Math. Ann. 324, 521–556 (2002) **Mathematische Annalen**

DOI: 10.1007/s00208-002-0351-3

Harnack inequalities and sub-Gaussian estimates for random walks

Alexander Grigor'yan · András Telcs

Received: 15 November 2001 / Revised version: 21 February 2002 /
Published online: 6 August 2002 –

Abstract. We show that the β-parabolic Harnack inequality for random walks on graphs is equivalent, on one hand, to the sub-Gaussian estimate for the transition probability and, on the other hand, to the conjunction of the elliptic Harnack inequality, the doubling volume property, and the fact that the mean exit time in any ball of radius R is of the order R^β. The latter condition can be replaced by a certain estimate of the resistance of annuli.

1 Introduction

In 1986, P.Li and S.-T.Yau [27] proved the following remarkable Gaussian estimate for the heat kernel $p_t(x, y)$ on any complete Riemannian manifold M with non-negative Ricci curvature:

$$p_t(x, y) \simeq \frac{1}{V(x, \sqrt{t})} \exp\left(-\frac{d^2(x, y)}{ct}\right). \tag{1.1}$$

Here $d(x, y)$ is the geodesic distance between points $x, y \in M$, $V(x, r)$ is the Riemannian volume of a geodesic ball $B(x, r)$ of radius r centered at x; the sign $\simeq$ means that the ratio of the two sides of (1.1) is bounded from above and below by two positive constants, for all $x, y \in M$ and $t > 0$ (the value of the constant c may be different for the upper and the lower bounds). This estimate reflects the fact that the Brownian motion X_t on the manifold M travels at distance $\approx \sqrt{t}$ over time t.

A. Grigor'yan
Department of Mathematics, London SW7 2BZ, UK 662
(e-mail: a.grigoryan@ic.ac.uk)

A. Telcs
IMC, Graduate School of Business, Zrinyi u. 14, Budapest, 1051, Hungary
(e-mail: h197tel@ella.hu)

Partially supported by a visiting grant of the London Mathematical Society.

Many efforts were made to understand what geometric properties of the manifold are responsible for (1.1). The first author [18] and L.Saloff-Coste [31] obtained (1.1) under the assumption that M satisfies *the volume doubling property*: for all $x \in M$ and $R > 0$

$$V(x, 2R) \leq CV(x, R), \tag{VD}$$

and a certain *Poincaré inequality*. Moreover, L.Saloff-Coste proved that (VD) and the Poincaré inequality are equivalent to (1.1).

For the purposes of the present paper it is important to mention that (1.1) is equivalent to a *parabolic Harnack inequality* (see for example [15]) which says the following: for any non-negative function $u(x,t)$ solving the heat equation $\frac{\partial u}{\partial t} = \Delta u$ in a cylinder $B(z, 2R) \times (0, 4R^2)$,

$$\sup_{B(z,R)\times(R^2,2R^2)} u(x,t) \leq C \inf_{B(z,R)\times(3R^2,4R^2)} u(x,t), \tag{PH_2}$$

where the constant C does not depend on z, R. Clearly, the same space/time scaling $time \approx distance^2$ appears here as well.

The parabolic Harnack inequality implies the *elliptic Harnack inequality*: for any non-negative harmonic function $u(x)$ in a ball $B(z, 2R)$,

$$\sup_{B(z,R)} u \leq C \inf_{B(z,R)} u. \tag{H}$$

Historically, the first significant results on elliptic and parabolic Harnack inequalities are due to J.Moser [28], [29], who proved them for solutions of uniformly elliptic and parabolic equations in the divergence form in $\mathbb{R}^n$. In the geometric terms the results of Moser mean that (PH_2) and (H) hold on any Riemannian manifold quasi-isometric to $\mathbb{R}^n$.

Moser's method of proving Harnack inequalities as well as the competing methods ([15], [18], [25], [31] and many others) yield (PH_2) and (H) under the same set of assumptions about M. For quite a long time it was not clear if (H) was actually weaker than (PH_2). A solution to this problem arose from a different field – analysis on fractals.

By fractals we mean sets like a Sierpinski carpet, which can be constructed by using certain self-similar procedures. Normally fractals can be equipped with a distance function d, a (Hausdorff) measure μ, and an energy functional $\mathcal{E}$. Denote by M such a metric-measure-energy space. Using methods of abstract potential theory, one defines a Hunt process X_t on M (see [16]), which in many interesting cases happens to be diffusion with a continuous heat kernel $p_t(x, y)$ (see [1]).

M. Barlow and E. Perkins [8], M. Barlow and R. Bass [3], [4], [5] showed that, for a large variety of fractal sets, the following heat kernel estimate takes place:

$$p_t(x, y) \simeq \frac{1}{t^{\alpha/\beta}} \exp\left(-\left(\frac{d^\beta(x, y)}{ct}\right)^{\frac{1}{\beta-1}}\right), \tag{1.2}$$

for a certain natural range of x, y, t. Here α, β are parameters related by $2 \leq \beta \leq \alpha + 1$. M. Barlow and R. Bass [5] also showed that on such fractals a β-*parabolic Harnack inequality* (PH_β) holds. This inequality is a generalization of (PH_2) which one obtains by replacing everywhere R^2 by R^β. The existence of spaces satisfying (PH_β) with $\beta > 2$ proves that (H) is actually weaker than (PH_2) because $(PH_\beta) \Longrightarrow (H)$ for *any* β whereas (PH_β) are not equivalent for different β. However, a new interesting question arises:

What is "the difference" between (PH_β) and (H)?

This question is very much related to obtaining criteria for the following sub-Gaussian estimate of the heat kernel:

$$p_t(x, y) \simeq \frac{1}{V(x, t^{1/\beta})} \exp\left(-\left(\frac{d^\beta(x, y)}{ct}\right)^{\frac{1}{\beta-1}}\right). \tag{1.3}$$

In the case of fractals satisfying (1.2) one has $V(x, r) \simeq r^\alpha$. Hence, (1.3) unifies (1.1) and (1.2). Observe that the parameter β governs the space/time scaling by the relation

$$time = distance^\beta.$$

In the case $\beta > 2$, the propagation of X_t is slower than in the Gaussian case $\beta = 2$.

The present paper answers the above questions in the setting of random walks on graphs (it is well understood that a large scale behavior of random walks exhibits the same phenomena as that of diffusions on manifolds or fractals). Note first that the volume doubling condition (VD) and the elliptic Harnack inequality (H) are necessary for (PH_β). These two conditions ensure a certain homogeneity of the space M but neither of them contains a parameter β. To recover β one needs a third condition. Let $E(x, R)$ be the *mean exit time* of the process X_t from the ball $B(x, R)$ provided $X_0 = x$. For example in $\mathbb{R}^n$ we have $E(x, R) \simeq R^2$. Consider the hypothesis

$$E(x, R) \simeq R^\beta. \tag{E_β}$$

Then our main result says that (in the setting of random walks on infinite graphs)

$$(1.3) \Longleftrightarrow (PH_\beta) \Longleftrightarrow (VD) + (H) + (E_\beta).$$

Alternatively, the condition (E_β) can be replaced by a certain estimate of the *resistance* of annuli (see Sections 2, 3 below). In many interesting cases, (VD) and $\left(E_\beta\right)$ (or the resistance condition) can be effectively verified. However, verifying the elliptic Harnack inequality (H) may be very difficult, and it is still an open problem to find optimal criteria for (H).

Technically speaking, the main novelty of this paper is in obtaining the *on-diagonal upper estimate*

$$p_t(x,x) \le \frac{C}{V(x,t^{1/\beta})} \tag{DUE_β}$$

from $(VD)+(H)+(E_\beta)$. For an off-diagonal upper bound as well as for a lower bound, we use previously known arguments. The major difficulty in proving (DUE_β) is that the right hand side depends on x. A celebrated method of Nash [30] that was successfully applied for proving x-independent estimates like

$$p_t(x,x) \le f(t) \tag{1.4}$$

(see for example [35], [9], [19], [11]), does not work in our case. The estimate of Li and Yau (1.1) can be proved using a certain parabolic mean value inequality (see [19], [26]). However, all known proofs of the latter are too much linked to the classical Gaussian time/space scaling and do not work in the case $\beta > 2$. This difficulty can be overcome in the strongly recurrent case by using the resolvent method – see for example [8], [33], [34] (note for comparison that $\mathbb{R}^n$ is strongly recurrent only for $n = 1$).

To prove (DUE_β) in the full generality, we develop in this paper a new method (partly inspired by [1] and [33]) that is based on a certain estimate (5.31) for *λ-polyharmonic functions*. This estimate can be considered as an extension of the maximum principle for harmonic functions, and it is proved by virtue of a higher order Feynman-Kac formula (5.25) (see Sect. 5).

As we have already mentioned, we prove all the results in the framework of random walks on infinite graphs. We believe that a certain modification of our method will work also in the setting of diffusions on manifolds and fractals. We intend to return to this problem elsewhere.

In Sect. 2 we describe the framework and give the background material. In Sect. 3 we state the main Theorem 3.1 and discuss some consequences and examples. Sections 4 – 6 contain the proofs. We use the letters c, C to denote positive constants whose values are unimportant and may change at each occurrence.

Acknowledgements. We are grateful to Martin Barlow and Thierry Coulhon for useful discussions. We are indebted to the unnamed referee for a number of suggestions that helped to improve the paper.

2 Preliminaries

The results cited here can be found in many places (cf. [10], [12], [22]).

Measure and distance. Let Γ be an infinite connected locally finite graph endowed with a *weight* μ_{xy}. The latter is a symmetric non-negative function on $\Gamma \times \Gamma$ such that $\mu_{xy} > 0$ if and only if x and y are connected by an edge (in which case we write $x \sim y$). The weight μ_{xy} on edges induces a weight $\mu(x)$ on vertices and hence a measure μ on subsets $A \subset \Gamma$ defined by

$$\mu(x) := \sum_{y:y\sim x} \mu_{xy} \quad \text{and} \quad \mu(A) := \sum_{x\in A} \mu(x).$$

Let $d(x, y)$ be the graph distance between the points $x, y \in \Gamma$, that is the minimal number of edges in any edge path connecting x and y. Denote metric balls and their measures as follows:

$$B(x, R) := \{y \in \Gamma : d(x, y) < R\} \quad \text{and} \quad V(x, R) := \mu(B(x, R)).$$

A weighted graph (Γ, μ) satisfies the *volume doubling property* if

$$V(x, 2R) \le C V(x, R) \quad \forall x \in \Gamma,\ \forall R > 0 \tag{VD}$$

for some constant C. It is known that (VD) self-improves to the following inequality

$$\frac{V(x, R)}{V(y, r)} \le C \left(\frac{d(x, y) + R}{r}\right)^{\alpha}, \tag{2.1}$$

for all pairs of balls $B(x, R)$ and $B(y, r)$ such that $r \le R$ and for some $\alpha, C > 0$.

