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Energy Scaling of Compressed Elastic Films – Three-Dimensional Elasticity and Reduced Theories

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Abstract

We derive an optimal scaling law for the energy of thin elastic films under isotropic compression, starting from three-dimensional nonlinear elasticity. As a consequence we show that any deformation with optimal energy scaling must exhibit fine-scale oscillations along the boundary, which coarsen in the interior. This agrees with experimental observations of folds which refine as they approach the boundary.

We show that both for three-dimensional elasticity and for the geometrically nonlinear Föppl-von Kármán plate theory the energy of a compressed film scales quadratically in the film thickness. This is intermediate between the linear scaling of membrane theories which describe film stretching, and the cubic scaling of bending theories which describe unstretched plates, and indicates that the regime we are probing is characterized by the interplay of stretching and bending energies.

Blistering of compressed thin films has previously been analyzed using the Föppl-von Kármán theory of plates linearized in the in-plane displacements, or with the scalar eikonal functional where in-plane displacements are completely neglected. The predictions of the linearized plate theory agree with our result, but the scalar approximation yields a different scaling.

1. Introduction

Blistering phenomena in compressed thin elastic films have recently attracted widespread interest in the mechanics [32, 16, 5], physics [29, 23, 1, 35, 34, 4, 9] and mathematics [3, 7, 17, 12] literature. Most studies so far have concentrated on the linearized Föppl-von Kármán (FvK) model [24, 10], in which the in-plane components of the displacement field are treated only to leading order. Here we study this problem starting from the full nonlinear three-dimensional theory of elasticity, and show that the energy per unit volume scales linearly in the film thickness h. In terms of the total energy referred to in the abstract, this corresponds to a quadratic scaling in h. In the following we always consider energies per unit volume.

The derivation of dimensionally reduced theories for thin elastic films has been studied by Euler, J. Bernoulli, Cauchy, Kirchhoff, Love, E. and F. Cosserat, Föppl, von Kármán and a great many modern authors (see [30, 10] and references therein). The only theories which have been derived rigorously from nonlinear three-dimensional elasticity by means of Γ convergence are membrane theory [25–27] and more recently bending theory ([13, 14], see also [33] for a derivation under more stringent assumptions); and the FvK plate theory in a different regime, characterized by an energy density proportional to the fourth power of the thickness [15]. Membrane theories account for deformations where the film is stretched in a way which is essentially uniform in the thickness, and give energy densities of order one in the film thickness. Bending theories instead focus on unstretched plates, i.e., they are limited to deformations where the stretching term vanishes and only consider higher-order corrections in the film thickness. The resulting energy densities are quadratic in the film thickness. In the regime of interest here, however, both stretching and bending are active and relevant. Indeed, membrane theories would predict zero energy, and bending theories would predict infinite energy – neither of which gives much insight into the material behavior. More precisely, since no zero-stretch deformation exists with the given boundary conditions and finite bending energy, we need to allow for finite stretching. The resulting patterns are therefore determined by the interplay of the two energy terms, and analogously the linear scaling we obtain for the energy density is intermediate between the two mentioned.

The study of blistering using the tools of calculus of variations was initiated in 1994 by ORTIZ & GIOIA [32], who noticed that many experimental observations could be reproduced with a scalar model, which can be obtained from the linearized FvK theory by neglecting the in-plane displacements. Since then, considerable mathematical effort [3, 7, 17, 12] has gone into the study of the resulting scalar functional, which constitutes the natural generalization to two-dimensional gradient fields of the classical Modica-Mortola functionals in the theory of Gamma-convergence, and had already been proposed with a different physical interpretation by AVILES & GIGA in 1987 [6].

Only recently has attention has been devoted to rigorous studies of the energetics of the vectorial FvK model. The solutions of the scalar problem correspond to an energy per unit volume of order $\varepsilon_*^2(1+\sigma)$, where ε_* is the compression ratio and σ is the rescaled film thickness, which constitutes the small parameter in this problem, defined after (1.1). (The corresponding total energy is linear in the film thickness.) Already in their 1994 paper [32], ORTIZ & GIOIA exhibited a construction which, after suitable smoothing, achieved an energy of order $\varepsilon_*^2\sigma^{1/2}$ in a half-plane geometry. In 1997 POMEAU & RICA also proposed [35] (see also [34]) that energy-minimizing states develop oscillations orthogonal to the boundary, and based on scaling estimates for the fold energies they proposed that oscillations would refine approaching the boundary. Recent rigorous results on energy scaling in FvK [8, 19, 18] were based on the ideas proposed by KOHN & MÜLLER in their mathematical theory of self-similar twin branching in martensites [21, 22] (see also

[11]). The correct energy scaling turned out to be $\varepsilon_*^2\sigma^{2/3}$ for the restricted problem considered in [19], and $\varepsilon_*^2\sigma$ for the full geometrically linear FvK problem [8,18]. In these three papers, fold-branching patterns are constructed which, at least in their gross features, are similar to the one proposed by Pomeau and Rica. A different $\sigma^{5/3}L^{1/3}$ energy scaling was instead obtained considering sharp ridges of length L which appear in situations, such as paper crumpling, in which the boundary condition is replaced by a volume constraint [29,28,23,9].

In this paper we focus on the problem of thin-film blistering and study the energetics of blistered thin films using both a geometrically nonlinear treatment of the FvK ansatz and directly the full three-dimensional elastic energy, which does not rely in any way on the FvK theory of plates. We show that the same estimates we had obtained for the energy in the geometrically linear model [8] hold also in this case, hence proving that – at least as far as energy scaling for compressed isotropic plates is concerned – the FvK approximation correctly reproduces the behavior of elasticity theory, and a geometrically linear treatment of the in-plane deformations is justified. As a consequence we show that any deformation with optimal energy scaling must exhibit fine-scale oscillations along the boundary, which coarsen in the interior. This agrees with experimental observations of folds which refine as they approach the boundary [2].

We consider a uniform, isotropic elastic film of given cross-section Ω and thickness h, subject to an isotropic in-plane compression $\varepsilon_* > 0$. We take the compressed state as reference, and impose no-displacement boundary conditions on $\partial\Omega$. Let $\phi : \Omega \times (0,h) \to \mathbb{R}^3$ be the deformation field, and $W_{3D}(F)$ be the elastic energy density, which vanishes for $F \in (1+\varepsilon_*)SO(3)$. We show (Theorems 3 and 4) that for smooth domains Ω the energy per unit thickness of a minimizer ϕ,

$$I_{3D}^{h,\varepsilon_*}[\phi,\Omega] = \frac{1}{h}\int_{\Omega\times(0,h)} W_{3D}(\nabla\phi), \tag{1.1}$$

scales as $\varepsilon_*^2\sigma$ when the scaled thickness $\sigma = h/\varepsilon_*^{1/2}$ is small. In proving this result we assume that the potential $W_{3D}(F)$ behaves as the squared distance from its null set $\mathcal{K} = (1+\varepsilon_*)SO(3)$ in a neighborhood of $\mathcal{K}$, and that it has at least quadratic growth at infinity. Boundary conditions are imposed only on the in-plane components at the lateral boundaries, i.e., $\phi_i(x) = x_i$ for $x \in (\partial\Omega)\times(0,h)$ and $i = 1,2$, with the out-of-plane component satisfying $|\phi_3 - x_3| \ll h$.

The FvK ansatz amounts to enslaving the deformation ϕ to its behavior on the mid-plane of the film (see Section 4.2 for details). The resulting variational problem is

$$I_{2D}^{h,\varepsilon_*}[\psi,\Omega] = \int_\Omega W_{2D}(\nabla\psi) + h^2|\nabla\nu|^2, \tag{1.2}$$

where $\psi(x_1,x_2) = \phi(x_1,x_2,h/2)$ is the deformation of the mid-plane,

$$\nu = \frac{\psi_{,1}\wedge\psi_{,2}}{|\psi_{,1}\wedge\psi_{,2}|} \tag{1.3}$$

is the normal to the surface defined by ψ, and $|\nabla\nu|^2 = |\partial_1\nu|^2 + |\partial_2\nu|^2$. The new potential $W_{2D}(a|b)$ is obtained from $W_{3D}(a|b|c)$ by optimizing over vectors

c parallel to $a \wedge b$, and it has invariance and growth properties similar to those of W_{3D}, with the group of three-dimensional linear isometries $SO(3)$ replaced by the group of linear orthogonal maps of $\mathbb{R}^2$ into $\mathbb{R}^3$, called $O(2,3)$. In particular, in a neighborhood of $(1+\varepsilon_*)O(2,3)$, W_{2D} is bounded from above and below by constants times $\text{dist}^2(F,(1+\varepsilon_*)O(2,3))$. The boundary conditions are now $\psi_i = x_i$ for $i = 1, 2$ and $\psi_3 = 0$ for $x \in \partial\Omega$. We show that both inf I_{2D} and inf I_{3D} scale as $\sigma\varepsilon_*^2 = h\varepsilon_*^{3/2}$ for small h and ε_*.

The construction for the upper bound is, in its general lines, analogous to the one we used for the geometrically linear problem [8], and is presented in Section 3. Since the FvK model is obtained from three-dimensional elasticity by assuming a specific form for the deformation, the bound on FvK leads naturally to a bound on the three-dimensional elastic energy (see Section 4.2). The lower bounds are presented separately, in Section 2 for FvK and in Section 4.1 for the three-dimensional case. In all constructions the natural length-scale for in-plane oscillations is given by $\sigma = h/\varepsilon_*^{1/2}$, and hence in the following we replace h^2 by $\sigma^2\varepsilon_*$ in the FvK functional (1.2).

2. Lower bound

In this section we prove the lower bound on the geometrically nonlinear two-dimensional Föppl-von Kármán energy I_{2D} defined in (1.2), under the assumption

$$W_{2D}(F) \geqq c\,\text{dist}^2(F,(1+\varepsilon_*)O(2,3)), \tag{2.1}$$

where $O(2,3)$ is the group of orthogonal linear maps from $\mathbb{R}^2$ to $\mathbb{R}^3$. Here and below, we denote by c a positive constant, which might change from line to line, but does not depend on the parameters of the problem (ε_* and h).

Theorem 1 (Lower bound, 2D**).** *Let Ω be an open, bounded, Lipschitz domain in $\mathbb{R}^2$, and assume* (2.1) *holds. Then, there is a constant $c_\Omega > 0$, depending only on Ω, such that*

$$I_{2D}^{h,\varepsilon_*}[\psi,\Omega] \geqq c_\Omega \min\left(h\varepsilon_*^{3/2}, \varepsilon_*^2\right) \tag{2.2}$$

for any ψ which obeys the boundary condition $\psi(x_1,x_2) = (x_1,x_2,0)$ on $\partial\Omega$.

Remark 1. We observe that no boundary condition is imposed on v. The same can be done in the geometrically linear case. Indeed, Theorem 1 in [8] holds, with minor changes in the proof, without the hypothesis $\nabla w = 0$ on $\partial\Omega$. Further, the boundary condition on ψ can be weakened as in (2.5) below, with no changes in the proof.

The basic idea in the proof is to show that, due to the boundary condition, in a neighborhood of the boundary of width σ the energy per unit area is at least of order ε_*^2, which is the energy of the identity map. Roughly speaking, a function with infinitesimal energy would have ∇v vanishing in L^2, hence its image would be contained in a plane. This allows us to pass, locally, from maps $\mathbb{R}^2 \to \mathbb{R}^3$ with gradient in $(1+\varepsilon_*)O(2,3)$ to maps $\mathbb{R}^2 \to \mathbb{R}^2$ with gradient in $(1+\varepsilon_*)SO(2)$.

But a gradient which is a rotation a.e. is a constant rotation, hence our hypothetical function with infinitesimal energy density is linear, and both singular values of its gradient equal $1 + \varepsilon_*$. This contradicts the no-stretch boundary condition.

To make this argument precise, we consider a subset of Ω of the form

$$U_{f,\sigma} = \{x \in \mathbb{R}^2 : x_2 \in (0,\sigma),\, f(x_2) \leqq x_1 \leqq f(x_2) + \sigma\}, \tag{2.3}$$

where $f : (0,\sigma) \to \mathbb{R}$ is a Lipschitz function with Lipschitz constant L. The $\{x_1 = f(x_2)\}$ part of the boundary of $U_{f,\sigma}$,

$$\Gamma_{f,\sigma} = \{(f(x_2), x_2) : x_2 \in (0,\sigma)\}, \tag{2.4}$$

is assumed to be part of $\partial\Omega$. For small enough σ, any given Lipschitz domain Ω has at least c/σ such disjoint subsets (modulo rotations and translations), where c depends on Ω. In each such domain, a lower bound for the energy is obtained by finding a fixed rotation $R \in SO(2)$ whose distance from the gradient field $\nabla\psi$ can be estimated in terms of the distance of $\nabla\psi$ from the set of all rotations $SO(2)$. Here we use a quantitative version of Reshetnyak's estimate recently obtained by Friesecke, James & Müller [13, 14], which is stated precisely in Proposition 1 below. The basic estimate in a domain $U_{f,\sigma}$ is given in the following Lemma

Lemma 1. *For any* $\psi : U_{f,\sigma} \to \mathbb{R}^3$ *such that*

$$\left|\psi(x_1,x_2) - \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}\right| \leqq \frac{1}{4} h \varepsilon_*^{1/2} \qquad \text{for} \quad (x_1,x_2) \in \Gamma_{f,\sigma}, \tag{2.5}$$

the energy is bounded from below by

$$I_{2\mathrm{D}}[\psi, U_{f,\sigma}] \geqq c_L \varepsilon_*^2 \sigma^2, \tag{2.6}$$

where c_L *depends only on the Lipschitz constant* L *of* f.

