$$\sum_{p=1}^q (m_{i_p} z_{i_p} - m_{i_p}^2 p(\beta^{(t)})) \leq -p(\beta^{(t)}).$$



If  $p(\beta^{(t)}) = 0$ , then  $z_{i_p} = 0$ , and the inequality is trivial. Thus suppose that  $p(\beta^{(t)}) = \ell > 0$ . Defining  $n_p = m_{i_p}$  for  $1 \leq p \leq q$ , we have  $z_{i_p} = n_{p-1}$  if  $p > 1$ , and  $z_{i_1} = 0$ . Thus the inequality becomes

$$\sum_{p=2}^q n_p n_{p-1} - \sum_{p=1}^q n_p^2 \ell \leq -\ell$$

or equivalently

$$(\ell - 1) \left( \sum_{p=1}^q n_p^2 \right) + \left( \sum_{p=1}^q n_p^2 - \sum_{p=2}^q n_p n_{p-1} \right) \geq \ell.$$

Now the first of the two terms on the left is clearly at least  $\ell - 1$ , and the second is at least 1, for it is the (positive definite) quadratic form for the Dynkin graph of type  $\mathbb{A}_q$  applied to the dimension vector  $(n_1, \dots, n_q)$ .  $\square$

In view of [6, Theorem 1.3], this has an immediate corollary.

**Corollary 6.4.** *The inverse image in  $\mu_\alpha^{-1}(0)$  of the stratum of representation type  $(k_1, \beta^{(1)}; \dots; k_r, \beta^{(r)})$  has dimension at most  $\alpha \cdot \alpha - 1 + p(\alpha) + \sum_t p(\beta^{(t)})$ .*

If  $Q$  has no oriented cycles and the  $\beta^{(t)}$  are the coordinate vectors, then the stratum consists only of the origin, and its inverse image in  $\mu_\alpha^{-1}(0)$  is the nilpotent cone. According to Lusztig [24] it is Lagrangian, so its dimension is equal to the bound in the corollary.

### 7. Normality of quotients

Let  $X$  be an irreducible affine scheme of finite type over  $K$ , let  $G$  be a reductive group acting on  $X$ , and let  $\pi : X \rightarrow X//G$  be the natural map. It is well known that if  $X$  is normal, then so is  $X//G$ . Here we find a different condition which still implies normality of the quotient.

Recall [13, §5.7] that a scheme  $X$  is said to have property  $(S_i)$  if

$$\text{depth}(\mathcal{O}_{X,Z}) \geq \min(i, \text{codim}_X Z)$$

for every irreducible closed subset  $Z$  of  $X$ . By [11, Theorem 3.8], for affine  $X$  it is equivalent that the local cohomology groups  $H_Z^j(X)$  vanish for all such  $Z$  and all  $j < \min(i, \text{codim}_X Z)$ .

**Theorem 7.1.** *Suppose that  $X$  and an open subscheme  $U$  of  $X//G$  have property  $(S_i)$ . If  $\pi^{-1}(X//G \setminus U)$  has codimension at least  $i$  in  $X$ , then  $X//G$  has  $(S_i)$ .*

*Proof.* Suppose that  $Z$  is an irreducible closed subset of  $X//G$ . Let  $V$  be the complement of  $U$ . By [11, Proposition 1.9] there is a long exact sequence

$$0 \rightarrow H_{Z \cap V}^0(X // G) \rightarrow H_Z^0(X // G) \rightarrow H_{Z \cap U}^0(X // G) \rightarrow H_{Z \cap V}^1(X // G) \rightarrow \dots,$$

so it suffices to prove that if  $j < \min(i, \text{codim}_{X // G} Z)$  then  $H_{Z \cap V}^j(X // G)$  and  $H_{Z \cap U}^j(X // G)$  both vanish.

Now  $H_{Z \cap V}^j(X // G)$  embeds in  $H_{\pi^{-1}(Z \cap V)}^j(X)$  by [14, Corollary 6.8], and this space vanishes since  $X$  has property  $(S_i)$  and  $\pi^{-1}(Z \cap V)$  has codimension at least  $i$  in  $X$ .

