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A Geometrical Approach to the Study of Unbounded Solutions of Quasilinear Parabolic Equations

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Abstract

In this article, we are interested in the existence and uniqueness of solutions for quasilinear parabolic equations set in the whole space $\mathbb{R}^N$. We consider, in particular, cases when there is no restriction on the growth or the behavior of these solutions at infinity. Our model equation is the mean-curvature equation for graphs for which Ecker and Huisken have shown the existence of smooth solutions for *any* locally Lipschitz continuous initial data. We use a geometrical approach which consists in seeing the evolution of the graph of a solution as a geometric motion which is then studied by the so-called "level-set approach". After determining the right class of quasilinear parabolic PDEs which can be taken into account by this approach, we show how the uniqueness for the original PDE is related to "fattening phenomena" in the level-set approach. Existence of solutions is proved using a local L^∞ bound obtained by using in an essential way the level-set approach. Finally we apply these results to convex initial data and prove existence and comparison results in full generality, i.e., without restriction on their growth at infinity.

1. Introduction

In a series of works (see [7] for an introductory paper, and [8, 6, 9]), we are investigating quasilinear parabolic equations set in the whole space $\mathbb{R}^N$ and, more precisely, existence and uniqueness properties for solutions with general growth at infinity.

This paper is the starting point, and our main motivation comes from a result of Ecker & Huisken [16] for the so-called mean-curvature equation for graphs

$$\frac{\partial u}{\partial t} - \Delta u + \frac{\langle D^2 u Du, Du\rangle}{1+|Du|^2} = 0 \quad \text{in } \mathbb{R}^N \times (0,\infty), \tag{1.1}$$

with the initial data

$$u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

where $u : \mathbb{R}^N \times [0,\infty) \to \mathbb{R}$ is the solution, Du and D^2u denote respectively the gradient and the Hessian matrix of u with respect to the space variable, $u_0 : \mathbb{R}^N \to \mathbb{R}$ is a given function and $|\cdot|$ (respectively $\langle \cdot\,,\cdot\rangle$) stands for the classical Euclidean norm (respectively inner product) in $\mathbb{R}^N$.

Ecker and Huisken proved the following very surprising result: for any initial data $u_0 \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N)$, there exists a solution u of (1.1), (1.2) in $C^\infty(\mathbb{R}^N \times (0,\infty)) \cap C(\mathbb{R}^N \times [0,\infty))$. This result was even extended to initial data in $C(\mathbb{R}^N)$ by ANGENENT [1]. The intriguing point is that no assumption is made on the growth of u_0 at infinity and therefore the solution u can have also an arbitrary behavior at infinity.

This result brings up a lot of challenging questions: the first one concerns the uniqueness of the solution they build. In general, the difficulty in obtaining a uniqueness result for a PDE comes from the fact that a notion of weak solution is used: this is not at all the case here since the solutions are known to be regular, even C^∞. The real difficulty lies in taking into account any behavior for the solution at infinity.

A second question is related to the existence result itself: Ecker and Huisken proved it by using differential geometry and the maximum principle and it would be interesting to have a purely analytical proof of it. Again the lack of prescribed behavior of the solutions at infinity creates an unusual difficulty. In particular, to get a local L^∞ bound on u is *a priori* a key point, but to obtain a local L^∞ bound on Du is also a rather difficult task.

Finally, we may wonder to which type of quasilinear parabolic equations the result of Ecker and Huisken can be extended. For the reader, this may seem to be a question to be investigated later but, in fact, in order to provide interesting results for (1.1), it is necessary to understand the main underlying structure of the equation which allows such a strange result to hold.

Our answer to this question is the geometrical interpretation of (1.1) by motion by mean curvature for graphs. Motions of hypersurfaces with general curvature-dependent velocities were studied recently by the so-called "level-set approach," a weak notion for the evolution which allows us to define these motions past the development of singularities. The level-set approach was first introduced by OSHER & SETHIAN [31] for numerical computations and then studied from a theoretical point of view by EVANS & SPRUCK [18] in the case of motion by mean curvature and by CHEN, GIGA & GOTO [13] for more general normal velocities. Later, more singular cases were investigated by ISHII [27], ISHII & SOUGANIDIS [28] and properties of the level-set approach were obtained by BARLES, SONER & SOUGANIDIS [10].

In the case of equation (1.1), as for any suitable quasilinear parabolic equations, the level-set approach arises when we consider the motion in dimension $N+1$. To do so, we have to introduce the function $v : \mathbb{R}^{N+1} \times [0,+\infty) \to \mathbb{R}$ defined by

$$v(x,y,t) = y - u(x,t).$$

For (1.1), the function v is a solution of

$$\frac{\partial v}{\partial t} - \Delta v + \frac{\langle D^2 v Dv, Dv\rangle}{|Dv|^2} = 0 \quad \text{in } \mathbb{R}^{N+1} \times (0,\infty), \tag{1.3}$$

which is the equation in the level-set approach corresponding to motion by mean curvature.

In order to give a suitable sense of solution for this singular equation and related ones in non-divergence form, we use the notion of viscosity solutions: we refer the reader to the User's guide of CRANDALL, ISHII & LIONS [15] or the books of FLEMING & SONER [20], BARDI & CAPUZZO DOLCETTA [2], BARLES [5] or BARDI *et al.* [3] for an introduction and/or a detailed presentation of this notion of solutions.

The most classical result concerning (1.3) is the well-posedness in the space of bounded uniformly continuous functions (BUC in short), more precisely: for any $v_0 \in BUC(\mathbb{R}^{N+1})$, there exists a unique solution v of (1.3) in $BUC(\mathbb{R}^{N+1} \times [0,T])$ for all $T > 0$ such that

$$w(x, y, 0) = v_0(x, y) \quad \text{in } \mathbb{R}^{N+1}.$$

At this point, it is worth remarking that boundedness is not an issue: indeed, one of the key property of (1.3) is to be invariant by every nondecreasing change of functions: if v is a solution of (1.3), then $\tanh(v)$, or more generally $\Psi(v)$ with $\Psi' > 0$, is a solution as well.

Therefore it can be thought that the study of (1.1), (1.2) just reduces to the study of (1.3) through the changes $v(x, y, t) = \tanh(y-u(x,t))$ and $v_0(x, y) = \tanh(y-u_0(x))$ and that all results follow easily from an extension of the above-mentioned well-posedness result to spaces of bounded continuous functions (denoted by C_b), v and v_0 being clearly in C_b but not in BUC in general. In particular, the uniqueness of a solution u of (1.1), (1.2) is an immediate consequence of a uniqueness result for solutions of (1.3) in C_b.

Unfortunately, we are unable to prove that the problem is well posed in C_b and even the extensions to the well-posedness in BUC are rather weak. The concrete consequences of this geometrical approach are, on the one hand, a local L^∞ bound for a large class of quasilinear parabolic PDEs whose proof is rather simple and natural and, on the other hand, a "generic" uniqueness result for the solutions of (1.1), (1.2) as well as for more general equations.

It is worth pointing out that the possible non-uniqueness feature for (1.1),(1.2) is related to the so-called "fattening phenomena" or "nonempty interior difficulty" for (1.3); despite the fact that it seems obvious that no interior can develop because, by the maximum principle, we have formally

$$\frac{\partial v}{\partial y}(x, y, 0) > 0 \;\text{ in } \mathbb{R}^{N+1} \Longrightarrow \frac{\partial v}{\partial y}(x, y, t) > 0 \;\text{ in } \mathbb{R}^{N+1} \times (0, +\infty),$$

but we are unable to prove this property even in a weaker sense.

Now, we turn to a more precise description of the contents of the present paper. It is devoted to the study of the geometrical approach, explained above in the special case of the mean-curvature equation, for more general PDEs like

$$\begin{aligned} \frac{\partial u}{\partial t} - \mathrm{Tr}\left[b(Du)D^2u\right] = 0 \quad \text{in } \mathbb{R}^N \times (0,\infty), \\ u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^N, \end{aligned} \tag{1.4}$$

where b is a continuous function from $\mathbb{R}^N$ into the space of the nonnegative symmetric matrices $\mathcal{S}_N^+$ and the initial data u_0 is any continuous function in $\mathbb{R}^N$. The first question we address is: when is (1.4) associated with a geometric PDE in dimension $N+1$ to which the level-set approach applies?

In Section 2, we derive formally a geometrical equation from (1.4) (see (2.1)). We then study this equation, distinguishing two cases: the "classical" one, to which the classical level-set approach applies, and the "very singular" one, for which the more sophisticated arguments of ISHII [27] are necessary. For the reader's convenience, we show how the classical results apply in the "classical" case (Section 3). In the "very singular case" (Section 4), our comparison result enters the framework of [27]. Nevertheless, the particular singular set we deal with allows a more elementary proof. It has, in particular, the advantage of using explicit test-functions and therefore allows the proof to be extended to the case when the function b in (1.4) depends on (x, t, Du). Here, to avoid to many technicalities, we restrict ourselves to b depending only on Du and address this more general case in the forthcoming paper [9].

Then we study the consequences of this geometrical approach for (1.4). At first, we prove in Section 5 that the level-set approach works. A local L^∞ bound for the solutions of (1.4) follows rather easily (see Section 7) and the existence of discontinuous viscosity solutions of (1.4) is an almost immediate consequence of it (see Theorem 8.1 in Section 8). The existence of smooth solutions requires a local gradient bound; we use the one of EVANS & SPRUCK [19] for (1.1) and the ones of CHOU & KWONG [14] for the more general equation (1.4) (see Section 9).

Uniqueness is an even more difficult issue and we were able to obtain it only in particular cases: of course the first "generic" uniqueness result we provide in Section 6 is not satisfactory and most of the results we obtain in this direction are proved by working directly on (1.4). However a striking application of the geometrical approach to uniqueness (and also existence) is the case of convex initial data (Section 10): under suitable assumptions on b (the same as those required for the geometrical approach to hold), we prove that there exists a unique solution u of (1.4), which is convex in the space variable at each time, and this for any convex initial data u_0 without any restriction on its growth at infinity. The proof relies strongly on the convexity-preserving property of GIGA, GOTO, ISHII & SATO [21], which we extend to our more singular case. Compared to the result we previously obtained in [7] by working directly on (1.1), we no longer assume u_0 to be coercive and we extend the result to equations like (1.4).

For completeness, we conclude this introduction by describing the results we obtain in the next two parts of this study. Two types of uniqueness results for the non-convex case are proved: the first ones [8] concern the case $N = 1$. We show, not only for (1.1) but for a larger class of equations, a uniqueness result without any growth assumptions on the solutions. Unfortunately, in general, this result is valid only in the class of classical solutions; however, in the case of (1.1), using the argument of Section 9 and in particular Remark 9.1, this uniqueness result for smooth solutions implies a comparison result between possibly discontinuous sub- and supersolutions.

The proof relies upon examining the PDE obtained by integrating in x. For (1.1), this PDE reads

$$w_t - \arctan(w_{xx}) = 0 \quad \text{in } \mathbb{R} \times (0, +\infty), \tag{1.5}$$

and the key point is that (1.5) enjoys uniqueness properties in $C(\mathbb{R})$, essentially because it is possible to use a "friendly giants" method, whose consequence is a general uniqueness property for (1.1). Of course, this method can be extended to far more general equations. We learned recently that related results were obtained independently and by rather different methods by Chou & Kwong [14].

In the second one [6], we use classical viscosity solutions arguments to prove the uniqueness for solutions of (1.4) and even more general equations: we obtain a comparison result for sub- and supersolutions with polynomial growth but, unfortunately, with a rather restrictive assumption on the initial data which reads, in the locally Lipschitz continuous case,

$$|Du_0(x)| \leqq C(1 + |x|^\nu) \quad \text{in } \mathbb{R}^N$$

for some constant $C > 0$ and $0 \leqq \nu < (1 + \sqrt{5})/2$. A strange feature of this result is that it can be obtained either by working directly on (1.4) or on the associated geometrical PDE and both proofs lead to the same condition on u_0. As we mentioned it above, in a forthcoming paper, we investigate more general equations, namely

$$\frac{\partial u}{\partial t} - \mathrm{Tr}\left[b(x, t, Du)D^2u\right] + H(x, t, Du) = 0 \quad \text{in } \mathbb{R}^N \times (0, \infty).$$

After we obtained most of the results described above, we learned that representation formulas for the mean-curvature equation (1.3) (and even more general geometrical equations) have been established independently by Soner & Touzi [33,34] and by Buckdahn, Cardaliaguet & Quincampoix [12]. We tried to prove uniqueness for (1.1) by showing that the "non-fattening phenomena" cannot occur for (1.3) in the case of graphs, but we failed. It is an intriguing question whether it is possible to prove such properties by using these formulas.

2. Derivation of a geometrical PDE

As explained in the introduction for the special case of equation (1.1), we associate a geometrical equation with the quasilinear equation (1.4) which allows us to use the level-set approach. This method has already been used by Evans [17] for the heat equation and by Giga & Sato [22] in the case of Hamilton-Jacobi equations.

When u is a solution of (1.4) we consider, for every $t \geqq 0$, $\mathrm{Graph}(u(\cdot\,,t))$ as an hypersurface in $\mathbb{R}^{N+1}$ and, to represent it, we follow the ideas of the level set approach, taking any function $v : \mathbb{R}^N \times \mathbb{R} \times [0,+\infty) \to \mathbb{R}$ such that

$$v(x,u(x,t),t) = 0 \quad \text{for every } (x,t) \in \mathbb{R}^N \times (0,+\infty).$$

Note that, for all $t \geqq 0$, $\mathrm{Graph}(u(\cdot\,,t)) \subset \Gamma_t$, where Γ_t is the 0-level set of $v(\cdot\,,\cdot\,,t)$. Differentiating formally the previous inequality, we obtain

$$\begin{aligned} D_y v\,\frac{\partial u}{\partial t} + \frac{\partial v}{\partial t} &= 0,\\ D_x v + D_y v\, Du &= 0,\\ D^2_{xx} v + 2D^2_{xy} v \otimes Du + D^2_{yy} v Du \otimes Du + D_y v D^2 u &= 0, \end{aligned}$$

and it follows that v has to solve, at least formally

$$\frac{\partial v}{\partial t} - \mathrm{Tr}\left[b\left(-\frac{D_x v}{D_y v}\right)\left(D^2_{xx} v - 2D^2_{xy} v \otimes \frac{D_x v}{D_y v} + D^2_{yy} v \frac{D_x v}{D_y v} \otimes \frac{D_x v}{D_y v}\right)\right] = 0 \tag{2.1}$$

in $\mathbb{R}^{N+1} \times (0,+\infty)$.

This new equation has strong discontinuities when the gradient of the solution, $Dv = (D_x v, D_y v)$, lies in the subset

$$\mathcal{D} = \{p = (p_1, \cdots, p_{N+1}) \in \mathbb{R}^{N+1} : p_{N+1} = 0\}, \tag{2.2}$$

but (2.1) satisfies the following first properties which is a motivation to study (1.4) via the level-set approach.

Lemma 2.1. *We have*

(i) *Equation* (2.1) *is degenerate parabolic outside* $\mathcal{D}$.
(ii) *If* $u \in C(\mathbb{R}^N \times [0,+\infty))$ *is a viscosity subsolution* (*a supersolution*) *of* (1.4) *with initial data* $u_0 \in C(\mathbb{R}^N)$, *then the function* $v(x,y,t) = y - u(x,t)$ *defined for* $(x,y,t) \in \mathbb{R}^N \times \mathbb{R} \times [0,+\infty)$ *is a viscosity supersolution* (*respectively subsolution*) *of* (2.1) *with initial data* $v_0(x,y) = y - u_0(x)$.
(iii) *Equation* (2.1) *is invariant under every monotone change of function* $v \to \Psi \circ v$, *where* $\Psi \in C(\mathbb{R})$ *is a monotone function.*

We skip the proofs of these three properties since they do not present any difficulty. Let us mention that (ii) and (iii) are obvious in the smooth case. Property (ii) is straightforward using the definition of viscosity solutions for the singular equation (2.1) we recall in Section 4.1. For (iii), we even prove a discontinuous version of it in Lemma 4.1. Finally it is worth pointing out that we choose to work with $v(x,y,t) = y - u(x,t)$ instead of $u(x,t) - y$ as usual.

