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Concentration and Lack of Observability of Waves in Highly Heterogeneous Media

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Abstract

We construct rapidly oscillating Hölder continuous coefficients for which the corresponding 1-dimensional wave equation lacks the classical observability property guaranteeing that the total energy of solutions may be bounded above by the energy localized in an open subset of the domain where the equation holds, if the observation time is large enough. The coefficients we build oscillate arbitrarily fast around two accumulation points. This allows us to build quasi-eigenfunctions for the corresponding eigenvalue problem that concentrate the energy away from the observation region as much as we wish. This example may be extended to several space dimensions by separation of variables and illustrates why the well-known controllability and dispersive properties for wave equations with smooth coefficients fail in the class of Hölder continuous coefficients. In particular we show that for such coefficients no Strichartz-type estimate holds.

1. Introduction and main result

Let us consider the following variable coefficient 1-dimensional wave equation

$$\begin{array}{ll} \rho(x)u_{tt} - u_{xx} = 0, & 0 < x < 1,\ 0 < t < T, \\ u(0,t) = u(1,t) = 0, & 0 < t < T, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & 0 < x < 1. \end{array} \tag{1.1}$$

We assume that ρ is measurable and that it is bounded above and below by finite, positive constants, i.e.,

$$0 < \rho_0 \leqq \rho(x) \leqq \rho_1 < \infty \text{ a.e. } x \in (0,1). \tag{1.2}$$

Under these conditions system (1.1) is well posed in the sense that for any pair of initial data $(u_0, u_1) \in H_0^1(0,1) \times L^2(0,1)$ there exists a unique solution

$$u \in C\left([0,T];\ H_0^1(0,1)\right) \cap C^1\left([0,T];\ L^2(0,1)\right). \tag{1.3}$$

Moreover, the energy of solutions

$$E(t) = \frac{1}{2}\int_0^1 \left[\rho(x) \mid u_t(x,t) \mid^2 + \mid u_x(x,t) \mid^2\right] dx \tag{1.4}$$

is constant in time. When $\rho \in BV(0,1)$, the following observability properties are known to hold:

1. **Boundary observability:** If $T > \sqrt{\rho_1}$, there exists $C(T) > 0$ such that

$$E(0) \leqq C \int_0^T \left[|u_x(0,t)|^2 + |u_x(1,t)|^2\right] dt \tag{1.5}$$

for every solution of (1.1).

2. **Internal observability:** For any subinterval $(\alpha, \beta) \subset (0,1)$, if $T > 2\sqrt{\rho_1}$ $\max(\alpha, 1-\beta)$, there exists $C > 0$ such that

$$E(0) \leqq C \int_0^T \int_\alpha^\beta \left[\rho(x) u_t^2 + u_x^2\right] dx dt \tag{1.6}$$

for every solution of (1.1).

These results may be proved easily using sidewise energy estimates for the wave equation in which the role of space and time are interchanged. We refer to [6] for the details of the proof. These observability estimates are relevant in the context of controllability. In fact they are equivalent to the controllability of the system with controls acting on the boundary or in the interior of the domain respectively (see [10]).

For a long time the problem of whether these estimates do hold for less regular coefficients (say $\rho \in L^\infty(0,1)$ or $\rho \in C([0,1])$) has been open. In [1] the problem of homogenization was considered. It was shown that, for a suitable ρ, due to concentration effects of high-frequency solutions, the constant C on the observability inequalities (1.5), (1.6) blows up when ρ is replaced by $\rho_\epsilon(x) = \rho(x/\epsilon)$ and $\epsilon \to 0$. This result shows that the constant C in (1.5), (1.6) does not only depend on the lower and upper bounds ρ_0 and ρ_1 of ρ. The results in [1] were evidence of the possible lack of observability for highly oscillatory density functions ρ. We refer to [4] for an in-depth spectral analysis of the low frequencies. But, up to now, there has been no proof of this negative result in the literature. In this paper we definitely answer the question by the negative. More precisely, we prove that the following holds:

Theorem 1. *There exist Hölder continuous density functions* $\rho \in C^{0,s}([0,1])$ *for all* $0 < s < 1$, *for which* (1.5) *and* (1.6) *fail for all* $T > 0$ *and for every subinterval* $(\alpha, \beta) \subset (0,1)$ $((\alpha, \beta) \neq (0,1))$.

Remark 1. 1. The density functions we build are in fact of class C^∞ everywhere in $(0,1)$ except at the extremes $x = 0$ and $x = 1$ of the interval.
2. Obviously, the density functions we obtain are not of finite total variation. The total variation blows up on the two extremes $x = 0, 1$.

3. A similar construction may be done by means of piecewise constant density functions (see Remark 3).
4. In fact, given any smooth density ρ in $[0, 1]$, we can perturb it in a subinterval of arbitrarily small length so that inequalities (1.5) and (1.6) fail for the new density $\tilde{\rho}$. More precisely, given a smooth ρ and a subinterval $[x_0, x_1]$ of $[0, 1]$ we can find a Hölder continuous function ϵ with support in (x_0, x_1) and L^∞-norm of arbitrarily small size, and such that the inequalities (1.5) and (1.6) fail for the new density $\tilde{\rho} = \rho + \epsilon$.

The proof of Theorem 1 is based on an argument introduced by Colombini & Spagnolo in [5] in a different context that allows us to construct a density ρ for which there exists a sequence of pairs $(\varphi_k(x), \lambda_k)$ satisfying

$$\varphi_k'' + \lambda_k^2 \rho(x)\varphi_k = 0, \tag{1.7}$$

and such that φ_k is exponentially concentrated on any given point of the closed interval $[0, 1]$. In fact we construct a double sequence so that part of it is concentrated on $x = 0$, while the other one is concentrated on $x = 1$. Let us explain the main idea behind this argument. We consider density functions $\rho(x)$ which oscillate more and more as x approaches the extremes of the interval $[0, 1]$. The sequence of pairs $(\varphi_k(x), \lambda_k)$ is constituted by functions $\varphi_k(x)$ which oscillate at the same order as ρ in a small region inside $[0, 1]$ close to $x = 0$ or $x = 1$. In this region a resonance-type phenomenum occurs and $\varphi_k(x)$ becomes exponentially larger than in the rest of the interval $[0, 1]$. Note that (1.7) together with

$$\varphi_k(0) = \varphi_k(1) = 0 \tag{1.8}$$

constitute the eigenvalue problem associated with (1.1). There is no reason for the functions φ_k above, exponentially concentrated near the boundary, to satisfy (1.8). However, by choosing in an appropriate way the point where the energy concentrates, close to one of the extremes $x = 0, 1$, the values of φ_k and φ_k' at $x = 0, 1$ may be guaranteed to be exponentially small. For this reason, these functions φ_k are referred to as quasi-eigenfunctions. These quasi-eigenfunctions allow us to construct a sequence of solutions u_k of (1.1) of the form

$$u_k(x, t) = e^{i\lambda_k t}\varphi_k(x) + \tilde{v}_k(x, t)$$

where $\tilde{v}_k(x, t)$ is a correction introduced to make u_k satisfy the boundary conditions. The solutions u_k concentrate in the interior of $(0, 1)$ along the time and therefore constitute an obstacle to the boundary observability. Recall that we are dealing with a double sequence of quasi-eigenfunctions and therefore with a double family u_k of solutions of (1.1), one of them being concentrated near $x = 0$ while the other one is concentrated near $x = 1$. In this way we can guarantee that neither (1.5) nor (1.6) hold, whatever the interval $(\alpha, \beta) \subset (0, 1)$ $((\alpha, \beta) \neq (0, 1))$ happens to be.

The rest of the paper is organized as follows: in Section 2 we state two ordinary-differential-equation lemmas introduced in [5] that we use to construct the density ρ. In section 3 we build the pathological density ρ and the sequence of quasi-eigenfunctions associated with ρ. In Sections 4 and 5 we prove the lack of boundary and interior observability respectively for this choice of ρ. In Section 6 we extend these results to the multi-dimensional case. In Section 7 we state the results on the lack of controllability that can be derived from Theorem 1. Finally, in Section 8 we comment on some related results. In particular we generalize our result to the case where the variable coefficient is in the principal part of the operator in the wave equation, and to the corresponding Schrödinger model.

2. Preliminary lemmas

In this section we recall the following two lemmas proved in [5].

Lemma 1. *There exists $\bar{\epsilon} > 0$ such that, for all $\epsilon \in (0, \bar{\epsilon})$, it is possible to find two even real functions, $\alpha_\epsilon(x)$ and $w_\epsilon(x)$, of class C^∞ on $\mathbb{R}$, satisfying*

$$\begin{aligned} & w_\epsilon'' + \alpha_\epsilon(x) w_\epsilon = 0, \\ & w_\epsilon(0) = 1, \qquad w_\epsilon'(0) = 0, \end{aligned} \tag{2.1}$$

in such a way that

$$\alpha_\epsilon(x) \text{ is } 1\text{-periodic on } x < 0 \text{ and on } x > 0, \tag{2.2}$$

$$\alpha_\epsilon(x) \equiv 4\pi^2 \text{ in a neighborhood of } x = 0, \tag{2.3}$$

$$|\alpha_\epsilon(x) - 4\pi^2| \leqq M\epsilon, \qquad |\alpha_\epsilon'(x)| \leqq M\epsilon, \tag{2.4}$$

$$\begin{cases} w_\epsilon(x) = p_\epsilon(x) e^{-\epsilon|x|} \\ \text{for some } p_\epsilon(x) \text{ 1-periodic on } x < 0 \text{ and on } x > 0, \end{cases} \tag{2.5}$$

$$|w_\epsilon| + |w_\epsilon'| + |w_\epsilon''| \leqq C, \tag{2.6}$$

$$\int_0^1 w_\epsilon(x) dx \geqq \gamma\epsilon, \qquad (\gamma > 0), \tag{2.7}$$

$$\int_0^1 |w_\epsilon|^2 dx \geqq \gamma, \tag{2.8}$$

where M, C and γ are constants independent of ϵ.

Remark 2. As a consequence of (2.5), (2.1) and (2.7), we have in particular for all integers $n \geqq 0$,

$$w_\epsilon(x) = e^{-\epsilon|x|}, \quad w_\epsilon'(x) = 0, \quad w_\epsilon''(x) = -4\pi^2 e^{-\epsilon|x|}, \quad \text{for } x = \pm n. \tag{2.9}$$

Remark 3. The parameter ϵ in (2.1) allows us to introduce a family of coefficients α_ϵ approaching a constant (see (2.4)), and for which we know explicitly the decay rate of the solution of (2.1) w_ϵ as $|x| \to \infty$ (see (2.5)). As we will see, this is important in order to guarantee the Hölder continuity of the density ρ that we construct in the next section.

Remark 4. For a fixed $\epsilon > 0$, Lemma 1 establishes the existence of a coefficient $\alpha(x)$ and a solution $w(x)$ of

$$\begin{aligned} & w'' + \alpha(x) w = 0, \\ & w(0) = 1, \qquad w'(0) = 0, \end{aligned} \tag{2.10}$$

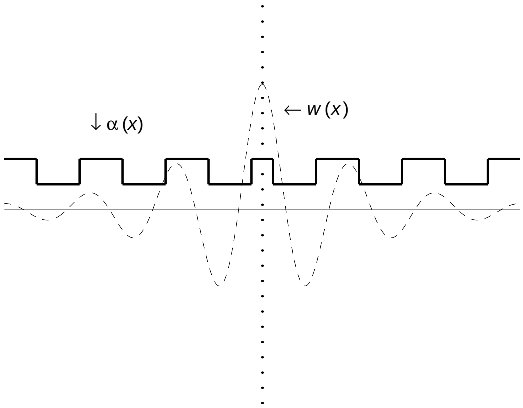

Fig. 1. Construction of α and w satisfying (2.11).

satisfying

$$\begin{aligned}&\alpha(x) \text{ is 1-periodic on } x<0 \text{ and on } x>0\\ &\begin{cases} w(x)=p(x)e^{-k|x|}\\ \text{for some } p(x) \text{ 1-periodic on } x<0 \text{ and on } x>0 \text{ and } k>0.\end{cases}\end{aligned} \tag{2.11}$$

Explicit examples of piecewise-constant functions α and solutions w with the above properties may be built easily (see [1]). The main idea is to consider a periodic coefficient α_1 such that the solution w_1 of (2.10) with $\alpha = \alpha_1$ satisfies $w_1(x) = p(x)e^{-kx}$ for some 1-periodic function p and $k > 0$. Let $x_s \in \mathbb{R}$ be a point where the solution satisfies $w_1'(x_s) = 0$. Then α_2, w_2, the even reflection of α_1 and w_1 with respect to x_s respectively, satisfies (2.10) and $w_2(x) = p(x)e^{-k|x-x_s|}$ with $p(x)$ periodic for $x > x_s$ and $x < x_s$. Finally we can translate α_2 and w_2 in order to have (2.11). This construction is illustrated in Fig. 1.

In [5] Lemma 1 is stated with α_ϵ being 2π-periodic. A straightforward change of variables shows that α_ϵ may be taken to be 1-periodic as well, as stated in Lemma 1.

