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Global Weak Solutions for the Two-Dimensional Motion of Several Rigid Bodies in an Incompressible Viscous Fluid

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Abstract

We consider the two-dimensional motion of several non-homogeneous rigid bodies immersed in an incompressible non-homogeneous viscous fluid. The fluid, and the rigid bodies are contained in a fixed open bounded set of $\mathbb{R}^2$. The motion of the fluid is governed by the Navier-Stokes equations for incompressible fluids and the standard conservation laws of linear and angular momentum rule the dynamics of the rigid bodies. The time variation of the fluid domain (due to the motion of the rigid bodies) is not known *a priori*, so we deal with a free boundary value problem. The main novelty here is the demonstration of the global existence of weak solutions for this problem. More precisely, the global character of the solutions we obtain is due to the fact that we do not need any assumption concerning the lack of collisions between several rigid bodies or between a rigid body and the boundary. We give estimates of the velocity of the bodies when their mutual distance or the distance to the boundary tends to zero.

1. Introduction

The aim of this paper is to prove an existence result for a coupled system of nonlinear partial and ordinary differential equations modelling the motion of several rigid bodies inside a fluid flow. The governing equations for the fluid flow are the classical Navier-Stokes system, whereas the motion of the rigid bodies is governed by the balance equations for linear and angular momentum (Newton's laws).

Let $\Omega \subset \mathbb{R}^2$ be an open bounded set representing the domain occupied by the fluid and by N rigid bodies. We denote by $F(t)$ the domain occupied by the fluid and by $S^i(t)$, $i = 1, \ldots, N$ the domains occupied by the rigid bodies at the instant (t). The full system of equations modelling the motion of the fluid and of the rigid bodies can be written as

$$\rho_F\,(\boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}) - \nu\Delta\boldsymbol{u} + \nabla p = \rho_F\,\boldsymbol{g}, \quad \boldsymbol{x}\in F(t), t\in[0,T], \tag{1.1}$$

$$\operatorname{div}\boldsymbol{u} = 0, \quad \boldsymbol{x}\in F(t), t\in[0,T], \tag{1.2}$$

$$\frac{\partial\rho_F}{\partial t} + \operatorname{div}(\rho_F\boldsymbol{u}) = 0, \quad \boldsymbol{x}\in F(t), t\in[0,T], \tag{1.3}$$

$$\boldsymbol{u} = 0, \quad \boldsymbol{x}\in\partial\Omega,\ t\in[0,T], \tag{1.4}$$

$$\boldsymbol{u} = \boldsymbol{h}_i' + \omega_i(\boldsymbol{x}-\boldsymbol{h}_i)^\perp, \quad \boldsymbol{x}\in\partial S^i(t),\ t\in[0,T],\ i=1,\dots,N, \tag{1.5}$$

$$M_i\boldsymbol{h}_i'' = -\int_{\partial S^i(t)} \mathbb{T}\boldsymbol{n}\,d\Gamma + \int_{S^i(t)} \rho_S\,\boldsymbol{g}\,dx, \quad t\in[0,T],\ i=1,\dots,N, \tag{1.6}$$

$$\begin{aligned} J_i\frac{d\omega_i}{dt} = &-\int_{\partial S^i(t)} (\boldsymbol{x}-\boldsymbol{h}_i)^\perp\cdot\mathbb{T}\boldsymbol{n}\,d\Gamma \\ &+\int_{S^i(t)} (\boldsymbol{x}-\boldsymbol{h}_i)^\perp\cdot\rho_S\,\boldsymbol{g}\,dx, \quad t\in[0,T],\ i=1,\dots,N, \end{aligned} \tag{1.7}$$

$$\boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}^0(\boldsymbol{x}), \quad \boldsymbol{x}\in F(0), \tag{1.8}$$

$$\rho_F(\boldsymbol{x},0) = \rho_F^0(\boldsymbol{x}), \quad \boldsymbol{x}\in F(0), \tag{1.9}$$

$$\rho_S(\boldsymbol{x},0) = \rho_S^0(\boldsymbol{x}), \quad \boldsymbol{x}\in\Omega\setminus\bar F(0), \tag{1.10}$$

$$S^i(0) = S^{i,0}, \tag{1.11}$$

$$\boldsymbol{h}_i(0) = \boldsymbol{h}_i^0\in\mathbb{R}^2, \quad \boldsymbol{h}_i'(0) = \boldsymbol{h}_i^1\in\mathbb{R}^2, \quad \omega_i(0) = \omega_i^0\in\mathbb{R}. \tag{1.12}$$

In the above system the unknowns are $\boldsymbol{u}(\boldsymbol{x},t)$ (the Eulerian velocity field of the fluid), $\rho_F(\boldsymbol{x},t)$ (the density field of the fluid), $\rho_S(\boldsymbol{x},t)$ (the density field of the rigid part), $\boldsymbol{h}_i(t)$, $i=1,\dots,N$ (the position of the gravity centres of the rigid bodies) and $\omega_i(t)$, $i=1,\dots,N$ (the angular velocities of the rigid bodies). For all $\boldsymbol{x} = \binom{x_1}{x_2}$, we denote by $\boldsymbol{x}^\perp$ the vector $\boldsymbol{x}^\perp = \binom{x_2}{-x_1}$.

Moreover we have denoted by $\partial\Omega$ the boundary of Ω, by $\partial S^i(t)$ the boundary of the i^{th}-rigid body at instant t, by $\boldsymbol{n}(\boldsymbol{x},t)$ the outwards unit vector field normal to $\partial F(t)$ and by $\boldsymbol{g}(\boldsymbol{x}\ t)$ the applied body forces (per unit mass). The constant $\nu > 0$ stands for the viscosity of the fluid. Further, we have denoted by M_i (by J_i) the mass (respectively, the inertia moment related to the mass centre) of the i^{th}-rigid body and by $\mathbb{T}$ the Cauchy stress tensor field in the fluid. The components $(T_{kl})_{k,l\in\{1,2\}}$ of $\mathbb{T}$ are related to the velocity field $\boldsymbol{u}$ by

$$T_{kl}(\boldsymbol{x},t) = -p(\boldsymbol{x},t)\delta_{kl} + \nu\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right), \quad k,l = 1,2. \tag{1.13}$$

The existence of weak solutions of (1.1)–(1.13) (in a sense which will be defined below) has already been studied. In [14] and [21] the authors prove a global existence result in the case of one body in a fluid filling the whole space. The problem in a bounded domain with several rigid bodies was considered in [2–5, 10, 12, 13]. In [4] and [5], the authors prove global existence up to collisions in the two-dimensional and in the three-dimensional cases, for both incompressible and compressible fluid. In [12] and [13], the authors show global existence for one rigid body in the presence of eventual collisions. The same type of result is obtained in [2, 3, 10] by different methods. The methods in the papers quoted above do not seem to be applicable in the case of several rigid bodies with eventual collisions. The stationary problem was studied in [21] and [8] (see also the references therein).

The main novelty of this paper is that we show a method of proving global existence in the presence of eventual collisions for the case of several rigid bodies immersed in a non-homogeneous fluid. Our results are valid in two space dimensions. The global existence (with collisions) for several rigid bodies seems to be an open question in the three-dimensional case.

Let us mention that a local (in time) existence result of strong solutions was proved in [9].

The plan of this paper is as follows: In Section 2 we introduce some notation and state the main results. In Section 3 we introduce a penalized problem and describe the main steps of the proof of the existence result. Section 4 contains some properties of a function space specific to our problem. In Section 5 we apply classical results of DiPerna and Lions in order to pass to the limit in the transport equation of the density. In Section 6 we derive several technical results which are then used, in Section 7, to prove the compactness of the sequence of approximated velocity fields. The main results are proved in Section 8.

2. Notation and main result

We first introduce some general notation.

Let $G \subset \mathbb{R}^2$ be a bounded open set with a $\mathscr{C}^2$ boundary.

If $\boldsymbol{v} \in L^2(G, \mathbb{R}^2)$ is a vector field we denote by $\mathrm{D}(\boldsymbol{v})$ the tensor field defined by

$$\mathrm{D}_{ij}(\boldsymbol{v}) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \quad i, j = 1, 2,$$

where the derivatives are calculated in the distributions sense, i.e., in $\mathscr{D}'(G)$.

We say that $\boldsymbol{v} \in L^2(G, \mathbb{R}^2)$ is a rigid velocity field if $\mathrm{D}_{ij}(\boldsymbol{v}) = 0$, in $\mathscr{D}'(G)$, for $i, j \in \{1, 2\}$.

We will use the following classical functional spaces:

$\mathscr{V}(G) = \left\{\boldsymbol{v} \in \mathscr{C}_0^\infty(G; \mathbb{R}^2) \,\middle|\, \operatorname{div} \boldsymbol{v} = 0\right\}$,

$V(G)$ is the closure of $\mathscr{V}(G)$ in $\left[H^1(G)\right]^2$,

$H(G)$ is the closure of $\mathscr{V}(G)$ in $\left[L^2(G)\right]^2$.

According to classical results (see, for instance, [22]) we have

$V(G) = \left\{\boldsymbol{v} \in \left[H_0^1(G)\right]^2 \middle| \operatorname{div} \boldsymbol{v} = 0 \text{ in } L^2(G)\right\}$,

$H(G) = \left\{\boldsymbol{v} \in \left[L^2(G)\right]^2 \middle| \operatorname{div} \boldsymbol{v} = 0 \text{ in } \mathscr{D}'(G),\ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ in } \mathrm{H}^{-1/2}(\partial G)\right\}$.

Moreover, we will use some non-standard function spaces specific to our problem.

Let $\Omega \subset \mathbb{R}^2$ be the fixed set representing the domain occupied by the fluid and by the solid bodies. We suppose that the boundary of Ω is of class $\mathscr{C}^2$.

If χ is the characteristic function of a subset of Ω, we define

$$K(\chi) = \left\{ \boldsymbol{v} \in V(\Omega) : \chi \mathrm{D}(\boldsymbol{v}) = 0 \text{ in } L^2(\Omega) \right\}, \tag{2.1}$$

and

$$S(\chi) = \{ \boldsymbol{x} \in \Omega : \chi(\boldsymbol{x}) = 1 \}.$$

The space $K(\chi)$ is clearly a closed subspace of $V(\Omega)$.

According to Lemma 1.1 in [23, p. 18], if $S(\chi)$ is an open connected subset of Ω, then, for every $\boldsymbol{v} \in K(\chi)$, there exist a vector $\boldsymbol{k}_v$ and a constant ℓ_v such that

$$\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{k}_v + \ell_v \boldsymbol{x}^\perp \quad \forall \boldsymbol{x} \in S(\chi). \tag{2.2}$$

If $\sigma > 0$ and $G \subset \mathbb{R}^2$ is an open set we denote by G_σ the σ-neighbourhood of G, i.e.,

$$G_\sigma = \left\{ \boldsymbol{x} \in \mathbb{R}^2 : d(\boldsymbol{x}, G) < \sigma \right\}, \tag{2.3}$$

and we define the function space

$$K_\sigma(\chi) = \left\{ u \in V(\Omega) \middle| \mathrm{D}(u)(\boldsymbol{x}) = 0 \ \forall \boldsymbol{x} \in S_\sigma(\chi) \right\}.$$

Moreover, we denote by $K_0(\chi)$ the closure of $\cup_{\sigma>0} K_\sigma(\chi)$ in $H^1(\Omega)$.

Let us now go back to the notation in problem (1.1)–(1.11). We suppose that the sets $S^i(t)$, $i = 1, \ldots, N$, representing the regions occupied by the solid bodies at instant t, are open and that, at the initial moment, the boundary of $S^i(0)$, for all $i = 1, \ldots, N$, is of class $\mathscr{C}^2$. Moreover, we suppose that $S^i(0) \cap S^j(0) = \emptyset$ for all $i, j = 1, \ldots, N$, $i \neq j$.

Due to the regularity assumptions above, it can be easily checked that the following result holds.

Proposition 2.1. *There exists $\delta > 0$ such that for all $i = 1, \ldots, N$, and for all $\boldsymbol{x} \in S^i(0)$ (for all $\boldsymbol{x} \in \mathbb{R}^2 \setminus \overline{\Omega}$) there exists a open disk B of radius δ included in $S^i(0)$ (respectively, in $\mathbb{R}^2 \setminus \overline{\Omega}$) and containing $\boldsymbol{x}$.*

Throughout this paper we fix $\delta > 0$ satisfying the conditions in the proposition above.

In the particular case when we choose, in (2.3), $\sigma = \delta$ we denote by G_{ext} the set G_δ. More precisely, we put

$$G_{\text{ext}} = \left\{ \boldsymbol{x} \in \mathbb{R}^2 \middle| d(\boldsymbol{x}, G) < \delta \right\},$$

and we denote by G_{int} the "δ-kernel" of G defined by

$$G_{\text{int}} = \left\{ \boldsymbol{x} \in \mathbb{R}^2 \middle| B(\boldsymbol{x}, \delta) \subset G \right\}.$$

We remark that, due to Proposition 2.1, the δ-neighbourhood of the G_{int} and the "δ-kernel" of G_{ext} are equal to G.

Moreover, if $f \in L^1_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$, then we denote by $\overline{f}$ the convolution of f by a radially symmetric regularizing kernel supported in $B(0, \delta)$. More precisely, we put

$$\overline{f} = w_\delta * f = \int_{\mathbb{R}^2} w_\delta(\boldsymbol{x} - \boldsymbol{y}) f(\boldsymbol{y})\, dy, \tag{2.4}$$

where w_δ is a symmetric kernel, i.e., $w_\delta \in \mathscr{D}(\mathbb{R}^2)$, $w_\delta(\boldsymbol{x}) = \widetilde{w}_\delta(|\boldsymbol{x}|)$, $\int_{\mathbb{R}^2} w_\delta(\boldsymbol{x})\, dx = 1$ and $\operatorname{supp} w_\delta \subset B(0, \delta)$.

The remark below, which can be checked by a simple calculation, plays an important role in the remaining part of this work.

Remark 2.1. If $\boldsymbol{u}$ is a rigid velocity field in the set G, then $\overline{\boldsymbol{u}}(\boldsymbol{x}) = \boldsymbol{u}(\boldsymbol{x})$ for all $\boldsymbol{x} \in G_{\mathrm{int}}$.

We denote by $\varphi^i(\cdot\,, t)$ the characteristic function of $S^i(t)$ and we put $\varphi = \sum_{i=1}^N \varphi^i$. We notice that φ is the characteristic function of the rigid part $S(t)$ of Ω at the instant t.

We denote by $\psi^i(\cdot\,, t)$ the characteristic function of the "δ-kernel" of $S^i(t)$, i.e.,

$$S(\psi^i) = \left(S^i(t)\right)_{\mathrm{int}}$$

and we put $\psi = \sum_{i=1}^N \psi^i$.

If $T > 0$, we denote by Q the cylinder $Q = \Omega \times [0, T]$ and we put

$$\mathrm{Char}(Q) = \{g : Q \to \{0, 1\}\}\,, \qquad \mathrm{Char}(\Omega) = \{g : \Omega \to \{0, 1\}\}\,,$$

i.e., $\psi \in \mathrm{Char}(Q)$ if and only if ψ is the characteristic function of some subset of Q.

If $\psi \in \mathrm{Char}(Q)$, we denote by $L^p(0, T; K(\psi))$ the space of functions $\boldsymbol{v} \in L^p(0, T; V(\Omega))$ such that $\boldsymbol{v}(t) \in K(\psi(\cdot\,, t))$ for almost all $t \in [0, T]$.