Remark 2.1. Concerning (ii), we wonder whether some kind of converse property is true: if v is a solution of (2.1) with initial data $v_0(x,y) = y - u_0(x)$, does there exist a solution u of (1.4) such that $v(x,y,t) = y - u(x,t)$? The answer is not clear and it is the main issue of our approach. We refer to Section 6 for related discussions and results.

We conclude this section by introducing some notation which is used throughout the paper. Every point z of $\mathbb{R}^{N+1}$ is written $z = (x, y)$ with $x \in \mathbb{R}^N$ and $y \in \mathbb{R}$. In a natural way, every vector p which has the meaning of a gradient is written $p = (p_x, p_y)$ with $p_x \in \mathbb{R}^N$ and $p_y = p_{N+1} \in \mathbb{R}$. We decompose every matrix $X \in \mathcal{S}_{N+1}$ into blocks in the following way

$$X = \left(\begin{array}{c|c} X_{xx} & X_{xy} \\ \hline X_{xy}^T & X_{yy} \end{array}\right),$$

where $X_{xx} \in \mathcal{S}_N$, $X_{xy} \in \mathbb{R}^N$, X_{xy}^T is the transpose of X_{xy} (i.e., the row vector whose coordinates are those of X_{xy}) and $X_{yy} \in \mathbb{R}$. With this notation, the nonlinearity involved in (2.1) can be written, for every $p \in \mathbb{R}^{N+1} - \mathcal{D}$ and $X \in \mathcal{S}_{N+1}$,

$$F(p, X) = -\mathrm{Tr}\left[b(q)\left(X_{xx} + 2X_{xy} \otimes q + X_{yy} q \otimes q\right)\right] = -\mathrm{Tr}[\tilde{b}(p)\, X], \quad (2.3)$$

where $q = -p_x/p_y$ and

$$\tilde{b}(p) = \left(\begin{array}{c|c} b(q) & b(q)q \\ \hline (b(q)q)^T & \langle b(q)q, q\rangle \end{array}\right).$$

3. The geometrical equation: the classical framework

A priori the nonlinearity F is discontinuous on $\mathcal{D}$ (see (2.2)). In this section, we provide assumptions on b ensuring that we are in the "classical framework", which means that the (classical) level-set approach applies readily to (2.1) (see [18, 13, 21, 10]). In this classical framework, F has to be continuous, except at $p = 0$. The typical example is the mean-curvature equation (see Example 3.1).

More precisely, we start by recalling the assumptions as they appear in [21]. In what follows, $\|\cdot\|$ is any norm on $\mathcal{S}_N$ and $S^{N-1} = \{\xi \in \mathbb{R}^N : |\xi| = 1\}$ is the unit sphere of $\mathbb{R}^N$.

Those assumptions are:

(F1) $F : (\mathbb{R}^{N+1} - \{0\}) \times \mathcal{S}_{N+1} \to \mathbb{R}$ is continuous;
(F2) $F(p, X + Y) \leqq F(p, X)$ for all $p \in \mathbb{R}^{N+1}$, $X, Y \in \mathcal{S}_{N+1}$, $Y \geqq 0$;
(F3) $-\infty < F_*(0, 0) = F^*(0, 0) < +\infty$ where F_* and F^* are the semicontinuous envelopes of F defined by $F_*(p, X) = \liminf_{(\rho,Y)\to(p,X)}\{F(\rho, Y) : \rho \neq 0\}$ and $F^* = -(-F)_*$;
(F4) for every $R > 0$, $\sup\{|F(p, X)| : |p| \leqq R, \|X\| \leqq R\} < +\infty$.

We have, the following classical result.

Theorem 3.1. *Under assumptions (F1)–(F4), for any initial data $v_0 \in UC(\mathbb{R}^{N+1})$, there exists a unique solution v of* (2.1) *which is in $UC(\mathbb{R}^{N+1} \times [0, T))$ for every $T > 0$.*

Notice that, if F is continuous, then (F1)–(F4) reduce to (F2) only.

We state now the assumptions on b which permit us to extend the F given by (2.3) by continuity in $(\mathbb{R}^{N+1} - \{0\}) \times \mathcal{S}_{N+1}$ in order to ensure that (F1)–(F4) hold.

(H1) There is a positive constant K_1 such that $\|b(q)\| \leqq K_1$ for all $q \in \mathbb{R}^N$.
(H2) There is a positive constant K_2 such that $|b(q)q| \leqq K_2$ for every $q \in \mathbb{R}^N$.
(H3) There is a positive constant K_3 such that $|\langle b(q)q, q\rangle| \leqq K_3$ for every $q \in \mathbb{R}^N$.
(H4) For every $q \in S^{N-1}$, $\lim_{\lambda\to+\infty} b(\lambda q)$ and $\lim\limits_{\lambda\to-\infty} b(\lambda q)$ exist and are equal. Moreover $b_\infty(q) := \lim_{\lambda\to\pm\infty} b(\lambda q)$ is continuous on S^{N-1}.
(H5) For every $q \in S^{N-1}$, $\lim_{\lambda\to+\infty} \lambda b(\lambda q)q$ and $\lim_{\lambda\to-\infty} \lambda b(\lambda q)q$ exist and are equal. Moreover $\zeta_\infty(q) := \lim_{\lambda\to\pm\infty} \lambda b(\lambda q)q$ is continuous on S^{N-1}.
(H6) For every $q \in S^{N-1}$, $\lim_{\lambda\to+\infty} \lambda^2\langle b(\lambda q)q, q\rangle$ and $\lim_{\lambda\to-\infty} \lambda^2\langle b(\lambda q)q, q\rangle$ exist and are equal. Moreover the function $\alpha_\infty(q) := \lim_{\lambda\to\pm\infty} \lambda^2\langle b(\lambda q)q, q\rangle$ is continuous on S^{N-1}.

Proposition 3.1. *Let F be defined by* (2.3) *with $b \in C(\mathbb{R}^N; \mathcal{S}_N^+)$. Then assumptions* (*H1*)–(*H6*) *are equivalent to assumptions* (*F1*)–(*F4*). *It follows that, under assumptions* (*H1*)–(*H6*), *Theorem 3.1 hold.*

Proof of Proposition 3.1. We use the notation of Section 2. We start by assuming (H1)–(H6). For every $((p_x, 0), X) \in (\mathcal{D} - \{0\}) \times \mathcal{S}_{N+1}$, we extend F by setting

$$\begin{aligned} &F((p_x, 0), X) \\ &\quad = -\mathrm{Tr}\left[b_\infty\left(\frac{p_x}{|p_x|}\right)X_{xx}\right] + 2\langle\zeta_\infty\left(\frac{p_x}{|p_x|}\right), X_{xy}\rangle + X_{yy}\alpha_\infty\left(\frac{p_x}{|p_x|}\right). \end{aligned}$$

From the assumed continuity of b_∞, ζ_∞ and α_∞ on S^{N-1}, the extended Hamiltonian F is clearly continuous in $\mathbb{R}^{N+1} - \{0\}$; thus (F1) holds. Assumption (F2) is an immediate consequence of Lemma 2.1 (i). Finally, from the boundedness conditions (H1), (H2) and (H3), it is obvious that $F_*(0,0) = F^*(0,0) = 0$ and $|F(p, X)| \leqq K_1R + 2K_2R + K_3R$ for $|p|$, $\|X\| \leqq R$. It shows that (F3) and (F4) hold.

Conversely, suppose that (F1)–(F4) hold. Assumption (F2) implies easily that $b(\xi)$ is positive for all $\xi \in \mathbb{R}^N$. Let $\xi \in S^{N-1}$; for any $\lambda \neq 0$, we have

$$\tilde{b}(q, 1/\lambda) = \left(\begin{array}{c|c} b(\lambda q) & \lambda b(\lambda q)q \\ \hline (\lambda b(\lambda)q)^T & \lambda^2\langle b(\lambda q)q, q\rangle \end{array}\right). \tag{3.1}$$

From (F4), we know that $\tilde{b}(q, 1/\lambda)$ is bounded for every $q \in S^{N-1}$ and $\lambda \geqq 1$. It follows that $\|b(\xi)\|$, $|b(\xi)\xi|$ and $|\langle b(\xi)\xi, \xi\rangle|$ are bounded for every $\xi \in \{\lambda q : \lambda \neq 0, q \in S^{N-1}\} = \mathbb{R}^N - \overline{B}(0, 1)$. Since these quantities are obviously bounded in $\overline{B}(0, 1)$, we get (H1)–(H3). From (F1), we know that $\tilde{b}$ is continuous in $\mathbb{R}^{N+1} - \{0\}$. On the one hand, it follows easily that b is continuous in $\mathbb{R}^N$. On the other hand, sending λ to $\pm\infty$ in (3.1), we see that b_∞, ζ_∞ and α_∞ are well defined:

$$\lim_{\lambda\to\pm\infty} \tilde{b}(q, 1/\lambda) = \tilde{b}(q, 0) = \left(\begin{array}{c|c} b_\infty(q) & \zeta_\infty(q) \\ \hline (\zeta_\infty(q))^T & \alpha_\infty(q)\rangle \end{array}\right). \tag{3.2}$$

Invoking again the continuity of $\tilde{b}$, (3.2) implies that b_∞, ζ_∞ and α_∞ are continuous on S^{N-1}; thus (H4)–(H6) hold. This ends the proof. $\square$

Proposition 3.1 applies of course in the case of the mean-curvature equation. The computations are developed in the following example.

Example 3.1. Mean curvature equation (1.1).

In this case,

$$b(q) = I - \frac{q \otimes q}{1 + |q|^2} \quad \text{for every } q \in \mathbb{R}^N. \tag{3.3}$$

Easy computations show that

$$F(p, X) = -\mathrm{Tr}\left[\left(I - \frac{p \otimes p}{|p|^2}\right) X\right] \quad \text{for every } (p, X) \in (\mathbb{R}^{N+1} - \{0\}) \times \mathcal{S}_{N+1}$$

and then, (2.1) associated with (3.3) is the classical geometric mean-curvature equation.

The checking of (H1)–(H6) consists of straightforward computations. We obtain $b_\infty(q) = I - q \otimes q$, $\zeta_\infty(q) = 0$ and $\alpha_\infty(q) = 1$ for every $q \in S^{N-1}$.

4. The geometrical equation: the very singular case

In this section, we study the case when the discontinuities of F on $\mathcal{D}$ cannot be reduced to a discontinuity at $p = 0$. This question was addressed by many authors: Goto [23], Ishii & Souganidis [28], Ishii [27] or Ohnuma & Sato [30].

Ishii [27] deals with the worst set of singularities. Our approach is strongly inspired by his work: Ishii extends the notion of viscosity by restricting the class of test-functions. His result applies but we provide a simpler proof which relies on the special form of our set of singularities.

We refer to the end of the section for examples of PDEs which are covered by our framework but which do not satisfy the assumptions of Section 3.

4.1. Definitions and first properties

We recall the definition of viscosity solutions for very singular equations as it appears in Ishii [27].

In what follows, the set of the upper-semicontinuous (or lower-semicontinuous) functions is denoted by USC (respectively LSC). For any locally bounded function v, v^* and v_* are respectively the upper- and lower-semicontinuous envelopes of v and $\mathcal{P}^{2,+}(v^*)$ and $\mathcal{P}^{2,-}(v_*)$ are its parabolic semijets (see [15] for a definition).

We define semicontinuous envelopes for F, which are *adapted* to the set of discontinuity $\mathcal{D}$, by, for every $(p, X) \in \mathbb{R}^{N+1} \times \mathcal{S}_{N+1}$,

$$F^*(p, X) = \limsup_{(\rho,Y)\to(p,X)} \{F(\rho, Y) : (\rho, Y) \in (\mathbb{R}^{N+1} - \mathcal{D}) \times \mathcal{S}_{N+1}\},$$
$$F_*(p, X) = \liminf_{(\rho,Y)\to(p,X)} \{F(\rho, Y) : (\rho, Y) \in (\mathbb{R}^{N+1} - \mathcal{D}) \times \mathcal{S}_{N+1}\}.$$

Clearly, F^* and F_* inherit the same properties as F: they are still degenerate elliptic and geometric.

Definition 1. A locally bounded function $v : \mathbb{R}^{N+1} \times (0, +\infty) \mapsto \mathbb{R}$ is said to be a viscosity subsolution (or supersolution) of (2.1) if and only if for any $(x, t) \in \mathbb{R}^{N+1} \times (0, +\infty)$,

$$a + F_*(p, X) \leqq 0 \text{ for all } (a, p, X) \in \mathcal{P}^{2,+}(v^*)(x, t)$$

(respectively

$$a + F^*(p, X) \geqq 0 \text{ for all } (a, p, X) \in \mathcal{P}^{2,-}(v_*)(x, t) \text{)}.$$

A discontinuous function v is a viscosity solution of (2.1) provided it is both a sub- and a supersolution.

With this definition, all the basic properties of classical viscosity solutions extend to this case. In particular, the classical stability result for viscosity solutions holds. The proof is the same as those in the classical references given in the introduction.

We continue with the *invariance lemma* which is a characteristic of geometric equations (cf. Section 2).

Lemma 4.1. *If* $v \in USC(\mathbb{R}^{N+1} \times [0, +\infty))$ (*or* $LSC(\mathbb{R}^{N+1} \times [0, +\infty))$ *is a viscosity subsolution* (*respectively supersolution*) *of* (2.1)*, then, for any nondecreasing function* $\Psi \in USC(\mathbb{R})$ (*respectively* $\Psi \in LSC(\mathbb{R})$) *the function* $\Psi \circ v$ *is a viscosity subsolution* (*respectively supersolution*) *of the same equation.*

Proof of Lemma 4.1. We will prove the assertion in the case of a subsolution; the proof for supersolutions is analogous. We proceed by approximation of Ψ.

We construct a non-increasing family $(\Psi_\varepsilon)_{\varepsilon>0}$ of smooth strictly increasing functions such that

$$\inf_{\varepsilon>0} \Psi_\varepsilon = \Psi.$$

Let ϕ be a C^2 function and (x_0, t_0) be a local maximum of $\Psi_\varepsilon(v) - \phi$. Without loss of generality, we can suppose that $(\Psi_\varepsilon(v) - \phi)(x_0, t_0) = 0$. It follows that, for every $x \in \mathbb{R}^{N+1}$ and $t \in [0, +\infty)$,

$$\Psi_\varepsilon(v)(x, t) \leqq \phi(x, t) \Longleftrightarrow v(x, t) \leqq \Phi_\varepsilon \circ \phi(x, t),$$

where we set $\Phi_\varepsilon = (\Psi_\varepsilon)^{-1}$. Thus (x_0, t_0) is a local maximum of $v - \Phi_\varepsilon \circ \phi$ and since v is a subsolution of (2.1), we get

$$\Phi_\varepsilon{}' \frac{\partial \phi}{\partial t}(x_0, t_0) + F_* \Big(\Phi_\varepsilon{}' D\phi(x_0, t_0), \Phi_\varepsilon{}' D^2\phi(x_0, t_0) + \Phi_\varepsilon{}'' D\phi \otimes D\phi\Big) \leqq 0.$$

Using the fact that F_* is geometric and dividing the last inequality by $\Phi_\varepsilon{}' > 0$, we get

$$\frac{\partial \phi}{\partial t}(x_0, t_0) + F_* \Big(D\phi(x_0, t_0), D^2\phi(x_0, t_0)\Big) \leqq 0$$

which proves that $\Psi_\varepsilon \circ v$ is a subsolution of (2.1). From classical results about viscosity solutions, we know that

$$\limsup_{\varepsilon\to 0}{}^* \Psi_\varepsilon(v) = \inf_{\varepsilon>0} \Psi_\varepsilon \circ v = \Psi \circ v$$

is a subsolution of (2.1), which is the desired result. □

4.2. Comparison result

We turn now to a comparison principle for (2.1).

Theorem 4.1. *Suppose that* (H1)–(H4) *hold and let* $v_0 \in UC(\mathbb{R}^{N+1})$. *If* $v_1 \in USC(\mathbb{R}^{N+1} \times [0, +\infty))$ *or* ($v_2 \in LSC(\mathbb{R}^{N+1} \times [0, +\infty))$) *is a subsolution* (*respectively a supersolution*) *of* (2.1), *and if* $v_1(\cdot\,, 0) \leqq v_0 \leqq v_2(\cdot\,, 0)$ *in* $\mathbb{R}^{N+1}$, *then* $v_1 \leqq v_2$ *in* $\mathbb{R}^{N+1} \times [0, +\infty)$.