Let us explain briefly the result in Lemma 1. If we restrict the first equation in (2.1) to $\mathbb{R}^+$ we obtain the Hill equation

$$w'' + \lambda q(x)w = 0, \quad x \in \mathbb{R}^+,\ \lambda \in \mathbb{R} \text{ and } q \text{ being 1-periodic.}$$

It is well known that for any periodic function q there exist some positive values of λ for which this equation has a solution of the form $w = p(x)e^{-ax}$ where p is 1-periodic and $a > 0$ (see [7]). The proof of Lemma 1 in [5] relies on a suitable choice of q, λ and w satisfying these properties. The values of α and w for $x < 0$ are obtained by even extension. Note that the condition (2.3) assures the regularity of this extension. The explicit choice of q, λ and w is as follows:

$$\begin{aligned}\lambda &= 4\pi^2,\\ q(x) &= 1 - 4\epsilon r(2\pi x)\sin(4\pi x) + 2\epsilon r'(2\pi x)\cos^2(2\pi x)\\ &\quad - 4\epsilon^2 r^2(2\pi x)\cos^4(2\pi x),\\ w(x) &= \cos(2\pi x)\exp\left(-2\epsilon\int_0^{2\pi x} r(s)\cos^2(s)ds\right),\end{aligned}$$

where $r(s) \geqq 0$ is a fixed 2π-periodic function, of class C^∞, vanishing in a neighborhood of $s = 0$ and satisfying the conditions

$$\int_0^{2\pi} r(s)\cos^2 s\; ds = \tfrac{1}{2}, \qquad \int_0^{2\pi} r(s)\cos^2 s \sin s\; ds > 0.$$

Lemma 2. *Let $\phi(x)$ be a solution of the equation*

$$\phi'' + h^2 a(x)\phi = 0, \quad x \in \mathbb{R},$$

where $h \in Z$ and $a(x)$ is a strictly positive function of class C^1, and let us consider the energy functions

$$\begin{aligned}E_\phi(x) &= 4\pi^2 h^2 |\phi(x)|^2 + |\phi'(x)|^2,\\ \tilde{E}_\phi(x) &= h^2 a(x)|\phi(x)|^2 + |\phi'(x)|^2.\end{aligned}$$

Then, for all t_1 and t_2, the following estimates hold:

$$E_\phi(x_2) \leqq E_\phi(x_1)\exp\left|h\int_{x_1}^{x_2} |4\pi^2 - a(x)|dx\right|, \tag{2.12}$$

$$\tilde{E}_\phi(x_2) \leqq \tilde{E}_\phi(x_1)\exp\left|\int_{x_1}^{x_2} \frac{|a'(x)|}{a(x)}dx\right|. \tag{2.13}$$

To prove this result it is sufficient to differentiate the energy functions and to apply Gronwall's Lemma.

3. Construction of the density and the quasi-eigenfunctions

In this section we make the main construction of the paper. We build simultaneously the density ρ and the sequence of quasi-eigenfunctions that exhibit the concentration effect we are looking for. Our construction is inspired by that in [5]. Let us consider the sequences

$$r_j = 2^{-j}, \quad h_j = 2^{2^{Nj}}, \quad \epsilon_j = h_j^{-1}(\log h_j)^2, \tag{3.1}$$

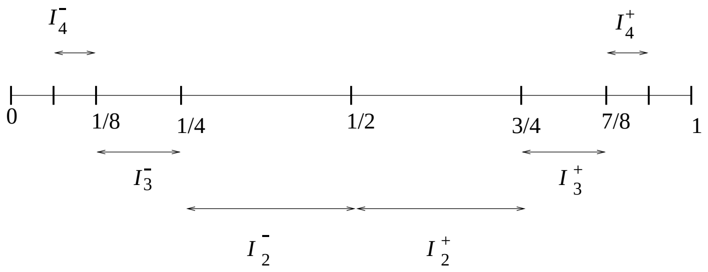

Fig. 2. Partition of the interval [0, 1] in the subintervals $I_j^{\pm}$.

where $N > 1$ is a fixed, large enough integer (with respect to the constant M in Lemma 1) so that the following inequalities hold:

$$\epsilon_k \leqq \frac{1}{2M} \tag{3.2}$$

$$4M \sum_{j=1}^{k-1} \epsilon_j h_j r_j \leqq \epsilon_k h_k r_k \tag{3.3}$$

$$2M \sum_{j=k+1}^{\infty} \epsilon_j r_j \leqq \epsilon_k r_k. \tag{3.4}$$

Note that such inequalities are true for large N due to the following: $\{\epsilon_k\}_{k\geqq 2}$ is a decreasing sequence for $N \geqq 2$ and $\epsilon_1 \to 0$ as $N \to \infty$; the sequence $\epsilon_j h_j r_j (\epsilon_k h_k r_k)^{-1} = (A_N)^{j-k}$ with A_N converging to infinity as $N \to \infty$ and finally $\epsilon_j r_j (\epsilon_k r_k)^{-1} = (\delta_N)^{j-k}$ for some $\delta_N \to 0$ for $N \to \infty$. With this choice of the sequence r_k we define a partition of the interval (0, 1):

$$(0, 1/2] = \bigcup_{j\geqq 2} I_j^-, \qquad [1/2, 1) = \bigcup_{j\geqq 2} I_j^+ \tag{3.5}$$

$$I_j^- = \left(m_j^- - \frac{r_j}{2}, m_j^- + \frac{r_j}{2}\right], \quad I_j^+ = \left[m_j^+ - \frac{r_j}{2}, m_j^+ + \frac{r_j}{2}\right), \quad j \geqq 2, \tag{3.6}$$

$$m_j^- = \frac{r_j}{2} + \sum_{k=j+1}^{\infty} r_k, \qquad m_j^+ = 1 - m_j^-, \qquad j \geqq 2. \tag{3.7}$$

We observe that $m_j^{\pm}$ is the center of the interval $I_j^{\pm}$ with length r_j (see Fig. 1). The super-index + (respectively −) indicates that the interval is to the right (respectively left) of $x = 1/2$. This notation conveniently distinguishes the two singularities, at $x = 0$ and $x = 1$, of the density that we are going to construct.

Now, we define the density ρ as follows

$$\rho(x) = \begin{cases} \alpha_{\epsilon_{2j}}(h_{2j}(x - m_{2j}^-)) & \text{for } x \in I_{2j}^-, \\ \alpha_{\epsilon_{2j+1}}(h_{2j+1}(x - m_{2j+1}^+)) & \text{for } x \in I_{2j+1}^+, \\ 4\pi^2 & \text{for } x \in [0, 1] \setminus \left(\bigcup_{j\geqq 1}(I_{2j}^- \cup I_{2j+1}^+)\right), \end{cases} \tag{3.8}$$

where α_{ϵ_j} are the functions introduced in Lemma 1. The density ρ oscillates near $x = 0$ and $x = 1$ but with different frequencies so that, in some sense, as we will see, these two oscillations do not interact.

Note that $\rho(x) \in C^\infty(0, 1)$ because $\alpha_\epsilon \in C^\infty(\mathbb{R})$ and

$$\rho(x) \equiv 4\pi^2 \text{ in a neighborhood of the extremes of } I_j^\pm, \tag{3.9}$$

since $\frac{h_j r_j}{2}$ is an integer and properties (2.2) and (2.3) hold. On the other hand, $\rho(x) \in C^{0,s}([0, 1])$ for all $0 < s < 1$. Indeed, by (2.2) and the bound (2.4) for $|\alpha'_\epsilon|$ we have

$$\begin{aligned} |\alpha_{\epsilon_j}|_{C^{0,s}(\mathbb{R})} &= \max_{x,y\in\mathbb{R}} \frac{|\alpha_{\epsilon_j}(x) - \alpha_{\epsilon_j}(y)|}{|x-y|^s} \leqq \max_{x,y\in[0,1]} \frac{|\alpha_{\epsilon_j}(x) - \alpha_{\epsilon_j}(y)|}{|x-y|^s} \\ &\leqq M\epsilon_j \max_{x,y\in[0,1]} |x-y|^{1-s} \leqq M\epsilon_j \end{aligned} \tag{3.10}$$

for $0 < s < 1$. Note that here $|\cdot|_{C^{0,s}(\mathbb{R})}$ represents the Hölder semi-norm. Therefore

$$|\rho|_{C^{0,s}(I_j^\pm)} \leqq M\epsilon_j h_j^s \quad \text{for } 0 < s < 1. \tag{3.11}$$

The term on the right is uniformly bounded in j in view of (3.1), i.e.,

$$\sup_j \epsilon_j h_j^s < \infty \quad \text{for } 0 < s < 1. \tag{3.12}$$

From (3.9)–(3.12) we conclude that $\rho(x) \in C^{0,s}([0, 1])$ for all $s < 1$. In fact, if we define I_x as the interval of the family $\{I_j^\pm\}_{j\geqq 2}$ such that $x \in I_x$ and l_x, r_x are the left and right extremes of I_x respectively, then

$$\begin{aligned} \max_{x,y\in[0,1]} \frac{|\rho(x)-\rho(y)|}{|x-y|^s} &\leqq \max_{0\leqq x\leqq y\leqq 1} \frac{|\rho(x)-\rho(r_x)| + |\rho(l_y)-\rho(y)|}{|x-y|^s} \\ &\leqq \max_{x\in[0,1]} \frac{|\rho(x)-\rho(r_x)|}{|x-r_x|^s} + \max_{y\in[0,1]} \frac{|\rho(l_y)-\rho(y)|}{|l_y-y|^s} \\ &\leqq 2\sup_{j\geqq 2} |\rho|_{C^{0,s}(I_j^\pm)}. \end{aligned}$$

Here we have used the fact that, in view of (3.9), $\rho(r_x) = \rho(l_y)$.

Finally, we observe that ρ is bounded above and below with positive constants in view of (2.4) and (3.2). In fact we have

$$2\pi^2 \leqq \rho(x) \leqq 8\pi^2. \tag{3.13}$$

Let us define two sequences of quasi-eigenfunctions $\{\varphi_{2j}^-\}_{j\geqq 1}$ and $\{\varphi_{2j+1}^+\}_{j\geqq 1}$ as the solutions of the following initial value problems:

$$\begin{aligned} &(\varphi_{2j}^-)'' + h_{2j}^2 \rho(x) \varphi_{2j}^- = 0, \qquad 0 < x < 1, \\ &\varphi_{2j}^-(m_{2j}^-) = 1, \qquad (\varphi_{2j}^-)'(m_{2j}^-) = 0 \end{aligned} \tag{3.14}$$

$$\begin{aligned} &(\varphi_{2j+1}^+)'' + h_{2j+1}^2 \rho(x) \varphi_{2j+1}^+ = 0, \qquad 0 < x < 1, \\ &\varphi_{2j+1}^+(m_{2j+1}^+) = 1, \qquad (\varphi_{2j+1}^+)'(m_{2j+1}^+) = 0. \end{aligned} \tag{3.15}$$

These are simply quasi-eigenfunctions since the boundary conditions at $x = 0, 1$ are not necessarily fulfilled. For example, there is no reason for φ_{2j}^{-} to vanish either at $x = 0$ or at $x = 1$. However, we will see that φ_{2j}^{-} is mainly concentrated in the interior of I_{2j}^{-} so that the values of φ_{2j}^{-} at $x = 0, 1$ are exponentially small. This justifies referring to φ_{2j}^{-} as quasi-eigenfunctions in the sense that the missing boundary conditions are almost satisfied. The same argument applies to φ_{2j+1}^{+}. To see that φ_{2j}^{-} is concentrated in the interior of I_{2j}^{-} we observe that it satisfies

$$\begin{aligned} &(\varphi_{2j}^{-})'' + h_{2j}^{2}\alpha_{\epsilon_{2j}}(h_{2j}(x - m_{2j}^{-}))\varphi_{2j}^{-} = 0, \qquad x \in I_{2j}^{-}, \\ &\varphi_{2j}^{-}(m_{2j}^{-}) = 1, \qquad (\varphi_{2j}^{-})'(m_{2j}^{-}) = 0. \end{aligned} \tag{3.16}$$

Therefore,

$$\varphi_{2j}^{-}(x) = w_{\epsilon_{2j}}(h_{2j}(x - m_{2j}^{-})), \tag{3.17}$$

where $w_{\epsilon_{2j}}$ is the function in Lemma 1 associated with $\alpha_{\epsilon_{2j}}$. Combining Remark 2 and the fact that $h_{2j}r_{2j}/2$ are integers we deduce that

$$\begin{aligned} \int_{I_{2j}^{-}} |\varphi_{2j}^{-}(x)|^{2}dx &= \frac{1}{h_{2j}}, \int_{-r_{2j}h_{2j}}^{r_{2j}h_{2j}} |w_{\epsilon_{2j}}(s)|^{2}ds \geqq \frac{1}{h_{2j}} \int_{0}^{1} |w_{\epsilon_{2j}}(s)|^{2}ds \\ &\geqq \frac{\gamma^{2}\epsilon_{2j}^{2}}{h_{2j}} \geqq \frac{C}{h_{2j}^{3}}, \end{aligned} \tag{3.18}$$

$$\left|\varphi_{2j}^{-}(m_{2j}^{-} - \frac{r_{2j}}{2})\right|^{2} + \left|(\varphi_{2j}^{-})'(m_{2j}^{-} - \frac{r_{2j}}{2})\right|^{2} = e^{-\epsilon_{2j}h_{2j}r_{2j}}, \tag{3.19}$$

$$\left|\varphi_{2j}^{-}(m_{2j}^{-} + \frac{r_{2j}}{2})\right|^{2} + \left|(\varphi_{2j}^{-})'(m_{2j}^{-} + \frac{r_{2j}}{2})\right|^{2} = e^{-\epsilon_{2j}h_{2j}r_{2j}}, \tag{3.20}$$

i.e., the L^2 norm of φ_{2j}^{-} in I_{2j}^{-} is of the order h_{2j}^{-3} but exponentially larger than the values of φ_{2j}^{-} at the extremes of I_{2j}^{-} in the sense that $h_{2j}^{3}e^{-\epsilon_{2j}h_{2j}r_{2j}} \leqq C_{p}h_{2j}^{-p}$ for all $p > 0$. In fact, for any $p > 0$,

$$h_{j}^{p}e^{-\epsilon_{j}h_{j}r_{j}} = h_{j}^{p}h_{j}^{-r_{j}\log(h_{j})} = h_{j}^{p-2^{-j+Nj}\log(2)} \to 0 \quad \text{as } j \to \infty. \tag{3.21}$$

Roughly speaking, we have checked that φ_{2j}^{-} is concentrated in the interior of I_{2j}^{-}. Now, we might expect the energy of φ_{2j}^{-} to still be small outside I_{2j}^{-}. We claim that this is the case, i.e., φ_{2j}^{-} is small at $x = 0, 1$. The analyses we shall make at the extremes ($x = 0$ and $x = 1$) will in differ in nature. Therefore, at this point we divide our analysis into these two cases.