Proof. We first rescale ψ according to

$$\psi_\sigma(x) = \frac{1}{\sigma}\psi(\sigma x), \tag{2.7}$$

which gives $\nabla\psi_\sigma(x) = (\nabla\psi)(\sigma x)$, $\nu_\sigma(x) = \nu(\sigma x)$, $\nabla\nu_\sigma(x) = \sigma(\nabla\nu)(\sigma x)$, and correspondingly $f_\sigma(x_2) = \frac{1}{\sigma} f(\sigma x_2)$, which does not change the Lipschitz constant. Hence

$$I_{2\mathrm{D}}^{h,\varepsilon_*}[\psi, U_{f,\sigma}] = \sigma^2 I_{2\mathrm{D}}^{\varepsilon_*^{1/2},\varepsilon_*}[\psi_\sigma, U_{f_\sigma,1}] \tag{2.8}$$

and it suffices to prove the result for $\sigma = 1$, i.e., $h^2 = \varepsilon_*$. We argue by contradiction, and suppose that the assertion is false. Then there exist sequences ψ_j and U_j given by uniformly Lipschitz functions f_j (and $\sigma = 1$) such that

$$\frac{1}{\varepsilon_j^2}\int_{U_j} W_{2\mathrm{D}}^{\varepsilon_j}(\nabla\psi_j) + \varepsilon_j|\nabla\nu_j|^2 dx \to 0, \tag{2.9}$$

with the boundary condition

$$\left|\psi_j(x) - (x_1, x_2, 0)^T\right| \leqq \frac{1}{4}\varepsilon_j \qquad \text{for } x \in \Gamma_j. \tag{2.10}$$

Let $n_j(y_1, y_2) = \nu_j(y_1 + f_j(y_2), y_2)$. Then,

$$\frac{1}{\varepsilon_j^{1/2}}||\nabla n_j||_{L^2((0,1)^2)} \leqq \frac{1+L}{\varepsilon_j^{1/2}}||\nabla \nu_j||_{L^2(U_j)} \to 0. \tag{2.11}$$

Hence we can find unit vectors $\bar{\nu}_j \in S^2$ such that

$$\frac{1}{\varepsilon_j^{1/2}}||\nu_j - \bar{\nu}_j||_{L^p(U_j)} = \frac{1}{\varepsilon_j^{1/2}}||n_j - \bar{\nu}_j||_{L^p((0,1)^2)} \to 0 \qquad \forall p < \infty. \tag{2.12}$$

Let now $Q_j \in SO(3)$ be a rotation such that $Q_j\bar{\nu}_j = e_3$, and define $\psi_j' = PQ_j\psi_j$, where $P = e_1^{(2)} \otimes e_1^{(3)} + e_2^{(2)} \otimes e_2^{(3)}$ is the canonical projection of $\mathbb{R}^3$ into $\mathbb{R}^2$ (see Appendix A for details on notation). The two-dimensional vector field ψ_j' is the projection of ψ_j on the plane orthogonal to $\bar{\nu}_j$, and therefore has gradient close to $(1+\varepsilon_j)SO(2)$. More precisely, from Lemma 12 of Appendix A applied to $F = Q_j\nabla\psi$ and $\nu_F = Q_j\nu_j$ we get, for any $x \in U_j$,

$$\operatorname{dist}(\nabla\psi_j', (1+\varepsilon_j)SO(2)) \leqq c\left[\operatorname{dist}(\nabla\psi, (1+\varepsilon_j)O(2,3)) + |\nu_j - \bar{\nu}_j|^2\right]. \tag{2.13}$$

Comparing this with (2.12) (with $p = 4$) we see that

$$\varepsilon_j^{-2}\int_{U_j}\operatorname{dist}^2(\nabla\psi_j', (1+\varepsilon_j)SO(2)) \to 0. \tag{2.14}$$

By Proposition 1 there exist rotations $R_j \in SO(2)$ such that

$$\frac{1}{\varepsilon_j}||\nabla\psi_j' - (1+\varepsilon_j)R_j||_{L^2(U_j)} \to 0. \tag{2.15}$$

We now consider the trace of ψ_j' on the boundary Γ_j. By the trace theorem and the Poincaré inequality there are constants c_j such that

$$\frac{1}{\varepsilon_j}||\psi_j' - (1+\varepsilon_j)R_jx - c_j||_{L^2(\Gamma_j)} \to 0. \tag{2.16}$$

But this contradicts the boundary condition (2.10). To see this, consider for example that (2.16) implies

$$\frac{1}{\varepsilon_j}\left[\psi_j'(x) - \psi_j'(y) - (1+\varepsilon_j)R_j(x-y)\right] \to 0 \qquad \text{in } L^2(\Gamma_j \times \Gamma_j). \tag{2.17}$$

From the boundary condition (and the fact that ψ' is an orthogonal projection of ψ) we get

$$\begin{aligned} \left|\psi_j'(x) - \psi_j'(y)\right| &\leqq \left|\psi_j(x) - \psi_j(y)\right| \\ &\leqq |x-y| + \left|\psi_j(x) - \tilde{x}\right| + \left|\psi_j(y) - \tilde{y}\right| \\ &\leqq |x-y| + \tfrac{1}{2}\varepsilon_j, \end{aligned} \tag{2.18}$$

where $\tilde{x} = (x_1, x_2, 0)$ and $\tilde{y}$ has similar meaning. But $|R_j(x-y)| = |x-y|$, because R_j is a rotation, hence

$$\frac{1}{\varepsilon_j}\left|\psi_j'(x) - \psi_j'(y) - (1+\varepsilon_j)R_j(x-y)\right| \geqq |x-y| - \frac{1}{2}, \tag{2.19}$$

which contradicts (2.17).

We now conclude the proof of the lower bound. The details of the evaluation of the distance of projected gradients from the rotation group are presented in Appendix A.

Proof of Theorem 1. If σ is small enough there are $c_L|\partial\Omega|/\sigma$ disjoint domains of the form (2.3) along the boundary of Ω. Since by Lemma 1, in each of them the energy is at least $c_L\varepsilon_*^2\sigma^2$, for $\sigma < \sigma_0$ we conclude

$$I_{2D}^{h,\varepsilon_*}[\psi,\Omega] \geqq c_L|\partial\Omega|\sigma\varepsilon_*^2. \tag{2.20}$$

If instead $\sigma = h/\varepsilon_*^{1/2} \geqq \sigma_0$, we observe that

$$I_{2D}^{h,\varepsilon_*}[\psi,\Omega] \geqq I_{2D}^{\sigma_0\varepsilon_*^{1/2},\varepsilon_*}[\psi,\Omega] \geqq c_L|\partial\Omega|\sigma_0\varepsilon_*^2. \tag{2.21}$$

Since σ_0 depends only on Ω, the result follows.

Remark 2. In general, we cannot expect to be able to express the constant c_Ω in (2.2) as $c_L|\partial\Omega|$. To see this, consider for example $\Omega = (0,a)\times(0,1-a)$, with a small. Then, the energy of the identity map $\psi(x) = (x_1, x_2, 0)^T$, which is proportional to $\varepsilon_*^2|\Omega| = \varepsilon_*^2 a(1-a)$, vanishes for $a \to 0$, whereas $c_L|\partial\Omega|$ does not depend on a (σ_0 does).

3. Upper bound

We now prove the upper bound for the geometrically nonlinear Föppl-von Kármán problem, by explicitly constructing a deformation field with the optimal energy scaling. The FvK ansatz will allow us to naturally extend the same construction to the three-dimensional problem (see Section 4.2). Throughout this section we assume that

$$W_{2D}(F) \leqq c\left|F^TF - (1+\varepsilon_*)^2\mathrm{Id}_2\right|^2. \tag{3.1}$$

Since in the construction $(F^TF)^{1/2} - \mathrm{Id}_2$ is bounded by constants times ε_*, this bound is needed only for F sufficiently close to the null set of W_{2D}. Our main result is the following

Theorem 2 (Upper bound**).** *Let Ω be an open, bounded subset of $\mathbb{R}^2$ with piecewise C^4 boundary, and let W_{2D} obey* (3.1). *Then, there is a positive constant $\bar{c}_\Omega$ such that for any $\varepsilon_* \in (0,1)$ and small enough h there is a deformation field $\psi : \Omega \to \mathbb{R}^3$ such that*

$$I_{2D}^{h,\varepsilon_*}[\psi,\Omega] \leqq c_\Omega \varepsilon_*^{3/2} h = c_\Omega \varepsilon_*^2 \sigma, \tag{3.2}$$

which obeys the boundary conditions $\psi(x_1,x_2) = (x_1,x_2,0)$, $\nabla\psi(x_1,x_2) = \nabla(x_1,x_2,0)$ for $x \in \partial\Omega$.

The main ideas in the construction are best explained by considering a part of Ω around a flat boundary. For greater clarity, we first discuss (Section 3.1) the construction in a square, and then show how it can be extended to generic domains (Section 3.2). The proof of Theorem 2 is then given at the end of Section 3, after all necessary ingredients have been presented. As mentioned above, the scaling of the constructed ψ is obtained more naturally in terms of the compression ratio ε_* and of the rescaled thickness $\sigma = h/\varepsilon_*^{1/2}$.

3.1. Straight-boundary problem

In this section we construct a small-energy map $\psi : [0,1]^2 \to \mathbb{R}^3$ on a square which satisfies $\psi(0,t) = (0,t,0)$, $\partial_s\psi(0,t) = (1,0,0)$. The construction is done in such a way that it can be adapted, by an appropriate choice of coordinates, to smoothly curved boundaries (see Section 3.2). To better illustrate the structure of the various bounds, we do not indicate explicit values for the numeric constants. In this section, c denotes global numeric constants, which can change from appearance to appearance but do not depend on any of the parameters of the problem. In the following sections, when dealing with curved boundaries, c will depend on the domain, but never on the small parameters ε_* or h. We will always tacitly assume that the strain ε_* is bounded.

3.1.1. Preliminaries and general scheme. We make the ansatz

$$\psi(s,t) = \psi_0(s,t) + \psi_1(s,t), \tag{3.3}$$

where

$$\psi_0(s,t) = \begin{pmatrix} s \\ t \\ \delta s \end{pmatrix} = te_2 + s(1+\delta^2)^{1/2} e_2 \wedge \tau \tag{3.4}$$

is a plane inclined with slope δ, and the unit vectors τ and $e_2 \wedge \tau$ are defined so that τ is the normal to the plane, and the triplet $\{e_2, \tau, e_2 \wedge \tau\}$ forms an orthonormal basis of $\mathbb{R}^3$ with the canonical orientation. In components, we have

$$\tau = \frac{1}{(1+\delta^2)^{1/2}} \begin{pmatrix} -\delta \\ 0 \\ 1 \end{pmatrix}, \qquad e_2 \wedge \tau = \frac{1}{(1+\delta^2)^{1/2}} \begin{pmatrix} 1 \\ 0 \\ \delta \end{pmatrix}. \tag{3.5}$$

The slope δ is chosen so that ψ_0 relaxes the compression in the e_1 direction, i.e., that $|\psi_{0,s}|^2 = (1+\varepsilon_*)^2$. This gives

$$1+\delta^2 = (1+\varepsilon_*)^2, \tag{3.6}$$

which for small ε_* amounts to $\delta = (2\varepsilon_*)^{1/2} + O(\varepsilon_*)$. In the following we use interchangeably δ or ε_*, with the understanding that they are always related by (3.6).

The oscillations around the plane ψ_0, which relax the compression in the e_2 direction, are included via

$$\psi_1(s,t) = a(s,t)e_2 + b(s,t)\tau + d(s,t)e_2 \wedge \tau. \tag{3.7}$$

The functions a, b and d, which are Cartesian components of the oscillatory part of the deformation, have smooth dependence on s and high-frequency oscillations in t, and are constructed in Sections 3.1.2 and 3.1.3.

To estimate the energy in terms of a, b and d we compute

$$\psi_{,s} = a_{,s}e_2 + b_{,s}\tau + \left(d_{,s} + (1+\delta^2)^{1/2}\right) e_2 \wedge \tau \tag{3.8}$$

and

$$\psi_{,t} = (1+a_{,t})e_2 + b_{,t}\tau + d_{,t}e_2 \wedge \tau, \tag{3.9}$$

which give

$$|\psi_{,s}|^2 - (1+\delta^2) = a_{,s}^2 + b_{,s}^2 + d_{,s}^2 + 2(1+\delta^2)^{1/2} d_{,s}, \tag{3.10}$$

$$\psi_{,s}\cdot\psi_{,t} = Q_2 + d_{,s}d_{,t}, \tag{3.11}$$

$$|\psi_{,t}|^2 - (1+\delta^2) = Q_1 + d_{,t}^2, \tag{3.12}$$

where

$$Q_1 = (1+a_{,t})^2 + b_{,t}^2 - (1+\delta^2) \tag{3.13}$$

and

$$Q_2 = a_{,s}(1+a_{,t}) + b_{,s}b_{,t} + (1+\delta^2)^{1/2} d_{,t}. \tag{3.14}$$

In writing (3.10)–(3.12) we have collected the "dangerous" terms into the two quantities Q_1 and Q_2, whereas the others are bounded by

$$R = a_{,s}^2 + b_{,s}^2 + 2(1+\varepsilon_*)|d_{,s}| + d_{,s}^2 + d_{,t}^2. \tag{3.15}$$

This separation is motivated by the fact that in our construction the t derivatives will be larger than the s derivatives, and d will be smaller than a and b (see, e.g., (3.35), or the construction in Lemma 3 for more details). Then, it is easy to check that

$$\begin{aligned} W_{2\mathrm{D}}(\nabla\psi) &\leqq c\left\{\left[|\psi_{,s}|^2 - (1+\varepsilon_*)^2\right]^2 + \left(\psi_{,s}\cdot\psi_{,t}\right)^2 + \left[|\psi_{,t}|^2 - (1+\varepsilon_*)^2\right]^2\right\} \\ &\leqq c\left(Q_1^2 + Q_2^2 + R^2\right). \end{aligned} \tag{3.16}$$

Further, we observe that

$$|Q_1| + |Q_2| + |R| \leqq \tfrac{1}{4} \tag{3.17}$$

implies that a, b and d are uniformly Lipschitz, and

$$|\psi_{,s} \wedge \psi_{,t}|^2 = \psi_{,s}^2 \psi_{,t}^2 - (\psi_{,s} \cdot \psi_{,t})^2 \geqq \tfrac{1}{4}, \tag{3.18}$$

hence – at least as long as (3.17) is satisfied – we can estimate $|\nabla v|$ by $|\nabla^2 \psi|$. We first construct functions which satisfy $Q_1 = 0$ and consider suitable rescalings to obtain highly oscillatory functions with small amplitude. The main point is then to estimate the energy in a box inside which the period of the oscillations (in t) doubles.