Remark 4.1. Note that “bounded” or “unbounded” solutions is not the point in this theorem. Since the equation is geometric, up to making a change of variable $v \mapsto \tanh(v)$ together with Lemma 4.1, we can suppose that the solutions are bounded. Another remark is that we are able to compare bounded continuous solutions with bounded uniformly continuous solutions. Of course, it gives uniqueness only in $UC(\mathbb{R}^{N+1} \times [0, +\infty))$.

The difficulty in proving such a result comes obviously from the unusual set of discontinuities $\mathcal{D}$. We begin with some arguments giving an idea of the proof.

Setting

$$\mathcal{S}(\mathcal{D}) = \left\{ \left(\begin{array}{c|c} X_{xx} & 0 \\ \hline 0 & 0 \end{array} \right) \ :\ X_{xx} \in \mathcal{S}_N \right\}, \tag{4.1}$$

we have

Lemma 4.2. *Assume* (H1)–(H4). Then, for all $p \in \mathcal{D} - \{0\}$ and $X \in \mathcal{S}(\mathcal{D})$, $F^*(p, X) = F_*(p, X)$. Moreover $F^*(0, 0) = 0 = F_*(0, 0)$.

This lemma is proved at the end of the section. It suggests that we may use in the proof of the theorem test-functions φ such that $D^2\varphi \in \mathcal{S}(\mathcal{D})$ when $D\varphi \in \mathcal{D}$. Doing things this way, we do not see the discontinuities of F in the proof.

Prof of Theorem 4.1. Without loss of generality, we can assume that v_1 and v_2 are bounded. We recall that we write z for a point $z = (x, y) \in \mathbb{R}^N \times \mathbb{R}$ and by $|z|$ we mean $(|x|^2 + y^2)^{1/2}$. We argue by contradiction, assuming that there exists $(z_0, t_0) \in \mathbb{R}^{N+1} \times [0, +\infty)$ such that $(v_1 - v_2)(z_0, t_0) > 0$. We introduce the function

$$\phi(z_1, z_2) = \frac{|x_1 - x_2|^4}{\varepsilon^4} + \frac{|y_1 - y_2|^4}{\varepsilon^4} = \varphi(z_1 - z_2)$$

which is chosen in order to ensure that $D^2\varphi(Z) \in \mathcal{S}(\mathcal{D})$ when $D\varphi(Z) \in \mathcal{D}$ (for the definitions of $\mathcal{D}$ and $\mathcal{S}(\mathcal{D})$, see (2.2) and (4.1)). We then set

$$M_{\varepsilon,\alpha,\eta} = \sup_{(\mathbb{R}^{N+1})^2\times[0,+\infty)} \left\{ v_1(z_1,t) - v_2(z_2,t) - \phi(z_1,z_2) - \alpha(|z_1|^2+|z_2|^2) - \eta t \right\}.$$

At first, it is clear that $M_{\varepsilon,\alpha,\eta} > 0$ for α,η sufficiently small since $\phi(z_0,z_0) = 0$. Moreover $M_{\varepsilon,\alpha,\eta}$ is achieved at some $(\bar z_1, \bar z_2, \bar t)$ by the boundedness and the semicontinuous properties of v_1 and v_2. Actually $\bar z_1$ and $\bar z_2$ depend on α,ε,η, but we omit this dependence in the notation for simplicity.

When $\bar t = 0$, we have

$$0 < M_{\varepsilon,\alpha,\eta} \leqq v_0(\bar z_1) - v_0(\bar z_2) - \frac{|\bar x_1 - \bar x_2|^4}{\varepsilon^4} - \frac{|\bar y_1 - \bar y_2|^4}{\varepsilon^4} - \alpha(|\bar z_1|^2 + |\bar z_2|^2)$$

which leads to a contradiction using the uniform continuity of v_0. Thus, there cannot exist a subsequence of parameters (ε,α) going to $(0,0)$ such that $\bar t = 0$. Therefore, we can suppose that $\bar t > 0$ for ε and α sufficiently small.

From the fundamental result of the User's guide to viscosity solutions [15, Theorem 8.3], for every $\rho > 0$, we get $a_1, a_2 \in \mathbb{R}$ and $X, Y \in \mathcal{S}_{N+1}$ such that

$$(a_1, D\varphi(\bar z_1 - \bar z_2) + 2\alpha \bar z_1, X + 2\alpha I) \in \bar{\mathcal{P}}^{2,+}(v_1)(\bar z_1, \bar t),$$

$$(a_2, D\varphi(\bar z_1 - \bar z_2) - 2\alpha \bar z_2, Y - 2\alpha I) \in \bar{\mathcal{P}}^{2,-}(v_2)(\bar z_2, \bar t)$$

and

$$-\left(\frac{1}{\rho} + \|A\|\right)\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leqq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leqq \begin{pmatrix} A + 2\rho A^2 & -(A+2\rho A^2) \\ -(A+2\rho A^2) & A + 2\rho A^2 \end{pmatrix} \quad (4.2)$$

for some $a_1 - a_2 = \eta$ and $A = D^2\varphi(\bar z_1 - \bar z_2)$. Writing that v_1 is a subsolution and v_2 a supersolution of (2.1), we have

$$\begin{aligned} \eta + F_*(D\varphi(\bar z_1 - \bar z_2) + 2\alpha \bar z_1, X + 2\alpha I)& \\ - F^*(D\varphi(\bar z_1 - \bar z_2) - 2\alpha \bar z_2, Y - 2\alpha I) &\leqq 0. \end{aligned} \quad (4.3)$$

We compute, for every $Z = (Z_x, Z_y) \in \mathbb{R}^N \times \mathbb{R}$,

$$D\varphi(Z) = \frac{4}{\varepsilon^4}(|Z_x|^2 Z_x, Z_y^3) \text{ and } D^2\varphi(Z) = \frac{4}{\varepsilon^4}\left(\begin{array}{c|c} 2Z_x \otimes Z_x + |Z_x|^2 I & 0 \\ \hline 0 & 3Z_y^2 \end{array}\right).$$

Since $M_{\varepsilon,\alpha,\eta} > 0$, we get

$$\phi(\bar z_1, \bar z_2, \bar t) + \alpha(|\bar z_1|^2 + |\bar z_2|^2) \leqq \|v_1\|_\infty + \|v_2\|_\infty$$

(recall that v_1 and v_2 are assumed to be bounded). It follows that

$$\lim_{\alpha\to 0^+} \alpha|\bar z_1|, \ \lim_{\alpha\to 0^+} \alpha|\bar z_2| = 0, \quad (4.4)$$

$$|\bar z_1 - \bar z_2| \text{ is bounded as } \alpha \text{ goes to } 0. \quad (4.5)$$

From (4.5) and (4.2), we find that X and Y are bounded when α goes to 0. Therefore, using (4.4), we can extract subsequences such that $\bar{z}_1 - \bar{z}_2 \to \bar{Z}$ and

$$(D\varphi(\bar{z}_1 - \bar{z}_2) + 2\alpha\bar{z}_1, X + 2\alpha I) \longrightarrow \left(D\varphi(\bar{Z}), \bar{X}\right),$$

$$(D\varphi(\bar{z}_1 - \bar{z}_2) - 2\alpha\bar{z}_2, Y - 2\alpha I) \longrightarrow \left(D\varphi(\bar{Z}), \bar{Y}\right),$$

when α goes to 0. Note that $\bar{X}, \bar{Y}$ satisfy also (4.2) with $A = D^2\varphi(\bar{Z})$. From (4.3), it follows that

$$\eta + F_*(D\varphi(\bar{Z}), \bar{X}) - F^*(D\varphi(\bar{Z}), \bar{Y}) \leqq 0. \tag{4.6}$$

Now, if $D\varphi(\bar{Z}) \notin \mathcal{D}$, then we are done since (4.2) implies that $\bar{X} \leqq \bar{Y}$ and since in this case $F^* = F_* = F$ is degenerate elliptic.

But, when $D\varphi(\bar{Z}) \in \mathcal{D}$, we need more information about $\bar{X}, \bar{Y}$ in order to get the contradiction. At first, $D\varphi(\bar{Z}) \in \mathcal{D}$ implies $\bar{Z}_y = 0$; thus

$$D^2\varphi(\bar{Z}) = \left(\begin{array}{c|c} 2\bar{Z}_x \otimes \bar{Z}_x + |\bar{Z}_x|^2 I & 0 \\ \hline 0 & 0 \end{array}\right) \in \mathcal{S}(\mathcal{D}). \tag{4.7}$$

At this stage, we would like to apply Lemma 4.2 but we need first to transfer to $\bar{X}, \bar{Y}$ the appropriate property of $D^2\varphi(\bar{Z})$, namely $D^2\varphi(\bar{Z}) \in \mathcal{S}(\mathcal{D})$. To this end, we state

Lemma 4.3. *If $D^2\varphi(\bar{Z}) \in \mathcal{S}(\mathcal{D})$, then there exist $X', Y' \in \mathcal{S}(\mathcal{D})$ such that*

$$\bar{X} \leqq X' \leqq Y' \leqq \bar{Y}.$$

Moreover, $X' = Y' = 0$ when $D\varphi(\bar{Z}) = 0$.

We postpone the proof and complete that of Theorem 4.1. Taking advantage of the ellipticity of F^* and F_* together with Lemma 4.2, we get from (4.6)

$$\eta + F^*(D\varphi(\bar{Z}), X') - F^*(D\varphi(\bar{Z}), Y') \leqq 0.$$

Since $X' \leqq Y'$, the ellipticity of F^* leads to a contradiction. It achieves the proof of the theorem. $\square$

We turn to the proof of the lemmas.

Proof of Lemma 4.2. Let us consider $((p_x, 0), X) \in \mathcal{D} \times \mathcal{S}(\mathcal{D})$. It is sufficient to see that $F(\rho, X + Y)$ has a limit when $(\rho, Y) \to ((p_x, 0), 0)$, $(\rho, Y) \in (\mathbb{R}^{N+1} - \mathcal{D}) \times \mathcal{S}_{N+1}$. Since $X \in \mathcal{S}(\mathcal{D})$, we have

$$\begin{aligned} F(\rho, X + Y) = F(\rho, X) + F(\rho, Y) = -\mathrm{Tr}\left[b\left(-\frac{\rho_x}{\rho_y}\right) X_{xx}\right] \\ - \mathrm{Tr}\left[b\left(-\frac{\rho_x}{\rho_y}\right)\left(Y_{xx} - 2Y_{xy} \otimes \frac{\rho_x}{\rho_y} + Y_{yy}\frac{\rho_x}{\rho_y} \otimes \frac{\rho_x}{\rho_y}\right)\right]. \end{aligned}$$

At first, from (H1), (H2) and (H3), when $\|Y\| \leqq \varepsilon$, we obtain

$$\left| Tr\left[b\left(-\frac{\rho_x}{\rho_y}\right)\left(Y_{xx} - 2Y_{xy}\otimes\frac{\rho_x}{\rho_y} + Y_{yy}\frac{\rho_x}{\rho_y}\otimes\frac{\rho_x}{\rho_y}\right)\right]\right| \leqq O(\varepsilon) \underset{\varepsilon\to 0}{\to} 0. \quad (4.8)$$

If $p_x = 0$, then $p = 0$ and (4.8) implies that $F^*(0,0) = 0 = F_*(0,0)$. If $p \in \mathcal{D} - \{0\}$, then $p_x \neq 0$ and from (H4), we get

$$\lim_{\rho\to(p_x,0)} \mathrm{Tr}\left[b\left(-\frac{\rho_x}{\rho_y}\right) X_{xx}\right] = \mathrm{Tr}\left[b_\infty\left(\frac{p_x}{|p_x|}\right) X_{xx}\right].$$

It achieves the proof of the lemma. □

Proof of Lemma 4.3. We set $A = D^2\varphi(\bar{Z})$, $B = A + 2\rho A^2$ (see (4.2)) and $B_\delta = B + \delta I$ for $\delta > 0$. Note that $B \in \mathcal{S}(\mathcal{D})$ and $B \geqq 0$ as we can see with the help of Formula (4.7). Moreover, we find from (4.2) that

$$\langle \bar{X}p, p\rangle - \langle \bar{Y}q, q\rangle \leqq \langle B(p-q), p-q\rangle < \langle B_\delta(p-q), p-q\rangle. \quad (4.9)$$

It shows in particular that $\bar{X} < B_\delta$ (and $-\bar{Y} < B_\delta$); thus $(\bar{X} - B_\delta)$ (respectively $(\bar{Y} + B_\delta)$) is invertible. We then obtain X' using a sup-convolution. We set, for every $p, r \in \mathbb{R}^{N+1}$ and $k > 1$,

$$F_r(p) = \langle \bar{X}p, p\rangle - \langle kB_\delta(p-r), p-r\rangle \quad (4.10)$$

and consider $\sup_{p\in\mathbb{R}^N} F_r(p)$ which is well defined. For every $h \in \mathbb{R}^N$,

$$\langle DF_r(p), h\rangle = 2\langle \bar{X}p, h\rangle - 2\langle kB_\delta(p-r), h\rangle$$

which means that the supremum is achieved for

$$p = (kB_\delta - \bar{X})^{-1}kB_\delta r$$

(note that $(kB_\delta - \bar{X})$ is invertible since $\bar{X} < B_\delta < kB_\delta$). Next, an explicit but tedious computation yields a matrix $X' \in \mathcal{S}_{N+1}$ such that

$$\sup_{p\in\mathbb{R}^{N+1}} F_r(p) = \langle X'r, r\rangle.$$

Taking successively the particular value $p = r$ and $p = 0$ in (4.10), we get $X' \geqq \bar{X}$ and $X' \geqq -kB_\delta$. Similarly, we can construct Y' by setting, for $k > 1$,

$$\inf_{p\in\mathbb{R}^{N+1}} \left\{\langle \bar{Y}p, p\rangle + \langle kB_\delta(p-s), p-s\rangle\right\} = \langle Y's, s\rangle.$$

We obtain a matrix Y' satisfying $Y' \leqq \bar{Y}$ and $Y' \leqq kB_\delta$. From (4.9), we get

$$\begin{aligned}\langle X'r, r\rangle - \langle Y's, s\rangle \leqq \sup_{p,q\in\mathbb{R}^{N+1}} \big\{&\langle B_\delta(p-q), p-q\rangle \\ &- \langle kB_\delta(p-r), p-r\rangle - \langle kB_\delta(q-s), q-s\rangle\big\}\end{aligned}$$

for any $r, s \in \mathbb{R}^{N+1}$. An explicit calculation of this supremum yields

$$\langle X'r, r\rangle - \langle Y's, s\rangle \leqq \langle B_\delta(r-s), r-s\rangle.$$

By taking $r = s$, we obtain $\langle X'r, r\rangle - \langle Y'r, r\rangle \leqq 0$ for every $r \in \mathbb{R}^{N+1}$. It follows that $-kB_\delta \leqq X' \leqq Y' \leqq kB_\delta$. Now, letting δ go to 0, up to extracting a subsequence, we get two matrices, still denoted by X', Y', such that

$$-kB \leqq X' \leqq Y' \leqq kB. \tag{4.11}$$

Recalling that $B \in \mathcal{S}(\mathcal{D})$, we find first that $X'_{yy} = Y'_{yy} = 0$. Then, from (4.11), for any $r \in \mathbb{R}^{N+1}$, we have

$$\langle X'r, r\rangle = \langle X'_{xx}r_x, r_x\rangle + 2\langle X'_{xy}, r_x\rangle r_y \leqq k\langle Br, r\rangle \leqq K|r_x|^2$$

for some positive constant K. Taking tr_x instead of r_x for $t \in \mathbb{R}$, we get

$$t^2\langle X'_{xx}r_x, r_x\rangle + 2t\langle X'_{xy}, r_x\rangle r_y \leqq Kt^2|r_x|^2$$

which provides $\langle X_{xy}, r_x\rangle = 0$ dividing by t and letting t go to 0^+ or 0^-. Since this holds for any $r_x \in \mathbb{R}^N$ we are done. The same arguments hold for Y'. Finally, if $D\varphi(\bar{Z}) = 0$, then $\bar{Z}_x = 0$, $\bar{Z}_y = 0$; it implies $A = B = 0$. From (4.11), we get $X' = Y' = 0$, which completes the proof of the lemma. □

4.3. Existence of solutions

Our result is the

Theorem 4.2. *Assume* (H1)–(H4). *For every* $v_0 \in UC(\mathbb{R}^{N+1})$, *there exists a unique* $v \in UC\left(\mathbb{R}^{N+1} \times [0, +\infty)\right)$ *solving* (2.1) *with initial data* v_0.