Also  $H_{Z \cap U}^j(X // G) \cong H_{Z \cap U}^j(U)$  by excision [11, Proposition 1.3], and this vanishes since  $U$  has  $(S_i)$ . □

Recall that a scheme has property  $(R_i)$  if its singular locus has codimension at least  $i$ . Serre’s criterion says that  $X$  is normal if and only if it has  $(R_1)$  and  $(S_2)$ .

**Corollary 7.2.** *Let  $U$  be an open subscheme of  $X // G$ . If  $U$  is normal, its complement  $Z$  has codimension at least two in  $X // G$ ,  $\pi^{-1}(Z)$  has codimension at least two in  $X$ , and  $X$  has  $(S_2)$ , then  $X // G$  is normal.*

*Proof.*  $X // G$  is regular in codimension 1 since  $U$  is normal and its complement has codimension at least 2. Thus we just need  $X // G$  to have property  $(S_2)$ , and this is a special case of the theorem. □

### 8. Completion of the proof

Recall that a dimension vector  $\alpha$  is *sincere* if  $\alpha_i > 0$  for all  $i$ . Given a quiver  $Q$  and a sincere dimension vector  $\alpha$ , we say that we are in the *nearly Kleinian* case if either

- (1)  $Q$  has only one vertex, say  $i$ , and  $\alpha_i = 1$ , or
- (2)  $Q$  is obtained from an extended Dynkin quiver by adjoining any number of loops at extending vertices, and  $\alpha = \delta$ , the minimal positive imaginary root for the extended Dynkin quiver. (Recall that  $i$  is an *extending vertex* if  $\delta_i = 1$ .)

**Lemma 8.1.**  *$N_Q(0, \alpha)$  is normal in the nearly Kleinian case.*

*Proof.* Observe that the preprojective relations involve commutators, and if  $a$  is a loop and the dimension vector at the corresponding vertex is 1, then  $aa^* - a^*a = 0$  automatically. Thus  $a$  and  $a^*$  can be removed from the relations. The result is now clear – in case (1)  $N_Q(0, \alpha)$  is an affine space, and in case (2) it is the product of a Kleinian singularity and an affine space. □

**Lemma 8.2.** *Let  $\alpha$  be a sincere dimension vector for a quiver  $Q$  with  $\alpha \in \Sigma_0$ , and suppose that we are not in the nearly Kleinian case. If  $(k_1, \beta^{(1)}; \dots; k_r, \beta^{(r)})$  is a representation type for dimension vector  $\alpha$ , then at least one of the following holds.*

- (i)  $\sum_i k_i^2 < \sum_i \alpha_i^2$ .
- (ii)  $p(\alpha) - \sum_i p(\beta^{(i)}) \geq 2$ .

*Proof.* We suppose that (i) and (ii) fail, and derive a contradiction. We have  $\sum_i \alpha_i^2 = \alpha \cdot \alpha = \sum_{s,t} k_s k_t \beta^{(s)} \cdot \beta^{(t)} \geq \sum_t k_t^2$ . Since (i) fails this must be an equality, which since the  $\beta^{(t)}$  have non-negative integer components, implies that the  $\beta^{(t)}$  must be distinct coordinate vectors. Since  $\alpha$  is sincere, they are exactly the coordinate vectors in some order, and the multiplicities  $k_t$  are the corresponding components of  $\alpha$ .

If  $\alpha = \beta^{(t)}$  for some  $t$  then we are in the nearly Kleinian case, a contradiction. Thus the decomposition of  $\alpha$  as a sum over  $t$  of  $k_t$  copies of each  $\beta^{(t)}$  has at least two terms, so since  $\alpha \in \Sigma_0$  we know that  $p(\alpha) > \sum_t k_t p(\beta^{(t)})$ . Since (ii) fails, it follows that  $p(\alpha) = 1 + \sum_t k_t p(\beta^{(t)}) = 1 + \sum_t p(\beta^{(t)})$ , and  $k_t = 1$  whenever  $p(\beta^{(t)}) > 0$ , that is, whenever there is a loop at the corresponding vertex.