Analysis at $x = 0$: We estimate, for $x \leqq m_{2j}^{-} - r_{2j}/2$, the energy function

$$E_{\varphi_{2j}^{-}}(x) = 4\pi^{2}h_{2j}^{2}\left|\varphi_{2j}^{-}(x)\right|^{2} + \left|(\varphi_{2j}^{-})'(x)\right|^{2}. \tag{3.22}$$

In view of (3.19) we have

$$E_{\varphi_{2j}^-}\left(m_{2j}^- - \frac{r_{2j}}{2}\right) \leqq 4\pi^2 h_{2j}^2 e^{-\epsilon_{2j} h_{2j} r_{2j}}. \tag{3.23}$$

On the other hand, taking (2.12) into account, the definition of ρ in (3.8) and the estimate for α_ϵ in (2.4), we have

$$\begin{aligned} E_{\varphi_{2j}^-}(x) &\leqq E_{\varphi_{2j}^-}\left(m_{2j}^- - \frac{r_{2j}}{2}\right) \exp\left(4\pi^2 h_{2j}^2 \int_x^{m_{2j}-r_{2j}/2} |4\pi^2 - \rho|\right) \\ &\leqq E_{\varphi_{2j}^-}\left(m_{2j}^- - \frac{r_{2j}}{2}\right) \exp\left(4\pi^2 M h_{2j} \sum_{k=j+1}^{\infty} \epsilon_{2k} r_{2k}\right). \end{aligned}$$

This last term can be estimated with the aid of (3.4) and (3.23) and therefore

$$\begin{aligned} E_{\varphi_{2j}^-}(x) &\leqq E_{\varphi_{2j}^-}\left(m_{2j}^- - \frac{r_{2j}}{2}\right) \exp\left(4\pi^2 M h_{2j} \sum_{k=j+1}^{\infty} \epsilon_{2k} r_{2k}\right) \\ &\leqq 4\pi^2 h_{2j}^2 \exp\left(-h_{2j}\epsilon_{2j} r_{2j} + \frac{h_{2j}\epsilon_{2j} r_{2j}}{2}\right) \\ &\leqq 4\pi^2 h_{2j}^2 \exp\left(-\frac{h_{2j}\epsilon_{2j} r_{2j}}{2}\right) \\ &\leqq 4\pi^2 h_{2j}^2 \exp\left(-2^{-2j-1}(\log h_{2j})^2\right) \end{aligned}$$

for all $x \leqq m_{2j}^- - r_{2j}$. Hence

$$\left|\varphi_{2j}^-(x)\right|^2 + \left|(\varphi_{2j}^-)'(x)\right|^2 \leqq C_p h_{2j}^{-p} \quad \forall p > 0, \quad \forall x \leqq m_{2j}^- - r_{2j}/2. \tag{3.24}$$

In particular (3.24) holds for $x = 0$.

Analysis at $x = 1$**:** Here we first estimate, for $m_{2j}^- + r_{2j}/2 \leqq x \leqq m_{2j+1}^+ - r_{2j+1}/2$, the energy function

$$\widetilde{E}_{\varphi_{2j}^-}(x) = h_{2j}^2 \rho(x) \left|\varphi_{2j}^-(x)\right|^2 + \left|(\varphi_{2j}^-)'(x)\right|^2. \tag{3.25}$$

In view of (3.19) we have

$$\tilde{E}_{\varphi_{2j}^-}\left(m_{2j}^- + \frac{r_{2j}}{2}\right) \leqq 4\pi^2 h_{2j}^2 e^{-\epsilon_{2j} h_{2j} r_{2j}}. \tag{3.26}$$

On the other hand, we get, by the aid of Lemma 2 (estimate (2.13)), the definition of ρ in (3.8), the estimate for α'_ϵ in (2.4) and the fact that $\rho(x) \geqq 2\pi^2$:

$$\begin{aligned} \widetilde{E}_{\varphi_{2j}^-}(x) &\leqq \widetilde{E}_{\varphi_{2j}^-}\left(m_{2j}^- + \frac{r_{2j}}{2}\right) \exp\left(\int_{m_{2j}^- + r_{2j}/2}^{x} \frac{|\rho'(s)|}{|\rho(s)|} ds\right) \\ &\leqq \widetilde{E}_{\varphi_{2j}^-}\left(m_{2j}^- + \frac{r_{2j}}{2}\right) \exp\left(M \sum_{k=1}^{2j-1} \epsilon_k r_k h_k\right). \end{aligned}$$

This last term can be estimated with (3.26) and (3.19) and therefore

$$\widetilde{E}_{\varphi_{2j}^-}(x) \leqq h_{2j}^2 \exp\left(-h_{2j}\epsilon_{2j}r_{2j} + \tfrac{1}{4}h_{2j}\epsilon_{2j}r_{2j}\right) = h_{2j}^2 \exp\left(-\tfrac{3}{4}h_{2j}\epsilon_{2j}r_{2j}\right)$$

for all $x \in [m_{2j}^- - r_{2j}/2, m_{2j+1}^+ - r_{2j+1}/2]$. In particular, taking (3.13) into account, we have

$$\begin{aligned} E_{\varphi_{2j}^-}(m_{2j+1}^+ - \tfrac{1}{2}r_{2j+1}) &\leqq 2\tilde{E}_{\varphi_{2j}^-}(m_{2j+1}^+ - \tfrac{1}{2}r_{2j+1}) \\ &\leqq 2h_{2j}^2 \exp\left(-\tfrac{3}{4}h_{2j}\epsilon_{2j}r_{2j}\right). \end{aligned}$$

For $x \geqq m_{2j+1}^+ - r_{2j+1}/2$ we use the energy function (3.22),

$$\begin{aligned} E_{\varphi_{2j}^-}(x) &\leqq E_{\varphi_{2j}^-}(m_{2j+1}^+ - \tfrac{1}{2}r_{2j+1}) \exp\left(4\pi^2 M h_{2j} \sum_{k=j}^{\infty} \epsilon_{2k+1}r_{2k+1}\right) \\ &\leqq 2h_{2j}^2 \exp\left(-\tfrac{3}{4}h_{2j}\epsilon_{2j}r_{2j} + \tfrac{1}{2}h_{2j}\epsilon_{2j}r_{2j}\right) \\ &\leqq 2h_{2j}^2 \exp\left(-\tfrac{1}{4}h_{2j}\epsilon_{2j}r_{2j}\right) \\ &\leqq 2h_{2j}^2 \exp\left(-2^{-2j-2}(\log h_{2j})^2\right) \end{aligned}$$

for all $x \geqq m_{2j+1}^+ - r_{2j+1}/2$. Therefore

$$\left|\varphi_{2j}^-(x)\right|^2 + \left|(\varphi_{2j}^-)'(x)\right|^2 \leqq C_p h_{2j}^{-p} \quad \forall p > 0, \quad \forall x \geqq m_{2j+1}^+ - \tfrac{1}{2}r_{2j+1}, \tag{3.27}$$

which holds in particular for $x = 1$.

We have proved the existence of a sequence of quasi-eigenfunctions φ_{2j}^- concentrated in the interior of I_{2j}^-, i.e., a sequence of solutions of (3.16) which satisfy

$$\int_{I_{2j}^-} |\varphi_{2j}^-(x)|^2 dx \geqq \frac{C}{h_{2j}^3}, \tag{3.28}$$

$$\left|\varphi_{2j}^-(0)\right|^2 + \left|(\varphi_{2j}^-)'(0)\right|^2 \leqq C_p h_{2j}^{-p} \quad \forall p > 0, \tag{3.29}$$

$$\left|\varphi_{2j}^-(1)\right|^2 + \left|(\varphi_{2j}^-)'(1)\right|^2 \leqq C_p h_{2j}^{-p} \quad \forall p > 0, \tag{3.30}$$

$$\int_\alpha^\beta \left[|(\varphi_{2j}^-)'(x)|^2 + h_{2j}^2|\varphi_{2j}^-(x)|^2\right] dx \leqq C_p h_{2j}^{-p} \quad \forall p > 0, \tag{3.31}$$

for all $(\alpha, \beta) \subset (0, 1)$ with $\alpha \neq 0$ and $j \geqq J$ large enough to have $I_{2j}^- \cap (\alpha, \beta) = \emptyset$ for all $j \geqq J$. A similar result can be obtained for φ_{2j+1}^+. More precisely,

$$\int_{I_{2j+1}^+} |\varphi_{2j+1}^+(x)|^2 dx \geqq \frac{C}{h_{2j+1}^3}, \tag{3.32}$$

$$\left|\varphi_{2j+1}^+(0)\right|^2 + \left|(\varphi_{2j+1}^+)'(0)\right|^2 \leqq C_p h_{2j+1}^{-p} \quad \forall p > 0, \tag{3.33}$$

$$\left|\varphi_{2j+1}^+(1)\right|^2 + \left|(\varphi_{2j+1}^+)'(1)\right|^2 \leqq C_p h_{2j+1}^{-p} \quad \forall p > 0, \tag{3.34}$$

$$\int_\alpha^\beta \left[|(\varphi_{2j+1}^+)'(x)|^2 + h_{2j}^2|\varphi_{2j+1}^+(x)|^2\right] dx \leqq C_p h_{2j}^{-p} \quad \forall p > 0, \tag{3.35}$$

for all $(\alpha, \beta) \subset (0, 1)$ with $\beta \neq 1$ and $j \geqq J$ large enough to have $I^+_{2j+1} \cap (\alpha, \beta) = \emptyset$ for all $j \geqq J$.

Remark 5. Let us describe how the claim of point 4 in Remark 1 may be proved. First of all, note that the construction above (in this section) may be done with a Hölder continuous density ϵ of arbitrarily small support $[l_1, l_2]$, with an arbitrary value of $\epsilon(l_1) = \epsilon(l_2)$ and with $\|\epsilon(x) - \epsilon(l_2)\|_{L^\infty(l_1,l_2)}$ arbitrarily small. On the other hand, for a given smooth density function ρ and any subinterval $[x_0, x_1]$ of $[0, 1]$ we can find a smooth function $\hat{\rho}$ such that

$$\hat{\rho} = \begin{cases} \rho_c \text{ constant, in a compact set } [l_1, l_2] \subset\subset (x_0, x_1), \\ \rho(x) \text{ in } [0, x_0] \cup [x_1, 1]. \end{cases}$$

We now choose ϵ as above with $\epsilon(l_2) = \rho_c$ and define

$$\tilde{\rho} = \begin{cases} \epsilon(x) \text{ in } [l_1, l_2] \subset\subset (x_0, x_1), \\ \hat{\rho}(x) \text{ in } [0, l_0) \cup (l_1, 1]. \end{cases}$$

In this way we obtain a Hölder continuous density function $\tilde{\rho}$ with localized quasi-eigenfunctions within the interval $[l_1, l_2]$ and such that $\rho = \tilde{\rho}$ outside $[x_0, x_1]$, and $\|\rho - \tilde{\rho}\|_{L^\infty(0,1)}$ is arbitrarily small.