The proof uses the classical *Perron method*, introduced in the framework of viscosity solutions by Ishii in [26] (see also [2,5,15]). The application of this method in our case does not present any special difficulties. Nevertheless, we provide a proof for the readers' convenience.

Proof of Theorem 4.2. The uniqueness part comes immediately from Theorem 4.1 and, because of Lemma 4.1, we can suppose that v_0 is bounded. We divide the proof into steps.

Step 1. We construct a solution $v \in C(\mathbb{R}^{N+1} \times [0, +\infty))$ when the initial data is smooth. Let $v_0 \in C^2(\mathbb{R}^{N+1}) \cap W^{2,\infty}(\mathbb{R}^{N+1})$ and define, for any $C > 0$, two functions $\underline{v}$, $\overline{v}$ by setting

$$\underline{v}(z,t) := -Ct + v_0(z) \quad \text{and} \quad \overline{v}(z,t) = Ct + v_0(z)$$

for any $(z,t) \in \mathbb{R}^{N+1} \times [0, +\infty)$. It follows from (H1)–(H3) that the nonlinearity F appearing in (2.1) is bounded on bounded subsets. Therefore, C may be chosen large enough in order that $\underline{v}$ and $\overline{v}$ are respectively sub- and super solution of (2.1).

Consider then the set $\mathcal{F}$ of subsolutions of (2.1) w such that $\underline{v} \leqq w \leqq \overline{v}$. Set then for every $(z,t) \in \mathbb{R}^{N+1} \times [0, +\infty)$, $v(z,t) = \sup_{w\in\mathcal{F}} w(z,t)$. The set $\mathcal{F}$

is nonempty and v is well defined. Thus, we find from the comparison result and classical arguments of the Perron method that v is a discontinuous solution of (2.1).

Now, we also have from the definition of v that $v^*(\cdot\,,0) = v_*(\cdot\,,0) = v_0$. Thus we deduce from the comparison result that $v^* = v_* = v$, which is the desired continuous solution.

Step 2. We show that the solution v we built in Step 1 is actually in $BUC(\mathbb{R}^{N+1} \times [0,+\infty))$. First of all, since the constant functions are smooth solutions of (2.1), the comparison result shows that v is bounded, namely $|v| \leqq |v_0|$ in $\mathbb{R}^{N+1} \times [0,+\infty)$. Next, from the definition of v, for all $h > 0$, we have

$$\underline{v}(\cdot\,,h) = v_0 - Ch \leqq v(\cdot\,,h) \leqq v_0 + Ch = \overline{v}(\cdot\,,h) \quad \text{in } \mathbb{R}^{N+1}. \tag{4.12}$$

From Theorem 4.1, and since the nonlinearity in (2.1) depends only on (Du, D^2u), the function $v(\cdot\,,\cdot+h)$ is a solution of (2.1) with initial data $v(\cdot\,,h)$ and $v \pm Ch$ are solutions of (2.1) with initial data $v_0 \pm Ch$. Thus Theorem 4.1, together with (4.12), yields

$$v - Ch \leqq v(\cdot\,,\cdot+h) \leqq v + Ch \quad \text{in } \mathbb{R}^{N+1} \times [0,+\infty).$$

It provides a modulus of continuity in the time variable which is independent of the space variable. Arguing in the same way with translations in space $v_0 \mapsto v_0(\cdot+\xi)$, $\xi \in \mathbb{R}^{N+1}$, we obtain a modulus of continuity for the space variable which is independent of the time variable. It proves that $v \in BUC(\mathbb{R}^{N+1} \times [0,+\infty))$.

Step 3 The general case when $v_0 \in BUC(\mathbb{R}^{N+1})$. Using a classical convolution procedure, we construct a sequence $(v_0^n)_{n\in\mathbb{N}}$ of functions $v_0^n \in C^2(\mathbb{R}^{N+1}) \cap W^{2,\infty}(\mathbb{R}^{N+1})$ such that $|v_0 - v_0^n| \leqq 1/n$ in $\mathbb{R}^{N+1}$. It follows that $-2/n + v_0^n \leqq v_0^m \leqq v_0^n + 2/n$ for $m \geqq n$. According to Steps 1 and 2, we can consider, for every $n \in \mathbb{N}$, the unique solution $v^n \in BUC(\mathbb{R}^{N+1} \times [0,+\infty))$ of (2.1) with initial data v_0^n. Proceeding as in Step 2, we deduce from the previous inequality that $-2/n + v^n \leqq v^m \leqq v^n + 2/n$ for $m \geqq n$. Thus $(v^n)_{n\in\mathbb{N}}$ converges uniformly in $\mathbb{R}^{N+1} \times [0,+\infty)$ to some function v which is still bounded uniformly continuous. From the stability result, v is a viscosity solution of (2.1) with initial data v_0. It achieves the proof of the Theorem. □

4.4. Examples

We give some examples of PDEs like (1.4) which are covered by the very singular case.

(1) In addition to the mean-curvature equation for graphs (1.1), we can deal with the non-geometric mean-curvature equation

$$\frac{\partial u}{\partial t} - \operatorname{div}\frac{Du}{\sqrt{1+|Du|^2}} = 0, \tag{4.13}$$

or equations like

$$\frac{\partial u}{\partial t} - \frac{\Delta u}{(1+|Du|^2)^{\alpha}} = 0, \quad \alpha \geqq 1. \tag{4.14}$$

These equations lead to a geometrical equation like (2.1) with a singularity only at $|Dv| = 0$ and they satisfy the assumptions of the classical framework.
(2) Consider a generalization of the mean-curvature equation for graphs, namely

$$\frac{\partial u}{\partial t} - \mathrm{Tr}\left[(I - g(Du)Du \otimes Du)D^2u\right] = 0, \tag{4.15}$$

where g is a continuous function from $\mathbb{R}^N$ into $\mathbb{R}$. In this case, $b(q) = I - g(q)q \otimes q$ is symmetric nonnegative and satisfies (H1)–(H3) if and only if there exists a positive constant C such that

$$\frac{1}{|q|^2}\left(1 - \frac{C}{|q|^2}\right) \leqq g(q) \leqq \frac{1}{|q|^2} \quad \text{for every } q \in \mathbb{R}^N. \tag{4.16}$$

Using the notation of Section 3, for every $q \in S^{N-1}$, we have $b_\infty(q) = I - q \otimes q$ and $\zeta_\infty(q) = 0$; thus (H4) and (H5) are fulfilled and this equation falls into our study area. Concerning (H6), we have

$$0 \leqq \langle b(\lambda q)\lambda q, \lambda q\rangle = \lambda^2(1 - \lambda^2 g(\lambda q)) \leqq C.$$

We cannot conclude from this that there is a limit for all functions g; it means that the last assumption does not hold in general and this equation is not covered by the classical framework in the whole generality. We relate in detail such a situation below.
(3) We turn to an explicit example of a PDE like (1.4) which leads to a geometrical equation whose set of singularities is exactly $\mathcal{D}$ and is not removable. Consider

$$\frac{\partial u}{\partial t} - \frac{f(Du)}{(1 + |Du|^2)^2}\langle D^2uDu, Du\rangle = 0 \quad \text{in } \mathbb{R}^N \times (0, T), \tag{4.17}$$

where $f : \mathbb{R}^N \to \mathbb{R}$ is any bounded, nonnegative function. In this case,

$$b(q) = \frac{f(q)}{(1 + |q|^2)^2} q \otimes q.$$

The assumptions (H1)–(H3) are obviously satisfied and, for every $q \in \mathbb{R}^N$, $b(\lambda q) \to 0$ as $\lambda \to \pm\infty$; thus (H4) holds. It follows that this equation is covered by "the very singular case" of this section. It leads to a geometrical equation like (2.1) with

$$\begin{aligned} &F(p, X) \\ &= -\mathrm{Tr}\left[f\left(-\frac{p_x}{p_y}\right)\frac{p_x \otimes p_x}{(|p_x|^2 + p_y^2)^2}(p_y^2 X_{xx} - 2p_y X_{xy} \otimes p_x + X_{yy} p_x \otimes p_x)\right], \end{aligned}$$

for every $p = (p_x, p_y) \in \mathbb{R}^{N+1}$ and $X \in \mathcal{S}_{N+1}$.

For simplicity, set $N = 1$ and $f(q) = 1 + \cos q$. It follows that $F^*((p_x, 0), X) = 0$ and $F_*((p_x, 0), X) = -2X_{yy}$, for every $p = (p_x, 0)$, $p_x \neq 0$ and $X \in \mathcal{S}_2$ such that $X_{yy} > 0$. Therefore, in general

$$F^* \neq F_* \quad \text{on } \mathcal{D} = \{p : p_y = 0\}.$$

This shows that we cannot remove the singularities of F outside 0. Thus (4.17) does not satisfy the assumptions of Section 3.

5. The level-set approach

In this section, for the sake of completeness, we recall the basic ideas of the level-set approach and we apply them to Equation (2.1) both in the classical and very singular framework. We refer to [18, 13, 10], etc. for a more complete description of this approach.

We are given a triplet $(\Gamma_0, \Omega_0^+, \Omega_0^-)$, where Ω_0^+, Ω_0^- are disjoint open subsets of $\mathbb{R}^{N+1}$ and $\Gamma_0 = \left(\Omega_0^+ \cup \Omega_0^-\right)^C$. In general, we have in mind $\Gamma_0 = \partial\Omega_0^+ = \partial\Omega_0^-$. Note that these sets form a partition of $\mathbb{R}^{N+1}$ and Γ_0 can be thought of as being an hypersurface.

Let v_0 be any uniformly continuous function whose 0-level set is exactly Γ_0, namely,

$$\Gamma_0 = \{z \in \mathbb{R}^{N+1} : v_0(z) = 0\}, \tag{5.1}$$

and such that

$$\{z \in \mathbb{R}^{N+1} : v_0(z) > 0\} = \Omega_0^+ \text{ and } \{z \in \mathbb{R}^{N+1} : v_0(z) < 0\} = \Omega_0^-. \tag{5.2}$$

This choice of signs defines an orientation of Γ_0 making it possible to distinguish an "interior", Ω_0^+, and an "exterior", Ω_0^-. Secondly, it is always possible to find such a function v_0 by taking, for example, the *signed-distance* to Γ_0 defined by

$$\mathrm{d}(z, \Gamma_0) := \begin{cases} +\mathrm{dist}(z, \Gamma_0) & \text{if } z \in \Omega_0^+, \\ -\mathrm{dist}(z, \Gamma_0) & \text{if } z \in \Omega_0^-, \end{cases} \tag{5.3}$$

where dist denotes the usual positive distance. Clearly $\mathrm{d}(\cdot\,, \Gamma_0)$ is Lipschitz continuous in $\mathbb{R}^{N+1}$.

We then define the *generalized evolution* of $(\Gamma_0, \Omega_0^+, \Omega_0^-)$ by the family $(\Gamma_t, \Omega_t^+, \Omega_t^-)_{t \geqq 0}$, using the

Theorem 5.1. *Under the assumptions of Theorem 3.1 or 4.2, there exists a unique solution v of* (2.1) *in $UC(\mathbb{R}^{N+1} \times (0, +\infty))$ with initial data v_0. Moreover, if $\tilde{v}_0 \in UC(\mathbb{R}^{N+1})$ satisfies*

$$\{\tilde{v}_0 = 0\} = \Gamma_0, \quad \{\tilde{v}_0 > 0\} = \Omega_0^+ \quad \textit{and} \quad \{\tilde{v}_0 < 0\} = \Omega_0^-,$$

and if $\tilde{v} \in UC(\mathbb{R}^{N+1} \times (0, +\infty))$ is the viscosity solution of (2.1) *with initial data $\tilde{v}_0$, then*

$$\begin{aligned} \{v(\cdot\,, t) > 0\} &= \{\tilde{v}(\cdot, t) > 0\} := \Omega_t^+, \\ \{v(\cdot\,, t) < 0\} &= \{\tilde{v}(\cdot, t) < 0\} := \Omega_t^-, \\ \{v(\cdot\,, t) = 0\} &= \{\tilde{v}(\cdot, t) = 0\} := \Gamma_t. \end{aligned}$$

This result implies that the family $(\Gamma_t, \Omega_t^+, \Omega_t^-)_{t\geqq 0}$ exists and is uniquely defined independently of the choice of the representation $v_0 \in UC(\mathbb{R}^{N+1})$ satisfying (5.1) and (5.2). The set $\bigcup_{t\geqq 0} \Gamma_t \times \{t\}$ is called the *front* associated with Γ_0 by (2.1) and Γ_t is the *front at time* t. Note that, at least formally, Γ_t evolves with a normal velocity equal to

$$\mathcal{V}_n(z) = -F(D\mathrm{d}(\cdot\,, \Gamma_t)(z), D^2\mathrm{d}(\cdot\,, \Gamma_t)(z)),$$

for $z \in \Gamma_t$.

Proof of Theorem 5.1. We give a proof inspired by Ishii's one (see [27]). We only show that if $\{v_0 > 0\} \subset \{\tilde{v}_0 > 0\}$, then this inclusion remains true for all $t \geqq 0$, i.e., $\{v(\cdot\,,t) > 0\} \subset \{\tilde{v}(\cdot\,,t) > 0\}$. The other inclusions are obtained by straightforward adaptations. From Theorem 3.1 or 4.2, $v, \tilde{v} \in UC(\mathbb{R}^{N+1} \times [0,+\infty))$ and we recall that the hyperbolic tangent function (denoted by tanh) is a bounded uniformly continuous increasing function. Then, using Lemma 4.1, we show that $\tanh(v)$ is a bounded uniformly continuous solution of (2.1) with the bounded initial data $\tanh(v_0)$. Next, we introduce the uniformly continuous increasing function $\theta^+(r) := \max(r,0)$ and claim, thanks to Lemma 4.1 once more, that $\theta^+ \circ \tilde{v}$ and $\theta^+ \circ \tanh(v)$ are both uniformly continuous solutions of (2.1). Finally, we introduce the lower-semicontinuous function

$$\theta(r) = \begin{cases} +2 & \text{if } r > 0, \\ 0 & \text{if } r \leqq 0, \end{cases}$$

and observe that $\theta \circ \theta^+ \circ \tilde{v}$ is a lower semicontinuous supersolution of (2.1). In fact, the previous changes are made in order to obtain the suitable initial condition

$$\theta \circ \theta^+ \circ \tilde{v}(\cdot\,,0) \geqq \theta^+ \circ \tanh(v(\cdot\,,0)),$$

which follows easily from the assumption $\{v_0 > 0\} \subset \{\tilde{v}_0 > 0\}$. Since $\theta^+ \circ \tanh(v)$ is uniformly continuous, we apply the comparison result 4.1 and get that, for all $t \geqq 0$,

$$\theta \circ \theta^+(\tilde{v}(\cdot\,,t)) \geqq \theta^+ \circ \tanh(v(\cdot\,,t)).$$

We obtain $\{v(\cdot\,,t) > 0\} \subset \{\tilde{v}(\cdot\,,t) > 0\}$, which ends the proof. □

6. Connection between geometrical and quasilinear PDEs. Application to uniqueness

In this section, we specify the connections between (1.4) and (2.1) initiated in Section 2, and in particular in terms of uniqueness for (1.4). Let u be a continuous viscosity solution of (1.4) with initial data $u_0 \in C(\mathbb{R}^N)$ and v be the solution of (2.1) with initial data $\mathrm{d}(\cdot\,, \mathrm{Graph}(u_0))$. The main question is whether or not

$$\mathrm{Graph}(u(\cdot\,,t)) = \{(x,y) \in \mathbb{R}^{N+1} : y - u(x,t) = 0\}$$

and the front

$$\Gamma_t = \{(x, y) \in \mathbb{R}^{N+1} : v(x, y, t) = 0\}$$

coincide for all $t \geqq 0$. If the answer is yes, this obviously provides a uniqueness result for (1.4) since the Γ_t's are uniquely determined because of Theorem 5.1. And it may be thought that the answer is indeed yes – by applying Theorem 5.1 together with Lemma 2.1 (ii) with initial data $\tilde{v}_0 = \tanh(y - u_0(x))$, which is a particular representation of $\mathrm{Graph}(u_0)$. Unfortunately, $\tilde{v}_0$ is not uniformly continuous if u_0 is not uniformly continuous and, as we pointed out in the introduction, we do not know how to prove Theorem 5.1 replacing "$\tilde{v}_0 \in UC(\mathbb{R}^{N+1})$" by "$\tilde{v}_0 \in C_{\mathrm{b}}(\mathbb{R}^{N+1})$"; we do not know even if such a result is true.

