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# *Variational Problems on Multiply Connected Thin Strips I: Basic Estimates and Convergence of the Laplacian Spectrum*

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**Abstract**

Let $M$ be a planar embedded graph whose arcs meet transversally at the vertices. Let $\mathcal{O}(\varepsilon)$ be a strip-shaped domain around $M$, of width $\varepsilon$ except in a neighborhood of the singular points. Assume that the boundary of $\mathcal{O}(\varepsilon)$ is smooth. We define comparison operators between functions on $\mathcal{O}(\varepsilon)$ and on $M$, and we derive energy estimates for the compared functions. We define a Laplace operator on $M$ which is in a certain sense the limit of the Laplace operator on $\mathcal{O}(\varepsilon)$ with Neumann boundary conditions. In particular, we show that the $p$-th eigenvalue of the Laplacian on $\mathcal{O}(\varepsilon)$ converges to the $p$-th eigenvalue of the Laplacian on $M$ as $\varepsilon$ tends to 0. A similar result holds for the magnetic Schrödinger operator.

## 1. Introduction

Modern techniques enable the manufacture of complex networks of mesoscopic strips, pipes and rings. Systems of this kind are of growing importance in several physical and technological areas, such as superconductivity [16], optics [11], semiconductors manufacturing [10], etc. Their domains are characterized by a small lateral dimension. Therefore we shall develop an asymptotic theory for a variety of variational problems there. In this paper we define the basic geometrical structures and derive a series of fundamental estimates. As an application we characterize the asymptotic limit of the Laplacian of such domains. In our next paper, [18], we study the asymptotic limit of the Ginzburg-Landau functional in such networks. Our work generalizes the results of [21] and [19] where single rings were considered.

Let $M$ be an embedded oriented graph in $\mathbb{R}^2$, and suppose that its arcs intersect transversally. Assume that all the arcs of $M$ are $C^2$ manifolds. Then, $M$ is a very singular manifold, which is however the limit of a sequence of sets $\mathcal{O}(\varepsilon)$ obtained by fattening $M$: assume that these fattened sets are of width $2\varepsilon$ except in the neighborhood of the singular points of $M$, where it is admitted that $\mathcal{O}(\varepsilon)$ lies at a distance at most $2\varepsilon$ from $M$. Then, the operator $\mathcal{A}^\varepsilon = -\Delta$ on $\mathcal{O}(\varepsilon)$ with Neumann boundary conditions is well defined; it is a self-adjoint operator in $L^2(\mathcal{O}(\varepsilon))$ defined via the energy quadratic form

$$E^\varepsilon(u,u) = \int_{\mathcal{O}(\varepsilon)} |\operatorname{grad} u|^2\, dy,$$

and its domain is the space of functions in the Sobolev space $H^2\big(\mathcal{O}(\varepsilon)\big)$ whose normal derivative vanishes at $\partial\mathcal{O}(\varepsilon)$.

The set $\mathcal{O}(\varepsilon)$ is homotopically equivalent to $M$; hence, all the topological invariants of $M$ can be obtained by a calculation on $\mathcal{O}(\varepsilon)$ which is a reasonably nice manifold with boundary. We show here that it is also possible to define a Laplace-Beltrami operator $\mathcal{A}$ on $M$, starting from the energy form on $M$

$$E(f,f) = \int_M |df|^2\, d\mathrm{vol}_M,$$

where $d\mathrm{vol}_M$ is the volume form on $M$. Of course, this makes sense if we define the differential of a function $f$ on $M$: this is done by taking the differential of $f$ restricted to the interior of the arcs. We find that the domain of $\mathcal{A}$ consists of all the functions which are continuous, whose restriction to the arcs is in $H^2$, and such that the sum of all the outgoing derivatives with respect to the arc-length at the singular point vanishes: this is a Kirchhoff-type condition. We prove that the above definitions make sense because the spectrum of $\mathcal{A}^\varepsilon$ converges to the spectrum of $\mathcal{A}$. The graph $M$ and appropriate function spaces over it are introduced in Section 2. The "lace" $\mathcal{O}(\varepsilon)$ is defined in Section 3. The convergence of the spectrum is proved in Section 8.