Random walk. The main object of our study is a random walk X_n on Γ, which is a reversible Markov chain with respect to μ, defined by the following one-step transition probability

$$P(x, y) = \frac{\mu_{xy}}{\mu(x)}.$$

We also consider $P(x, y)$ as a Markov operator which acts on functions on Γ by

$$Pf(x) = \sum_{y} P(x, y) f(y).$$

For any non-negative integer n, the n-step transition probability P_n is defined by $P_n(x, y) = \mathbb{P}_x(X_n = y)$. Alternatively, P_n is the n-th convolution power of the operator P. Define the *heat kernel* of (Γ, μ) by

$$p_n(x, y) = \frac{P_n(x, y)}{\mu(y)}. \tag{2.2}$$

It is easy to see that $p_n(x,y) = p_n(y,x)$, and for any $y \in \Gamma$, the function $u_n(x) = p_n(x,y)$ satisfies the *heat equation*

$$u_{n+1}(x) - u_n(x) = \Delta u_n. \tag{2.3}$$

Here Δ is the Laplace operator of the graph (Γ, μ) defined by $\Delta = P - \mathrm{I}$, that is

$$\Delta u(x) = \sum_y P(x,y)u(y) - u(x) = \frac{1}{\mu(x)} \sum_y (u(y) - u(x))\, \mu_{xy}.$$

For any parameter $\beta > 1$, consider the following estimates of the heat kernel, which in general may be true or not: for all $x, y \in \Gamma$

$$p_n(x,y) \le \frac{C}{V(x, n^{1/\beta})} \exp\left[-\left(\frac{d(x,y)^\beta}{Cn}\right)^{\frac{1}{\beta-1}}\right], \ \forall n \ge 1, \tag{UE_β}$$

$$(p_n + p_{n+1})(x,y) \ge \frac{c}{V(x, n^{1/\beta})} \exp\left[-\left(\frac{d(x,y)^\beta}{cn}\right)^{\frac{1}{\beta-1}}\right], \ \forall n \ge d(x,y) \vee 1. \tag{LE_β}$$

The conjunction $\left(UE_\beta\right) + \left(LE_\beta\right)$ is a discrete analogue of the estimate (1.3) from the Introduction. The obvious distinctions between (UE_β) and $\left(LE_\beta\right)$ – the restriction $n \ge d(x,y)$ and the term $p_n + p_{n+1}$ instead of p_n – reflect the discreteness of the time (see [22, Section 14]). Although a priori β is any number > 1, in fact $\left(UE_\beta\right) + \left(LE_\beta\right)$ imply $\beta \ge 2$.

In some parts of this paper, we assume that the following condition is satisfied: for some positive p_0

$$P(x,y) \ge p_0 \quad \text{for all } x \sim y. \tag{p_0}$$

In particular, (p_0) implies that each point $x \in \Gamma$ has a uniformly bounded number of edges.

Green function and killed random walk. For any finite non-empty subset $A \subset \Gamma$, denote by $c_0(A)$ the set of all functions on A extended to be 0 outside A. Define operator Δ^A on $c_0(A)$ as follows: for any $f \in c_0(A)$

$$\Delta^A f(x) = \begin{cases} \Delta f(x), & x \in A, \\ 0, & x \notin A. \end{cases}$$

The operator $-\Delta^A$ is positive definite and has the inverse $G^A = \left(-\Delta^A\right)^{-1}$ which is called the *Green function*. Respectively, the *Green kernel* is defined by

$$g^A(x,y) = \frac{G^A(x,y)}{\mu(y)}.$$

Note that $g^A(x, y)$ is non-negative and vanishes if one of the points x, y is outside A. Also, $g^A(x, y) = g^A(y, x)$, and $g^A(\cdot, y)$ satisfies the equation

$$\Delta g^A = -\delta_y$$

where $\delta_y(x) = 0$ if $x \neq y$ and $\delta_y(y) = 1/\mu(y)$.

Let P^A be restriction of the Markov operator P to the set A. The iterates of P^A are denoted by P_n^A and define the random walk X_n^A with the killing condition outside A. Then we have

$$G^A(x, y) = \sum_{n=0}^{\infty} P_n^A(x, y) \quad \text{and} \quad g^A(x, y) = \sum_{n=0}^{\infty} p_n^A(x, y),$$

where p_n^A is the density of P_n^A.

Mean exit time. The exit time from a set $A \subset \Gamma$ is defined as

$$T_A := \min\{n \geq 0 : X_n \notin A\}.$$

Its expectation $\mathbb{E}_x(T_A)$ is called the *mean exit time* and will be denoted also by $E_x(A)$. If $A = B(x, R)$ then we write

$$T_{x,R} := T_{B(x,R)} \quad \text{and} \quad E(x, R) := \mathbb{E}_x \left(T_{B(x,R)}\right).$$

The mean exit time is related to the Green function by the identity

$$E_x(A) = \sum_y G^A(x, y) = \sum_y g^A(x, y)\mu(y). \tag{2.4}$$

Denote

$$\overline{E}(A) := \sup_z E_z(A)$$

and

$$\overline{E}(x, R) := \overline{E}\left(B(x, R)\right) = \sup_{z \in B(x,R)} \mathbb{E}_z \left(T_{B(x,R)}\right). \tag{2.5}$$

The following two conditions will be frequently used:

$$E(x, R) \simeq R^{\beta}, \quad \text{for all } x \in \Gamma \text{ and } R \geq 1, \tag{E_β}$$

and

$$\overline{E}(x, R) \leq CE(x, R), \quad \text{for all } x \in \Gamma \text{ and } R > 0. \tag{$\overline{E}$}$$

If $\left(E_\beta\right)$ is satisfied then we have also

$$\overline{E}(x,R) \simeq R^{\beta}, \quad \text{for all } x \in \Gamma \text{ and } R \geq 1, \tag{2.6}$$

which in particular implies $(\overline{E})$ (see [22, Proposition 6.1]).

Resistance. For any function f on Γ define its energy by

$$\mathcal{E}(f) = \frac{1}{2} \sum_{x,y:\, x\sim y} (f(x) - f(y))^2 \mu_{xy}.$$

For any two sets $A \subset B \subset \Gamma$ the *resistance* $\rho(A,B)$ is defined by

$$\rho^{-1}(A,B) = \inf\left\{\mathcal{E}(f) : f|_A = 1 \text{ and } f|_{B^c} = 0\right\}, \tag{2.7}$$

where $B^c := \Gamma \setminus B$. The following condition is an analog of (E_β) for resistance:

$$\rho\left(B(x,R), B(x,MR)\right) \simeq \frac{R^{\beta}}{V(x,R)}, \quad \text{for all } x \in \Gamma \text{ and } R \geq 1, \tag{ρ_β}$$

for some fixed (large) number M whose value is unimportant.

Harnack inequalities. A function u is said to be *harmonic* in a set $A \subset \Gamma$ if u is defined in $\overline{A}$ (that consists of all points in A and all their neighbors) and if $\Delta u(x) = 0$ for any $x \in A$.

Definition 2.1. We say that (Γ, μ) satisfies an *elliptic Harnack inequality* if, for all $x \in \Gamma$, $R > 0$, and for any non-negative harmonic function u in $B(x, 2R)$, the following inequality holds

$$\max_{B(x,R)} u \leq C \min_{B(x,R)} u. \tag{H}$$

Definition 2.2. Given $\beta > 1$, we say that (Γ, μ) satisfies a β*-parabolic Harnack inequality* if, for all $x \in \Gamma$, $R \geq 1$ and for any non-negative function $u_n(y)$ defined for $n \in [0, 4N]$, $y \in \overline{B(x, 2R)}$ and satisfying the heat equation (2.3) in $[0, 4N) \times B(x, 2R)$, the following inequality holds

$$\max_{\substack{n\in[N,2N) \\ y\in B(x,R)}} u_n(y) \leq C \min_{\substack{n\in[3N,4N) \\ y\in B(x,R)}} \left(u_n(y) + u_{n+1}(y)\right), \tag{PH_β}$$

where N is a positive integer such that $N \simeq R^{\beta}$ and $N \geq 2R$.

Since any harmonic function satisfies also the heat equation, $\left(PH_\beta\right) \Longrightarrow (H)$ for any β.

Remark 2.1. If the graph (Γ, μ) satisfies the condition (p_0) then the Harnack inequality (H) automatically holds for all balls with a *bounded* range of the radius R and for all non-negative *superharmonic* functions u in $B(x, 2R)$ (see [22, Proposition 3.2]); in this case the constant C in (H) depends on the upper bound of the radius. The main point of Definition 2.1 (and Definition 2.2) is that (H) (resp., $\left(PH_\beta\right)$) holds for arbitrarily large R with *the same* constant C.

3 Main result

Theorem 3.1. *If graph* (Γ, μ) *satisfies* (p_0) *then, for any* $\beta \geq 2$*, the following conditions are equivalent:*

(i) $(UE_\beta) + (LE_\beta)$
(ii) (PH_β)
(iii) $(VD) + (H) + (\rho_\beta)$
(iv) $(VD) + (H) + (E_\beta)$

Some of the implications in Theorem 3.1 were already known and are included here for completeness as for instance, $(i) \Longleftrightarrow (ii)$ – see [15], [13] for the case $\beta = 2$, and [23] for any β (see also [5]). The implication $(ii) \Longrightarrow (iii)$ was proved in [33, Section 7].

The implication $(iii) \Longrightarrow (iv)$ is proved in the present paper in Sect. 4.3. In the view of (ρ_β) and (E_β), this can be regarded as the proof of the following relation

$$E(x, R) \simeq \rho\left(B(x, R), B(x, MR)\right) V(x, R),$$

which we call *Einstein's relation*.

The major part of this paper is devoted to the proof of (UE_β) in the implication $(iv) \Longrightarrow (i)$. The proof consists of many steps and goes via an upper estimate (G_β) for the resolvent and a diagonal upper bound (DUE_β) for the heat kernel as is shown on the diagram:

$$\left.\begin{matrix}(VD)\\(H)\\(E_\beta)\end{matrix}\right\} \overset{Thm.5.9}{\Longrightarrow} \left.\begin{matrix}(VD)\\(G_\beta)\end{matrix}\right\} \overset{Thm.6.1}{\Longrightarrow} \left.\begin{matrix}(VD)\\(DUE_\beta)\\(E_\beta)\end{matrix}\right\} \overset{Thm.6.2}{\Longrightarrow} (UE_\beta)$$

(see also Sect. 6.3). The derivation of (LE_β) from $(VD) + (H) + (E_\beta)$ is practically contained in [22, Section 13] and is briefly outlined in Sect. 6.3.

Note that the equivalence

$$(PH_\beta) \Longleftrightarrow (VD) + (H) + (E_\beta)$$

is new even for $\beta = 2$. For the case $\beta = 2$, we have a further result. Denote by $\lambda(x, R)$ the smallest eigenvalue of $-\Delta^{B(x,R)}$ and consider the following hypothesis:

$$\lambda(x, R) \geq \frac{c}{R^2}, \quad \text{for all } x \in \Gamma \text{ and } R \geq 1. \tag{λ}$$

Corollary 3.2. *If* (Γ, μ) *satisfies* (p_0) *then* $(PH_2) \Longleftrightarrow (VD) + (H) + (\lambda)$.

Indeed, the direction " $\Longrightarrow$ " follows from Theorem 3.1(iv) using (2.6) and the inequality

$$\lambda_1(x, R) \geq \overline{E}(x, R)^{-1}$$

(see [22, Propositions 6.1,6.2]), whereas the direction " $\Longleftarrow$ " follows from Theorem 3.1(iii) and the inequalities

$$\frac{R^2}{V(x, 2R)} \leq \rho(B(x, R), B(x, 2R)) \leq \frac{1}{\lambda(x, 2R)V(x, R)}$$

(see [33]).