Nevertheless, the inclusion used in Section 2 to derive the geometrical PDE is always true.

Theorem 6.1. *Suppose that* (H1)–(H4) *hold. Let u be a viscosity subsolution* (*or supersolution*) *of* (1.4) *with initial data $u_0 \in C(\mathbb{R}^N)$ and v be a viscosity solution of* (2.1) *with initial data* $\mathrm{d}(\cdot\,, \mathrm{Graph}(u_0))$. *For every $t \in [0, +\infty)$, we have*

$$\mathrm{Graph}(u(\cdot\,, t)) \subset \{(x, y) \in \mathbb{R}^{N+1} : v(x, y, t) \leqq 0\}$$

$$(\text{respectively} \quad \mathrm{Graph}(u(\cdot\,, t)) \subset \{(x, y) \in \mathbb{R}^{N+1} : v(x, y, t) \geqq 0\}).$$

If u is a solution of (1.4), *then*

$$\mathrm{Graph}(u(\cdot\,, t)) \subset \Gamma_t \quad \textit{for all } t \in [0, +\infty),$$

where $(\Gamma_t)_{t \geqq 0}$ is the generalized evolution associated with $\Gamma_0 = \mathrm{Graph}(u_0)$.

Proof of Theorem 6.1. Suppose that u is a subsolution. Define the nondecreasing function $\theta^+(r) := \max(r, 0)$. For all $z = (x, y) \in \mathbb{R}^N \times \mathbb{R}$, we have,

$$\tanh\left[\theta^+(y - u_0(x))\right] \geqq \tanh\left[\mathrm{d}(z, \mathrm{Graph}(u_0))\right],$$

since, on the one hand, $|y - u_0(x)| \geqq \mathrm{dist}(z, \mathrm{Graph}(u_0))$; and, on the other hand, if $y \leqq u_0(x)$, then $\mathrm{d}(z, \mathrm{Graph}(u_0)) \leqq 0$ (for the definition of d, see (5.3)). From Lemma 2.1 (ii) and from the invariance of supersolutions of (2.1) under nondecreasing changes of variables (see Lemma 4.1), we know that the function $\tanh[\theta^+(y - u(x, t))]$ is a supersolution of (2.1) with initial data $\tanh[\theta^+(y - u_0(x))]$. Moreover, the function $\tanh(v)$ is a solution (thus a subsolution) of (2.1) with initial data $\tanh(v_0)$. Applying Theorem 4.1 (see Remark 4.1), we get

$$\tanh\left[\theta^+\big(y - u(x, t)\big)\right] \geqq \tanh[v(z, t)].$$

Thus, $y = u(x, t)$ implies that $v(z, t) \leqq 0$, which proves the first inclusion. If u is a supersolution, we repeat the same arguments with $\theta_-(r) := \min(r, 0)$. We get the other inclusion. In order to prove the last statement of the theorem it suffices to notice, on the one hand, that u is a solution provided that u is both a sub- and a supersolution and, on the other hand, that

$$\Gamma_t = \{z \in \mathbb{R}^{N+1} : v(z,t) \leqq 0\} \cap \{z \in \mathbb{R}^{N+1} : v(z,t) \geqq 0\}.$$

This achieves the proof of the theorem. $\square$

In fact, the uniqueness for (1.4) and the so-called "fattening phenomena" for the front are closely related as shown by the

Theorem 6.2. *Assume that* (H1)–(H4) *hold and let* $u_0 \in C(\mathbb{R}^N)$. *Suppose that the front* $\bigcup_{t\geq 0} \Gamma_t \times \{t\}$ *associated with* $\mathrm{Graph}(u_0)$ *has empty interior in* $\mathbb{R}^{N+1} \times [0,+\infty)$. *Then* (1.4) *has at most one continuous viscosity solution with initial data* u_0.

We point out that Theorem 6.2 provides uniqueness only in the class of continuous functions. But discontinuous solutions may also exist if the front looks like a rake for instance. This result says nothing about the existence of solutions; it may be possible taht the front contains no continuous graph.

In the literature, the "fattening phenomena" may have different meanings. In Theorem 6.2, we use the standard topological meaning. We first want to remark that assuming that Γ_t has empty interior in $\mathbb{R}^{N+1}$ for all $t \geqq 0$ is stronger than assuming that $\bigcup_{t\geqq 0} \Gamma_t \times \{t\}$ has empty interior in $\mathbb{R}^{N+1} \times [0,T]$. In fact, under our assumptions it turns out to be equivalent. The proof of this equivalence comes from the preservation of inclusion of sets under motions governed by (2.1) and the fact that, if the front at a time t contains a ball, this ball cannot shrink instantaneously. We skip the proof and refer to the one of Theorem 7.1 which is similar.

In Barles & Souganidis [11], a "no-interior condition" is considered, namely

$$\bigcup_{t\geqq 0} \Gamma_t \times \{t\} = \partial\left(\bigcup_{t\geqq 0} \Omega_t^+ \times \{t\}\right) = \partial\left(\bigcup_{t\geqq 0} \Omega_t^- \times \{t\}\right). \tag{6.1}$$

This condition is stronger than the topological one. When it is satisfied, we have a better result.

Theorem 6.3. *Assume* (H1)–(H4) *and let* $u_0 \in C(\mathbb{R}^N)$. *Suppose that* (6.1) *holds for the front associated with* $\mathrm{Graph}(u_0)$. *If* u *and* $\tilde{u}$ *are* (*possibly discontinuous*) *viscosity solutions of* (1.4)*, then* $u_* = \tilde{u}_*$ *and* $u^* = \tilde{u}^*$ *in* $\mathbb{R}^N \times [0,+\infty)$.

Contrarily to Theorem 6.2, this theorem provides a "weak" uniqueness result for discontinuous viscosity solutions of (1.4). It is worth pointing out that stronger results providing equalities like $u^* = \tilde{u}_*$ and $u_* = \tilde{u}^*$ in $\mathbb{R}^N \times [0,+\infty)$ cannot be obtained by such a geometrical approach since a discontinuity of u or $\tilde{u}$ can appear or, on the contrary, be removed by a slight rotation of the axis in $\mathbb{R}^{N+1}$ and therefore such discontinuities have no real geometrical meaning. We refer the reader to Barles, Soner & Souganidis [10], Ilmanen [24] and Soner [32] for a more complete discussion and results about the "fattening phenomena" or "nonempty interior difficulty".

We turn to the proofs.

Proof of Theorem 6.2. Suppose that there exist two solutions $u_1, u_2 \in C(\mathbb{R}^N \times [0, +\infty))$ of (1.4) with initial data u_0 and define v to be the solution of (2.1) with initial data $v_0 = \mathrm{d}(\cdot\,, \mathrm{Graph}(u_0))$. From the level-set approach, we have $\Gamma_t = \{v(\cdot\,, t) = 0\}$. We will see that, if u_1 and u_2 are different, then the front has nonempty interior in $\mathbb{R}^{N+1} \times [0, +\infty)$. If $u_1 \neq u_2$, we can suppose that there exists $(x_0, t_0) \in \mathbb{R}^N \times [0, +\infty)$ such that

$$u_1(x_0, t_0) - u_2(x_0, t_0) = \varepsilon > 0.$$

By continuity of u_1 and u_2, there exists some ball $B(x_0, \rho)$, $\rho > 0$ and some $\tau > 0$ such that

$$u_1(x, t) - u_2(x, t) \geqq \frac{\varepsilon}{2} > 0 \quad \text{in } B(x_0, \rho) \times [t_0, t_0 + \tau]. \tag{6.2}$$

But, from Theorem 6.1,

$$\mathrm{Graph}(u_1(\cdot\,, t)), \mathrm{Graph}(u_2(\cdot\,, t)) \subset \{v(\cdot\,, t) = 0\} \quad \text{for all } t \geqq 0.$$

To conclude, it suffices to show that

$$B(x_0, \rho) \times [t_0, t_0 + \tau] \subset \bigcup_{t \geqq 0} \Gamma_t \times \{t\}.$$

We need a lemma whose proof is postponed.

Lemma 6.1. *Under Assumptions* (H1)–(H4)*, let* $v \in UC(\mathbb{R}^{N+1} \times [0, +\infty))$ *be a solution of* (2.1) *with initial data* v_0*. If* $y \mapsto v_0(x, y)$ *is nondecreasing for all* $x \in \mathbb{R}^N$*, then* $y \mapsto v(x, y, t)$ *is nondecreasing for all* $(x, t) \in \mathbb{R}^N \times [0, +\infty)$*. In particular, the result holds if* v *is the solution associated with the initial condition* $v_0 = \mathrm{d}(\cdot\,, \mathrm{Graph}(u_0))$ *where* $u_0 \in C(\mathbb{R}^N)$.

From this lemma, we obtain

$$v(x, y, t) = 0 \quad \text{for all } (x, y, t) \in B(x_0, \rho) \times [u_2(x, t_0), u_1(x, t_0)] \times [t_0, t_0 + \tau].$$

Using (6.2), we find that $B(x_0, \rho) \times [u_2(x, t_0), u_1(x, t_0)] \times [t_0, t_0 + \tau]$ has nonempty interior in $\mathbb{R}^{N+1} \times [0, +\infty)$ which ends the proof. $\square$

Proof of Lemma 6.1. By assumption, $v_0(x, y+h) \geqq v_0(x, y)$ for all $x \in \mathbb{R}^{N+1}$, $y \in \mathbb{R}$ and $h > 0$. From the comparison result (see Theorem 4.1) it follows that $v(\cdot\,, \cdot + h, t) \geqq v(\cdot\,, \cdot\,, t)$ for all $t \geqq 0$, since $v(\cdot\,, \cdot + h, \cdot)$ is a solution of (2.1) with initial data $v_0(\cdot\,, \cdot + h)$. This proves the first part of the lemma.

It remains to show that the function $(x, y) \mapsto \mathrm{d}((x, y), \mathrm{Graph}(u_0))$ is nondecreasing in the y variable when u_0 is continuous. To this end, consider $x \in \mathbb{R}^N$ and $y_2 \geqq y_1$. We suppose that $y_2 \geqq y_1 \geqq u_0(x)$. Indeed, the case $u_0(x) \geqq y_2 \geqq y_1$ can be treated in the same way with straightforward adaptations and the case $y_2 \geqq u_0(x) \geqq y_1$ is obvious. Assume for contradiction that

$$0 \leqq r_2 := \mathrm{d}((x, y_2), \mathrm{Graph}(u_0)) < \mathrm{d}((x, y_1), \mathrm{Graph}(u_0)) =: r_1,$$

and define $u_{0|\overline{B}(x,r_1)}$ as the restriction of u_0 to the ball $\overline{B}(x, r_1)$. From the definition of d as an infimum, the hypothesis $y_1 \geqq u_0(x)$ and the continuity of u_0, it follows that

$$\mathrm{Graph}(u_{0|\overline{B}(x,r_1)}) \cap \overline{B}((x, y_1), r_1) \subset \partial^- B((x, y_1), r_1),$$

where $\partial^- B((x, y_1), r_1)$ stands for the part $\partial B((x, y_1), r_1)$ of the boundary of $\overline{B}((x, y_1), r_1)$ lying in the half-space $\{y \leqq y_1\}$. Since $y_2 \geqq y_1$ and $r_2 < r_1$, we get

$$\mathrm{Graph}(u_{0|\overline{B}(x,r_1)}) \cap B((x, y_2), r_2) = \emptyset$$

which gives a contradiction. □

Proof of Theorem 6.3. Let us show that $u_* = \tilde{u}_*$. We argue by contradiction, assuming that there exists $(\bar{x}, \bar{t}) \in \mathbb{R}^N \times [0, +\infty)$ such that $u_*(\bar{x}, \bar{t}) > \tilde{u}_*(\bar{x}, \bar{t})$. From Theorem 6.1, we have, for all $t \geqq 0$, $\mathrm{Graph}(\tilde{u}(\cdot, t)) \subset \Gamma_t = \{v(\cdot, t) = 0\}$, where v is the solution of (2.1) with initial data $\mathrm{d}(\cdot, \mathrm{Graph}(u_0))$. It follows from (6.1) that

$$\bigcup_{t \geqq 0} \mathrm{Graph}(\tilde{u}(\cdot, t)) \times \{t\} \subset \overline{\bigcup_{t \geqq 0} \Omega_t^+ \times \{t\}}\ ;$$

thus, there exists a sequence $(x_n, y_n, t_n) \in \mathbb{R}^{N+1} \times [0, +\infty)$ such that $(x_n, y_n, t_n) \to (\bar{x}, \tilde{u}_*(\bar{x}, \bar{t}), \bar{t})$ when $n \to +\infty$ and $v(x_n, y_n, t_n) > 0$ for all $n \geqq 0$. From the non-decrease of v in y (Lemma 6.1), we have $y_n > u(x_n, t_n)$ since $v(x_n, u(x_n, t_n), t_n) = 0$. It follows that

$$u_*(x, t) \leqq \liminf_{n \to +\infty} u(x_n, t_n) \leqq \liminf_{n \to +\infty} y_n = \tilde{u}_*(x, t)$$

which is a contradiction. We prove $u^* = \tilde{u}^*$ in the same way. □

The last result of this section is related to the empty interior condition of Theorem 6.2 and is inspired by the related results of Evans & Spruck [19] in the mean-curvature case.

If $u_0 \in C(\mathbb{R}^N)$ and if $v_0 = \mathrm{d}(\cdot, \mathrm{Graph}(u_0))$, then the subsets $\{v_0 = \lambda\}$, $\lambda \in \mathbb{R}$ are the graphs of functions $u_0^\lambda \in C(\mathbb{R}^N)$. More precisely, for $\lambda \geqq 0$, the function $\omega(x, \lambda) := u_0^\lambda(x)$ (or $\omega(x, \lambda) := u_0^{-\lambda}(x)$) is the unique viscosity solution of

$$\frac{\partial \omega}{\partial \lambda} - \sqrt{1 + |D\omega|^2} = 0 \quad \text{in } \mathbb{R}^N \times (0, +\infty),$$

(respectively

$$\frac{\partial \omega}{\partial \lambda} + \sqrt{1 + |D\omega|^2} = 0 \quad \text{in } \mathbb{R}^N \times (0, +\infty)\ ).$$

We refer to Barles [4] for a simple proof of this claim. Our result is the

Proposition 6.1. *Assume* (H1)–(H4). *Except for a countable subset of values of* λ, *the fronts associated with the evolution of* $\mathrm{Graph}(u_0^\lambda)$ *have empty interior in* $\mathbb{R}^{N+1} \times [0, +\infty)$. *In particular, there exists at most one continuous viscosity solution* u^λ *of* (1.4) *with initial data* u_0^λ.

We may interpret this result by saying that non-uniqueness for (1.4) is a "rare" event.

Noticing that $u_0^\lambda \downarrow u_0$ in $C(\mathbb{R}^N)$ as $\lambda \downarrow 0^+$, we find that we can approach any $u_0 \in C(\mathbb{R}^N)$ in a monotone way by a sequence of u_0^λ for which (1.4) has at most one continuous solution. The interesting thin about this result is that the u_0^λ's have in general the same behavior as u_0. It means that we actually have uniqueness for a large class of initial data including functions with arbitrary growth.

Proof of Proposition 6.1. Let v be the unique solution of (2.1) with initial data $v_0 = \mathrm{d}(\cdot\,, \mathrm{Graph}(u_0))$. Since, for every $\lambda \in \mathbb{R}$, $v - \lambda$ is the unique uniformly continuous solution of (2.1) with initial data $v_0 - \lambda$, at each time t, the front Γ_t^λ associated with $\mathrm{Graph}(u_0^\lambda)$ coincides with $\{v(\cdot\,, t) = \lambda\}$. In particular, the fronts are disjoint for different values of λ and it follows, from there, that the family of values of λ such that $\bigcup_{t \geqq 0} \Gamma_t^\lambda \times \{t\}$ has nonempty interior is countable. To conclude, it is sufficient to apply Theorem 6.2. $\square$

7. A local L^∞ *a priori* bound

In this section, we use the relations between (1.4) and (2.1) to provide a local L^∞ bound for the solutions of (1.4).