The idea of the proof is to construct a mapping $Q^\varepsilon$ from $H^1(M)$ to $H^1\big(\mathcal{O}(\varepsilon)\big)$, and a mapping $P^\varepsilon$ from $H^1\big(\mathcal{O}(\varepsilon)\big)$ to $H^1(M)$, and to compare the relevant Rayleigh quotients. Away from singular points, $Q^\varepsilon f$ is extended as a constant along the normal, and $P^\varepsilon u$ is the normal average of $u$. All the difficulty is concentrated at the singular points: for instance, if $v$ is a singular point of $M$, $Q^\varepsilon f$ will be taken equal to $f(v)$ in a neighborhood of $v$ of size $O(\varepsilon)$ in $\mathcal{O}(\varepsilon)$. Before we define a normal extension, we have first to transform $f$ into an $\hat{f}$ which is constant around the singular points of $M$, and whose $L^2$ and $H^1$ norm are very close to those of $f$; this can be done by a piecewise affine transformation of the coordinates; then the idea of constant normal extension works. The definition of $P^\varepsilon$ uses different scales: we take $f$ to be the normal average of $u$ at a finite distance from the singular points of $M$; at a distance at most $C\varepsilon$ of the junctions, we take $P^\varepsilon u$ to be the average of the normal averages along the arms of the junction; in between, we correct the normal average by a linear interpolation.

The comparison of the Rayleigh quotient for $P^\varepsilon u$ to the Rayleigh quotient for $u$ relies on a careful estimate for the difference between normal averages on different branches close to a singular point. In order to prove such an estimate, we construct bounded differential forms on subpipes joining two particular branches, while remaining inside $\mathcal{O}(\varepsilon)$. Because of the particular geometry involved in our problem, we term those estimates "plumber shop estimates". We also show a precise

Poincaré inequality on junctions where the right-hand side contains the square of the average of the boundary values instead of the $L^2$ norm of the boundary values. This result illustrates the fact that the details of the smoothing of the set $\mathcal{O}(\varepsilon)$ at the junction are unimportant. It relies on the construction of a Lipschitz continuous bijection, whose inverse is Lipschitz continuous and which sends a junction into a fixed set. These results are derived in Sections 5, 6 and 7, *where the main results are summarized in Theorem 1 and Theorem 2.*

The comparison of the eigenvalues (performed in Section 8) relies on the inf-sup principle.

In Section 9, we generalize the comparison techniques and results to the case of the Schrödinger operator with a magnetic potential. The results are analogous, and the relevant estimates are obtained through simple modifications of the techniques of Sections 5 to 8.

In [18] we prove the convergence (in a suitable sense) of the Ginzburg-Landau functional on $\mathcal{O}(\varepsilon)$ to a Ginzburg-Landau functional defined over $M$. The result is useful for a large class of problems in superconductivity and quantum mechanics. In Section 4 of [18] we summarize the results of the current paper and [18].

The limiting process we carry out is reminiscent of works of Colin de Verdière [7] and Colbois & Anné [1]. Carlson [3] has recently studied the spectrum of the Laplacian on an integer lattice graph as a limit of the associated networks. There is also an extensive literature on approximating thin elastic structures by lower dimensional related structures, e.g., [6] and the references therein.

Several authors developed theories concerning the Laplacian and more generally linear differential operators on a graph, as for instance Carlson in articles [5] and [4] and Kostrykin & Schrader in [14] and [15]; beyond describing all the transmission conditions that lead to a self-adjoint extension of the Laplace operator, they have also studied the scattering matrix, with a view towards the applications to quantum computing. They also mention that it would be useful to justify mathematically the passages to the limit on thin domains.