4. Lack of boundary observability

In this section we prove that (1.5) fails for all $T > 0$ for the density function ρ we have built in Section 3. Consider the sequence of quasi-eigenfunctions $\{\varphi^-_{2j}\}_{j\geqq 1}$. We can construct the following sequence of solutions of the first equation in (1.1)

$$v_j(x, t) = e^{ih_{2j}t}\varphi^-_{2j}(x). \tag{4.1}$$

Note that v_j does not satisfy the boundary conditions in (1.1) due to the fact that φ^-_{2j} are not true eigenfunctions, i.e., they do not vanish at $x = 0, 1$. However, we can correct v_j with a function $\widetilde{v}_j$ in such a way that

$$u_j = v_j + \widetilde{v}_j \tag{4.2}$$

satisfies all the equations in system (1.1). To this end we define $\widetilde{v}_j$ as the unique solution of

$$\begin{array}{ll} \rho(x)\widetilde{v}_{tt} - \widetilde{v}_{xx} = 0, & 0 < x < 1,\ 0 < t < T, \\ \widetilde{v}(0, t) = -v_j(0, t) = -e^{ih_{2j}t}\varphi^-_{2j}(0), & 0 < t < T, \\ \widetilde{v}(1, t) = -v_j(1, t) = -e^{ih_{2j}t}\varphi^-_{2j}(1), & 0 < t < T, \\ \widetilde{v}(x, 0) = \widetilde{v}_t(x, 0) \quad = 0, & 0 < x < 1. \end{array} \tag{4.3}$$

In the rest of the section we prove that (1.5) fails for the sequence u_j, i.e.,

$$\lim_{j\to\infty} \frac{\int_0^1 \left[|u_{j,x}(x, 0)|^2 + \rho(x)|u_{j,t}(x, 0)|^2\right] dx}{\int_0^T \left[|u_{j,x}(0, t)|^2 + |u_{j,x}(1, t)|^2\right] dt} = \infty. \tag{4.4}$$

We first estimate the numerator in (4.4):

$$\begin{aligned}
&\int_0^1 \left[|u_{j,x}(x,0)|^2 + \rho(x)|u_{j,t}(x,0)|^2\right] dx \\
&\qquad = \int_0^1 \left[|v_{j,x}(x,0)|^2 + \rho(x)|v_{j,t}(x,0)|^2\right] dx \\
&\qquad = \int_0^1 \left[|(\varphi_{2j}^-)'|^2 + h_{2j}^2 \rho(x)|\varphi_{2j}^-|^2\right] dx \\
&\qquad \geqq h_{2j}^2 \rho_m \int_0^1 |\varphi_{2j}^-|^2 dx \geqq h_{2j}^2 \rho_m \int_{I_{2j}^-} |\varphi_{2j}^-|^2 dx. \qquad (4.5)
\end{aligned}$$

Concerning the denominator in (4.4) we have

$$\begin{aligned}
&\int_0^T \left[|u_{j,x}(0,t)|^2 + |u_{j,x}(1,t)|^2\right] dt \\
&\qquad \leqq 2\int_0^T \left[|v_{j,x}(0,t)|^2 + |v_{j,x}(1,t)|^2\right] dt \\
&\qquad\quad + 2\int_0^T \left[|\widetilde{v}_{j,x}(0,t)|^2 + |\widetilde{v}_{j,x}(1,t)|^2\right] dt \\
&\qquad = 2T\left[|(\varphi_{2j}^-)'(0)|^2 + |(\varphi_{2j}^-)'(1)|^2\right] \\
&\qquad\quad + 2\int_0^T \left[|\widetilde{v}_{j,x}(0,t)|^2 + |\widetilde{v}_{j,x}(1,t)|^2\right] dt. \qquad (4.6)
\end{aligned}$$

The last term can be estimated applying the following result:

Proposition 1. *Consider the following system:*

$$\begin{cases}
\rho(x)v_{tt} - v_{xx} = 0, & 0 < x < 1,\ 0 < t < T, \\
v(0,t) = f_1(t), \quad v(1,t) = f_2(t), & 0 < t < T, \\
v(x,0) = v_t(x,0) = 0, & 0 < x < 1,
\end{cases} \qquad (4.7)$$

where $\rho \in L^\infty(0,1)$, $0 < \rho_0 \leqq \rho(x) \leqq \rho_1 < \infty$. *Given* $T > 0$, *there exists* $C(T) > 0$ *such that*

$$\begin{aligned}
&\int_0^T \int_0^1 \left[|v_x|^2 + \rho(x)|v_t|^2\right] dx dt \\
&\qquad \leqq C(T)||\rho||_\infty \left(||f_1||^2_{W^{2,\infty}(0,T)} + ||f_2||^2_{W^{2,\infty}(0,T)}\right), \qquad (4.8) \\
&\int_0^T \left[|v_x(0,t)|^2 + |v_x(1,t)|^2\right] \\
&\qquad \leqq C(T)||\rho||_\infty \left(||f_1||^2_{W^{3,\infty}(0,T)} + ||f_2||^2_{W^{3,\infty}(0,T)}\right). \qquad (4.9)
\end{aligned}$$

We prove this proposition at the end of the section.

When applying Proposition 1 to $\tilde{v}$ in (4.6) we obtain

$$\int_0^T \left[|u_{j,x}(0,t)|^2 + |u_{j,x}(1,t)|^2\right] dt \\ \leqq 2C(T)h_{2j}^6 \left[|(\varphi_{2j}^-)'(0)|^2 + |\varphi_{2j}^-(0)|^2 + |(\varphi_{2j}^-)'(1)|^2 + |\varphi_{2j}^-(1)|^2\right]. \quad (4.10)$$

Finally, combining (4.5), (4.10) and the estimates for φ_{2j}^- of the previous section we easily obtain

$$\frac{\int_0^1 \left[|u_{j,x}(x,0)|^2 + \rho(x)|u_{j,t}(x,0)|^2\right] dx}{\int_0^T \left[|u_{j,x}(0,t)|^2 + |u_{j,x}(1,t)|^2\right] dt} \geqq \frac{h_{2j}^{-1}\rho_m C}{2T(1+h_{2j}^6)C_p h_{2j}^{-p}}$$

which converges to infinity for $j \to \infty$ and $p > 7$.

Remark 6. The lack of boundary observability that we have proved above relies on the existence of a unique sequence of quasi-eigenfunctions $\left\{\varphi_{2j}^-\right\}_{j\geqq 1}$. Therefore, the double sequence of eigenfunctions $\left\{\varphi_{2j}^-\right\}$ and $\left\{\varphi_{2j-1}^+\right\}_{j\geqq 1}$ constructed in Section 3 is not necessary here. It will be used later to prove the lack of interior observability.

Proof of Proposition 1. We introduce

$$h(x,t) = (1-x)f_1(t) + xf_2(t). \quad (4.11)$$

Then,

$$w(x,t) = v(x,t) - h(x,t) \quad (4.12)$$

satisfies

$$\begin{array}{ll} \rho(x)w_{tt} - w_{xx} = -\rho(x)\left[(1-x)f_1''(t) + xf_2''(t)\right], & 0<x<1,\ 0<t<T, \\ w(0,t) = w(1,t) = 0, & 0<t<T, \\ w(x,0) = -\left[(1-x)f_1(0) + xf_2(0)\right], & 0<x<1, \\ w_t(x,0) = -\left[(1-x)f_1'(0) + xf_2'(0)\right], & 0<x<1. \end{array} \quad (4.13)$$

Classical estimates on non-homogeneous wave equations show that

$$\int_0^T \int_0^1 \left[|w_x|^2 + \rho(x)|w_t|^2\right] dxdt \leqq C(T)||\rho||_\infty \left(||f_1''||_\infty^2 + ||f_2''||_\infty^2\right). \quad (4.14)$$

On the other hand,

$$\int_0^T \int_0^1 \left[|h_x|^2 + \rho(x)|h_t|^2\right] dxdt \\ \leqq 2T||\rho||_\infty \left(||f_1'||_\infty^2 + ||f_1||_\infty^2 + ||f_2'||_\infty^2 + ||f_2||_\infty^2\right). \quad (4.15)$$

This last estimate and (4.14) allow us to obtain easily the first inequality (4.8).

For the boundary inequality (4.9), we first obtain a corresponding boundary estimate for system (4.13). The classical procedure for such estimates is to multiply the first equation in (4.13) by the multiplier xw_x and integrate:

$$0 = \int_0^T \int_0^1 \rho(x) w_{tt} x w_x dx dt - \int_0^T \int_0^1 w_{xx} x w_x dx dt + \int_0^T \int_0^1 \rho(x) \left[(1-x) f_1''(t) + x f_2''(t)\right] x w_x. \tag{4.16}$$

Now we integrate by parts in the second term of the right hand side,

$$\int_0^T \int_0^1 w_{xx} x w_x dx dt = \int_0^T \int_0^1 \frac{x}{2} \frac{d}{dx} |w_x|^2 dx dt = -\int_0^T \int_0^1 \frac{|w_x|^2}{2} dx dt + \int_0^T \frac{|w_x(1,t)|^2}{2} dt. \tag{4.17}$$

Combining (4.16) and (4.17) we easily find the following estimate:

$$\int_0^T |w_x(1,t)|^2 dt \leqq CT \|\rho\|_\infty \left(\|w_{tt}\|^2_{L^2(0,1)} + \|w\|^2_{H_0^1(0,1)} + ||f_1''||^2_\infty + ||f_2''||^2_\infty \right). \tag{4.18}$$

Here we have to remove the L^2 norm of w_{tt} from the right-hand side. To do this we observe that $\tilde{w} = w_t$ satisfies the system

$$\begin{aligned}
\rho(x)\tilde{w}_{tt} - \tilde{w}_{xx} &= -\rho(x)\left[(1-x) f_1'''(t) + x f_2'''(t)\right], && 0 < x < 1,\\
\tilde{w}(0,t) &= \tilde{w}(1,t) = 0, && 0 < t < T,\\
\tilde{w}(x,0) &= w_t(x,0) = -\left[(1-x) f_1'(0) + x f_2'(0)\right], && 0 < x < 1,\\
\tilde{w}_t(x,0) &= \frac{1}{\rho(x)} \left[w_{xx}(x,0) - \rho(x)\left((1-x) f_1''(0) + x f_2''(0)\right)\right] \\
&= -\left((1-x) f_1''(0) + x f_2''(0)\right) && 0 < x < 1.
\end{aligned} \tag{4.19}$$

Once again the energy estimate for the non-homogeneous problem provides

$$\int_0^T \int_0^1 \left[|\tilde{w}_x|^2 + \rho(x) |\tilde{w}_t|^2 \right] dx dt \leqq C(T) ||\rho||_\infty \left(||f_1||^2_{W^{3,\infty}(0,T)} + ||f_2||^2_{W^{3,\infty}(0,T)} \right). \tag{4.20}$$

This inequality allows us to estimate the term with $w_{tt} = \tilde{w}_t$ in (4.18). Then we have

$$\int_0^T |w_x(1,t)|^2 dt \leqq CT \|\rho\|_\infty \left(\|w\|^2_{H_0^1(0,1)} + ||f_1||^2_{W^{3,\infty}(0,T)} + ||f_2||^2_{W^{3,\infty}(0,T)} \right), \tag{4.21}$$

for some constant $C > 0$.

A similar estimate can be obtained for the L^2 norm of $w_x(0,t)$. Therefore

$$\int_0^T \left[|w_{j,x}(0,t)|^2 + |w_{j,x}(1,t)|^2\right] dt$$
$$\leqq CT\|\rho\|_\infty \left(\|w\|^2_{H_0^1(0,1)} + ||f_1||^2_{W^{3,\infty}(0,T)} + ||f_2||^2_{W^{3,\infty}(0,T)}\right). \quad (4.22)$$

On the other hand,

$$\int_0^T \left[|h_x(0,t)|^2 + |h_x(1,t)|^2\right] \leqq T\left(||f_1||^2_\infty + ||f_2||^2_\infty\right).$$

Combining this last estimate with (4.22) we easily obtain the inequality (4.9) for $v = h + w$. □

Remark 7. We recall that when $\rho \in W^{1,\infty}$ we may proceed differently in the proof of Proposition 1. Indeed, the term

$$I = \int_0^T \int_0^1 \rho(x) w_{tt} x w_x \, dxdt$$

can be bounded as follows. Integrating by parts with respect to time we obtain

$$I = -\int_0^T \int_0^1 \rho(x) w_t x w_{xt} \, dxdt + \int_0^1 \rho(x) w_t x w_x \, dx\Big|_0^T .$$

The first integral in this identity may be rewritten as

$$\int_0^T \int_0^1 \rho(x) w_t x w_{xt} \, dxdt = \tfrac{1}{2} \int_0^T \int_0^1 \rho(x) \left(|w_t|^2\right)_x dxdt$$

and, after integrating by parts,

$$\int_0^T \int_0^1 \rho(x) w_t x w_{xt} dxdt = -\tfrac{1}{2} \int_0^T \int_0^1 \rho_x |w_t|^2 dxdt.$$

Obviously this argument cannot be applied in our case since ρ_x is not bounded.