Now we have

$$\begin{aligned} q(\alpha) - \sum_i k_i q(\beta^{(i)}) &= \sum_t q(k_t \beta^{(t)}) + \frac{1}{2} \sum_{\substack{s,t \\ s \neq t}} (k_s \beta^{(s)}, k_t \beta^{(t)}) - \sum_t k_t q(\beta^{(t)}) \\ &= \sum_t (k_t^2 - k_t) q(\beta^{(t)}) + \frac{1}{2} \sum_{\substack{s,t \\ s \neq t}} (k_s \beta^{(s)}, k_t \beta^{(t)}) \\ &= \frac{1}{2} \sum_t k_t \left( (k_t - 1) (\beta^{(t)}, \beta^{(t)}) + \sum_{\substack{s \\ s \neq t}} (k_s \beta^{(s)}, k_t \beta^{(t)}) \right) \\ &= \frac{1}{2} \sum_t k_t (\beta^{(t)}, \alpha - \beta^{(t)}). \end{aligned}$$

As observed above, we have  $\beta^{(t)} \neq \alpha$ , so  $(\beta^{(t)}, \alpha - \beta^{(t)}) = -2 - d_t$  for some  $d_t \geq 0$  by [6, Corollary 5.7]. Then

$$p(\alpha) - \sum_t k_t p(\beta^{(t)}) = 1 - \sum_t k_t + \frac{1}{2} \sum_t k_t (2 + d_t) = 1 + \frac{1}{2} \sum_t k_t d_t.$$

Now since (ii) fails we have  $d_t = 0$  for all  $t$ , that is, that  $(\beta^{(t)}, \alpha - \beta^{(t)}) = -2$ . Equivalently

$$(k_t - 1) (\beta^{(t)}, \beta^{(t)}) + \sum_{\substack{s \\ s \neq t}} k_s (\beta^{(s)}, \beta^{(t)}) = -2.$$

Let  $Q''$  be the quiver obtained from  $Q$  by deleting all loops. Since the  $\beta^{(t)}$  are distinct coordinate vectors we have

$$\begin{aligned} (\beta^{(t)}, \alpha)_{Q''} &= k_t (\beta^{(t)}, \beta^{(t)})_{Q''} + \sum_{\substack{s \\ s \neq t}} k_s (\beta^{(t)}, \beta^{(s)})_{Q''} \\ &= 2k_t + \sum_{\substack{s \\ s \neq t}} k_s (\beta^{(t)}, \beta^{(s)})_Q = 2k_t - 2 - (k_t - 1) (\beta^{(t)}, \beta^{(t)})_Q. \end{aligned}$$

Now this is zero, for either  $(\beta^{(t)}, \beta^{(t)})_Q = 2$ , or there are loops in  $Q$  at the vertex, in which case  $k_t = 1$ .

Since  $\alpha$  is a vector for  $Q''$  with positive integer components, connected support, and the bilinear form with each coordinate vector gives 0, it follows that  $Q''$  is an extended Dynkin quiver and  $\alpha$  is a multiple of the minimal positive imaginary root  $\delta$  (see [15, §1.2]). Now if  $Q$  contains any loops then some  $k_t = 1$ , so that  $\alpha = \delta$ . Otherwise  $Q = Q''$  is extended Dynkin itself, and the fact that  $\alpha \in \Sigma_0$  implies that  $\alpha = \delta$ . Thus we are in the nearly Kleinian case, a contradiction.  $\square$

We can now prove our main result.

**Theorem 8.3.**  $N_Q(\lambda, \alpha)$  is a normal variety for any  $\lambda$  and  $\alpha$ .

*Proof.* In view of the etale local structure result, it suffices to prove normality for  $\lambda = 0$ . We prove it for all quivers and dimension vectors  $\alpha$  by induction on  $\sum_i \alpha_i^2$ . Since a product of normal varieties is normal, and a symmetric product of copies of a normal variety is normal, by the main result of [7] we may assume that  $\alpha \in \Sigma_0$ . Clearly we may suppose that  $\alpha$  is sincere. Also, we may assume that we are not in the nearly Kleinian case, since that has been dealt with separately.