Evans & Saito have treated a closely related problem in [9], and Saito in [20]. Their limiting process is different from ours; it introduces a singularity in the approximation process that disappears in the limit. More comments on that question are given in Remark 1.

Finally, it should be observed that the planar character of the graph is wholly irrelevant; we could have taken a graph embedded in $\mathbb{R}^n$, and a fattening in $\mathbb{R}^n$; then the results would have been completely identical, and the proofs would need only very slight modifications.

## 2. The singular manifold $M$

We start with a definition of the class of singular manifolds on which we work. Intuitively, $M$ can be drawn as an electric circuit with a finite number of nodes and twice continuously differentiable branches. The nodes constitute a singular set, while the branches intersect transversally at the nodes. The graph $M$ can also be seen as the embedding of a finite planar graph, with smooth arcs, and transversal intersections at the nodes.

The graph $M$ is an embedded finite planar graph; it has a set of edges $\mathcal{E}$ identified with curves in $\mathbb{R}^2$, and a set of vertices $\mathcal{V}$. Each edge is numbered by $j \in \{1, \dots, |\mathcal{E}|\}$ and is parametrized by an injective mapping $\psi_j$ of class $C^2$ and of rank one from an interval $(a_j, b_j)$ to $\mathbb{R}^2$; we assume that $\psi_j$, $\psi_j'$, and $\psi_j''$ can be extended as continuous bounded functions over $[a_j, b_j]$, and that $\psi_j'$ is bounded away from 0 over $[a_j, b_j]$. Thus, without loss of generality, the parameter is the arc length. Loops are not forbidden. It is convenient to denote by $M_j = \psi_j\big((a_j, b_j)\big)$ the arc of $M$ indexed by $j$. We will abuse notation and identify $M$ with the union of the closed edges $\bar{M}_j$, $\mathcal{E}$ with the union of the edges $M_j$ and $\mathcal{V}$ with the union of the boundaries $\bar{M}_j \setminus M_j$. We require that for all pairs of distinct indices $j$ and $k$, $M_j$ and $\bar{M}_k$ have an empty intersection.

We need to describe all the arcs leaving or entering any vertex of $M$ with enough information to write Kirchhoff-like transmission conditions. This is done by introducing for each $v \in \mathcal{V}$ the set $J(v)$ defined as follows:

$$J(v) = \big\{(j, a_j, +1) : \psi_j(a_j) = v\} \cup \{(j, b_j, -1) : \psi_j(b_j) = v\big\}. \tag{1}$$

If $\zeta$ belongs to $J(v)$, its components are denoted by $(\zeta[1], \zeta[2], \zeta[3])$. There are $|J(v)|$ curves which start or end at any vertex $v \in \mathcal{V}$; an arc might start and end at $v$ if it is a loop.

We define the outgoing tangent vectors $e_\kappa$ at $v$ as $e_\kappa = \kappa[3]\psi'_{\kappa[1]}(\kappa[2])$. The transversality condition we impose on $M$ can be formulated precisely now: for every $v \in \mathcal{V}$, and for every distinct elements $\kappa$ and $\lambda$ of $J(v)$,

$$e_\kappa \neq e_\lambda. \tag{2}$$

At $x = \psi_j(\theta)$, the unit tangent vectors are $\pm\psi_j'(\theta)$; the tangent space is $T_x M_j = \mathbb{R}\psi_j'(\theta)$. If $\phi$ is any function from a neighborhood of 0 in $\mathbb{R}$ to $M \setminus \mathcal{V}$ which maps 0 to $x$, then $\phi'(0)$ is a tangent vector at $x$ to $M$.