In order to define weak solutions of (1.1)–(1.13) we follow the ideas in [12] and [4]. This weak formulation is global in the sense that the unknown functions are defined on the whole domain Ω. More precisely, instead of considering separately the velocity (density) fields of the fluid and the rigid bodies, we consider only one velocity field $\boldsymbol{u}$ (respectively, one density field ρ) defined in $\Omega \times [0, T]$. For every $i = 1, \ldots, N$, the restriction of $\boldsymbol{u}(\cdot\,, t)$ to $S^i(t)$ is a rigid velocity field. Weak solutions of our problem can be defined as follows:

Definition 1. Let $\boldsymbol{u}^0 \in H(\Omega)$, $\rho^0 \in L^\infty(\Omega)$ and $\varphi^{i,0}$ be the characteristic functions of $S^i(0)$, $i = 1, \ldots, N$. A set of functions $\left\{\boldsymbol{u}, \rho, \varphi^i,\ i = 1, \ldots, N\right\}$ such that

$$\boldsymbol{u} \in L^\infty(0, T; H(\Omega)) \cap L^2(0, T; K(\varphi)), \tag{2.5}$$

$$\varphi^i \in \mathrm{Char}(Q) \cap \mathscr{C}^{0,1/p}(0, T; L^p(\Omega)), \quad 1 \leqq p < \infty, \tag{2.6}$$

$$\varphi = \sum_{i=1}^N \varphi^i \in \mathrm{Char}(Q), \tag{2.7}$$

$$\rho \in L^\infty(Q), \tag{2.8}$$

is said to be a weak solution of (1.1)–(1.13) if the equalities

$$\int_Q \big(\rho \boldsymbol{u}(\boldsymbol{\xi}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{\xi}) - \nu \mathrm{D}(\boldsymbol{u}) : \mathrm{D}(\boldsymbol{\xi})\big)\,dx\,dt = -\int_\Omega \rho^0 \boldsymbol{u}^0 \cdot \boldsymbol{\xi}(\boldsymbol{x},0)dx - \int_Q \rho \boldsymbol{g}\cdot\boldsymbol{\xi}\,dx\,dt, \tag{2.9}$$

$$\int_Q \rho\,(\eta_t + (\boldsymbol{u}\cdot\nabla)\eta)\;dx\,dt = -\int_\Omega \rho^0 \cdot \eta(\boldsymbol{x},0)\,dx, \tag{2.10}$$

$$\int_Q \varphi^i\,(\eta_t + (\boldsymbol{u}\cdot\nabla)\eta)\;dx\,dt = -\int_\Omega \varphi^{i,0} \cdot \eta(\boldsymbol{x},0)\,dx, \quad i = 1,\dots,N \tag{2.11}$$

hold for any functions $\boldsymbol{\xi} \in H^1(Q) \cap L^2(0,T;K(\varphi))$, $\boldsymbol{\xi}(T) = 0$, $\eta \in \mathscr{C}^1(Q)$, $\eta(T) = 0$.

The main result of this paper is

Theorem 2.1. *If* $\boldsymbol{u}^0 \in H(\Omega)$, $\boldsymbol{g} \in L^2(Q)$, $\rho^0 \in L^\infty(\Omega)$, $\rho^0 \geqq m_0 > 0$ *for some constant* m_0 *and the boundaries* $\partial\Omega$, $\partial S^i(0)$, $i = 1,\dots,N$ *are of class* $\mathscr{C}^2$, *then there exists at least one weak solution of* (1.1)–(1.13). *Moreover, this solution satisfies the energy estimate*

$$\int_\Omega \rho|\boldsymbol{u}|^2 dx + \int_Q \nu\,|D(\boldsymbol{u})|^2\;dx\,dt \leqq C\left\{\int_\Omega \rho^0|\boldsymbol{u}^0|^2\,dx + \|\boldsymbol{g}\|^2_{L^2(Q)}\right\} \tag{2.12}$$

for some constant $C > 0$.

Finally, there exists a family of isometries $\{\mathscr{A}^i_{s,t}\}_{s,t\in[0,T],i\in\{1,\dots,N\}}$ *of* $\mathbb{R}^2$ *such that*

$$S(\varphi^i(t)) = \mathscr{A}^i_{s,t}\big(S(\varphi^i(s))\big) \quad \forall s,t \in [0,T],\ \forall i = 1,\dots,N \tag{2.13}$$

and $\mathscr{A}^i_{s,t}$ *are Lipschitz-continuous with respect to* s *and* t.

Remark 2.2. The theorem above combined with (2.7) implies that $S(\varphi^i(t)) \cap S(\varphi^j(t)) = \emptyset$ for all $i,j = 1,\dots,N$, $i \neq j$ and for all $t \in [0,T]$. Since the sets $S(\varphi^i(t))$, $i,j = 1,\dots,N$ are open, this fact does not exclude eventual touching of the boundaries of different bodies or of a boundary of a body and the boundary $\partial\Omega$.

Theorem 2.2. *Let* $\{\boldsymbol{u},\rho,\varphi^i,\ i = 1,\dots,N\}$ *be a weak solution of* (1.1)–(1.13) *and* $h_{ij}(t) = \mathrm{dist}\,\big(S(\varphi^i(t)), S(\varphi^j(t))\big)$, $h_{0i}(t) = \mathrm{dist}\,\big(\partial\Omega, S(\varphi^i(t))\big)$. *Then the following assertions hold:*

(1) *If* $E_{0i} = \{t \in [0,T] : h_{0i}(t) = 0\}$, *then* $\boldsymbol{u}(\boldsymbol{x},t) = 0$ *as* $\boldsymbol{x} \in S(\varphi^i(t))$ *for almost all* $t \in E_{0i}$; *if* $E_{ij} = \{t \in [0,T] : h_{ij}(t) = 0\}$ *then there exists a rigid velocity field* $\boldsymbol{v}(\boldsymbol{x},t)$ *such that* $\boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{v}(\boldsymbol{x},t)$ *for all* $\boldsymbol{x} \in S(\varphi^i(t)) \cup S(\varphi^j(t))$ *and for almost all* $t \in E_{ij}$.

(2) *For all* $i \neq j$ *and for all* $t_0 \in [0, T]$ *with* $h_{ij}(t_0) = 0$,

$$\lim_{t \to t_0} \frac{h_{ij}(t)}{|t - t_0|^2} = 0.$$

Remark 2.3. The theorem above allows collisions of different solids or collisions between solids and the boundary, but only with vanishing relative velocity and relative acceleration. This fact might seem to be a new paradox in fluid mechanics. A possible explanation is that the concept of weak solutions "à la Leray" of the Navier-Stokes incompressible model (see for instance [16] and [22]) is not appropriate for describing collisions with a non-zero relative velocity.

3. Main steps of the proof Theorem 2.1

The first step in the proof of Theorem 2.1 is to approximate the rigid bodies by very viscous fluids. In this way we introduce a penalized problem. More precisely, for given $n \in \mathbb{N}$, $\boldsymbol{u}^0 \in H(\Omega)$, $\rho^0 \in L^\infty(\Omega)$, and $\psi^{i,0} \in L^\infty(\Omega) \cap \mathrm{Char}(\Omega)$, we consider the following penalized problem.

Find a set of functions $\left\{\boldsymbol{u}_n,\ \rho_n,\ \varphi_n,\ \varphi_n^i,\ \psi_n^i, \quad i = 1, \dots, N\right\}$ such that

$$\boldsymbol{u}_n \in L^\infty(0, T; H(\Omega)) \cap L^2(0, T; V(\Omega)), \tag{3.1}$$

$$\psi_n^i,\ \varphi_n^i \in \mathrm{Char}(Q) \cap \mathscr{C}^{0,1/p}(0, T; L^p(\Omega)),\ 1 \leqq p < \infty,\ i = 1, \dots, N, \tag{3.2}$$

$$\varphi_n = \sum_{i=1}^N \varphi_n^i, \tag{3.3}$$

$$\rho_n \in L^\infty(Q), \tag{3.4}$$

$$S(\varphi_n^i) = \left(S(\psi_n^i)\right)_{\text{ext}}, \quad i = 1, \dots, N \tag{3.5}$$

and such that relations

$$\int_Q \big(\rho_n \boldsymbol{u}_n \big(\boldsymbol{\xi}_t + (\boldsymbol{u}_n \cdot \nabla)\boldsymbol{\xi}\big) - (\nu + n\varphi_n)\mathrm{D}(\boldsymbol{u}_n) : \mathrm{D}(\boldsymbol{\xi})\big)\, dx\, dt$$
$$= -\int_\Omega \rho^0 \boldsymbol{u}^0 \cdot \boldsymbol{\xi}(\cdot\,, 0)\, dx - \int_Q \rho_n \boldsymbol{g} \cdot \xi\, dx\, dt, \tag{3.6}$$

$$\int_Q \rho_n\, (\eta_t + (\boldsymbol{u}_n \cdot \nabla)\eta)\ dx\, dt = -\int_\Omega \rho^0 \cdot \eta(\cdot\,, 0)\, dx, \tag{3.7}$$

$$\int_0^T \int_{\Omega_{\text{ext}}} \psi_n^i\, (\gamma_t + (\overline{\boldsymbol{u}_n} \cdot \nabla)\gamma)\, dx\, dt \tag{3.8}$$
$$= -\int_{\Omega_{\text{ext}}} \psi^{i,0} \cdot \gamma(\cdot\,, 0)\, dx, \quad i = 1, \dots, N,$$

hold for any functions $\boldsymbol{\xi} \in H^1(Q) \cap L^2(0,T;V(\Omega))$, $\boldsymbol{\xi}(\cdot\,,T) = 0$, $\eta \in \mathscr{C}^1(Q)$, $\eta(\cdot\,,T) = 0$, $\gamma \in \mathscr{C}^1((0,T)\times\Omega_{\text{ext}})$, $\gamma(\cdot\,,T) = 0$.

The function $\overline{\boldsymbol{u}_n}(\cdot\,,t)$ in (3.8) is defined as in (2.4) after extending $\boldsymbol{u}$ by zero outside Ω. The replacement of $\boldsymbol{u}_n$ by $\overline{\boldsymbol{u}_n}$ in (3.8) (which is much smoother) allows the application of some standard results on ordinary differential equations and on characteristics of transport equations. Moreover, due to Remark 2.1 we will obtain a rigid motion when $n \to \infty$, without passing to the limit with respect to δ.

The result below asserts the existence of weak solutions for (3.1)–(3.8). This result can be proved following step by step the classical methods of investigation of the Navier-Stokes equations for non-homogeneous fluids (see [1] or [19]). This is why we omit the proof.

Theorem 3.1. *For any* $n \in \mathbb{N}$, $\boldsymbol{u}^0 \in H(\Omega)$, $\rho^0 \in L^\infty(\Omega)$, $\psi^{i,0} \in L^\infty(\Omega) \cap$ *Char*(Ω) *there exists at least a solution of the penalized problem* (3.1)–(3.8). *This solution has the following properties:*

$$\int_\Omega \rho_n|\boldsymbol{u}_n|^2\,dx + \int_Q (\nu + n\varphi_n)\big|D(\boldsymbol{u}_n)\big|^2\,dx\,dt \leqq C\left\{\int_\Omega \rho^0|\boldsymbol{u}^0|^2\,dx + \|\boldsymbol{g}\|^2_{L^2(Q)}\right\}, \tag{3.9}$$

for some constant $C > 0$, $\rho(\boldsymbol{x},t) \geqq m_0$ *for a.e.* $\boldsymbol{x} \in \Omega, t \in [0,T]$,

$$\|\rho_n(t)\|_{L^p(\Omega)} = \left\|\rho^0\right\|_{L^p(\Omega)}, \qquad 1 \leqq p \leqq \infty, \tag{3.10}$$

$$\left\|\psi_n^i(t)\right\|_{L^p(\Omega_{\text{ext}})} = \left\|\psi^{i,0}\right\|_{L^p(\Omega_{\text{ext}})}, \quad 1 \leqq p \leqq \infty, i = 1,\dots,N. \tag{3.11}$$

Moreover, for all $t \in [0,T]$ *the functions* $\psi_n^i(\cdot\,,t)$ *take, a.e. in* Ω, *only two values:* 0 *and* 1.

According to Theorem 3.1 the sequences $\{\boldsymbol{u}_n\}$, $\{\rho_n\}$, $\left\{\psi_n^i\right\}$ have subsequences (which we also denote by $\{\boldsymbol{u}_n\}$, $\{\rho_n\}$, $\left\{\psi_n^i\right\}$) such that

$$\boldsymbol{u}_n \to \boldsymbol{u} \quad \text{in } L^2(0,T;V(\Omega)) \text{ weakly and in } L^\infty(0,T;H(\Omega)) \text{ weakly}^*, \tag{3.12}$$

$$\rho_n \to \rho \quad \text{in } L^\infty(Q) \text{ weakly}^*, \tag{3.13}$$

$$\psi_n^i \to \psi^i \text{ in } L^\infty(0,T,L^\infty(\Omega_{\text{ext}})) \text{ weakly}^*. \tag{3.14}$$

Moreover, denote by φ^i, $i = 1,\dots,N$ the characteristic functions of $\left(S(\psi^i)\right)_{\text{ext}}$ and by $\varphi = \sum_{i=1}^N \varphi^i$.

The second step of the proof consists in showing that the weak limits defined above satisfy the transport equations. More precisely, we will show that the following result holds true.

Proposition 3.1. *The functions* $\boldsymbol{u}$, ρ *and* φ *defined above satisfy relations* (2.5)–(2.8), (2.10) *and* (2.11).

The proof of this result, which is based on the results of DiPerna and Lions (see [6] and [19]), is given in Section 5.

The third and the most technical step of the proof consists in proving the following result, which is proved in Section 7.

Theorem 3.2. *The sequence* $\{\boldsymbol{u}_n\}$ *in* (3.12) *converges strongly to* $\boldsymbol{u}$ *in* $L^2(Q)$.

The last step consists in combining Proposition 3.1 and Theorem 3.2 in order to prove our main existence theorem.

4. Some properties of the space $K(\chi)$

In this section we give some properties of the space $K(\chi)$ defined by (2.1).

Let χ be the characteristic function of an open subset of Ω. Throughout this section we assume that χ satisfies the following assumptions:

(A1) The characteristic function $\chi = \sum_{i=1}^{N} \chi^i$, where χ_i is the characteristic function of an open connected set $S(\chi^i)$, for $i = 1, \dots, N$.
(A2) The sets $S(\chi^i)$, $i = 1, \dots, N$, have smooth boundaries (say $\mathscr{C}^2$).

Assumption (A1) implies that $S(\chi^i) \cap S(\chi^j) = \emptyset$ for all $i, j = 1, \dots, N$, $i \neq j$. This fact does not exclude the case where $\partial S(\chi^i) \cap \partial S(\chi^j) \neq \emptyset$ for some values of i and j with $i \neq j$.

Moreover, the assumptions above clearly imply that

$$K(\chi) = \bigcap_{i=1}^{N} K(\chi^i).$$

If we consider now $\boldsymbol{u} \in K(\chi)$, then the restriction of $\boldsymbol{\xi}$ to each of the sets $S(\chi^i)$ is a rigid velocity field. The result below gives information on the behaviour of $\boldsymbol{u} \in K(\chi)$ in the case when the boundaries of $S(\chi^i)$ and $S(\chi^j)$, $i \neq j$, have common points.

Proposition 4.1. *Suppose that* $i, j \in \{1, \dots, N\}$, $i \neq j$, *are such that* $\partial S(\chi^i) \cap \partial S(\chi^j) \neq \emptyset$. *Then, for any* $\boldsymbol{u} \in K(\chi)$, *there exists a rigid velocity field* $\boldsymbol{w}$ *such that* $\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{w}(\boldsymbol{x})$ *for all* $\boldsymbol{x} \in S(\chi^i) \cup S(\chi^j)$.

Proof. Since $\boldsymbol{u} \in K(\chi^i)$, there exists a rigid function $\boldsymbol{w}$ such that $\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{w}(\boldsymbol{x})$ for all $\boldsymbol{x} \in S(\chi^i)$. Let us introduce the function $\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{w}$. We have to prove that

$$\boldsymbol{v}(\boldsymbol{x}) = 0 \quad \forall \boldsymbol{x} \in S(\chi^j).$$

Let us suppose that $M \in \partial S(\chi^i) \cap \partial S(\chi^j)$. Since $\boldsymbol{v} \in K(\chi^j)$, we have the representation:

$$\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{a} + \omega(\boldsymbol{x} - \boldsymbol{x}_M)^{\perp} \quad \forall \boldsymbol{x} \in S(\chi^j), \tag{4.1}$$

where $\boldsymbol{a} \in \mathbb{R}^2$ and $\omega \in \mathbb{R}$ are constants.

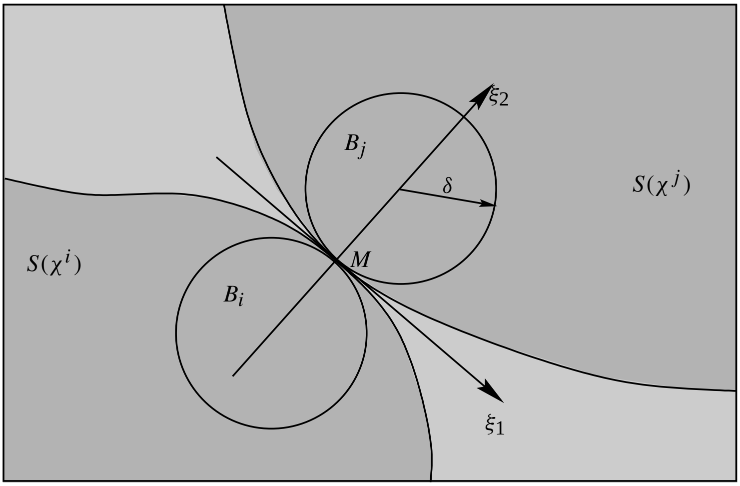

Fig. 4.1. Contact between solids $S(\chi^i)$ and $S(\chi^j)$.

