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Homogenization of Non-Uniformly Bounded Operators: Critical Barrier for Nonlocal Effects

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Abstract

This paper deals with the homogenization of a class of highly oscillating monotonic operators which are uniformly elliptic but non-uniformly bounded. We determine an asymptotic barrier below which we obtain a classical behaviour and above which nonlocal effects appear. We also prove that this condition is optimal in the case of a fibre-reinforced medium.

1. Introduction

Let Ω be a bounded open set of $\mathbb{R}^d$ and let f be a function in $L^{p'}(\Omega;\mathbb{R}^n)$ (we can also take f in $W^{-1,p'}(\Omega;\mathbb{R}^n)$ without restriction), where $p, p' > 1$ are two conjugate numbers. In this paper we study the asymptotic behaviour of the class of the nonlinear Dirichlet problems parametrized by a small positive number ε:

$$\begin{aligned} -\operatorname{div}\left(a_\varepsilon(x,\nabla u_\varepsilon)\right) &= f \ \text{ in } \Omega, \\ u_\varepsilon &= 0 \ \text{ on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $a_\varepsilon : \mathbb{R}^d \times \mathbb{R}^{nd} \longrightarrow \mathbb{R}^{nd}$ is a highly oscillating monotonic operator defined by $a_\varepsilon(x,\lambda) := A_\varepsilon\left(\frac{x}{\varepsilon},\lambda\right)$. The operator $A_\varepsilon(y,\lambda)$ is a Carathéodory-type function, which is Y-periodic (Y is the unit cube of $\mathbb{R}^d$) with respect to $y \in \mathbb{R}^d$ and strictly monotonic with respect to $\lambda \in \mathbb{R}^{nd}$. Moreover A_ε is uniformly elliptic and satisfies the boundedness condition

$$\text{for a.e. } y \in \mathbb{R}^d,\ \forall\, \lambda \in \mathbb{R}^{nd}, \quad |A_\varepsilon(y,\lambda)| \leqq \beta_\varepsilon(y)\,(1+|\lambda|^{p-1}),$$

where β_ε is a positive function in $L^\infty(Y)$ such that $\|\beta_\varepsilon\|_{L^\infty(Y)} \to +\infty$.

The difficulty of the asymptotic analysis comes from the dependence of A_ε with respect to ε combined with the non-uniform boundedness of β_ε.

The first, nowadays classical result concerning the homogenization of oscillating monotonic operators is due to TARTAR [11]. When A_ε does not depend on ε, he proved that the sequence u_ε strongly converges in $W_0^{1,p}(\Omega;\mathbb{R}^n)$ to the solution u_0 of the Dirichlet problem

$$\begin{aligned} -\operatorname{div}(a_0(\nabla u_0)) &= f \ \text{ in } \Omega, \\ u_0 &= 0 \ \text{ on } \partial\Omega, \end{aligned} \tag{1.2}$$

where the (homogenized) operator a_0 can be explicitely computed thanks to a local problem. Then by the Γ-convergence theory, CARBONE & SBORDONE [5] and BUTTAZZO & DAL MASO [4] extended the Tartar result to non-uniformly bounded (convex) operators a_ε such that

$$\beta_\varepsilon \text{ is bounded in } L^1(Y) \text{ and equi-integrable.} \tag{1.3}$$

More recently MOSCO [9] obtained a similar result for $p = 2$ by using the theory of Dirichlet forms.

However, when condition (1.3) does not hold, the limit behaviour of the Dirichlet problem (1.1) can be different from the classical one (1.2). Indeed nonlocal effects may appear. The first examples of such a phenomenon were obtained by FENCHENKO & KHRUSLOV [6], KHRUSLOV [7] in the linear case. In the same framework MOSCO [9] deduced from the Beurling-Deny representation formula [2], a general representation for the Γ-limit of the quadratic forms associated with the linear problem (1.1); this representation naturally contains a nonlocal term. BELLIEUD & BOUCHITTÉ [1] extended these nonlocal homogenization results to particular nonlinear functionals.

Our aim is to determine an asymptotic barrier for A_ε, below which the classical behaviour (1.2) holds true, and above which nonlocal effects may appear. The main result of the paper is then the following.

Under the boundedness assumption

$$\forall \lambda \in \mathbb{R}^{nd}, \quad \int_Y A_\varepsilon(y,\lambda)\cdot\lambda\, dy \leqq c|\lambda|^p,$$

combined with some technical assumptions, we prove that the condition

$$\begin{aligned} &\forall\,\lambda \in \mathbb{R}^{nd}, \quad \varepsilon^p\, C_\lambda(\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} 0, \\ &C_\lambda(\varepsilon) := \sup_{\substack{V\in W^{1,p}(Y;\mathbb{R}^n)\setminus\{0\} \\ \int_Y V=0}} \frac{\displaystyle\int_Y A_\varepsilon(y,\lambda)\cdot\lambda\,|V|^p}{\displaystyle\int_Y A_\varepsilon(y,\nabla V)\cdot\nabla V} \end{aligned} \tag{1.4}$$

implies a classical behaviour (1.2) for problem (1.1) (see Theorem 2.1). We also prove that in contrast with condition (1.3), condition (1.4) is optimal to avoid nonlocal effects for a particular example. The optimality is shown thanks to the fibre-reinforced model of [1], for which we give the precise asymptotic behaviour of $C_\lambda(\varepsilon)$ (see Proposition 2.4).

In [3] we obtained an optimal result of convergence in the "opposite" case, i.e., when the sequence A_ε is uniformly bounded but non-uniformly elliptic. In this context the analysis is less intricate since condition (1.4) is replaced by a more simple one without the weight $A_\varepsilon(y,\lambda)\cdot\lambda$.

2. Statement of the results

2.1. Statement of the problem

In the following:

- Ω is a bounded domain of $\mathbb{R}^d$;
- ε denotes a positive parameter, n a non-negative integer and $p, p' > 1$ are two conjugate numbers, i.e., $\frac{1}{p}+\frac{1}{p'}=1$;
- $\cdot$ denotes the scalar product and $|\cdot|$ the Euclidian norm in $\mathbb{R}^n$ or in $\mathbb{R}^{nd}$;
- for any $\lambda \in \mathbb{R}^{nd}$, $x \in \mathbb{R}^d$, λ is considered as a $n\times d$ matrix, $\lambda\, x$ denotes the vector of $\mathbb{R}^n$ defined by $(\lambda\, x)_i = \sum_{j=1}^d \lambda_{ij}\, x_j$;
- for any $u \in \mathbb{R}^n$, $x \in \mathbb{R}^d$, $u \otimes x$ denotes the $n\times d$ matrix defined by $(u\otimes x)_{ij} := u_i\, x_j$;
- Y denotes the unit cube $[-\frac{1}{2},\frac{1}{2}[^d$;
- $W^{1,p}_\#(Y;\mathbb{R}^n)$ denotes the Y-periodic vector-valued functions in $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^n)$.

Let $A_\varepsilon : \mathbb{R}^d \times \mathbb{R}^{nd} \longrightarrow \mathbb{R}^{nd}$ be a vector-valued function satisfying the following conditions:

- A_ε is a Carathéodory function, i.e.,

$$\begin{aligned}&\text{for any } \lambda \in \mathbb{R}^{nd},\ y \longmapsto A_\varepsilon(y,\lambda) \text{ is measurable,}\\ &\text{for a.e. } y \in \mathbb{R}^d,\ \ \lambda \longmapsto A_\varepsilon(y,\lambda) \text{ is continuous;}\end{aligned} \tag{2.1}$$

- A_ε is Y-periodic with respect to the first variable, and strictly monotonic with respect to the second one, i.e.,

$$\text{for a.e. } y \in \mathbb{R}^d,\ \forall\,\lambda,\mu \in \mathbb{R}^{nd},\ \lambda \neq \mu,\ \ (A_\varepsilon(y,\lambda) - A_\varepsilon(y,\mu))\cdot(\lambda-\mu) > 0; \tag{2.2}$$

- A_ε is uniformly elliptic, i.e., there exists a positive constant α such that

$$\text{for a.e. } y \in \mathbb{R}^d,\ \forall\,\lambda \in \mathbb{R}^{nd},\quad A_\varepsilon(y,\lambda)\cdot\lambda \geqq \alpha\,|\lambda|^p; \tag{2.3}$$

- A_ε satisfies the boundedness condition

$$\text{for a.e. } y \in \mathbb{R}^d,\ \forall\,\lambda \in \mathbb{R}^{nd},\quad |A_\varepsilon(y,\lambda)| \leqq \beta_\varepsilon(y)\,(1+|\lambda|^{p-1}), \tag{2.4}$$

where β_ε is a function in $L^\infty(Y)$, with $\|\beta_\varepsilon\|_{L^\infty(Y)} \to +\infty$ as $\varepsilon \to 0$.

We define the highly oscillating monotonic function

$$a_\varepsilon(x, \lambda) := A_\varepsilon\left(\frac{x}{\varepsilon}, \lambda\right), \quad x \in \Omega, \lambda \in \mathbb{R}^{nd}, \tag{2.5}$$

and we consider the nonlinear Dirichlet problem

$$\begin{aligned} -\operatorname{div}(a_\varepsilon(x, \nabla u_\varepsilon)) &= f \ \text{ in } \Omega, \\ u_\varepsilon &= 0 \ \text{ on } \partial\Omega. \end{aligned} \tag{2.6}$$

By the theory of monotonic operators (see, e.g., Theorem 2.1 p. 171 of [8]) there exists a unique solution $u_\varepsilon \in W_0^{1,p}(\Omega; \mathbb{R}^n)$ of the Dirichlet problem (2.6).

Our aim is to give an optimal condition on A_ε in order to obtain the following asymptotic behaviour of problem (2.6): for every given f in $L^{p'}(\Omega; \mathbb{R}^n)$, we have

$$u_\varepsilon \rightharpoonup u_0 \quad \text{weakly in } W_0^{1,p}(\Omega; \mathbb{R}^n), \tag{2.7}$$

where u_0 is the solution of the Dirichlet problem

$$\begin{aligned} -\operatorname{div}(a_0(\nabla u_0)) &= f \ \text{ in } \Omega, \\ u_0 &= 0 \ \text{ on } \partial\Omega. \end{aligned} \tag{2.8}$$

Let us first describe in a formal way the limit operator a_0 of (2.8). For that we apply the result of the classical case [11], i.e., the case where A_ε does not depend on ε. Therefore we fix ε and, for any $\lambda \in \mathbb{R}^{nd}$, we define $X_\varepsilon(\cdot, \lambda)$ as the solution in $W_\#^{1,p}(Y; \mathbb{R}^n)$ with zero averaged value on Y, of the variational cell problem

$$\begin{aligned} &\forall\, V \in W_\#^{1,p}(Y; \mathbb{R}^n), \quad \int_Y A_\varepsilon(y, \nabla_y W_\varepsilon(y, \lambda)) \cdot \nabla V(y)\, dy = 0, \\ &\text{where} \quad W_\varepsilon(y, \lambda) := \lambda\, y - X_\varepsilon(y, \lambda). \end{aligned} \tag{2.9}$$

Then by [11] the homogenized operator of the oscillating operator $A_\varepsilon(\frac{x}{\delta}, \lambda)$ as $\delta \to 0$ and ε is fixed, is defined by

$$A_\varepsilon^0(\lambda) := \int_Y A_\varepsilon(y, \nabla_y W_\varepsilon(y, \lambda))\, dy. \tag{2.10}$$

It is thus natural to think that the function a_0 of (2.8) is obtained as the limit

$$\forall\, \lambda \in \mathbb{R}^{nd}, \quad A_\varepsilon^0(\lambda) \underset{\varepsilon \to 0}{\longrightarrow} a_0(\lambda). \tag{2.11}$$

Of course limit (2.11) is a “natural” idea for obtaining the homogenized problem (2.8) but is far from being sufficient. Indeed nonlocal effects may appear in the limit behaviour as shown in [7], [6] and [1]. These are due to non-uniform boundedness of the sequence β_ε of (2.4), when $\|\beta_\varepsilon\|_{L^\infty(Y)} \to +\infty$. In fact these nonlocal effects naturally enter into the homogenization process when the operators are not uniformly bounded. Indeed in the linear case MOSCO [9] established the link between homogenization and the Beurling-Deny [2] representation of the Dirichlet forms, which contains a nonlocal term.

The problem is to determine a critical barrier above which nonlocal effects appear and below which the homogenization of problem (2.6) leads to the classical problem (2.8). The condition (1.3) on β_ε is sufficient to have the classical behaviour (2.8) but this condition is far from being optimal (see Section 2.3). In the next subsection we will give an optimal condition which separates the classical case from the case where nonlocal effects appear.