In order to state the main result of this section, we introduce, for any function $u_0 \in C(\mathbb{R}^N)$,

$$M_{u_0}(x, R) = \max_{y \in \overline{B}(x,R)} u_0(y) \quad \text{and} \quad m_{u_0}(x, R) = \min_{y \in \overline{B}(x,R)} u_0(y).$$

We have the following

Theorem 7.1. *Under assumptions* (H1)–(H4)*, if* $u \in C(\mathbb{R}^N \times [0, +\infty))$ *is a solution of* (1.4) *with initial data* $u_0 \in C(\mathbb{R}^N)$*, then there exists a positive constant* C *such that, for all* $x \in \mathbb{R}^N$*,* $t \geqq 0$*,*

$$m_{u_0}(x, \sqrt{2Ct}) - \sqrt{2Ct} \leqq u(x, t) \leqq M_{u_0}(x, \sqrt{2Ct}) + \sqrt{2Ct}.$$

Remark 7.1. This local L^∞ bound is a direct consequence of the level-set approach and it justifies the fact that we need at least some kind of degeneracy on b in the gradient variable, as implied by (H1)–(H4); indeed, clearly, such a bound does not hold for the heat equation and therefore we cannot hope that such an approach applies for this equation.

Proof of Theorem 7.1. The basic idea is that the geometrical evolution governed by (2.1) preserves the inclusion of sets. Thus we can expect that the evolution of balls initially put "under" (or "over") the graph of a solution of (1.4) will provide some control on the growth. This fact is illustrated in Fig. 7.1 in the case of the mean-curvature equation.

We take $v_0(z) = \mathrm{d}(z, \mathrm{Graph}(u_0)) \in UC(\mathbb{R}^{N+1})$ (where d is defined by (5.3)) and let v be the unique uniformly continuous solution of (2.1) with initial data v_0. In order to prove the result, we aim at comparing v with subsolutions like those which appear in the following Lemma, whose proof is postponed.

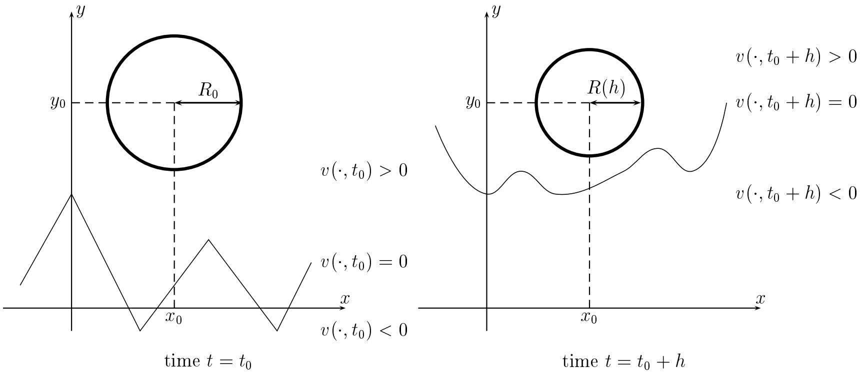

Fig. 7.1. Evolution of a graph with a ball which is put above it

Lemma 7.1. *We suppose that* (H1), (H2) *and* (H3) *hold. Fix* $R_0 > 0$, $x_0 \in \mathbb{R}^N$ *and* $y_0 \in \mathbb{R}$. *Let* $\Psi : \mathbb{R} \to \mathbb{R}$ *be any smooth nondecreasing function. Then the function* φ, *defined for every* $(x, y, t) \in \mathbb{R}^N \times \mathbb{R} \times [0, +\infty)$, *by*

$$\varphi(x, y, t) = \Psi(R_0^2 - 2Ct - |x - x_0|^2 - (y - y_0)^2),$$

where $C = N(K_1 + K_2 + K_3) + 1$, *is a* (*classical*) *strict subsolution of* (2.1).

Let $x_0 \in \mathbb{R}^N$, $t_0 \in (0, +\infty)$ and $y_0 = M_{u_0}(x_0, \sqrt{2Ct_0}) + \sqrt{2Ct_0}$, where C is taken as in Lemma 7.1. It follows that

$$\overline{B}((x_0, y_0), \sqrt{2Ct_0}) \subset \{(x, y) \in \mathbb{R}^{N+1} : y > u_0(x)\},$$

which implies

$$\mathrm{d}\left(\cdot\,, \overline{B}((x_0, y_0), \sqrt{2Ct_0})\right) \leqq \mathrm{d}(\cdot\,, \mathrm{Graph}(u_0)) = v_0. \tag{7.1}$$

Let us define

$$\varphi(x, y, t) = \Psi(2Ct_0 - 2Ct - |x - x_0|^2 - (y - y_0)^2),$$

with $\Psi(z) = z/2\sqrt{2Ct_0}$ for all $z \in \mathbb{R}$. The function Ψ satisfies the assumptions of Lemma 7.1. For clarity, we set $r = (|x - x_0|^2 + (y - y_0)^2)^{1/2}$. From (7.1), we get

$$\begin{aligned}\varphi(\cdot\,, \cdot\,, 0) &= \frac{\sqrt{2Ct_0} + r}{2\sqrt{2Ct_0}}(\sqrt{2Ct_0} - r) \leqq \sqrt{2Ct_0} - r \\ &= \mathrm{d}\left(\cdot\,, \overline{B}((x_0, y_0), \sqrt{2Ct_0})\right) \leqq v_0.\end{aligned}$$

Now, since φ is a function with quadratic growth at infinity and v is uniformly continuous, we know that

$$\min_{\mathbb{R}^{N+1} \times [0,T]} \{v - \varphi\}$$

is achieved for every $T > 0$ if we assume that it is positive. Using Lemma 7.1, φ is a strict smooth subsolution of (2.1); thus the minimum is necessarily achieved at

$t = 0$ which is a contradiction. Since the previous arguments hold for every $T > 0$, we get finally

$$\varphi \leqq v \quad \text{in } \mathbb{R}^{N+1} \times [0, +\infty). \tag{7.2}$$

By Lemma 6.1, we have

$$\operatorname{Graph}(u(\cdot\,, t)) \subset \Gamma_t(v) \quad \text{for every } t \geqq 0. \tag{7.3}$$

From (7.2) and (7.3), it follows that, for all $t \geqq 0$,

$$\{\varphi(\cdot\,, t) \geqq 0\} \subset \{v(\cdot\,, t) \geqq 0\} \subset \{(x, y) \in \mathbb{R}^{N+1} : y \geqq u(x, t)\}.$$

But

$$\varphi(x, y, t) \geqq 0 \Longleftrightarrow (x, y) \in \overline{B}((x_0, y_0), \sqrt{2C(t_0 - t)}).$$

By letting $t \to t_0$ and by using the assumed continuity of u, we obtain

$$u(x_0, t_0) \leqq y_0 = M_{u_0}(x_0, \sqrt{2Ct_0}) + \sqrt{2Ct_0}.$$

The opposite inequality is obtained with straightforward adaptations. □

We end the section with the proof of the lemma and an example.

Proof of Lemma 7.1. Without loss of generality, we can suppose that $x_0 = 0$ and $y_0 = 0$. Moreover, from Lemma 4.1, we can suppose that $\Psi(z) = z$ for every $z \in \mathbb{R}$. From (H1), (H2) and (H3), we get

$$|F_*(D\varphi, D^2\varphi)| \leqq N(K_1 + K_2 + K_3)|D^2\varphi| \leqq 2(C - 1)$$

where we set $C = N(K_1 + K_2 + K_3) + 1$. It follows that

$$\frac{\partial \varphi}{\partial t} + F_*(D\varphi, D^2\varphi) \leqq -2C + 2(C - 1) \leqq -2 < 0,$$

which achieves the proof. □

Example 7.1. Evolution of balls in the case of the mean-curvature equation (1.1). We recall that, in the case of (1.1), b is given by (3.3). Following the computations of Lemma 7.1, we have

$$\frac{\partial \varphi}{\partial t} - \operatorname{Tr}\left[b\left(-\frac{D_x\varphi}{D_y\varphi}\right)\left(D^2_{xx}\varphi - 2D^2_{xy}\varphi \otimes \frac{D_x\varphi}{D_y\varphi} + D^2_{yy}\varphi \frac{D_x\varphi}{D_y\varphi} \otimes \frac{D_x\varphi}{D_y\varphi}\right)\right] = -2(C - N).$$

By taking $C = N$, we find that φ is in fact a classical solution of (2.1). Thus, by the level-set approach, it follows that the 0-level set of φ evolves according to its mean curvature. An easy computation shows that

$$\Omega_0^+ = \{\varphi(\cdot\,, 0) > 0\} = B((x_0, y_0), R_0), \quad \Gamma_0 = \partial B((x_0, y_0), R_0),$$

and, for every $t \geqq 0$,

$$\Omega_t^+ = \{\varphi(\cdot\,, t) > 0\} = B((x_0, y_0), R(t)), \quad \Gamma_t = \partial B((x_0, y_0), R(t)),$$

where $R(t) = (R_0^2 - 2Nt)^{1/2}$. We recover by this method the well-known result of EVANS & SPRUCK [18, Section 7.1]: balls remain balls for the mean-curvature motion and they shrink into a point for $t^* = R_0^2/2N$.

8. The boundary of the front. Existence of discontinuous solutions

Theorem 6.1 provides the first connections between the front and the graphs of the solutions of (1.4) when they exist. In this section, we describe more precisely the structure of the front and obtain the existence of discontinuous solutions to (1.4).

For any continuous function u_0, we consider the generalized evolution $(\Gamma_t)_{t \geqq 0}$ of Graph(u_0) and the uniformly continuous solution v of (2.1) with initial data $v_0 = \mathrm{d}(\cdot\,, \mathrm{Graph}(u_0))$. For every $(x, t) \in \mathbb{R}^N \times [0, +\infty)$, we define

$$u^+(x, t) := \sup\{y \in \mathbb{R} : v(x, y, t) \leqq 0\}$$

and

$$u^-(x, t) := \inf\{y \in \mathbb{R} : v(x, y, t) \geqq 0\}.$$

Note that the functions $u^+(\cdot\,, t)$ and $u^-(\cdot\,, t)$ are defined such that their graphs are the "upper-boundary" and the "lower-boundary" of the front Γ_t at each time t: see Fig. 8.1. We have the first properties

Lemma 8.1. *Under assumptions* (H1)–(H4)*, the functions u^+ and u^- are locally bounded in $\mathbb{R}^N \times [0, +\infty)$). Moreover $u^+ \in USC(\mathbb{R}^N \times [0, +\infty))$ and $u^- \in LSC(\mathbb{R}^N \times [0, +\infty))$.*

Proof of Lemma 8.1. We give the proof for u^+, the one for u^- being similar. We start by proving that u^+ is well defined and locally bounded. Looking at the proof

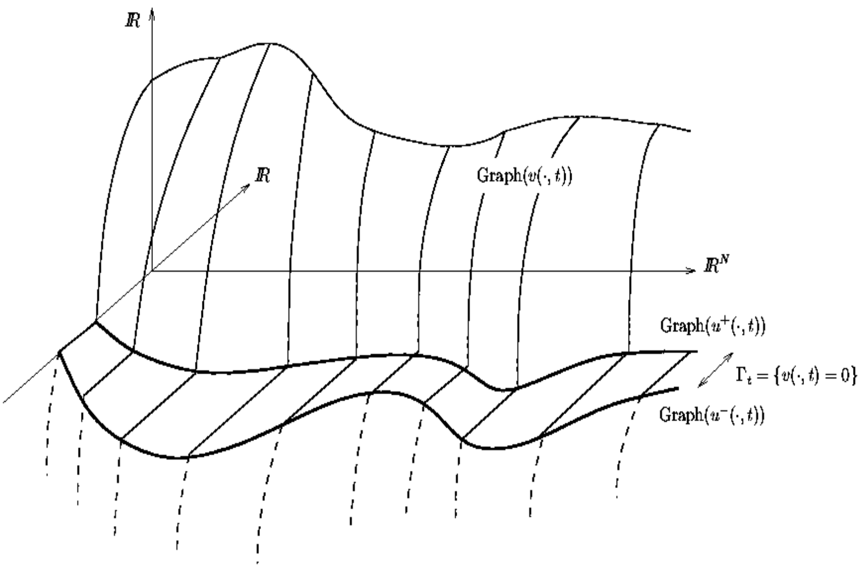

Fig. 8.1. Front which fattens at time $t > 0$

of Theorem 7.1, we see that inequality (7.2) implies that, for every $(x_0, y_0, t_0) \in \mathbb{R}^N \times \mathbb{R} \times [0, +\infty)$, there exists a constant $M > 0$ such that

$$v > 0 \quad \text{in } \overline{B}((x_0, y_0), M) \times [0, t_0/2].$$

Note that M does not depend on y_0 in the sense that $v > 0$ in every $\overline{B}((x_0, y), M) \times [0, t_0/2]$ with $y \geqq y_0$, by non-decrease of $y \mapsto v(x, y, t)$ for every (x, t) (see Lemma 6.1). It proves that $u^+ \leqq y_0$ in a neighborhood of $(x_0, t_0/2)$. The same reasoning holds with straightforward adaptations to prove that u^+ is locally bounded from below.

We turn to the proof of the upper-semicontinuity of u^+. Consider any sequence of points $((x_n, y_n, t_n))_{n\in\mathbb{N}}$ such that $(x_n, y_n, t_n) \in \mathcal{H} := \{(x, y, t) \in \mathbb{R}^{N+1} \times [0, +\infty) : y \leqq u^+(x, t)\}$ and $(x_n, y_n, t_n) \to (x, y, t)$ as $n \to +\infty$. For every n, we have $v(x_n, y_n, t_n) \leqq 0$. Since v is continuous, by sending n to infinity, we get $v(x, y, t) \leqq 0$ which proves that $(x, y, t) \in \mathcal{H}$; thus $\mathcal{H}$ is closed. This ends the proof. □

Theorem 8.1. *Suppose that* (H1)–(H4) *hold. Let* $u_0 \in C(\mathbb{R}^N)$ *and* v *be the solution of* (2.1) *associated with the initial data* $v_0 = \mathrm{d}(\cdot\,, \mathrm{Graph}(u_0))$. *Then* u^+ *and* u^- *are* (*possibly discontinuous*) *viscosity solutions of* (1.4) *with initial data* u_0. *Moreover,* u^+ *and* u^- *are respectively the maximal subsolution and the minimal supersolution of* (1.4) *with initial data* u_0.

We refer to Fig. 8.1 for an illustration of this theorem. Let us point out that this result provides only the existence of a discontinuous viscosity solution to (1.4) for any continuous initial data. We refer to Section 9 for optimal results of regularity of u^+ and u^-. We give a geometrical proof of the theorem using the fact that characteristic functions of sets which evolve are discontinuous solutions of (2.1).

We first introduce some notation. For any subset $A \subset \mathbb{R}^{N+1}$, $\mathrm{int}(A)$ denotes the interior of A in $\mathbb{R}^{N+1}$ and $\mathbb{1}_A$ is the characteristic function of A defined, for any $(x, y) \in \mathbb{R}^{N+1}$, by $\mathbb{1}_A(x, y) = 1$ if $(x, y) \in A$ and 0 otherwise. For the sake of simplicity of notation, when the set $A = A_t$ depends on t, we will denote by $\mathbb{1}_{A_t}$ the function $(x, y, t) \mapsto \mathbb{1}_{A_t}(x, y)$. We need the following lemma due to BARLES, SONER & SOUGANIDIS [10].

Lemma 8.2. *Define* u_0 *and* v *as in Theorem 8.1 and consider*

$$\Omega_t^+ = \{v(\cdot\,, t) > 0\} \quad \text{and} \quad \Upsilon_t = \mathrm{int}\big(\{v(\cdot\,, t) \geqq 0\}\big).$$

Then the functions $\mathbb{1}_{\Omega_t}$ *and* $\mathbb{1}_{\Upsilon_t}$ *are* (*discontinuous*) *viscosity solutions of* (2.1).