3.1.2. Construction of periodic solutions. We first construct small-energy functions (a, b, d) which are periodic in t, and do not depend on s. The main requirement on such functions is that Q_1, as defined in (3.13), be zero. This gives a differential relation between a and b, which is best dealt with by collecting the two into a single vectorial function, γ (see (3.29)). In the definition of γ we also impose periodicity conditions which will be helpful in the branching construction of Section 3.1.3. Whereas in the geometrically linear case we could exhibit an explicit solution for the periodic profile, here an implicit characterization turns out to be simpler, and it is given by the following

Lemma 2. *For every $\varepsilon_* > 0$ there exists a C^∞ curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ with the properties:*

$$|\gamma'| = 1 + \varepsilon_*, \tag{3.19}$$

$$\gamma_1' \geqq 0, \tag{3.20}$$

$$\gamma(t+1) = \gamma(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{3.21}$$

$$\gamma(-t) = -\gamma(t), \tag{3.22}$$

$$\gamma\left(\tfrac{1}{2}\right) = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}, \tag{3.23}$$

and satisfying the bound

$$|\gamma_1 - t| + \left|\gamma_1' - 1\right| + \left|\gamma_1''\right| \leqq c\varepsilon_*, \qquad |\gamma_2| + \left|\gamma_2'\right| + \left|\gamma_2''\right| \leqq c\varepsilon_*^{1/2}, \tag{3.24}$$

where c does not depend on ε_*.

Proof. We construct γ by rescaling the curves

$$\tilde{\gamma}(\tau) = \begin{pmatrix} \tau \\ \frac{A}{2\pi} \sin 2\pi\tau \end{pmatrix}, \tag{3.25}$$

which obey (3.20)–(3.23) for any A. To achieve (3.19) we first choose A so that the length of $\tilde{\gamma}([0, 1])$ is $1 + \varepsilon_*$, and then reparametrize proportionally to arc length. More specifically, consider the rescaled length of the $\tilde{\gamma}([0, \tau])$ curve,

$$\Lambda(\tau) = \frac{1}{1+\varepsilon_*} \int_0^\tau |\tilde{\gamma}'(\tau)|\, d\tau = \frac{1}{1+\varepsilon_*} \int_0^\tau \sqrt{1 + A^2 \cos^2 2\pi\tau}\, d\tau. \tag{3.26}$$

Since $\Lambda(1)$, is a smooth monotone function of A, there exists a unique A_* such that $\Lambda(1) = 1$. Fix $A = A_*$. Then Λ is smooth and odd, $\Lambda' \geqq 1/(1 + \varepsilon_*)$, and $\Lambda(\tau + 1) = \Lambda(\tau) + 1$. Thus the inverse function Λ^{-1} is smooth, odd and has the property $\Lambda^{-1}(t + 1) = \Lambda^{-1}(t) + 1$. Hence $\gamma = \tilde{\gamma} \circ \Lambda^{-1}$ has the properties (3.19)–(3.23). To check (3.24) we consider the small-ε_* limit. From (3.26) we get

$$(1 + \varepsilon_*)\Lambda(\tau) = \tau + \frac{1}{4}A^2\left(\tau + \frac{1}{4\pi}\sin 4\pi\tau\right) + O(A^4), \qquad (3.27)$$

hence the condition $\Lambda(1) = 1$ gives $A_* = 2\varepsilon_*^{1/2} + O(\varepsilon_*)$. Then,

$$\gamma_1(t) = t - \frac{\varepsilon_*}{4\pi}\sin 4\pi t + O(\varepsilon_*^2), \qquad \gamma_2(t) = \frac{\varepsilon_*^{1/2}}{\pi}\sin 2\pi t + O(\varepsilon_*), \quad (3.28)$$

which proves the bounds on $|\gamma_1 - t|$ and $|\gamma_2|$ in (3.24). The remaining bounds are easily established by computing the derivatives of both sides of the relation $\tilde{\gamma} = \gamma \circ \Lambda$, which gives $\tilde{\gamma}' = \gamma'\Lambda'$ and $\tilde{\gamma}'' = \gamma''\Lambda'^2 + \gamma'\Lambda''$.

Given $\gamma(t)$, we define odd, one-periodic functions by

$$\hat{a}(t) = \gamma_1(t) - t, \qquad \hat{b}(t) = \gamma_2(t) \qquad (3.29)$$

which satisfy

$$\hat{a}\left(-\tfrac{1}{2}\right) = \hat{a}(0) = \hat{a}\left(\tfrac{1}{2}\right) = 0, \qquad (3.30)$$

$$\hat{b}\left(-\tfrac{1}{2}\right) = \hat{b}(0) = \hat{b}\left(\tfrac{1}{2}\right) = 0. \qquad (3.31)$$

Moreover, for $t \in [-1, 1]$,

$$\hat{d}(t) = \frac{1}{2(1 + \varepsilon_*)}\left[t^2(1 + \varepsilon_*)^2 - \gamma_1^2(t) - \gamma_2^2(t)\right], \qquad (3.32)$$

which is even and satisfies

$$(1 + \varepsilon_*)\hat{d}' + (1 + \hat{a}')(\hat{a} - t\hat{a}') + \hat{b}'(\hat{b} - t\hat{b}') = 0. \qquad (3.33)$$

The bounds (3.24) on γ are directly translated into

$$|\hat{a}| + |\hat{a}'| + |\hat{a}''| + |\hat{d}| + |\hat{d}'| + |\hat{d}''| \leqq c\varepsilon_*, \qquad |\hat{b}| + |\hat{b}'| + |\hat{b}''| \leqq c\varepsilon_*^{1/2}. \qquad (3.34)$$

To conclude this section, we display a periodic solution with period h,

$$a(s, t) = h\hat{a}\left(\frac{t}{h}\right), \quad b(s, t) = h\hat{b}\left(\frac{t}{h}\right), \quad d(s, t) = 0. \qquad (3.35)$$

(The use of h here and below to denote periods should not be confused with the use of h as film thickness in the introduction.) It is easy to check that $W_{2\mathrm{D}}(\nabla\psi) = 0$ and $|\nabla^2\psi|^2 \leqq c\varepsilon_* h^{-2}$, hence the energy in a given box is bounded by

$$I_{2\mathrm{D}}[\psi, (0, l) \times (0, h)] \leqq c\varepsilon_*^2 hl\frac{\sigma^2}{h^2}. \qquad (3.36)$$

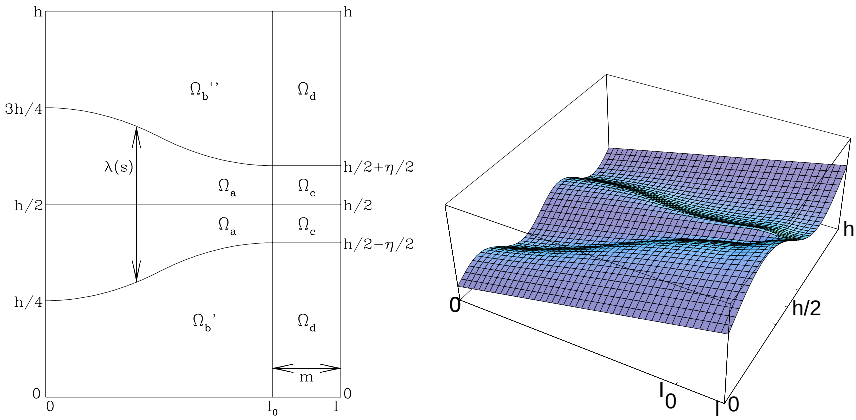

Fig. 1. Subdivision of the domain B used in the construction of Lemma 3 (*left*) and representation of the constructed ψ (*right*).

Remark 3. In the small-ε_* limit the present construction reduces to the one used for the geometrically linear theory in [8]. Indeed, comparing (A.6) and (3.2) of [8] with (3.7) above, we see that to leading order in ε_* the triplet (a, b, d) used here coincides with $(2\varepsilon_* v, (2\varepsilon_*)^{1/2} w, 2\varepsilon_* z)$ in the notation of [8]. Then, the small-ε_* limit of (3.28) gives exactly (3.4) and (3.5) of [8]. Further, (3.32) and (3.33) correspond to (3.12) and (3.13) of [8].

3.1.3. Fold branching. We now show how the oscillation period of the construction (3.35) can change with s. In particular, we take (3.35) with some period h for $s = 0$, the same profile with double period at some distance $s = l$, and construct the deformation in the intermediate region, where the width of the central fold decreases to zero. In order to keep the stretching energy $W_{2D}(\nabla\psi)$ small we use the third component of the deformation, which is described by the function d. Since a change in the slope of order 1 in a distance less that σ would determine high bending energy $|\nabla v|^2$, we can smoothly reduce the width of the inner fold only down to some finite value (called η below), and then we must decrease its amplitude instead of its width (see (3.64)). The decreasing width of the central fold in the first part of the construction is described by means of a function $\phi : \mathbb{R} \to [0, 1]$ such that $\phi(t) = 0$ for $t \leqq 0$, $\phi(t) + \phi(1 - t) = 1$, with its first, second and third derivative bounded, and $\phi(t) \geqq t^3$ in $(0, 1)$. The bending term will be bounded using the estimate

$$\int_0^\xi \frac{dt}{\lambda + \zeta\phi(t)} \leqq \int_0^\infty \frac{dt}{\lambda + \zeta t^3} \leqq \frac{c}{\lambda^{2/3}\zeta^{1/3}}, \tag{3.37}$$

where $\xi \in (0, 1)$, and $\lambda, \zeta > 0$.

Lemma 3. *There exists a constant $\bar{c} > 0$ such that in any rectangle $B = (0, l) \times (0, h)$ with $l \geqq \bar{c}h$, $h \geqq \sigma$, there is a deformation ψ which satisfies*

$$a = b = d = 0, \qquad a_{,t} = \hat{a}'(0), \quad b_{,t} = \hat{b}'(0), \quad d_{,t} = 0 \quad \text{for } t = 0 \text{ and } t = h, \tag{3.38}$$

$$a = \frac{h}{2}\hat{a}\left(\frac{t}{h/2}\right), \quad b = \frac{h}{2}\hat{b}\left(\frac{t}{h/2}\right), \quad a_{,s} = b_{,s} = d = d_{,s} = 0 \qquad \text{for } s = 0, \tag{3.39}$$

$$a = h\hat{a}\left(\frac{t}{h}\right), \quad b = h\hat{b}\left(\frac{t}{h}\right), \quad a_{,s} = b_{,s} = d = d_{,s} = 0 \qquad \text{for } s = l, \tag{3.40}$$

with energy bounded by

$$\int_B Q_1^2 + Q_2^2 + R^2 + \varepsilon_* \sigma^2 |\nabla^2 (a|b|d)| \leqq c\varepsilon_*^2 \left[\frac{h^5}{l^3} + \sigma l \left(\frac{\sigma}{h}\right)^{1/2} + \sigma h \left(\frac{\sigma}{h}\right)^{1/8}\right], \tag{3.41}$$

and with the pointwise bounds $|Q_1| + |Q_2| + |R| \leqq \frac{1}{4}$,

$$|a| + |d| \leqq c\varepsilon_* h, \qquad |\nabla a| + |\nabla d| \leqq c\varepsilon_*, \qquad \sigma|\nabla^2 a| + \sigma|\nabla^2 d| \leqq c\varepsilon_*, \tag{3.42}$$

$$|b| \leqq c\varepsilon_*^{1/2} h, \qquad |\nabla b| \leqq c\varepsilon_*^{1/2}, \qquad \sigma|\nabla^2 b| \leqq c\varepsilon_*^{1/2}. \tag{3.43}$$

Remark 4. Using (3.16) we can immediately transform the present result (3.41) into a bound on $I_{2\text{D}}$,

$$I_{2\text{D}}[\psi, B] \leqq c\varepsilon_*^2 \left[\frac{h^5}{l^3} + \sigma l \left(\frac{\sigma}{h}\right)^{\frac{1}{2}} + \sigma h \left(\frac{\sigma}{h}\right)^{1/8}\right]. \tag{3.44}$$

The statement of the lemma just presented allows for a simpler extension to the curvilinear case (see (3.98)).

Proof. We first decompose the domain into the part of length l_0 where the inner fold smoothly decreases its width from $h/2$ to η, and the one where it disappears by interpolation, of length $m = l - l_0$ (see Fig. 1). The values of m and η will be chosen below, now we merely assume the ordering $\eta \leqq m \leqq h \leqq l$ which leads to many simplifications.

For $s \in [0, l_0]$ the width of the inner fold is given by

$$\lambda(s) = \frac{h}{2}\left[1 - \phi\left(\frac{s}{l_0}\right)\right] + \eta\phi\left(\frac{s}{l_0}\right), \tag{3.45}$$

which smoothly decreases from $h/2$ to η. The construction is done in three separate pieces: Ω_a is the region $|y - h/2| \leqq \lambda(s)/2$ occupied by the "small" fold, Ω_b' is the region $0 \leqq y \leqq [h - \lambda(s)]/2$ occupied by the first half of the "large" fold, and Ω_b'' is the one occupied by the other half (see Fig. 1).