2.2. Statement of the result

For technical reasons we only consider the class of monotonic operators which satisfy both restrictive conditions:

- A_ε is $(p-1)$-homogeneous with respect to the second variable

$$\text{for a.e. } y \in Y,\ \forall t \in \mathbb{R},\ \forall \lambda \in \mathbb{R}^{nd}, \quad A_\varepsilon(y, t\lambda) = |t|^{p-2} t\, A_\varepsilon(y, \lambda); \tag{2.12}$$

- A_ε satisfies a Hölder-type inequality

$$\begin{aligned} &\text{for a.e. } y \in Y,\ \forall\, \lambda, \mu \in \mathbb{R}^{nd}, \\ &|A_\varepsilon(y,\lambda)\cdot\mu| \leqq C\,(A_\varepsilon(y,\lambda)\cdot\lambda)^{\frac{1}{p'}}\,(A_\varepsilon(y,\mu)\cdot\mu)^{\frac{1}{p}} \end{aligned} \tag{2.13}$$

where C is a positive constant. These conditions are not so restrictive since they are satisfied in the linear case $A_\varepsilon(y,\lambda) := B_\varepsilon(y)\,\lambda$, where B_ε is a matrix-valued function, and in the p-Laplacian case $A_\varepsilon(y,\lambda) := \beta_\varepsilon(y)\,|\lambda|^{p-2}\,\lambda$.

The main result of the paper is the following.

Theorem 2.1. *Let A_ε be a sequence of operators satisfying properties* (2.1)–(2.4), (2.12), (2.13) *and the L^1-boundedness condition*

$$\forall\,\lambda \in \mathbb{R}^{nd}, \quad \int_Y A_\varepsilon(y,\lambda)\cdot\lambda\,dy \leqq c\,|\lambda|^p, \tag{2.14}$$

where c is a positive constant. We also assume that

$$\forall\,\lambda \in \mathbb{R}^{nd}, \quad \varepsilon^p \sup_{\substack{V \in W^{1,p}(Y;\mathbb{R}^n)\setminus\{0\} \\ \int_Y V = 0}} \frac{\displaystyle\int_Y A_\varepsilon(y,\lambda)\cdot\lambda\,|V|^p}{\displaystyle\int_Y A_\varepsilon(y,\nabla V)\cdot\nabla V} \underset{\varepsilon\to 0}{\longrightarrow} 0. \tag{2.15}$$

Finally we assume that there exists a continuous function $a_0 : \mathbb{R}^{nd} \longrightarrow \mathbb{R}^{nd}$ which is strictly monotonic and such that limit (2.11) *holds, i.e., $A^0_\varepsilon(\lambda)$ defined by* (2.10) *converges to $a_0(\lambda)$ for any λ.*

Then the solution u_ε of problem (2.6) *weakly converges in $W^{1,p}_0(\Omega;\mathbb{R}^n)$ to the solution u_0 of the classical Dirichlet problem* (2.8).

Remark 2.2.

- By the boundedness condition (2.4) and the ellipticity condition (2.3) combined with the Poincaré-Wirtinger inequality in $W^{1,p}(Y;\mathbb{R}^n)$, there exists, for any $\lambda \in \mathbb{R}^{nd}$ and any $\varepsilon > 0$, an optimal constant $C_\lambda(\varepsilon) > 0$ such that

$$\begin{aligned}&\forall\, V \in W^{1,p}(Y;\mathbb{R}^n),\\ &\int_Y A_\varepsilon(y,\lambda)\cdot\lambda\,|V - {\textstyle\int_Y} V|^p \leqq C_\lambda(\varepsilon)\int_Y A_\varepsilon(y,\nabla V)\cdot\nabla V.\end{aligned} \tag{2.16}$$

  Inequality (2.16) is a weighted Poincaré-Wirtinger inequality with the weight $A_\varepsilon(\cdot,\lambda)\cdot\lambda$. Then condition (2.15) is equivalent to

$$\forall\,\lambda \in \mathbb{R}^{nd},\quad \varepsilon^p\, C_\lambda(\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} 0. \tag{2.17}$$

  In the particular case where $A_\varepsilon(y,\lambda) := \beta_\varepsilon(y)\,|\lambda|^{p-2}\lambda$ condition (2.15) reduces to

$$\varepsilon^p \sup_{\substack{V\in W^{1,p}(Y;\mathbb{R}^n)\setminus\{0\}\\ \int_Y V=0}} \frac{\displaystyle\int_Y \beta_\varepsilon\,|V|^p}{\displaystyle\int_Y \beta_\varepsilon\,|\nabla V|^p} \underset{\varepsilon\to 0}{\longrightarrow} 0. \tag{2.18}$$

- On the one hand the L^1-boundedness condition (2.14) prevents the appearance of a zero order term in the limit problem (see [7] and [1]). In the linear case this term corresponds to the so called *killing measure* in the Beurling-Deny representation formula of the Dirichlet forms (see [9]).
  On the other hand condition (2.15) prevents the appearance of nonlocal effects. In the linear case these nonlocal effects are associated with the so-called *jumping measure* in the Beurling-Deny formula (see [9]).

Remark 2.3. At the end of the subsection, we will prove that the function A^0_ε satisfies the estimates

$$\forall\,\lambda \in \mathbb{R}^{nd},\quad c^{-1}\,|\lambda|^p \leqq A^0_\varepsilon(\lambda)\cdot\lambda \quad\text{and}\quad |A^0_\varepsilon(\lambda)| \leqq c\,|\lambda|^{p-1}, \tag{2.19}$$

where $c \geqq 1$ is a constant. Therefore the assumption (2.11) on the convergence of the homogenized function A^0_ε (2.10) to the function a_0, is partly justified (for a dense subset of $\mathbb{R}^d$ and for a subsequence of ε) by the previous estimates and by a diagonal extraction. However, we have to assume the continuity of a_0 as well as its strict monotonicity. Thanks to (2.19) the limit a_0 of A^0_ε also satisfies the same estimates

$$\forall\,\lambda \in \mathbb{R}^{nd},\quad c^{-1}\,|\lambda|^p \leqq a_0(\lambda)\cdot\lambda \quad\text{and}\quad |a_0(\lambda)| \leqq c\,|\lambda|^{p-1}, \tag{2.20}$$

which combined with the strict monotonicity of a_0 ensure the existence and the uniqueness of the limit Dirichlet problem (2.8).

Let us prove estimates (2.19). These estimates will be useful in the proof of Theorem 2.1.

Proof of (2.19). By the definition of the local problem (2.9) we have

$$\int_Y A_\varepsilon(y, \nabla_y W_\varepsilon(y,\lambda)) \cdot \nabla_y W_\varepsilon(y,\lambda) = \int_Y A_\varepsilon(y, \nabla_y W_\varepsilon(y,\lambda)) \cdot \lambda .$$

By applying (2.13) to the right-hand side, the Hölder inequality and the boundedness (2.14) we obtain

$$\begin{aligned}\int_Y A_\varepsilon(y, \nabla_y W_\varepsilon) \cdot \nabla_y W_\varepsilon &\leqq \int_Y C \left(A_\varepsilon(y, \nabla_y W_\varepsilon) \cdot \nabla_y W_\varepsilon\right)^{\frac{1}{p'}} \left(A_\varepsilon(y,\lambda)\cdot\lambda\right)^{\frac{1}{p}} \\ &\leqq C \left(\int_Y A_\varepsilon(y, \nabla_y W_\varepsilon) \cdot \nabla_y W_\varepsilon\right)^{\frac{1}{p'}} \\ &\quad \times \left(\int_Y A_\varepsilon(y,\lambda)\cdot\lambda\right)^{\frac{1}{p}} \\ &\leqq C' \, |\lambda| \left(\int_Y A_\varepsilon(y, \nabla_y W_\varepsilon) \cdot \nabla_y W_\varepsilon\right)^{\frac{1}{p'}} ,\end{aligned}$$

whence the estimate

$$\int_Y A_\varepsilon(y, \nabla_y W_\varepsilon(y,\lambda)) \cdot \nabla_y W_\varepsilon(y,\lambda) \leqq c\,|\lambda|^p . \tag{2.21}$$

From now on, we denote by c any positive constant whose exact value does not matter.

By applying again inequality (2.13), the Hölder inequality and estimate (2.21) we have, for any $\mu \in \mathbb{R}^{nd}$,

$$\left| \int_Y A_\varepsilon(y, \nabla_y W_\varepsilon(y,\lambda)) \cdot \mu \right| \leqq c\,|\lambda|^{p-1}|\mu|$$

which implies the second estimate of (2.19).

On the other hand the ellipticity (2.3) of A_ε and the definition (2.10) of A^0_ε imply that, for any $\lambda \in \mathbb{R}^d$,

$$\begin{aligned}\|\nabla_y W_\varepsilon(\cdot,\lambda)\|_{L^p(Y)} &\leqq c \left(\int_Y A_\varepsilon(y, \nabla_y W_\varepsilon(y,\lambda)) \cdot \nabla_y W_\varepsilon(y,\lambda)\right)^{\frac{1}{p}} \\ &= c \left(A^0_\varepsilon(\lambda)\cdot\lambda\right)^{\frac{1}{p}} .\end{aligned} \tag{2.22}$$

Then estimate (2.22) combined with the second estimate of (2.19) yield

$$\|\nabla_y W_\varepsilon(\cdot,\lambda)\|_{L^p(Y)} \leqq c\,|\lambda|,$$

and thanks to the Poincaré-Wirtinger inequality we obtain

$$\|W_\varepsilon(\cdot,\lambda)\|_{W^{1,p}(Y)} \leqq c\,|\lambda|. \tag{2.23}$$

Assume now that the uniform ellipticity of A_ε^0 does not hold. Then there exists a sequence λ_ε of $\mathbb{R}^{nd}$ with $|\lambda_\varepsilon| = 1$, which converges to λ with $|\lambda| = 1$, and such that $A_\varepsilon^0(\lambda_\varepsilon) \cdot \lambda_\varepsilon \to 0$. Whence by estimate (2.22) we have

$$\|\nabla_y W_\varepsilon(\cdot, \lambda_\varepsilon)\|_{L^p(Y)} \leqq c\big(A_\varepsilon^0(\lambda_\varepsilon) \cdot \lambda_\varepsilon\big)^{\frac{1}{p}} \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

By (2.23) we can also assume that $X_\varepsilon(\cdot, \lambda_\varepsilon)$ weakly converges in $W_\#^{1,p}(Y; \mathbb{R}^n)$ to a function X_0. Then by the semi-lower continuity of the L^p norm we have

$$\|\lambda - \nabla X_0\|_{L^p(Y)} \leqq \liminf_{\varepsilon\to 0} \|\nabla_y W_\varepsilon(\cdot, \lambda_\varepsilon)\|_{L^p(Y)} = 0,$$

whence $\nabla X_0 = \lambda$ with $|\lambda| = 1$, which contradicts the Y-periodicity of X_0. The first estimate of (2.19) is thus proved. $\square$

2.3. *Optimality of the condition on A_ε*

Let us now show the optimality of the asymptotic condition (2.15). We will give an example in which nonlocal effects appear as soon as condition (2.15) is violated. For that we consider the case of a fibre-reinforced medium, studied by BELLIEUD & BOUCHITTÉ [1]. In [1] the function A_ε is defined as follows for $n = 1$ and $d = 3$:

Let Q_ε be the cylinder in $Y := [-\frac{1}{2}, \frac{1}{2}[^3$ of axis y_3 and radius $r_\varepsilon \to 0$, and let A_ε be the function defined by

$$A_\varepsilon(y, \lambda) := \beta_\varepsilon(y)\,|\lambda|^{p-2}\lambda, \quad \text{where} \quad \beta_\varepsilon := \mathbf{1}_{Y\setminus Q_\varepsilon} + \gamma_\varepsilon\,\mathbf{1}_{Q_\varepsilon}, \quad \gamma_\varepsilon \to +\infty. \tag{2.24}$$

In this case the weighted Poincaré-Wirtinger inequality (2.16) can be written

$$\forall\, V \in W^{1,p}(Y; \mathbb{R}), \quad \int_Y \beta_\varepsilon\,|V - {\textstyle\int_Y} V|^p \leqq C(\varepsilon) \int_Y \beta_\varepsilon\,|\nabla V|^p, \tag{2.25}$$

where $C(\varepsilon)$ is the optimal constant. Then condition (2.15) is equivalent to (2.18) and hence to $\varepsilon^p\, C(\varepsilon) \to 0$ as $\varepsilon \to 0$.

The following result yields the precise asymptotic behaviour of $C(\varepsilon)$.