Proof of Theorem 8.1. We give the proof for u^+, calling it u for clarity of notation. The same reasoning holds with easy adaptations for u^-. Let us start by showing that u is a subsolution. Remembering that u is an upper-semicontinuous function by Lemma 8.1, we consider a smooth function $\phi(x, t)$ such that $u - \phi$ achieves a global maximum of 0 at $(\bar{x}, \bar{t}) \in \mathbb{R}^N \times (0, +\infty)$. It follows that $u \leqq \phi$ and $u(\bar{x}, \bar{t}) = \phi(\bar{x}, \bar{t})$. Set $\psi(x, y, t) = \tanh(y - \phi(x, t))$.

We claim that $(\mathbb{1}_{\Omega_t})_* - \psi$ achieves a global minimum 0 at $(\bar{x}, u(\bar{x}, \bar{t}), \bar{t})$. By continuity of v, we have $v(\bar{x}, u(\bar{x}, \bar{t}), \bar{t}) = 0$; thus $(\mathbb{1}_{\Omega_t})_*(\bar{x}, u(\bar{x}, \bar{t}), \bar{t}) = 0$.

This implies $((\mathbb{1}_{\Omega_t})_* - \psi)(\bar{x}, u(\bar{x}, \bar{t}), \bar{t}) = 0$. It remains to check that $((\mathbb{1}_{\Omega_t})_* - \psi)(x, y, t) \geqq 0$ for every (x, y, t). If $(\mathbb{1}_{\Omega_t})_*(x, y, t) = 0$, then $(x, y, t) \in \{v \leqq 0\}$; thus, from Lemma 6.1, we have $y \leqq u(x, t) \leqq \phi(x, y, t)$. We obtain $y - \phi(x, t) \leqq 0$ and $((\mathbb{1}_{\Omega_t})_* - \psi)(x, y, t) \geqq 0$ in this case. Now, if $(\mathbb{1}_{\Omega_t})_*(x, y, t) = 1$, then the same inequality holds since $\tanh \leqq 1$. This proves the claim.

We compute the derivatives of ψ and get,

$$\frac{\partial \psi}{\partial t} = -\tanh' \cdot \tfrac{\partial \phi}{\partial t}, \quad D_x\psi = -\tanh' \cdot D_x\phi, \quad D_y\psi = \tanh',$$

$$D^2_{xx}\psi = \tanh'' \cdot D_x\phi \otimes D_x\phi - \tanh' \cdot D^2_{xx}\phi, \quad D^2_{yy}\psi = \tanh'',$$

$$D^2_{xy}\psi = -\tanh'' \cdot D_x\phi.$$

By Proposition 8.2, $(\mathbb{1}_{\Omega_t})_*$ is a supersolution of (2.1); writing the viscosity inequality at the point $(\bar{x}, u(\bar{x}, \bar{t}), \bar{t})$, a calculation leads to

$$\frac{\partial \phi}{\partial t} - \operatorname{Tr}\left[b(D_x\phi) D^2_{xx}\phi\right] \leqq 0,$$

which shows that u is a viscosity subsolution.

We continue by proving that u_* is a supersolution. Consider a smooth function ϕ such that $u_* - \phi$ achieves a global minimum of 0 at $(\bar{x}, \bar{t}) \in \mathbb{R}^N \times (0, +\infty)$. We claim first that $(\mathbb{1}_{\Omega_t})^*(\bar{x}, u_*(\bar{x}, \bar{t}), t) = 1$. Otherwise, $(\bar{x}, u_*(\bar{x}, \bar{t}), t)$ lies in the interior of $\bigcup_{t \geq 0} \Gamma_t \times \{t\}$; it means that there exists $\varepsilon > 0$ such that

$$v(x, y, t) = 0 \text{ for all } (x, y, t) \in \overline{B}(\bar{x}, \varepsilon) \times [u_*(\bar{x}, \bar{t}) - \varepsilon, u_*(\bar{x}, \bar{t}) + \varepsilon] \times [\bar{t} - \varepsilon, \bar{t} + \varepsilon].$$

By definition of u, it follows that $u(x, t) \geqq u_*(\bar{x}, \bar{t}) + \varepsilon$ for every $x \in \overline{B}(\bar{x}, \varepsilon)$ and $t \in [\bar{t} - \varepsilon, \bar{t} + \varepsilon]$. It leads to a contradiction and proves the claim.

Defining ψ as above, we observe that the function $(\mathbb{1}_{\Omega_t})^* - \psi$ achieves a global maximum point at $\big((\bar{x}, u_*(\bar{x}, \bar{t})), \bar{t}\big)$. Indeed, if $(\mathbb{1}_{\Omega_t})^*(x, y, t) = 0$, then $(\mathbb{1}_{\Omega_t})^* - \psi \leqq 1$. If $(\mathbb{1}_{\Omega_t})^*(x, y, t) = 1$, then $(x, y, t) \in \overline{\{v > 0\}} = \{y \geqq u_*(x, t)\}$, since u_* is lower-semicontinuous. It follows that $y \geqq \phi(x, t)$ and $\tanh(y - \phi(x, t)) \geqq 0$; thus $(\mathbb{1}_{\Omega_t})^* - \psi \leqq 1$ and we are done in any case. Using the fact that $(\mathbb{1}_{\Omega_t})^*$ is a subsolution of (2.1) by Proposition 8.2, we conclude as above.

It remains to check that the initial condition holds. On the one hand, from the continuity of v, we have $v(x, u(x, 0), 0) = 0$. It implies $u(x, 0) = u_0(x)$ since Γ_0 is exactly the graph of the continuous function u_0. On the other hand, looking at the proof of the supersolution, we see $(x, u_*(x, t), t) \in \overline{\Omega_t}$ for every $(x, t) \in \mathbb{R}^N \times [0, +\infty)$. By continuity of v, $v(x, u_*(x, t), t) = 0$. For $t = 0$, it means $u_*(x, 0) = u_0(x)$. Finally, note that from Theorem 6.1, the graphs of all subsolutions of (2.1) lie in $\{v \leqq 0\}$ and in the same way the graphs of all supersolutions of (2.1) lie in $\{v \geqq 0\}$. Therefore, u^- is the minimal subsolution and u^+ is the maximal supersolution of (2.1); and the proof of the theorem is complete. $\square$

Remark 8.1. If the “no-interior condition” (6.1) holds, then Theorem 6.3 implies $(u^+)_* = u^-$ and $(u^-)^* = u^+$ in $\mathbb{R}^N \times [0, +\infty)$. But even in this case, we cannot conclude that a continuous viscosity solution exists, since the front may look like a Heaviside function for instance.

9. Fronts with more regularity

As mentioned before, the extremal solutions u^+ and u^- have no regularity in general. We give below some conditions under which they are smooth. It is the case, when the front is associated with (1.4) and a locally Lipschitz initial data u_0, as soon as the solutions u of this equation satisfy local L^∞ and gradient bounds. On one hand, these bounds allow the construction of smooth solutions for any continuous initial data. On the other hand, using approximation methods, we see that this regularity holds for the extremal solutions. We start with a more precise result in the case of the mean-curvature equation.

Theorem 9.1. *Let $u_0 \in C(\mathbb{R}^N)$. Then the extremal solutions u^+ and u^- of* (1.1) *with initial data u_0 are in $C^\infty(\mathbb{R}^N \times (0, +\infty)) \cap C(\mathbb{R}^N \times [0, +\infty))$.*

We recall that the smooth existence for $u_0 \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N)$ is proved in ECKER & HUISKEN [16] (see also CHOU & KWONG [14]) using a gradient estimate. Here, following ANGENENT [1], we take advantage of an interior gradient estimate of EVANS & SPRUCK [18] to prove the result for initial data u_0 which are merely continuous.

In the general case, we have

Theorem 9.2. *Assume that b satisfies assumptions* (H1)–(H4) *and*

$$b(q) \geqq \Lambda(|q|)Id \tag{9.1}$$

for some nonnegative continuous function Λ in $\mathbb{R}^N$. Suppose that for any $u_0 \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N)$ there exists a smooth solution of (1.4) *with initial data u_0 satisfying a local gradient bound, namely*

$$\|Du\|_{\infty,\Omega_{R,T}} \leqq K,$$

where $\Omega_{R,T} := \overline{B}(0, R) \times [0, T]$ and K is a positive constant which depends only on R, T, $\|u\|_{\infty,\bar{\Omega}_{R,T}}$ and $\|Du_0\|_{\infty,\overline{B}(0,R')}$, with $R' = R'(R, T) > 0$. Then the extremal solutions u^+ and u^- are smooth.

The above theorem applies to more general quasilinear equations than (1.1) (see examples at the end of the section) but it requires the initial data to be locally Lipschitz continuous.

Remark 9.1. Under the assumptions of Theorem 9.1 or 9.2, if Γ_t has empty interior in $\mathbb{R}^{N+1}$ for all $t \geqq 0$ (or equivalently the front $\bigcup_{t \geqq 0} \Gamma_t \times \{t\}$ has empty interior in $\mathbb{R}^{N+1} \times [0, T]$, see Section 6), then the smoothness of the extremal solutions together with Theorem 6.2 implies $u^+ = u^-$ in $\mathbb{R}^N \times [0, +\infty)$. It follows that

$\Gamma_t = \partial\Omega_t^+ = \partial\Omega_t^-$ for all $t \geqq 0$. In this case, we have in particular uniqueness and comparison for the discontinuous solutions of (1.4). Moreover, the weak notion of propagation given by the level-set approach coincides with the classical notion in differential geometry.

Before turning to the proof of the theorems, we state a lemma concerning the time regularity of solutions for which the space regularity is already known. We recall that a function $m : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a modulus of continuity if $m(0^+) := \lim_{s\to 0^+} = 0$ and $m(s+t) \leqq m(s) + m(t)$ for any $s, t \geqq 0$.

Lemma 9.1. *Let $R > 0$, $0 \leqq t_0 < T$, $x_0 \in \mathbb{R}^N$ and $u \in C(\overline{B}(x_0, R) \times [t_0, T])$ be a viscosity solution of the equation*

$$\frac{\partial u}{\partial t} + G(x, t, Du, D^2u) = 0 \text{ in } \Omega_{R,t_0,T} = B(x_0, R) \times (t_0, T), \tag{9.2}$$

where $G \in C(\overline{B}(x_0, R) \times [t_0, T] \times \mathbb{R}^N \times \mathcal{S}_N)$ is degenerate elliptic. If m denotes a modulus of continuity of $u(\cdot, t_0)$, i.e. if, for every $x, y \in \overline{B}(x_0, R)$,

$$|u(y, t_0) - u(x, t_0)| \leqq m(|y - x|),$$

then there exists a modulus of continuity $\tilde{m}$ depending only on G, m and $\|u\|_{\infty,\overline{\Omega}_{R,t_0,T}}$ such that, for every $t \in [t_0, T]$ and $x \in \overline{B}(x_0, R/2)$,

$$|u(x, t) - u(x, t_0)| \leqq \tilde{m}(|t - t_0|). \tag{9.3}$$

Moreover, if $m(r) = Lr$ for some $L \geqq 0$ and if

$$|G(x, t, p, X)| \leqq M(1 + |X|) \quad \text{on} \quad \overline{B}(x_0, R) \times [t_0, T] \times \overline{B}(0, L) \times \mathcal{S}_N \tag{9.4}$$

for some constant $M \geqq 0$, then there exists $\tilde{L} = \tilde{L}(L, M, \|u\|_{\infty,\overline{\Omega}_{R,t_0,T}}) > 0$ such that $\tilde{m}(r) = \tilde{L}r^{1/2}$.

Of course, the key point in Lemma 9.1 is the fact that $\tilde{m}$ depends only on G, m and $\|u\|_{\infty,\overline{\Omega}_{R,t_0,T}}$. As a by-product of this result, it is clear that a local L^∞ bound together with a time-uniform space-local modulus of continuity for the solutions of equations like (9.2) implies a uniform local modulus of continuity in time. In the statement of Lemma 9.1, for the sake of simplicity of formulation, we do not make precise the dependence with respect to G, except in the second part of the result; this dependence will appear clearly in the proof.

Proof of Theorem 9.1. We divide the proof into two steps.

Step 1. We construct a smooth solution for any continuous initial data. Let $u_0 \in C(\mathbb{R}^N)$ and $(u_0^R)_{R>0}$ be a sequence of uniformly continuous functions converging to u_0, uniformly on every compact subset. Thanks to classical results for viscosity solutions (see [13] and references therein), we associate with each u_0^R a continuous viscosity solution u^R of (1.1) with initial data u_0^R. But the u^R satisfy the L^∞ local bound of Theorem 7.1, and, from Evans & Spruck [19], we learn that the u^R are in fact smooth and satisfy the interior local gradient bounds proved in [19]. From

Lemma 9.1 we get then an interior local modulus of continuity for the u^R; therefore, up to an extraction argument, we can suppose that the family $(u^R)_{R>0}$ converge locally uniformly in $\mathbb{R}^N \times (0, +\infty)$ to a function $u \in C(\mathbb{R}^N \times (0, +\infty))$ which is, by a classical stability result, a viscosity solution of (1.1) in $\mathbb{R}^N \times (0 + \infty)$.

It remains to check that the initial condition is continuously satisfied. In view of Lemma 9.1, the u^R admit the same modulus of continuity at time $t = 0$ and it follows that u is continuous at time $t = 0$ with $u(\cdot\,, 0) = u_0$. Finally, from [19] we find that $u \in C^\infty(\mathbb{R}^N \times (0, +\infty))$ since it is a continuous solution of (1.1).

Step 2. We show that u^+ is smooth; the proof for u^- is the same with straightforward adaptations. Let $u_0 \in C(\mathbb{R}^N)$. Consider, for any $\lambda > 0$, the function u_0^λ defined by

$$\mathrm{Graph}(u_0^\lambda) = \{\mathrm{d}(\cdot\,, \mathrm{Graph}(u_0)) = \lambda\},$$

and the unique uniformly continuous solution v of (1.3) with initial data $\mathrm{d}(\cdot\,, \mathrm{Graph}(u_0))$ (we recall that (1.3) is the geometrical equation associated with (1.1)). By Step 1, we associate with each $\lambda > 0$ a smooth solution u^λ of (1.1) which satisfies, from Theorem 6.1, the condition that, for all $t \geqq 0$,

$$\mathrm{Graph}(u^\lambda(\cdot\,, t)) \subset \{v(\cdot\,, t) = \lambda\} \subset \{v(\cdot\,, t) > 0\}. \tag{9.5}$$

Now, as in Step 1, the family $(u^\lambda)_{\lambda>0}$ satisfies the interior gradient bound of [19]; thus, using Lemma 9.1 and the same arguments as above, we can assume that u^λ converges locally uniformly to a solution u of (1.1) with initial data u_0. From (9.5), we find that $u \geqq u^+$ and thus $u = u^+$. It follows that u^+ is continuous and therefore smooth, thanks again to [19]. □

Proof of Theorem 9.2. Since the proof is close to the previous one, we only give a sketch of it. We use arguments of Step 2 in the proof of Theorem 9.1. The only change is that, using the ellipticity condition (9.1), we get, in addition to the gradient bound, local bounds for high order derivatives of the u_λ (see LADYZENSKAJA, SOLONNIKOV & URAL'CEVA [29]). It follows that, up to an extraction, we can assume that the family $(u_\lambda)_{\lambda>0}$ converges locally uniformly to a smooth function u which is also a solution of (1.4). We conclude as in the proof of Theorem 9.1, that $u = u^+$ is actually smooth. □

It remains to give the proof of the lemma.

Proof of Lemma 9.1. The main step in the proof consists in showing that, for any $\eta > 0$, we can find positive constants $C, K > 0$ large enough, depending only on η, G, m and $\|u\|_{\infty,\bar{\Omega}_{R,t_0,T}}$ such that, for any $x \in B(x_0, R/2)$,

$$u(y,t) - u(x,t_0) \leqq \eta + C|y-x|^2 + K(t-t_0) \ \text{ for every } (y,t) \in \bar{\Omega}_{R,t_0,T}, \tag{9.6}$$

and

$$u(y,t) - u(x,t_0) \geqq -\eta - C|y-x|^2 - K(t-t_0) \ \text{ for every } (y,t) \in \bar{\Omega}_{R,t_0,T}. \tag{9.7}$$

We only prove (9.6), the inequality (9.7) being proved in an analogous way. In what follows, x is fixed in $B(x_0, R/2)$.