5. Lack of internal observability

This section is devoted to proving the lack of observability from any subinterval (α, β) of $(0,1)$, provided $(\alpha,\beta) \neq (0,1)$. We first assume that $\alpha > 0$ and consider the sequence of quasi-eigenfunctions $\{\varphi^-_{2j}\}_{j\geqq 1}$ from which we can construct the following sequence of solutions of the first equation in (1.1):

$$v_j(x) = e^{ih_{2j}t} \varphi^-_{2j}(x). \quad (5.1)$$

Note that v_j does not satisfy the boundary conditions in (1.1) due to the fact that φ^-_{2j} are not true eigenfunctions.

As in the previous case, we correct v_j with a function $\widetilde{v}_j$ in such a way that

$$u_j = v_j + \widetilde{v}_j \tag{5.2}$$

satisfies all the equations in system (1.1). To this end we define $\widetilde{v}_j$ as the unique solution of (4.3).

Now we prove that for the sequence u_j, (1.6) fails, i.e.,

$$\lim_{j\to\infty} \frac{\int_0^1 \left[|u_{j,x}(x,0)|^2 + \rho(x)|u_{j,t}(x,0)|^2\right] dx}{\int_0^T \int_\alpha^\beta \left[|u_{j,x}(x,t)|^2 + |u_{j,t}(x,t)|^2\right] dxdt} = \infty. \tag{5.3}$$

The numerator in (5.3) can be estimated by (4.5).

Concerning the denominator in (5.3) we have

$$\begin{aligned}
&\int_0^T \int_\alpha^\beta \left[|u_{j,x}(x,t)|^2 + |u_{j,t}(x,t)|^2\right] dxdt \\
&\quad \leqq 2\int_0^T \int_\alpha^\beta \left[|v_{j,x}(x,t)|^2 + |v_{j,t}(x,t)|^2\right] dxdt \\
&\qquad + 2\int_0^T \int_\alpha^\beta \left[|\widetilde{v}_{j,x}(x,t)|^2 + |\widetilde{v}_{j,t}(x,t)|^2\right] dxdt \\
&\quad \leqq 2T\int_\alpha^\beta \left[|(\varphi_{2j}^-)'(x)|^2 + h_{2j}^2|\varphi_{2j}^-(x)|^2\right] dx \\
&\qquad + 2\int_0^T \int_0^1 \left[|\widetilde{v}_{j,x}(x,t)|^2 + |\widetilde{v}_{j,t}(x,t)|^2\right] dxdt.
\end{aligned}$$

Then, by Proposition 1 applied to $\tilde{v}$

$$\begin{aligned}
&\int_0^T \int_\alpha^\beta \left[|u_{j,x}(x,t)|^2 + |u_{j,t}(x,t)|^2\right] dxdt \\
&\quad \leqq 2T\int_\alpha^\beta \left[|(\varphi_{2j}^-)'(x)|^2 + h_{2j}^2|\varphi_{2j}^-(x)|^2\right] dx \\
&\qquad + C(T)h_{2j}^4\left[|\varphi_{2j}^-(0)|^2 + |\varphi_{2j}^-(1)|^2\right].
\end{aligned}$$

Finally we obtain

$$\begin{aligned}
&\frac{\int_0^1 \left[|u_{j,x}(x,0)|^2 + \rho(x)|u_{j,t}(x,0)|^2\right] dx}{\int_0^T \int_\alpha^\beta \left[|u_{j,x}(x,t)|^2 + |u_{j,t}(x,t)|^2\right] dxdt} \\
&\geqq \frac{h_{2j}^2\rho_m \int_{I_{2j}^-} |\varphi_{2j}^-|^2 dx}{2T\int_\alpha^\beta \left[|(\varphi_{2j}^-)'(x)|^2 + h_{2j}^2|\varphi_{2j}^-(x)|^2\right] dx + C(T)h_{2j}^4\left[|\varphi_{2j}^-(0)|^2 + |\varphi_{2j}^-(1)|^2\right]} \\
&\geqq \frac{h_{2j}^{-1}\rho_m C}{2(T + C(T)h_{2j}^4)C_p h_{2j}^{-p}},
\end{aligned}$$

which converges to infinity for $j \to \infty$ when $p > 5$.

When $\alpha = 0$ we have $\beta \neq 1$ and we can argue in a similar way with the sequence of quasi-eigenfunctions φ^{+}_{2j+1}which concentrates near $x = 1$ instead of $x = 0$.

6. The multi-dimensional case

In this section we show that the result of Theorem 1 can be easily extended to higher dimensional wave equations. The main idea is that, based on the 1-dimensional construction above, we can construct densities ρ in separated variables, which oscillate in a neighborhood of any point of the domain or of the boundary. For these densities we also construct a sequence of quasi-eigenfunctions concentrated inside the domain.

Let Ω be an open set of $\mathbb{R}^d$, $d \geqq 2$, with boundary $\partial\Omega$ of class C^3 and consider the wave equation

$$\begin{array}{lll} \rho(x)u_{tt} - \Delta u = 0, & & x \in \Omega, \quad 0 < t < T, \\ u(x,t) = 0, & & x \in \partial\Omega, \ 0 < t < T, \\ u(x,0) = u_0(x), & u_t(x,0) = u_1(x) & x \in \Omega. \end{array} \tag{6.1}$$

The energy of the system is given by

$$E(t) = \frac{1}{2}\int_\Omega \left[\rho(x)|u_t(x,t)|^2 + |\nabla u(x,t)|^2\right] dx, \tag{6.2}$$

and the boundary and internal observability properties read:

$$E(0) \leqq C\int_0^T \int_{\partial\Omega} \left|\frac{\partial u}{\partial n}(x,t)\right|^2 d\sigma dt, \tag{6.3}$$

$$E(0) \leqq C\int_0^T \int_\omega \left[\rho(x)|u_t(x,t)|^2 + |\nabla u(x,t)|^2\right] dx dt, \tag{6.4}$$

respectively. Here $\omega \subset \Omega$ is an open subset and $\frac{\partial}{\partial n}$ represents the normal derivative.

When $\rho \in C^2(\bar{\Omega})$ and Ω is of class C^3, inequalities (6.3) and (6.4) hold provided a geometric control condition is fulfilled (see [2] and [3]). This condition requires that every ray of geometric optics enters the set where the observation is being made (the boundary $\partial\Omega$ in (6.3) and the open subset $\omega \subset \Omega$ in (6.4)) in time less than T. When $\rho \in C^1(\Omega)$ the existence of rays is guaranteed but uniqueness fails in general and the analysis of inequalities (6.3) and (6.4) remains to be done in this more general setting, except for the space dimension $n = 1$ in which we know that these inequalities hold even when the coefficient is in BV.

In this section we show how to construct Hölder continuous density functions ρ such that the above observability inequalities fail for a large class of subsets ω. In fact, the density functions we build are such that there exists a sequence of solutions for which the energy can be concentrated around a given point as much as we wish and for time intervals of arbitrary length. However, this result cannot be easily interpreted in terms of geometric optics, since we need C^1 coefficients in order to build solutions of the Hamiltonian system that yields the bicharacteristic rays. In any case, since for the density functions we build there exists a sequence of solutions that concentrates its energy around a point in space as much as we wish, the only possibility of getting inequalities of the form (6.3), (6.4) is to have this point belonging to the observed region. Thus, this is in contrast with the microlocal results that apply for density functions that are in C^2. In the latter case, the total energy of solutions along the ray is captured by measuring the energy at any point of the ray.

To simplify things, we construct here a density ρ with only one singular point at the boundary, although we could also construct densities with a finite number of singular points. This particular choice provides the following negative result:

Theorem 2. *Given any point $x_s \in \partial\Omega$, there exist Hölder continuous density functions $\rho \in C^{0,s}(\Omega)$ for all $0 < s < 1$, for which (6.3) and (6.4) fail for every $T > 0$ and for every subset $\omega \subset \Omega$ such that x_s does not belong to the closure of ω.*

Remark 8. As we mentioned in the introduction, in one space dimension, the lack of observability may also be shown for piecewise constant coefficients oscillating arbitrarily fast between two given values at some point (or several points). By separation of variables, as in the proof of Theorem 2 we present below, this can be extended to several space dimensions.

However, in dimensions $d \geqq 2$, the lack of observability for piecewise constant densities is not new. Indeed, following Snell's Law, we can show that, when the interface between two media with different speeds of propagation has a suitable geometry, there exist rays of geometric optics that are trapped in one of these media. This shows that observability fails when making measurements in the other medium. We refer to [11] for the technical details.

Nevertheless, the counterexamples we give here are more dramatic since solutions are not concentrated along rays but, in some sense, at a standing point in space.

Proof. We assume, without loss of generality, the following two conditions for the singular point x_s:

$$\begin{aligned} &x_s = 0 \in \partial\Omega, \\ &(0, a)^d \subset \Omega \quad \text{for a small enough } a > 0. \end{aligned} \tag{6.5}$$

For the first condition to hold it is sufficient to translate Ω so that the singular point is at the origin. The second one holds after a suitable rotation.

With the notation introduced in Section 3 we define

$$\rho(x) = \widehat{\rho}(x_1) + \widehat{\rho}(x_2) + \cdots + \widehat{\rho}(x_d), \tag{6.6}$$

where

$$\widehat{\rho}(s) = \begin{cases} \alpha_{\varepsilon_j}(h_j(s - m_j^-)) & \text{for } s \in I_j^-,\ j \geqq 2, \\ 4\pi^2 & \text{for } s \in K \backslash (\cup_{j \geqq 2} I_j^-), \end{cases} \tag{6.7}$$

where K is a large compact set with $\Omega \subset K^d$. Observe that ρ is defined in separated variables from a one-dimensional function $\widehat{\rho}$ as the one introduced in Section 3 for the one-dimensional case. The only difference is that $\widehat{\rho}$ oscillates around a unique point $s = 0$ instead of two points, which is the case for the density function in Section 3.

Note that $\rho(x) \in C^{0,s}(K^d)$ and it is bounded above and below by positive constants that we will refer to as ρ_0 and ρ_1 respectively.

We now construct the sequence of quasi-eigenfunctions $\varphi_j(x)$. Consider

$$\varphi_j(x) = \widehat{\varphi}_j(x_1)\widehat{\varphi}_j(x_2)\cdots\widehat{\varphi}_j(x_d) \tag{6.8}$$

where $\widehat{\varphi}_j$ is the solution of

$$\begin{aligned} &(\widehat{\varphi}_j)''(s) + h_j^2\widehat{\rho}(s)\widehat{\varphi}_j(s) = 0, \qquad s \in K, \\ &\widehat{\varphi}_j(m_j^-) = 1, \qquad (\widehat{\varphi})_j'(m_j^-) = 0. \end{aligned} \tag{6.9}$$

Therefore, φ_j satisfies

$$\begin{aligned} \Delta\varphi_j + h_j^2\rho(x)\varphi_j &= 0, \qquad x \in K^d, \\ \varphi_j(m_j^-, m_j^-, \cdots, m_j^-) &= 1, \\ \nabla\varphi_j(m_j^-, m_j^-, \cdots, m_j^-) &= 0. \end{aligned} \tag{6.10}$$

We refer to φ_j as quasi-eigenfunctions of (6.1). They are not true eigenfunctions because their restrictions to Ω do not satisfy the boundary condition

$$\varphi_j(x) = 0 \qquad \text{on } \partial\Omega. \tag{6.11}$$

However, we show that φ_j is mainly concentrated in $(I_j^-)^d$. This is a consequence of the fact that $\widehat{\varphi}_j$ is mainly concentrated in I_j^-. In fact, we can argue as in Section 3 for the one-dimensional case to obtain

$$\begin{aligned} \int_{I_{2j}^-} |\widehat{\varphi}_j(s)|^2 ds &\geqq C h_j^{-3}, \\ \left|\widehat{\varphi}_j\left(m_j^- - \frac{r_j}{2}\right)\right|^2 + \left|(\widehat{\varphi}_j)'\left(m_j^- - \frac{r_j}{2}\right)\right|^2 &= e^{-\varepsilon_j h_j r_j}, \\ \left|\widehat{\varphi}_j\left(m_j^- + \frac{r_j}{2}\right)\right|^2 + \left|(\widehat{\varphi}_j)'\left(m_j^- + \frac{r_j}{2}\right)\right|^2 &= e^{-\varepsilon_j h_j r_j}. \end{aligned}$$

Following the ideas in Section 3 we now estimate $\widehat{\varphi}_j$ in any compact interval to the left and to right of I_j^- from the estimates in the extremes of I_j^-. In particular we have

$$\left|\widehat{\varphi}_j(s)\right|^2 + \left|(\widehat{\varphi}_j)'(s)\right|^2 \leqq C_p h_j^{-p} \qquad \forall p > 0, \quad s \in K \backslash I_j^-. \tag{6.12}$$

Using these estimates for $\widehat{\varphi}_j$ we prove that $\varphi_j(x) = \widehat{\varphi}_j(x_1)\widehat{\varphi}_j(x_2)\cdots\widehat{\varphi}_j(x_d)$ is concentrated in $(I_j^-)^d$, i.e.,

$$\begin{aligned} &\int_{(I_{2j}^-)^d} |\varphi_j(x)|^2 dx \geqq C h_j^{-3d}, \\ &\left|\varphi_j(x)\right|^2 + \left|\nabla\varphi_j(x)\right|^2 \leqq C_p h_j^{-p} \qquad \forall p > 0, \quad x \in K^d \backslash (I_j^-)^d. \end{aligned} \tag{6.13}$$

Once we have constructed the density and a sequence of quasi-eigenfunctions we introduce the following sequence of solutions of the first equation in (6.1):

$$v_j(x,t) = e^{ih_j t}\varphi_j(x). \tag{6.14}$$

Note that v_j does not satisfy the boundary conditions in (6.1). However, we correct v_j with a new function $\tilde{v}_j$ in such a way that

$$u_j = v_j + \tilde{v}_j \tag{6.15}$$

satisfies all the equations in (6.1). Therefore, we define $\tilde{v}_j$ as the unique solution of

$$\begin{aligned} \rho(x)\tilde{v}_{tt} - \Delta\tilde{v} &= 0, && x \in \Omega, \quad 0 < t < T, \\ \tilde{v}(x,t) = -v_j(x,t) &= -e^{ih_j t}\varphi_j(x), && x \in \partial\Omega, \ 0 < t < T, \\ \tilde{v}(x,0) = \tilde{v}_t(x,0) &= 0 && x \in \Omega. \end{aligned} \tag{6.16}$$

We claim that for the sequence u_j observability inequality (6.3) fails. Moreover, (6.4) fails as well for those observability zones ω for which x_s does not belong to $\bar{\omega}$. The proof is a straightforward generalization of the one-dimensional case discussed in Sections 4 and 5. However, for the sake of completeness we give the proof for the boundary observability case.