A function $f$ on $M_j$ is of class $C^r$ ($r \leqq 2$) if $f \circ \psi_j$ is of class $C^r$ on $(a_j, b_j)$. We will use repeatedly the notation $f_j = f \circ \psi_j$, for all functions $f$ defined on $M$. The differential of $f$ of class C¼$^1$ at $x = \psi_j(\theta) \in M_j$ is a linear form on $T_x M_j$ defined as follows: if $\phi$ is as above, then

$$df(x)\phi'(0) = \big(f \circ \phi\big)'(0). \tag{3}$$

The differential of $f$ is an example of a cotangent vector, i.e., an element of $T_x^* M_j$, the dual of $T_x M_j$. The definition of the differential of a function is completely independent of the metric structure. If a continuous function $f$ from $M$ to $\mathbb{R}$ is of class $C^1$ on all the arcs $M_j$, then its differential is defined on $M \setminus \mathcal{V}$ by (3).

For all continuous functions $f$ from $M$ to $\mathbb{R}$, the integral of $f$ over $M$ is defined by

$$\int_M f \, d\mathrm{vol}_M = \sum_{1 \leqq j \leqq |\mathcal{E}|} \int_{a_j}^{b_j} f_j(\theta)\, d\theta,$$

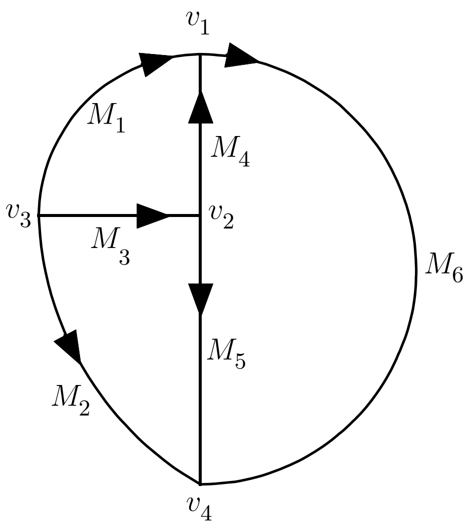


**Fig. 1.** An example of an embedded graph; here, $J(v_1) = \{(1, b_1, -), (4, b_4, -), (6, a_6, +)\}$.

where we have used the notation (2). The standard extension procedure yields a Radon measure on $M$. For this measure, $\mathcal{V}$ is a null set so that the spaces $L^p(M)$ are well defined.

It will be convenient from time to time to reparametrize the arcs out of a given vertex $v$, so that they are all locally outgoing: thus we define, for all $\kappa \in J(v)$,

$$\psi_\kappa(\theta) = \psi_{\kappa[1]}(\kappa[2] + \kappa[3]\theta), \quad \theta \in [0, b_{\kappa[1]} - a_{\kappa[1]}].$$

The convention is that Latin indices indicate a fixed edge numbering and that Greek indices indicate outgoing numbering out of a given vertex. There exists a strictly positive number $\ell$ such that for all $v$ the images of $(0, \ell)$ by the mappings $\psi_\kappa$ are pairwise disjoint; we can take for $\ell$ a number strictly inferior to the minimum of the lengths $(b_j - a_j)/2$. The last observation is that $e_\kappa = \psi'_\kappa(0)$.

## 3. The Laplace operator on $\mathcal{M}$

Let $C^1(M)$ be the space of continuous functions $f$ on $M$ whose restriction to $M_j$ is of class $C^1$ and such that $df$ has limits at the endpoints of $M_j$: this means that $f'_j$ has limits at $a_j$ and $b_j$. For all $f$ and $g$ in $C^1(M)$, we define an energy bilinear form by

$$E(f, g) = \int_{M \setminus \mathcal{V}} df(x)\, dg(x) d\mathrm{vol}_M\, .$$

In local coordinates,

$$E(f, g) = \sum_{j=1}^{|\mathcal{E}|} \int_{a_j}^{b_j} f'_j g'_j \, d\theta. \tag{4}$$

Denote by $H^1(M)$ the completion of $C^1(M)$ with respect to the pre-Hilbertian norm

$$|f|_{H^1} = \left(|f|_{L^2}^2 + E(f, f)\right)^{1/2}.$$

The space $H^1(M)$ can also be described as the space

$$\left\{f \in C^0(M) : f_j \in H^1(a_j, b_j), \forall\, j \in \{1, \dots, \mathcal{V}\}\right\}.$$