Since the boundaries $\partial S(\chi^i)$ and $\partial S(\chi^j)$ are of class $\mathscr{C}^2$, there exist $\delta > 0$ and the open disks B_i and B_j of radius δ such that

$$B_i \subset S(\chi^i), \quad B_j \subset S(\chi^j), \quad \overline{B_i} \cap \overline{B_j} = \{M\},$$

where $\overline{B_i}$ is the closure of B_i in $\mathbb{R}^2$.

Let us introduce a system of coordinates with the origin in M and having $M\xi_1$ and $M\xi_2$ as coordinate axis (see Fig. 4.1). With respect to this new system of coordinates, (4.1) becomes

$$\boldsymbol{v}(\boldsymbol{\xi}) = \boldsymbol{a} + \omega \boldsymbol{\xi}^{\perp} \tag{4.2}$$

for all $\boldsymbol{\xi} \in \mathbb{R}^2$ such that $\boldsymbol{\xi} + \boldsymbol{x}_M \in S(\chi^j)$.

The equations of the boundaries ∂B_i and ∂B_j with respect to this system of coordinates are

$$\xi_2 = \pm c(\xi_1),\ \ \xi_1 \in [-\delta, \delta]$$

where $c(\xi_1) = \delta - \sqrt{\delta^2 - \xi_1^2}$ for all $\xi_1 \in [-\delta, \delta]$. Notice that

$$\frac{\xi_1^2}{2\delta} \leqq c(\xi_1) \leqq \frac{\xi_1^2}{\delta} \quad \forall\, \xi_1 \in (-\delta, \delta). \tag{4.3}$$

First we prove that $\boldsymbol{a} = \mathbf{0}$. Let us fix an arbitrary positive number $\tau < \delta$ and consider the domain

$$D_\tau = \left\{(\xi_1, \xi_2) : -\tau < \xi_1 < \tau,\ \frac{|\xi_1|}{\tau} c(\tau) < \xi_2 < c(\tau)\right\}. \tag{4.4}$$

Since $D_\tau \subset B_j$, we have

$$\int_{D_\tau} \boldsymbol{v} d\xi_1 d\xi_2 = \boldsymbol{a} \int_{D_\tau} d\xi_1\, d\xi_2 + \omega \int_{D_\tau} \boldsymbol{\xi}^{\perp} d\xi_1\, d\xi_2.$$

Hence

$$|\boldsymbol{a}|\mu(D_\tau) \leqq \int_{D_\tau} |\boldsymbol{v}|\, d\xi_1\, d\xi_2 + |\omega| \int_{D_\tau} |\boldsymbol{\xi}|\, d\xi_1\, d\xi_2 \\ \leqq [\mu(D_\tau)]^{1/2} \, \|\boldsymbol{v}\|_{L^2(D_\tau)} + |\omega|\sqrt{\tau^2 + c^2(\tau)}\mu(D_\tau), \tag{4.5}$$

where μ is the two-dimensional Lebesgue measure.

Since $\boldsymbol{v}$ vanishes in B_i we can apply the Poincaré inequality to get

$$\|\boldsymbol{v}\|_{L^2(D_\tau)} \leqq 2c(\tau)\, \|\nabla \boldsymbol{v}\|_{L^2(\Omega)} . \tag{4.6}$$

Relations (4.5) and (4.6) imply that

$$|\boldsymbol{a}| \leqq 2\, \|\nabla \boldsymbol{v}\|_{L^2(\Omega)}\, c(\tau)\mu(D_\tau)^{-1/2} + |\omega|\sqrt{\tau^2 + c^2(\tau)}. \tag{4.7}$$

Since $\mu(D_\tau) = \tau c(\tau)$, from (4.3) and (4.7) we have

$$|\boldsymbol{a}| \leqq C\tau^{1/2} \quad \forall \tau < \delta,$$

where

$$C = \max\left\{ \frac{2}{\sqrt{\delta}}\, \|\nabla \boldsymbol{v}\|_{L^2(\Omega)} , \sqrt{2\delta}|\omega| \right\}$$

is a constant independent of τ. By passing to the limit when $\tau \to 0$ we find that $\boldsymbol{a} = 0$.

Let us now prove that $\omega = 0$. For any positive real number $r < \delta$ let us consider the set G_r defined by

$$G_r = \left\{ (\xi_1, \xi_2) : 0 < \xi_1 < r,\ |\xi_2| < \frac{c(r)}{r}\xi_1 \right\}.$$

We notice that

$$\partial G_r = \overline{\Gamma_r^i} \cup \overline{\Gamma_r^j} \cup \Gamma_r,$$

where we used the notation:

$$\Gamma_r^i = \left\{ (\xi_1, \xi_2) \in \partial G_r : 0 < \xi_1 < r \text{ and } \xi_2 = -\frac{c(r)}{r}\xi_1 \right\},$$
$$\Gamma_r^j = \left\{ (\xi_1, \xi_2) \in \partial G_r : 0 < \xi_1 < r \text{ and } \xi_2 = \frac{c(r)}{r}\xi_1 \right\}$$

and

$$\Gamma_r = \{ (\xi_1, \xi_2) \in \partial G_r : \xi_1 = r \} .$$

It is clear that $\Gamma_r^i \subset B_i$ and $\Gamma_r^j \subset B_j$.

Since $\operatorname{div} \boldsymbol{v} = 0$ in Ω, we have

$$\int_{\partial G_r} \boldsymbol{v} \cdot \boldsymbol{n}\, ds = 0.$$

Using the fact that $\boldsymbol{v} = 0$ on Γ_r^i, we see that the relation above yields

$$\int_{\Gamma_r^j} \boldsymbol{v} \cdot \boldsymbol{n}\, ds + \int_{\Gamma_r} \boldsymbol{v} \cdot \boldsymbol{n}\, ds = 0.$$

From (4.2) and the fact that $\boldsymbol{a} = \mathbf{0}$ and $\Gamma_r^j \subset B_j$, we obtain

$$\left| \int_{\Gamma_r^j} \boldsymbol{v} \cdot \boldsymbol{n}\, ds \right| = \left| \int_0^{\sqrt{r^2+c^2(r)}} \omega\tau\, d\tau \right| = \frac{|\omega|}{2}(r^2 + c^2(r)) \geqq \frac{|\omega| r^2}{2}.$$

Hence, for all $r \in (0, \delta)$, we have the inequality

$$\frac{|\omega| r^2}{2} \leqq \left| \int_{\Gamma_r} \boldsymbol{v} \cdot \boldsymbol{n}\, ds \right| \leqq \int_{\Gamma_r} |\boldsymbol{v}|\, ds \leqq \sqrt{2c(r)}\, \|\boldsymbol{v}\|_{L^2(\Gamma_r)},$$

which gives the estimate

$$\frac{\omega^2 r^4}{8c(r)} \leqq \|\boldsymbol{v}\|^2_{L^2(\Gamma_r)} \quad \forall r \in (0, \delta).$$

By using (4.3), we get

$$\frac{\delta\omega^2 r^2}{8} \leqq \|\boldsymbol{v}\|^2_{L^2(\Gamma_r)} \quad \forall r \in (0, \delta).$$

Integrating this inequality with respect to r from 0 to an arbitrary $\gamma \in (0, \delta)$ we obtain

$$\frac{\delta\omega^2}{8} \frac{\gamma^3}{3} \leqq \|\boldsymbol{v}\|^2_{L^2(G_\gamma)}. \tag{4.8}$$

On the other hand, by the Poincaré inequality, we have

$$\|\boldsymbol{v}\|^2_{L^2(G_\gamma)} \leqq (2c(\gamma))^2\, \|\nabla \boldsymbol{v}\|^2_{L^2(G_\gamma)}. \tag{4.9}$$

Inequalities (4.3), (4.8) and (4.9) imply that

$$\omega^2 \leqq 96 \frac{\gamma}{\delta^3} \|\nabla \boldsymbol{v}\|^2_{L^2(G_\gamma)} \quad \forall \gamma \in (0, \delta).$$

By passing to the limit when $\gamma \to 0$, the relations above imply that $\omega = 0$, i.e., $\boldsymbol{v} = 0$ in $S(\chi^j)$. Proposition 4.1 is now proved.

The method used in the proof of Proposition 4.1 can be easily extended to the case when one of the sets of $S(\chi^i)$ touches the boundary of Ω. In this case, the behaviour of $\boldsymbol{u} \in K(\chi)$ is given in the result below, which is stated without proof.

Proposition 4.2. *Suppose that $\partial S(\chi^i) \cap \partial\Omega \neq \emptyset$ for some $i = 1, \dots, N$. Then, any $\boldsymbol{u} \in K(\chi)$ satisfies the condition $\boldsymbol{u}(\boldsymbol{x}) = 0$ for all $\boldsymbol{x} \in S(\chi^i)$.*

We next state and prove a result showing that the union of the spaces $K_\sigma(\chi)$, $\sigma > 0$, is dense in the space $K(\chi)$.

Proposition 4.3. *For any* $\boldsymbol{\xi} \in K(\chi)$*, there exists a sequence of functions* $\{\boldsymbol{\xi}_\sigma\}_{\sigma>0} \subset K(\chi)$ *satisfying the conditions:* $\boldsymbol{\xi}_\sigma \in K_\sigma(\chi)$ *for all* $\sigma > 0$ *and* $\boldsymbol{\xi}_\sigma \to \boldsymbol{\xi}$ *in* $H^1(\Omega)$ *as* $\sigma \to 0$.

Proof. We prove this assertion by using the notion of a stream function. Let $\psi \in H^2(\Omega)$ be the stream function of $\boldsymbol{\xi}$ (i.e., $\boldsymbol{\xi} = \nabla^\perp \psi$, where $\nabla^\perp \psi = \begin{pmatrix} \partial\psi/\partial x_2 \\ -\partial\psi/\partial x_1 \end{pmatrix}$). It clearly suffices to prove that there exists a sequence of functions $\{\psi_\sigma\}_{\sigma>0} \subset H^2(\Omega)$ satisfying the conditions

$$\lim_{\sigma\to 0} \psi_\sigma = \psi \text{ in } H^2(\Omega) \quad \text{and} \quad \boldsymbol{\xi}_\sigma = \nabla^\perp \psi_\sigma \in K_\sigma(\chi) \quad \forall\, \sigma > 0. \tag{4.10}$$

We remark that the general form of the stream function of a rigid velocity field is

$$\psi(\boldsymbol{x}) = a + \boldsymbol{b} \cdot \boldsymbol{x} + c|\boldsymbol{x}|^2, \tag{4.11}$$

where $a, c \in \mathbb{R}$ and $\boldsymbol{b} \in \mathbb{R}^2$ are some constants.

Let us divide the sets $S(\chi^0) = \mathbb{R}^2 \setminus \Omega$ and $S(\chi^i)$, $i = 1, \ldots, N$, into several groups as follows: if $\partial S(\chi^i) \cap \partial S(\chi^j) \neq \emptyset$, then $S(\chi^i)$ and $S(\chi^j)$ belong to the same group. Let m be the number of groups. We clearly have $m \leqq N + 1$. Denote by ϕ^k, $k = 1, \ldots, m$, the sum of functions χ^i included in the group number k and let us define r by

$$r = \min_{k\neq\ell} \operatorname{dist} \left(S(\phi^k), S(\phi^\ell)\right).$$

We clearly have $r > 0$.

According to Proposition 4.1 there exist the rigid functions $\boldsymbol{u}_k$, $k = 1, \ldots, m$, such that $\boldsymbol{\xi}(\boldsymbol{x}) = \boldsymbol{u}_k(\boldsymbol{x})$ as $\boldsymbol{x} \in S(\phi^k) \cap \Omega$. Let ψ_k, $k = 1, \ldots, m$, be the corresponding stream functions (which have the form (4.11)). Each of these functions is determined up to a constant. We choose these constants such that $\psi_k(\boldsymbol{x}) = \psi(\boldsymbol{x})$ for all $\boldsymbol{x} \in S(\phi^k) \cap \Omega$.

Moreover, we introduce an auxiliary function $\widetilde{\psi} \in H^2(\Omega)$ such that $\widetilde{\psi}(\boldsymbol{x}) = \psi_k(\boldsymbol{x})$ for all $\boldsymbol{x} \in S_{r/3}(\phi^k) \cap \Omega$. With this auxiliary function we can write ψ as $\psi = \widetilde{\psi} + w$, where $w \in H^2(\Omega)$ and $w(\boldsymbol{x}) = 0$ for all $\boldsymbol{x} \in \bigcup_{k=0}^N S(\chi^k) \cap \Omega$.

Suppose that $\{w^\sigma\} \subset H^2(\Omega)$ is a sequence such that

$$w^\sigma(\boldsymbol{x}) = 0 \quad \forall\, \boldsymbol{x} \in \bigcup_{k=0}^{N} S_\sigma(\chi^k) \cap \Omega, \quad \text{and} \quad \lim_{\sigma\to 0} w_\sigma = w \text{ in } H^2(\Omega), \tag{4.12}$$

then the sequence $\{\psi_\sigma\}$ defined by $\psi_\sigma = \widetilde{\psi} + w_\sigma$ for all $\sigma > 0$ clearly satisfies (4.10).

In order to construct $\{w^\sigma\}$ satisfying (4.12) for any $\sigma > 0$, we consider the sequence of functions $\{\eta_\sigma\}_{\sigma>0}$ such that $\eta_\sigma : \mathbb{R}^+ \to \mathbb{R}^+$, $\eta_\sigma(s) = 0$ if $s \in [0, \sigma]$,

$\eta_\sigma(s) = 1$ if $s \geqq 2\sigma$ and $\left|\eta''_\sigma(s)\right| = 4\sigma^{-2}$ if $s \in (\sigma, 2\sigma)$. We can take for instance

$$\eta_\sigma(s) = \begin{cases} 0 & \text{if } s \leqq \sigma, \\ 2(\sigma^{-1}s - 1)^2 & \text{if } \sigma < s \leqq \frac{3}{2}\sigma, \\ 1 - 4(2 - \sigma^{-1}s)^2 & \text{if } \frac{3}{2}\sigma < s < 2\sigma, \\ 1 & \text{if } s \geqq 2\sigma. \end{cases}$$

Denote by $\eta^k_\sigma(\boldsymbol{x})$ the function $\eta_\sigma(d_k(\boldsymbol{x}))$, where $d_k(\boldsymbol{x})$ denotes the distance from $\boldsymbol{x}$ to $S(\chi^k)$. After some calculation we conclude that the function

$$w^\sigma(\boldsymbol{x}) = w(\boldsymbol{x}) \prod_{k=0}^{N} \eta^k_\sigma(\boldsymbol{x})$$

converges to w in $H^2(\Omega)$ as $\sigma \to 0$. The proposition is proved.

As a consequence of the result above we obtain

Corollary 4.1. *The spaces $K(\chi)$ and $K_0(\chi)$ coincide.*

Proof. For any $\sigma > 0$, we have $K_\sigma(\chi) \subset K(\chi)$. Hence

$$\bigcup_{\sigma>0} K_\sigma(\chi) \subset K(\chi).$$

Since $K(\chi)$ is a closed subspace of H^1_0,

$$K_0(\chi) = \overline{\bigcup_{\sigma>0} K_\sigma(\chi)} \subset K(\chi).$$

The opposite inclusion follows directly from Proposition 4.3, thus the result is proved.

5. Compactness of the density field

5.1. Some background on the transport equation

In this subsection we gather, for easy reference, some basic facts about transport equations and, in particular, those concerning compactness of weak solutions. We do not give proofs, we only refer to the relevant literature.

Let us consider the problem of finding $\psi \in L^\infty(Q)$ such that

$$\frac{\partial \psi}{\partial t} + \operatorname{div}(\psi \boldsymbol{v}) = 0, \qquad \text{in } \mathscr{D}'(Q), \tag{5.1}$$

$$\psi(\boldsymbol{x}, 0) = \psi_0(\boldsymbol{x}), \quad \text{in } L^\infty(\Omega), \tag{5.2}$$

where $\boldsymbol{v}$ is a given vector field $\boldsymbol{v} \in L^2(0, T; V(\Omega))$ and $\psi_0 \in L^\infty(\Omega)$. We recall the following result of DIPERNA & LIONS (see [6]).