We claim that

$$\lim_{j\to\infty} \frac{\int_\Omega \left[|\nabla u_j(x,0)|^2 + \rho(x)|u_{j,t}(x,0)|^2\right] dx}{\int_0^T \int_{\partial\Omega} |\frac{\partial u_j}{\partial n}(x,t)|^2 d\sigma dt} = \infty. \tag{6.17}$$

Indeed the numerator in (6.17) can be bounded below as in (4.5)

$$\int_\Omega \left[|\nabla u_j(x,0)|^2 + \rho(x)|u_{j,t}(x,0)|^2\right] dx \geqq h_j^2 \rho_m \int_{(I_j^-)^d} |\varphi_j|^2 dx. \tag{6.18}$$

Concerning the denominator, we proceed as in (4.6) to obtain

$$\int_0^T \int_{\partial\Omega} \left|\frac{\partial u_j}{\partial n}(x,t)\right|^2 d\sigma dt \leqq 2T \int_{\partial\Omega} \left|\frac{\partial \varphi_j}{\partial n}\right|^2 d\sigma + 2\int_0^T \int_{\partial\Omega} \left|\frac{\partial \tilde{v}_j}{\partial n}(x,t)\right|^2 d\sigma dt. \tag{6.19}$$

To estimate the last term we introduce the generalization of Proposition 1 to the d-dimensional case. □

Proposition 2. *Consider the system*

$$\begin{aligned} \rho(x)v_{tt} - \Delta v &= 0, && x \in \Omega, \quad 0 < t < T, \\ v(x,t) &= f(x)e^{ith}, && x \in \partial\Omega, \ 0 < t < T, \\ v(x,0) = v_t(x,0) &= 0, && x \in \Omega, \end{aligned} \tag{6.20}$$

where $\rho \in L^\infty(\Omega)$, $0 < \rho_m \leqq \rho(x) \leqq \rho_M < \infty$ *a.e. and* f *is the restriction to* $\partial\Omega$ *of a function* $F(x) \in H^2(\Omega)$. *Given* $T > 0$, *there exists* $C(T) > 0$ *such that*

$$\int_0^T \int_\Omega \left[|\nabla v|^2 + \rho(x)|v_t|^2 \right] dxdt \leqq C(T) \left(\|h^2 \rho F + \Delta F\|^2_{L^2(\Omega)} + \|F\|^2_{H^1(\Omega)} + h^2 \|F\|^2_{L^2(\Omega)} \right),$$

$$\int_0^T \int_{\partial\Omega} |\nabla v|^2 \, dxdt \leqq C(T) h \left(\|h^2 \rho F + \Delta F\|^2_{L^2(\Omega)} + \|F\|^2_{H^1(\Omega)} + h^2 \|F\|^2_{L^2(\Omega)} \right). \quad (6.21)$$

We prove this result at the end of the section.

We apply Proposition 2 to the solution $\tilde{v}$ of (6.16). Note that $\varphi_j\big|_{\partial\Omega}$ satisfies the hypothesis of the proposition because it is the restriction to $\partial\Omega$ of a function $\varphi_j \in H^2(\Omega)$. However, the estimates in (6.21) depend on the extension we choose for the boundary values to the interior of Ω. It would be natural to choose φ_j itself as the extension but this is not convenient since this function exhibits a concentration of energy inside Ω.

To avoid this problem we remove from φ_j its energy concentrated in $(I_j^-)^d$ with a suitable cutoff function.

Let us introduce the cutoff functions $\psi_j \in H^2(\Omega)$ with the following conditions:

$$\psi_j(x) = \begin{cases} 1 \text{ if } x \in \Omega \backslash (I_{j-1}^- \cup I_j^- \cup I_{j+1}^-)^d, \\ 0 \text{ if } x \in (I_j^-)^d, \end{cases}$$

$$|\psi_j(x)| \leqq 1, \quad |\nabla \psi_j(x)| \leqq C h_j, \quad |\Delta \psi_j(x)| \leqq C h_j^2 \text{ for all } x \in \Omega,$$

for some constant $C > 0$. The sequence ψ_j with the above properties can be, for example, $\psi_j(x) = \psi(x r_j + m_j^-)$ where $x r_j + m_j^- = (x_1 r_j + m_j^-, \ldots, x_d r_j + m_j^-)$ and $\psi \in H^2(\mathbb{R}^d)$ is a fixed function satisfying

$$\psi(x) = \begin{cases} 1 \text{ if } x \in \mathbb{R}^d \backslash (-3/4, 5/2)^d, \\ 0 \text{ if } x \in (-1/2, 1/2)^d, \end{cases} \qquad |\psi(x)| \leqq 1, \text{ for all } x \in \mathbb{R}^d.$$

We apply Proposition 2 to the solution $\tilde{v}$ of (6.16) with $\psi_j \varphi_j$ as the extension of $\varphi_j\big|_{\partial\Omega}$ to Ω. Note that

$$\begin{aligned} \|h_j^2 \rho \psi_j \varphi_j + \Delta \left(\psi_j \varphi_j\right)\|^2_{L^2(\Omega)} &= \|2\nabla\psi_j \cdot \nabla\varphi_j + \varphi_j \Delta\psi_j\|^2_{L^2(\Omega\backslash(I_j^-)^d)} \\ &\leqq C \left(\|\nabla\psi_j\|^2_{L^\infty(\Omega)} + \|\Delta\psi_j\|^2_{L^\infty(\Omega)} \right) \|\varphi_j\|^2_{H^1(\Omega\backslash(I_j^-)^d)} \\ &\leqq C h_j^4 \|\varphi_j\|^2_{H^1(\Omega\backslash(I_j^-)^d)}, \\ \|\psi_j\varphi_j\|^2_{H^1(\Omega)} h_j^2 &\leqq \|\psi_j\|^2_{H^1(\Omega)} \|\varphi_j\|^2_{H^1(\Omega\backslash(I_j^-)^d)} h_j^2 \leqq C \|\varphi_j\|^2_{H^1(\Omega\backslash(I_j^-)^d)} h_j^4, \\ \|\psi_j\varphi_j\|^2_{L^2(\Omega)} &\leqq \|\psi_j\|^2_{L^2(\Omega)} \|\varphi_j\|^2_{L^2(\Omega\backslash(I_j^-)^d)} \leqq C \|\varphi_j\|^2_{H^1(\Omega\backslash(I_j^-)^d)}, \end{aligned}$$

for some constant $C > 0$. Therefore we deduce that

$$\int_0^T \int_{\partial\Omega} \left| \frac{\partial \tilde{v}_j}{\partial n}(x,t) \right|^2 d\sigma dt \leqq C(T) h_j^4 \|\varphi_j\|^2_{H^1(\Omega \setminus (I_j^-)^d)}. \tag{6.22}$$

Finally, combining (6.18) and (6.22) we have

$$\frac{\int_\Omega \left[|\nabla u_j(x,0)|^2 + \rho(x) |u_{j,t}(x,0)|^2 \right] dx}{\int_0^T \int_{\partial\Omega} \left| \frac{\partial u_j}{\partial n}(x,t) \right|^2 d\sigma dt} \geqq C(\rho_0, T) \frac{\int_{(I_j^-)^d} |\varphi_j|^2 dx}{h_j^2 \int_{\partial\Omega} \left| \frac{\partial \varphi_j}{\partial n} \right|^2 d\sigma + h_j^4 \|\varphi_j\|^2_{H^1(\Omega \setminus (I_j^-)^d)}}. \tag{6.23}$$

Now, taking into account the estimates (6.13) for φ_j we find that the above quantity converges to infinity as $j \to \infty$.

Proof of Proposition 2. We generalize the proof of Proposition 1. Let us introduce

$$g(x,t) = F(x) e^{iht}. \tag{6.24}$$

Then,

$$w(x,t) = v(x,t) - g(x,t) \tag{6.25}$$

satisfies

$$\begin{cases} \rho(x) w_{tt} - \Delta w = \rho(x) h^2 e^{iht} F(x) + e^{iht} \Delta F(x), & x \in \Omega, \quad 0 < t < T, \\ w(x,t) = 0, & x \in \partial\Omega, \ 0 < t < T, \\ w(x,0) = -F(x), & x \in \Omega, \\ w_t(x,0) = -ihF(x), & x \in \Omega. \end{cases} \tag{6.26}$$

Classical estimates on non-homogeneous wave equations show that

$$\int_0^T \int_\Omega \left[|\nabla w|^2 + \rho(x) |w_t|^2 \right] dx dt \leqq C(T) \left(\|h^2 \rho F + \Delta F\|^2_{L^2(\Omega)} + \|F\|^2_{H^1(\Omega)} + h^2 \|F\|^2_{L^2(\Omega)} \right). \tag{6.27}$$

On the other hand,

$$\int_0^T \int_\Omega \left[|\nabla g|^2 + \rho(x) |g_t|^2 \right] dx dt \leqq 2T ||\rho||_\infty \left(\|F\|^2_{H_0^1(\Omega)} + h^2 \|F\|^2_{L^2(\Omega)} \right). \tag{6.28}$$

This last estimate and (6.27) allow us to obtain easily the first inequality in (6.21).

For the boundary inequality in (6.21) we first obtain a boundary estimate for system (6.26). The classical procedure for getting such estimates is to multiply the first equation in (6.26) by the multiplier $\nu \cdot \nabla w$, where $\nu \in C^1(\bar{\Omega})^d$ is a vector field which coincides with the outward normal of Ω at the boundary, i.e., $\nu = n$ on $\partial\Omega$.

The existence of such a vector field is proved in [10] (Lemma 3.1, Chapter 1). Then we have

$$0=\int_0^T\int_\Omega \rho(x)w_{tt}\nu\cdot\nabla w\,dxdt-\int_0^T\int_\Omega \Delta w\nu\cdot\nabla w\,dxdt + \int_0^T\int_\Omega \rho(x)h^2F\nu\cdot\nabla w\,dxdt. \tag{6.29}$$

Now we integrate by parts in the second term of the right-hand side,

$$\int_0^T\int_0^1 \Delta w\nu\cdot\nabla w\,dxdt=-\frac{1}{2}\int_0^T\int_\Omega \operatorname{div}\nu\,|\nabla w|^2\,dxdt + \frac{1}{2}\int_0^T\int_{\partial\Omega}|\nabla w|^2\,d\sigma dt. \tag{6.30}$$

Combining (6.29) and (6) we easily find the following estimate:

$$\int_0^T\int_{\partial\Omega}|\nabla w|^2\,d\sigma dt \leqq CT\|\rho\|_\infty\left(\|w_{tt}\|^2_{L^2(0,1)}+\|w\|^2_{H_0^1(0,1)}+h^2||F||^2_{L^2(\Omega)}\right). \tag{6.31}$$

Here the constant $C>0$ only depends on $\|\nu\|_{W^{1,\infty}}$, i.e., the geometry of the domain.