We start the construction from the lower side of the rectangle. In Ω_b' we set

$$a(s,t) = [h - \lambda(s)]\hat{a}\left(\frac{t}{h - \lambda(s)}\right), \tag{3.46}$$

$$b(s,t) = [h - \lambda(s)]\hat{b}\left(\frac{t}{h - \lambda(s)}\right), \tag{3.47}$$

$$d(s,t) = -\lambda(s)'[h - \lambda(s)]\hat{d}\left(\frac{t}{h - \lambda(s)}\right). \tag{3.48}$$

Since $\hat{a}$ and $\hat{b}$ are 1-periodic, $\lambda(0) = h/2$ and $\lambda'(0) = \lambda''(0) = 0$, the boundary condition (3.39) is satisfied. It is easily checked that also (3.38) is satisfied for $t = 0$. On the upper boundary $t = (h - \lambda)/2$ we get

$$\begin{aligned} a = b = 0, \qquad & d = -\lambda'(h-\lambda)\hat{d}\big(\tfrac{1}{2}\big), \\ a_{,t} = \hat{a}'\big(\tfrac{1}{2}\big), \quad b_{,t} = \hat{b}'\big(\tfrac{1}{2}\big), \quad & d_{,t} = -\lambda'\hat{d}'\big(\tfrac{1}{2}\big). \end{aligned} \tag{3.49}$$

Now consider the central region Ω_a, which corresponds to $|t - h/2| \leqq \lambda/2$ (see Fig. 1). We set

$$a(s,t) = \lambda(s)\hat{a}\left(\frac{t - h/2}{\lambda(s)}\right), \tag{3.50}$$

$$b(s,t) = \lambda(s)\hat{b}\left(\frac{t - h/2}{\lambda(s)}\right), \tag{3.51}$$

$$d(s,t) = \lambda'(s)\left[\lambda(s)\hat{d}\left(\frac{t - h/2}{\lambda(s)}\right) + \zeta(s)\right], \tag{3.52}$$

where $\zeta(s)$ will be chosen later. The boundary condition (3.39) for $s = 0$ is again automatically satisfied. Since $\hat{a}$ and $\hat{b}$ are odd and $\hat{d}$ is even, for $t = (h - \lambda)/2$ we obtain again (3.49), provided that

$$-\lambda'(h-\lambda)\hat{d}\big(\tfrac{1}{2}\big) = \lambda'\left[\lambda\hat{d}\big(\tfrac{1}{2}\big) + \zeta\right] \tag{3.53}$$

which is satisfied if we choose $\zeta = -h\hat{d}(1/2)$. For $t = (h + \lambda)/2$ only the sign of the derivative of d changes,

$$\begin{aligned} a = b = 0, \qquad & d = \lambda'\lambda(\hat{d}\big(\tfrac{1}{2}\big) + \zeta), \\ a_{,t} = \hat{a}'\big(\tfrac{1}{2}\big), \quad b_{,t} = \hat{b}'\big(\tfrac{1}{2}\big), \quad & d_{,t} = \lambda'\hat{d}'\big(\tfrac{1}{2}\big). \end{aligned} \tag{3.54}$$

Finally in Ω_b'' we set

$$a(s,t) = [h - \lambda(s)]\hat{a}\left(\frac{t-h}{h - \lambda(s)}\right), \tag{3.55}$$

$$b(s,t) = [h - \lambda(s)]\hat{b}\left(\frac{t-h}{h - \lambda(s)}\right), \tag{3.56}$$

$$d(s,t) = -\lambda(s)'[h - \lambda(s)]\hat{d}\left(\frac{t-h}{h - \lambda(s)}\right). \tag{3.57}$$

It is a simple check that the boundary conditions (3.39) [for $s = 0$], (3.54) [for $t = (h+\lambda)/2$] and (3.38) [for $t = h$] are satisfied. This concludes the construction in the region $s \in [0, l_0]$. To estimate the energy of this test function, we compute Q_1, Q_2 and R and use (3.16). We give details of the computation only for the region Ω_a, since the other ones need only minor changes. First we observe that

$$Q_1 = (1 + a_{,t})^2 + b_{,t}^2 - (1 + \delta^2) = (1 + \hat{a}')^2 + (\hat{b}')^2 - (1 + \varepsilon_*)^2 = 0, \quad (3.58)$$

where $\hat{a}$ and $\hat{b}$ are evaluated at $\xi = (t - h/2)/\lambda(s)$. The s-derivatives have the form $a_{,s} = \lambda'(s)(\hat{a} - \xi\hat{a}')(\xi)$. Then, we get

$$Q_2 = a_{,s}(1 + a_{,t}) + b_{,s}b_{,t} + (1 + \varepsilon_*)d_{,t} \quad (3.59)$$

$$= \lambda' \left[(\hat{a} - \xi\hat{a}')(1 + \hat{a}') + (\hat{b} - \xi\hat{b}')\hat{b}' + (1 + \varepsilon_*)\hat{d}' \right] = 0 \quad (3.60)$$

(from (3.33)). Finally, from (3.34) we get

$$R \leqq c\varepsilon_* \left(\lambda'^2 + h\lambda''\right) \leqq c\varepsilon_* \frac{h^2}{l_0^2} \quad (3.61)$$

which gives $W_{2D} \leqq c\varepsilon_*^2 h^4/l^4$. Since we assumed $h \leqq l/\bar{c}$, with a suitable choice of $\bar{c}$ we can enforce $R \leqq 1/8$ (and use (3.18)).

Now we compute the second gradient. The most dangerous term is the ∂_t^2 derivative in Ω_a, which diverges as $\varepsilon_*^{1/2}/\lambda$. However the area is only of size λ, and we get, using (3.37),

$$\int_0^{l_0} ds \int_{-\lambda/2}^{\lambda/2} dt \frac{\varepsilon_*}{(\lambda(s))^2} = \int_0^{l_0} ds \frac{\varepsilon_*}{\lambda(s)} \leqq \frac{cl\varepsilon_*}{\eta^{2/3}h^{1/3}}. \quad (3.62)$$

The other terms in $|\nabla^2 a|^2 + |\nabla^2 b|^2 + |\nabla^2 d|^2$ are bounded by $c\varepsilon_*/h^2 + c\varepsilon_*/l^2$, and after integration give the lower-order contributions $c\varepsilon_* l/h + c\varepsilon_* h/l$. Hence

$$I_{2D}[\psi, \Omega_a \cup \Omega_b' \cup \Omega_b''] \leqq c\varepsilon_*^2 \left[\frac{h^5}{l^3} + \sigma^2 \frac{l}{h^{1/3}\eta^{2/3}} \right]. \quad (3.63)$$

It remains to construct ψ in the region $s \in [l_0, l]$. This is done by using a smooth interpolation between the values at $s = l_0$ and and $s = l$,

$$\psi(s, t) = \psi(l_0, t) \left[1 - \phi\left(\frac{s}{m}\right) \right] + \psi(l, t)\phi\left(\frac{s}{m}\right), \quad (3.64)$$

where $\psi(l, t)$ is given by (3.40). This has small energy because the two values between which we are interpolating differ significantly only in the small set $\Omega_c = [l_0, l] \times [(h - \eta)/2, (h + \eta)/2]$, whereas in the larger set $\Omega_d = [l_0, l] \times \{\eta/2 \leqq |t - h/2| \leqq h/2\}$ they are similar and both have small energy. More precisely, in Ω_c we have $|\nabla b| \leqq c\varepsilon_*^{1/2}$, $|\nabla a| \leqq c\varepsilon_*$, $d = 0$ and hence $|Q_1| + |Q_2| + |R| \leqq c\varepsilon_*^2$. The second gradient is controlled by $|\nabla^2\psi| \leqq c\varepsilon_*^{1/2}(\eta/m^2 + 1/\eta)$. Thus

$$I_{2D}[\psi, \Omega_c] \leqq c\varepsilon_*^2[m\eta + \sigma^2 m/\eta]. \quad (3.65)$$

In Ω_d, however,

$$|a(l_0,t)-a(l,t)| \leqq c\varepsilon_*\eta, \qquad |a_{,t}(l_0,t)-a_{,t}(l,t)| \leqq c\varepsilon_*\frac{\eta}{h} \tag{3.66}$$

and

$$|b(l_0,t)-b(l,t)| \leqq c\varepsilon_*^{1/2}\eta, \qquad |b_{,t}(l_0,t)-b_{,t}(l,t)| \leqq c\varepsilon_*^{1/2}\frac{\eta}{h} \tag{3.67}$$

since $a(l,\cdot)$ and $a(l_0,\cdot)$ have been defined as different rescalings of the same smooth function $\hat{a}$ (and similarly for b), and $d=0$. It follows that

$$Q_1+Q_2 \leqq c\varepsilon_*\frac{\eta}{m}, \qquad R \leqq c\varepsilon_*\frac{\eta^2}{m^2} \tag{3.68}$$

which give $\int_{\Omega_c} W_{2D} \leqq c\varepsilon_*^2 h\eta^2/m$. The bending term is bounded by $|\nabla^2\psi| \leqq c\varepsilon_*^{1/2}(1/h+\eta/m^2+\eta/mh)$. Hence

$$I_{2D}[\psi,\Omega_d] \leqq c\varepsilon_*^2[h\eta^2/m+\sigma^2 m/h+\sigma^2 h\eta^2/m^3]. \tag{3.69}$$

Collecting the various terms in (3.63), (3.65) and (3.69), and dropping the irrelevant ones, we get

$$I_{2D}[\psi,B] \leqq c\varepsilon_*^2\left[\frac{h^5}{l^3}+\frac{\sigma^2 l}{\eta^{2/3}h^{1/3}}+\frac{\eta^2 h}{m}+m\eta+\sigma^2\frac{m}{\eta}+\sigma^2\frac{h\eta^2}{m^3}\right]. \tag{3.70}$$

We finally fix $\eta=\sigma^{3/4}h^{1/4}$ and $m=\bar{c}(\eta h)^{1/2}=\bar{c}\sigma^{3/8}h^{5/8}$ and obtain (3.41). The bound on $|Q_1|+|Q_2|+|R|$ is then obtained by choosing $\bar{c}$ (which determines m) large enough with respect to all other constants entering the estimates above.

3.1.4. Global solution in a square and qualitative discussion. To better illustrate the role of the previous results in the global construction, we now show how they can be used to obtain a test function to a model problem in the unit square, as outlined at the beginning of Section 3.1. The construction is done in two steps: first we obtain a deformation field with small energy which obeys the boundary condition on $s=0$ only approximately, i.e., such that $|\psi(s,t)-(0,t,0)| \leqq c\varepsilon_*^{1/2}\sigma$, and then modify it at small s to obtain an exact solution. The resulting ψ is illustrated in Fig. 2.

The approximate boundary condition is clearly satisfied by an oscillatory solution of (3.35) with period $h=\sigma$. We take this solution in the region $s\in[0,\sigma]$, where we shall later interpolate. This function has energy ε_*^2 per unit area, hence we cannot use it in a region of area larger than σ. In the interior part of the sample we use coarser oscillations, which are obtained through a number of period-doubling transitions (as constructed in Section 3.1.3). Let $h_i=2^i\sigma$ be the period at step i, l_i be the width of the i-th period-doubling step, and $L_i=\sum_{k\leq i} l_k$ be the cumulative width up to the i-th step. Then, for each i, in each of the rectangles

$$R_{ij}=(L_{i-1},L_i)\times(jh_i,(j+1)h_i), \qquad 0\leqq j\leqq\frac{1}{h_i}, \tag{3.71}$$

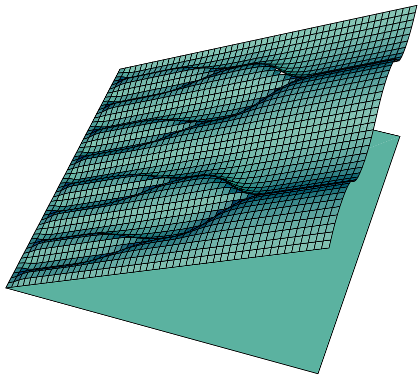

Fig. 2. Construction of ψ on a square with boundary conditions on one side.

we define $\bar{\psi}$ as in Lemma 3 with $l = l_i$ and $h = h_i$. The energy of $\bar{\psi}$ is estimated by

$$I_{2D}[\bar{\psi}, (0,1)^2] \leqq c\varepsilon_*^2 \sum_i \frac{1}{h_i}\left[\frac{h_i^5}{l_i^3} + \sigma l_i \left(\frac{\sigma}{h_i}\right)^{1/2} + \sigma h_i \left(\frac{\sigma}{h_i}\right)^{1/8}\right]. \tag{3.72}$$

We substitute in this expression $h_i = \sigma 2^i$ and $l_i = \sigma 2^{\alpha i}$, and obtain

$$I_{2D}[\bar{\psi}, (0,1)^2] \leqq c\varepsilon_*^2 \sigma \sum_i 2^{(4-3\alpha)i} + 2^{\left(\alpha - \frac{3}{2}\right)i} + 2^{-i/8} \tag{3.73}$$

All geometric series converge provided that $\frac{4}{3} < \alpha < \frac{3}{2}$, hence for all such α we get the desired bound. The constructed ψ obeys the bounds

$$|\nabla \bar{\psi} - \mathrm{Id}| \leqq c\varepsilon_*^{1/2}, \qquad |\nabla^2 \bar{\psi}| \leqq \frac{c\varepsilon_*^{1/2}}{\sigma}, \tag{3.74}$$

and for $s < \sigma$ also

$$\left|\bar{\psi} - \begin{pmatrix} s \\ t \\ 0 \end{pmatrix}\right| \leqq c\varepsilon_*^{1/2}\sigma. \tag{3.75}$$

Finally, we smooth $\bar{\psi}$ close to the boundary, and define

$$\psi(s,t) = \phi\left(\frac{s}{\sigma}\right)\bar{\psi}(s,t) + \left[1 - \phi\left(\frac{s}{\sigma}\right)\right]\begin{pmatrix} s \\ t \\ 0 \end{pmatrix}, \tag{3.76}$$

where ϕ is a smoothed step function, as defined before (3.37). For $s \geqq \sigma$, $\psi = \bar{\psi}$. For $s = 0$, ψ agrees with the identity map $(s, t, 0)$ up to the first derivative, hence the boundary condition is satisfied. In the region $s \leqq \sigma$, the bounds (3.74) hold also for ψ. Since $W_{2D}(\mathrm{Id} + G) \leqq c(\varepsilon_*^2 + |G|^2)$, we obtain the desired estimate

$$I_{2D}[\psi, (0,1)^2] \leqq c\varepsilon_*^2\sigma = c\varepsilon_*^{3/2}h. \tag{3.77}$$

This concludes the construction in the case of a square with boundary conditions only on one side.