The proof of Theorem 3.1 contains the following new implication (see Remark 6.4):

$$(VD) + (MV) + (E_\beta) \Longrightarrow \left(UE_\beta\right), \tag{3.1}$$

where (MV) stands for the elliptic mean value inequality defined in Sect. 5.2. It is interesting to recall that (UE_2) is equivalent to (VD)+a *parabolic* version of (MV) (see [12], [26]). Theorem 3.1 recovers also the results obtained in [22] and [33].

In conclusion of this section, we give some examples highlighting the relations between the hypotheses in questions.

Example 3.1. Let us observe that $(VD) + (H)$ does not imply (E_β) since there are graphs with $(VD) + (H) + (E_\beta)$ with different β (see for example [2], [6], [24]).

Let us sketch an example where $(VD) + (E_\beta)$ holds but (H) does not. Indeed, take two copies Γ_1 and Γ_2 of the same graph that satisfies $(VD) + (H) + (E_\beta)$ and such that the random walk on it is transient. Create a new graph Γ whose vertex set is the disjoint union of those of Γ_1 and Γ_2, and the edge set consists of those of Γ_1 and Γ_2 plus one additional edge connecting a vertex in Γ_1 to the corresponding vertex in Γ_2. Then it is not difficult to prove that Γ satisfies $(VD) + (E_\beta)$ but not (H) (see [21] for a manifold analogue of this construction). In addition, it is possible to prove that (UE_β) holds on Γ; hence (LE_β) must fail.

Probably, $(H) + \left(E_\beta\right)$ does not imply (VD) but we do not have an example for that.

Example 3.2. Here we describe a weighted graph satisfying $(VD) + (H) + (E_\beta)$ and such that $V(x, R)$ substantially depends on x. Let Γ be the Vicsek tree (embedded in $\mathbb{R}^2$) that is the union of the increasing sequence of blocks $\{\Gamma_k\}_{k=1}^{\infty}$ – see Fig. 1. Here $\Gamma_0 = \{o\}$, and Γ_{k+1} consists of Γ_k and its four copies translated and glued in an obvious way.

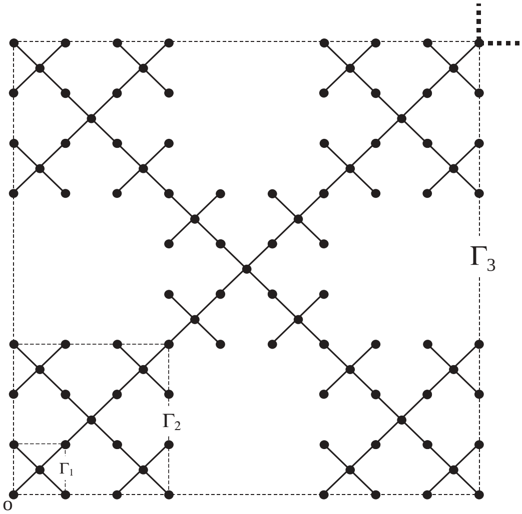

Fig. 1. Blocks $\Gamma_1, \Gamma_2, \Gamma_3, \ldots$ in the Vicsek tree

Fix $a \geq 1$ and define weight μ_{xy} for any edge $\overline{xy}$ by $\mu_{xy} = a^k$ where k is the minimal index such that Γ_k contains x or y. Since $d(x, o) \simeq 3^k$ for any $x \in \Gamma_k \setminus \Gamma_{k-1}$, this implies for all $x \neq o$,

$$\mu(x) \simeq d(x, o)^{\delta} \tag{3.2}$$

where $\delta = \log_3 a$.

Let x_k be the symmetry center of Γ_k and set $R_k = 3^{k-1} + \frac{1}{2}$; then $\Gamma_k = B(x_k, R_k)$. Clearly,

$$|B(x_k, R_k)| = |\Gamma_k| \simeq 5^k \simeq R_k^{\alpha}$$

where $|\cdot|$ is the cardinality of a set and $\alpha = \log_3 5$. It is not difficult to see that the same relation holds for all balls $B(x, R)$ in Γ with $R \geq 1$, that is

$$|B(x, R)| \simeq R^{\alpha}. \tag{3.3}$$

From (3.2) and (3.3), one easily obtains, for all $x \in \Gamma$ and $R \geq 1$,

$$V(x, R) = \mu\left(B(x, R)\right) \simeq R^{\alpha}\left(R + d\left(x, o\right)\right)^{\delta}, \tag{3.4}$$

which in particular proves (VD) for (Γ, μ).

Due to the tree structure of Γ, it is easy to compute the Green kernel $g^{\Gamma_k}(x_k, \cdot)$ (cf. [7, Section 4]). The set $\overline{\Gamma_k} \setminus \Gamma_k$ consists of a single point; denote it by z_k (see Fig. 2).

Let $g_k(y)$ be a function on Γ satisfying the following conditions:

- g_k vanishes outside Γ_k, in particular at z_k;

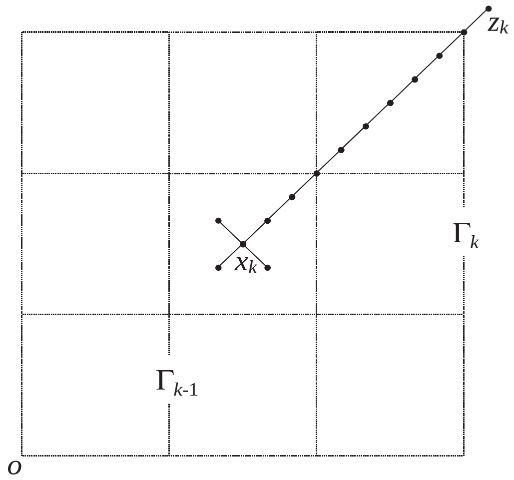

Fig. 2. Points x_k and z_k

- g_k increases linearly along the path from z_k to x_k with a constant increment c_k at each step;
- g_k remains constant along all other paths in Γ_k.

These conditions uniquely determine g_k, up to the choice of c_k. Clearly, g_k is harmonic in $\Gamma_k \setminus x_k$. At point x_k, we have $\Delta g_k = -c_k/4$. Therefore, if we take $c_k := 4/\mu(x_k) = a^{-k}$ then we obtain for all $y \in \Gamma_k$

$$\Delta g_k(y) = -\delta_{x_k}(y)$$

and hence $g_k \equiv g^{\Gamma_k}(x_k, \cdot)$.

For any point y on the paths from z_k to x_k, we have $g_k(y) = c_k d(y, z_k)$. In particular, for any $y \in B(x_k, \frac{1}{3}R_k)$,

$$g_k(y) \simeq \left(\frac{3}{a}\right)^k \quad \text{and} \quad g_k(y)\mu(y) \simeq 3^k.$$

Therefore, by (2.4)

$$E(x_k, R_k) = \sum_{y \in \Gamma_k} g_k(y)\mu(y) \simeq 3^k |\Gamma_k| \simeq 15^k \simeq R_k^\beta$$

where $\beta = \log_3 15$. It is easy to show that the same relation $E(x, R) \simeq R^\beta$ holds for all $x \in \Gamma$ and $R \geq 1$, which proves (E_β).

The Green kernel $g_k = g^{\Gamma_k}(x_k, \cdot)$ constructed above is nearly radial[1]. A similar argument shows that the same is true for all balls in Γ, which implies (H) by [22, Proposition 10.1].

[1] More precisely this means that the Green kernel satisfies a certain condition (HG) described below in Sect. 4.1.

Hence, (Γ, μ) satisfies $(VD) + (H) + \left(E_\beta\right)$ and, by Theorem 3.1, (Γ, μ) satisfies $\left(UE_\beta\right) + \left(LE_\beta\right)$ with the volume function (3.4).

4 Harnack inequality and resistance

4.1 Harnack inequality for Green function

Definition 4.1. We say (Γ, μ) satisfies a *ball covering* property (BC) if, for all $\varepsilon > 0$, any ball $B(x, R)$ can be covered by $N = N(\varepsilon)$ balls of radii εR.

It is well known that (VD) implies (BC).

Definition 4.2. We say that (Γ, μ) satisfies a *Harnack inequality for the Green function* if, for some (large) constants M, C, for all $x \in \Gamma$ and $R > 0$ and for any finite set $U \supset B(x, MR)$,

$$\sup_{y \notin B(x,R)} g^U(x, y) \leq C \inf_{z \in B(x,R)} g^U(x, z). \tag{HG}$$

It is possible to show that $(HG) \Longrightarrow (H)$ (cf. [22, Proposition 10.1]). Here we need a converse statement.

Proposition 4.1. *Assume that* (Γ, μ) *satisfies* (p_0). *Then* $(BC) + (H) \Longrightarrow (HG)$. *In particular,* $(VD) + (H) \Longrightarrow (HG)$.

The main part of the proof is contained in the following lemma.

Lemma 4.2. *Assume that* (Γ, μ) *satisfies* (p_0), (BC), (H). *Let* y, z *be two points in a ball* $B(x, R)$ *such that the shortest path*

$$y = \xi_0 \sim \xi_1 \sim ... \sim \xi_k = z \tag{4.1}$$

connecting y *and* z *in* Γ, *does not intersect* $B(x, \varepsilon R)$ *for some* $\varepsilon \in (0, 1)$. *Then, for any finite set* U *containing* $B(x, 3R)$,

$$g^U(x, y) \leq C_\varepsilon g^U(x, z). \tag{4.2}$$

Proof. If R is in a bounded range then (4.2) follows from the hypothesis (p_0) (cf. Remark 2.1 or [22, Proposition 3.2]). Assume in the sequel that R is large enough, and observe that for any ξ_i

$$d(x, \xi_i) \leq d(x, y) + d(y, \xi_i) \quad \text{and} \quad d(x, \xi_i) \leq d(x, z) + d(\xi_i, z),$$

whence

$$d(x, \xi_i) \leq \frac{d(x, y) + d(x, z) + d(y, z)}{2} \leq d(x, y) + d(x, z) < 2R.$$

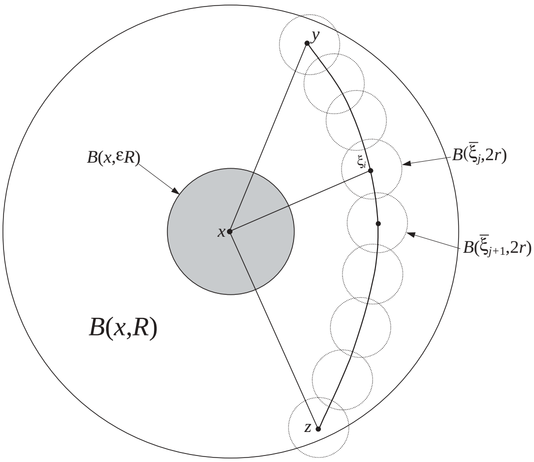

Fig. 3. Covering the path from y to z by balls $B(\overline{\xi}_j, 2r)$

In particular, we have $\xi_i \in B(x, 2R)$.

By (BC), the ball $B(x, 2R)$ can be covered by at most $N = N(\varepsilon/16)$ balls of radius $r = \frac{\varepsilon}{8}R$. Select out of them only those balls which contain at least one point ξ_i, and denote their centers by o_j, $j = 0, 1, ..., n$, where $n \le N$. Among the points ξ_i which belong to the ball $B(o_j, r)$ select one and denote it by $\overline{\xi}_j$, making sure that the points $y = \xi_0$ and $z = \xi_k$ are selected. Since the balls $B(o_j, r)$ cover the path (4.1) and $B(o_j, r) \subset B(\overline{\xi}_j, 2r)$, the balls $B(\overline{\xi}_j, 2r)$ also cover this path. Let us rearrange the points $\overline{\xi}_j$ in the order of increasing $d(y, \overline{\xi}_j)$ (see Fig. 3). Then $y = \overline{\xi}_0$, $z = \overline{\xi}_n$, and

$$d(\overline{\xi}_j, \overline{\xi}_{j+1}) < 4r, \tag{4.3}$$

for all $0 \le j < n$.