By the lemma, for any representation type  $(k_1, \beta^{(1)}; \dots; k_r, \beta^{(r)})$ , conditions (i) or (ii) of the lemma hold.

By the etale local structure, a point in the stratum of this representation type has a neighbourhood which is isomorphic in the etale topology to a neighbourhood of 0 in  $N_{Q'}(0, \kappa)$ . By the remark before Theorem 4.9, the points of strata where (i) holds are normal by induction.

On the other hand, if a stratum satisfies (ii), then Corollary 6.4 and [6, Theorem 1.2] imply that the inverse image of the stratum in  $\mu_\alpha^{-1}(0)$  has codimension at least 2.

We now apply Corollary 7.2. Let  $U$  be the normal locus of  $N_Q(0, \alpha)$ , an open subset. By the observations above, its complement has inverse image in  $\mu_\alpha^{-1}(0)$  of codimension at least 2. Thus in fact  $N_Q(0, \alpha)$  is normal.  $\square$

Using Geometric Invariant Theory quotients it is possible to construct resolutions of singularities. If  $\theta \in \mathbb{Z}^I$  satisfies  $\theta \cdot \alpha = 0$ , then there are notions of  $\theta$ -stable and  $\theta$ -semistable elements of  $\mu_\alpha^{-1}(\lambda)$ , there is a GIT quotient  $\mu_\alpha^{-1}(\lambda) // (GL(\alpha), \theta)$ , and a semisimplification map

$$\mu_\alpha^{-1}(\lambda) // (GL(\alpha), \theta) \rightarrow N_Q(\lambda, \alpha)$$

which is a projective morphism, see [17]. Moreover we have:

(1) If  $\alpha \in \Sigma_\lambda$ , then  $\mu_\alpha^{-1}(\lambda)$  is irreducible by [6], and the general element is a simple representation of  $\Pi^\lambda$ , hence  $\theta$ -stable. Thus the semisimplification map is a birational morphism of irreducible varieties.

(2) If  $\alpha$  is indivisible, that is, its components have no common divisor, then one can choose a weight  $\theta$  such that every  $\theta$ -semistable element is  $\theta$ -stable. Since

any  $\theta$ -stable element has trivial endomorphism algebra as a  $\Pi^\lambda$ -module, the fibre  $\mu_\alpha^{-1}(\lambda)$  is smooth (see for example [4, Lemma 10.3]), and then  $N_Q(\lambda, \alpha)$  is smooth by Luna’s Slice Theorem [23, Corollary 1, p98].

Thus, if (1) and (2) hold, then the semisimplification map is a resolution of singularities. As usual, the Grauert-Riemenschneider Theorem gives the following consequence (cf. [16, p50]).

**Corollary 8.4.** *If  $\alpha \in \Sigma_\lambda$  is indivisible, then  $N_Q(\lambda, \alpha)$  has rational singularities.*

*Proof.* It suffices to observe that  $N_Q(\lambda, \alpha)$  is normal and that if  $\theta$  is chosen so that  $\theta$ -semistables are  $\theta$ -stable, then since  $\mu_\alpha^{-1}(\lambda) // (GL(\alpha), \theta)$  is a symplectic quotient, it has a symplectic form, and hence has trivial canonical class.  $\square$

Incidentally, if  $\alpha \in \Sigma_0$ , then the moment map  $\mu_\alpha$  is flat by [6], and  $\alpha \in \Sigma_\lambda$  for any  $\lambda$  with  $\lambda \cdot \alpha = 0$ . Letting  $\mathfrak{h} = \{\lambda \in K^I \mid \lambda \cdot \alpha = 0\}$ , there is a flat family  $\mu_\alpha^{-1}(\mathfrak{h}) // GL(\alpha) \rightarrow \mathfrak{h}$  whose fibre over 0 is  $N_Q(0, \alpha)$ . In case  $\alpha$  is also indivisible, the general fibre of this family is smooth, and the whole family has a simultaneous resolution of singularities  $\mu_\alpha^{-1}(\mathfrak{h}) // (GL(\alpha), \theta)$ . If  $Q$  is extended Dynkin and  $\alpha = \delta$ , then the family is the semi-universal deformation, or rather its lift through a Weyl group action.