Proposition 5.1. *The problem* (5.1), (5.2) *has a unique weak solution* $\psi \in L^\infty(Q) \cap \mathscr{C}\left([0,T]; L^1(\Omega)\right)$*, in the sense that there exists a unique* $\psi \in L^\infty(Q) \cap \mathscr{C}\left([0,T]; L^1(\Omega)\right)$ *such that*

$$\int_Q \psi\,(\eta_t + (\boldsymbol{v}\cdot\nabla)\eta)\;dx\,dt = -\int_\Omega \psi_0\eta(\cdot\,,0)\,dx \quad \forall\,\eta \in \mathscr{C}^1(Q),\ \eta(\cdot\,,T) = 0.$$

Furthermore, if the data satisfies $\psi_0(\boldsymbol{x}) \in \{0,1\}$ *a.e. in* Ω*, then* $\psi(\boldsymbol{x},t) \in \{0,1\}$ *a.e. in* Q*.*

For a proof of Proposition 5.1, we refer to [6]. Let us only point out that the previous problem need not be complemented by boundary conditions because the velocity field $\boldsymbol{v}$ vanishes on $\partial\Omega$.

We will essentially use the following compactness result, also due to DiPerna & Lions (see for instance [19]).

Theorem 5.1. *Let* $\{\psi_n\}_{n>0}$ *and* $\{\boldsymbol{v}_n\}_{n>0}$ *be two sequences such that*

$$\{\psi_n\} \subset \mathscr{C}\left([0,T]; L^1(B_R)\right) \text{ for all } R > 0,$$
$$\{\boldsymbol{v}_n\} \subset L^2\left(0,T; V(\Omega)\right).$$

If the sequence $\{\psi_n\}$ *is bounded in* $L^\infty(Q)$*, the sequence* $\{\boldsymbol{v}_n\}$ *is bounded in* $L^2\left(0,T; V(\Omega)\right)$ *and*

$$\frac{\partial\psi_n}{\partial t} + \operatorname{div}(\psi_n\boldsymbol{v}_n) = 0 \qquad \text{in } \mathscr{D}'(Q),$$
$$\psi_n(0) \to \psi_0 \quad \text{in } L^1(\Omega),$$
$$\boldsymbol{v}_n \rightharpoonup \boldsymbol{v} \qquad \text{weakly in } L^2(0,T;V(\Omega)),$$

for some $\psi_0 \in L^\infty(\Omega)$*,* $\psi_0 \geqq 0$ *a.e., then* $\{\psi_n\}$ *converges strongly in the space* $\mathscr{C}\left([0,T]; L^p(\Omega)\right)$ *for all* $1 \leqq p < \infty$ *to the unique solution* $\psi \in L^\infty(Q) \cap \mathscr{C}\left([0,T]; L^1(\Omega)\right)$ *of the problem*

$$\frac{\partial\psi}{\partial t} + \operatorname{div}(\psi\boldsymbol{v}) = 0 \qquad \text{in } \mathscr{D}'(Q),$$
$$\psi(\boldsymbol{x},0) = \psi_0(\boldsymbol{x}) \quad \text{a.e. in } \Omega.$$

5.2. Passage to the limit in the transport equations

In this subsection we apply the results in the previous subsection to the sequences of solutions of the penalized problem (3.1)–(3.8).

In order to prove Proposition 3.1 we first notice that, by Theorem 5.1, we have the following result.

Lemma 5.1. *The sequences* $\{\rho_n\}$, $\left\{\psi_n^i\right\}$ *contain subsequences* (*which we also denote by* $\{\rho_n\}$, $\left\{\psi_n^i\right\}$) *such that*
$\rho_n \to \rho$ *strongly in* $\mathscr{C}\left([0,T]; L^p(\Omega)\right)$, $(1 \leqq p < \infty)$,
$\psi_n^i \to \psi^i$ *strongly in* $L^p(\Omega_{\text{ext}}\times]0,T[)$, $(1 \leqq p < \infty)$.

Corollary 5.1. *The corresponding subsequences of* $\{\varphi_n^i\}$ *and* $\{\varphi_n\}$ *(which we also denote by* $\{\varphi_n^i\}$ *and* $\{\varphi_n\}$*) converge respectively to* φ^i *and* φ *strongly in* $L^p(\Omega_{\text{ext}}\times]0,T[)$, $(1 \leqq p < \infty)$.

We can obtain more information about the convergence of ψ_n^i, φ_n^i, φ_n by using the regularity of the vector field $\overline{\boldsymbol{u}_n}$. In order to obtain this information we recall some classical notions on ordinary differential equations and characteristics of transport equations.

Let us consider the following Cauchy problem:

$$\begin{aligned} \frac{dX(t)}{dt} &= \overline{\boldsymbol{u}_n}(X,t), \\ X(s) &= \boldsymbol{y}, \end{aligned} \tag{5.3}$$

where $\boldsymbol{y} \in \Omega_{\text{ext}}$ and $s \in [0,T]$ are given. Since for almost all $t \in [0,T]$, $\overline{\boldsymbol{u}_n}(\cdot\,,t) \in \mathscr{D}(\mathbb{R}^2)$ and $\overline{\boldsymbol{u}_n}(\boldsymbol{x},\cdot) \in L^\infty(0,T;\mathbb{R}^2)$ for all $\boldsymbol{x} \in \Omega_{\text{ext}}$, it follows from classical results (see for instance [20, Section 68]) that (5.3) admits a unique solution defined in $[0,T]$. Moreover, since $\overline{\boldsymbol{u}_n}|_{\partial\Omega_{\text{ext}}} = 0$, it follows that $X(t) \in \Omega_{\text{ext}}$ for all $t \in [0,T]$. Let us denote by $\mathscr{M}_{s,t}^n(\boldsymbol{y})$ this unique solution.

The properties of the family of mappings $\mathscr{M}_{s,t}^n(\boldsymbol{y})$ can be summarized by the following result.

Lemma 5.2. (a) *The set of functions*

$$\boldsymbol{y} \to \mathscr{M}_{s,t}^n(\boldsymbol{y})$$

is bounded in $\mathscr{C}^2(\Omega_{\text{ext}};\mathbb{R}^2)$*, uniformly with respect to* $s,t \in [0,T]$ *and* $n > 0$.

(b) *The set of functions*

$$s \to \mathscr{M}_{s,t}^n(\boldsymbol{y})$$

is bounded in $W^{1,\infty}(0,T;\mathbb{R}^2)$*, uniformly with respect to* $t \in [0,T]$*,* $\boldsymbol{y} \in \Omega_{\text{ext}}$ *and* $n > 0$*. Moreover, the set of functions*

$$t \to \mathscr{M}_{s,t}^n(\boldsymbol{y})$$

is bounded in $W^{1,\infty}(0,T;\mathbb{R}^2)$*, uniformly with respect to* $t \in [0,T]$*,* $\boldsymbol{y} \in \Omega_{\text{ext}}$ *and* $n > 0$.

(c) *Also* $\det\left(\frac{\partial\mathscr{M}_{s,t}^n(\boldsymbol{y})}{\partial\boldsymbol{y}}\right) = 1$ *for any* $\boldsymbol{y} \in \Omega_{\text{ext}}$, $s,t \in [0,T]$, $n > 0$.

Proof. The boundedness of $\mathscr{M}_{s,t}(\cdot)$ in $\mathscr{C}(\Omega_{\text{ext}})$ is a direct consequence of the fact that $\mathscr{M}_{s,t}(\boldsymbol{y}) \in \Omega_{\text{ext}}$ for all $\boldsymbol{y} \in \Omega_{\text{ext}}$. Moreover, according to Theorem 1A in [17, p. 57] (see also [20, Section 69]), for each fixed $(s,t) \in [0,T]\times[0,T]$, the function $\mathscr{M}_{s,t}(\cdot)$ is $\mathscr{C}^1(\Omega_{\text{ext}})$ and the functions $t \to \frac{\partial\mathscr{M}_{s,t}(\boldsymbol{y})}{\partial y_i}$, $i = 1,2$, are absolutely continuous in t and they satisfy the linear initial value problem

$$\frac{d}{dt}\left(\frac{\partial\mathscr{M}_{s,t}(\boldsymbol{y})}{\partial y_i}\right) = \nabla_{\boldsymbol{x}}\overline{\boldsymbol{u}_n}(\mathscr{M}_{s,t}(\boldsymbol{y}),t)\frac{\partial\mathscr{M}_{s,t}(\boldsymbol{y})}{\partial y_i} \text{ a.e. in } [0,T] \tag{5.4}$$

$$\frac{\partial\mathscr{M}_{s,s}(\boldsymbol{y})}{\partial y_1} = \begin{pmatrix}1\\0\end{pmatrix}, \quad \frac{\partial\mathscr{M}_{s,s}(\boldsymbol{y})}{\partial y_2} = \begin{pmatrix}0\\1\end{pmatrix}. \tag{5.5}$$

Since $\overline{\boldsymbol{u}_n}$ is bounded in $L^\infty(0,T;\mathscr{C}^2(\Omega_{\text{ext}}))$, relations (5.4), (5.5) above imply that assertion (a) of the lemma is true.

In order to prove assertion (b) we first notice that the boundedness of the functions $t \to \mathscr{M}_{s,t}(\boldsymbol{y})$ and $s \to \mathscr{M}_{s,t}(\boldsymbol{y})$ in $\mathscr{C}(\Omega_{\text{ext}})$ is a direct consequence of the fact that $\mathscr{M}_{s,t}(\boldsymbol{y}) \in \Omega_{\text{ext}}$ for all $\boldsymbol{y} \in \Omega_{\text{ext}}$ and $t, s \in [0,T]$. Moreover, according to (5.3), it is clear that $t \to \frac{\partial \mathscr{M}_{s,t}(\boldsymbol{y})}{\partial t}$ is bounded in $L^\infty(0,T;\mathbb{R}^2)$. Concerning the function $s \to \frac{\partial \mathscr{M}_{s,t}(\boldsymbol{y})}{\partial s}$, we notice that it is absolutely continuous in t and satisfies the linear initial value problem

$$\frac{d}{dt}\left(\frac{\partial \mathscr{M}_{s,t}(\boldsymbol{y})}{\partial s}\right) = \nabla_{\boldsymbol{x}} \overline{\boldsymbol{u}_n}(\mathscr{M}_{s,t}(\boldsymbol{y}),t)\frac{\partial \mathscr{M}_{s,t}(\boldsymbol{y})}{\partial s} \text{ a.e. in } [0,T] \tag{5.6}$$

$$\left.\frac{\partial \mathscr{M}_{s,t}(\boldsymbol{y})}{\partial s}\right|_{t=s} = -\overline{\boldsymbol{u}_n}(\boldsymbol{y},s). \tag{5.7}$$

Since $\overline{\boldsymbol{u}_n}$ is bounded in $L^\infty(0,T;\mathscr{C}^1(\Omega_{\text{ext}}))$, relations (5.6), (5.7) above imply that the function $s \to \mathscr{M}_{s,t}(\boldsymbol{y})$ is bounded in $W^{1,\infty}(0,T;\mathbb{R}^2)$. This ends the proof of assertion (b).

In order to prove assertion (c) it suffices to notice that relations (5.4), (5.5) above and the classical Liouville theorem imply that

$$\text{Det}\left(\frac{\partial \mathscr{M}_{s,t}(\boldsymbol{y})}{\partial \boldsymbol{y}}\right) = \exp \int_s^t \text{div}\left(\overline{\boldsymbol{u}_n}(\mathscr{M}_{s,\eta}(\boldsymbol{y}),\eta)\right) d\eta \tag{5.8}$$

and to use the fact that $\text{div}\overline{\boldsymbol{u}_n} = 0$.

From the lemma above we can conclude the following corollaries.

Corollary 5.2. *The sequence* $\{\mathscr{M}^n\}$ *converges to* $\mathscr{M}$ *in*

$$\mathscr{C}^{0,\alpha}\big([0,T]\times[0,T];\mathscr{C}^1(\Omega_{\text{ext}})\big),$$

$\alpha < 1$, *as* $n \to \infty$, *where* $\mathscr{M}_{s,t}(\boldsymbol{y})$ *is the unique solution of the Cauchy problem*

$$\frac{dX(t)}{dt} = \overline{\boldsymbol{u}}(X,t),$$
$$X(s) = \boldsymbol{y} \in \Omega_{\text{ext}}.$$

Simple calculations show that the solution of the transport equation (3.8) is

$$\psi_n^i(\boldsymbol{x},t) = \psi^{i,0}\left(\mathscr{M}_{t,0}^n(\boldsymbol{x})\right). \tag{5.9}$$

The relation above, Corollary 5.2 and the dominated convergence theorem imply the following result.

Corollary 5.3. *The functions* ψ^i *in Lemma 5.1 satisfy the condition*

$$\psi^i(\boldsymbol{x},t) = \psi^{i,0}\left(\mathscr{M}_{t,0}(\boldsymbol{x})\right) \quad \forall\, \boldsymbol{x} \in \Omega_{\text{ext}},\ \forall\, t \in [0,T]. \tag{5.10}$$

Proof of Proposition 3.1. From (3.9) and (5.12) we conclude that $\varphi^i \mathrm{D}(\boldsymbol{u}) = 0$ for $i = 1, \dots, N$. This fact implies that there exist rigid functions $\boldsymbol{v}^i$ such that $\boldsymbol{v}^i(\boldsymbol{x}, t) = \boldsymbol{u}(\boldsymbol{x}, t)$ for $\boldsymbol{x} \in S(\varphi^i(t))$. It follows from Remark 2.1 that

$$\overline{\boldsymbol{u}}(\boldsymbol{x}, t) = \boldsymbol{v}^i(\boldsymbol{x}, t) \quad \forall \boldsymbol{x} \in S(\psi^i(t)). \tag{5.11}$$

Let us define $\mathscr{A}^i_{s,t}(\boldsymbol{y})$ as the unique solution of the problem:

$$\begin{aligned} \frac{dX(t)}{dt} &= \boldsymbol{v}^i(X, t), \\ X(s) &= \boldsymbol{y} \in \mathbb{R}^2. \end{aligned} \tag{5.12}$$

If $\boldsymbol{y} \in S(\psi^i(s))$, then, by (5.11), we know that $\mathscr{A}^i_{s,t}(\boldsymbol{y}) = \mathscr{M}_{s,t}(\boldsymbol{y})$, so relation (5.10) can be rewritten as

$$\psi^i(\boldsymbol{x}, t) = \psi^{i,0}\big(\mathscr{A}^i_{t,0}(\boldsymbol{x})\big) \quad \forall \boldsymbol{x} \in \Omega_{\text{ext}}, \ \forall t \in [0, T]. \tag{5.13}$$

Since $\mathscr{A}^i_{t,0}(\boldsymbol{x})$ is a rigid displacement, relation (5.13) implies that

$$\varphi^i(\boldsymbol{x}, t) = \varphi^{i,0}\big(\mathscr{A}^i_{t,0}(\boldsymbol{x})\big) \quad \forall \boldsymbol{x} \in \Omega_{\text{ext}}, \ \forall t \in [0, T]. \tag{5.14}$$

The relation above implies that

$$\frac{\partial \varphi^i(\boldsymbol{x}, t)}{\partial t} + \operatorname{div}\big(\varphi^i(\boldsymbol{x}, t)\boldsymbol{v}^i(\boldsymbol{x}, t)\big) = 0 \text{ in } \mathscr{D}'(\Omega_{\text{ext}} \times [0, T)).$$

In other words we showed that

$$\int_0^T \int_{\Omega_{\text{ext}}} \varphi^i(\eta_t + (\boldsymbol{v}^i \cdot \nabla)\eta)\, dx\, dt = -\int_{\Omega_{\text{ext}}} \varphi^{i,0}\eta_0\, dx$$

for all $\eta \in \mathscr{C}^1(Q)$, $\eta(T) = 0$.

Moreover, since $\varphi^i \boldsymbol{v}^i = \varphi^i \boldsymbol{u}$ and $\boldsymbol{u}(\boldsymbol{x}, t) = 0$ for $\boldsymbol{x} \in \Omega_{\text{ext}} \setminus \Omega$,

$$\int_0^T \int_{\Omega} \varphi^i(\eta_t + (\boldsymbol{u} \cdot \nabla)\eta)\, dx\, dt = -\int_{\Omega} \varphi^{i,0}\eta_0\, dx \tag{5.15}$$

for all $\eta \in \mathscr{C}^1(Q)$, $\eta(T) = 0$.

We have thus proved that φ^i and $\boldsymbol{u}$ satisfy (2.11). Moreover, from Proposition 5.1 we know that $\varphi^i \in \mathrm{Char}(Q)$.