By hypothesis, we have

$$d(x, \overline{\xi}_j) \ge \varepsilon R = 8r,$$

so that the point x is outside the ball $B(\overline{\xi}_j, 8r)$. Since $\overline{\xi}_j \in B(x, 2R)$ and $8r < R$, we see that

$$B(\overline{\xi}_j, 8r) \subset B(x, 3R) \subset U.$$

Hence, the function $g^U(x, \cdot)$ is harmonic in $B(\overline{\xi}_j, 8r)$. By (4.3), we have $\overline{\xi}_{j+1} \in B(\overline{\xi}_j, 4r)$, and the Harnack inequality (H) applied in the ball $B(\overline{\xi}_j, 8r)$, yields

$$g^U(x, \overline{\xi}_j) \le C g^U(x, \overline{\xi}_{j+1}).$$

Iterating this inequality n times, we obtain (4.2).

Proof of Proposition 4.1. The following argument is due to T.Delmotte (private communication). Let y and z be the points where the sup and inf in (HG) are attained, respectively. As follows from the maximum principle,

$$y \in B(x, R+1) \setminus B(x, R) \quad \text{and} \quad z \in B(x, R) \setminus B(x, R-1).$$

We need to prove that

$$g^U(x, y) \le C g^U(x, z) \tag{4.4}$$

where $U \supset B(x, MR)$ for some M. In fact, any $M \ge 11$ will do as we will see below.

For a bounded range of R, (4.4) follows from (p_0) (cf. Remark 2.1). Assume in the sequel that R is large enough, connect y and z by the shortest path in Γ as in (4.1), and consider two cases.

Case 1. All points ξ_i are outside $B(x, \frac{1}{4}R)$. Then (4.4) follows from Lemma 4.2 since $y, z \in B(x, 2R)$ and $U \supset B(x, 6R)$.

Case 2. One of the points ξ_i is in the ball $B(x, \frac{1}{4}R)$. Let us show that the shortest path connecting x and z in Γ does not intersect $B(y, \frac{1}{4}R)$. Indeed, set $\xi = \xi_i$ so that $d(x, \xi) < \frac{1}{4}R$. By the triangle inequality, we have

$$\begin{aligned} d(y, z) &= d(y, \xi) + d(z, \xi) \\ &\ge (d(x, y) - d(x, \xi)) + (d(x, z) - d(x, \xi)) \\ &> d(x, y) + d(x, z) - \frac{R}{2}, \end{aligned}$$

whence

$$d(y, z) - d(x, z) > d(x, y) - \frac{R}{2} \ge \frac{R}{2} > \frac{R}{4}. \tag{4.5}$$

For any vertex η on the shortest path between x and z, we obtain by (4.5)

$$d(y, \eta) \ge d(y, z) - d(\eta, z) \ge d(y, z) - d(x, z) > \frac{R}{4},$$

which means that the shortest path from x to z lies outside the ball $B(y, \frac{1}{4}R)$. Since $x, z \in B(y, 3R)$ and

$$U \supset B(x, 11R) \supset B(y, 9R),$$

we obtain by Lemma 4.2

$$g^U(y, x) \le C g^U(y, z).$$

Similarly, connecting x to y, we obtain

$$g^U(z, y) \le C g^U(z, x).$$

Multiplying these inequalities and using the symmetry of the Green kernel, we obtain (4.4). □

4.2 Green function and resistance

Proposition 4.3. *Assume that the graph (Γ, μ) satisfies (p_0) and (HG). Then for any ball $B(x, R)$ and for any $0 < r \le R/M$, we have*

$$\sup_{y \notin B(x,r)} g^{B(x,R)}(x, y) \simeq \rho(B(x, r), B(x, R)) \simeq \inf_{y \in B(x,r)} g^{B(x,R)}(x, y). \tag{4.6}$$

Proof. For an arbitrary graph (Γ, μ), the following is true: if A, B are finite subsets of Γ such that $\overline{A} \subset B$ then for any $x \in A^o$

$$\sup_{y \notin A^o} g^B(x, y) \ge \rho(A, B) \ge \inf_{y \in \overline{A}} g^B(x, y), \tag{4.7}$$

where A^o consists of the points in A which have neighbors only in A (see [20, Proposition 4.1] for a continuous version of (4.7)). Applying (4.7) for $A = B(x, r)$, $B = B(x, R)$ and combining with (HG) and (p_0) we obtain (4.6).

Proposition 4.4. *Assume that the graph (Γ, μ) satisfies (p_0) and (HG). Fix any ball $B(x, r)$ and denote $B_k = B(x, M^k r)$ for $k = 0, 1, \ldots$. Then for all integers $n > m \ge 0$,*

$$\sup_{y \notin B_m} g^{B_n}(x, y) \simeq \sum_{k=m}^{n-1} \rho(B_k, B_{k+1}) \simeq \inf_{y \in B_m} g^{B_n}(x, y). \tag{4.8}$$

Proof. The following general property of resistance follows directly from the variational definition (2.7):

$$\sum_{k=m}^{n-1} \rho(B_k, B_{k+1}) \le \rho(B_m, B_n).$$

Together with Proposition 4.3, it implies the lower bound for $\inf g^{B_n}$ in (4.8).

To obtain the upper bound for $\sup g^{B_n}$ observe that the difference

$$g^{B_{k+1}}(x, \cdot) - g^{B_k}(x, \cdot)$$

is a harmonic function in B_k. The maximum principle implies for any $y \in \Gamma$

$$g^{B_{k+1}}(x, y) - g^{B_k}(x, y) \le \sup_{z \notin B_k} g^{B_{k+1}}(x, z).$$

By Proposition 4.3, we obtain

$$g^{B_{k+1}}(x, y) - g^{B_k}(x, y) \le C\rho(B_k, B_{k+1}). \tag{4.9}$$

For any $y \notin B_m$, Proposition 4.3 yields

$$g^{B_{m+1}}(x, y) \le C\rho(B_m, B_{m+1}). \tag{4.10}$$

For such y, adding up (4.10) with (4.9) for $m < k < n$, we obtain the upper bound of $\sup g^{B_n}$ in (4.8).

4.3 Proof of $(iii) \Longrightarrow (iv)$ in Theorem 3.1

We need to prove that

$$(VD) + (H) + \left(\rho_\beta\right) \Longrightarrow \left(E_\beta\right).$$

Using (2.4), (4.6), $\left(\rho_\beta\right)$ and (VD), we obtain for $r = R/M$

$$E(x, R) \geq \sum_{y \in B(x,r)} g^{B(x,R)}(x, y)\mu(y) \geq c\rho(B(x, r), B(x, R))V(x, r) \geq cR^\beta.$$

For the upper bound, denote $r_k = M^k$, $B_k = B(x, r_k)$ and let n be the minimal integer so that $R < r_n$. Then we have

$$\begin{aligned} E(x, R) \leq E(x, r_n) &= \sum_{y \in B_n} g^{B_n}(x, y)\mu(y) \qquad (4.11)\\ &= \sum_{y \in B_0} g^{B_n}(x, y)\mu(y) + \sum_{m=0}^{n-1} \sum_{y \in B_{m+1} \setminus B_m} g^{B_n}(x, y)\mu(y). \end{aligned}$$

As follows from (p_0) (see Remark 2.1) the first term in the right hand side of (4.11) – the sum over B_0 – is majorized by a multiple of a similar sum over $B_1 \setminus B_0$, which is a part of the second term. Estimating g^{B_n} by Proposition 4.3 and applying $\left(\rho_\beta\right)$, we obtain

$$\begin{aligned} E(x, R) &\leq C \sum_{m=0}^{n-1} \left[\sum_{k=m}^{n-1} \rho(B_k, B_{k+1})\right] \mu\left(B_{m+1} \setminus B_m\right)\\ &\leq C \sum_{k=0}^{n-1} \left[\sum_{m=0}^{k} \mu\left(B_{m+1} \setminus B_m\right)\right] \rho\left(B_k, B_{k+1}\right)\\ &\leq C \sum_{k=0}^{n-1} \mu(B_{k+1})\rho(B_k, B_{k+1}) \leq C \sum_{k=0}^{n-1} r_{k+1}^\beta \leq CR^\beta. \end{aligned}$$

5 The resolvent

5.1 Definition of λ, m-resolvent

For any non-empty finite set $B \subset \Gamma$, for all real $\lambda \geq 0$ and integer $m \geq 0$, define the λ, m-resolvent of Δ in B by

$$G^B_{\lambda,m} = \left(\lambda \mathrm{I} - \Delta^B\right)^{-m}. \qquad (5.1)$$

Since the spectrum of $-\Delta^B$ is strictly positive, the operator $G^B_{\lambda,m}$ is well-defined. It follows from the definition that $G^B_{\lambda,0} = \mathrm{I}$ and $G^B_{0,1} = G^B$, where G^B is the usual Green function in B. Clearly, we have also

$$G^B_{\lambda,m} = \left(G^B_{\lambda,1}\right)^m . \tag{5.2}$$

Since $\Delta^B = P^B - \mathrm{I}$, we obtain from (5.1)

$$G^B_{\lambda,m} = \left((\lambda+1)\,\mathrm{I} - P^B\right)^{-m} = \omega^m \left(\mathrm{I} - \omega P^B\right)^{-m} .$$

where $\omega = (\lambda+1)^{-1}$. The binomial formula yields

$$\begin{aligned} G^B_{\lambda,m} &= \omega^m \left(\mathrm{I} + m\omega P^B + \frac{m(m+1)}{2}\omega^2 \left(P^B\right)^2 + ...\right) \\ &= \sum_{n=0}^{\infty} Q_m(n)\omega^{n+m} P^B_n , \end{aligned} \tag{5.3}$$

where $Q_0(0) = 1$, $Q_0(n) = 0$ for $n \geq 1$, and for $m \geq 1$

$$Q_m(n) = \binom{n+m-1}{m-1} = \frac{(n+m-1)\,(n+m-2)\,...\,(n+1)}{(m-1)!}. \tag{5.4}$$

By (5.3) we extend the definition of $G^B_{\lambda,m}$ to infinite sets B. If $\lambda > 0$ then the series in (5.3) always converges. If $\lambda = 0$ then $G^B_{\lambda,m}$ may be equal to ∞ for infinite B.

The λ, m-resolvent has a symmetric kernel defined by

$$g^B_{\lambda,m}(x,y) = \frac{G^B_{\lambda,m}(x,y)}{\mu(y)} = \sum_{n=0}^{\infty} Q_m(n)\omega^{n+m} p^B_n(x,y). \tag{5.5}$$

If B is finite and $m \geq 1$ then (5.1) implies

$$(\Delta - \lambda)\, g^B_{\lambda,m} = -g^B_{\lambda,m-1} \quad \text{in } B. \tag{5.6}$$

5.2 Upper bound for $0, m$*-resolvent*

Definition 5.1. We say that the *mean-value inequality* (MV) holds on (Γ, μ) if, for any ball $B(x,r)$ and for any non-negative harmonic function u in $B(x,r)$,

$$u(x) \leq \frac{C}{V(x,r)} \sum_{y \in B(x,r)} u(y)\mu(y) . \tag{MV}$$

Clearly, $(H) + (VD) \Longrightarrow (MV)$ because

$$u(x) \leq C \inf_{B(x,r/2)} u \leq \frac{C}{V(x,r/2)} \sum_{y \in B(x,r/2)} u(y)\mu(y).$$

Lemma 5.1. *If (Γ, μ) satisfies $(VD) + (MV)$ then for all $x \in \Gamma$, $R > 0$, and $y \neq x$,*

$$g^{B(x,R)}(x, y) \leq C \frac{\overline{E}(x, 2R)}{V(x, d)}, \tag{5.7}$$

where $d = d(x, y)$.