Proposition 2.4. *Assume that*

$$\gamma_\varepsilon\,|Q_\varepsilon| \underset{\varepsilon\to 0}{\longrightarrow} k \in]0, +\infty[. \tag{2.26}$$

Then the optimal constant $C(\varepsilon)$ of (2.25) *satisfies the following asymptotic behaviours:*

- *if $1 < p < 2$, $C(\varepsilon) \to +\infty$ and*

$$C(\varepsilon) \underset{\varepsilon\to 0}{\sim} \frac{1}{2\pi}\left(\frac{p-1}{2-p}\right)^{p-1} k\, r_\varepsilon^{p-2}; \tag{2.27}$$

• *if* $p = 2$, $C(\varepsilon) \to +\infty$ *and*

$$C(\varepsilon) \underset{\varepsilon \to 0}{\sim} \frac{1}{2\pi} k \, |\ln r_\varepsilon|; \tag{2.28}$$

• *if* $p > 2$, $C(\varepsilon)$ *is bounded.*

Moreover we have, for any $\lambda \in \mathbb{R}^3$,

$$A_\varepsilon^0(\lambda) \underset{\varepsilon \to 0}{\longrightarrow} a_0(\lambda) := |\lambda|^{p-2}\lambda + k \, |\lambda_3|^{p-2}\lambda_3 \, \nu_3, \tag{2.29}$$

where k *is given by* (2.26) *and* (ν_1, ν_2, ν_3) *denotes the canonic basis of* $\mathbb{R}^3$.

On the other hand, Bellieud & Bouchitté [1] obtain the limit behaviour of the Dirichlet problem (2.6) when A_ε is defined by (2.24), and prove that, under assumption (2.26), the homogenized problem contains a nonlocal term if and only if

$$\begin{aligned} &\liminf_{\varepsilon \to 0} \; \varepsilon^p \, r_\varepsilon^{p-2} \; > 0 \text{ for } 1 < p < 2, \\ &\liminf_{\varepsilon \to 0} \; \varepsilon^2 \, |\ln r_\varepsilon| > 0 \text{ for } p = 2. \end{aligned}$$

Then by the results of Proposition 2.4 these conditions are equivalent to

$$\liminf_{\varepsilon \to 0} \; \varepsilon^p \, C(\varepsilon) > 0.$$

Therefore, for this particular example, the quantity $\varepsilon^p \, C(\varepsilon)$ is the desired critical barrier, below which the homogenization is classical, and above which the homogenization is nonlocal. For this example, boundedness assumption (2.14) is an immediate consequence of (2.26) and the value of the homogenized operator a_0 is given by (2.29) and thus satisfies the conditions of Theorem 2.1.

We can also note that the equi-integrability condition (1.3) is equivalent to

$$\gamma_\varepsilon \, |Q_\varepsilon| \underset{\varepsilon \to 0}{\longrightarrow} 0. \tag{2.30}$$

Indeed condition (2.30) ensures the boundedness of the sequence β_ε in $L^1(\Omega)$. Moreover (2.30) implies that the integral of β_ε over any measurable set E tends to 0 as $|E| \to 0$ uniformly with respect to ε, since the "bad" sets are the subsets of Q_ε, whence the equi-integrability of the sequence β_ε.

We know (see Introduction) that (1.3), or equivalently (2.30), is sufficient to obtain the classical homogenization result (2.8) of problem (2.6). However Theorem 2.1 and Proposition 2.4 show that this condition is far from recovering all the cases of classical behaviour. In particular in the case $p > 2$, the limit is always classical without extra assumption.

3. Proof of the homogenization theorem

The proof of Theorem 2.1 follows the so-called method of oscillating test functions due to TARTAR [11]. In order to obtain the limit of the flow $\xi_\varepsilon := a_\varepsilon(\cdot, \nabla u_\varepsilon)$ we consider the family of periodic test functions $W_\varepsilon(\cdot, \lambda)$, $\lambda \in \mathbb{R}^{nd}$, from the auxiliary cell problem (2.9).

In a first step (Section 3.1), we obtain some *a priori* estimates satisfied by u_ε and ξ_ε.

In a second step (Section 3.2), by using the monotonicity of a_ε with u_ε and the rescaled test function $\varepsilon\, W_\varepsilon(\frac{x}{\varepsilon}, \lambda)$ as well as the homogeneity of a_ε, we obtain the inequality (3.8) satisfied by ξ_ε and the function $\eta_\varepsilon(\frac{x}{\varepsilon}, \lambda)$, where $\eta_\varepsilon(y, \lambda) := A_\varepsilon(y, \nabla_y W_\varepsilon(y, \lambda))$. To pass to the limit in this inequality we essentially need the following limits

$$\begin{aligned} \xi_\varepsilon \cdot \nabla_y W_\varepsilon(\tfrac{x}{\varepsilon}, \lambda) &\rightharpoonup \xi_0 \cdot \lambda\, x \quad && \text{weakly in } \mathcal{D}'(\Omega), \\ \eta_\varepsilon(\tfrac{x}{\varepsilon}, \lambda) \cdot \nabla u_\varepsilon &\rightharpoonup a_0(\lambda) \cdot \nabla u_0 && \text{weakly in } \mathcal{D}'(\Omega), \end{aligned} \tag{3.1}$$

where ξ_0 is the weak limit of ξ_ε in the weak sense of the Radon measures. In the classical framework, namely when A_ε is uniformly bounded, limits (3.1) are a simple consequence of the div-curl Lemma of Murat, TARTAR [11]. In our context the situation is more delicate since ξ_0 is only a Radon measure at this point. Then assumption (2.15) is the key-ingredient used to compensate for the lack of compactness in these products of weakly converging sequences.

For the first limit of (3.1) condition (2.15) naturally appears after integrating by parts and using Hölder-type inequalities. The case of the second limit is more complicated and it exploits the oscillations due to the periodicity (see Lemma 3.1). Since $\eta_\varepsilon(\frac{x}{\varepsilon}, \lambda)$ has a zero divergence by definition (2.9), an integration by parts leads us to study the product $\eta_\varepsilon(\frac{x}{\varepsilon}, \lambda)\, u_\varepsilon$. At this level the idea (see the proof of Lemma 3.1 in Section 3.4) is to replace the oscillating sequence $\eta_\varepsilon(\frac{x}{\varepsilon}, \lambda)$ by its averaged value $\int_Y \eta_\varepsilon(y, \lambda)$, which is equal to $A^0_\varepsilon(\lambda)$ by definition (2.10). For that we show the existence of an oscillating function h_ε which satisfies

$$\eta_\varepsilon(\tfrac{x}{\varepsilon}, \lambda) - A^0_\varepsilon(\lambda) = -\,\varepsilon\, \mathrm{div}\,(a_\varepsilon(x, \nabla h_\varepsilon))$$

and whose L^p norm of the gradient is controlled by the constant $C_\lambda(\varepsilon)$ defined in (2.16). We then obtain the second limit of (3.1) thanks to several integrations by parts combined with assumption (2.15).

In a third step (Section 3.3), we determine the value of the limit flow ξ_0. From the inequality (3.6) obtained in the previous step (Section 3.2) and a technical result (Lemma 3.2) we deduce that ξ_0 belongs in fact to $L^{p'}(\Omega)$. Then by using similar monoticity inequalities of type (3.20) and homogeneity arguments, we prove the equality $\xi_0 = a_0(\nabla u_0)$ which concludes the proof of Theorem 2.1.

3.1. A priori estimates

Let $u_\varepsilon \in W_0^{1,p}(\Omega; \mathbb{R}^n)$ be the solution of the Dirichlet problem (2.6). By the uniform ellipticity (2.3) of A_ε we have

$$\begin{aligned}\alpha \, \|\nabla u_\varepsilon\|^p_{L^p(\Omega)} &\leqq \int_\Omega a_\varepsilon(x, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon \\ &= \int_\Omega f \cdot u_\varepsilon \\ &\leqq \|f\|_{L^{p'}(\Omega)} \, \|u_\varepsilon\|_{L^p(\Omega)} \\ &\leqq c_f \, \|\nabla u_\varepsilon\|_{L^p(\Omega)}.\end{aligned}$$

Then the sequence u_ε is bounded in $W_0^{1,p}(\Omega; \mathbb{R}^n)$ and convergence (2.7) holds up to a subsequence. We also have

$$a_\varepsilon(\cdot, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon \text{ is bounded in } L^1(\Omega). \tag{3.2}$$

On the other side inequality (2.13) combined with the Hölder inequality implies that, for any $\mu \in \mathbb{R}^{nd}$,

$$\int_\Omega |a_\varepsilon(x, \nabla u_\varepsilon) \cdot \mu| \leqq C \left(\int_\Omega a_\varepsilon(x, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon \right)^{\frac{1}{p'}} \left(\int_\Omega a_\varepsilon(x, \mu) \cdot \mu \right)^{\frac{1}{p}},$$

whence by estimates (2.14) and (3.2) $a_\varepsilon(\cdot, \nabla u_\varepsilon)$ is bounded in $L^1(\Omega; \mathbb{R}^{nd})$ and thus satisfies, up to a subsequence still denoted by ε, the weak convergence in the Radon measures sense

$$\xi_\varepsilon := a_\varepsilon(\cdot, \nabla u_\varepsilon) \rightharpoonup \xi_0 \quad \text{weakly in } \mathcal{M}(\Omega; \mathbb{R}^{nd}) \, *, \tag{3.3}$$

i.e., for any continuous function $\Phi \in C_c(\Omega; \mathbb{R}^{nd})$ with compact support in Ω,

$$\int_\Omega a_\varepsilon(x, \nabla u_\varepsilon) \cdot \Phi \underset{\varepsilon \to 0}{\longrightarrow} \int_\Omega \xi_0 \cdot \Phi. \tag{3.4}$$

Since by definition $-\operatorname{div}(\xi_\varepsilon) = f$, the vector-valued measure ξ_0 satisfies the equality in the distributions sense

$$-\operatorname{div}(\xi_0) = f \quad \text{in } \mathcal{D}'(\Omega, \mathbb{R}^n). \tag{3.5}$$

Therefore the proof of Theorem 2.1 consists in proving the equality $\xi_0 = a_0(\nabla u_0)$ where ξ_0 is the limit defined by (3.3) and a_0 is the limit of the homogenized operator A_ε^0 defined by (2.10). The main difficulty of the proof comes from the weakness of convergence (3.3).

We proceed in two steps, which are shown in the following two subsections. In the first step we prove a variational inequality satisfied by the measure ξ_0. In the second step we deduce from this inequality that ξ_0 is a function in $L^{p'}(\Omega; \mathbb{R}^{nd})$; then we show the desired equality.

3.2. An inequality satisfied by ξ_0

We will prove that for any $\lambda \in \mathbb{R}^{nd}$ and any $\varphi \in C_c(\Omega)$ (continuous with compact support in Ω) the following inequality holds:

$$\int_\Omega f \cdot u_0 - \int_\Omega \xi_0 \cdot \lambda\, \varphi - \int_\Omega a_0(\lambda) \cdot \nabla u_0\, |\varphi|^{p-2}\varphi + \int_\Omega a_0(\lambda) \cdot \lambda\, |\varphi|^p \geqq 0. \tag{3.6}$$

The starting point of the proof is an inequality of monotonicity as in the classical framework [11]. Let us define

$$\eta_\varepsilon(y,\lambda) := A_\varepsilon\left(y, \nabla_y W_\varepsilon(y,\lambda)\right), \quad y \in Y,\ \lambda \in \mathbb{R}^{nd}, \tag{3.7}$$

where W_ε is defined by (2.9), and let ξ_ε be defined by (3.3). The monotonicity (2.2) of A_ε implies that

$$\int_\Omega \left(\xi_\varepsilon - \eta_\varepsilon(\tfrac{x}{\varepsilon}, \lambda\,\varphi)\right) \cdot \left(\nabla u_\varepsilon - \nabla_y W_\varepsilon(\tfrac{x}{\varepsilon}, \lambda\,\varphi)\right) \geqq 0.$$

On the other hand, by the homogeneity condition (2.12) and by the uniqueness of the local problem (2.9), for a.e. $y \in Y$, the function $W_\varepsilon(y,\cdot)$ is 1-homogeneous and the function $\eta_\varepsilon(y,\cdot)$ is $(p-1)$-homogeneous. Then u_ε being solution of problem (2.6), the previous inequality can be written

$$\begin{aligned} \int_\Omega f \cdot u_\varepsilon - \int_\Omega \xi_\varepsilon \cdot \nabla_y W_\varepsilon(\tfrac{x}{\varepsilon}, \lambda)\,\varphi& \\ - \int_\Omega \eta_\varepsilon(\tfrac{x}{\varepsilon}, \lambda) \cdot \nabla u_\varepsilon\, |\varphi|^{p-2}\varphi& \\ + \int_\Omega \eta_\varepsilon(\tfrac{x}{\varepsilon}, \lambda) \cdot \nabla_y W_\varepsilon(\tfrac{x}{\varepsilon}, \lambda)\, |\varphi|^p \geqq 0.& \end{aligned} \tag{3.8}$$