The simple geometry of this example enables us to discuss the qualitative aspects of the construction. First, we observe that the (11) component of the compressive stress is simply relaxed by the tilting of the horizontal plane to the plane described by ψ_0. Therefore the problem is essentially equivalent to a uniaxial compression in the (22) direction, which cannot be relaxed by large-scale deformations, due to the boundary conditions. This is the reason for the appearance of the oscillations. Interesting analogies can be drawn with the phenomenon of twin-branching near an austenite–martensite interface studied in [21, 22, 11]. In our problem, oscillation branching arises because the boundary condition enforces short-scale oscillations at small s, whereas the bending term favors large-scale oscillations in the bulk. At variance with the martensitic problem, there is no preferred slope in this problem, and only a constraint in the total length. This explains why we have smooth oscillations here, whereas there are flat regions separated by thin interfaces in the case of martensitic twins. It is also interesting to observe the similarity of the straight-twin constructions used in the study of martensites before the branching analysis of KOHN & MÜLLER [21, 22] and the straight-fold construction used by ORTIZ & GIOIA [32] for the present problem.

3.2. Domains with curved boundary

In this section we show how the preceding construction can be adapted to generic smooth domains. The mapping from the straight boundary of Section 3.1 to a curved one is done by using as coordinates arc length and distance to the boundary, see Section 3.2.1. Then, in Section 3.2.2 we show how to construct a low-energy test function ψ in a curvilinear triangle which, via triangulation of the original domain, suffices to prove Theorem 2. In the following we denote by c a generic constant that depends only on the domain.

3.2.1. Neighborhood of a curved boundary. Consider a C^4 curve $\alpha : (0, L) \to \mathbb{R}^2$ (which later will be part of the boundary of the domain where we perform the construction), parametrized by arc length. Let $n = (\alpha_2', -\alpha_1')$ be a unit normal of α (later the inward normal to the boundary of Ω). Then $n' = -\kappa\alpha'$, where κ is the curvature of α, and

$$\Phi : (s, t) \to (x_1, x_2) = \alpha(t) + sn(t) \tag{3.78}$$

defines a diffeomorphism of a rectangle $(0, H) \times (0, L)$ to a (one-sided) tubular neighborhood of the curve α, provided H is sufficiently small (in the following, we always assume that $H|\kappa| \leqq \frac{1}{2}$).

To extend our construction from rectangles to curved domains we adapt our ansatz (3.3) as follows. We set

$$\tilde{\psi}(x_1, x_2) = \tilde{\psi}_0(x_1, x_2) + \tilde{\psi}_1(x_1, x_2) \tag{3.79}$$

where the plane (3.4) is replaced by

$$\tilde{\psi}_0(x_1, x_2) = \begin{pmatrix} x_1 \\ x_2 \\ \delta \operatorname{dist}(x, \alpha) \end{pmatrix} \tag{3.80}$$

and $\operatorname{dist}(x, \alpha) = s$ is the distance to the curve described by α. The normal to $\tilde{\psi}_0$, expressed in (s, t) coordinates, is

$$\tau(t) = \frac{1}{(1 + \delta^2)^{1/2}}(-\delta n(t) + e_3). \tag{3.81}$$

The generalization of the oscillatory part (3.7) is then

$$\psi_1(s, t) = (1 - s\kappa(t)) \left[a(s, t)\alpha'(t) + b(s, t)\tau(t)\right] + (1 - s\kappa(t))^2 d(s, t)\alpha'(t) \wedge \tau. \tag{3.82}$$

Here and below, we denote by $\tilde{\psi}(x_1, x_2)$ a deformation field expressed in Cartesian coordinates, and by $\psi = \tilde{\psi} \circ \Phi$ the same deformation field expressed as a function of the boundary-adapted coordinates (s, t). For example, the field $\tilde{\psi}_1(x_1, x_2)$ entering (3.79) is obtained from $\psi(s, t)$ defined in (3.82) by $\tilde{\psi}_1 = \psi_1 \circ \Phi^{-1}$, and analogously

$$\tilde{\psi}_0 \circ \Phi = \psi_0(s, t) = \alpha(t) + sn(t) + \delta s e_3. \tag{3.83}$$

To proceed, we first express the strain energy in the new coordinates. Then we bound it in terms of a, b and d (and their derivatives) alone. More precisely we establish a bond solely in the quantities Q_1, Q_2 and R introduced in (3.13)–(3.15). This will allow us to apply the estimates derived for the rectangle to a generic domain without changes. The corresponding estimate for the second derivatives is straightforward.

Lemma 4. *Let ψ be of the form* (3.79)–(3.82)*, where a, b and d satisfy*

$$|a| + |d| \leqq c\varepsilon_* q, \qquad |\nabla a| + |\nabla d| \leqq c\varepsilon_*, \qquad |b| \leqq c\varepsilon_*^{1/2} q, \qquad |\nabla b| \leqq c\varepsilon_*^{1/2} \tag{3.84}$$

for some $q > 0$. Then, at any point $(x_1, x_2) = \Phi(s, t)$,

$$W_{2D}(\nabla_x \tilde{\psi}) = W_{2D}\left(\psi_{,s} \left| \frac{\psi_{,t}}{1 - s\kappa(t)}\right.\right) \leqq c\left(Q_1^2 + Q_2^2 + R^2 + \varepsilon_*^2 q^2\right) \tag{3.85}$$

where Q_1, Q_2 and R have been defined in (3.13)–(3.15)*, and*

$$|\nabla_x^2 \tilde{\psi}|^2 \leqq c\left[\varepsilon_* + q^2 \varepsilon_* + |\nabla_{s,t}^2(a, b, d)|^2\right]. \tag{3.86}$$

Proof. To prove the first equality, we compute the gradient of the diffeomorphism Φ,

$$\frac{\partial(x_1, x_2)}{\partial(s,t)} = \nabla\phi = n \otimes n + (1 - s\kappa)\alpha' \otimes \alpha', \tag{3.87}$$

and observe that W_{2D} is invariant under rotations. Hence the change from (e_1, e_2) components to (n, α') components leaves W_{2D} unchanged, and only the factor $(1 - s\kappa)$ needs to be taken explicitly into account.

To prove the upper bound in (3.85), we first compute

$$\psi_{0,s} = n(t) + \delta e_s = (1 + \varepsilon_*)\alpha' \wedge \tau \tag{3.88}$$

and

$$\psi_{0,t} = \alpha'(t) + sn'(t) = (1 - s\kappa)\alpha'. \tag{3.89}$$

In computing the derivatives of the oscillatory part ψ_1 we can avoid explicit consideration of all terms bounded by $\varepsilon_* q$, i.e., all terms where either a or d is not differentiated. Terms where b is not differentiated can be ignored only if there is an additional factor of $\varepsilon_*^{1/2}$, which is the case, e.g., for the terms where $\partial_t\tau = -\delta n'/(1 + \varepsilon_*)$ enters as a factor. The result is

$$\begin{aligned} \psi_{,s} = (1 - s\kappa)\left[a_{,s}\alpha' + \left(b_{,s} - \frac{\kappa}{1 - s\kappa} b\right)\tau\right] \\ + \left[(1 + \varepsilon_*) + (1 - s\kappa)^2 d_{,s}\right]\alpha' \wedge \tau + O(\varepsilon_* q) \end{aligned} \tag{3.90}$$

and

$$\frac{\psi_{,t}}{1 - s\kappa} = (1 + a_{,t})\alpha' + \left(b_{,t} - \frac{s\kappa'}{1 - s\kappa} b\right)\tau + (1 - s\kappa)d_{,t}\alpha' \wedge \tau + O(\varepsilon_* q), \tag{3.91}$$

where $O(\varepsilon_* q)$ represents terms which can be bounded in absolute value by $c\varepsilon_* q$. Then, we compute, analogously to (3.10)–(3.12),

$$\begin{aligned} |\psi_{,s}|^2 - (1 + \varepsilon_*)^2 = (1 - s\kappa)^2\left(a_{,s}^2 + b_{,s}^2\right) + (1 - s\kappa)^4 d_{,s}^2 \\ +2(1 + \varepsilon_*)(1 - s\kappa)^2 d_{,s} + O(\varepsilon_* q), \end{aligned} \tag{3.92}$$

$$\frac{\psi_{,s} \cdot \psi_{,t}}{1 - s\kappa} = (1 - s\kappa)Q_2 + (1 - s\kappa)^2 d_{,s} d_{,t} + O(\varepsilon_* q), \tag{3.93}$$

$$\frac{|\psi_{,t}|^2}{(1 - s\kappa)^2} - (1 + \varepsilon_*)^2 = Q_1 + (1 - s\kappa)d_{,t}^2 + O(\varepsilon_* q). \tag{3.94}$$

The two terms containing $b\tau$ have become also of order $\varepsilon_* q$ since they are squared or multiplied by derivatives of b in computing the scalar products. All factors $1 - s\kappa$ are uniformly bounded since we assumed $s|\kappa| \leqq \frac{1}{2}$. This concludes the proof of (3.85). To prove (3.86) we observe that $\tilde{\psi}$ differs from the identity $\tilde{\psi}_{\mathrm{id}}(x_1, x_2) = (x_1, x_2, 0)$ only by quantities which are bounded by $|(a, b, d)| + \delta \leqq c\varepsilon_*^{1/2}$. Since we are considering second derivatives, we can replace $\tilde{\psi}$ with $\tilde{\psi} - \tilde{\psi}_{\rm id}$. From the previous definitions we have

$$\tilde{\psi} - \tilde{\psi}_{\rm id} = (\delta s e_3 + \psi_1) \circ \Phi^{-1}. \tag{3.95}$$

In computing the second gradient of (3.95), we differentiate both a, b and d (which enter ψ_1) and the α-dependent quantities [$n(t)$, $\alpha'(t)$, $\kappa(t)$, etc.]. The latter have at least two bounded derivatives. The only term which does not contain any of the three functions, $\delta s e_3$, is proportional to δ. Hence

$$|\nabla_x^2 \tilde{\psi}| \leqq c \left(\delta + |(a, b, d)| + |\nabla_{s,t}(a, b, d)| + |\nabla_{s,t}^2(a, b, d)|\right)^2, \tag{3.96}$$

and the thesis follows.

We observe that, since the Jacobian $1 - s\kappa$ of the transformation Φ is bounded, the same estimates hold for the integrated quantities.

We now show how the estimates we had obtained for the half-plane problem are extended to the case of curved boundaries. First, fix a period h and consider the periodic solution of (3.35) in some rectangle $A = (0, l) \times (0, h)$. Then $Q_1 = Q_2 = R = 0$, (3.84) is satisfied with $q = h$, and $|\nabla^2(a|b|d)| \leqq c\varepsilon_*^{1/2}/h$. Hence from the two previous Lemmas we get

$$I_{2{\rm D}}[\tilde{\psi}, \Phi(A)] \leqq c\varepsilon_*^2 h l \left(h^2 + \frac{\sigma^2}{h^2}\right). \tag{3.97}$$

Remark 5. Comparing (3.97) with the analogous estimate (3.36) holding for the case of a straight boundary, we notice that the first term proportional to h^3 is new (and it is simple to check, from the proof of Lemma 4, that its coefficient vanishes when κ, κ' tend to zero). While in the case of a straight boundary the energy bound is decreasing with increasing h (and the only reason for using small h is the boundary condition), (3.97) shows that, for a curved boundary, the optimal h is of order $\sigma^{1/2}$. This implies that, for curved boundaries, period-doubling of the folding pattern will stop when the period has reached a value of order $\sigma^{1/2}$, which gives the optimal energy per unit area, of order $\varepsilon_*^2\sigma$.

We now extend to curved boundaries the result of Lemma 3 on the period-doubling box. From (3.42) and (3.43) we see that we can take $q = h$, and from (3.41) and the two previous Lemmas we obtain

$$I_{2{\rm D}}[\tilde{\psi}, \Phi(B)] \leqq c\varepsilon_*^2 \left[\frac{h^5}{l^3} + \sigma l \left(\frac{\sigma}{h}\right)^{\frac{1}{2}} + \sigma h \left(\frac{\sigma}{h}\right)^{1/8} + h^3 l + \sigma^2 h l\right]. \tag{3.98}$$

This is the direct extension of (3.44) to the curvilinear case. The last two terms in (3.98) are however new, and they have the same origin and role as the new term in (3.97).

3.2.2. Construction in piecewise smooth domains, via triangulation. We now put together the various pieces of analysis presented above, following a strategy similar to the one illustrated in Section 3.1.4, to obtain a construction on curvilinear triangles. We start by showing how to handle a region close to one of the edges.

Lemma 5. *Let $\Omega \subset \mathbb{R}^2$ and $\alpha \subset \partial\Omega$ be such that $U = \Phi^{-1}(\Omega)$ has the form*

$$\{(s,t) : t \in [0,L], 0 \leqq s \leqq f(t)\} \tag{3.99}$$

where f is uniformly Lipschitz, and

$$\sup_{(s,t)\in U} s|\kappa(t)| \leqq \frac{1}{2} \tag{3.100}$$

where κ is the curvature of α. Then, there is a constant c such that for any (ε_*, h) there is $\psi : \Omega \to \mathbb{R}^3$ such that

$$I_{2D}[\psi,\Omega] \leqq c\varepsilon_*^2\sigma, \tag{3.101}$$

with

$$\left|\tilde\psi(x) - \begin{pmatrix} x_1 \\ x_2 \\ \delta\,\mathrm{dist}(x,\alpha)\end{pmatrix}\right| \leqq c\varepsilon_*^{1/2}\sigma \qquad \text{on } \partial\Omega, \tag{3.102}$$

and with $|\nabla\tilde\psi - \mathrm{Id}_2| \leqq c\varepsilon_*^{1/2}$, $\sigma|\nabla^2\tilde\psi| \leqq c\varepsilon_*^{1/2}$ in Ω.