Proof. If $d > R$ then $g^{B(x,R)}(x, y) = 0$ and there is nothing to prove. Otherwise, consider the function

$$u(z) := g^{B(x,2R)}(x, z).$$

This function is non-negative and harmonic in the ball $B(y, d) \subset B(x, 2R)$. Hence, by (MV), (2.4), and $(\overline{E})$, we obtain

$$u(y) \leq \frac{C}{V(y, d)} \sum_{z \in B(y,d)} u(z)\mu(z) \leq \frac{C}{V(x, d)} \overline{E}(x, 2R).$$

Finally, (5.7) follows from $g^{B(x,R)} \leq g^{B(x,2R)}$.

Lemma 5.2. *For any set $B \subset \Gamma$, for all integers $m \geq 0$ and reals $\lambda \geq 0$, we have*

$$\|G^B_{\lambda,m}\| \leq \overline{E}(B)^m, \tag{5.8}$$

where $\|G^B_{\lambda,m}\|$ is the operator norm in the space $c_0(B)$ endowed with the sup*-norm.*

Proof. Consider first the case $\lambda = 0$, $m = 1$ when we have $G^B_{0,1} = G^B$. For any $f \in c_0(B)$ and any $x \in B$, (2.4) and (2.5) imply

$$G^B f(x) = \sum_y G^B(x, y) f(y) \leq \sum_y G^B(x, y)\|f\| \leq \overline{E}(B)\|f\|,$$

where $\|f\| := \sup |f|$. Therefore, we obtain

$$\|G^B\| = \sup_{f \in c_0(B)\setminus\{0\}} \frac{\|G^B f\|}{\|f\|} \leq \overline{E}(B)\,. \tag{5.9}$$

Iterating (5.9) and using (5.2) we obtain

$$\|G^B_{0,m}\| \leq \|G^B_{0,1}\|^m \leq \overline{E}(B)^m,$$

which proves (5.8) for the case $\lambda = 0$ and any $m \geq 1$.

It easily follows from (5.5) that $g^B_{\lambda,m} \leq g^B_{0,m}$ which implies

$$\|G^B_{\lambda,m}\| \leq \|G^B_{0.m}\|,$$

whence (5.8) follows for any $\lambda \geq 0$.

Lemma 5.3. *If* (Γ, μ) *satisfies* $(VD)+(MV)$ *then for any* $m \geq 1$*, for all* $x \in \Gamma$*,* $R > 0$*, and* $y \neq x$*,*

$$g_{0,m}^{B(x,R)}(x, y) \leq C \frac{\overline{E}(x, 5R)^m}{V(x, d)}, \tag{5.10}$$

where $d = d(x, y)$ *and the constant* C *depends on* m *and on the constants from the hypotheses.*

Proof. The case $m = 1$ follows from Lemma 5.1, so we can assume $m \geq 2$ and argue by induction in m. Denote for simplicity $B = B(x, R)$, $G_m^B := G_{0,m}^B$ and $g_m^B := g_{0,m}^B$. Assuming $y \in B(x, R)$, let us set $r = d(x, y)/2$ and observe that the balls $B(x, r)$ and $B(y, r)$ do not intersect. Therefore, using $G_m^B = G_{m-1}^B \circ G_1^B$, we obtain

$$\begin{aligned} g_m^B(x, y) &= \sum_z g_{m-1}^B(x, z) g_1^B(z, y)\mu(z) \\ &\leq \left(\sum_{z \notin B(x,r)} + \sum_{z \notin B(y,r)} \right) g_{m-1}^B(x, z) g_1^B(z, y)\mu(z). \end{aligned}$$

Denoting

$$f(z) := g_1^B(z, y)\mathbf{1}_{\{z \notin B(y,r)\}} \quad \text{and} \quad h(z) := g_{m-1}^B\,(x, z)\,\mathbf{1}_{\{z \notin B(x,r)\}}$$

we obtain

$$g_m^B(x, y) \leq G_{m-1}^B f(x) + G_1^B h(y). \tag{5.11}$$

Since the Green kernels in questions are symmetric and increase with B, we obtain by (5.7) and (VD),

$$\begin{aligned} \|f\| = \sup_{z \notin B(y,r)} g_1^{B(x,R)}(z, y) &\leq \sup_{z \notin B(y,r)} g_1^{B(y,2R)}(y, z) \\ &\leq C \frac{\overline{E}(y, 4R)}{V(y, r)} \leq C \frac{\overline{E}(x, 5R)}{V(x, r)}, \end{aligned}$$

and by the inductive hypothesis

$$\|h\| = \sup_{z \notin B(x,r)} g_{m-1}^{B(x,R)}\,(x, z) \leq C \frac{\overline{E}(x, 5R)^{m-1}}{V(x, r)}.$$

Combining together (5.11), (5.8), and the above estimates for $\|f\|$, $\|g\|$, we obtain (5.10).

5.3 Upper bound for λ-harmonic functions

Lemma 5.4. *Assume that (Γ, μ) satisfies $\left(E_\beta\right)$. Let $B = B(x_0, R)$ be an arbitrary ball on Γ, and let f be a non-negative function in $\overline{B}$, which satisfies in B the equation $\Delta f - \lambda f = 0$ with a constant $0 < \lambda < 1$. Then*

$$f(x_0) \le C \exp\left(-c\lambda^{1/\beta} R\right) \max_{\overline{B}\setminus B} f, \tag{5.12}$$

where the constants $C, c > 0$ depend on the constants in hypothesis $\left(E_\beta\right)$.

This lemma was essentially proved in [22, Lemma 7.4]. Since it plays an important role in the proof of Theorem 3.1, we reproduce the proof below, with minor improvements.

Let us start with a weaker version of Lemma 5.4.

Lemma 5.5. *Assume that the hypothesis $(\overline{E})$ holds on (Γ, μ). Let $A = B(x_0, r)$ be an arbitrary ball on Γ, and let f be a non-negative function in $\overline{A}$, which satisfies in A the equation $\Delta f - \lambda f = 0$ with a constant λ such that*

$$\lambda \ge (\overline{E}_A)^{-1}. \tag{5.13}$$

Then

$$f(x_0) \le (1 - \varepsilon) \max_{\overline{A}} f, \tag{5.14}$$

where $\varepsilon > 0$ depends on the constants in hypothesis $(\overline{E})$.

Proof. Without loss of generality, we can assume $\max_{\overline{A}} f = 1$. As follows from (2.4), the function $u(x) := E_A(x)$ satisfies in A the equation $\Delta u = -1$; besides, u vanishes outside A. Set

$$\lambda_0 := (\overline{E}_A)^{-1} = \frac{1}{\max u},$$

and consider the function $w = 1 - \frac{\lambda_0}{2} u$. Clearly, we have $\frac{1}{2} \le w \le 1$ and

$$\Delta w = \frac{\lambda_0}{2} \le \lambda_0 w \le \lambda w \quad \text{in } A.$$

Since $f \le 1 = w$ in $\overline{A} \setminus A$, the comparison principle for the operator $\Delta - \lambda$ implies that $f \le w$ in A. In particular,

$$f(x_0) \le w(x_0) = 1 - \frac{\lambda_0}{2} u(x_0) \le 1 - \frac{u(x_0)}{2 \max u}.$$

The hypothesis $(\overline{E})$ implies

$$\frac{u(x_0)}{\max u} = \frac{E(x_0, r)}{\overline{E}(x_0, r)} \ge c,$$

whence (5.14) follows.

Proof of Lemma 5.4. By [22, Proposition 6.1], the hypothesis (E_β) implies $(\overline{E})$, which will enable us to use Lemma 5.5.

Without loss of generality, we can assume $\max_{\overline{B}\setminus B} f = 1$. The function f is subharmonic in B, which implies by the maximum principle that

$$\max_B f \le \max_{\overline{B}\setminus B} f = 1. \tag{5.15}$$

If $\lambda^{1/\beta} R$ is bounded by a (large but fixed) constant then the right hand side of (5.12) can be made > 1 just by adjusting the constant C. Since $f(x_0) \le 1$, in this case (5.12) is trivially satisfied.

Assume in the sequel that

$$\lambda^{1/\beta} R > 2K^{1/\beta}, \tag{5.16}$$

where $K > 1$ is a large enough constant to be chosen below. Since $\lambda < 1$, we obtain from (5.16)

$$R > 2K^{1/\beta} > 2$$

and

$$\lambda > K \left(\frac{R}{2}\right)^{-\beta} > K (R-1)^{-\beta}. \tag{5.17}$$

By the hypothesis $\left(E_\beta\right)$, we can choose the constant K so large that

$$E(x,r) \ge K^{-1} r^\beta \quad \text{for all } x \in \Gamma \text{ and } r \ge 1. \tag{5.18}$$

Find a number r from the equation

$$\lambda = K r^{-\beta}. \tag{5.19}$$

Clearly, we have for this r

$$r < R-1 \quad \text{and} \quad r = \left(\frac{K}{\lambda}\right)^{1/\beta} > 1.$$

It follows from (5.18) and (5.19) that

$$\lambda \ge \frac{1}{E(x,r)} \ge \frac{1}{\overline{E}(x,r)},$$

so that Lemma 5.5 applies in any ball of radius r.

For any $i = 1, 2, ...,$ let x_i be a point of maximum of the function f in the ball $B(x_0, ir')$ where $r' := r + 1$. Set $m_i = f(x_i)$ and $m_0 = f(x_0)$. Consider the ball $A_i = B(x_i, r)$ for any $i = 0, 1, 2, ..., k-1$, where

$$k := \lfloor R/r' \rfloor \ge 1$$

(see Fig. 4).