We will determine the limit of any integral of (3.8). By the weak limit (2.7) we have

$$\int_\Omega f \cdot u_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \int_\Omega f \cdot u_0. \tag{3.9}$$

Limit of the second term of (3.8). Let us define the function

$$w_\varepsilon^\lambda(x) := \varepsilon\, W_\varepsilon\left(\frac{x}{\varepsilon}, \lambda\right), \tag{3.10}$$

where W_ε is defined by (2.9). Let $\varphi \in \mathcal{D}(\Omega) := C_0^\infty(\Omega)$. By integrating by parts and by putting the function $\varphi\, w_\varepsilon^\lambda$ in (2.6) we obtain

$$\int_\Omega \xi_\varepsilon \cdot \nabla_y W_\varepsilon(\tfrac{x}{\varepsilon}, \lambda)\,\varphi = \int_\Omega f \cdot w_\varepsilon^\lambda\, \varphi - \int_\Omega \xi_\varepsilon \cdot (w_\varepsilon^\lambda \otimes \nabla\varphi). \tag{3.11}$$

By estimate (2.23) it is clear that $w_\varepsilon^\lambda(x)$ strongly converges to $\lambda\, x$ in $L^p(\Omega;\mathbb{R}^n)$, whence

$$\int_\Omega f \cdot w_\varepsilon^\lambda\, \varphi \underset{\varepsilon\to 0}{\longrightarrow} \int_\Omega f \cdot (\lambda\, x)\, \varphi.$$

For the last term of (3.11) we need assumption (2.15). Let $(\nu_i)_{1\leqq i\leqq d}$ be the canonic basis of $\mathbb{R}^d$ and $(e_k)_{1\leqq k\leqq n}$ be the canonic basis of $\mathbb{R}^n$, we have

$$\int_\Omega \xi_\varepsilon \cdot (w_\varepsilon^\lambda \otimes \nabla\varphi) = \int_\Omega \xi_\varepsilon \cdot (\lambda\, x \otimes \nabla\varphi) - \sum_{i=1}^d \int_\Omega \varepsilon\, \xi_\varepsilon \cdot (X_\varepsilon(\tfrac{x}{\varepsilon},\lambda) \otimes \nu_i)\, \frac{\partial\varphi}{\partial x_i},$$

and by the weak convergence (3.3)

$$\int_\Omega \xi_\varepsilon \cdot (\lambda\, x \otimes \nabla\varphi) \underset{\varepsilon\to 0}{\longrightarrow} \int_\Omega \xi_0 \cdot (\lambda\, x \otimes \nabla\varphi).$$

On the other hand, by inequality (2.13) combined with the Hölder inequality we have, for any $\nu \in \mathbb{R}^d$ and with $X_\varepsilon^k := X_\varepsilon \cdot e_k$,

$$\begin{aligned}
&\int_\Omega \left|\xi_\varepsilon \cdot (X_\varepsilon(\tfrac{x}{\varepsilon},\lambda) \otimes \nu)\right| \\
&\quad \leqq \sum_{k=1}^n \int_\Omega |\xi_\varepsilon \cdot (e_k \otimes \nu)|\, |X_\varepsilon^k(\tfrac{x}{\varepsilon},\lambda)| \\
&\quad \leqq \sum_{k=1}^n C \left(\int_\Omega \xi_\varepsilon \cdot \nabla u_\varepsilon\right)^{\frac{1}{p'}} \left(\int_\Omega a_\varepsilon(x, e_k \otimes \nu) \cdot (e_k \otimes \nu)\, |X_\varepsilon^k(\tfrac{x}{\varepsilon},\lambda)|^p\right)^{\frac{1}{p}}
\end{aligned}$$

$$\begin{aligned}
&|X_\varepsilon^k| \leqq |X_\varepsilon| \\
&\quad \leqq \sum_{k=1}^n c \left(\int_Y A_\varepsilon(y, e_k \otimes \nu) \cdot (e_k \otimes \nu)\, |X_\varepsilon(y,\lambda)|^p\right)^{\frac{1}{p}} \quad \text{by (3.2).}
\end{aligned}$$

Moreover by applying successively the Poincaré inequality (2.16) estimates (2.14), (2.21) and condition (2.17), we obtain for any $e \in \mathbb{R}^n$ and $\mu := e \otimes \nu$,

$$\begin{aligned}
&\int_Y A_\varepsilon(y,\mu) \cdot \mu\, |X_\varepsilon(y,\lambda)|^p \\
&\qquad \leqq \int_Y (A_\varepsilon(y,\mu)\cdot\mu)\, 2^p \left(|2\,\lambda|^p + |W_\varepsilon(y,\lambda) - \textstyle\int_Y W_\varepsilon|^p\right) \\
&\qquad \leqq c + C_\mu(\varepsilon) \int_Y A_\varepsilon(y, \nabla_y W_\varepsilon(y,\lambda)) \cdot \nabla_y W_\varepsilon(y,\lambda) \\
&\qquad = o(\varepsilon^{-p}).
\end{aligned}$$

We thus deduce from the previous estimates that

$$\int_\Omega \varepsilon\, \xi_\varepsilon \cdot (X_\varepsilon(\tfrac{x}{\varepsilon},\lambda) \otimes \nabla\varphi) \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Passing to the limit in equality (3.11) then yields

$$\int_\Omega \xi_\varepsilon \cdot \nabla_y W_\varepsilon(\tfrac{x}{\varepsilon},\lambda)\, \varphi \underset{\varepsilon\to 0}{\longrightarrow} \int_\Omega f \cdot (\lambda\, x)\, \varphi - \int_\Omega \xi_0 \cdot (\lambda\, x \otimes \nabla\varphi).$$

Finally, when we put the function $(\lambda\, x)\, \varphi$ in (3.5), the previous limit becomes

$$\int_\Omega \xi_\varepsilon \cdot \nabla_y W_\varepsilon(\tfrac{x}{\varepsilon}, \lambda)\, \varphi \underset{\varepsilon\to 0}{\longrightarrow} \int_\Omega \xi_0 \cdot \lambda\, \varphi \quad \text{for any } \varphi \in \mathcal{D}(\Omega). \tag{3.12}$$

Limit (3.12) also holds for any $\varphi \in C_c(\Omega)$. Indeed inequality (2.13) and the Hölder inequality combined with estimates (3.2) and (2.21) imply that

$$\begin{aligned} &\int_\Omega |\xi_\varepsilon \cdot \nabla_y W_\varepsilon(\tfrac{x}{\varepsilon}, \lambda)| \\ &\qquad \leqq C \left(\int_\Omega \xi_\varepsilon \cdot \nabla u_\varepsilon\right)^{\frac{1}{p'}} \left(\int_Y A_\varepsilon(y, \nabla_y W_\varepsilon(y, \lambda)) \cdot \nabla_y W_\varepsilon(y, \lambda)\right)^{\frac{1}{p}} \\ &\qquad \leqq c\, |\lambda| \end{aligned}$$

which shows that the sequence $\xi_\varepsilon \cdot \nabla_y W_\varepsilon(\frac{x}{\varepsilon}, \lambda)$ is bounded in $L^1(\Omega)$.

Limit of the third term of (3.8). We need the following result which is essentially based on condition (2.15).

Lemma 3.1. *Let v_ε be a sequence in $W_0^{1,p}(\Omega; \mathbb{R}^n)$ such that*

$$\begin{aligned} &v_\varepsilon \rightharpoonup v_0 \ \text{ weakly in } W_0^{1,p}(\Omega; \mathbb{R}^n), \\ &a_\varepsilon(\cdot, \nabla v_\varepsilon) \cdot \nabla v_\varepsilon \ \text{ is bounded in } L^1(\Omega), \end{aligned} \tag{3.13}$$

and let G_ε be a sequence of Y-periodic functions in $L^p_\#(Y; \mathbb{R}^{nd})$ such that

$$\int_Y A_\varepsilon(y, G_\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} b_0 \in \mathbb{R}^{nd} \quad \text{and} \quad A_\varepsilon(\cdot, G_\varepsilon) \cdot G_\varepsilon \ \text{ is bounded in } L^1(Y). \tag{3.14}$$

Then the following convergence holds in the distributions sense:

$$\begin{aligned} &\forall\, \theta \in \mathcal{D}(\Omega; \mathbb{R}^d), \\ &\int_\Omega a_\varepsilon(x, g_\varepsilon) \cdot (v_\varepsilon \otimes \theta) \underset{\varepsilon\to 0}{\longrightarrow} \int_\Omega b_0 \cdot (v_0 \otimes \theta), \quad g_\varepsilon(x) := G_\varepsilon\left(\frac{x}{\varepsilon}\right). \end{aligned} \tag{3.15}$$

Lemma 3.1 will be proved in the last subsection. Thanks to Lemma 3.1 we will obtain the limit of the third term of (3.8).

Let $\psi \in \mathcal{D}(\Omega)$. By definitions (3.7) and (2.9) we have $\mathrm{div}_y(\eta_\varepsilon(y, \lambda)) = 0$ in the distribution sense; indeed, since (2.9) holds true in the torus sense it also holds in the distribution sense thanks to the periodicity, by using a test function of type $V(y) := \sum_{\kappa \in \mathbb{R}^d} \psi(y+\kappa)$. Then owing to an integration by parts we obtain

$$\int_\Omega \eta_\varepsilon(\tfrac{x}{\varepsilon}, \lambda) \cdot \nabla u_\varepsilon\, \psi = -\int_\Omega \eta_\varepsilon(\tfrac{x}{\varepsilon}, \lambda) \cdot (u_\varepsilon \otimes \nabla \psi).$$

The sequence $v_\varepsilon := u_\varepsilon$ satisfies the conditions (3.13) of Lemma 3.1 by (2.7), (3.2), and the sequence $G_\varepsilon := \nabla_y W_\varepsilon(\cdot, \lambda)$ satisfies the conditions (3.14) of Lemma 3.1 by (2.11), (2.21) with $b_0 := a_0(\lambda)$. Then convergence (3.15) yields

$$\int_\Omega \eta_\varepsilon(\tfrac{x}{\varepsilon}, \lambda) \cdot (u_\varepsilon \otimes \nabla \psi) \underset{\varepsilon\to 0}{\longrightarrow} \int_\Omega a_0(\lambda) \cdot (u_0 \otimes \nabla \psi) = -\int_\Omega a_0(\lambda) \cdot \nabla u_0\, \psi$$

since $\nabla(\psi\, u_0) = u_0 \otimes \nabla\psi + \nabla u_0\, \psi$. We thus obtain the limit

$$\int_\Omega \eta_\varepsilon(\tfrac{x}{\varepsilon}, \lambda) \cdot \nabla u_\varepsilon\, \psi \ \underset{\varepsilon\to 0}{\longrightarrow} \ \int_\Omega a_0(\lambda) \cdot \nabla u_0\, \psi,$$

for any $\psi \in \mathcal{D}(\Omega)$. This limit also holds for any $\psi \in C_c(\Omega)$ since the inequality (2.13) combined with estimates (3.2), (2.21) implies that the sequence $\eta_\varepsilon(\frac{x}{\varepsilon}, \lambda) \cdot \nabla u_\varepsilon$ is bounded in $L^1(\Omega)$. In particular we have for any $\varphi \in C_c(\Omega)$,

$$\int_\Omega \eta_\varepsilon(\tfrac{x}{\varepsilon}, \lambda) \cdot \nabla u_\varepsilon\, |\varphi|^{p-1}\varphi \ \underset{\varepsilon\to 0}{\longrightarrow} \ \int_\Omega a_0(\lambda) \cdot \nabla u_0\, |\varphi|^{p-1}\varphi. \tag{3.16}$$

Limit of the fourth term of (3.8). By the definition of W_ε in (2.9) and by estimate (2.23) the sequence $v_\varepsilon(x) := \varepsilon\, W_\varepsilon(\frac{x}{\varepsilon}, \lambda)$ weakly converges to $\lambda\, x$ in $W^{1,p}(\Omega; \mathbb{R}^n)$. Therefore, by proceeding as in the previous step, owing to Lemma 3.1, we obtain the following limit:

$$\int_\Omega \eta_\varepsilon(\tfrac{x}{\varepsilon}, \lambda) \cdot \nabla_y W_\varepsilon(\tfrac{x}{\varepsilon}, \lambda)\, |\varphi|^p \ \underset{\varepsilon\to 0}{\longrightarrow} \ \int_\Omega a_0(\lambda) \cdot \lambda\, |\varphi|^p. \tag{3.17}$$

Finally, passing to the limit in inequality (3.8), thanks to limits (3.9), (3.12), (3.16) and (3.17), yields the desired inequality (3.6).