Concerning the function $\varphi = \sum_{i=1}^N \varphi^i$, we notice that it satisfies the equation

$$\int_0^T \int_{\Omega} \varphi(\eta_t + (\boldsymbol{u} \cdot \nabla)\eta)\, dx\, dt = -\int_{\Omega} \varphi_0\eta_0\, dx,$$

where $\varphi_0 = \sum \varphi^{i,0}$ takes only two values: 1 and 0. By Proposition 5.1 it follows that φ takes also only two values: 1 and 0, i.e., (2.7) holds true. This fact implies that $S(\varphi^i) \cap S(\varphi^j) = \emptyset$ if $i \neq j$.

According to Theorem 5.1, the function $\rho \in L^\infty(Q)$ satisfies (2.10) and

$$0 < m_0 \leqq \rho \leqq \|\rho^0\|_{L^\infty(\Omega)}.$$

6. Some technical results

In this section we give several technical results which are an essential ingredient of the proof of the compactness of the velocity field.

If $\sigma > 0$ and $G \subset \mathbb{R}^2$, we denote by G_σ the σ-neighbourhood of G, i.e.,

$$G_\sigma = \left\{ \boldsymbol{x} \in \mathbb{R}^2 : d(\boldsymbol{x}, G) < \sigma \right\}.$$

Notice that if we take $\sigma = \delta$, where $\delta > 0$ is the number fixed in Section 2, then $G_\delta = G_{\text{ext}}$.

Let $\chi : \Omega \to \{0, 1\}$ be the characteristic function of a subset $S(\chi)$ of Ω. We suppose that the boundary of $S(\chi)$ is of class $\mathscr{C}^2$. Let us introduce the function spaces

$V^s(\Omega)$ the closure of $\mathscr{V}(\Omega)$ in $H^s(\Omega)$, $0 < s \leqq 1$,

$K^s(\chi)$ the closure of $K(\chi)$ in $H^s(\Omega)$, $0 \leqq s \leqq 1$,

where the spaces $\mathscr{V}(\Omega)$ and $K(\chi)$ were introduced in Section 2. We note that $V^1(\Omega) = V(\Omega)$ and $K^1(\chi) = K(\chi)$, where the space $V(\Omega)$ was also introduced in Section 2.

Moreover we define several projection operators.

First we denote by $P^s(\chi)$, the orthogonal projector of $H^s(\Omega)$ onto $K^s(\chi)$, $0 \leqq s \leqq 1$.

If $\sigma > 0$, we denote by $P^s_\sigma(\chi)$ the orthogonal projector of $H^s(\Omega)$ onto the space of functions which are rigid velocity fields in a σ-neighbourhood of $S(\chi)$. More precisely, for $0 \leqq s < 1$, we set $P^s_\sigma(\chi) = P^s(\mathbb{1}_{S_\sigma(\chi)})$, where $\mathbb{1}_{S_\sigma(\chi)}$ is the characteristic function of $S_\sigma(\chi)$.

Lemma 6.1. *For any $\sigma > 0$ there exists $n_0 > 0$ (depending only on σ) such that*

$$S(\varphi^i_n(t)) \subset S_\sigma(\varphi^i(t)) \textit{ and } S(\varphi^i(t)) \subset S_\sigma(\varphi^i_n(t))$$

for all $n > n_0$, for all $t \in [0, T]$ and for all $i = 1, \dots, N$.

Proof. According to Corollary 5.2, we have $\mathscr{M}^n \to \mathscr{M}$ in $\mathscr{C}([0, T] \times [0, T] \times \Omega)$. This fact, combined with (5.9) and (5.10) implies that, for any $\sigma > 0$, there exists $n_0 > 0$ such that

$$S(\psi^i_n(t)) \subset S_\sigma(\psi^i(t)) \text{ and } S(\psi^i(t)) \subset S_\sigma(\psi^i_n(t)) \tag{6.1}$$

for all $n > n_0$, $t \in [0, T]$, $i = 1, \dots, N$. By considering the δ-neighbourhood of the sets above and by using (3.5) we find that for all $\sigma > 0$ we have the relations

$$\begin{aligned} S_\delta(\psi^i_n(t)) &= S(\varphi^i_n), & S_{\sigma+\delta}(\psi^i) &= S_\sigma(\varphi^i), \\ S_\delta(\psi^i(t)) &= S(\varphi^i) \quad \text{and} & S_{\sigma+\delta}(\psi^i_n) &= S_\sigma(\varphi^i_n). \end{aligned} \tag{6.2}$$

Relations (6.1) and (6.2) imply the proof of the lemma.

The next result of this section is

Proposition 6.1. *Let $\boldsymbol{u}$ and φ be the functions considered in Proposition 3.1; then*

$$\lim_{\sigma\to 0}\int_0^T \left\| P_\sigma^s(\varphi(\cdot\,,t))\boldsymbol{u}(\cdot\,,t)-\boldsymbol{u}(\cdot\,,t)\right\|^2_{L^2(\Omega)}\,dt=0. \tag{6.3}$$

Proof. For almost every $t\in[0,T]$ we have $\boldsymbol{u}(t)\in K(\varphi(\cdot\,,t))$. Then, by Proposition 4.3 there exists a sequence $\{\boldsymbol{u}_\sigma\}_{\sigma>0}$ that converges to $\boldsymbol{u}(t)$ in $K(\varphi(\cdot\,,t))$, and such that $\boldsymbol{u}_\sigma\in K_\sigma(\varphi(\cdot\,,t))$ for all $\sigma>0$.

Then we have

$$\begin{aligned}\left\| P_\sigma^s(\varphi(\cdot\,,t))\boldsymbol{u}(\cdot\,,t)-\boldsymbol{u}(\cdot\,,t)\right\|_{L^2(\Omega)} &\leqq \left\| P_\sigma^s(\varphi(\cdot\,,t))\boldsymbol{u}(\cdot\,,t)-\boldsymbol{u}(\cdot\,,t)\right\|_{V^s(\Omega)}\\ &\leqq \|\boldsymbol{u}_\sigma-\boldsymbol{u}(\cdot\,,t)\|_{H_0^1(\Omega)}\,.\end{aligned}$$

We conclude that the sequence of functions

$$f_\sigma(t)=\left\| P_\sigma^s(\varphi(\cdot\,,t))\boldsymbol{u}(\cdot\,,t)-\boldsymbol{u}(\cdot\,,t)\right\|^2_{L^2(\Omega)}$$

converges to zero for a.e. $t\in[0,T]$. Since $\{f_\sigma\}$ is bounded from above by the function $g\in L^1(0,T)$ defined by $g(t)=\|\boldsymbol{u}(\cdot\,,t)\|^2_{H_0^1(\Omega)}$, by using the Lebesgue dominated convergence theorem we conclude that assertion (6.3) holds true.

Let us introduce a family of open sets $\{E_\sigma\}_{\sigma>0}$ in the following way: for any $\sigma\in(0,1)$ we first define the sets

$$E_\sigma^{ij}=\left\{t\in[0,T]:0<\operatorname{dist}\big(S(\varphi^i(t)),S(\varphi^j(t))\big)<\sigma^{1/4}\right\} \tag{6.4}$$

for all $i,j=0,\dots,N$, and then we denote

$$E_\sigma=\bigcup_{i,j=1}^N E_\sigma^{ij}. \tag{6.5}$$

Proposition 6.2. *The family of sets $\{E_\sigma\}_{\sigma>0}$ defined above satisfies*

$$\lim_{\sigma\to 0}\mu(E_\sigma)=0, \tag{6.6}$$

where μ denotes the Lebesgue measure in $\mathbb{R}$.

Proof. Since, by (5.14), $S(\varphi^i(t))=\mathscr{A}^i_{0,t}\left(S(\varphi^{i,0})\right)$ and, for all $\boldsymbol{y}\in\Omega$, the functions $t\to\mathscr{A}^i_{0,t}(\boldsymbol{y})$ defined by (5.12) are continuous, we deduce that the real function

$$t\to d\big(S(\varphi^i(t)),S(\varphi^j(t))\big)$$

is also continuous in $[0,T]$. By applying a classical measure-theory result (see, for instance, Theorem E in [11, p. 38]) we find that $\mu(E_\sigma)\to 0$ when $\sigma\to 0$. Thus the sequence $\{E_\sigma\}_{\sigma>0}$ satisfies (6.6).

We now introduce some notation close to the notation used in the proof of Proposition 4.3, for the time-independent case. More precisely, for $t \in [0, T]$, we divide the sets $S(\varphi^0(\cdot\,, t)) = \mathbb{R}^2 \setminus \Omega$ and $S(\varphi^i(\cdot\,, t))$, $i = 1, \ldots, N$, into several groups as follows: if $\partial S(\varphi^i(\cdot\,, t)) \cap \partial S(\varphi^j(\cdot\,, t)) \neq \emptyset$, then $S(\varphi^i(\cdot\,, t))$ and $S(\varphi^j(\cdot\,, t))$ belong to the same group. Let $m(t)$ be the number of groups at time t. We clearly have $m(t) \leqq N + 1$, for all $t \in [0, T]$. Let us introduce the following notation:

– $J_k(t) = \left\{i \in \{0, \ldots, N\} : S(\varphi^i(\cdot\,, t)) \text{ belong to the group number } k\right\}$,
– $\phi^k(\cdot\,, t)$, $k = 1, \ldots, m(t)$ is the characteristic function of the set

$$\bigcup_{i \in J_k(t)} S\left(\varphi^i(\cdot\,, t)\right)$$

and
– $\phi_n^k(\cdot\,, t)$, $k = 1, \ldots, m(t)$ is the characteristic function of the set

$$\bigcup_{i \in J_k(t)} S\left(\varphi_n^i(\cdot\,, t)\right).$$

Remark 6.1. An alternative definition of the groups is the following: if $J \subset \{0, \ldots, N\}$, we say that the domains $S\left(\varphi^i(\cdot\,, t)\right)$, $i \in J$ form one group if $\bigcup_{i \in J} S_\gamma\left(\varphi^i(\cdot\,, t)\right)$ is a connected domain for every $\gamma > 0$ and for every $j \in \{0, \ldots, N\} \setminus J$ there exists $\gamma > 0$ such that $\bigcup_{i \in J} S_\gamma\left(\varphi^i(\cdot\,, t)\right) \bigcup S_\gamma\left(\varphi^j(\cdot\,, t)\right)$ is not connected.

Lemma 6.2. *For almost every $t \in [0, T]$ there exists a subsequence of $\{\boldsymbol{u}_n\}$ (also denoted by $\{\boldsymbol{u}_n\}$) and the corresponding sequences of rigid velocity fields $\left\{\boldsymbol{v}_n^k\right\}$, $k = 1, \ldots, m(t)$ such that,*

$$\lim_{n \to \infty} \left\| \boldsymbol{u}_n(\cdot\,, t) - \boldsymbol{v}_n^k(\cdot\,, t) \right\|_{W^{1,p}\left(S(\phi^k(\cdot\,,t))\right)} = 0 \quad \forall\, p \in [1, 2) \quad \forall\, k = 1, \ldots, m(t). \tag{6.7}$$

Moreover, if $0 \in J_k(t)$, then $\boldsymbol{v}_n^k(\cdot\,, t) = 0$.

Proof. Let us define

$$F_n(t) = \int_\Omega \varphi_n(\boldsymbol{x}, t)\, |D(\boldsymbol{u}_n(\boldsymbol{x}, t))|^2 \, dx.$$

If $p < 2$ then, by Hölder inequality, we have

$$\begin{aligned}
&\int_{S(\varphi(t))} |D(\boldsymbol{u}_n(\boldsymbol{x}, t))|^p \, dx \\
&\quad \leqq \int_{S(\varphi_n(t))} |D(\boldsymbol{u}_n(\boldsymbol{x}, t))|^p \, dx \\
&\qquad + \int_{S(\varphi(t)) \setminus S(\varphi_n(t))} |D(\boldsymbol{u}_n(\boldsymbol{x}, t))|^p \, dx \\
&\quad \leqq \mu\left(S(\varphi_n(t))\right)^{\frac{2-p}{2}} \|D(\boldsymbol{u}_n(\boldsymbol{x}, t))\|^p_{L^2(S(\varphi_n(t)))} \\
&\qquad + \mu\left(S(\varphi(t)) \setminus S(\varphi_n(t))\right)^{\frac{2-p}{2}} \|D(\boldsymbol{u}_n(\boldsymbol{x}, t))\|^p_{L^2(\Omega)} \\
&\quad \leqq C_1 F_n(t)^{\frac{p}{2}} + \|D(\boldsymbol{u}_n(\boldsymbol{x}, t))\|^p_{L^2(\Omega)}\, \mu\left(S(\varphi(t)) \setminus S(\varphi_n(t))\right)^{\frac{2-p}{2}}.
\end{aligned} \tag{6.8}$$

From (3.9) it follows that the family $\{nF_n\}$ is bounded in $L^1(0,T)$, so we know that $F_n \to 0$ strongly in $L^1(0,T)$. On the other hand, from (3.9) and Lemma 6.1 it follows that the second term in the right-hand side of (6.8) tends to zero in $L^1(0,T)$.

Let us denote by $A_n(t)$ the expression in the right-hand side of (6.8). We have proved that

$$\lim_{n\to\infty} \|A_n(t)\|_{L^1(0,T)} = 0,$$

so, up to the extraction of a subsequence, we have

$$\lim_{n\to\infty} A_n(t) = 0 \quad \text{for almost all } t \in (0,T). \tag{6.9}$$

Let $\varepsilon > 0$. By Egorov's theorem (see, for instance, [15]), there exists a subset M_ε of $[0,T]$ such that:

(H1) $A_n \overset{n\to\infty}{\longrightarrow} 0$ uniformly in M_ε;
(H2) $\|\boldsymbol{u}_n(\cdot\,,t)\|_{H^1(\Omega)} \leqq \frac{1}{\varepsilon}$, for all $n \in \mathbb{N}$ and for all $t \in M_\varepsilon$;
(H3) $\mu\left([0,T] \setminus M_\varepsilon\right) \overset{\varepsilon\to 0}{\longrightarrow} 0$.

Let us consider a fixed t such that $t \in M_\varepsilon$. By applying a version of the Korn inequality (see, for instance, [23, p. 20, p. 118]) it follows that for each $i \in \{0,\ldots,N\}$ and for each $p \in (1,2)$ there exists a a rigid velocity field denoted by $\overline{\boldsymbol{v}}_n^i(\cdot\,,t)$ such that

$$\left\|\boldsymbol{u}_n(\cdot\,,t) - \overline{\boldsymbol{v}}_n^i(\cdot\,,t)\right\|^p_{W^{1,p}(S(\varphi^i(\cdot\,,t)))} \leqq C\,\|D(\boldsymbol{u}_n(\cdot\,,t))\|^p_{L^p(S(\varphi^i(\cdot\,,t)))} \leqq C A_n(t). \tag{6.10}$$

Since $\overline{\boldsymbol{v}}_n^i(\cdot\,,t)$ is a rigid velocity field, it can be extended in a unique manner to a rigid velocity field defined on $\mathbb{R}^2$. For the sake of simplicity, this rigid velocity field is also denoted by $\overline{\boldsymbol{v}}_n^i(\cdot\,,t)$. Moreover, by using the properties (H1) and (H2) of M_ε, we obtain

$$\left\|\overline{\boldsymbol{v}}_n^i(\cdot\,,t)\right\|_{W^{1,p}(S(\varphi^i(\cdot\,,t)))} \leqq \frac{2}{\varepsilon} \ \text{ for all } n \in \mathbb{N} \ \text{ and for all } t \in M_\varepsilon. \tag{6.11}$$

For every $i = 0,\ldots,N$ one of the following assertions holds true:

- We have $d\left(S(\varphi^i(\cdot\,,t)), S(\varphi^j(\cdot\,,t))\right) > 0$ for all $j \neq i$. In this case, there exists an index $k(i)$ such that $S(\varphi^i(\cdot\,,t)) = S(\phi^{k(i)}(\cdot\,,t))$ and relation (6.7) follows directly from (6.10) and (6.9).
- There exists an index $j \neq i$, such that that $S(\varphi^i(\cdot\,,t))$ and $S(\varphi^j(\cdot\,,t))$ belong to the same group with $d\left(S(\varphi^i(\cdot\,,t)), S(\varphi^j(\cdot\,,t))\right) = 0$. (We remark that two sets can belong to the same group even if their mutual distance is strictly greater than zero.)

Consequently it suffices to consider the second case. Let us fix the indexes i and j such that $S(\varphi^i(\cdot\,,t))$ and $S(\varphi^j(\cdot\,,t))$ belong to the same group and $d\big(S(\varphi^i(\cdot\,,t)), S(\varphi^j(\cdot\,,t))\big) = 0$.