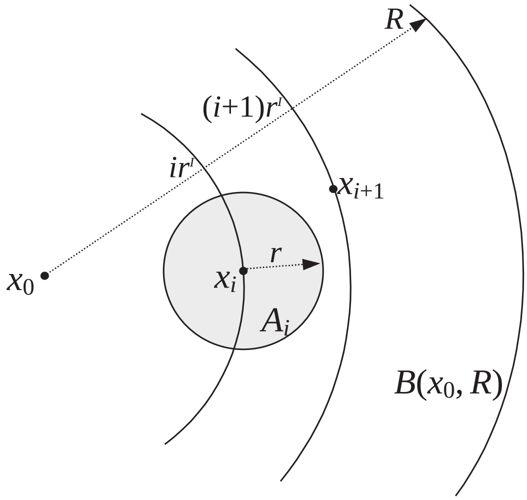

Fig. 4. The point x_i and the ball $A_i = B(x_i, r)$

Then we have

$$\overline{A}_i \subset B(x_i, r') \subset B(x_0, (i+1)r') \subset B(x_0, R),$$

whence we see that

$$\Delta f - \lambda f = 0 \text{ in } A_i \quad \text{and} \quad \max_{\overline{A}_i} f \leq m_{i+1}.$$

Applying Lemma 5.5 to the function f in the ball A_i, we obtain

$$m_i \leq (1-\varepsilon)m_{i+1}.$$

Iterating this inequality k times and using $m_k \leq 1$ (which follows from (5.15)) we conclude

$$f(x_0) = m_0 \leq (1-\varepsilon)^k. \tag{5.20}$$

Finally, by the choice of k and r, we have

$$k \simeq \frac{R}{r} \simeq \lambda^{1/\beta} R,$$

so that (5.20) implies (5.12). □

5.4 Feynman-Kac formula for λ-polyharmonic functions

Let f be a function on Γ such that

$$\Delta f - \lambda f = 0 \quad \text{in } B, \tag{5.21}$$

for a finite non-empty set $B \subset \Gamma$. Then the function $v = G^B_{\lambda,m} f$ satisfies

$$(\Delta - \lambda)^{m+1}\, v = 0 \quad \text{in } B \tag{5.22}$$

so that v_m is a λ*-polyharmonic function.* Moreover, v satisfies the following boundary conditions outside B:

$$v = 0,\ (\Delta - \lambda)\, v = 0,\ (\Delta - \lambda)^2\, v = 0,\ \ldots,\ (\Delta - \lambda)^m\, v = (-)^m f, \tag{5.23}$$

so that v can be regarded as a solution to the boundary value problem (5.22)-(5.23). The following statement provides a probabilistic representation of such a solution.

Lemma 5.6. *(A Feynman-Kac formula) Let f be a function on Γ satisfying (5.21). Then for all $x \in B$,*

$$f(x) = \mathbb{E}_x\left[\omega^T f(X_T)\right], \tag{5.24}$$

where $T = T_B$ is the first exit time from B and $\omega = (1 + \lambda)^{-1}$.

Furthermore, for all $m \geq 0$, $\lambda \geq 0$, and $x \in B$,

$$G^B_{\lambda,m} f(x) = \mathbb{E}_x\left[Q_{m+1}(T)\omega^{T+m} f(X_T)\right]. \tag{5.25}$$

Proof. For any integer $m \geq -1$, denote

$$v_m(x) = \mathbb{E}_x\left[Q_{m+1}(T)\omega^{T+m} f(X_T)\right]. \tag{5.26}$$

Clearly, $v_{-1} = 0$ in B, $v_0 = f$ in B^c and $v_m = 0$ in B^c for all $m \geq 1$. Let us prove that, for all $m \geq 0$,

$$\Delta v_m - \lambda v_m = -v_{m-1} \quad \text{in } B. \tag{5.27}$$

Indeed, for any $x \in B$, the Markov property implies

$$\begin{aligned} \mathbb{E}_x\left[Q_{m+1}(T-1)\omega^{T-1+m} f(X_T)\right] &= \sum_{y\sim x} P(x,y)\mathbb{E}_y\left[Q_{m+1}(T)\omega^{T+m} f(X_T)\right] \\ &= P v_m(y). \end{aligned} \tag{5.28}$$

By the property of the binomial coefficients, we have for all $m \geq 0$ and $T \geq 1$

$$Q_{m+1}(T-1) = Q_{m+1}(T) - Q_m(T).$$

Hence, the left hand side of (5.28) is equal to

$$\begin{aligned}&\mathbb{E}_x\left[Q_{m+1}(T)\omega^{T-1+m}f(X_T)\right]-\mathbb{E}_x\left[Q_m(T)\omega^{T-1+m}f(X_T)\right]\\&=\omega^{-1}v_m(x)-v_{m-1}(x).\end{aligned}$$

Substituting this into (5.28) and using $\omega^{-1}=1+\lambda$, we obtain (5.27).

For $m=0$ we obtain from (5.27) $\Delta v_0-\lambda v_0=0$. Since $v_0=f$ outside B, we conclude $v_0=f$ also in B. Therefore, (5.24) follows from (5.26) for $m=0$. If $m\geq 1$ then solving (5.27) with the boundary condition $v_m=0$ outside B, we obtain $v_m=G_\lambda^B v_{m-1}$. Therefore $v_m=G_{\lambda,m}^B f$, whence (5.25) follows.

Corollary 5.7. *For any non-empty finite set $B\subset\Gamma$ and for any non-negative function f in Γ such that $\Delta f-\lambda f=0$ in B,*

$$\left(G_{\lambda,m}^B f(x)\right)^2\leq c_m f(x)G_{\lambda,2m}^B f(x),\tag{5.29}$$

for all $x\in\Gamma$, $m\geq 0$, $\lambda\geq 0$.

Proof. Using the notation (5.26) we have by the Cauchy-Schwarz inequality

$$v_m(x)^2\leq\mathbb{E}_x\left[\omega^T f(X_T)\right]\mathbb{E}_x\left[Q_{m+1}^2(T)\omega^{T+2m}f(X_T)\right].\tag{5.30}$$

By (5.4) we obtain

$$Q_{m+1}^2(T)\leq c_m Q_{2m+1}(T).$$

Hence, (5.30) implies

$$v_m^2\leq c_m v_0 v_{2m}\,,$$

which by Lemma 5.6 coincides with (5.29).

Corollary 5.8. *Under the hypotheses of Lemma 5.4, we have for any integer $k\geq 0$*

$$G_{\lambda,k}^B f(x_0)\leq CR^{\beta k}\exp\left(-c\lambda^{1/\beta}R\right)\max_{\overline{B}\setminus B}f.\tag{5.31}$$

Proof. Indeed, by (5.29), (5.8), and (5.15), we obtain

$$\begin{aligned}\left[G_{\lambda,k}^B f(x_0)\right]^2&\leq Cf(x_0)G_{\lambda,2k}^B f(x_0)\\&\leq Cf(x_0)\|G_{\lambda,2k}^B\|\max_B f\\&\leq Cf(x_0)\overline{E}(B)^{2k}\max_{\overline{B}\setminus B}f.\end{aligned}$$

Using the estimate (5.12) for $f(x_0)$ and the hypothesis $\left(E_\beta\right)$, we conclude the proof.

5.5 Upper bound for λ, m-resolvent

Theorem 5.9. *Assume that (Γ, μ) satisfies $(VD) + (MV) + \left(E_\beta\right)$. Then for a large enough $m > 1$ and for all $0 < \lambda < 1$, $x \in \Gamma$*

$$g_{\lambda,m}(x,x) \le C \frac{\lambda^{-m}}{V(x, \lambda^{-1/\beta})}, \qquad (G_\beta)$$

where the constant C depends on the constants in the hypotheses as well as on m.

We start with a lemma.

Lemma 5.10. *Assume that (Γ, μ) satisfies $(VD) + (MV) + \left(E_\beta\right)$. Then, for all $x \in \Gamma$ and positive r, R such that $R \ge r + 1$, the following estimate holds*

$$g_{\lambda,m}^{B(x,R)}(x,x) - g_{\lambda,m}^{B(x,r)}(x,x) \le C \frac{R^{\beta m}}{V(x,r)} \exp\left(-c\lambda^{1/\beta} r\right),$$

where the constants $C, c > 0$ depend on the constants in the hypotheses, and C depends also on m.

Proof. Fix a point $x \in \Gamma$, set $A = B(x,r)$, $B = B(x,R)$, and consider the functions

$$v_m(y) := g_{\lambda,m}^B(x,y) - g_{\lambda,m}^A(x,y).$$

Clearly, we have $v_0 = 0$ and for all $m \ge 1$

$$\Delta v_m - \lambda v_m = -v_{m-1} \quad \text{in } A.$$

Therefore,

$$v_m = G_\lambda^A v_{m-1} + u_m \quad \text{in } A, \qquad (5.32)$$

where u_m solves the following boundary value problem

$$\begin{cases} \Delta u_m - \lambda u_m = 0 \text{ in } A, \\ u_m|_{A^c} = v_m. \end{cases}$$

Iterating (5.32) and using $v_0 = 0$ we obtain

$$v_m = G_{\lambda,m-1}^A u_1 + G_{\lambda,m-2}^A u_2 + \dots + u_m. \qquad (5.33)$$

Since $\overline{A} \subset B$, we have by Corollary 5.8

$$G_{\lambda,l}^A u_k(x) \le C r^{\beta l} \exp\left(-c\lambda^{1/\beta} r\right) \max_{B \setminus A} v_k, \qquad (5.34)$$

and by Lemma 5.3

$$\max_{B\setminus A} v_k = \max_{y\in B\setminus A} g^B_{\lambda,k}(x,y) \le C\frac{R^{\beta k}}{V(x,r)}. \tag{5.35}$$

Therefore, we obtain from (5.33), (5.34), and (5.35)

$$\begin{aligned} v_m(x) &= \sum_{k=1}^{m} G^A_{\lambda,m-k} u_k(x) \\ &\le C\sum_{k=1}^{m} r^{\beta(m-k)} \exp\left(-c\lambda^{1/\beta} r\right) \frac{R^{\beta k}}{V(x,r)} \\ &\le C\frac{R^{\beta m}}{V(x,r)} \exp\left(-c\lambda^{1/\beta} r\right), \end{aligned}$$

which was to be proved.

Proof of Theorem 5.9. Set $B_k = B(x, 2^k)$ for $k = 0, 1, 2, \ldots$. Obviously, we have

$$g_{\lambda,m}(x,x) = g^{B_0}_{\lambda,m}(x,x) + \sum_{k=0}^{\infty}\left(g^{B_{k+1}}_{\lambda,m}(x,x) - g^{B_k}_{\lambda,m}(x,x)\right).$$

The first term $g^{B_0}_{\lambda,m}(x,x)$ is estimated as follows, using (5.5):

$$\begin{aligned} g^{B_0}_{\lambda,m}(x,x) &= \sum_{n=0}^{\infty} Q_m(n)\omega^{n+m} p^{B(x,1)}_n(x,x) \\ &= Q_m(0)\omega^m p^{B(x,1)}_0(x,x) \\ &= \frac{\omega^m}{V(x,1)} \le \frac{1}{V(x,1)}. \end{aligned}$$

Applying Lemma 5.10 with $r = 2^k$ and $R = 2^{k+1}$, we obtain for any $k \ge 0$

$$g^{B_{k+1}}_{\lambda,m}(x,x) - g^{B_k}_{\lambda,m}(x,x) \le C\frac{2^{k\beta m}}{V(x,2^k)} \exp\left(-c\lambda^{1/\beta} 2^k\right),$$

whence

$$g_{\lambda,m}(x,x) \le \frac{1}{V(x,1)} + C\sum_{k=0}^{\infty} \exp\left(-c\lambda^{1/\beta}2^k\right)\frac{2^{k\beta m}}{V(x,2^k)}.$$

Set $r = \lambda^{-1/\beta}$ and rewrite this inequality as follows

$$g_{\lambda,m}(x,x) \le \frac{1}{V(x,1)} + C\sum_{k=0}^{\infty} \exp\left(-c\frac{2^k}{r}\right)\left(\frac{2^k}{r}\right)^{\beta m}\frac{V(x,r)}{V(x,2^k)}\frac{r^{\beta m}}{V(x,r)}. \tag{5.36}$$

Let us choose m so large that

$$\beta m > \alpha,$$

where α is the exponent from (2.1). If $2^k \le r$ then by (2.1)

$$\frac{V(x,r)}{V(x,2^k)} \le C\left(\frac{r}{2^k}\right)^{\alpha}, \tag{5.37}$$

and the k-th term in the sum (5.36) is estimated from above by

$$\left(\frac{2^k}{r}\right)^{\beta m-\alpha} \frac{r^{\beta m}}{V(x,r)}. \tag{5.38}$$

The sequence of these numbers is an increasing geometric series in k; hence, the sum of all the terms in (5.36) with k such that $2^k \le r$, is bounded by a multiple of the largest term in (5.38); that is by

$$C\frac{r^{\beta m}}{V(x,r)}. \tag{5.39}$$

If $2^k > r$ then the k-th term in the sum (5.36) is bounded by

$$\exp\left(-c\frac{2^k}{r}\right)\left(\frac{2^k}{r}\right)^{\beta m} \frac{r^{\beta m}}{V(x,r)} \le C\exp\left(-\frac{c}{2}\frac{2^k}{r}\right)\frac{r^{\beta m}}{V(x,r)}.$$

The sequence of these numbers decreases in k faster than a geometric series; hence, the sum of all the terms in (5.36) with k such that $2^k > r$ is again bounded by (5.39).