3.3. Proof of the equality $\xi_0 = a_0(\nabla u_0)$

We have to first prove that the vector-valued measure ξ_0 defined by (3.5) is a function in $L^{p'}(\Omega; \mathbb{R}^{nd})$. This is a consequence of inequality (3.6) and the following result.

Lemma 3.2. *Let μ be a Radon measure on Ω and let F be a mapping from $L^p(\Omega)$ into $\mathbb{R}$ which is continuous at the point* 0. *Assume that*

$$\forall\, \varphi \in C_c(\Omega), \quad F(\varphi) + \int_\Omega \varphi\, \mu \geqq 0. \tag{3.18}$$

Then μ belongs to $L^{p'}(\Omega)$.

Lemma 3.2 will be proved in the last subsection.

For any $\lambda \in \mathbb{R}^{nd}$, let us apply Lemma 3.2 with $\mu := -\,\xi_0 \cdot \lambda$ and

$$F(\varphi) := \int_\Omega \Big(f \cdot u_0 - a_0(\lambda) \cdot \nabla u_0\, |\varphi|^{p-2}\varphi + a_0(\lambda) \cdot \lambda\, |\varphi|^p \Big).$$

The mapping F is clearly continuous on $L^p(\Omega)$ and assumption (3.18) holds thanks to inequality (3.6). Therefore $\xi_0 \cdot \lambda \in L^{p'}(\Omega)$ for any $\lambda \in \mathbb{R}^{nd}$, and equivalently ξ_0 is a function in $L^{p'}(\Omega; \mathbb{R}^{nd})$.

Let us now prove the equality $\xi_0 = a_0(\nabla u_0)$. Let $(\varphi_k)_{1\leqq k\leqq m}$ be a family of m functions in $C_c(\Omega)$ with disjoint support, and let $(\lambda_k)_{1\leqq k\leqq m}$ be a family of m vectors in $\mathbb{R}^{nd}$. We define the function

$$\Phi_m := \sum_{k=1}^m \varphi_k\, \lambda_k \in C_c(\Omega; \mathbb{R}^{nd}), \tag{3.19}$$

and we consider the monotonicity inequality

$$\int_\Omega \left(\xi_\varepsilon - \eta_\varepsilon(\tfrac{x}{\varepsilon}, \Phi_m)\right) \cdot \left(\nabla u_\varepsilon - \nabla_y W_\varepsilon(\tfrac{x}{\varepsilon}, \Phi_m)\right) \geqq 0,$$

where ξ_ε, W_ε and η_ε are defined by (3.3), (2.9) and (3.7). Since the functions φ_k have disjoint support and, for a.e. $y \in Y$, the functions $W_\varepsilon(y, \cdot)$, $\eta_\varepsilon(y, \cdot)$ are homogeneous, the previous inequality can be written

$$\int_\Omega \left(\xi_\varepsilon - \sum_{k=1}^m \eta_\varepsilon(\tfrac{x}{\varepsilon}, \lambda_k)\, |\varphi_k|^{p-2}\varphi_k\right) \cdot \left(\nabla u_\varepsilon - \sum_{k=1}^m \nabla_y W_\varepsilon(\tfrac{x}{\varepsilon}, \lambda_k)\, \varphi_k\right) \geqq 0.$$

Now we proceed as in the previous subsection in which we deduced inequality (3.6) from inequality (3.8). Passing to the limit in the previous inequality then yields

$$\int_\Omega f \cdot u_0 - \int_\Omega \xi_0 \cdot \Phi_m - \int_\Omega \sum_{k=1}^m a_0(\lambda_k) \cdot \nabla u_0\, |\varphi_k|^{p-2}\varphi_k + \int_\Omega \sum_{k=1}^m a_0(\lambda_k) \cdot \lambda_k\, |\varphi_k|^p \geqq 0,$$

whence again by the homogeneity of a_0 and the disjointness of the supports of φ_k we deduce the inequality

$$\int_\Omega f \cdot u_0 - \int_\Omega \xi_0 \cdot \Phi_m - \int_\Omega a_0(\Phi_m) \cdot \nabla u_0 + \int_\Omega a_0(\Phi_m) \cdot \Phi_m \geqq 0. \tag{3.20}$$

On the other hand, by the density of $C_c(\Omega; \mathbb{R}^{nd})$ in $L^p(\Omega; \mathbb{R}^{nd})$, there exists, for any $\Phi \in L^p(\Omega; \mathbb{R}^{nd})$, a sequence $(\Phi_m)_{m \geqq 0}$ of functions of type (3.19) such that Φ_m strongly converges to Φ in $L^p(\Omega; \mathbb{R}^{nd})$. Moreover since a_0 is continuous on $\mathbb{R}^{nd}$ and satisfies the boundedness condition (2.20), a_0 also defines a continuous mapping from $L^p(\Omega; \mathbb{R}^{nd})$ into $L^{p'}(\Omega; \mathbb{R}^{nd})$. Therefore passing to the limit $\Phi_m \to \Phi$ in the last inequality implies that

$$\int_\Omega f \cdot u_0 - \int_\Omega \xi_0 \cdot \Phi - \int_\Omega a_0(\Phi) \cdot \nabla u_0 + \int_\Omega a_0(\Phi) \cdot \Phi \geqq 0. \tag{3.21}$$

Finally by putting u_0 in equality (3.5) we obtain

$$\int_\Omega \xi_0 \cdot \nabla u_0 = \int_\Omega f \cdot u_0$$

and inequality (3.21) becomes

$$\int_\Omega (\xi_0 - a_0(\Phi)) \cdot (\nabla u_0 - \Phi) \geqq 0.$$

Let $\Psi \in L^p(\Omega; \mathbb{R}^{nd})$ and $\Phi_t := \nabla u_0 - t\,\Psi$, $t > 0$. We thus have

$$\int_\Omega (\xi_0 - a_0(\Phi_t)) \cdot \Psi \geqq 0.$$

Then, by passing to the limit $t \to 0$ and by using the continuity of a_0 on $L^p(\Omega;\mathbb{R}^{nd})$, we obtain

$$\int_\Omega (\xi_0 - a_0(\nabla u_0)) \cdot \Psi \geqq 0, \quad \text{for any } \Psi \in L^p(\Omega;\mathbb{R}^{nd}),$$

which implies the equality $\xi_0 = a_0(\nabla u_0)$ and concludes the proof of Theorem 2.1.

3.4. Proof of the lemmas

Proof of Lemma 3.1. To prove convergence (3.15) we will first introduce an auxiliary function H_ε^ν. Let us define $B_\varepsilon := A_\varepsilon(\cdot, G_\varepsilon)$ and $\overline{B}_\varepsilon$ its averaged value on Y. Let $\nu \in \mathbb{R}^d$, thanks to the strict monotonicity of A_ε combined with the continuity of the mapping $\left(V \longmapsto \int_Y (B_\varepsilon - \overline{B}_\varepsilon)\,\nu \cdot V\right)$ in the quotient space $W^{1,p}_\#(Y;\mathbb{R}^n)/\mathbb{R}$ provided with the norm $\|\nabla V\|_{L^p(Y)}$ (consequence of the Poincaré-Wirtinger inequality in Y), there exists a unique solution H_ε^ν in $W^{1,p}_\#(Y;\mathbb{R}^n)$ with zero averaged value on Y, of the variational cell problem

$$\begin{aligned}&\forall\, V \in W^{1,p}_\#(Y;\mathbb{R}^n),\\ &\qquad \int_Y A_\varepsilon(y, \nabla H_\varepsilon^\nu(y)) \cdot \nabla V(y)\,dy = \int_Y (B_\varepsilon(y) - \overline{B}_\varepsilon)\,\nu \cdot V(y)\,dy.\end{aligned} \tag{3.22}$$

Putting the function H_ε^ν in (3.22) yields

$$\int_Y A_\varepsilon(y, \nabla H_\varepsilon^\nu) \cdot \nabla H_\varepsilon^\nu = \int_Y B_\varepsilon \cdot (H_\varepsilon^\nu \otimes \nu)$$

since H_ε^ν has a zero averaged value. Moreover by using successively inequality (2.13), the Hölder inequality, the weighted Poincaré-Wirtinger inequality (2.16) and the estimate of (3.14), we have

$$\begin{aligned}&\left|\int_Y B_\varepsilon \cdot (H_\varepsilon^\nu \otimes \nu)\right|\\ &\quad \leqq \sum_{k=1}^n \int_Y |B_\varepsilon \cdot (e_k \otimes \nu)\, H_\varepsilon^{\nu,k}|\\ &\quad \leqq \sum_{k=1}^n \int_Y C\,(A_\varepsilon(y, G_\varepsilon)\cdot G_\varepsilon)^{\frac{1}{p'}} \left(A_\varepsilon(y, e_k \otimes \nu)\cdot(e_k\otimes\nu)\,|H_\varepsilon^\nu|^p\right)^{\frac{1}{p}}\\ &\quad \leqq \sum_{k=1}^n C\left(\int_Y A_\varepsilon(y, G_\varepsilon)\cdot G_\varepsilon\right)^{\frac{1}{p'}} \left(\int_Y A_\varepsilon(y, e_k \otimes \nu)\cdot(e_k\otimes\nu)\,|H_\varepsilon^\nu|^p\right)^{\frac{1}{p}}\\ &\quad \leqq \sum_{k=1}^n c\left(C_{e_k\otimes\nu}(\varepsilon)\int_Y A_\varepsilon(y, \nabla H_\varepsilon^\nu)\cdot \nabla H_\varepsilon^\nu\right)^{\frac{1}{p}}.\end{aligned}$$

We thus deduce from the previous inequality and condition (2.17) the following estimate:

$$\int_Y A_\varepsilon(y, \nabla H_\varepsilon^\nu)\cdot\nabla H_\varepsilon^\nu = o\big(\varepsilon^{-p'}\big). \tag{3.23}$$

Let $\theta \in \mathcal{D}(\Omega;\mathbb{R}^d)$. By rescaling equation (3.22) with $h_\varepsilon^\nu(x) := \varepsilon\, H_\varepsilon^\nu(\frac{x}{\varepsilon})$ we obtain

$$\begin{aligned}\int_\Omega a_\varepsilon(x,g_\varepsilon)\cdot(v_\varepsilon\otimes\theta) &= \sum_{i=1}^d \int_\Omega (a_\varepsilon(x,g_\varepsilon)\,\nu_i)\cdot(\theta_i\, v_\varepsilon)\\ &= \int_\Omega \overline{B}_\varepsilon\cdot(v_\varepsilon\otimes\theta) + \sum_{i=1}^d\int_\Omega \varepsilon\, a_\varepsilon(x,\nabla h_\varepsilon^{\nu_i})\cdot\nabla(\theta_i\, v_\varepsilon).\end{aligned}$$

Furthermore, by the limits of (3.13) and (3.14) we have

$$\int_\Omega \overline{B}_\varepsilon\cdot(v_\varepsilon\otimes\theta) \underset{\varepsilon\to0}{\longrightarrow} \int_\Omega b_0\cdot(v_0\otimes\theta).$$

Therefore to prove convergence (3.15) we are led to prove that

$$\int_\Omega \varepsilon\, a_\varepsilon(x,\nabla h_\varepsilon^\nu)\cdot\nabla(\varphi\, v_\varepsilon) \underset{\varepsilon\to0}{\longrightarrow} 0$$

for any $\nu\in\mathbb{R}^d$ and any $\varphi\in\mathcal{D}(\Omega)$. To prove the latter limit it is enough to prove that, for any $\nu,\nu'\in\mathbb{R}^d$,

$$\int_\Omega \varepsilon\,|a_\varepsilon(x,\nabla h_\varepsilon^\nu)\cdot\nabla v_\varepsilon| \underset{\varepsilon\to0}{\longrightarrow} 0 \quad\text{and}\quad \int_\Omega \varepsilon\,|a_\varepsilon(x,\nabla h_\varepsilon^\nu)\cdot(v_\varepsilon\otimes\nu')| \underset{\varepsilon\to0}{\longrightarrow} 0. \tag{3.24}$$