One may wonder whether  $N_Q(0, \alpha)$  is actually singular. We have a partial answer based on the following lemma.

**Lemma 8.5.** *Suppose that  $\pi : \tilde{X} \rightarrow X$  is a projective, birational morphism between smooth irreducible varieties. If  $\tilde{X}$  has trivial canonical class, and  $\pi^{-1}(U) \rightarrow U$  is an isomorphism for some open  $U \subseteq X$  whose complement  $Z$  has codimension  $\geq 2$ , then  $\pi$  is an isomorphism.*

*Proof.* Let  $j : U \rightarrow X$  be the inclusion, and let  $i : U \rightarrow \tilde{X}$  be the composition of the isomorphism  $U \rightarrow \pi^{-1}(U)$  and the inclusion  $\pi^{-1}(U) \rightarrow \tilde{X}$ .

Since  $X$  is smooth, the codimension condition implies that any locally free sheaf  $\mathcal{F}$  on  $X$  is  $Z$ -closed [13, Theorem 5.10.5], so the map  $j^* : \Gamma(X, \mathcal{F}) \rightarrow \Gamma(U, \mathcal{F}|_U)$  is an isomorphism by [13, Proposition 5.9.8]. Similarly, if  $\mathcal{G}$  is a locally free sheaf on  $\tilde{X}$  then  $i^* : \Gamma(\tilde{X}, \mathcal{G}) \rightarrow \Gamma(U, \mathcal{G}|_U)$  is injective by  $Z$ -purity [13, Proposition 5.10.2].

Let  $X$  and  $\tilde{X}$  have dimension  $n$ . Since  $\tilde{X}$  has trivial canonical class, there is a nowhere vanishing  $n$ -form  $\sigma$  on  $\tilde{X}$ . Then  $i^*\sigma$  is such a form on  $U$ , and since  $j^* : \Gamma(X, \omega_X) \rightarrow \Gamma(U, \omega_U)$  is an isomorphism, there is an  $n$ -form  $\tau$  on  $X$  with  $j^*\tau = i^*\sigma$ . Now  $i^*\pi^*\tau = j^*\tau = i^*\sigma$ , so the fact that  $i^* : \Gamma(\tilde{X}, \omega_{\tilde{X}}) \rightarrow \Gamma(U, \omega_U)$  is injective implies that  $\pi^*\tau = \sigma$ .

In particular  $\pi^*\tau$  is a nowhere vanishing  $n$ -form on  $\tilde{X}$ . Now if  $x \in \tilde{X}$  then  $(\pi^*\tau)_x$  is the composition

$$\wedge^n T_x \tilde{X} \xrightarrow{\wedge^n d\pi_x} \wedge^n T_{\pi(x)} X \xrightarrow{\tau_{\pi(x)}} K$$

where  $d\pi_x : T_x \tilde{X} \rightarrow T_{\pi(x)} X$  is the induced map on tangent spaces. Since this is nonzero,  $d\pi_x$  must be invertible. Thus  $\pi$  is étale, and hence it has finite fibres. Then by Zariski's Main Theorem,  $\pi$  is bijective. Since it is also birational, it is an isomorphism.  $\square$

**Proposition 8.6.** *Let  $Q$  have no loops at vertices, and let  $\alpha \in \Sigma_0$  be indivisible. If  $\alpha$  is an imaginary root (equivalently, not a coordinate vector), then  $N_Q(0, \alpha)$  is singular.*

*Proof.* Since  $N_Q(0, \alpha)$  does not depend on the orientation of  $Q$ , see for example [9, Lemma 2.2], and since  $Q$  has no loops, we can reorient it so that it has no oriented cycles.