Finally, by (5.37) with $k=0$, we obtain

$$\frac{1}{V(x,1)} \le \frac{Cr^{\alpha}}{V(x,r)} \le \frac{Cr^{\beta m}}{V(x,r)},$$

which means that the term $\frac{1}{V(x,1)}$ in (5.36) is also bounded by (5.39). Hence, $g_{\lambda,m}(x,x)$ is bounded by (5.39), whence $\left(G_\beta\right)$ follows by substituting $r=\lambda^{-1/\beta}$. □

6 Estimates of the heat kernel

6.1 Diagonal upper estimate

Theorem 6.1. *If (Γ,μ) satisfies $(VD)+\left(G_\beta\right)$ then, for all $x,y\in\Gamma$ and $n\ge 1$,*

$$p_n(x,x) \le \frac{C}{V(x,n^{1/\beta})} \tag{DUE_β}$$

and

$$p_n(x,y) \le \frac{C}{\sqrt{V(x,n^{1/\beta})V(y,n^{1/\beta})}}. \tag{PUE_β}$$

Proof. Let us first prove that, for $\lambda = n^{-1}$ and any $m \geq 0$,

$$p_{2n}(x, x) \leq C\lambda^m g_{\lambda,m}(x, x). \tag{6.1}$$

Indeed, $p_{2k}(x, x)$ is non-increasing in k. Therefore, for $\lambda = n^{-1}$ we have

$$\begin{aligned} g_{\lambda,m}(x, x) &= \sum_{k=0}^{\infty} Q_m(k)\omega^{k+m} p_k(x, x) \geq \sum_{k=0}^{\infty} Q_m(2k)\omega^{2k+m} p_{2k}(x, x) \\ &\geq c \sum_{k=1}^{n} k^{m-1}\omega^{2k+m} p_{2k}(x, x) \geq cn^m p_{2n}(x, x), \end{aligned}$$

(where we have used $\omega^{2k+m} \geq (1 + 1/n)^{-2n-m} \geq e^{-2}2^{-m}$) whence (6.1) follows. By the hypothesis $\left(G_\beta\right)$ we have for some m

$$g_{\lambda,m}(x, x) \leq \frac{C}{\lambda^m V(x, \lambda^{-1/\beta})},$$

which together with (6.1) implies $\left(DUE_\beta\right)$ for even n.

Using the semigroup property and the Cauchy-Schwarz inequality, we obtain (PUE_β) for even n:

$$\begin{aligned} p_{2n}(x, y) &= \sum_z p_n(x, z)p_n(z, y)\mu(z) \qquad (6.2) \\ &\leq \sqrt{p_{2n}(x, x)p_{2n}(y, y)} \qquad (6.3) \\ &\leq \frac{C}{\sqrt{V(x, n^{1/\beta})V(y, n^{1/\beta})}}. \end{aligned}$$

Again by the semigroup property, we obtain

$$p_{2n+1}(x, y) = \sum_{z\sim x} p_1(x, z)p_{2n}(z, y)\mu(z) \leq \sup_{z\sim x} p_{2n}(z, y) \tag{6.4}$$

whence

$$p_{2n+1}(x, y) \leq \sup_{z\sim x} p_{2n}(z, y) \leq \sup_{z\sim x} \frac{C}{\sqrt{V(z, n^{1/\beta})V(y, n^{1/\beta})}}.$$

By (VD),

$$V(z, n^{1/\beta}) \simeq V(x, n^{1/\beta})$$

for all $z \sim x$, whence (PUE_β) and (DUE_β) follow for odd $n > 1$. Finally, (PUE_β) and (DUE_β) for $n = 1$ follow directly from the definition (2.2) of $p_n(x, y)$.

6.2 Off-diagonal upper estimate

Theorem 6.2. *For any graph (Γ, μ)*

$$(VD) + (DUE_\beta) + \left(E_\beta\right) \Longrightarrow \left(UE_\beta\right).$$

The following lemma plays a crucial role in the proof.

Lemma 6.3. *Let (Γ, μ) satisfy $\left(E_\beta\right)$ and let f be a bounded non-negative function on Γ which vanishes in a ball $B(x_0, R)$. Then for any $n \geq 1$ the function $P_n f$ admits the following estimate at the point x_0:*

$$P_n f(x_0) \leq C \exp\left[-c\left(\frac{R^\beta}{n}\right)^{\frac{1}{\beta-1}}\right] \sup_\Gamma f, \tag{6.5}$$

where the constants $C, c > 0$ depend on the constants in $\left(E_\beta\right)$.

This lemma can be deduced from [22, Lemma 7.4] but we give here a self-contained proof of it based on Lemma 5.4.

Proof of Lemma 6.3. If $R > n$ then we have $P_n f(x_0) = 0$ and there is nothing to prove. Assume in the sequel $R \leq n$. Without loss of generality, we can also assume $\sup f = 1$.

Set $A = B(x_0, R)$, fix some $\lambda \in (0, 1)$ and find a function $h(x)$ on $\overline{A}$ solving the boundary value problem

$$\begin{cases} \Delta h = \lambda h \text{ in } A, \\ h = 1 \quad\ \text{ in } \overline{A} \setminus A. \end{cases}$$

Then the function $u_n(x) := (1+\lambda)^n h(x)$ solves the heat equation (2.3) in $\mathbb{N} \times A$ and satisfies the following initial boundary conditions:

$$\begin{aligned} u_0(x) &\geq 0 \text{ in } A, \\ u_n(x) &\geq 1 \text{ in } \overline{A} \setminus A. \end{aligned}$$

Comparing with the function $P_n f(x)$ that also solves the heat equation, we conclude by the parabolic comparison principle that

$$P_n f(x) \leq u_n(x),$$

for all $x \in A$ and $n \geq 0$.

On the other hand, by Lemma 5.4, we have

$$h(x_0) \leq C \exp\left(-c\lambda^{1/\beta} R\right),$$

whence

$$P f_n(x_0) \leq (1+\lambda)^n h(x_0) \leq C \exp\left(\lambda n - c\lambda^{1/\beta} R\right). \tag{6.6}$$

Now choose λ from the condition $c\lambda^{1/\beta}R = 2\lambda n$; that is,

$$\lambda = \left(\frac{cR}{2n}\right)^{\frac{\beta}{\beta-1}}. \tag{6.7}$$

Of course, we can always assume $c < 1$. Then the assumption $R \le n$ implies $\lambda < 1$ so that this λ can be used in (6.6). Therefore, we obtain

$$Pf_n(x_0) \le C\exp(-\lambda n) = C\exp\left(-c'\left(\frac{R^\beta}{n}\right)^{\frac{1}{\beta-1}}\right),$$

which finishes the proof. □

Proof of Theorem 6.2. We can assume that (PUE_β) holds because, as was shown in the proof of Theorem 6.1, (PUE_β) follows from $(VD) + (DUE_\beta)$.

Step 1. Let us prove that $(PUE_\beta) + (VD) + \left(E_\beta\right)$ implies, for all $R \ge 0$,

$$I_n(x,R) := \sum_{z\notin B(x,R)} p_n^2(x,z)\mu(z) \le \frac{C}{V(x,n^{1/\beta})}\exp\left[-\left(\frac{R^\beta}{Cn}\right)^{\frac{1}{\beta-1}}\right]. \tag{6.8}$$

By (2.1) we have for all $x, y \in \Gamma$, $n > 0$, $\varepsilon > 0$

$$\frac{V(x,n^{1/\beta})}{V(y,n^{1/\beta})} \le C\left(1+\frac{d(x,y)}{n^{1/\beta}}\right)^\alpha \le C_\varepsilon \exp\left[\varepsilon\left(\frac{d^\beta(x,y)}{n}\right)^{\frac{1}{\beta-1}}\right]. \tag{6.9}$$

Therefore, by (PUE_β) and (6.9),

$$\begin{aligned} p_n(x,y) &\le \frac{C}{V(x,n^{1/\beta})}\left[\frac{V(x,n^{1/\beta})}{V(y,n^{1/\beta})}\right]^{1/2} \\ &\le \frac{C_\varepsilon}{V(x,n^{1/\beta})}\exp\left[\varepsilon\left(\frac{d^\beta(x,y)}{n}\right)^{\frac{1}{\beta-1}}\right]. \end{aligned} \tag{6.10}$$

If $R^\beta \le Cn$ then (6.8) follows from (6.3) and (DUE_β). Assuming in the sequel $R^\beta > Cn$, denote $R_k = 2^k R$ where $k = 0, 1, 2, \ldots$. Splitting the summation in (6.8) to annuli

$$A_k := B(x,R_k) \setminus B(x,R_{k-1}),$$

denoting $f_k = \mathbf{1}_{A_k} p_n(x,\cdot)$, and using Lemma 6.3, we obtain

$$\begin{aligned} I_n(x,R) &= \sum_{k=1}^{\infty}\sum_{z\in A_k} p_n^2(x,z)\mu(z) \\ &= \sum_{k=1}^{\infty} P_n f_k(x) \le C\sum_{k=1}^{\infty}\exp\left[-c\left(\frac{R_{k-1}^\beta}{n}\right)^{\frac{1}{\beta-1}}\right]\sup f_k\,. \end{aligned} \tag{6.11}$$

By (6.10), we obtain

$$\sup f_k \le \sup_{y \notin B(x,R_k)} p_n(x,y) \le \frac{C_\varepsilon}{V(x,n^{1/\beta})} \exp\left[\varepsilon \left(\frac{R_k^\beta}{n}\right)^{\frac{1}{\beta-1}}\right].$$

Substituting this estimate in (6.11) and estimating the sum in (6.11) for small enough ε, we obtain (6.8).