By inequality (2.13), the Hölder inequality and estimates (3.23) and (3.13), we have

$$\begin{aligned}&\int_\Omega \varepsilon\,|a_\varepsilon(x,\nabla h_\varepsilon^\nu)\cdot\nabla v_\varepsilon|\\ &\qquad\leqq C\,\varepsilon\left(\int_\Omega a_\varepsilon(x,\nabla h_\varepsilon^\nu)\cdot\nabla h_\varepsilon^\nu\right)^{\frac{1}{p'}}\left(\int_\Omega a_\varepsilon(x,\nabla v_\varepsilon)\cdot\nabla v_\varepsilon\right)^{\frac{1}{p}}\\ &\qquad= o(1),\end{aligned}$$

whence the first limit of (3.24). Similarly we have

$$\begin{aligned}&\int_\Omega \varepsilon\,|a_\varepsilon(x,\nabla h_\varepsilon^\nu)\cdot(v_\varepsilon\otimes\nu')|\\ &\qquad\leqq \sum_{i=1}^d\int_\Omega \varepsilon\left|a_\varepsilon(x,\nabla h_\varepsilon^\nu)\cdot(e_i\otimes\nu')\right|\,|v_\varepsilon^i|\end{aligned}$$

$$\begin{aligned}&\lambda_i := e_i\otimes\nu'\\ &\qquad\leqq \sum_{i=1}^d C\,\varepsilon\left(\int_\Omega a_\varepsilon(x,\nabla h_\varepsilon^\nu)\cdot\nabla h_\varepsilon^\nu\right)^{\frac{1}{p'}}\left(\int_\Omega a_\varepsilon(x,\lambda_i)\cdot\lambda_i\,|v_\varepsilon|^p\right)^{\frac{1}{p}}\\ &\qquad= \sum_{i=1}^d o(1)\left(\int_\Omega a_\varepsilon(x,\lambda_i)\cdot\lambda_i\,|v_\varepsilon|^p\right)^{\frac{1}{p}}.\end{aligned}$$

Therefore the second limit of (3.24) holds true if

$$\int_\Omega a_\varepsilon(x,\lambda)\cdot\lambda\,|v_\varepsilon|^p \quad \text{is bounded for any } \lambda\in\mathbb{R}^{nd}. \tag{3.25}$$

For proving estimate (3.25) we consider the piece-wise constant function defined by

$$\overline{v}_\varepsilon(x) := \fint_{\varepsilon\kappa+\varepsilon Y} v_\varepsilon \quad \text{if } x\in(\varepsilon\kappa+\varepsilon Y), \quad \kappa\in\mathbb{Z}^d,$$

where the function v_ε is extended by 0 outside Ω. Then by rescaling the weighted Poincaré-Wirtinger inequality (2.16) with each function $V(y) := v_\varepsilon(\varepsilon\kappa+\varepsilon y)$ and by the homogeneity of A_ε, we obtain the inequality

$$\int_\Omega a_\varepsilon(x,\lambda)\cdot\lambda\,|v_\varepsilon-\overline{v}_\varepsilon|^p \leqq \varepsilon^p\, C_\lambda(\varepsilon)\int_\Omega a_\varepsilon(x,\nabla v_\varepsilon)\cdot\nabla v_\varepsilon,$$

whence by condition (2.17) combined with the estimate of (3.13)

$$\int_\Omega a_\varepsilon(x,\lambda)\cdot\lambda\,|v_\varepsilon-\overline{v}_\varepsilon|^p \underset{\varepsilon\to 0}{\longrightarrow} 0. \tag{3.26}$$

On the other hand, by considering the paving $(\varepsilon\kappa+\varepsilon Y)_{\kappa\in\mathbb{Z}^d}$ we also have

$$\begin{aligned}\int_\Omega a_\varepsilon(x,\lambda)\cdot\lambda\,|\overline{v}_\varepsilon|^p &= \sum_{\kappa\in\mathbb{Z}^d}\int_{\varepsilon\kappa+\varepsilon Y} a_\varepsilon(x,\lambda)\cdot\lambda\left|\fint_{\varepsilon\kappa+\varepsilon Y} v_\varepsilon\right|^p\\ &\leqq \sum_{\kappa\in\mathbb{Z}^d}\varepsilon^d\int_Y A_\varepsilon(y,\lambda)\cdot\lambda\fint_{\varepsilon\kappa+\varepsilon Y}|v_\varepsilon|^p\\ &= \int_Y A_\varepsilon(y,\lambda)\cdot\lambda\int_\Omega |v_\varepsilon|^p\end{aligned}$$

which is bounded thanks to estimate (2.14) and the boundedness of v_ε in $W_0^{1,p}(\Omega;\mathbb{R}^n)$. The previous estimate combined with limit (3.26) thus imply the desired estimate (3.25). Lemma 3.1 is proved. □

Proof of Lemma 3.2. By the duality L^p-$L^{p'}$ the measure μ belongs to $L^{p'}(\Omega)$ if and only if there exists a positive constant $c>0$ such that

$$\forall\,\varphi\in C_c(\Omega), \qquad \left|\int_\Omega \varphi\,\mu\right| \leqq c\,\|\varphi\|_{L^p(\Omega)}.$$

Let us now proceed by contradiction. Assume that the previous estimate does not hold. Then, for any non-zero real number c_0, there exists a sequence $(\varphi_h)_{h\in\mathbb{N}}$ in $C_c(\Omega)$ such that

$$\forall\,h\in\mathbb{N}, \quad \int_\Omega \varphi_h\,\mu = c_0 \quad \text{and} \quad \lim_{h\to+\infty}\|\varphi_h\|_{L^p(\Omega)} = 0.$$

By passing to the limit $h\to+\infty$ in inequality (3.18) with φ_h, and by the continuity of F, we deduce the inequality $F(0)+c_0 \geqq 0$. Since c_0 is arbitrary, it thus remains to choose c_0 such that $F(0)+c_0<0$ in order to obtain the desired contradiction. □

4. Proof of the optimality result

Let us first present the strategy concerning the proof of the optimality result. The best constant $C(\varepsilon)$ in (2.25) is defined by the supremum

$$C(\varepsilon) = \sup_{\substack{V\in W^{1,p}(Y)\setminus\{0\} \\ \int_Y V=0}} \frac{\displaystyle\int_Y \beta_\varepsilon\,|V|^p}{\displaystyle\int_Y \beta_\varepsilon\,|\nabla V|^p}, \tag{4.1}$$

where the maximiser V_ε satisfies the Euler-Lagrange equation

$$\forall\, V \in W^{1,p}(Y), \quad \int_Y \beta_\varepsilon\,|\nabla V_\varepsilon|^{p-2}\nabla V_\varepsilon \cdot \nabla V = \int_Y \beta_\varepsilon\,|V_\varepsilon|^{p-2} V_\varepsilon \left(V - \textstyle\fint_Y V\right). \tag{4.2}$$

We aim at constructing an explicit “almost maximizer” $\hat{V}_\varepsilon^R$ sufficiently close (up to multiplicative and additive constants) to V_ε for small ε so that the small error could be controlled both from above and below. The function $\hat{V}_\varepsilon^R$, $R > 0$, is defined by $\hat{V}_\varepsilon^R := 0$ in Q_ε, $\hat{V}_\varepsilon^R := 1$ in $Y \setminus Q_R$ (where Q_R is the ball of radius R) and $\hat{V}_\varepsilon^R$ solves the p-Laplace equation in the annulus $Q_R \setminus Q_\varepsilon$. By choosing the parameter R small enough, we can ensure that $\hat{V}_\varepsilon^R$ is also a strong approximation of 1 in $L^2(Y)$.

On one hand, putting $\hat{V}_\varepsilon^R - \fint_Y V_\varepsilon^R$ as test function in (4.1) provides an estimate of $C(\varepsilon)$ from below. On the other hand, the function $\hat{V}_\varepsilon^R$ allows us to obtain an explicit approximation of the left-hand side of (4.2) (see Lemma 4.1 below) from which we deduce an estimate of $C(\varepsilon)$ from above.

Let us now prove Proposition 2.4 by following the above scheme. We consider the function A_ε defined by (2.24) and the Poincaré-Wirtinger inequality (2.25) defined with the weight β_ε.

Thanks to the compact imbedding $W^{1,p}(Y) \hookrightarrow L^p(Y)$ combined with the semi-lower continuity of the mapping $\left(V \longmapsto \int_Y |\nabla V|^p\right)$ in $W^{1,p}(Y)$, there exists a maximizer V_ε in $W^{1,p}(Y)$ related to the supremum (4.1) such that

$$\int_Y V_\varepsilon = 0 \quad \text{and} \quad \int_Y \beta_\varepsilon\,|V_\varepsilon|^p = 1. \tag{4.3}$$

We then have

$$\frac{1}{C(\varepsilon)} = \int_Y \beta_\varepsilon\,|\nabla V_\varepsilon|^p. \tag{4.4}$$

The proof of Proposition 2.4 is based on equality (4.4) combined with the following result.

Lemma 4.1. *Let Q_R be the closed cylinder in $Y := [-\frac{1}{2}, \frac{1}{2}[^3$, of axis y_3 and of radius $R < \frac{1}{2}$. Let $\hat{V}_\varepsilon^R$ be the function defined by ($r^2 := y_1^2 + y_2^2$)*

$$
\begin{aligned}
&\hat{V}_\varepsilon^R(y) := 0 && \text{if } y \in Q_\varepsilon, \\
&\hat{V}_\varepsilon^R(y) := 1 && \text{if } y \in Y \setminus Q_R, \\
&\hat{V}_\varepsilon^R(y) := \frac{r^q - r_\varepsilon^q}{R^q - r_\varepsilon^q} && \text{if } y \in Q_R \setminus Q_\varepsilon \text{ and } p \neq 2, \quad q := \tfrac{p-2}{p-1}, \\
&\hat{V}_\varepsilon^R(y) := \frac{\ln r - \ln r_\varepsilon}{\ln R - \ln r_\varepsilon} && \text{if } y \in Q_R \setminus Q_\varepsilon \text{ and } p = 2.
\end{aligned}
\tag{4.5}
$$

Then there exists a positive constant C_R such that, for any $V \in W^{1,p}(Y)$,

$$
\begin{aligned}
&\left| \int_Y \beta_\varepsilon \, |\nabla \hat{V}_\varepsilon^R|^{p-2} \nabla \hat{V}_\varepsilon^R \cdot \nabla V - \hat{\delta}_R(\varepsilon) \left(\fint_{Y \setminus Q_R} V - \fint_{Q_\varepsilon} V \right) \right| \\
&\qquad \leqq C_R \, \hat{\delta}_R(\varepsilon) \left(\|\nabla V\|_{L^p(Y \setminus Q_R)} + r_\varepsilon^{-\frac{2}{p}} \|\nabla V\|_{L^p(Q_\varepsilon)} \right),
\end{aligned}
\tag{4.6}
$$

$$
\text{where} \quad \hat{\delta}_R(\varepsilon) := \int_Y \beta_\varepsilon \, |\nabla \hat{V}_\varepsilon^R|^p = \int_{Q_R \setminus Q_\varepsilon} |\nabla \hat{V}_\varepsilon^R|^p. \tag{4.7}
$$

Lemma 4.1 will be proved in the last subsection.

For obtaining the asymptotic behaviours of $C(\varepsilon)$ defined by (4.4), the cases where $p < 2$ and $p = 2$ are quite similar. So in the first subsection we will prove the asymptotic behaviours (2.27) and (2.28); then in the second subsection we will study the case $p > 2$. In the third subsection we will determine the limit operator a_0 defined by (2.11) in the particular case (2.24). The fourth subsection will be devoted to the proof of the technical lemmas.

4.1. The case $1 < p \leqq 2$

Estimate of $\mathbf{C}(\varepsilon)$ from below. The function $\hat{V}_\varepsilon^R$ defined by (4.5) satisfies the estimate

$$
\|\hat{V}_\varepsilon^R - 1\|_{L^p(Y)} = 1 + o_{R,\varepsilon}(1),
$$

where $o_{R,\varepsilon}(1)$ denotes a term which tends to 0 as $R \to 0$ uniformly with respect to ε. In the following we will first pass to the limit (liminf) $\varepsilon \to 0$ then to the limit $R \to 0$ using the uniform convergence of $o_{R,\varepsilon}(1)$ with respect to ε.

Putting the function $V := \hat{V}_\varepsilon^R - \int_Y \hat{V}_\varepsilon^R$ in inequality (2.25) yields

$$
C(\varepsilon) \geqq \frac{\displaystyle\int_{Q_\varepsilon} \beta_\varepsilon |V|^p}{\displaystyle\int_Y \beta_\varepsilon |\nabla V|^p} = \frac{\gamma_\varepsilon \, |Q_\varepsilon|}{\hat{\delta}_R(\varepsilon)} \big(1 + o_{R,\varepsilon}(1)\big). \tag{4.8}
$$

Moreover we easily deduce from definitions (4.5) and (4.7) the following asymptotic behaviours for any fixed $R < \frac{1}{2}$:

$$\begin{aligned} \hat{\delta}_R(\varepsilon) &\underset{\varepsilon\to 0}{\sim} \hat{\delta}(\varepsilon) := 2\pi\, |q|^{p-1}\, r_\varepsilon^{2-p} \text{ if } 1 < p < 2, \\ \hat{\delta}_R(\varepsilon) &\underset{\varepsilon\to 0}{\sim} \hat{\delta}(\varepsilon) := 2\pi\, |\ln r_\varepsilon|^{-1} \qquad \text{if } p = 2. \end{aligned} \tag{4.9}$$

Then estimates (4.8), (4.9) combined with limit (2.26) imply that

$$\frac{\hat{\delta}(\varepsilon)\, C(\varepsilon)}{\gamma_\varepsilon\, |Q_\varepsilon|} \geqq \frac{\hat{\delta}(\varepsilon)}{\hat{\delta}_R(\varepsilon)} \left(1 + o_{R,\varepsilon}(1)\right),$$

and by passing to the liminf (for any subsequence of ε) in both sides of the inequality for a fixed small R, we obtain thanks to (4.9),

$$\liminf_{\varepsilon\to 0} \left(\frac{\hat{\delta}(\varepsilon)\, C(\varepsilon)}{\gamma_\varepsilon\, |Q_\varepsilon|} \right) \geqq 1 + o_R(1), \tag{4.10}$$

where $o_R(1)$ tends to 0 as $R \to 0$. In particular we have $C(\varepsilon) \to +\infty$.