Define

$$\boldsymbol{w}_n^\ell(\boldsymbol{x},t) = \boldsymbol{u}(\boldsymbol{x},t) - \overline{\boldsymbol{v}}_n^\ell(\boldsymbol{x},t), \quad \boldsymbol{x} \in S(\varphi^\ell(\cdot\,,t)),\ \ell \in \{i,j\}, \tag{6.12}$$

where $\overline{\boldsymbol{v}}_n^\ell$ are the rigid velocity fields introduced in (6.10). By relation (6.10) it follows that

$$\left\| \boldsymbol{w}_n^\ell(\cdot\,,t) \right\|^p_{W^{1,p}(S(\varphi^\ell(\cdot\,,t)))} \leqq C A_n(t), \quad \ell \in \{i, j\}. \tag{6.13}$$

By a classical extension result, the function $\boldsymbol{w}_n^j(\cdot\,,t)$ can be extended to a function $\widetilde{\boldsymbol{w}}_n^j(\cdot\,,t) \in W^{1,p}(\Omega)$ such that

$$\left\| \widetilde{\boldsymbol{w}}_n^j(\cdot\,,t) \right\|_{W^{1,p}(\Omega)} \leqq C \left\| \boldsymbol{w}_n^j(\cdot\,,t) \right\|_{W^{1,p}(S(\varphi^j(\cdot\,,t)))}, \quad \operatorname{div} \widetilde{\boldsymbol{w}}_n^j = 0 \text{ in } L^p(\Omega),$$

where C is a constant independent of n.

We introduce the function

$$\widetilde{\boldsymbol{u}}_n(\boldsymbol{x},t) = \boldsymbol{u}_n(\boldsymbol{x},t) - \overline{\boldsymbol{v}}_n^j(\boldsymbol{x},t) - \widetilde{\boldsymbol{w}}_n^j(\boldsymbol{x},t) \quad \forall\, \boldsymbol{x} \in \Omega.$$

It is clear that

$$\begin{aligned} &\widetilde{\boldsymbol{u}}_n(\boldsymbol{x},t) = \overline{\boldsymbol{v}}_n^i(\boldsymbol{x},t) - \overline{\boldsymbol{v}}_n^j(\boldsymbol{x},t) + \boldsymbol{w}_n^i(\boldsymbol{x},t) - \widetilde{\boldsymbol{w}}_n^j(\boldsymbol{x},t) \quad \forall\, \boldsymbol{x} \in S\big(\varphi^i(\cdot\,,t)\big), \\ &\widetilde{\boldsymbol{u}}_n(\boldsymbol{x},t) = 0 \quad \forall\, \boldsymbol{x} \in S\big(\varphi^j(\cdot\,,t)\big). \end{aligned}$$

Define $\boldsymbol{v}_n(\boldsymbol{x},t) = \overline{\boldsymbol{v}}_n^i(\boldsymbol{x},t) - \overline{\boldsymbol{v}}_n^j(\boldsymbol{x},t)$ and $\boldsymbol{w}_n(\boldsymbol{x},t) = \boldsymbol{w}_n^i(\boldsymbol{x},t) - \widetilde{\boldsymbol{w}}_n^j(\boldsymbol{x},t)$. Since $\boldsymbol{v}_n(\boldsymbol{x},t)$ is a rigid velocity field, there exist $\boldsymbol{a}_n(t) \in \mathbb{R}^2$ and $\omega_n(t) \in \mathbb{R}$ such that we can write

$$\boldsymbol{v}_n(\boldsymbol{x},t) = \boldsymbol{a}_n(t) + \omega_n(t)(\boldsymbol{x} - \boldsymbol{x}_M)^\perp,$$

where $M \in \partial S(\varphi^i(\cdot\,,t)) \cap \partial S(\varphi^j(\cdot\,,t))$.

By (6.11) it follows that there exists a constant $C > 0$ such that

$$|\boldsymbol{a}_n(t)| + |\omega_n(t)| \leqq \frac{C}{\varepsilon} \quad \forall\, n \in \mathbb{N},\ \forall\, t \in M_\varepsilon. \tag{6.14}$$

On the other hand, by repeating the entire procedure in the proof of Proposition 4.1 we find that for all $\tau \in (0\,,\delta)$ and for all $p \in (1,2)$ we have

$$\begin{aligned} |\boldsymbol{a}_n(t)| \leqq{}& 2c(\tau)\,[\mu(D_\tau)]^{-1/p}\, \|\nabla \widetilde{\boldsymbol{u}}_n(\cdot\,,t)\|_{L^p(\Omega)} + |\omega_n(t)|\sqrt{\tau^2 + c^2(\tau)} \\ &+ \frac{1}{\mu(D_\tau)} \int_{D_\tau} |\boldsymbol{w}_n(\boldsymbol{x},t)| dx \ \ \forall\, t \in [0,T], \end{aligned} \tag{6.15}$$

where the set D_τ was introduced in (4.4) and the function c was also introduced in the proof of Proposition 4.1.

By using (6.14) and (6.13), relation (6.15) implies that

$$\limsup_{n\to\infty} |\boldsymbol{a}_n(t)| \leqq \frac{C}{\varepsilon} \tau^{\frac{2p-3}{p}} \quad \forall\, \tau \in (0,\delta), \tag{6.16}$$

where C is a constant independent of τ. If we first suppose that $p \in \left(\frac{3}{2}, 2\right)$ and use the fact that (6.16) is valid for all $\tau \in (0,\delta)$ and that $\boldsymbol{a}_n(t)$ is independent of τ, we obtain

$$\lim_{n\to\infty} |\boldsymbol{a}_n(t)| = 0.$$

Similarly,

$$\limsup_{n\to\infty} |\omega_n(t)|^p \leqq C\gamma^{2p-3} \limsup_{n\to\infty} \|\nabla\widetilde{\boldsymbol{u}}_n\|^p_{L^p(G_\gamma)} \leqq C\frac{\gamma^{2p-3}}{\varepsilon} \quad \forall\,\gamma\in(0,\delta), \tag{6.17}$$

for some constant $C > 0$, where the domain G_γ was defined in the proof of Proposition 4.1. By passing to the limit when $\gamma \to 0$, the estimate above implies, provided that $p \in \left(\frac{3}{2}, 2\right)$, the relation $\lim_{n\to\infty} \omega_n(t) = 0$.

Thus, we can say that, if $p \in \left(\frac{3}{2}, 2\right)$, then

$$\lim_{n\to\infty} \left\|\overline{\boldsymbol{v}}^i_n(\boldsymbol{x},t) - \overline{\boldsymbol{v}}^j_n(\boldsymbol{x},t)\right\|_{W^{1,p}(\Omega)} = 0$$

for all $t \in M_\varepsilon$ and consequently for almost all $t \in [0,T]$, since $\varepsilon > 0$ is arbitrary. Therefore, for almost all $t \in [0,T]$, we have

$$\lim_{n\to\infty} \left\|\boldsymbol{u}_n(\cdot\,,t) - \overline{\boldsymbol{v}}^i_n(\cdot\,,t)\right\|_{W^{1,p}\left(S(\varphi^i(\cdot\,,t))\cup S(\varphi^j(\cdot\,,t))\right)} = 0.$$

If k is a number such that $i \in J_k(t)$, then the relation above is also valid for all $j \in J_k(t)$. This conclusion allows us to take the function $\overline{\boldsymbol{v}}^i_n$ as the function $\boldsymbol{v}^k_n$ in the formulation of the lemma. Thus relation (6.7) is proved for $p \in \left(\frac{3}{2}, 2\right)$ and consequently for all $p \in [1,2)$. Finally, if $0 \in J_k(t)$, then we can take $\boldsymbol{v}^k_n = \overline{\boldsymbol{v}}^0_n$ and $\overline{\boldsymbol{v}}^0_n = \boldsymbol{0}$. The lemma is proved.

The main result of this section is

Proposition 6.3. *Let* $\{E_\sigma\}_{\sigma>0}$ *be the family of sets defined by* (6.4), (6.5). *Then for all* $s \in [0,1)$ *we have*

$$\lim_{\sigma\to 0}\lim_{n\to\infty} \left\| P^s_\sigma(\varphi(\cdot\,,t))\boldsymbol{u}_n - \boldsymbol{u}_n \right\|_{L^2([0,T]\setminus E_\sigma;V^s(\Omega))} = 0. \tag{6.18}$$

Proof. Let us first suppose that, for an arbitrary $\sigma_0 > 0$, there exists a family of functions $\left(\boldsymbol{u}^\sigma_n(\cdot\,,t)\right)_{n,\sigma}$ such that $\boldsymbol{u}^\sigma_n(\cdot\,,t) \in K^s_\sigma(\varphi(\cdot\,,t))$ and

$$\lim_{\sigma\to 0}\lim_{n\to\infty} \left\|\boldsymbol{u}^\sigma_n(\cdot\,,t) - \boldsymbol{u}_n(\cdot\,,t)\right\|_{V^s(\Omega)} = 0 \tag{6.19}$$

for almost all $t \in [0,T] \setminus E_{\sigma_0}$. Since

$$\left\|P^s_\sigma(\varphi(\cdot\,,t))\boldsymbol{u}_n(\cdot\,,t) - \boldsymbol{u}_n(\cdot\,,t)\right\|_{V^s(\Omega)} \leqq \left\|\boldsymbol{u}^\sigma_n(\cdot\,,t) - \boldsymbol{u}_n(\cdot\,,t)\right\|_{V^s(\Omega)},$$

relation (6.19) still holds if we replace $\boldsymbol{u}^\sigma_n(\cdot\,,t)$ by $P^s_\sigma(\varphi(\cdot\,,t))\boldsymbol{u}_n(\cdot\,,t)$. Moreover, the function $t \to \left\|\boldsymbol{w}^\sigma_n(\cdot\,,t)\right\|_{V^s(\Omega)}$, where

$$\boldsymbol{w}^\sigma_n(\cdot\,,t) = P^s_\sigma(\varphi(\cdot\,,t))\boldsymbol{u}_n(\cdot\,,t) - \boldsymbol{u}_n(\cdot\,,t),$$

is measurable, and

$$\begin{aligned}\int_0^T \left\| \boldsymbol{w}_n^\sigma(\cdot\,,t)\right\|_{V^s(\Omega)}^{2/s} dt &\leqq \int_0^T \|\boldsymbol{u}_n(\cdot\,,t)\|_{V^s(\Omega)}^{2/s} dt \\ &\leqq C\int_0^T \|\boldsymbol{u}_n(\cdot\,,t)\|_{L^2(\Omega)}^{2(1-s)/s} \|\boldsymbol{u}_n(\cdot\,,t)\|_{V^1(\Omega)}^2 dt \qquad (6.20)\\ &\leqq C\int_0^T \|\boldsymbol{u}_n(\cdot\,,t)\|_{V^1(\Omega)}^2 dt \leqq C\end{aligned}$$

due to the energy estimate (3.9).

Relations (6.19) and (6.20) yield (6.18).

Thus in order to finish the proof of the proposition we have only to construct a family of functions $\left(\boldsymbol{u}_n^\sigma(\cdot\,,t)\right)_{n,\sigma}$ such that $\boldsymbol{u}_n^\sigma(\cdot\,,t) \in K_\sigma^s(\varphi(\cdot\,,t))$, for all $n \geqq 1$, and which satisfies (6.19).

By Lemma 6.1, for any $\gamma > 0$ there exists $n_0 > 0$ such that, for all $n > n_0$, $t \in [0,T]$ and $k = 1,\ldots,m(t)$, we have

$$S\left(\phi_n^k(\cdot\,,t)\right) \subset S_\gamma\left(\phi^k(\cdot\,,t)\right) \qquad (6.21)$$

and

$$\mu\left(S_\gamma\left(\phi^k(\cdot\,,t)\right)\setminus S\left(\phi_n^k(\cdot\,,t)\right)\right) < \gamma.$$

Let us fix an arbitrary $t \in E\setminus E_\sigma$, where $E \subset [0,T]$ is the set for which the assertion of Lemma 6.2 holds. (In particular we have that $\mu(E) = T$.) By (6.4) and (6.5) we have

$$d\left(S(\phi^k(\cdot\,,t)),\, S(\phi^\ell(\cdot\,,t))\right) \geqq \sigma^{1/4}. \qquad (6.22)$$

Let us introduce the stream functions Ψ_n, $\widetilde{\Psi}_n^k$, $k = 1,\ldots,m(t)$, such that

$$\begin{aligned}&\nabla^\perp\Psi_n = \boldsymbol{u}_n, \qquad \Psi_n|_{\partial\Omega} = \left.\frac{\partial\Psi_n}{\partial n}\right|_{\partial\Omega} = 0\\ &\nabla^\perp\widetilde{\Psi}_n^k = \boldsymbol{v}_n^k.\end{aligned}$$

By Lemma 6.2, the stream functions $\widetilde{\Psi}_n^k$ can be chosen such that

$$\lim_{n\to\infty}\left\|\Psi_n(\cdot\,,t) - \widetilde{\Psi}_n^k(\cdot\,,t)\right\|_{W^{2,p}(S(\phi^k(\cdot\,,t)))} = 0.$$

This implies that, for any $\gamma > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$ we have

$$\left\|\Psi_n(\cdot\,,t) - \widetilde{\Psi}_n^k(\cdot\,,t)\right\|_{W^{2,p}(S(\phi^k(\cdot\,,t)))}^p \leqq \gamma. \qquad (6.23)$$

By (6.22) there exists a function $\widetilde{\Psi}_n(\cdot\,,t) \in W^{2,p}(\Omega)$ such that $\widetilde{\Psi}_n(\boldsymbol{x},t) = \widetilde{\Psi}_n^k(\boldsymbol{x},t)$ for $\boldsymbol{x} \in S_{2\sigma}(\phi^k(\cdot\,,t))$, $k = 1,\ldots,m(t)$. Denote by $\bar{\eta}_\sigma(\boldsymbol{x},t)$ the function

$\prod_{i=0}^{N} \eta_\sigma(d_i(\boldsymbol{x},t))$, where $d_i(\boldsymbol{x},t)$ is the distance from $\boldsymbol{x}$ to $S(\varphi^i(\cdot\,,t))$ and the function $\eta_\sigma(\cdot)$ has been defined in the proof of Proposition 4.3.

Let us now introduce the sequence of functions $\{\Psi_n^\sigma\}$ defined by

$$\Psi_n^\sigma = (1-\bar\eta_\sigma)\widetilde\Psi_n + \bar\eta_\sigma \Psi_n.$$

Then we have:

$$\begin{aligned}
&\left\|\Psi_n^\sigma(\cdot\,,t) - \Psi_n(\cdot\,,t)\right\|^p_{W^{2,p}(\Omega)} \\
&\qquad = \left\|(1-\bar\eta_\sigma)\big(\widetilde\Psi_n(\cdot\,,t) - \Psi_n(\cdot\,,t)\big)\right\|^p_{W^{2,p}(\Omega)} \\
&\qquad = \sum_{k=1}^{m(t)} \left\|(1-\bar\eta_\sigma)\big(\widetilde\Psi_n^k(\cdot\,,t) - \Psi_n(\cdot\,,t)\big)\right\|^p_{W^{2,p}\left(S_{2\sigma}(\phi^k(\cdot\,,t))\right)} \qquad (6.24)\\
&\qquad \leqq \sum_{k=1}^{m(t)} \sum_{i\in J_k(t)} \left\|(1-\bar\eta_\sigma)\big(\widetilde\Psi_n^k(\cdot\,,t) - \Psi_n(\cdot\,,t)\big)\right\|^p_{W^{2,p}\left(S_{2\sigma}(\varphi^i(\cdot\,,t))\right)}.
\end{aligned}$$

The classical trace theorem and (6.23) imply that

$$\begin{aligned}
&\left\|\widetilde\Psi_n^k(\cdot\,,t) - \Psi_n(\cdot\,,t)\right\|^p_{L^p(\partial S(\varphi^i(\cdot\,,t)))} \\
&\qquad + \left\|\nabla\widetilde\Psi_n^k(\cdot\,,t) - \nabla\Psi_n(\cdot\,,t)\right\|^p_{L^p(\partial S(\varphi^i(\cdot\,,t)))} \leqq C\gamma \qquad (6.25)
\end{aligned}$$

for $i \in J_k(t)$.