To remove the L^2 norm of w_{tt} from the right-hand side we observe that $\tilde{w}=w_t$ satisfies the system

$$\begin{aligned}
\rho(x)\tilde{w}_{tt}-\Delta\tilde{w} &= \rho(x)ih^3e^{iht}F(x)+ihe^{iht}\Delta F(x), && x\in\Omega,\quad 0<t<T,\\
\tilde{w}(x,t) &= 0, && x\in\partial\Omega,\ 0<t<T,\\
\tilde{w}(x,0) &= w_t(x,0)=-ihF(x), && x\in\Omega,\\
\tilde{w}_t(x,0) &= w_{tt}(x,0),\\
&= \frac{1}{\rho(x)}\left[\Delta w(x,0)+\rho(x)h^2F(x)+\Delta F\right]\\
&= h^2F(x), && x\in\Omega.
\end{aligned} \tag{6.32}$$

Once again the energy estimate for the non-homogeneous problem provides

$$\int_0^T\int_\Omega\left[|\tilde{w}_x|^2+\rho(x)|\tilde{w}_t|^2\right]dxdt \leqq C(T)h\left(\|h^2\rho F+\Delta F\|^2_{L^2(\Omega)}+\|F\|^2_{H^1(\Omega)}+h^2\|F\|^2_{L^2(\Omega)}\right). \tag{6.33}$$

This inequality allows us to estimate the term $w_{tt}=\tilde{w}_t$ in (6.31). Then we have

$$\int_0^T\int_{\partial\Omega}|\nabla w|^2\,d\sigma dt \leqq C(T)h\left(\|h^2\rho F+\Delta F\|^2_{L^2(\Omega)}+\|F\|^2_{H^1(\Omega)}+h^2\|F\|^2_{L^2(\Omega)}\right) \tag{6.34}$$

for some constant $C(T)>0$.

On the other hand,

$$\int_0^T \int_{\partial\Omega} |\nabla g|^2 \, d\sigma dt \leqq T||F||^2_{H^2(\Omega)}.$$

Combining this last estimate with (6) we easily obtain the inequality (6.21) for $v = g + w$. □

7. On the lack of controllability of the wave equation

Consider the controlled system:

$$\begin{aligned} &\rho(x)u_{tt}(x,t) - u_{xx}(x,t) = f(x,t)\chi_{(\alpha,\beta)}, \ 0 < x < 1, \quad 0 < t < T, \\ &u(0,t) = g_1(t), \quad u(1,t) = g_2(t), \qquad\qquad 0 < t < T, \\ &u(x,0) = u_0(x), \ u_t(x,0) = u_1(x). \end{aligned} \tag{7.1}$$

Here $\chi_{(\alpha,\beta)}$ represents the characteristic function of the interval $(\alpha, \beta) \subset (0, 1)$, f is an internal control acting on (α, β) and g_1, g_2 are boundary controls acting on the extremes $x = 0, 1$ respectively. When $\rho \in BV(0, 1)$ and $0 < \rho_0 \leqq \rho(x) \leqq \rho_1 < \infty$ a.e. the following controllability result holds: *Given* $T > T_m = \sqrt{\rho_1} \max\{1 - \beta, \alpha\}$ *and the initial data* $(u_0, u_1) \in L^2(0, 1) \times H^{-1}(0, 1)$ *there exists* $f \in L^2(0, T; H^{-1}(\alpha, \beta))$ *and* $g_1, g_2 \in L^2(0, T)$ *such that the solution of* (7.1) *satisfies*

$$u(x,t) = u_t(x,t) = 0 \qquad \forall t \geqq T. \tag{7.2}$$

In fact, only one among the controls f, g_1, g_2 is sufficient to guarantee the controllability if T is sufficiently large and ρ is BV. The proof of this result is a consequence of a corresponding observability inequality for the uncontrolled equation and the well-known HUM (Hilbert Uniqueness Method) method (see [10]).

The non-observability result stated in Theorem 1 provides Hölder continuous density functions $\rho \in C^{0,s}([0, 1])$ for all $0 < s < 1$, for which the above controllability property does not hold.

To simplify things we restrict ourselves to the boundary controllability case which is the most delicate one. We prove the following

Theorem 3. *There exist Hölder continuous functions* $\rho \in C^{0,s}([0, 1])$ *for all* $0 < s < 1$ *with* $0 < \rho_0 \leqq \rho(x) \leqq \rho_1 < \infty$ *such that, for any* $T > 0$, *there exists initial data* (u_0, u_1), *in the class*

$$(u_0, \rho u_1) \in L^2(0, 1) \times H^{-1}(0, 1), \tag{7.3}$$

such that, for any $g_2 \in L^2(0, T)$, *the solution* u *of* (7.1) (*with* $f = 0$ *and* $g_1 = 0$) *does not satisfy* (7.2).

Remark 9. The proof of Theorem 3 that we present here can be adapted to the higher dimensional case, taking into account the lack of internal observability stated in Theorem 2.

Proof. We argue by contradiction. Assume that for $T > 0$ and any initial data (u_0, u_1) in the class (7.3) there exists a control $g_2 \in L^2(0,T)$ such that u reaches the equilibrium at time $t = T$. We are going to show that, when this controllability property holds, the corresponding observability inequality for the adjoint system will hold as well, which is in contradiction with the negative result of Theorem 1.

We proceed in two steps:

Step 1. We prove that the following operator is linear and continuous:

$$\begin{aligned} S : L^2(0,1) \times H^{-1}(0,1) &\to L^2(0,T) \\ (u_0, \rho(x) u_1) &\to g_2, \end{aligned}$$

where g_2 is the control with minimal L^2 norm which makes u reach the equilibrium in time $t = T$.

Note that S is well defined because we are assuming that the controllability property holds and the control of minimal L^2 norm is unique due to the convexity of the norm.

Next we prove that S is linear. Let us introduce $N \subset L^2(0,T)$, the subset of controls of the trivial initial data $(0,0)$. The subset N can be characterized as follows: $g \in N$ if and only if

$$\int_0^T g(t) v_x(1,t) = 0 \quad \forall (v_0, v_1) \in H^2 \cap H_0^1(0,1) \times H_0^1(0,1), \tag{7.4}$$

where v is the solution of the adjoint system with initial data (v_0, v_1):

$$\begin{aligned} \rho(x) v_{tt} - v_{xx} = 0, \qquad & 0 < x < 1,\ 0 < t < T, \\ v(0,t) = v(1,t) = 0, \qquad & 0 < t < T, \\ v(x,0) = v_0(x),\ v_t(x,0) = v_1(x),\ & 0 < x < 1. \end{aligned} \tag{7.5}$$

Indeed, (7.4) is equivalent to $u(T) \equiv u_t(T) \equiv 0$. To see this we multiply by v in (7.1) and integrate by parts to obtain

$$\begin{aligned} &\langle \rho u_t(T), v(T) \rangle_{H_0^1} - \int_0^1 \rho(x) u(x,T) v_t(x,T) dx \\ &- \langle \rho u_1, v_0 \rangle_{H_0^1} + \int_0^1 \rho(x) u_0(x) v_1(x) dx + \int_0^T g_2(t) v_x(1,t) = 0. \end{aligned} \tag{7.6}$$

Taking into account that the initial data are $(0,0)$, we see that (7.4) is equivalent to $u(T) \equiv u_t(T) \equiv 0$.

When ρ is regular we can consider less regular initial data $(v_0, v_1) \in H_0^1(0,1) \times L^2(0,1)$ in (7.4), because solutions with these initial data satisfy the extra regularity property $v_x(1,t) \in L^2(0,T)$.

It is easy to see that N is a non-empty closed linear subset of $L^2(0,T)$ and therefore we can decompose $L^2(0,T)$ in a direct sum as follows:

$$L^2(0,T) = N + N^{\perp}.$$

Then, any L^2 control $\hat{g}_2$ of the initial data (u_0, u_1) can be uniquely decomposed as $\hat{g}_2 = g_2 + n$, where $g_2 \in N^\perp$ is the minimal L^2 norm control and $n \in N$. We deduce that $S(u_0, \rho u_1)$ is the unique control g_2 of (u_0, u_1) satisfying $g_2 \in N^\perp$.

Now we assume that $S(u_0, \rho u_1) = g_2$ and $S(v_0, \rho v_1) = h_2$. Then $g_2+h_2 \in N^\perp$ and it is a control of (u_0+v_0, u_1+v_1). Therefore $S(u_0+v_0, \rho u_1+\rho v_1) = g_2+h_2$ and S is linear.

Finally, to prove the continuity of S we use the closed-graph theorem. Let us consider a sequence of initial data $(u_{0,j}, u_{1,j})$ and a sequence of associated minimal L^2 controls $g_{2,j}$ such that

$$(u_{0,j}, \rho(x)u_{1,j}) \to (u_0, \rho(x)u_1) \quad \text{in } L^2(0,1) \times H^{-1}(0,1), \tag{7.7}$$

$$g_{2,j} \to g_2 \quad \text{in } L^2(0,T). \tag{7.8}$$

Observe that $g_2 \in N^\perp$ because $\{g_{2,j}\}_{j\in N} \subset N^\perp$ and $N^\perp$ is closed. On the other hand, as $g_{2,j}$ is a control of $(u_{0,j}, u_{1,j})$, we have

$$-\langle \rho u_{1,j}, v(x,0)\rangle_{H_0^1} + \int_0^1 \rho(x)u_{0,j}(x)v_t(x,0)dx + \int_0^T g_j(t)v_x(1,t) = 0 \tag{7.9}$$

for all $(v_0, v_1) \in H^2 \cap H_0^1(0,1) \times H_0^1(0,1)$. Indeed, in (7.9) we are simply writing that $u(T) \equiv u_t(T) \equiv 0$ in a weak form.

Passing to the limit in (7.9) we find that g_2 is a control for (u_0, u_1) and therefore $S(u_0, \rho u_1) = g_2$.

We have proved that S is a linear operator with a closed graph. By the closed graph theorem it is continuous, i.e., there exists a constant $C > 0$ such that

$$\|g_2\|_{L^2(0,T)} \leqq C\, \|(u_0, \rho(x)u_1)\|_{L^2(0,1)\times H^{-1}(0,1)}. \tag{7.10}$$

Step 2. We prove that (7.10) is equivalent to the corresponding observability inequality for the adjoint system, i.e.,

$$E(0) \leqq C \int_0^T |v_x(1,t)|^2 dt,$$

where v is a solution of (7.5) with initial data (v_0, v_1).

From (7.6) and taking into account that $u(T) \equiv u_t(T) \equiv 0$, we have

$$\left| -\langle \rho u_1, v_0\rangle_{H_0^1} + \int_0^1 \rho(x)u_0(x)v_1(x)dx \right| \leqq C\, \|(u_0, \rho(x)u_1)\|_{L^2(0,1)\times H^{-1}(0,1)}\, \|v_x(1,t)\|_{L^2(0,T)}$$

for all $(u_0, \rho(x)u_1) \in L^2(0,T) \times H^{-1}(0,T)$, and therefore

$$\|(v_0, v_1)\|_{H_0^1(0,1)\times L^2(0,1)} \leqq C \|v_x(1,t)\|_{L^2(0,T)}. \qquad \square$$

8. On the lack of dispersive properties and Strichartz inequalities for the wave equation

In this section we consider the wave equation in the whole space

$$\begin{aligned} \rho(x)u_{tt} - \Delta u = 0, \qquad x \in \mathbb{R}^d, \quad t > 0, \\ u(x,0) = u_0, \quad u_t(x,0) = u_1(x), \; x \in \mathbb{R}^d. \end{aligned} \tag{8.1}$$

For ρ constant and $d \geqq 2$, the Strichartz estimates establish space-time integrability properties of the solutions of this system due to dispersive effects. One version of these estimates is

$$\|u\|_{L^2_t(\mathbb{R};L^q_x(\mathbb{R}^d))} \leqq c\,\|u_0\|_{H^r(\mathbb{R}^d)} + \|u_1\|_{H^{r-1}(\mathbb{R}^d)} \tag{8.2}$$

provided that

$$q = \frac{2d}{d-2r-1}, \quad \frac{2(d+1)}{d-1} \leqq q < \infty. \tag{8.3}$$

When ρ is smooth the above estimates (8.2) hold locally in time and they are sharp (see [12]). For low regularity coefficients, $\rho \in C^{1,s}$, there exist weakened versions of estimates (8.2) (see [13] and [14]).

Note that the above estimates cannot be obtained by classical Sobolev embeddings. In fact, due to the conservation of energy we easily deduce that the solutions of (8.1) belong to the class $u \in C(0,\infty; H^r(\mathbb{R}^d))$. Therefore, the Sobolev embeddings in space allow us to obtain:

$$u \in C(0,\infty; L^{\frac{2d}{d-2r}}(\mathbb{R}^d)). \tag{8.4}$$

For ρ smooth, estimates (8.2) establish, in particular, that the solution $u(\cdot,t) \in L^q(\mathbb{R}^d)$ with $\frac{2d}{d-2r} < q \leqq \frac{2d}{d-2r-1}$ almost everywhere in $t \in \mathbb{R}$.

According to the constructions of the previous sections, for the $C^{0,s}$ density function we have built, and due to the existence of a sequence of solutions that concentrates its energy around a point as much as we wish, no (8.2) nor any weakened version of it may hold except of course for the integrability properties that Sobolev's embeddings provide. Therefore, we may say that in the class of $C^{0,s}$ density functions there are no Strichartz-type estimates even locally in space-time and for weaker integrability requirements.