Estimate of $\mathbf{C}(\varepsilon)$ from above. By putting the function V_ε defined by (4.1), (4.3) in inequality (4.6), we have thanks to limit (2.26), the equality (4.4) and the estimate (4.8) satisfied by $C(\varepsilon)$,

$$\begin{aligned} &\left| \int_Y \beta_\varepsilon\, |\nabla \hat{V}_\varepsilon^R|^{p-2} \nabla \hat{V}_\varepsilon^R \cdot \nabla V_\varepsilon - \hat{\delta}_R(\varepsilon) \left(\fint_{Y\setminus Q_R} V_\varepsilon - \fint_{Q_\varepsilon} V_\varepsilon \right) \right| \\ &\quad \leqq C_R\, \hat{\delta}_R(\varepsilon) \left[\left(\int_{Y\setminus Q_R} \beta_\varepsilon\, |\nabla V_\varepsilon|^p \right)^{\frac{1}{p}} + (\gamma_\varepsilon\, r_\varepsilon^2)^{-\frac{1}{p}} \left(\int_{Q_\varepsilon} \beta_\varepsilon\, |\nabla V_\varepsilon|^p \right)^{\frac{1}{p}} \right] \\ &\quad \leqq c_R\, \hat{\delta}_R(\varepsilon)\, C(\varepsilon)^{-\frac{1}{p}} \leqq c_R\, \hat{\delta}_R(\varepsilon)^{(1+\frac{1}{p})}, \end{aligned}$$

where c_R is a constant which only depends on R. Moreover since V_ε is bounded in $L^p(Y)$ by (4.3), we have

$$\int_{Y\setminus Q_R} V_\varepsilon = \int_Y V_\varepsilon - \int_{Q_R} V_\varepsilon = -\int_{Q_R} V_\varepsilon = o_{R,\varepsilon}(1).$$

Both previous estimates imply that

$$\left| \fint_{Q_\varepsilon} V_\varepsilon \right| - c_R\, \hat{\delta}_R(\varepsilon)^{\frac{1}{p}} + o_{R,\varepsilon}(1) \leqq \hat{\delta}_R(\varepsilon)^{-1} \left| \int_Y \beta_\varepsilon\, |\nabla \hat{V}_\varepsilon^R|^{p-2} \nabla \hat{V}_\varepsilon^R \cdot \nabla V_\varepsilon \right|,$$

whence, by the Hölder inequality combined with the values (4.7) of $\hat{\delta}_R(\varepsilon)$ and (4.4) of $C(\varepsilon)$,

$$\left| \fint_{Q_\varepsilon} V_\varepsilon \right| - c_R\, \hat{\delta}_R(\varepsilon)^{\frac{1}{p}} + o_{R,\varepsilon}(1) \leqq \left(\hat{\delta}_R(\varepsilon)\, C(\varepsilon) \right)^{-\frac{1}{p}}.$$

Or equivalently, by using assumption (2.26),

$$\left(\frac{\hat{\delta}_R(\varepsilon)}{\hat{\delta}(\varepsilon)}\right)^{\frac{1}{p}} \left[(\gamma_\varepsilon\,|Q_\varepsilon|)^{\frac{1}{p}} \left| \fint_{Q_\varepsilon} V_\varepsilon \right| - c_R\,\hat{\delta}_R(\varepsilon)^{\frac{1}{p}} + o_{R,\varepsilon}(1) \right] \leqq \left(\frac{\hat{\delta}(\varepsilon)\,C(\varepsilon)}{\gamma_\varepsilon\,|Q_\varepsilon|}\right)^{-\frac{1}{p}}. \tag{4.11}$$

It thus remains to obtain an estimate of the averaged value $\fint_{Q_\varepsilon} V_\varepsilon$. For that we will use the following result.

Lemma 4.2. *The following Poincaré-Wirtinger inequality holds in Q_ε:*

$$\forall\, V \in W^{1,p}(Q_\varepsilon), \quad \int_{Q_\varepsilon} \left| V - \fint_{Q_\varepsilon} V \right|^p \leqq c \int_{Q_\varepsilon} |\nabla V|^p, \tag{4.12}$$

where c is a positive constant independent of any small enough ε.

On the one hand, putting the function V_ε in equality (4.12) yields by (4.4)

$$\fint_{Q_\varepsilon} \left| V_\varepsilon - \fint_{Q_\varepsilon} V_\varepsilon \right|^p \leqq c\,(\gamma_\varepsilon\,|Q_\varepsilon|)^{-1} \int_{Q_\varepsilon} \beta_\varepsilon\,|\nabla V_\varepsilon|^p \leqq c\,(\gamma_\varepsilon\,|Q_\varepsilon|\,C(\varepsilon))^{-1},$$

whence by limit (2.26) combined with $C(\varepsilon) \to +\infty$

$$\fint_{Q_\varepsilon} \left| V_\varepsilon - \fint_{Q_\varepsilon} V_\varepsilon \right|^p \underset{\varepsilon\to 0}{\longrightarrow} 0. \tag{4.13}$$

On the other hand, by using the inequality

$$\forall\, a, b \in \mathbb{R}, \quad \big||a|^p - |b|^p\big| \leqq p\,|a-b|\,\left(|a|^{p-1} + |b|^{p-1}\right)$$

and the Hölder inequality, we have

$$\begin{aligned}
\left|\left|\fint_{Q_\varepsilon} V_\varepsilon\right|^p - \fint_{Q_\varepsilon} |V_\varepsilon|^p\right| &\leqq p \fint_{Q_\varepsilon} \left|V_\varepsilon - \fint_{Q_\varepsilon} V_\varepsilon\right| \left(|V_\varepsilon|^{p-1} + \left|\fint_{Q_\varepsilon} V_\varepsilon\right|^{p-1}\right) \\
&\leqq p \fint_{Q_\varepsilon} \left|V_\varepsilon - \fint_{Q_\varepsilon} V_\varepsilon\right| \left(|V_\varepsilon|^{p-1} + \left|\fint_{Q_\varepsilon} |V_\varepsilon|^p\right|^{\frac{1}{p'}}\right) \\
&\leqq c_p \left(\fint_{Q_\varepsilon} \left|V_\varepsilon - \fint_{Q_\varepsilon} V_\varepsilon\right|^p\right)^{\frac{1}{p}} \left(\fint_{Q_\varepsilon} |V_\varepsilon|^p\right)^{\frac{1}{p'}}.
\end{aligned}$$

The previous inequality combined with (4.3), (2.26) and (4.13) yields

$$\left|\fint_{Q_\varepsilon} V_\varepsilon\right|^p - \fint_{Q_\varepsilon} |V_\varepsilon|^p \underset{\varepsilon\to 0}{\longrightarrow} 0. \tag{4.14}$$

Moreover, by (4.4) ∇V_ε strongly converges to 0 in $L^p(Y;\mathbb{R}^3)$. The Poincaré-Wirtinger inequality in Y thus implies that V_ε strongly converges to 0 in $L^p(Y)$ since V_ε has a zero averaged value on Y. Then since

$$1 = \int_Y \beta_\varepsilon\,|V_\varepsilon|^p = \int_{Y\setminus Q_\varepsilon} |V_\varepsilon|^p + \gamma_\varepsilon\,|Q_\varepsilon| \fint_{Q_\varepsilon} |V_\varepsilon|^p,$$

we deduce from the strong convergence of V_ε the asymptotic behaviour

$$\fint_{Q_\varepsilon} |V_\varepsilon|^p \underset{\varepsilon\to 0}{\sim} \frac{1}{\gamma_\varepsilon\,|Q_\varepsilon|} \underset{\varepsilon\to 0}{\longrightarrow} k^{-1} > 0,$$

which combined with limit (4.14) yields

$$\left|\fint_{Q_\varepsilon} V_\varepsilon\right| \underset{\varepsilon\to 0}{\sim} (\gamma_\varepsilon\,|Q_\varepsilon|)^{-\frac{1}{p}}. \tag{4.15}$$

Finally, since by (4.9) $\hat{\delta}_R(\varepsilon) \to 0$ as $\varepsilon \to 0$, and thanks to the asymptotic behaviours (4.9), (4.15), passing to the liminf (for any subsequence of ε) in inequality (4.11) for a fixed small R, implies that

$$1 + o_R(1) \leqq \liminf_{\varepsilon\to 0} \left(\frac{\hat{\delta}(\varepsilon)C(\varepsilon)}{\gamma_\varepsilon |Q_\varepsilon|}\right)^{-\frac{1}{p}},$$

where $o_R(1)$ tends to 0 as $R \to 0$, or equivalently

$$\limsup_{\varepsilon\to 0} \left(\frac{\hat{\delta}(\varepsilon)C(\varepsilon)}{\gamma_\varepsilon |Q_\varepsilon|}\right) \leqq (1 + o_R(1))^{-p}. \tag{4.16}$$

From inequalities (4.10) and (4.16) we easily deduce that

$$C(\varepsilon) \underset{\varepsilon\to 0}{\sim} \frac{\pi\gamma_\varepsilon r_\varepsilon^2}{\hat{\delta}(\varepsilon)},$$

which combined with the definition (4.9) of $\hat{\delta}(\varepsilon)$ yields (2.27) and (2.28).

4.2. The case $p > 2$

By taking any non-zero test function V in the supremum (4.1), we find that $C(\varepsilon)$ is bounded from below by a positive constant. In order to prove that $C(\varepsilon)$ is bounded from above, we proceed by contradiction.

Assume that $C(\varepsilon) \to +\infty$. By definition (4.7) it is easy to check that there exists, for any $R < \frac{1}{2}$, a constant $c_R \geqq 1$ such that

$$c_R^{-1} \leqq \hat{\delta}_R(\varepsilon) \leqq c_R.$$

Then by the Hölder inequality, the definitions (4.7) of $\hat{\delta}_R(\varepsilon)$ and (4.4) of $C(\varepsilon)$ we have

$$\left|\int_Y \beta_\varepsilon\,|\nabla\hat{V}_\varepsilon^R|^{p-2}\nabla\hat{V}_\varepsilon^R \cdot \nabla V_\varepsilon\right| \leqq \hat{\delta}_R(\varepsilon)^{\frac{1}{p'}}\,C(\varepsilon)^{-\frac{1}{p}} \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Moreover by equality (4.4) and limit (2.26) we have

$$\|\nabla V_\varepsilon\|_{L^p(Y\setminus Q_R)} + (r_\varepsilon)^{-\frac{2}{p}}\,\|\nabla V_\varepsilon\|_{L^p(Q_\varepsilon)} = O\left(C(\varepsilon)^{-\frac{1}{p}}\right) \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Therefore passing to the limit in estimate (4.6), with $V := V_\varepsilon$ thanks to the previous limits, yields

$$\fint_{Y\setminus Q_R} V_\varepsilon - \fint_{Q_\varepsilon} V_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} 0. \tag{4.17}$$

On the other hand, as in the previous case the assumption $C(\varepsilon) \to +\infty$ implies that V_ε strongly converges to 0 in $L^p(Y)$ as well as the asymptotic behaviour (4.15). The strong convergence of V_ε combined with limit (4.17) then implies the limit

$$\fint_{Q_\varepsilon} V_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} 0$$

which contradicts (4.15) because of limit (2.26). Therefore $C(\varepsilon)$ is bounded.