Proof. We construct ψ on U, and then obtain the estimate for $\tilde\psi = \psi\circ\Phi^{-1}$ using the two preceding lemmas. We divide the domain U into strips $t \in [i\gamma, (i+1)\gamma]$ where γ is the maximum oscillation period which, as discussed after (3.97), will be of order $\sigma^{1/2}$. The precise value of γ will be given below. On the lines $t = i\gamma$ we impose as boundary conditions

$$a = b = d = 0, \qquad a_{,t} = \hat a'(0),\quad b_{,t} = \hat b'(0),\quad d_{,t} = 0 \tag{3.103}$$

as in Lemma 3. On the vertical boundaries we impose

$$a = h\hat a\left(\frac{t}{h}\right),\quad b = h\hat b\left(\frac{t}{h}\right),\quad a_{,s} = b_{,s} = d = d_{,s} = 0, \tag{3.104}$$

where h is the local period. In each strip, for $s = 0$ the functions a and b oscillate with period σ. Then, there is a sequence of period-doubling steps, up to period γ, and finally the period decreases again down to σ while approaching $s = f(t)$. The construction is hence done composing branching pieces, similar to those constructed in Lemma 3, and flat pieces, where the oscillations do not depend on s (see Fig. 3).

More specifically, we take the same sequences of widths $l_i = \sigma 2^{\alpha i}$ and heights $h_i = \sigma 2^i$ for the branching boxes as in Section 3.1.4. Since we want to stop at h_i of order $\sigma^{1/2}$, we fix a maximum number of branching steps N as the integer part of the solution of $2^x = \sigma^{-1/2}$, and define $\gamma = h_N = \sigma 2^N$. The construction is done iteratively, starting from $i = 1$ up to $i = N$ (possibly stopping earlier for small domains, see below). At step i, the strip is divided into 2^{N-i} intervals $T_{ij} = (jh_i, (j+1)h_i)$, $j = 0, \ldots, 2^{N-i} - 1$, and each j is considered separately. The appropriate width for a branching construction which reaches period h_i in the j-th substrip is

$$S_{ij} = \inf_{t \in T_{ij}} f(t). \tag{3.105}$$

If

$$S_{ij} \geqq 2L_i, \tag{3.106}$$

where $L_i = \sum_{k \leqq i} l_k$ is the cumulative width of all boxes up to the i-th one, we can use the construction of Lemma 3 in the box

$$B_{ij} = (L_{i-1}, L_i) \times T_{ij}. \tag{3.107}$$

This gives the increase in period at small s, while the symmetric one (with $s \to -s$) in the box

$$\bar{B}_{ij} = (S_{ij} - L_i, S_{ij} - L_{i-1}) \times T_{ij}, \tag{3.108}$$

gives the decrease in period at large s. We observe that S_{ij} is defined as the largest value of s for which $\bar{B}_{ij}$ can be placed inside U without overlapping with $\bar{B}$ boxes at smaller i, and that the condition (3.106) ensures that all $\bar{B}$ and B boxes used so far are disjoint. It is also clear that the next step (i.e., $i' = i + 1$) will influence only the region $L_i \leqq s \leqq S_{ij} - L_i$, $t \in T_{ij}$, which is nonempty only if (3.106) is satisfied. In the regions which have not yet been defined, and which will not be touched by the next step, we take the periodic solution (3.104) with period $h = h_i/2$. This means that we take the periodic solution both in the points $s \geqq S_{ij} - L_{i-1}$, $t \in T_{ij}$ where ψ has never been defined before, and in the entire $L_{i-1} \leqq s \leqq S_{ij} - L_{i-1}$, $f \in T_{ij}$ region if (3.106) was not satisfied. This concludes step i of the construction, which is then iterated up to $i = N$.

If, after reaching $i = N$, there is still a central region where ψ has not been defined (i.e., $S_{N0} \geqq 2L_N$), we use the periodic solution (3.104) with period $\gamma = h_N$. This completes the construction in the entire domain U. To check smoothness, we observe that only boundaries parallel to the coordinate axis have been introduced; all horizontal ones satisfy (3.103), all vertical ones satisfy (3.104), with period h_i at step i. The construction is displayed in Fig. 3.

We now estimate the energy of the constructed ψ. At step i, we have 2^{N-i+1} branching boxes, each with energy bounded by (3.98). The additional periodic pieces have width bounded by h_i times the Lipschitz constant of f, and the energy is there bounded by (3.97). The central region with period h_N has possibly area of order 1. The total energy is then controlled by

$$\begin{aligned} I_{2D}[\tilde{\psi}, \Omega] \leqq c\varepsilon_*^2 \sum_{i=1}^{N} \Bigg[\frac{h_i^4}{l_i^3} + l_i \left(\frac{\sigma}{h_i}\right)^{3/2} + \sigma \left(\frac{\sigma}{h_i}\right)^{1/8} + h_i^2 l_i \\ + h_i \operatorname{Lip} f \left(h_i^2 + \frac{\sigma^2}{h_i^2}\right)\Bigg] + c\varepsilon_*^2 \sup |f| \left(h_N^2 + \frac{\sigma^2}{h_N^2}\right) \end{aligned} \tag{3.109}$$

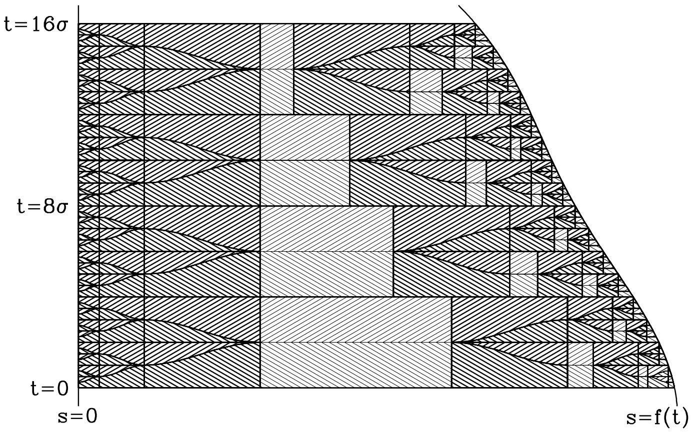

Fig. 3. Construction in Lemma 5.

(the last term in the series corresponds to the small periodic pieces, the term outside the series corresponds to the central region). By taking $l_i = \sigma 2^{\alpha i}$, as in Section 3.1.4, and $h_N = \sigma^{1/2}$ we get

$$I_{2\mathrm{D}}[\tilde{\psi}, \Omega] \leqq c\varepsilon_*^2\sigma \sum_{i=1}^{N} \left[2^{(4-3\alpha)i} + 2^{\left(\alpha-\frac{3}{2}\right)i} + 2^{-i/8} + \sigma^2 2^{(2+\alpha)i} + 2^{-i} \right] + c\varepsilon_*^2\sigma, \tag{3.110}$$

where the dependence on f has been included in the constant c. All geometric series except the fourth one converge provided that $\frac{4}{3} < \alpha < \frac{3}{2}$. The fourth one gives, after summation, $\sigma^2 2^{(2+\alpha)N} \leqq \sigma^{2-(2+\alpha)/2} \leqq \sigma^{1/4}$, hence we get (3.101).

The bounds on $\nabla\psi$ and $|\nabla^2\psi|$ follow directly from (3.42) and (3.43). Also (3.102) follows from the same equations and from the fact that all the rectangles which touch the boundary have $h = \sigma$. This concludes the proof.

We turn now to curvilinear triangles, which can be handled by applying Lemma 5 three times, and by smoothly matching the three resulting constructions.

Lemma 6. *Let Ω be a bounded domain such that its boundary is the union of three C^4 curves which join at angles less than $\pi/2$, and whose radius of curvature is always larger than twice the diameter of Ω. Then, there is a constant c_Ω such that, for sufficiently small ε_* and σ, there is $\tilde{\psi} : \Omega \to \mathbb{R}^3$ such that*

$$I_{2\mathrm{D}}[\tilde{\psi}, \Omega] \leqq c\varepsilon_*^2\sigma, \tag{3.111}$$

with $\tilde{\psi}(x_1, x_2) = (x_1, x_2, 0)$, $\nabla\tilde{\psi}(x_1, x_2) = \nabla(x_1, x_2, 0)$ on $\partial\Omega$. Further, $\tilde{\psi}$ obeys the bound

$$|\nabla\tilde{\psi}| + \sigma|\nabla^2\tilde{\psi}| \leqq c\varepsilon_*^{1/2}. \tag{3.112}$$

Proof. The construction is based on using Lemma 5 around each of the three sides. A smooth matching between the different pieces is obtained by using a smoothed distance function as skeleton of the construction. More specifically, let η be a smooth mollifier with support in the ball of radius $\frac{1}{2}$, and let $\eta_\sigma(x) = \sigma^{-2}\eta(x/\sigma)$. Define

$$W_d(x) = \text{dist}(x, \partial\Omega)\phi\left(\frac{\text{dist}(x, \partial\Omega)}{\sigma} - 1\right), \tag{3.113}$$

and

$$w_d = \eta_\sigma * W_d, \tag{3.114}$$

where $\phi : \mathbb{R} \to [0, 1]$ is a smooth function with $\phi(t) = 0$ for $t \leqq 0$, $\phi(t) = 1$ for $t \geqq 1$. It is easy to verify that $|\nabla w_d| \leqq c$, $|\nabla^2 w_d| \leqq c/\sigma$, $|w_d - \text{dist}(x, \partial\Omega)| \leqq c\sigma$, and $w_d = \nabla w_d = 0$ on $\partial\Omega$. Hence $\tilde{\psi}_d(x) = (x_1, x_2, \delta w_d)$ has bounded energy density, and fulfills the prescribed boundary conditions.

We now use Lemma 5 to reduce the energy in the three parts of Ω where the distance function is smooth. Let γ_i, $i = 1, 2, 3$, be the three smooth curves which form $\partial\Omega$, and let ω_i be the set of points of Ω which are closer to γ_i than to the other two. Each set ω_i obeys the hypothesis of Lemma 5, with $\alpha = \gamma_i$, hence there is $\tilde{\psi}_i(x)$ with energy of order $\varepsilon_*^2\sigma$ which differs from $\tilde{\psi}_d$ at most by $c\varepsilon_*^{1/2}\sigma$ on $\partial\omega_i$. The boundary of ω_i is in turn composed by three smooth parts, which we call $\gamma_i^{(k)}$, one of which coincides with γ_i. We use the function

$$\phi_i(x) = \prod_{k=1}^{3} \phi\left(\frac{\text{dist}(x, \gamma_i^{(k)})}{\sigma}\right), \tag{3.115}$$

to interpolate between $\tilde{\psi}_i$ and $\tilde{\psi}_d$. More precisely, we set

$$\tilde{\psi}(x) = \tilde{\psi}_i(x)\phi_i(x) + \tilde{\psi}_d(x)\,[1 - \phi_i(x)] \tag{3.116}$$

for all $x \in \omega_i$. It is clear that $\tilde{\psi} \neq \tilde{\psi}_i$ only in a region of measure $c\sigma$, where it has energy density bounded by $c\varepsilon_*^2$, as in (3.76), which implies that $\tilde{\psi}$ still has energy bounded by $c\sigma\varepsilon_*^2$. Further, $\tilde{\psi}$ agrees with $\tilde{\psi}_d$ up to the first gradient along $\partial\omega_i$, hence it satisfies the given boundary conditions and joins smoothly along internal boundaries. This concludes the proof.

Finally, we generalize this construction to generic piecewise C^4 domains.

Proof of Theorem 2. Any piecewise C^4 domain can be divided into finitely many curvilinear triangles satisfying the hypothesis of Lemma 6. The result follows by applying Lemma 6 to each of those triangles, and observing that the imposed boundary conditions guarantee smooth matching at all interfaces.

4. Three-dimensional elasticity

In this section we extend the previous results to the full three-dimensional elasticity theory, i.e., to the functional I_{3D} defined in (1.1). The elastic potential $W_{3D}(F) : M^{3\times3} \to \mathbb{R}$ is nonnegative, vanishes on $(1+\varepsilon_*)SO(3)$, and obeys the bounds

$$c_1 \mathrm{dist}^2(F, (1+\varepsilon_*)SO(3)) \leqq W_{3D}(F) \leqq c_2 \mathrm{dist}^2(F, (1+\varepsilon_*)SO(3)), \qquad (4.1)$$

where c_1 and c_2 are numerical constants. Since in all our constructions $\mathrm{dist}(F, (1+\varepsilon_*)SO(3))$ is bounded by a constant times ε_*, the upper bound in (4.1) is needed only in a neighborhood of $SO(3)$.