Now it is possible to define the Laplace operator $\mathcal{A}$ on $M$: in the space $L^2(M)$, $E$ is a closed bilinear form defined on its domain $H^1(M) \times H^1(M)$; it is clear that $H^1(M)$ is dense in $L^2(M)$, and that $E$ is symmetric and nonnegative so that, in particular, it is sectorial. Therefore we define Lap as the operator derived from the sesquilinear form $E$ (Theorem VI.2.1 of [13]). More precisely, the domain of Lap is the set of elements $f$ of $H^1(M)$ for which there exists a constant $C$ such that

$$\forall\, g \in H^1(M), \quad |E(f, g)| \leqq C|g|_{L^2(M)}.$$

The value of Lap $f$, for $f \in D(\mathcal{A})$ is given by

$$(\text{Lap}\, f, g) = E(f, g) \quad \forall\, g \in H^1(M).$$

Let us describe the domain of Lap: if $f$ belongs to $D(\text{Lap})$, and if $g$ has compact support in $M_j$ and is of class $C^1$, the inequality

$$\left| \int_{a_j}^{b_j} f_j' g_j'\, d\theta \right| \leqq C \left( \int_{a_j}^{b_j} |g_j|^2\, d\theta \right)^{1/2}$$

implies that the distribution $f_j''$ belongs to $L^2(a_j, b_j)$, and that

$$\left(\text{Lap}\, f\right) \circ \psi_j = -f_j''. \tag{5}$$

Thus, $f_j$ is in $H^2(a_j, b_j)$. Let us integrate by parts the relation $(\text{Lap}\, f, g) = E(f, g)$ for all $g$ in $C^1(M)$, with the help of the relation (5). We find that

$$\sum_{j=1}^{|\mathcal{E}|} f_j' g_j \,\Big|_{a_j}^{b_j} = 0,$$

for all $g$ in $H^1(M)$. We reorder this equality by summing on $\mathcal{V}$; as we can choose arbitrarily the value of $g(x)$ for $v \in \mathcal{V}$, we obtain

$$\sum_{\kappa \in J(v)} \lim_{\theta \downarrow 0} df(\psi_\kappa(\theta)) e_\kappa = 0,$$

or equivalently

$$\sum_{\kappa \in J(v)} \kappa[3] f_{\kappa[1]}'(\kappa[2]) = 0. \tag{6}$$

This condition is reminiscent of Kirchhoff's laws for currents in electric circuits: the sum of algebraic currents leaving any node has to vanish. It will be convenient to write $D(\text{Lap}) = H^2(M)$ by analogy with the usual Sobolev spaces.

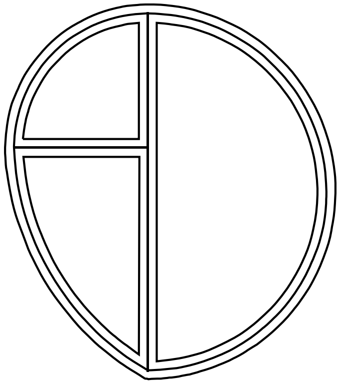

**Fig. 2.** The set $\hat{\mathcal{O}}(M, \varepsilon)$.

## 4. The set $\mathcal{O}(\varepsilon)$

In this section we describe the fattening of $M$ into an open set $\mathcal{O}(\varepsilon)$ with $C^2$ boundary and width $2\varepsilon$ around $M$, except close to the vertices of $M$, where we have to perform an adequate smoothing.

We start by studying $M$ near the vertices. Define the offset of $M$ by

$$\hat{\mathcal{O}}(M, \varepsilon) = \left\{x \in \mathbb{R}^2 : d(x, M) < \varepsilon\right\}.$$

The boundary of this set is not of class $C^2$ if $M$ is anything more complicated than the $C^2$ diffeomorphic image of a circle.