4.3. Determination of a_0

We have to prove that the sequence A_ε^0 defined by (2.10) simply converges to the function a_0 defined by (2.29). For any $\lambda \in \mathbb{R}^3$, let $\hat{X}_\varepsilon(\cdot,\lambda)$ be the Y-periodic function defined by (see the definition (4.5) of $\hat{V}_\varepsilon^R$)

$$\hat{X}_\varepsilon(y,\lambda) := \left(1 - \hat{V}_\varepsilon^{2r_\varepsilon}(y)\right)(\lambda_1 y_1 + \lambda_2 y_2), \quad y \in Y,$$

and extended by periodicity in $\mathbb{R}^3$, and let $\hat{W}_\varepsilon(\cdot,\lambda)$ be the function defined by

$$\hat{W}_\varepsilon^\lambda(y) := \lambda\cdot y - \hat{X}_\varepsilon(y,\lambda), \quad y \in Y. \tag{4.18}$$

By the definition (4.5) of $\hat{V}_\varepsilon^R$ and the Cauchy-Schwarz inequality

$$|\lambda_1 y_1 + \lambda_2 y_2| \leqq |\lambda|\, r, \quad \text{where} \quad r^2 := y_1^2 + y_2^2,$$

we obtain

$$|\nabla \hat{W}_\varepsilon^\lambda| \leqq |\lambda| \left(2 + r\,|\nabla \hat{V}_\varepsilon^{2r_\varepsilon}(y)|\,\mathbf{1}_{Q_{2r_\varepsilon}\setminus Q_\varepsilon}(y)\right) \leqq 2\,|\lambda| \left(1 + r_\varepsilon\,|\nabla \hat{V}_\varepsilon^{2r_\varepsilon}(y)|\right),$$

whence the estimate

$$|\nabla \hat{W}_\varepsilon^\lambda| \leqq c\,|\lambda| \tag{4.19}$$

which holds for any value of $p > 1$.

Let us prove that the function $\hat{W}_\varepsilon^\lambda$ defined by (4.18) is a good approximation of the function $W_\varepsilon(\cdot,\lambda)$ defined by (2.9) in the following sense:

$$\forall\, V \in W_\#^{1,p}(Y), \quad \int_Y \beta_\varepsilon |\nabla \hat{W}_\varepsilon^\lambda|^{p-2}\nabla \hat{W}_\varepsilon^\lambda \cdot \nabla V = o(1)\|\nabla V\|_{L^p(Y)}. \tag{4.20}$$

Let $V \in W_\#^{1,p}(Y)$. By definitions (4.5) and (4.18) we have

$$\int_Y \beta_\varepsilon\,|\nabla \hat{W}_\varepsilon^\lambda|^{p-2}\nabla \hat{W}_\varepsilon^\lambda \cdot \nabla V = \int_{Y\setminus Q_{2r_\varepsilon}} |\lambda|^{p-2}\lambda\cdot\nabla V + \int_{Q_\varepsilon} \gamma_\varepsilon\,|\lambda_3|^{p-2}\lambda_3\,\frac{\partial V}{\partial y_3}$$
$$+ \int_{Q_{2r_\varepsilon}\setminus Q_\varepsilon} |\nabla \hat{W}_\varepsilon^\lambda|^{p-2}\nabla \hat{W}_\varepsilon^\lambda \cdot \nabla V$$

which, combined with the equalities

$$\int_Y \nabla V = 0 \quad \text{and} \quad \int_{Q_\varepsilon} \frac{\partial V}{\partial y_3} = 0$$

as well as estimate (4.19), yields the desired result (4.20).

By a similar computation and thanks to limit (2.26) we also obtain

$$\hat{F}_\varepsilon(\lambda) := \int_Y \beta_\varepsilon \, |\nabla \hat{W}_\varepsilon^\lambda|^p \underset{\varepsilon\to 0}{\longrightarrow} |\lambda|^p + k\,|\lambda_3|^p. \tag{4.21}$$

Moreover the definition (2.10) of A_ε^0 implies that

$$A_\varepsilon^0(\lambda)\cdot\lambda = \min_{V\in W_\#^{1,p}(Y)} \int_Y \beta_\varepsilon\,|\lambda - \nabla V|^p \leqq \hat{F}_\varepsilon(\lambda). \tag{4.22}$$

On the other hand the convexity of $|\cdot|^p$ yields

$$A_\varepsilon^0(\lambda)\cdot\lambda \geqq \hat{F}_\varepsilon(\lambda) + p\int_Y \beta_\varepsilon |\nabla \hat{W}_\varepsilon^\lambda|^{p-2}\nabla \hat{W}_\varepsilon^\lambda \cdot (\nabla W_\varepsilon(\cdot,\lambda) - \nabla \hat{W}_\varepsilon^\lambda). \tag{4.23}$$

From estimate (2.21) combined with the uniform ellipticity (2.3) of A_ε, we deduce that $\nabla W_\varepsilon(\cdot,\lambda)$ is bounded in $L^p(Y)$, as well as $\nabla \hat{W}_\varepsilon^\lambda$ by (4.19). Whence the integral term in (4.23) tends to zero when we put the periodic function $V := W_\varepsilon(\cdot,\lambda) - \hat{W}_\varepsilon^\lambda$ in estimate (4.20). Therefore inequalities (4.22), (4.23) and limit (4.21) imply that

$$A_\varepsilon^0(\lambda)\cdot\lambda \underset{\varepsilon\to 0}{\longrightarrow} |\lambda|^p + k\,|\lambda_3|^p. \tag{4.24}$$

Finally, again by the convexity of $|\cdot|^p$ we obtain the inequality

$$\forall\,\mu\in\mathbb{R}^3, \quad A_\varepsilon^0(\lambda+\mu)\cdot(\lambda+\mu) \geqq A_\varepsilon^0(\lambda)\cdot\lambda + p\,A_\varepsilon^0(\lambda)\cdot\mu.$$

This combined with (4.24) yields the desired limit (2.29), which concludes the proof of Proposition 2.4.

4.4. Proof of the lemmas

Proof of Lemma 4.1. The cases $p<2$, $p=2$ and $p>2$ are quite similar. Let us thus prove estimate (4.6) for the case $p<2$. Let $V\in C^1(\overline{Y})$. We denote by $\overline{V}^Z$ the averaged value of V on the subset Z of Y. Since $|\nabla \hat{V}_\varepsilon^R|^{p-2}\nabla \hat{V}_\varepsilon^R$ has a zero divergence in $Q_R\setminus Q_\varepsilon$ and $\hat{V}_\varepsilon^R$ is radial, an integration by parts yields

$$\begin{aligned}\int_Y |\nabla \hat{V}_\varepsilon^R|^{p-2}\nabla \hat{V}_\varepsilon^R\cdot\nabla\Big(V - \overline{V}^{Y\setminus Q_R}\,\hat{V}_\varepsilon^R - \overline{V}^{Q_\varepsilon}\,(1-\hat{V}_\varepsilon^R)\Big)\\ = \int_{\Gamma_R} |\nabla \hat{V}_\varepsilon^R|^{p-2}\nabla \hat{V}_\varepsilon^R\cdot\nu\Big(V - \overline{V}^{Y\setminus Q_R}\Big)\\ + \int_{\Gamma_\varepsilon} |\nabla \hat{V}_\varepsilon^R|^{p-2}\nabla \hat{V}_\varepsilon^R\cdot\nu\Big(V - \overline{V}^{Q_\varepsilon}\Big),\end{aligned} \tag{4.25}$$

where $\Gamma_R, \Gamma_\varepsilon$ denote the side boundaries of the cylinders Q_R, Q_ε and ν denotes the (radial) outer unit normal to $\Gamma_R, \Gamma_\varepsilon$. Therefore we have to prove that the right-hand side of equality (4.25) is bounded by the right-hand side of inequality (4.6) since the left-hand side of (4.25) is equal to the left-hand side of (4.6).

By the definition (4.5) of $\hat{V}_\varepsilon^R$ and by the imbedding from $W^{1,p}(Y)$ into $L^p(\Gamma_R)$ combined with the Poincaré-Wirtinger inequality in $Y \setminus Q_R$, we have

$$\begin{aligned}\left| \int_{\Gamma_R} |\nabla \hat{V}_\varepsilon^R|^{p-2} \nabla \hat{V}_\varepsilon^R \cdot \nu \left(V - \overline{V}^{Y\setminus Q_R} \right) \right| &\leqq c_R\, r_\varepsilon^{2-p}\, \|\nabla V\|_{L^p(Y\setminus Q_R)} \\ \text{by (4.9)} \quad &\leqq c_R\, \hat{\delta}_R(\varepsilon)\, \|\nabla V\|_{L^p(Y\setminus Q_R)}.\end{aligned} \tag{4.26}$$

Similarly for the integral on Γ_ε we have

$$\left| \int_{\Gamma_\varepsilon} |\nabla \hat{V}_\varepsilon^R|^{p-2} \nabla \hat{V}_\varepsilon^R \cdot \nu \left(V - \overline{V}^{Q_\varepsilon} \right) \right| \leqq c_R\, \hat{\delta}_R(\varepsilon) \int_P \left| V(r_\varepsilon, \theta, y_3) - \overline{V}^{Q_\varepsilon} \right|, \tag{4.27}$$

where the function V is expressed in polar coordinates

$$(r, \theta, y_3) \in P_\varepsilon :=]0, r_\varepsilon[\times P \quad \text{with} \quad P :=]0, 2\pi[\times]-\tfrac{1}{2}, \tfrac{1}{2}[.$$

To estimate the right-hand side of (4.27) we can assume that $\overline{V}^{Q_\varepsilon} = 0$. By integrating the equality

$$r\, V(r_\varepsilon, \theta, y_3) = r\, V(r, \theta, y_3) + \int_r^{r_\varepsilon} r\, \frac{\partial V}{\partial \rho}(\rho, \theta, y_3)\, d\rho$$

on the set P_ε, we obtain the inequality

$$\frac{1}{2} r_\varepsilon^2 \int_P |V(r_\varepsilon, \theta, y_3)| \leqq \int_{P_\varepsilon} r\, |V(r, \theta, y_3)| + \int_{P_\varepsilon} r \int_r^{r_\varepsilon} \left| \frac{\partial V}{\partial \rho}(\rho, \theta, y_3) \right| d\rho.$$

Then the Hölder inequality implies that

$$r_\varepsilon^2 \int_P |V(r_\varepsilon, \theta, y_3)| \leqq c\, r_\varepsilon^{\frac{2}{p'}} \left[\left(\int_{P_\varepsilon} r\, |V|^p \right)^{\frac{1}{p}} + \left(\int_{P_\varepsilon} r \left(\int_r^{r_\varepsilon} \left| \frac{\partial V}{\partial \rho} \right| d\rho \right)^p \right)^{\frac{1}{p}} \right]$$

and for any $r \leqq r_\varepsilon$,

$$\begin{aligned}\left(\int_r^{r_\varepsilon} \left| \frac{\partial V}{\partial \rho} \right| d\rho \right)^p &\leqq \left(\int_r^{r_\varepsilon} \rho^{-\frac{p'}{p}}\, d\rho \right)^{\frac{p}{p'}} \int_0^{r_\varepsilon} \rho \left| \frac{\partial V}{\partial \rho} \right|^p \\ &\leqq c\, r^{p-2} \int_0^{r_\varepsilon} \rho \left| \frac{\partial V}{\partial \rho} \right|^p \quad \text{since } p < 2,\end{aligned}$$

whence the estimate

$$r_\varepsilon^2 \int_P |V(r_\varepsilon, \theta, y_3)| \leqq c\, r_\varepsilon^{\frac{2}{p'}} \left(\|V\|_{L^p(Q_\varepsilon)} + r_\varepsilon\, \|\nabla V\|_{L^p(Q_\varepsilon)} \right).$$

Therefore the previous inequality combined with the inequality (4.12) of Lemma 4.2 implies that

$$\int_P |V(r_\varepsilon, \theta, y_3)| \leqq c\, r_\varepsilon^{-\frac{2}{p}}\, \|\nabla V\|_{L^p(Q_\varepsilon)},$$

for any function V with zero averaged value on Q_ε. Finally, by combining estimates (4.26), (4.27) and the previous one, we obtain the desired result (4.6). $\square$

Proof of Lemma 4.2. Let $V \in C^1(\overline{Y})$ with zero averaged value on Q_ε and let W be the function defined by

$$W(y_1, y_2) := \int_{-\frac{1}{2}}^{\frac{1}{2}} V(y_1, y_2, y_3)\, dy_3.$$

It follows immediately that

$$\|V - W\|_{L^p(Q_\varepsilon)} \leqq \left\| \frac{\partial V}{\partial y_3} \right\|_{L^p(Q_\varepsilon)}.$$

On the other side W is defined on the disk D_ε of centre O and radius r_ε and has a zero averaged value on D_ε. Then by r_ε-rescaling the Poincaré-Wirtinger inequality in the disk of radius 1 we obtain the inequality

$$\|W\|_{L^p(D_\varepsilon)} \leqq c\, r_\varepsilon\, \|\nabla W\|_{L^p(D_\varepsilon)} \leqq c\, r_\varepsilon\, \|\nabla V\|_{L^p(Q_\varepsilon)},$$

where c is a positive constant. Finally both previous estimates imply (4.12). $\square$

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