It is well known (see, for instance, [7]), that the inequality:

$$\begin{aligned}
&\|f\|^p_{L^p\left(S_{2\sigma}(\varphi^i(\cdot\,,t))\setminus S(\varphi^i(\cdot\,,t))\right)} \\
&\qquad \leqq C\Big(\|f\|^p_{L^p(\partial S(\varphi^i(\cdot\,,t)))} + \sigma^p \|\nabla f\|^p_{L^p\left(S_{2\sigma}(\varphi^i(\cdot\,,t))\setminus S(\varphi^i(\cdot\,,t))\right)}\Big)
\end{aligned}$$

holds for all $f \in W_p^1\left(S_{2\sigma}(\varphi^i(\cdot\,,t)) \setminus S(\varphi^i(\cdot\,,t))\right)$, $i = 0,\dots N$, and for all σ small enough. This fact combined with (6.23), (6.24) and (6.25) implies that

$$\left\|\Psi_n^\sigma(\cdot\,,t) - \Psi_n(\cdot\,,t)\right\|^p_{W^{2,p}(\Omega)} \leqq C\sigma^{-2p}\gamma + C\sigma^{(2-p)/2}\|\boldsymbol{u}_n(\cdot\,,t)\|^p_{H^1(\Omega)}. \qquad (6.26)$$

Let us take $\boldsymbol{u}_n^\sigma(\cdot\,,t) = \nabla^\perp\Psi_n^\sigma(\cdot\,,t)$ and $\gamma = \sigma^{(2+3p)/2}$. Then (6.26) yields

$$\left\|\boldsymbol{u}_n^\sigma(\cdot\,,t) - \boldsymbol{u}_n(\cdot\,,t)\right\|^p_{W_p^1(\Omega)} \leqq C\sigma^{(2-p)/2}, \qquad (6.27)$$

where the constant C depends on t. Due to the classical embedding theorems, if $p \geqq \frac{2}{2-s}$, then (6.27) implies (6.19). This concludes the proof of the proposition.

7. Proof of Theorem 3.2

In order to prove Theorem 3.2 we need two results. The first one is

Proposition 7.1. *For any* $s \in (0,1)$ *there exists a* $\sigma_0 > 0$ *such that, for any* $\sigma \in (0, \sigma_0)$*, we have*

$$\lim_{n\to\infty} \int_Q \rho_n \boldsymbol{u}_n P^s_\sigma(\varphi(\cdot\,,t))(\boldsymbol{u}_n)\,dx\,dt = \int_Q \rho \boldsymbol{u} P^s_\sigma(\varphi(\cdot\,,t))(\boldsymbol{u})\,dx\,dt. \tag{7.1}$$

Proof. Consider an arbitrary $\sigma > 0$. Due to Lemma 6.1, there exists $n_0 > 0$ such that

$$S(\varphi_n(t)) \subset S_{\sigma/2}(\varphi(t)) \quad \forall\, t \in [0,T]$$

for all $n > n_0$.

Moreover, if we divide the interval $[0,T]$ into N_T subintervals $I_1 = [0,\tau]$, $I_2 = [\tau, 2\tau]$,. . .,$I_{N_T} = [(N_T-1)\tau, N_T\tau]$, where $\tau = T/N_T$, then the regularity of the function $t \to \mathscr{A}_{0,t}(\boldsymbol{y})$ implies that there exists $\tau > 0$ (depending on σ) such that

$$S_{\sigma/2}(\varphi(t)) \subset S_\sigma(\varphi(k\tau)), \tag{7.2}$$
$$S_{\sigma/2}(\varphi(k\tau)) \subset S_\sigma(\varphi(t)) \tag{7.3}$$

for all $t \in I_k$, and for all $k = 1, \ldots, N_T$.

More precisely, if L is the Lipschitz constant of the function $t \to \mathscr{A}_{0,t}(\boldsymbol{y})$, then there exist

$$\tau \in \left[\frac{\sigma}{\sigma_0/T + 2(L+1)}, \frac{\sigma}{2(L+1)}\right],$$

satisfying (7.2) and (7.3). In particular, it follows that there exists a constant $C > 0$ such that for all $\sigma \in (0,\sigma_0)$ there exist $\tau \geqq C\sigma$ satisfying (7.2) and (7.3).

Let us take one of the intervals I_k, $k = 1, \ldots, N_T$. In (3.6) we consider a test function $\boldsymbol{\xi}$, which is equal to zero if $t \notin I_k$ and such that $\boldsymbol{\xi}(\cdot\,,t) \in K_{\sigma/2}(\varphi(\cdot\,,k\tau))$ for all $t \in I_k$. In this case (3.6) implies, by using classical estimates on the Navier-Stokes equations (see for instance [18, pp. 70–71]), that there exists a constant $C > 0$ such that

$$\left| \int_{I_k} \int_\Omega \rho_n \boldsymbol{u}_n \boldsymbol{\xi}_t\,dx\,dt \right| \leqq C\,\|\boldsymbol{\xi}\|_{L^2(I_k;V(\Omega))} \quad \forall\, n > n_0.$$

The relation above implies that the sequence $\left\{\frac{d}{dt}\left(P^0_{\sigma/2}(\varphi(\cdot\,,k\tau))(\rho_n\boldsymbol{u}_n)\right)\right\}$ is bounded in $L^2(I_k;\left[K_{\sigma/2}(\varphi(\cdot\,,k\tau))\right]^*)$, where $\left[K_{\sigma/2}(\varphi(\cdot\,,k\tau))\right]^*$ is the dual space of $K_{\sigma/2}(\varphi(\cdot\,,k\tau))$ with respect to the pivot space $K^0_{\sigma/2}(\varphi(\cdot\,,k\tau))$. Moreover, from (3.9) and (3.10) it follows that the sequence $\{\rho_n\boldsymbol{u}_n\}$ is bounded in $L^2(I_k \times \Omega)$, so the sequence $\left\{P^0_{\sigma/2}(\varphi(\cdot\,,k\tau))(\rho_n\boldsymbol{u}_n)\right\}$ is also bounded in $L^2(I_k; K^0_{\sigma/2}(\varphi(\cdot\,,k\tau)))$.

Since, for all $s > 0$, the inclusion $K^0_{\sigma/2}(\varphi(\cdot\,,k\tau)) \subset \left[K^s_{\sigma/2}(\varphi(\cdot\,,k\tau))\right]^*$ is compact, it follows from Aubin's theorem that the sequence $\left\{P^0_{\sigma/2}(\varphi(\cdot\,,k\tau))(\rho_n\boldsymbol{u}_n)\right\}$

is relatively compact in $L^2\big(I_k;\,\big[K^s_{\sigma/2}(\varphi(\cdot\,,k\tau))\big]^*\big)$. Moreover, by Lemma 5.1 and (3.12) we have $\rho_n \boldsymbol{u}_n \rightharpoonup \rho \boldsymbol{u}$ weakly in $L^2(I_k;\,L^2(\Omega))$. It follows that

$$P^0_{\sigma/2}(\varphi(\cdot\,,k\tau))(\rho_n \boldsymbol{u}_n) \to P^0_{\sigma/2}(\varphi(\cdot\,,k\tau))(\rho \boldsymbol{u}) \tag{7.4}$$

in $L^2\big(I_k;\,\big[K^s_{\sigma/2}(\varphi(\cdot\,,k\tau))\big]^*\big)$ strongly for all $s > 0$.

On the other hand, by (7.3) we also have

$$P^0_{\sigma/2}(\varphi(\cdot\,,k\tau))P^s_\sigma(\varphi(\cdot\,,t)) = P^s_\sigma(\varphi(\cdot\,,t)) \quad \forall\, t \in I_k \text{ and } \forall\, s \geqq 0.$$

Using the relation above and the fact that $P^0_{\sigma/2}(\varphi(\cdot\,,k\tau))$ is self-adjoint in $L^2(\Omega)$ we obtain

$$\begin{aligned}
&\int_{I_k} \big\langle \rho_n \boldsymbol{u}_n, P^s_\sigma(\varphi(\cdot\,,t))(\boldsymbol{u}_n)\big\rangle_{L^2(\Omega)}\,dt \\
&\qquad = \int_{I_k} \big\langle P^0_{\sigma/2}(\varphi(\cdot\,,k\tau))(\rho_n \boldsymbol{u}_n),\, P^s_\sigma(\varphi(\cdot\,,t))(\boldsymbol{u}_n)\big\rangle_{L^2(\Omega)}\,dt \\
&\qquad = \int_{I_k} \big\langle P^0_{\sigma/2}(\varphi(\cdot\,,k\tau))(\rho_n \boldsymbol{u}_n),\, P^s_\sigma(\varphi(\cdot\,,t))(\boldsymbol{u}_n)\big\rangle_{[K^s_{\sigma/2}]^*,K^s_{\sigma/2}}\,dt.
\end{aligned}$$

By using (7.4) it follows that

$$\begin{aligned}
&\lim_{n\to\infty} \int_{I_k} \big\langle \rho_n \boldsymbol{u}_n,\, P^s_\sigma(\varphi(\cdot\,,t))(\boldsymbol{u}_n)\big\rangle_{L^2(\Omega)}\,dt \\
&\qquad = \int_{I_k} \big\langle P^0_{\sigma/2}(\varphi(\cdot\,,k\tau))(\rho \boldsymbol{u}),\, P^s_\sigma(\varphi(\cdot\,,t))(\boldsymbol{u})\big\rangle_{L^2(\Omega)} dt \\
&\qquad = \int_{I_k} \big\langle \rho \boldsymbol{u},\, P^s_\sigma(\varphi(\cdot\,,t))(\boldsymbol{u})\big\rangle_{L^2(\Omega)}\,dt \quad \forall\, k = 1,\dots,N_T.
\end{aligned}$$

By summing up the relations above, from $k = 1$ to $k = N_T$, we obtain the assertion of Proposition 7.1.

We also need the following result.

Proposition 7.2. *The sequences* $\{\rho_n\}$, $\{\boldsymbol{u}_n\}$ *defined above satisfy the relation*

$$\lim_{n\to\infty} \int_Q \rho_n \boldsymbol{u}_n^2 = \int_Q \rho \boldsymbol{u}^2.$$

Proof. We clearly have

$$\begin{aligned}
\int_Q \rho_n \boldsymbol{u}_n^2 - \int_Q \rho \boldsymbol{u}^2 = {} & \int_0^T \int_\Omega \big(\rho_n \boldsymbol{u}_n \cdot\ P^s_\sigma(\varphi(\cdot\,,t))[\boldsymbol{u}_n] \\
& - \rho \boldsymbol{u} \cdot\ P^s_\sigma(\varphi(\cdot\,,t))[\boldsymbol{u}]\big)\,dx\,dt \\
& + \int_0^T \int_\Omega \rho_n \boldsymbol{u}_n \cdot \big(\boldsymbol{u}_n - P^s_\sigma(\varphi(\cdot\,,t))[\boldsymbol{u}_n]\big)\,dx\,dt \\
& + \int_0^T \int_\Omega \rho \boldsymbol{u} \cdot \big(P^s_\sigma(\varphi(\cdot\,,t))[\boldsymbol{u}] - \boldsymbol{u}\big)\,dx\,dt.
\end{aligned} \tag{7.5}$$

In order to estimate the last integral in the right-hand side of of (7.5) we notice that

$$\left|\int_Q \rho \boldsymbol{u} \cdot \left(P_\sigma^s(\varphi(\cdot\,,t))[\boldsymbol{u}] - \boldsymbol{u}\right)\right| \leqq C \int_0^T \left\| P_\sigma^s(\varphi(\cdot\,,t))[\boldsymbol{u}] - \boldsymbol{u}\right\|_{L^2(\Omega)} dt, \quad (7.6)$$

where $C = \|\rho \boldsymbol{u}\|_{L^\infty(0,T;L^2(\Omega))}$. On the other hand, by Propositions 6.1, 6.2 and 6.3, for every $\gamma > 0$ there exists $\sigma_0 > 0$ such that for every $\sigma \in (0, \sigma_0)$ we have:

$$\mu(E_\sigma) \leqq \gamma, \quad (7.7)$$

$$\lim_{n\to\infty} \left\| P_\sigma^s(\varphi(\cdot\,,t))\boldsymbol{u}_n - \boldsymbol{u}_n\right\|_{L^2([0,T]\setminus E_\sigma;L^2(\Omega))} \leqq \gamma, \quad (7.8)$$

$$\left\| P_\sigma^s(\varphi(\cdot\,,t))\boldsymbol{u} - \boldsymbol{u}\right\|_{L^2([0,T]\setminus E_\sigma;L^2(\Omega))} \leqq \gamma. \quad (7.9)$$

For σ satisfying the conditions above, relations (7.6), (7.7) and (7.9) imply that

$$\begin{aligned}\left|\int_Q \rho \boldsymbol{u} \cdot \left(P_\sigma^s(\varphi)[\boldsymbol{u}] - \boldsymbol{u}\right)\right| &\leqq C \int_{[0,T]\setminus E_\sigma} \left\| P_\sigma^s(\varphi)[\boldsymbol{u}] - \boldsymbol{u}\right\|_{L^2(\Omega)} dt \\ &\quad + C \int_{E_\sigma} \left\| P_\sigma^s(\varphi)[\boldsymbol{u}] - \boldsymbol{u}\right\|_{L^2(\Omega)} dt \\ &\leqq C\gamma + C\mu\,(E_\sigma)^{\frac{1}{2}} \left(\int_{E_\sigma} \left\| P_\sigma^s(\varphi)[\boldsymbol{u}] - \boldsymbol{u}\right\|_{L^2(\Omega)}^2 dt\right)^{\frac{1}{2}}\end{aligned}$$

and, therefore, that

$$\left|\int_Q \rho \boldsymbol{u} \cdot \left(P_\sigma^s(\varphi)[\boldsymbol{u}] - \boldsymbol{u}\right)\right| \leqq C_1 \gamma^{\frac{1}{2}},$$

where $C_1 > 0$ is another constant. The second integral in the right-hand side of (7.5) can be estimated in a very similar manner, by using (7.8) instead of (7.9). Moreover, by Proposition 7.1 the first integral in the right-hand side of (7.5) tends to zero when $n \to \infty$. Since $\gamma > 0$ is arbitrary we obtain the conclusion of the proposition.

Proof of Theorem 3.2. We first notice that

$$\left|\int_Q \rho\left(\boldsymbol{u}_n^2 - \boldsymbol{u}^2\right)\right| \leqq \left|\int_Q \left(\rho_n \boldsymbol{u}_n^2 - \rho \boldsymbol{u}^2\right)\right| + \left|\int_Q (\rho_n - \rho)\,\boldsymbol{u}_n^2\right|. \quad (7.10)$$

Since $\boldsymbol{u}_n$ is bounded in $L^\infty\left(0,T;L^2(\Omega)\right)$ and in $L^2(0,T;H^1(\Omega))$ we can easily deduce that $\boldsymbol{u}_n$ is bounded in $L^4(Q)$. Moreover, by Lemma 5.1, we have $\rho_n \to \rho$ strongly in $L^2(Q)$, so the second term in the right-hand side of (7.10) tends to zero when $n \to \infty$. Since, by Proposition 7.2, the first term in the right-hand side of (7.10) also tends to zero, we conclude that

$$\lim_{n\to\infty} \int_Q \rho\left(\boldsymbol{u}_n^2 - \boldsymbol{u}^2\right) = 0. \quad (7.11)$$

Moreover,

$$\int_Q |\boldsymbol{u}_n - \boldsymbol{u}|^2 \leqq \frac{1}{m_0}\left(\int_Q \rho\big(\boldsymbol{u}_n^2 - \boldsymbol{u}^2\big) + \int_Q 2\rho\boldsymbol{u}\cdot(\boldsymbol{u}-\boldsymbol{u}_n)\right), \tag{7.12}$$

where m_0 is defined in Theorem 2.1. The right-hand side of (7.12) tends to zero by (7.11) and from the fact that $\boldsymbol{u}_n \to \boldsymbol{u}$ in $L^2(Q)$ weakly. We have thus proved the strong convergence of $\boldsymbol{u}_n$ to $\boldsymbol{u}$ in $L^2(Q)$.

8. Proof of the main results

8.1. Proof of Theorem 2.1.

By Proposition 3.1 (proved in Section 5), the functions $\boldsymbol{u}, \rho, \varphi^i,\ i = 1, \dots, N$ and φ satisfy relations (2.5)–(2.8), (2.10) and (2.11). So, in order to prove the existence of at least one weak solution of (1.1)–(1.13) we have only to prove that (2.9) is also satisfied. Due to Theorem 3.2 and to Lemma 5.1, we can say that, up to the extraction of a subsequence,

$$\boldsymbol{u}_n \overset{n\to\infty}{\longrightarrow} \boldsymbol{u} \quad \text{in } L^2(Q) \text{ strongly}, \tag{8.1}$$

$$\rho_n \overset{n\to\infty}{\longrightarrow} \rho \quad \text{in } \mathscr{C}\left([0,T]; L^p(\Omega)\right), \quad (1 \leqq p < \infty) \text{ strongly}. \tag{8.2}$$

Let σ be an arbitrary positive number. We choose the test function $\boldsymbol{\xi}$ in (3.6) such that $\boldsymbol{\xi} \in H^1(Q) \cap L^2(0,T; K_\sigma(\varphi))$. Then, due to Lemma 6.1, there exists $n_0 > 0$ (depending only on σ) such that

$$\varphi_n \mathrm{D}(\boldsymbol{\xi}) = 0 \quad \text{in } L^2(Q) \quad \forall n > n_0$$

and, consequently,

$$\int_Q \varphi_n \mathrm{D}(\boldsymbol{u}_n) : \mathrm{D}(\boldsymbol{\xi})\, dx\, dt = 0 \quad \forall n > n_0.$$

By using (8.1) and (8.2) we can pass to the limit in (3.6) to show that (2.9) holds for any $\boldsymbol{\xi} \in H^1(Q) \cap L^2(0,T; K_\sigma(\varphi))$. By using Proposition 4.3 and Corollary 4.1, it follows that (2.9) holds for any $\boldsymbol{\xi} \in H^1(Q) \cap L^2(0,T; K(\varphi))$.