4.1. Lower bound

Theorem 3 (Lower bound, 3D**).** *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded Lipschitz domain. Then, there are constants* c_Ω, c'_Ω *such that for sufficiently small* h *and* $\varepsilon_* > 0$ *and any* $\phi : \Omega \times (0, h) \to \mathbb{R}^3$ *such that*

$$|\phi(x) - x| \leqq \min\left(\tfrac{1}{4} h \varepsilon_*^{1/2}, c'_\Omega \varepsilon_*\right) \qquad \textit{for } x \in (\partial\Omega) \times (0, h) \qquad (4.2)$$

the lower bound on the energy is given by

$$\frac{1}{h} \int_{\Omega\times(0,h)} W_{3D}(\nabla\phi) d^3x \geqq c_\Omega \min\left(\varepsilon_*^{3/2} h, \varepsilon_*^2\right). \qquad (4.3)$$

For simplicity of exposition we first consider the case of ε_* finite, where we just have to show that for any Ω and ε_* there is a constant c_{Ω,ε_*} such that (4.3) holds for any h. Here, we further assume that the boundary of Ω contains a straight part and that $W(F) \geqq c|F|^3$ for large $|F|$. As in the two-dimensional case, we need only consider cubes of size $(0, h)^3$, with one face on the boundary of Ω. The argument for one cube is based on the well-known fact that a gradient vector field which is a rotation almost everywhere is a constant rotation (in a connected set). More precisely,

Lemma 7. *Let* f_j *be equibounded in* $W^{1,n}(\Omega, \mathbb{R}^n)$, *with* Ω *an open, bounded, Lipschitz subset of* $\mathbb{R}^n$, *and satisfy*

$$\int_\Omega \mathrm{dist}(\nabla f_j, SO(n)) \to 0. \qquad (4.4)$$

Then, there is one matrix $R \in SO(n)$ *such that, for a subsequence,*

$$\int_\Omega \left|\nabla f_j - R\right|^n \to 0. \qquad (4.5)$$

Proof. We follow closely the proof of Theorem 2.4 of [31] (see also [20]). Since f_j is equibounded in $W^{1,n}$, there is a subsequence weakly converging to $f \in W^{1,n}$. The polyconvex function

$$g(F) = |F|^n - n^{n/2} \det F \tag{4.6}$$

is nonnegative and vanishes only on matrices which are scalar multiples of matrices in $SO(n)$ (by isotropy it is enough to consider diagonal matrices, the result follows from the arithmetic-geometric mean inequality). Since $g(\nabla f_j) \to 0$ in L^1, by weak lower semicontinuity $g(\nabla f) = 0$ a.e. Since $\int \det \nabla f_j$ is continuous, we get $|\nabla f_j| \to |\nabla f|$ in L^n, hence convergence is strong, and $\nabla f \in SO(n)$ a.e. Now we show that $\nabla f \in SO(n)$ a.e. implies that ∇f is constant. For any gradient field, $\operatorname{div} \operatorname{cof} \nabla f = 0$. But $\operatorname{cof} \nabla f = \nabla f$, hence $\operatorname{div}\nabla f = \Delta f = 0$, i.e., f is harmonic and smooth. Moreover,

$$\Delta|\nabla f|^2 = (f_{k,j} f_{k,j})_{,ii} = 2 f_{k,iij} f_{k,j} + 2 f_{k,ij} f_{k,ij}. \tag{4.7}$$

But the left-hand side is zero because $|\nabla f|^2 = n$, and the first term in the right-hand side is zero because f is harmonic. It follows that the last term, $2|\nabla^2 f|^2$, also vanishes, hence f is affine.

With this result, we can now prove the following

Lemma 8. *Let $Q_h = (0, h)^3$, $\varepsilon_* > 0$, and assume that $W(F) \geqq c|F|^3$ for large $|F|$. Then, there is a positive constant c_{ε_*} such that if $\phi : Q_h \to \mathbb{R}^3$ obeys $\phi_1 = 0$, $\phi_2 = x_2$ on $\{0\} \times (0, h)^2$, then*

$$\int_{Q_h} W_{3D}(\nabla\phi) \geqq c_{\varepsilon_*} h^3. \tag{4.8}$$

Proof. The statement is invariant under rescaling in h. Hence we take $h = 1$, and proceed by contradiction. If the thesis is false, there is a sequence ϕ^j which obeys the boundary conditions such that $W_{3D}(\nabla\phi^j) \to 0$ in $L^1(Q_1)$. Then, ϕ^j is equibounded in $W^{1,3}$, and $\operatorname{dist}(\nabla\phi^j, (1 + \varepsilon_*)O(3)) \to 0$. By Lemma 7 there is a subsequence such that

$$\nabla\phi^j \to (1 + \varepsilon_*)R \qquad \text{in } L^3(Q_1), \tag{4.9}$$

with $R \in SO(3)$ a fixed rotation. By the trace theorem the boundary condition holds also for the limit. This implies $R_{12} = R_{13} = R_{23} = 0$, and $R_{22} = 1/(1 + \varepsilon_*)$. But this is a contradiction, since the only triangular matrix in $SO(3)$ is the identity (modulo sign changes of the diagonal elements).

From the previous lemma it follows immediately that, if $\partial\Omega$ has a straight part $\gamma \subset \partial\Omega$, then for small enough h any ϕ which obeys the boundary condition (4.2) has

$$\frac{1}{h} \int_{\Omega\times(0,h)} W_{3D}(\nabla\phi) d^3x \geqq c_{\varepsilon_*} |\gamma| h, \tag{4.10}$$

where the constant c_{ε_*} depends only on ε_*. This concludes the proof in the simplified case.

In order to include the dependence on ε_*, we use the quantitative version of Lemma 7 proved in [13, 14]. We state here their result in the case of Lipschitz domains of the special form we need.

Proposition 1 ([13, 14]**).** *Let $f : \mathbb{R}^{n-1} \to \mathbb{R}$ be a Lipschitz function with Lipschitz constant L, and let*

$$U_{f,h} = \left\{ (x_1, \bar{x}) \in \mathbb{R}^n : f(\bar{x}) < x_1 < f(\bar{x}) + h,\ \bar{x} \in (0, h)^{n-1} \right\}. \tag{4.11}$$

Then, there is a constant c_L, depending only on the Lipschitz constant L of f and on the dimensionality n, such that for any $u : U_{f,h} \to \mathbb{R}^n$,

$$\inf_{Q \in SO(n)} \int_{U_{f,h}} |\nabla u - Q|^2 \leqq c_L \int_{U_{f,h}} \mathrm{dist}^2(\nabla u, SO(n)). \tag{4.12}$$

In order to keep the dependence on ε_* explicit, we shall need to partition $U_{f,h}$, which has dimensions $\sigma \times \sigma \times h$, into subsets which have characteristic size of order h in all directions, and then apply the Proposition above to each of them. Given a real function f with Lipschitz constant L, we consider domains of the form $\Phi_f[(0, \sigma)^2] \times (0, h)$, where

$$\Phi_f(z) = (z_1 + f(z_2), z_2). \tag{4.13}$$

Lemma 9. *Let $\phi : \Phi[(0, \sigma)^2] \times (0, h) \to \mathbb{R}^3$ obey the boundary condition*

$$|\phi(x) - x| \leqq \tfrac{1}{4} h \varepsilon_*^{1/2} \qquad \text{for } x \in \Phi[\{0\} \times (0, \sigma)] \times (0, h). \tag{4.14}$$

Then

$$\int_{\Phi[(0,\sigma)^2] \times (0,h)} W_{3\mathrm{D}}^{\varepsilon_*}(\nabla \phi) \geqq c_L \sigma^2 \varepsilon_*^{5/2}, \tag{4.15}$$

where c_L depends only on the Lipschitz constant L of f.

Proof. As in the two-dimensional case, the statement is invariant under the rescaling $\tilde{\phi}(x) = \sigma \phi(\sigma x)$, hence we need only consider the case $\sigma = 1$. We argue by contradiction, and assume that there are sequences ε_j, ϕ_j and f_j such that

$$\varepsilon_j^{-5/2} \int_{\Phi_j[(0,1)^2] \times (0, \varepsilon_j^{1/2})} W_{3\mathrm{D}}^{\varepsilon_j}(\nabla \phi_j) \to 0, \tag{4.16}$$

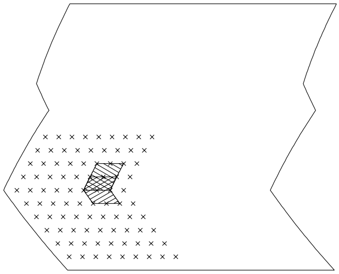

Fig. 4. Domain used in the proof of Lemma 9. The crosses mark the points in the transformed grid $\Phi_f(G_j)$. The two hatched areas represent two neighboring domains C_r and $C_{r'}$, (with $(r,r') \in G_j^{\mathrm{NN}}$), which overlap over half of their area.

with $|\phi_j - x| \leqq \varepsilon_j^{1/2}/4$ on the left boundary. For $a \in (0,3/4)^2$, let $C_a = \Phi_j\left[a + \left(0, \varepsilon_j^{1/2}\right)^2\right]$. By the quantitative Reshetnyak estimate of Proposition 1 applied to the domain $C_a \times (0, \varepsilon_j^{1/2})$, for every a there is a rotation $Q_{a,j} \in SO(3)$ such that

$$\int_{C_a \times (0,\varepsilon_j^{1/2})} |\nabla\phi_j - (1+\varepsilon_j)Q_{a,j}|^2 \leqq c_L \int_{C_a \times (0,\varepsilon_j^{1/2})} W_{3\mathrm{D}}^{\varepsilon_j}(\nabla\phi_j). \tag{4.17}$$

Let G_j be a square grid with spacing $\frac{1}{2}\varepsilon_j^{1/2}$ in $(0,\frac{1}{2})^2$ and G_j^{NN} be the set of all nearest–neighbor pairs in $G_j \times G_j$, respectively $G_j = (0,\frac{1}{2})^2 \cap \frac{1}{2}\varepsilon_j^{1/2}\mathbb{Z}^2$ and $G_j^{\mathrm{NN}} = \{(r,r') \in G_j^2 : |r - r'| = \frac{1}{2}\varepsilon_j^{1/2}\}$. Then, since if $(r,r') \in G_j^{\mathrm{NN}}$ the two domains C_r and $C_{r'}$ overlap in half of their area (see Fig. 4) and each point is covered by at most 4 of them, we get

$$\frac{1}{\varepsilon_j} \sum_{(r,r') \in G_j^{\mathrm{NN}}} |Q_{r,j} - Q_{r',j}|^2 \leqq c\varepsilon_j^{-5/2} \int_{\Phi_j[(0,1)^2] \times (0,\varepsilon_j^{1/2})} W_{3\mathrm{D}}^{\varepsilon_j}(\nabla\phi_j) \to 0. \tag{4.18}$$

By the discrete Sobolev embedding (see Lemma 10 below for details) there is a unique matrix Q_j such that

$$\frac{1}{\varepsilon_j} \sum_{r \in G_j} |Q_{r,j} - Q_j|^4 \leqq c \left(\frac{1}{\varepsilon_j} \sum_{(r,r') \in G_j^{\mathrm{NN}}} |Q_{r,j} - Q_{r',j}|^2 \right)^2 \to 0. \tag{4.19}$$

Since $Q_{r,j} \in SO(3)$ for all r, we can further assume $W_j \in SO(3)$. Now define $\psi_j : \Phi_j[(0,\frac{1}{2})^2] \to \mathbb{R}^3$ as the average of ϕ_j over x_3,

$$\psi_j(x_1,x_2) = \frac{1}{\varepsilon_j^{1/2}} \int_0^{\varepsilon_j^{1/2}} \phi_j(x_1,x_2,x_3)dx_3. \tag{4.20}$$

Then, by (4.17) in each domain C_a we have

$$\varepsilon_j^{1/2} \int_{C_a} |\nabla \psi_j - (1+\varepsilon_j) Q_{a,j} P^T|^2 \leqq \int_{C_a \times (0,\varepsilon_j^{1/2})} |\nabla \phi_j - (1+\varepsilon_j) Q_{a,j}|^2$$
$$\leqq c \int_{C_a \times (0,\varepsilon_j^{1/2})} W_{3D}^{\varepsilon_j}(\nabla \phi_j), \tag{4.21}$$

where $P = e_1^{(2)} \otimes e_1^{(3)} + e_2^{(2)} \otimes e_2^{(3)}$ is the canonical immersion of $\mathbb{R}^3$ into $\mathbb{R}^2$. We proceed as in Lemma 1, and define $\psi_j' = P Q_j^T \psi_j$ as the projection of ψ_j on the plane which better approximates the surface described by ψ_j itself. Then, since $\nabla \psi_j$ is close to $(1+\varepsilon_j) Q_{a,j}$, $\nabla \psi_j' = P Q_j^T \nabla \psi_j$ is close to $(1+\varepsilon_j) SO(2)$. More precisely, for all $r \in G_j$ and all $x \in C_r \times (0, \varepsilon_j^{1/2})$, we get

$$\begin{aligned} \operatorname{dist}(\nabla \psi_j', (1+\varepsilon_j) SO(2)) &\leqq |\nabla \psi_j' - (1+\varepsilon_j) P Q_j^T Q_{r,j} P^T| \\ &\quad + (1+\varepsilon_j) \operatorname{dist}(P Q_j^T Q_{r,j} P^T, SO(2)) \\ &\leqq |\nabla \psi_j - (1+\varepsilon_j) Q_{r,j} P^T| + (1+\varepsilon_j) |Q_j - Q_{r,j}|^2, \end{aligned} \tag{4.22}$$

where the last term has been estimated using Lemma 11. Now we square (4.22), integrate over $x \in C_r \times (0, \varepsilon_j^{1/2})$ and sum over $r \in G_j$. Using (4.16), (4.19), (4.21), we see that

$$\frac{1}{\varepsilon_j^2} \int_{\Phi_j[(0,1)^2]} \operatorname{dist}^2(\nabla \psi_j', (1+\varepsilon_j) SO(2)) \to 0 \tag{4.23}$$

as $\varepsilon_j \to 0$. This is exactly equivalent to (2.14). It is also clear from (4.14) that ψ_j obeys the same boundary conditions as in the two-dimensional case. The proof is then concluded as in Lemma 1.

We are now ready to prove the global lower bound.

Proof of Theorem 3. The proof is almost identical to that of Theorem 1. The only difference is in the case $h \geqq \sigma_0 \varepsilon_*^{1/2}$. Then, (2.21) should be replaced by

$$I_{3D}^{h,\varepsilon_*}[\phi, \Omega] \geqq \sum_{i=1}^{\lfloor h/\sigma_0 \varepsilon_*^{1/2} \rfloor} I_{3D}^{\sigma_0 \varepsilon_*^{1/2}, \varepsilon_*}[\phi^{(i)}, \Omega] \geqq c_L |\partial\Omega| \sigma_0 \varepsilon_*^2, \tag{4.24}$$

where $\phi^{(i)}$ represents the restriction of ϕ to the i-th slice of thickness $\sigma_0 \varepsilon_*^{1/2}$. The boundary condition (4.2) gives then (4.14) on each slice, provided that we choose $c_\Omega' \leqq \sigma_0/4$.

We finally present a short proof of the discrete Sobolev embedding used to obtain (4.19).

Lemma 10. *Let $A_N = [1, N] \cap \mathbb{Z}$ denote the positive integers from 1 to N. For any $f_r : A_N \times A_N \to \mathbb{R}^n$, there is $\bar{f} \in \mathbb{R}^n$ such that*

$$\sum_{r \in A_N^2} |f_r - \bar{f}|^4 \leqq cN^2 \left(\sum_{\substack{r,r' \in A_N^2 \\ |r-r'|=1}} |f_r - f_{r'}|^2 \right)^2 , \tag{4.25}$$

where c depends only on n.