We need a local description of $\hat{\mathcal{O}}(M, \varepsilon)$ close to the vertices of $M$. Let $\rho$ be the rotation of $+\pi/2$ in $\mathbb{R}^2$. Define $C^1$ maps $\Psi_j$ and $\Psi_\kappa$ by

$$\Psi_j(\theta, s) = \psi_j(\theta) + s\rho\psi_j'(\theta), \quad \theta \in [a_j, b_j], \qquad s \in \mathbb{R},$$
$$\Psi_\kappa(\theta, s) = \psi_\kappa(\theta) + s\rho\psi_\kappa'(\theta), \quad \theta \in [0, b_{\kappa[1]} - a_{\kappa[1]}], \quad s \in \mathbb{R}.$$

The two notations correspond to distinct parametrizations of the arcs.

For $\varepsilon$ small enough, $\Psi_j$ is a diffeomorphism from $(a_j, a_j + \ell) \times (-\varepsilon, \varepsilon)$ onto its image, and from $(b_j - \ell, b_j) \times (-\varepsilon, \varepsilon)$ onto its image; if $M_j$ happens to be a loop, $\Psi_j$ cannot be a diffeomorphism from $(a_j, b_j) \times (-\varepsilon, \varepsilon)$ onto its image, unless $M_j$ is diffeomorphic to a circle.

The following result is geometrically obvious, save for the exact expression of $R_0$. Thus we do not spell out the details.

**Lemma 1.** *Let $\bar{\beta}(v)$ be the absolute value of the minimum of the half-angle between two distinct edges leaving $v$, i.e.,*

$$\sin\bar{\beta}(v) = \min\left\{\sqrt{(1 - e_\kappa \cdot e_\lambda)/2} : \kappa \in J(v), \lambda \in J(v) \setminus \{\kappa\}\right\},$$

*and let*

$$R_0 = \min\left\{1/\sin(\bar{\beta}(v)) : v \in \mathcal{V}\right\}. \tag{7}$$

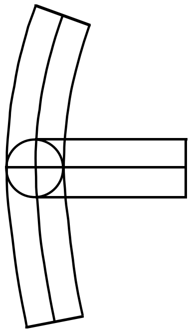

**Fig. 3.** The local shape of $\hat{\mathcal{O}}(M, \varepsilon)$ of Fig. 1 around vertex $v_3$.

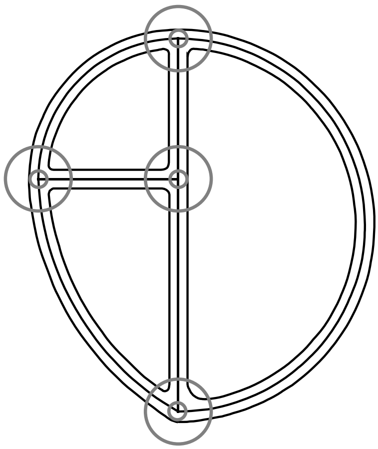

**Fig. 4.** The set $\mathcal{O}(\varepsilon)$.

*Then for all $R > R_0$, there exists $\varepsilon_0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ all the angular points of the boundary of $\hat{\mathcal{O}}(M, \varepsilon)$ belong to the union of the balls of radius $R\varepsilon$ centered at the vertices of $M$.*

For future reference, we introduce some notation. Assume that $e_\kappa + e_\lambda \neq 0$. Assume further, without loss of generality, that the difference of arguments $\arg e_\kappa - \arg e_\lambda$ belongs to the open interval $(0, \pi)$. Write

$$e = e_\kappa, \quad \hat{e} = e_\lambda, \quad \Psi = \Psi_\kappa, \quad \hat{\Psi} = \Psi_\lambda. \tag{8}$$

We can define now $\mathcal{O}(\varepsilon)$, which is smooth fattening of $M$: we fix $R > R_0$, and $\varepsilon$ as in Lemma 1. Define, for $t \in [R\varepsilon, \ell/2]$,

$$M_j(t) = \psi_j\big((a_j + t, b_j - t)\big) \tag{9}$$

and the fattened branch

$$\mathcal{B}_j(t, \varepsilon) = \Psi_j\big((a_j + t, b_j - t) \times (-\varepsilon, \varepsilon)\big).$$