More precisely, we have the following result:

Theorem 4. *Given any point $x_s \in \mathbb{R}^d$, there exist Hölder continuous density functions $\rho \in C^{0,s}(\mathbb{R}^d)$ for all $0 < s < 1$, and a sequence of solutions u_j of (8.1) for which*

$$\lim_{j\to\infty} \frac{\left(\int_{I^d} |u_j(\cdot,t)|^p dx\right)^{1/p}}{\left\|u_j(\cdot,0)\right\|_{H^r(\mathbb{R}^d)} + \left\|\partial_t u_j(\cdot,0)\right\|_{H^{r-1}(\mathbb{R}^d)}} = \infty \tag{8.5}$$

for any $p > \frac{2d}{d-2r}$, $t \in \mathbb{R}$ and for all d-dimensional cubes $I^d = [\,x_s, x_s+\delta]^d$ with $\delta > 0$.

Proof. We assume without loss of generality that $x_s = 0$. Consider the density ρ introduced in (6.6) and defined by (6.6) and (6.7), that we extend to $\mathbb{R}^d \setminus K^d$ by the constant $4\pi^2$.

As we have seen, there exists a sequence (h_j^2, φ_j) of eigenpairs of (8.1) concentrated around $x = 0$ associated with ρ, i.e., solutions of

$$\Delta\varphi_j + h_j^2 \rho(x)\varphi_j = 0, \; x \in K^d, \tag{8.6}$$

satisfying (6.13). Note that

$$u_j(x,t) = e^{ih_j t}\varphi_j(x) \tag{8.7}$$

constitute a sequence of solutions of (8.1) over K^d that we can extend to $\mathbb{R}^d$ in such a way that $(u_j, \partial_t u_j)$ is uniformly bounded in $H^r \times H^{r-1}$ as $j \to \infty$. We prove that, due to the concentration of energy of φ_j near $x = 0$, the sequence u_j satisfies (8.5).

Observe that $|u_j(\cdot,t)| = |\varphi_j(\cdot)|$ for all t and $x \in K^d$. Therefore it is sufficient to prove that

$$\lim_{j\to\infty} \frac{\left(\int_{I^d} |\varphi_j(x)|^p dx\right)^{1/p}}{\|\varphi_j\|_{H^r(K^d)} + h_j \|\varphi_j\|_{H^{r-1}(K^d)}} = \infty \quad \forall p > \frac{2d}{d-2r}. \tag{8.8}$$

Recall that $\varphi_j(x)$ is defined in separated variables over each I_j^- as follows:

$$\varphi_j(x) = \widehat{\varphi}_j(x_1)\cdots\widehat{\varphi}_j(x_d) = w_{\varepsilon_j}(h_j(x_1 - m_j^-))\cdots w_{\varepsilon_j}(h_j(x_d - m_j^-)).$$

Therefore, the change of variables $y_\alpha = h_j(x_\alpha - m_j^-)$ in (8.8) provides

$$\begin{aligned} &\frac{\left(\int_{I^d} |\varphi_j(x)|^p dx\right)^{1/p}}{\|\varphi_j\|_{H^r(K^d)} + h_j \|\varphi_j\|_{H^{r-1}(K^d)}} \\ &\geqq h_j^{-\frac{d}{q}+\frac{d}{2}-r} \frac{\left(\int_{\overline{I_j}} |w_{\varepsilon_j}(y)|^p dy\right)^{d/p}}{\|w_{\varepsilon_j}\|^d_{H^r(\overline{I_j})} + h_j \|w_{\varepsilon_j}\|^d_{H^{r-1}(\overline{I_j})} + \mathcal{O}(h_j^{-p})}, \end{aligned} \tag{8.9}$$

where the interval $\overline{I}_j = h_j(I_j^- - m_j^-)$, and for all $p \geqq 0$.

For j sufficiently large, $[0,1] \subset \overline{I}_j$ and the numerator in (8.9) can be bounded below by a constant $C(d,p)$ which does not depend on ε_j, i.e.,

$$\left(\int_{\overline{I_j}} |w_{\varepsilon_j}(y)|^p dy\right)^{d/p} \geqq \left(\int_0^1 |w_{\varepsilon_j}(y)|^p dy\right)^{d/p} \geqq C(d,p) > 0$$

in view of (2.8). On the other hand, the denominator in (8.9) is bounded above, by a constant which does not depend on ϵ_j, due to the properties of w_ε in Lemma 1.

It follows that the left-hand side of (8.9) cannot be uniformly bounded in $j \to \infty$ for $p > \frac{2d}{d-2r}$. $\square$

9. Comments

In this section we mention a number of applications and remarks related to the result stated in Theorem 1.

1. Density functions. The density function that we have constructed is singular at both extremes in the sense that it oscillates more and more as we approach $x = 0$ and $x = 1$. Similar constructions can be done to obtain densities with singularities at one interior point or a finite number of interior and boundary points. We have chosen to present the construction above since it is the simplest one that provides at the same time a non-observability result for the boundary case and for the interior one when we restrict ourselves to connected intervals (α, β) for the observation.

Note that a more general result, i.e., a density function for which any observability inequality fails without any restriction on the observability zone, would imply a construction of the density with infinitely many singular points localized in a dense set of $[0, 1]$. Whether such density functions exist is still an open problem.

2. Variable coefficients in the principal part of the operator. Theorem 1 can be easily adapted to systems of the form

$$\begin{aligned} u_{tt} - (a(x)u_x)_x = 0, \qquad & 0 < x < 1,\ 0 < t < T, \\ u(0, t) = u(1, t) = 0, \qquad & 0 < t < T, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x),\ & 0 < x < 1. \end{aligned} \tag{9.10}$$

In this case the energy of the system is given by

$$E(t) = \frac{1}{2} \int_0^1 \Big[|u_t(x, t)|^2 + a(x)|u_x(x, t)|^2 \Big] dx, \tag{9.11}$$

and the boundary and internal observability properties read:

$$E(0) \leqq C \int_0^T \Big[|u_x(0, t)|^2 + |u_x(1, t)|^2 \Big] dx, \tag{9.12}$$

$$E(0) \leqq C \int_0^T \int_\alpha^\beta \Big[|u_t(x, t)|^2 + a(x)|u_x(x, t)|^2 \Big] dx, \tag{9.13}$$

respectively.

Once again, when $a \in BV(0, 1)$ and $0 < a_0 \leqq a(x) \leqq a_1 < \infty$ a.e. $x \in (0, 1)$, both observability inequalities hold when T is large enough.

Theorem 5. *There exist continuous functions* $a \in C^{0,s}([0, 1])$ *for all* $0 < s < 1$ *with* $0 < a_0 \leqq a(x) \leqq a_1 < \infty$ *for which* (9.12) *and* (9.13) *fail for all* $T > 0$ *and for every subinterval* $(\alpha, \beta) \subset (0, 1)$ $((\alpha, \beta) \neq (0, 1))$.

Proof. We center our attention in the interior observability inequality (9.13) since the other one is similar. Consider the following change of variables

$$\begin{aligned} & y(x) = \int_0^x \frac{dr}{a(r)} \Big/ \int_0^1 \frac{dr}{a(r)}, \qquad \rho(y) = \left(\int_0^1 \frac{dr}{a(r)} \right)^2 a(x(y)) \\ & v(y, t) = u(x(y), t), \\ & v(y, 0) = u_0(x(y)), \qquad v_t(y, 0) = u_1(x(y)) \end{aligned} \tag{9.14}$$

where $x(y)$ represents the inverse function of $y(x)$. Note that $y(x) \in C^1(0,1)$ and $y(x)$ is invertible because $y'(x) \neq 0$ for all $x \in [0,1]$. Hence, $x(y) \in C^1(0,1)$ and $\rho(y)$ has the same regularity as $a(x)$.

With the above change of variables (9.10) is transformed into

$$\begin{aligned} \rho(y)v_{tt} - v_{yy} &= 0, && 0 < y < 1,\ 0 < t < T,\\ v(0,t) = v(1,t) &= 0, && 0 < t < T,\\ v(y,0) = v_0(y), \quad v_t(y,0) &= v_1(y), && 0 < y < 1. \end{aligned} \tag{9.15}$$

For this system we can apply Theorem 1 and find ρ for which the internal observability fails. More precisely, there exists a sequence v_j such that

$$\lim_{j\to\infty} \frac{\int_0^1 \left[|v_{j,y}(y,0)|^2 + \rho(y)|v_{j,t}(y,0)|^2\right] dy}{\int_0^T \int_{y(\alpha)}^{y(\beta)} \left[|v_{j,y}(y,t)|^2 + |v_{j,t}(y,t)|^2\right] dydt} = \infty \tag{9.16}$$

for any $0 < \alpha < \beta < 1$. Coming back to the original variables we easily obtain the following

$$\lim_{j\to\infty} \frac{\int_0^1 \left[a^2(x)|u_{j,x}(x,0)|^2 + a(x)|u_{j,t}(x,0)|^2\right] dx}{\int_0^T \int_\alpha^\beta \left[a^2(x)|u_{j,x}(x,t)|^2 + \widehat{a}^2|u_{j,t}(x,t)|^2\right] dxdt} = \infty, \tag{9.17}$$

where $\widehat{a} = \left(\int_0^1 \frac{1}{a(x)} dx\right)^{-1}$. Note that $a(x)$ is bounded above and below with positive constants because $\rho(y)$ satisfies this property. This fact, combined with (9.17), provides the result in Theorem 5. $\square$

3. Schrödinger equation. The singular densities introduced for the wave equation produce lack of observability for the linear Schrödinger equation

$$\begin{aligned} i\rho(x)u_t + \Delta u &= 0, && x \in \Omega, \quad 0 < t < T,\\ u(x,t) &= 0, && x \in \partial\Omega,\ 0 < t < T,\\ u(x,0) &= u_0(x), && x \in \Omega, \end{aligned} \tag{9.18}$$

where Ω is a bounded domain of R^d with boundary $\partial\Omega$ of class C^3 and $T > 0$.

The energy of the system is given by

$$E(t) = \frac{1}{2} \int_\Omega |\nabla u(x,t)|^2 dx, \tag{9.19}$$

and the boundary and internal observability properties read:

$$E(0) \leqq C \int_0^T \int_{\partial\Omega} |\frac{\partial u}{\partial n}(x,t)|^2 dx, \tag{9.20}$$

$$E(0) \leqq C \int_0^T \int_\omega |\nabla u(x,t)|^2 dx, \tag{9.21}$$

respectively. Here $\frac{\partial}{\partial n}$ represents the normal derivative.

When $\rho \in C^2(\Omega)$ and ω satisfies the geometrical control condition, the above observability inequalities hold for any $T > 0$. We refer to [9] for the proof of this result.

Theorem 2 can be also adapted for the Schrödinger equation:

Theorem 6. *Given any point $x_s \in \partial\Omega$, there exist Hölder continuous density functions $\rho \in C^{0,s}(\Omega)$ for all $0 < s < 1$, for which* (9.20) *and* (9.21) *fail for all $T > 0$ and for every subset $\omega \subset \Omega$ such that x_s does not belong to the closure of ω.*

The proof of this theorem is a straightforward generalization of the proof of Theorem 2.

In this result the density ρ has only one singular point x_s in the sense that it oscillates very much near this point. We observe that, as in the wave equation, more general results can be proved with densities oscillating around several different points.

Note that the Strichartz-type estimates fail also in (9.18).

4. Heat equation. In this section we observe that the singular densities introduced for the wave equation do not produce lack of observability for the linear heat equation

$$\begin{aligned} \rho(x)u_t - \Delta u = 0, \qquad & x \in \Omega, \quad 0 < t < T, \\ u(x,t) = 0, \qquad & x \in \partial\Omega, \ 0 < t < T, \\ u(x,0) = u_0(x), \ & x \in \Omega, \end{aligned} \tag{9.22}$$

where Ω is a bounded domain of $\mathbb{R}^d$ with boundary $\partial\Omega$ of class C^3 and $T > 0$.

The energy of the system is given by

$$E(t) = \frac{1}{2}\int_\Omega |u(x,t)|^2 dx, \tag{9.23}$$

and the boundary and internal observability properties read:

$$E(T) \leqq C \int_0^T \int_{\partial\Omega} |\frac{\partial u}{\partial n}(x,t)|^2 dx, \tag{9.24}$$

$$E(T) \leqq C \int_0^T \int_\omega |\nabla u(x,t)|^2 dx, \tag{9.25}$$

respectively.

When $\rho \in C^2(\Omega)$ the above observability inequalities hold for any $T > 0$ (see [8]).

We briefly sketch why our construction, which can be perfectly adapted to the heat equation, does not produce lack of observability.

Observe that the quasi-eigenfunctions φ_j that we constructed for the wave equation in (6.10) are also quasi-eigenfunctions of the heat equation in the sense that

$$v_j(x,t) = e^{-h_j t^2}\varphi_j(x)$$

satisfy the differential equation in (9.22) and they are exponentially concentrated in the interior of Ω.

However, due to the non-reversibility of the heat equation, in (9.24) and (9.25) we must estimate the energy of the solution in time $T > 0$. Therefore we have to estimate a much smaller function than for the wave equation since the solution of the heat equation at time T is multiplied by the factor $e^{-h_j T^2}$.

To our knowledge, there is no example in the literature of L^∞ coefficients for linear parabolic equations for which the observability inequalities (9.24) and (9.25) fail. This is an interesting open problem.

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