This Lemma can be obtained as a direct consequence of the usual embedding of $W^{1,2}$ into L^4, using multilinear finite elements. For variety we include a self-contained discrete proof.

Proof. We first prove the one-dimensional version of (4.25),

$$\sum_{i=1}^{N} |f_i - \bar{f}|^4 \leqq N^3 \left(\sum_{i=1}^{N-1} |f_i - f_{i+1}|^2 \right)^2 . \tag{4.26}$$

For any i and k, $|f_i - f_k| \leqq \sum_{j=1}^{N-1} |f_j - f_{j+1}|$. Since the bound does not depend on k, the same holds for $|f_i - \bar{f}|$, where $\bar{f}$ is the average. We get therefore

$$|f_i - \bar{f}| \leqq \sum_{j=1}^{N-1} |f_j - f_{j+1}| \leqq \left[N \sum_{j=1}^{N-1} (f_j - f_{j+1})^2 \right]^{1/2} . \tag{4.27}$$

Taking the fourth power and summing over i gives (4.26).

To prove (4.25) we need some notation. Let $\bar{f}_i$, be the average of $f_{i,j}$ over j, $\bar{f}'_j$ the average over i, and $\bar{f}$ the average over both i and j. Then,

$$\begin{aligned} |f_{i,j} - \bar{f}|^2 &\leqq c|f_{i,j} - \bar{f}_i|^2 + c|\bar{f}_i - \bar{f}|^2 \\ &\leqq cN \sum_j |f_{i,j} - f_{i,j+1}|^2 + cN \sum_i |\bar{f}_i - \bar{f}_{i+1}|^2, \end{aligned} \tag{4.28}$$

where we have used (4.27) twice. The last term is bounded by

$$N \sum_i |\bar{f}_i - \bar{f}_{i+1}|^2 = N \sum_i \left| \frac{1}{N} \sum_j f_{i,j} - f_{i+1,j} \right|^2 \leqq \sum_{i,j} \left| f_{i,j} - f_{i+1,j} \right|^2 = \Delta. \tag{4.29}$$

The first term in the right-hand side of (4.28) depends only on i, and its sum over i is also $N\Delta$. An analogous estimate can be obtained using averages over j, and it results in a bound which depends only on j. Multiplying the two, and summing over i and j, we prove the thesis.

4.2. Upper bound

In this section we prove the upper bound in the case of three-dimensional elasticity. The proof is based on the previous two-dimensional result, which is extended to 3D with the usual Föppl-von Kármán ansatz, as was done, e.g., in Appendix B of [8] for the (in-plane) geometrically linear case.

Theorem 4 (Upper bound, 3D**).** *Let $\Omega \subset \mathbb{R}^2$ be a bounded piecewise C^4 domain. Then, there is a constant c_Ω such that, for any $\varepsilon_* \in (0,1)$ and sufficiently small h, there is $\phi : \Omega \times (0,h) \to \mathbb{R}^3$ such that $\phi(x) = x$ on $(\partial\Omega) \times (0,h)$ and*

$$\frac{1}{h}\int_{\Omega\times(0,h)} W_{3\mathrm{D}}(\nabla\phi)d^3x \leqq c_\Omega \varepsilon_*^{3/2} h. \tag{4.30}$$

Proof. Let $\psi : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ be the two-dimensional deformation defined in Theorem 2, and let ν be the normal to the surface generated by ψ. We construct ϕ with the usual Föppl-von Kármán ansatz,

$$\phi(x_1, x_2, x_3) = \psi(x_1, x_2) + z_3\nu(x_1, x_2), \tag{4.31}$$

where $z_3 = (1+\varepsilon_*)(x_3 - h/2)$ is the distance to the mid-plane in the reference configuration, rescaled to its equilibrium value. Then,

$$\nabla\phi = (\partial_1\psi|\partial_2\psi|(1+\varepsilon_*)\nu) + z_3(\partial_1\nu|\partial_2\nu|0), \tag{4.32}$$

where $(\alpha|\beta|\gamma)$ denotes the matrix whose columns are α, β and γ. Moreover,

$$\begin{aligned}\nabla\phi^T\nabla\phi - (1+\varepsilon_*)\mathrm{Id}_3 = \nabla\psi^T\nabla\psi - (1+\varepsilon_*)\mathrm{Id}_2 + z_3(\nabla\psi^T\nabla\nu \\ + \nabla\nu^T\nabla\psi) + z_3^2\nabla\nu^T\nabla\nu,\end{aligned} \tag{4.33}$$

where 2×2 matrices are extended to 3×3 by including zeroes. We note that $\nabla\psi^T$ is close to a projection to the plane tangent to the graph of ψ, hence it does not change the norm of $\nabla\nu$ by more than a factor. More precisely, from

$$|\nabla\psi^T\nabla\nu| \leqq |\nabla\psi^T||\nabla\nu| \tag{4.34}$$

we deduce

$$\begin{aligned}\left|\nabla\phi^T\nabla\phi - (1+\varepsilon_*)\mathrm{Id}_3\right|^2 \leqq 2\left|\nabla\psi^T\nabla\psi - (1+\varepsilon_*)\mathrm{Id}_2\right|^2 \\ + 4z_3^2|\nabla\psi|^2|\nabla\nu|^2 + 2z_3^4|\nabla\nu|^4\end{aligned} \tag{4.35}$$

which gives

$$W_{3\mathrm{D}}(\nabla\phi) \leqq cW_{2\mathrm{D}}(\nabla\psi) + cz_3^2|\nabla\nu|^2 + cz_3^4|\nabla\nu|^4. \tag{4.36}$$

The last term can be absorbed in the last but one term, since $|z_3| \leqq \sigma\varepsilon_*^{1/2}$ and in our construction $\sigma|\nabla\nu|$ is bounded.

Appendix A. Distance of projected gradients from $SO(n)$

For the convenience of the reader we now include a proof of two estimates which have been used to establish the lower bounds. More precisely, Lemma 12 has been used in the two-dimensional lower bound to obtain (2.13), and Lemma 11 has been used in the three-dimensional lower bound to obtain (4.22). We start with some notation. Let $\{e_i^{(n)}\}_{i=1,\dots,n}$ be the canonical basis of $\mathbb{R}^n$. The nonnegative real numbers $\{\lambda_i^{(F)}\}_{i=1,\dots,n}$ denote the singular values of the $n \times n$ matrix F, i.e., $\left(\lambda_i^{(F)}\right)^2$ are the eigenvalues of $F^T F$. Every $n \times n$ matrix can be written as $F = QI\Lambda Q'$, where $Q, Q' \in SO(n)$, $\Lambda = \operatorname{diag}(\lambda_1^{(F)}, \dots, \lambda_n^{(F)})$, and $I = \mathrm{Id}_n$ if $\det F \geqq 0$, $I = \operatorname{diag}(-1, 1, \dots, 1)$ otherwise. The distance of a matrix F from a set K is defined by

$$\operatorname{dist}^2(F, K) = \inf_{G \in K} |F - G|^2, \tag{A.1}$$

where $|\cdot|$ denotes the matrix norm, $|F|^2 = \operatorname{Tr} F^T F = \sum F_{ij}^2$.

The distance of a matrix from the set $O(n)$ of the orthogonal ones can be represented in terms of its singular values,

$$\operatorname{dist}^2(F, O(n)) = \sum_i \left(\lambda_i^{(F)} - 1\right)^2. \tag{A.2}$$

By rotational invariance it is enough to prove (A.2) for diagonal matrices $F = \operatorname{diag}(\lambda_1^{(F)}, \dots, \lambda_n^{(F)})$. Then, for any $Q \in O(n)$, we have

$$|F - Q|^2 = |F|^2 + |Q|^2 - 2\sum_i \lambda_i^{(F)} Q_{ii}. \tag{A.3}$$

The first two terms do not depend on the choice of Q, the last one is clearly minimized by $Q = \mathrm{Id}_n$. This concludes the proof of (A.2). The same argument shows that

$$\operatorname{dist}(F, O(n)) = \operatorname{dist}(F, SO(n)) \qquad \text{for all } F \text{ such that } \det F \geqq 0. \tag{A.4}$$

Further, we claim that

$$\operatorname{dist}(F, SO(n)) \geqq 1 \qquad \text{for all } F \text{ such that } \det F \leqq 0. \tag{A.5}$$

If $\det F = 0$ at least one of the singular values vanishes, hence (A.2) is at least 1. If $\det F < 0$, take $Q \in SO(n)$ and consider the matrices $F_\mu = \mu F + (1 - \mu)Q$. Clearly $|F_\mu - Q| = \mu|F - Q|$. By continuity there is $\mu^* \in (0, 1)$ such that $\det F_{\mu^*} = 0$. But then $|F - Q| = |F_{\mu^*} - Q|/\mu^* \geqq 1$. This concludes the proof of (A.5).

Finally, from the definition of the singular values it is immediately seen that

$$|F^T F - \mathrm{Id}_n|^2 = \sum_{i=1}^n \left(\left(\lambda_i^{(F)}\right)^2 - 1\right) \tag{A.6}$$

which implies

$$\operatorname{dist}(F, SO(n)) \leqq \operatorname{dist}(F, O(n)) \leqq \left|F^T F - \mathrm{Id}_n\right| \leqq (1 + |F|)\operatorname{dist}(F, O(n)). \tag{A.7}$$

We are now ready to state our first result, which regards the distance from $SO(2)$ of the projection of matrices close to $SO(3)$. Since the proof does not depend on the dimensionality, we state it for general n.

Lemma 11. *Let $R \in SO(n)$, and $P = \sum_{i=1}^{n-1} e_i^{(n-1)} \otimes e_i^{(n)}$. Then,*

$$\operatorname{dist}\left(PRP^T, SO(n-1)\right) \leqq \left|(R - \mathrm{Id}_n)P^T\right|^2 \leqq |R - \mathrm{Id}_n|^2. \tag{A.8}$$

Proof. First observe that

$$\operatorname{dist}(PRP^T, SO(n-1)) \leqq |PRP^T - \mathrm{Id}_{n-1}| \leqq |(R - \mathrm{Id}_n)P^T| \tag{A.9}$$

since P is a projection operator and $P\mathrm{Id}_n P^T = \mathrm{Id}_{n-1}$. This concludes the proof of the Lemma if the right-hand side is larger than 1. Otherwise, (A.9) and (A.5) imply that $\det PRP^T \geqq 0$. Then, by (A.4) we get

$$\begin{aligned}\operatorname{dist}\left(PRP^T, SO(n-1)\right) &= \operatorname{dist}\left(PRP^T, O(n-1)\right)\\ &\leqq \left|\left(PRP^T\right)^T\left(PRP^T\right) - \mathrm{Id}_{n-1}\right|.\end{aligned} \tag{A.10}$$

But $P^T P = \mathrm{Id}_n - e_n^{(n)} \otimes e_n^{(n)}$, and $R^T R = \mathrm{Id}_n$. Hence

$$\left(PRP^T\right)^T\left(PRP^T\right) = P\mathrm{Id}_n P^T - (PR^T e_n^{(n)}) \otimes (PR^T e_n^{(n)}) \tag{A.11}$$

which gives

$$\begin{aligned}\operatorname{dist}\left(PRP^T, SO(n-1)\right) \leqq \left|PR^T e_n^{(n)}\right|^2 &= \left|P(R^T - \mathrm{Id}_n)e_n^{(n)}\right|^2\\ &\leqq \left|(R - \mathrm{Id}_n)P^T\right|^2\end{aligned} \tag{A.12}$$

since $P\mathrm{Id}_n e_n^{(n)} = 0$.

We now consider the projection of the 3×2 matrix $\nabla\psi$ which arises in the two-dimensional problem. Given $F \in M^{3\times 2}$, the normal to the plane generated by F is

$$\nu_F = \frac{F_1 \wedge F_2}{|F_1 \wedge F_2|}, \tag{A.13}$$

where $F_i = Fe_i^{(2)}$. If the denominator vanishes, i.e., F_1 and F_2 are linearly dependent, we can take as ν_F any unit vector orthogonal to both. Then, the following estimate holds:

Lemma 12. *There is a constant* c *such that, for any* $F \in M^{3\times 2}$*, one has*

$$\operatorname{dist}(PF, SO(2)) \leqq c\left[\operatorname{dist}(F, O(2,3)) + \left|\nu_F - e_3^{(3)}\right|^2\right], \tag{A.14}$$

where $P = e_1^{(2)} \otimes e_1^{(3)} + e_2^{(2)} \otimes e_2^{(3)}$.

Proof. Since $\operatorname{dist}(PF, SO(2)) \leqq 2 + |PF| \leqq 4 + \operatorname{dist}(F, O(2,3))$, we only need to consider the case $\operatorname{dist}(F, O(2,3)) + \left|\nu_F - e_3^{(3)}\right|^2 \leqq 1$. This implies

$$\det PF = \det(F_1|F_2|e_3) = F_1 \wedge F_2 \cdot e_3 = |F_1 \wedge F_2|\, \nu_F \cdot e_3 \geqq \tfrac{1}{2}|F_1 \wedge F_2|, \tag{A.15}$$

where in the last step we used $\nu_F \cdot e_3 = 1 - \frac{1}{2}|\nu_F - e_3|^2 \geqq \frac{1}{2}$. Since $P^T P = \mathrm{Id}_3 - e_3^{(3)} \otimes e_3^{(3)}$, in analogy to (A.10)–(A.12) and using (A.4) we get

$$\begin{aligned} \operatorname{dist}(PF, SO(2)) = \operatorname{dist}(PF, O(2)) &\leqq \left|F^T P^T P F - \mathrm{Id}_2\right| \\ &\leqq \left|F^T F - \mathrm{Id}_2\right| + \left|F^T e_3^{(3)}\right|^2. \end{aligned} \tag{A.16}$$

But $|F^T e_3^{(3)}| \leqq |F||\nu_F - e_3^{(3)}| \leqq 3|\nu_F - e_3^{(3)}|$, and the proof is concluded.

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