Step 2. Let us deduce $\left(UE_\beta\right)$ from (6.8). Denote $R = \frac{1}{2}d(x,y)$ and observe that Γ is covered by the union of $B(x,R)^c$ and $B(y,R)^c$. Therefore, we have by (6.2)

$$p_{2n}(x,y) \le \sum_{z \notin B(x,R)} p_n(x,z)p_n(z,y)\mu(z) + \sum_{z \notin B(y,R)} p_n(x,z)p_n(z,y)\mu(z). \tag{6.12}$$

By the Cauchy-Schwarz inequality, the first sum in (6.12) is dominated by $\sqrt{I_n(x,R)I_n(y,0)}$, and the second sum is dominated by $\sqrt{I_n(y,R)I_n(x,0)}$. Applying (6.8) we obtain

$$p_{2n}(x,y) \le \frac{C}{\sqrt{V(x,n^{1/\beta})V(y,n^{1/\beta})}} \exp\left[-\left(\frac{d^\beta(x,y)}{Cn}\right)^{\frac{1}{\beta-1}}\right],$$

which together with (6.9) yields $\left(UE_\beta\right)$ for even n. Finally, $\left(UE_\beta\right)$ for odd n follows from (6.4). □

6.3 Proof of $(iv) \Longrightarrow (i)$ *in Theorem 3.1*

Let us prove that

$$(VD) + (H) + \left(E_\beta\right) \Longrightarrow \left(UE_\beta\right).$$

As was shown in Sect. 5.2

$$(VD) + (H) \Longrightarrow (MV).$$

By Theorem 5.9

$$(VD) + (MV) + \left(E_\beta\right) \Longrightarrow \left(G_\beta\right)$$

and by Theorem 6.1

$$(VD) + \left(G_\beta\right) \Longrightarrow (DUE_\beta).$$

By Theorem 6.2

$$(VD) + (DUE_\beta) + \left(E_\beta\right) \Longrightarrow \left(UE_\beta\right),$$

which finishes the proof. □

Remark 6.4. Clearly, the hypothesis (H) can be replaced here by (MV) so that we obtain (3.1). Note also that the lower bound in the hypothesis $\left(E_\beta\right)$ was required only for Lemma 5.4 and for its consequence Lemma 6.3. In all other places we used only the upper bound $E(x, R) \le CR^\beta$.

Let us prove that

$$(VD) + (H) + \left(E_\beta\right) \Longrightarrow \left(LE_\beta\right).$$

By [22, Propositions 7.1,9.1], $(VD) + \left(E_\beta\right)$ implies

$$p_{2n}^{B(x,R)}(x, x) \ge \frac{c}{V(x, n^{1/\beta})}, \quad \text{for all } n \le \varepsilon R^\beta, \tag{6.13}$$

for some $c, \varepsilon > 0$. By Theorem 6.1

$$(VD) + (H) + \left(E_\beta\right) \Longrightarrow (DUE_\beta).$$

By [22, Proposition 13.1], [33], assuming (p_0), the conditions

$$(VD) + (H) + \left(E_\beta\right) + (DUE_\beta) + (6.13)$$

imply

$$p_n(x, y) + p_{n+1}(x, y) \ge \frac{c}{V(x, n^{1/\beta})} \quad \text{whenever } d(x, y) \le \delta n^{1/\beta}, \tag{6.14}$$

for some $\delta > 0$. Finally, using (p_0) and arguing as in [22, Proposition 13.2] or [13, Theorem 3.8] or [33], we obtain

$$(VD) + (6.14) \Longrightarrow \left(LE_\beta\right).$$

Remark 6.5. Alternatively, one can deduce $\left(LE_\beta\right)$ from $(VD) + (H) + \left(UE_\beta\right)$ using a modification of the method described in [22, Remark 15.1] (in the continuous setting, this was done in [23]). In particular, we see that $(VD)+(H)+\left(UE_\beta\right)$ is equivalent to each of the conditions $(i) - (iv)$ of Theorem 3.1.

7 Appendix: the lettered conditions

The conditions (H), $\left(PH_\beta\right)$, (HG), (MV) can be found in Definitions 2.1, 2.2, 4.2, 5.1, respectively. Here is the list of most of the other conditions used in this paper:

$$P(x, y) \ge p_0, \quad \forall x \sim y \tag{p_0}$$

$$V(x, 2R) \le CV(x, R), \quad \forall x \in \Gamma \quad \forall R > 0 \tag{VD}$$

$$\overline{E}(x,R) \le C E(x,R), \quad \forall x \in \Gamma \quad \forall R > 0 \qquad (\overline{E})$$

$$E(x,R) \simeq R^{\beta}, \quad \forall x \in \Gamma \quad \forall R \ge 1 \qquad (E_\beta)$$

$$\rho\left(B(x,R), B(x,MR)\right) \simeq \frac{R^{\beta}}{V(x,R)}, \quad \forall x \in \Gamma \quad \forall R \ge 1 \qquad (\rho_\beta)$$

$$p_n(x,y) \le \frac{C}{V(x,n^{1/\beta})} \exp\left[-\left(\frac{d(x,y)^{\beta}}{Cn}\right)^{\frac{1}{\beta-1}}\right], \ \forall x,y \in \Gamma \quad \forall n \ge 1 \qquad (UE_\beta)$$

$$(p_n + p_{n+1})(x,y) \ge \frac{c}{V(x,n^{1/\beta})} \exp\left[-\left(\frac{d(x,y)^{\beta}}{cn}\right)^{\frac{1}{\beta-1}}\right], \quad \forall n \ge d(x,y) \vee 1 \qquad (LE_\beta)$$

$$p_n(x,x) \le \frac{C}{V(x,n^{1/\beta})}, \quad \forall x \in \Gamma \quad \forall n \ge 1 \qquad (DUE_\beta)$$

$$p_n(x,y) \le \frac{C}{\sqrt{V(x,n^{1/\beta})V(y,n^{1/\beta})}} \quad \forall x,y \in \Gamma \quad \forall n \ge 1 \qquad (PUE_\beta)$$

$$g_{\lambda,m}(x,x) \le C\frac{\lambda^{-m}}{V(x,\lambda^{-1/\beta})}, \quad \forall x \in \Gamma \quad \forall \lambda \in (0,1). \qquad (G_\beta)$$

References

1. Barlow, M. T.: Diffusions on fractals. In: Lectures on Probability Theory and Statistics, Ecole d'été de Probabilités de Saint-Flour XXV – 1995, Lecture Notes Math. 1690, Springer, 1998. 1–121
2. Barlow, M. T.: Which values of the volume growth and escape time exponents are possible for graphs?. preprint
3. Barlow, M. T., Bass, R. F.: The construction of the Brownian motion on the Sierpisnki carpet. Ann. Inst. H. Poincaré, **25** (1989) 225–257
4. Barlow, M. T., Bass, R. F.: Transition densities for Brownian motion on the Sierpinski carpet. Probab. Th. Rel. Fields, **91** (1992) 307–330
5. Barlow, M. T., Bass, R. F.: Brownian motion and harmonic analysis on Sierpinski carpets. Canad. J. Math., **54** (1999) 673–744
6. Barlow, M. T., Bass, R. F.: Random walks on graphical Sierpinski carpets. In: Random walks and discrete potential theory (Cortona, Italy, 1997), Ed. Picardello, M., Woess, W., Symposia Math. 39, Cambridge Univ. Press, Cambridge, 1999. 26–55

7. Barlow, M.T., Coulhon, T., Grigor'yan, A.: Manifolds and graphs with slow heat kernel decay. Invent. Math., **144** (2001) 609–649
8. Barlow, M.T., Perkins, E.A.: Brownian motion on the Sierpinski gasket. Probab. Th. Rel. Fields, **79** (1988) 543–623
9. Carlen, E.A., Kusuoka S., Stroock, D.W.: Upper bounds for symmetric Markov transition functions. Ann. Inst. H. Poincaré, Prob. et Stat., (1987) suppl. au no.2, 245–287
10. Chung, F.R.K.: "Spectral Graph Theory", CBMS Regional Conference Series in Mathematics 92, AMS publications, 1996.
11. Coulhon, T.: Ultracontractivity and Nash type inequalities. J. Funct. Anal., **141** (1996) no.2, 510–539
12. Coulhon T., Grigor'yan, A.: Random walks on graphs with regular volume growth. Geom. and Funct. Analysis, **8** (1998) 656–701
13. Delmotte, T.: Parabolic Harnack inequality and estimates of Markov chains on graphs. Revista Matemática Iberoamericana, **15** (1999) no.1, 181–232
14. Einstein, A.: On the motion of small particles suspended in liquids at rest required by the molecular-kinetic theory of heat. Annalen der Physik, 1905. See also "Einstein's miraculous year", Ed. Stachel, J., Princeton University Press, 1998. 71–85
15. Fabes E.B., Stroock D.W.: A new proof of Moser's parabolic Harnack inequality via the old ideas of Nash. Arch. Rat. Mech. Anal., **96** (1986) 327–338
16. Fukushima, M., Oshima, Y., Takeda, M.: "Dirichlet forms and symmetric Markov processes", Studies in Mathematics 19, De Gruyter, 1994.
17. Goldstein, S.: Random walks and diffusion on fractals. In: Percolation theory and ergodic theory on infinite particle systems, Ed. Kesten, H., Springer, New York, 1987. 121–129
18. Grigor'yan, A.: The heat equation on non-compact Riemannian manifolds. (in Russian) Matem. Sbornik, **182** (1991) no.1, 55–87 Engl. transl. Math. USSR Sb., **72** (1992) no.1, 47–77
19. Grigor'yan, A.: Heat kernel upper bounds on a complete non-compact manifold. Revista Matemática Iberoamericana, **10** (1994) no.2, 395–452
20. Grigor'yan, A.: Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds. Bull. Amer. Math. Soc., **36** (1999) 135–249
21. Grigor'yan, A., Saloff-Coste, L.: Heat kernel on connected sums of Riemannian manifolds. Math. Research Letters, **6** (1999) no.3-4, 307–321
22. Grigor'yan, A., Telcs, A.: Sub-Gaussian estimates of heat kernels on infinite graphs. Duke Math. J., **109** (2001) no.3, 452–510
23. Hebisch, W., Saloff-Coste, L.: On the relation between elliptic and parabolic Harnack inequalities. Ann. Inst. Fourier, **51** (2001) no.5, 1437–1481
24. Jones O.D.: Transition probabilities for the simple random walk on the Sierpinski graph. Stoch. Proc. Appl., **61** (1996) 45–69
25. Kondrat'ev, V.A., Landis, E.M.: Qualitative theory of linear second-order partial differential equations. (in Russian) Itogi Nauki i Techniki, serija Sovremennye Problemy Matematiki, Fundamental'nye Napravlenija 32, VINITI, Moscow, 1988. 99–215 Engl. transl. In: Partial Differential Equations III, Encyclopedia of Math. Sci. 32, Springer, 1990.
26. Li, P., Wang, J.: Mean value inequalities. Indiana Univ. Math. J., **48** (1999) 1257–1283
27. Li, P., Yau, S.-T.: On the parabolic kernel of the Schrödinger operator. Acta Math., **156** (1986) no.3-4, 153–201
28. Moser, J.: On Harnack's theorem for elliptic differential equations. Comm. Pure Appl. Math., **14** (1961) 577–591
29. Moser, J.: A Harnack inequality for parabolic differential equations. Comm. Pure Appl. Math., **17** (1964) 101–134 Correction: Comm. Pure Appl. Math., **20** (1967) 231-236
30. Nash, J.: Continuity of solutions of parabolic and elliptic equations. Amer. J. Math., **80** (1958) 931–954

31. Saloff-Coste, L.: A note on Poincaré, Sobolev, and Harnack inequalities. Duke Math. J., **65** no.3, Internat. Math. Res. Notices, **2** (1992) 27–38
32. Telcs, A.: Random walks on graphs, electric networks and fractals. J. Prob. Th. Rel. Fields, **82** (1989) 435–449
33. Telcs, A.: Local sub-Gaussian transition probability estimates, the strongly recurrent case. to appear in Elec. J. Prob.
34. Telcs, A.: Volume and time doubling of graphs and random walks, the strongly recurrent case. to appear in Comm. Pure Appl. Math.
35. Varopoulos, N. Th.: Hardy-Littlewood theory for semigroups. J. Funct. Anal., **63** (1985) no.2, 240–260