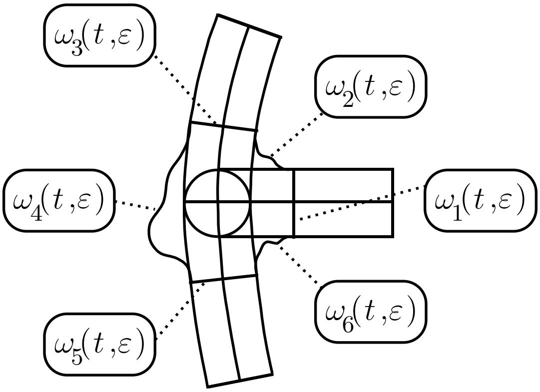


**Fig. 5.** The local smoothing at the junctions.

We write henceforth

$$t_1 = (R+1)\varepsilon, \tag{10}$$

and we choose now $\mathcal{O}(\varepsilon)$ in such a way that

$$\bigcup_{j=1}^{|\mathcal{E}|} \mathcal{B}_j(0,\varepsilon) \cup \bigcup_{v\in\mathcal{V}} B(v,\varepsilon) \subset \mathcal{O}(\varepsilon) \subset \bigcup_{j=1}^{|\mathcal{E}|} \mathcal{B}_j(0,\varepsilon) \cup \bigcup_{v\in\mathcal{V}} B(v, R\varepsilon).$$

These inclusions imply that outside the union of balls of radius $t_1$ centered around the vertices of $M$, $\mathcal{O}(\varepsilon)$ coincides with $\hat{\mathcal{O}}(M,\varepsilon)$. We require that the boundary of $\mathcal{O}(\varepsilon)$ be of class $C^2$ inside each $B(v,t_1)$. The details of the smoothing process will be shown to be irrelevant, provided that the boundary of $\mathcal{O}(\varepsilon)$ is well behaved.

The boundary of a junction is made out of line segments which are flat ends of fattened branches and of arcs which join one corner of a fattened branch to the next corner. These arcs are of class $C^2$, they are parametrized by arc length; their parametrizations are indexed by even numbers, and they satisfy some scaled estimates. The segments are also parametrized by arc length and their parametrizations are indexed by odd numbers.

The description of a well-behaved boundary close to the junctions is given now precisely. If there are $p/2 = |J(v)|$ arcs out of $v$, there will be $p/2$ arcs joining the arms of $\mathcal{O}(\varepsilon)$. These arcs are parametrized by the arc length as mappings $\omega_{2j}(\cdot,\varepsilon)$ of class $C^2$ from $[0,\alpha_j(\varepsilon)]$ to $\mathbb{R}^2$. We require that there exist a number $C_M$ such that for all $\varepsilon$ and all $j$

$$\left|\frac{\omega_{2j}(s,\varepsilon)-v}{\varepsilon}\right| + \left|\frac{\partial\omega_{2j}(s,\varepsilon)}{\partial s}\right| + \varepsilon\left|\frac{\partial^2\omega_{2j}(s,\varepsilon)}{\partial s^2}\right| \leqq C_M, \tag{11}$$

$$\alpha_j(\varepsilon) \leqq C_M\varepsilon. \tag{12}$$

We also require that a curve parametrized by $\omega_{2j}$ start at one end of a segment $\Psi_\kappa(t_1,\varepsilon)$ and end at one end of another segment $\Psi_\lambda(R\varepsilon,\varepsilon)$. We will parametrize the segment joining $\omega_{2j}(\alpha_j(\varepsilon), \varepsilon)$ to $\omega_{2j+2}(0, \varepsilon)$ by $\omega_{2j+1}$, the parameter running in $(0, 2\varepsilon)$. Then it is geometrically obvious that there exists a constant $C$ such that