Let us fix $i \in \{1, \dots, N\}$ and consider the function $\varphi^i \in C(0,T; L_p(\Omega)) \cap \mathrm{Char}(Q)$, $1 \leqq p \leqq \infty$. Since $S(\varphi^i(t)) = \mathscr{A}^i_{s,t}(S(\varphi^i(s)))$ and $\mathscr{A}^i_{s,t}$ is Lipschitz-continuous with respect to s and t, there exists a constant C such that for all $s, t \in [0,T]$ we have

$$S(\varphi^i(t)) \subset S_\gamma(\varphi^i(s)), \quad S(\varphi^i(s)) \subset S_\gamma(\varphi^i(t)),$$

where $\gamma = C\,|t-s|$.

Therefore,

$$\max\left\{\mu\big(S(\varphi^i(t))\setminus S(\varphi^i(s))\big),\ \mu\big(S(\varphi^i(s))\setminus S(\varphi^i(t))\big)\right\}$$
$$\leqq \max\left\{\mu\big(S_\gamma(\varphi^i(s))\setminus S(\varphi^i(s))\big),\ \mu\big(S_\gamma(\varphi^i(t))\setminus S(\varphi^i(t))\big)\right\}$$
$$\leqq C\left|\partial S(\varphi^i(t))\right|\,|t-s|,$$

where $\left|\partial S(\varphi^i(t))\right| = \left|\partial S(\varphi^i(0))\right|$ is the length of the boundary of the body which is bounded since $\partial S(\varphi^i(0))$ is of the class C^2. Hence

$$\|\varphi(t)-\varphi(s)\|_{L_p(\Omega)} = \mu\big(S_\gamma(\varphi^i(t))\triangle S(\varphi^i(s))\big)^{1/p} \leqq C\,|t-s|^{1/p}, \quad 1\leqq p<\infty,$$

where $A\triangle B = (A\cup B)\setminus(A\cap B)$.

This ends the proof of the existence of at least one weak solution of (1.1)–(1.13). The energy estimate (2.12) follows directly from (3.9).

Finally, representation (2.13) is already obtained in the proof of Proposition 3.1 (see relation (5.14)).

The theorem is entirely proved.

8.2. Proof of Theorem 2.2.

The first assertion of the theorem follows immediately from Proposition 4.1. Let us prove the second assertion. The arguments below are similar to those used in the proof of Proposition 4.1.

Since, for $i = 0, \dots, N$, the boundary $\partial S(\varphi^i)$ is a curve of class $\mathscr{C}^2$, then for all $\boldsymbol{x}\in\partial S(\varphi^i)$ there exists a closed ball of radius δ, containing $\boldsymbol{x}$ and included in the closure of $S(\varphi^i)$.

Let us consider two bodies $S(\varphi^i)$ and $S(\varphi^j)$ and suppose that $h_{ij}(t_0) = 0$ for some $t_0\in(0,T)$. Let us take two closed disks $B(P_i(t),\delta)\subseteq\overline{S(\varphi^i(t))}$ and $B(P_j(t),\delta)\subseteq\overline{S(\varphi^j(t))}$ of radius δ centred at points $P_i(t)$ and $P_j(t)$, respectively, and such that at time t_0 we have $B(P_i(t_0),\delta)\cap B(P_j(t_0),\delta)\neq\emptyset$. These disks move together with the bodies. Denote by $H(t)$ the distance between the two disks. We clearly have (see Fig. 8.1)

$$H(t) = \operatorname{dist}\big(P_i(t), P_j(t)\big) - 2\delta \ \text{ and } \ H(t_0) = 0.$$

Let $Q_i(t)\in\partial B(P_i(t),\delta)$ and $Q_j(t)\in\partial B(P_j(t),\delta)$ be two points such that $H(t) = \operatorname{dist}(Q_i(t), Q_j(t))$. Let us fix a moment of time $t\in(0,T)$ and introduce a new system of coordinates $\boldsymbol{\xi} = (\xi_1,\xi_2)$ with the origin in $Q_j(t)$. The axis ξ_1 is tangential to $\partial B(P_j(t),\delta)$ and the axis ξ_2 is orthogonal to it.

Let us consider the domain

$$G_\gamma = \left\{\boldsymbol{\xi}\in R^2 \mid -c(\xi_1) < \xi_2 < H(t) + c(\xi_1),\ -\gamma<\xi_1<\gamma\right\},$$

where $c(\gamma) = \delta - \sqrt{\delta^2-\gamma^2}$ is the function already considered in the proof of Proposition 4.1 (see Fig. 8.2).

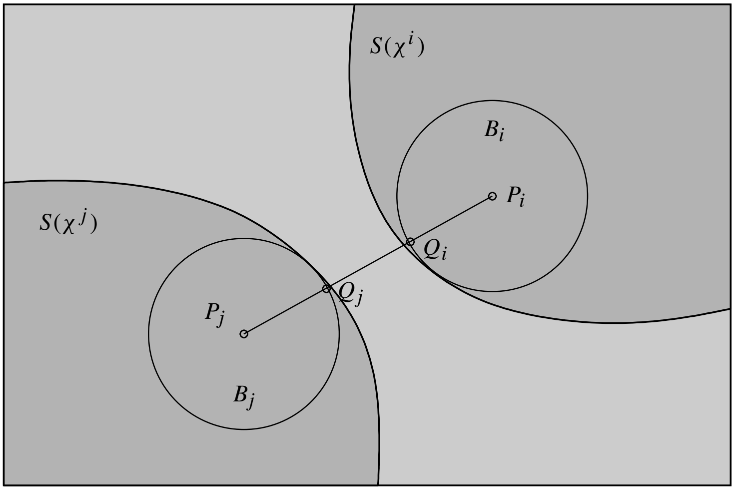

Fig. 8.1.

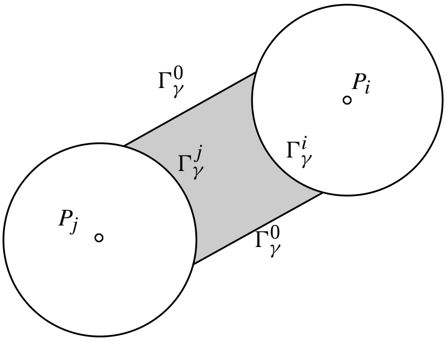

Fig. 8.2. Set G_γ.

The condition $\operatorname{div} \boldsymbol{u} = 0$ implies that

$$\int_{\partial G_\gamma} \boldsymbol{u} \cdot \boldsymbol{n} \, ds = 0,$$

where $\boldsymbol{n}$ is the outward unit vector field normal to ∂G_γ. Notice that $\partial G_\gamma = \Gamma_\gamma^i \cup \Gamma_\gamma^j \cup \Gamma_\gamma^0$, where $\Gamma_\gamma^k = \partial G_\gamma \cap \partial B(P_k(t), \delta),\ k = i, j$. We know that the restrictions of $\boldsymbol{u}$ to $S(\varphi^i(t))$ and $S(\varphi^j(t))$ are rigid velocity fields. We denote these rigid velocity fields by $\boldsymbol{v}_i$ and $\boldsymbol{v}_j$, so we have $\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{v}_k(\boldsymbol{x}, t)$ as $\boldsymbol{x} \in S(\varphi^k(t))$, $k = i, j$.

For the function $\boldsymbol{v}_k$ the following representation holds:

$$\boldsymbol{v}_k(\boldsymbol{x}, t) = \boldsymbol{a}_k(t) + \omega_k(t)(\boldsymbol{x} - \boldsymbol{x}_k(t))^\perp,\ \boldsymbol{x} \in B(P_k(t), \delta),\ k = i, j,$$

where $\boldsymbol{x}_k(t)$ is the position vector of the point $P_k(t)$. We notice also that

$$\frac{dH(t)}{dt} = (\boldsymbol{a}_i - \boldsymbol{a}_j) \cdot \boldsymbol{e}_2,$$

where $\boldsymbol{e}_2$ is the unit vector directed along the axis ξ_2.

Define $\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{v}_j$. This function is divergence-free and equal to zero in $S(\varphi_j(t))$. Thus we have:

$$\int_{G_\gamma} \boldsymbol{v} \cdot \boldsymbol{n}\, ds = \int_{\Gamma_\gamma^i} \boldsymbol{v} \cdot \boldsymbol{n}\, ds + \int_{\Gamma_\gamma^0} \boldsymbol{v} \cdot \boldsymbol{n}\, ds = 0. \tag{8.3}$$

It is not difficult to calculate that (8.3) implies

$$2\gamma(\boldsymbol{a}_i - \boldsymbol{a}_j) \cdot \boldsymbol{e}_2 = -\int_{\Gamma_\gamma^0} \boldsymbol{v} \cdot \boldsymbol{n}\, ds.$$

The relation above implies that

$$2\gamma \left| \frac{dH}{dt}(t) \right| \leqq \int_{\Gamma_\gamma^0} |\boldsymbol{v}|\, ds.$$

By integrating this inequality with respect to γ from 0 to some $r > 0$, we obtain:

$$r^2 \left| \frac{dH}{dt}(t) \right| \leqq \int_{G_r} |\boldsymbol{v}|\, d\xi_1 d\xi_2. \tag{8.4}$$

By using the Hölder and Poincaré inequalities, we can estimate the integral in the right-hand side of (8.4) to get:

$$\begin{aligned} \int_{G_r} |\boldsymbol{v}|\, d\xi_1 d\xi_2 &\leqq \mu(G_r)^{1/2}\ \|\boldsymbol{v}\|_{L_2(G_r)} \\ &\leqq C\mu(G_r)^{1/2}\,(2c(r) + H(t))\,\|\nabla \boldsymbol{v}\|_{L^2(\Omega)}\,. \end{aligned}$$

Since $\mu(G_r) \leqq 2r(2c(r) + H(t))$, it follows that

$$r^2 \left| \frac{dH}{dt}(t) \right| \leqq Cr^{1/2} \big(2c(r) + H(t)\big)^{3/2}\ \|\nabla \boldsymbol{v}\|_{L^2(\Omega)}\,.$$

Since $H(t_0) = 0$ and $H(t)$ is sufficiently small in a neighbourhood of the point t_0, we can take $r = H^{1/2}(t)$ in the relation above and, using (4.3), we find that

$$\left| \frac{dH}{dt}(t) \right| \leqq C\, H^{3/4}(t)\, z(t), \tag{8.5}$$

where $z(t) = \|\nabla \boldsymbol{v}(t)\|_{L^2(\Omega)}$.

Since the function z is in $L^2(0, T)$, and H is Lipschitz-continuous in t, we deduce that inequality (8.5) is valid for almost all $t \in [0, T]$.

If $s, t \in [0, T]$, by integrating (8.5) from s to t we obtain

$$H^{1/4}(s) - C \left| \int_s^t z(p)\, dp \right| \leqq H^{1/4}(t) \leqq H^{1/4}(s) + C \left| \int_s^t z(p)\, dp \right|. \tag{8.6}$$

Since $H(t_0) = 0$, then the relation above implies that

$$H^{1/4}(t) \leqq C \left| \int_{t_0}^{t} z(p)\, dp \right|$$

for all $t \in [0, T]$.

By applying Hölder's inequality and the fact that $z \in L^2(0, T)$, we obtain the estimate

$$H(t) \leqq C \left| \int_{t_0}^{t} \|\nabla \boldsymbol{v}(p)\|^2_{L^2(\Omega)}\, dp \right|^2 |t - t_0|^2.$$

The relation above and the obvious inequality $h_{ij}(t) \leqq H(t)$ imply that

$$\lim_{t \to t_0} h_{ij}(t)\, |t - t_0|^{-2} = 0.$$

This ends the proof of the theorem.

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References

1. ANTONTSEV, S. N., KAZHIKHOV, A. V. & MONAKHOV, V. N.: *Boundary value problems in mechanics of nonhomogeneous fluids*, North-Holland Publishing Co., Amsterdam, 1990. Translated from the Russian.
2. CONCA, C., SAN MARTÍN, J. A., & TUCSNAK, M.: Motion of a rigid body in a viscous fluid, *C. R. Acad. Sci. Paris Sér. I Math.* **328** (1999), 473–478.
3. CONCA, C., SAN MARTÍN, H. J., & TUCSNAK, M.: Existence of solutions for the equations modelling the motion of a rigid body in a viscous fluid, *Comm. Partial Differential Equations* **25** (2000), 1019–1042.
4. DESJARDINS, B. & ESTEBAN, M. J.: Existence of weak solutions for the motion of rigid bodies in a viscous fluid, *Arch. Rational Mech. Anal.* **146** (1999), 59–71.
5. DESJARDINS, B. & ESTEBAN, M. J.: On weak solutions for fluid-rigid structure interaction: compressible and incompressible models, *Comm. Partial Differential Equations* **25** (2000), 1399–1413.
6. DI PERNA, R.-J. & LIONS, P.-L.: Ordinary differential equations, transport theory and Sobolev spaces, *Invent. Math.* **9**8 (1989), 511–547
7. FUJITA, H. & SAUER, N.: On existence of weak solutions of the Navier-Stokes equations in regions with moving boundaries, *J. Fac. Sci. Univ. Tokyo Sec. 1A*, **17** (1970), 403–420.
8. GALDI, G. P.: On the steady self-propelled motion of a body in a viscous incompressible fluid, *Arch. Rational Mech. Anal.* **148** (1999), 53–88.
9. GRANDMONT, C. & MADAY, Y.: Existence de solutions d'un problème de couplage fluide-structure bidimensionel instationnaire, *C. R. Acad. Sci Paris Sér. I Math.* **326** (1998), 525–530.
10. GUNZBURGER, M. D. AND LEE, H.-C. & SEREGIN, G. A.: Global existence of weak solutions for viscous incompressible flows around a moving rigid body in three dimensions, *J. Math. Fluid Mech.* **2** (2000), 219–266.

11. HALMOS, P. R.: *Measure Theory* Van Nostrand Company, Inc., New York, NY, 1950.
12. HOFFMANN, K.-H. & STAROVOITOV, V. N.: On a motion of a solid body in a viscous fluid. Two-dimensional case, *Adv. Math. Sci. Appl.* **9** (1999), 633–648.
13. HOFFMANN, K.-H. & STAROVOITOV, V. N.: Zur Bewegung einer Kugel in einer zähen Flüssigkeit, *Doc. Math.* **5**, (2000), 15–21 (electronic).
14. JUDAKOV, N. V.: The solvability of the problem of the motion of a rigid body in a viscous incompressible fluid, *Dinamika Splošn. Sredy*, **18** (1974), 249–253.
15. KOLMOGOROV, A. N. & FOMIN, S. V.: *Introductory real analysis*, Prentice-Hall, Emglewood Cliffs, 1970.
16. LADYZHENSKAYA, O. A.: *The mathematical theory of viscous incompressible flows*, Gordon and Breach, New York, 1969.
17. LEE, E. B. & MARKUS, L.: *Foundations of Optimal Control Theory*, John Wiley & Sons, New York, 1967.
18. LIONS, J.-L.: *Quelques méthodes de résolution des problèmes aux limites non linéaires,* Dunod, 1969.
19. LIONS, P.-L.: *Mathematical Topics in Fluid Mechanics*, Oxford Science Publications, 1996.
20. MCSHANE, J. E.: *Integration*, Princeton University Press, 1944.
21. SERRE, D.: Chute libre d'un solide dans un fluide visqueux incompressible. Existence, *Japan J. Appl. Math.* **4** (1987), 99–110.
22. TEMAM, R.: *Navier-Stokes equations. Theory and numerical analysis,* North-Holland Publishing Co., Amsterdam, (1977). Studies in Mathematics and its Applications, Vol. 2.
23. TEMAM, R.: *Problèmes mathématiques en plasticité,* Gauthier-Villars, Paris, 1983.

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