Choosing  $\theta$  so that any  $\theta$ -semistable is  $\theta$ -stable, the semisimplification map is a resolution of singularities, and the quotient  $\mu_\alpha^{-1}(\lambda) // (GL(\alpha), \theta)$  classifies  $GL(\alpha)$ -orbits of  $\theta$ -stable elements of  $\mu_\alpha^{-1}(0)$ . By the lemma, if  $N_Q(0, \alpha)$  is smooth, the semisimplification map is a bijection. Thus, considering the fibre over 0, the set of nilpotent  $\theta$ -stable elements of  $\mu_\alpha^{-1}(0)$  must form a single orbit, so must have dimension  $\dim GL(\alpha) - 1$  (taking into account the fact that  $GL(\alpha)$  has a subgroup isomorphic to  $K^*$  which acts trivially).

Since  $Q$  has no oriented cycles, by [24] the nilpotent elements of  $\mu_\alpha^{-1}(0)$  form an equidimensional subvariety of  $Rep(\overline{Q}, \alpha)$  of dimension  $\dim Rep(Q, \alpha)$ . Since the nilpotent elements which are  $\theta$ -stable form an open subset of this, we must have

$$\dim GL(\alpha) - 1 = \dim Rep(Q, \alpha).$$

In other words  $q(\alpha) = 1$ , so that  $\alpha$  is a real root. The equivalence with coordinate vectors follows from the results in [6, Section 7].  $\square$

## 9. Appendix

We show that Kraft and Procesi's construction of conjugacy class closures [18] works in more than just the nilpotent case. Let  $C$  be a conjugacy class in  $Mat(n, K)$ , and choose  $t \geq 1$  and elements  $\xi_1, \dots, \xi_t \in K$  such that  $\prod_{j=1}^t (M - \xi_j 1) = 0$  for  $M \in C$ . For  $i \leq t$  let  $n_i$  be the rank of  $\prod_{j=i+1}^t (M - \xi_j 1)$ , so  $n_t = n$  and  $n_0 = 0$ . Define  $Z$  as in the introduction, but with the relations  $B_1 A_1 = (\xi_1 - \xi_2) 1$  and  $B_i A_i = A_{i-1} B_{i-1} + (\xi_i - \xi_{i+1}) 1$  for  $2 \leq i \leq t - 1$ . Let  $H$  be as in the introduction.

**Lemma 9.1.** *The map  $\Theta : Z \rightarrow Mat(n, K)$  sending  $(A_i, B_i)$  to  $A_{t-1} B_{t-1} + \xi_t 1$  induces an isomorphism  $Z // H \cong \overline{C}$ .*

*Proof.* Clearly  $\Theta$  is constant on orbits of  $H$  and equivariant for the conjugation action of  $GL(n, K)$ . The First Fundamental Theorem of Invariant Theory and an

induction on  $t$  shows that  $\Theta$  induces a closed embedding  $Z//H \rightarrow \text{Mat}(n, K)$ , see [18, §2]. The image of  $\Theta$  contains  $C$ , since if  $M \in C$  one can identify  $K^{n_i}$  with the image of  $\prod_{j=i+1}^t (M - \xi_j 1)$ , and take  $B_i = M - \xi_{i+1} 1$  and  $A_i$  to be the inclusion. In [18] the proof is completed using the Gerstenhaber-Hesselink Theorem, and perhaps our lemma could be proved in a similar way; here we use the Marsden-Weinstein reductions. Clearly it suffices to prove that  $Z//H$  is irreducible of dimension

$$\dim C = n^2 - \sum_{i=1}^t d_i^2$$

where  $d_i = n_i - n_{i-1}$ . By deframing,  $Z//H \cong N_Q(\lambda, \alpha)$  for the quiver  $Q$  illustrated in the introduction,  $\alpha = (n_1, \dots, n_{t-1}, 1)$  and  $\lambda = (\xi_2 - \xi_1, \dots, \xi_t - \xi_{t-1}, \nu)$ , where  $\nu$  is chosen so that  $\lambda \cdot \alpha = 0$ . Clearly

$$p(\alpha) = \sum_{i=1}^{t-1} (n_i n_{i+1} - n_i^2) = \frac{1}{2} \left( n^2 - \sum_{i=1}^t d_i^2 \right),$$

so by [6, Corollary 1.4] it suffices to show that  $\alpha \in \Sigma_\lambda$ .