First, if we take

$$C \geqq \frac{8\|u\|_{\infty,\bar{\Omega}_{R,t_0,T}}}{R^2}, \tag{9.8}$$

then (9.6) is clearly fulfilled on $\partial B(x_0, R) \times [t_0, T]$, for every $\eta, K > 0$ and for every $x \in B(x_0, R/2)$. It is worth noticing that C may be taken independent of x.

Next, we would like to ensure that (9.6) holds for $t = t_0$. To this end, we argue by contradiction assuming there exists $\eta > 0$ such that, for every $C > 0$, there exists $y_C \in \overline{B}(x_0, R)$ such that

$$u(y_C, t_0) - u(x, t_0) > \eta + C|y_C - x|^2. \tag{9.9}$$

It follows that

$$|y_C - x| \leqq \sqrt{\frac{2\|u\|_{\infty,\Omega_{R,t_0,T}}}{C}}. \tag{9.10}$$

Thus $|y_C - x| \to 0$ when $C \to \infty$. Coming back to (9.9), we get

$$m(|y_C - x|) \geqq u(y_C, t_0) - u(x, t_0) > \eta + C|y_C|^2 \geqq \eta.$$

Using (9.10), the inequality $m(|y_C - x|) \geqq \eta$ leads to a contradiction as soon as we choose C large enough and this choice depends only on η, $\|u\|_{\infty,\Omega_{R,t_0,T}}$ and m. Therefore, by choosing C large enough, we ensure that (9.6) is satisfied on the parabolic boundary $(\partial B(x_0, R) \times [t_0, T]) \cup (\overline{B}(x_0, R) \times \{t_0\})$. Finally, using the continuity of G, we can take K large enough in order that the function $(y, t) \mapsto u(x, t_0) + \eta + C|y - x|^2 + K(t - t_0) := \chi(y, t)$ is a (smooth) strict supersolution of (9.2). Thus, since u is a viscosity subsolution of (9.2), by using only the definition of viscosity subsolution, it is clear that $\max_{\bar{\Omega}_{R,t_0,T}}\{u - \chi\}$ is necessarily achieved on the parabolic boundary of $\Omega_{R,t_0,T}$. And (9.6) follows.

The first part of the lemma follows by observing that all our constants depend when η is fixed, on G, m and $\|u\|_{\infty,\bar{\Omega}_{R,t_0,T}}$ but not on $x \in B(x_0, R/2)$.

If we assume that $m(r) = Lr$ for some positive constant L, the condition (9.6) at time $t = t_0$ reads

$$u(y, t_0) - u(x, t_0) \leqq L|x - y| \leqq \eta + C|y - x|^2,$$

for every $y \in \overline{B}(x_0, R)$. Writing that the discriminant of $C|y - x|^2 + L|y - x| + \eta$ is nonpositive, we find that it holds if

$$C \geqq \frac{L^2}{4\eta}.$$

Using (9.4), χ is a supersolution if $K \geqq M(1 + 2C)$. Introducing these estimates in (9.6), we finally obtain, for $y = x$,

$$u(x, t) - u(x, t_0) \leqq \eta + M\left(1 + \frac{L^2}{2\eta}\right)(t - t_0)$$

for all $t \in [t_0, T]$. An easy optimization with respect to η of the right-hand side of the previous inequality shows that, for all $t \in [t_0, T]$,

$$u(x,t) - u(x,t_0) \leqq \tilde{L}\sqrt{t-t_0},$$

for some positive constant $\tilde{L}$ depending on M and L. This concludes the proof of the Lemma. □

We conclude this section with examples of equations satisfying the assumptions of Theorem 9.2. The following equations come from the paper of CHOU & KWONG [14] (see Section 4.4 for the precise statement of the equations).
(1) The non-geometric mean-curvature equation (4.13) and equation (4.14) are associated with a front with smooth boundary when u_0 is locally Lipschitz continuous.
(2) Consider equation (4.15) with $g(q) = g(|q|)$ continuous in $\mathbb{R}^N$. Suppose that

$$\frac{1}{r^2}\left(1 - \frac{C}{r^2}\right) \leqq g(r) < \frac{1}{r^2} \quad \text{for every } r > 0, \tag{9.11}$$

and, in addition, that g is a C^1 function such that

$$2g + rg' \leqq \frac{C}{(1+r^2)^{3/2}},$$

for every $r \geqq 0$. Then CHOU & KWONG [14] gives the gradient bound and the conclusion of Theorem 9.2 holds. Note that (9.11) is nothing but (4.16) with a strict inequality on the right-hand side. This strict inequality ensures that the condition of ellipticity (9.1) is satisfied.

10. Application to convex solutions

In this section, we are interested in convex solutions of (1.4). We also derive some properties for the generalized evolution of convex sets.

Our main result is

Theorem 10.1. *Assume* (H1)–(H4) *and let u_0 be a convex function in $\mathbb{R}^N$.*

(i) *Suppose that $u_1 \in USC(\mathbb{R}^N \times [0,+\infty))$ (or $u_2 \in LSC(\mathbb{R}^N \times [0,+\infty))$) is a viscosity subsolution (respectively supersolution) of* (1.4). *If $u_1(\cdot,0) \leqq u_0 \leqq u_2(\cdot,0)$ in $\mathbb{R}^N$, then $u_1 \leqq u_2$ in $\mathbb{R}^N \times [0,+\infty)$.*
(ii) *There exists a unique continuous viscosity solution u to* (1.4) *with initial data u_0. Moreover $u(\cdot,t)$ is convex for all $t \geqq 0$.*

It is worth pointing out that, in the previous theorem, the existence and comparison properties hold without any restriction on the growth at infinity of the solutions or the initial data. Moreover, they hold both in the classical and very singular framework. Therefore, we have a complete answer in this case.

In the particular case of the mean-curvature equation, the solution is in addition smooth.

Theorem 10.2. *If the initial data $u_0 \in C(\mathbb{R}^N)$ is convex, then there exists a unique continuous solution u of the mean-curvature equation for graphs* (1.1)*; moreover $u \in C^\infty(\mathbb{R}^N \times (0, +\infty)) \cap C(\mathbb{R}^N \times [0, +\infty))$ and $u(\cdot, t)$ is convex for any $t \geqq 0$.*

Theorem 10.1 and 10.2 strongly justify the geometrical approach to the study of (1.4): in the case of convex solutions, the existence of solutions follows (rather) easily from the L^∞ bound of Theorem 7.1 since it implies also a gradient bound; the existence proof can be done either using Theorem 8.1 or directly on (1.4) as in [7]. For the comparison result, we point out that working on (2.1) as we do here, in particular for the mean-curvature equation, provides better results: in [7], we obtain a comparison result working directly on (1.4) by using a Kružkov change ($u \mapsto -\exp(-u)$) but with stronger assumptions on b and u_0, which was assumed to be coercive.

Below, we will give a proof which is simpler and essentially based on the preservation of convexity for geometric motions governed by (2.1); more precisely, we have

Theorem 10.3. *Suppose that* (H1)–(H4) *hold. Let $v_0 \in UC(\mathbb{R}^N)$ be a convex* (*or concave*) *function and v be the associated solution of* (2.1)*. Then $v(\cdot, t)$ is convex* (*respectively concave*) *for any $t \geqq 0$.*

Proof of Theorem 10.3. This result is a consequence of the one established by Giga, Goto, Ishii & Sato in [21] that we extend to the very singular case by using an approximation argument.

Step 1. We define, for any $\varepsilon > 0$ and $(p, M) \in \mathbb{R}^{N+1} \times \mathcal{S}_{N+1}$,

$$F_\varepsilon(p, M) = \begin{cases} \varphi_\varepsilon(1/p_y) F(p, M) & \text{if} \quad p_y \neq 0, \\ 0 & \text{if} \quad p_y = 0, \end{cases}$$

where F appears in (2.1) and φ_ε is a smooth nonnegative real-valued function with compact support in $[-2/\varepsilon, 2/\varepsilon]$ and such that $\varphi_\varepsilon(r) = 1$ for $r \in [-1/\varepsilon, 1/\varepsilon]$. The F_ε's satisfy assumptions (F1)–(F4) and thus we can apply for each ε the results of Giga, Goto, Ishii & Sato [21] and get, for any $T > 0$, a solution $v_\varepsilon \in UC(\mathbb{R}^{N+1} \times (0, T))$ of

$$\frac{\partial v_\varepsilon}{\partial t} + F_\varepsilon(Dv_\varepsilon, D^2 v_\varepsilon) = 0 \ \text{ in } \ \mathbb{R}^{N+1} \times (0, T) \tag{10.1}$$

with initial data v_0. Moreover, since v_0 is convex and F_ε remains linear in the Hessian, we learn also from [21] that $v_\varepsilon(\cdot, t)$ is convex for any $t \geqq 0$.

Step 2. Our aim is now to show that the family $(v_\varepsilon)_{\varepsilon>0}$ is locally bounded. To this end, we introduce

$$\chi(z, t) = a|z|^2 + b + Ct.$$

Since $v_0 \in UC(\mathbb{R}^{N+1})$ there exist $a, b \in \mathbb{R}$ such that $v_0 \leqq \chi(\cdot, 0)$, and since v_ε has at most linear growth, χ is greater than v_ε at infinity for all $\varepsilon > 0$.

Moreover, if follows from (H1) that F is bounded on bounded set and so are the F_ε's uniformly in ε. Then, an easy computation of the derivatives of χ shows that, up to taking C sufficiently large independent of ε, the function $\chi(z, t) = a|z|^2 + b + Ct$

is a smooth supersolution of (10.1). It follows that $v_\varepsilon \leqq \chi$ for all $\varepsilon > 0$. Reasoning in the same way with a subsolution, we find that the family $(v_\varepsilon)_{\varepsilon>0}$ is locally bounded independently of ε.

Step 3. From the previous step we are able to introduce the "half-relaxed-limits" $\overline{v}$ and $\underline{v}$ of the family $(v_\varepsilon)_{\varepsilon>0}$. Since (F_ε) tends to F locally uniformly on $(\mathbb{R}^{N+1} - \mathcal{D}) \times \mathcal{S}_{N+1}$, we find, from the stability result, that they are respectively sub- and supersolution of (2.1) with initial data v_0. Finally, from the comparison result of Theorem 4.1, we learn that $\overline{v} = \underline{v} = v$ and therefore that the family $(v_\varepsilon)_{\varepsilon>0}$ converges locally uniformly to v as ε tends to 0. It follows that $v(\cdot\,, t)$ is convex for any $t \in [0, T)$. Since we can repeat the arguments for any $T \geqq 0$, this completes the proof. □

We continue with

Lemma 10.1. *Let* $\Gamma_0 = \mathrm{Graph}(u_0)$. *If* u_0 *is convex in* $\mathbb{R}^N$, *then the signed-distance* $\mathrm{d}(\cdot\,, \Gamma_0)$ (*see* (5.3) *for a definition*) *is concave.*

Proof of Lemma 10.1. It is sufficient to show that, for any $z_i = (x_i, y_i) \in \mathbb{R}^{N+1}$, $i \in \{1, 2\}$, we have

$$v_0\left(\frac{z_1 + z_2}{2}\right) \geqq \frac{v_0(z_1) + v_0(z_2)}{2}. \tag{10.2}$$

To this end, we set $z = (z_1 + z_2)/2$ and denote by $\mathcal{P}_z$ the hyperplane which contains $z' \in \Gamma_0$ such that $v_0(z) = \pm\,|z - z'|$ and is orthogonal to $z - z'$. At this stage, we have to distinguish many cases depending on the position of z_1, z_2 relatively to Γ_0. Since their studies are similar, we provide the proof of (10.2) only in the case $z_1 \in \{v_0 \geqq 0\}$, $z_2 \in \{v_0 \leqq 0\}$ and $v_0(z) = -|z - z'|$. In this case, since u_0 is assumed to be convex, $\mathcal{P}_z \subset \{v_0 \leqq 0\}$; thus we have

$$v_0(z_1) = \mathrm{d}(z_1, \Gamma_0) \leqq \mathrm{dist}(z_1, \mathcal{P}_z), \quad v_0(z_2) = \mathrm{d}(z_2, \Gamma_0) \leqq -\mathrm{dist}(z_2, \mathcal{P}_z). \tag{10.3}$$

But, using the orthogonal projection on $\mathcal{P}_z$, we see that

$$|z - z'| = \frac{\mathrm{d}(z_2, \mathcal{P}_z) - \mathrm{d}(z_1, \mathcal{P}_z)}{2} = -v_0(z) \tag{10.4}$$

and combining (10.3) and (10.4), we get (10.2) □

We are now able to give the proof of the main result.

Proof of Theorem 10.1. We begin with (i). Let u_0 be a convex function on $\mathbb{R}^N$ and $v_0 = d(\cdot\,, \mathrm{Graph}(u_0))$. From Lemma 10.1, v_0 is concave. Therefore, applying Theorem 10.3, the associated solution v of (2.1) is also concave with respect to the space variable at any time $t \geqq 0$. Assume then by contradiction that $y_2 = u_2(x, t) < u_1(x, t) = y_1$ for some $(x, t) \in \mathbb{R}^N \times [0, +\infty)$. It follows from Theorem 6.1 that $v(x, y_2, t) \geqq 0$ and $v(x, y_1, t) \leqq 0$. Thus from Lemma 6.1, $v(x, \cdot\,, t) \equiv 0$ on $[y_2, y_1]$. Since it is a concave function, it implies that $v(x, \cdot\,, t) \equiv 0$ on $[y_2, +\infty)$ which is a contradiction with Lemma 8.1.

We turn to the proof of (ii). Applying the previous comparison result to the extremal solutions u^+ and u^- we find that they are equal to the same continuous function u, such that $\{v(\cdot\,,t) = 0\} = \mathrm{Graph}(u(\cdot\,,t))$ for all $t \geqq 0$, which turns out to be the unique continuous viscosity solution of (1.4). Since $v(\cdot\,,t)$ is concave, $u(\cdot\,,t)$ is convex for any $t \geqq 0$. This completes the proof of (ii). □

We conclude this section with some consequences of Theorem 10.3 for the geometrical evolution of sets in the convex case.

Theorem 10.4. *Let Ω_0^+ be any open convex subset of $\mathbb{R}^{N+1}$ with boundary Γ_0 and let $(\Omega_t^+, \Omega_t^-, \Gamma_t)_{t\geq 0}$ be the generalized evolution of $(\Omega_0^+, \Omega_0^-, \Gamma_0)$ in the sense of Section 5. Then, while $\Omega_t^+ \neq \emptyset$, it remains convex and Γ_t is its boundary. In particular, Γ_t has empty interior in $\mathbb{R}^{N+1}$.*

This result is known in the case of motion by mean curvature of compact convex sets (see Evans & Spruck [18], Soner [32], Ilmanen [24]). Here, the result holds for possibly noncompact hypersurfaces (like graphs for instance) and for general motions governed by (2.1). Note that we get immediate properties of regularity of the front at each time t before extinction: in the general case the front is locally a Lipschitz continuous graph; in the case of the mean-curvature equation, the front is even a smooth hypersurface (for the regularity issue, see Evans & Spruck [19] and Imbert [25]).

Proof of Theorem 10.4. Let v be the unique solution of (2.1) with initial data $v_0 = \mathrm{d}(\cdot\,,\Gamma_0)$ associated with Γ_0 via the level-set approach. Let $t \geqq 0$ be such that $\Omega_t^+ \neq \emptyset$. From Theorem 10.3, $v(\cdot\,,t)$ is concave; thus Ω_t^+ is convex. To prove that $\partial\Omega_t^+ = \Gamma_t$, we argue by contradiction, assuming there exists $z_0 \in \Gamma_t$ and $r > 0$ such that $B(z_0, r) \cap \Omega_t^+ = \emptyset$. We have $v(z_0, t) = 0$ and $v(\cdot\,,t) \leqq 0$ on $B(z_0, r)$. Since $v(\cdot\,,t)$ is concave, it follows that $v(\cdot\,,t) \leqq 0$ in $\mathbb{R}^{N+1}$ which is a contradiction with $\Omega_t^+ \neq \emptyset$. □

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