Any decomposition of  $\alpha$  as a sum of positive roots must include one term whose component at vertex  $t$  is nonzero, say  $\beta = (m_1, \dots, m_{t-1}, 1)$  with  $0 \leq m_1 \leq \dots \leq m_{t-1} \leq n$ . All other terms must be of the form  $\gamma^{(jk)}$  with  $1 \leq j \leq k \leq t-1$ , where

$$\gamma_i^{(jk)} = \begin{cases} 1 & (\text{if } j \leq i \leq k) \\ 0 & (\text{else}). \end{cases}$$

The decompositions involved in the definition of  $\Sigma_\lambda$  are those in which the summands satisfy  $\lambda \cdot \gamma^{(jk)} = 0$ , that is,  $\xi_j = \xi_{k+1}$ .

Let  $e_i = m_i - m_{i-1}$  (with  $m_t = n$  and  $m_0 = 0$ ). Clearly the sequence  $e_1, \dots, e_t$  is obtained from  $d_1, \dots, d_t$  by decreasing  $d_j$  by one, and increasing  $d_{k+1}$  by one, for each term of the form  $\gamma^{(jk)}$  in the decomposition of  $\alpha$ .

Since  $p(\gamma^{(jk)}) = 0$ , to show that  $\alpha \in \Sigma_\lambda$  we need  $p(\alpha) > p(\beta)$  for a nontrivial decomposition, or equivalently  $\sum_{i=1}^t d_i^2 < \sum_{i=1}^t e_i^2$ . By including additional terms of the form  $\gamma^{(jk)}$  in the decomposition, we may reorder the  $e_i$ , and hence assume that  $e_j \leq e_\ell$  whenever  $\xi_j = \xi_\ell$ . This will reduce the size of  $\beta$ , but not change the value of  $\sum_{i=1}^t e_i^2$ . We now partition the sums according to the value of  $\xi_i$ . If  $\mu \in K$  and  $\{i \mid \xi_i = \mu\} = \{i_1, i_2, \dots, i_p\}$  with  $i_1 < i_2 < \dots < i_p$ , then  $d_{i_{p-q}}$  is the number of Jordan blocks of eigenvalue  $\mu$  of size  $> q$ , so  $d_{i_1} \leq d_{i_2} \leq \dots \leq d_{i_p}$ . It follows that  $\sum_{j=1}^p d_{i_j}^2 \leq \sum_{j=1}^p e_{i_j}^2$ , strict if not all  $e_{i_j} = d_{i_j}$ . This is just a numerical condition, but to avoid writing out a proof, observe that it is implicit in the Gerstenhaber-Hesselink Theorem (see [18, §1.3]).  $\square$

*Remark 9.2.* In the definition  $N_Q(\lambda, \alpha) = \mu_\alpha^{-1}(\lambda) // GL(\alpha)$  we have only considered Marsden-Weinstein reductions associated to fixed points of the dual of the Lie algebra. One could more generally consider  $\mu_\alpha^{-1}(\bar{C}) // GL(\alpha)$  for an arbitrary coadjoint orbit  $C$ . Here, by taking the closure of the coadjoint orbit, the inverse image is an affine variety, so the quotient is well-defined. However, the coadjoint orbit is determined by fixing a conjugacy class in  $Mat(\alpha_i, K)$  for each vertex  $i$ , and by using Lemma 9.1 one sees that the more general Marsden-Weinstein reduction is isomorphic  $N_{Q'}(\lambda', \alpha')$  for a quiver  $Q'$  which is obtained from  $Q$  by attaching a linear quiver to each vertex of  $Q$ .

The same comment applies to the Marsden-Weinstein reduction  $N(A, M, \omega, \zeta)$  considered in Section 3.

Actually this construction is already implicit in [27, §1], where  $Q$  consists of one vertex and a single loop, and  $Q'$  is obtained by attaching a new vertex to the original one by a single edge.

## References

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