$$\alpha(\varepsilon) \geqq C t_1. \tag{13}$$

We define a function $\phi(\cdot, \varepsilon)$ by

$$\begin{aligned}
l_0(\varepsilon) &= 0,\\
l_1(\varepsilon) &= 2\varepsilon,\\
\phi(s, \varepsilon) &= \omega_1(s, \varepsilon) \text{ over } [l_0(\varepsilon), l_1(\varepsilon)],\\
l_2(\varepsilon) &= (C_M + 2)\varepsilon,\\
\phi(s, \varepsilon) &= \omega_2\big((s - 2\varepsilon)\alpha_1(\varepsilon)/(C_M\varepsilon), \varepsilon\big) \text{ over } [l_1(\varepsilon), l_2(\varepsilon)],
\end{aligned}$$

and inductively

$$\begin{aligned}
l_{2j+1}(\varepsilon) &= l_{2j}(\varepsilon) + 2\varepsilon,\\
\phi(s, \varepsilon) &= \omega_{2j+1}(s - l_{2j(\varepsilon)}, \varepsilon) \text{ over } [l_{2j}(\varepsilon), l_{2j+1}(\varepsilon)],\\
l_{2j+2}(\varepsilon) &= l_{2j+1}(\varepsilon) + 2C_M\varepsilon,\\
\phi(s, \varepsilon) &= \omega_{2j+2}\big((s - l_{2j+1}(\varepsilon))\alpha_{j+1}(\varepsilon)/(C_M\varepsilon), \varepsilon\big) \text{ over } [l_{2j+1}(\varepsilon), l_{2j+2}(\varepsilon)].
\end{aligned}$$

We extend $\phi(\cdot, \varepsilon)$ as a periodic function of period $l_p(\varepsilon)$ on $\mathbb{R}$, and we require the following global condition: there exists $m > 0$ such that

$$\big|\phi(s, \varepsilon) - \phi(s', \varepsilon)\big| \geqq m \min\Big(m\varepsilon, \min_{k\in\mathbb{Z}} |s - s' + kl_p(\varepsilon)|\Big). \tag{14}$$

It will be convenient to write, for all $j$ and $k$ in $\mathbb{Z}$, $l_{j+pk} = l_j + kl_p(\varepsilon)$. Relations (12) and (13) imply that $\phi(\cdot, \varepsilon)$ is of class $C^2$ on each interval $(l_j(\varepsilon), l_{j+1}(\varepsilon))$ and satisfies the following estimate for all $s \notin \{l_j(\varepsilon),\ j \in \mathbb{Z}\}$:

$$\left|\frac{\phi(s, \varepsilon)}{\varepsilon}\right| + \left|\frac{\partial\phi(s, \varepsilon)}{\partial s}\right| + \varepsilon\left|\frac{\partial^2\phi(s, \varepsilon)}{\partial s^2}\right| \leqq C_M \tag{15}$$

with possibly a different $C_M$ from the one in (11).

For $t_1 \leqq t \leqq \ell$, $\mathcal{B}_j(t, \varepsilon)$ is diffeomorphic to $(a_j + t, b_j - t) \times (-\varepsilon, \varepsilon)$, and for $j \neq k$, $\mathcal{B}_j(t, \varepsilon)$ does not intersect $\mathcal{B}_k(t, \varepsilon)$; thus the open set

$$\mathcal{O}(\varepsilon) \setminus \bigcup_{j=1}^{|\mathcal{E}|} \overline{\mathcal{B}_j(t, \varepsilon)}$$

is a disjoint union of open sets which can be seen as the junctions between the pipes. Each of these junctions $U(t, \varepsilon, v)$ contains exactly one point $v$ of $\mathcal{V}$; it has $|J(v)|$ arms of width $2\varepsilon$ outside the circle of radius $t_1$ about $v$ and contains the open ball of radius $\varepsilon$ about $v$ (see Fig. 6).

The operator $\mathrm{Lap}^\varepsilon$ is defined as follows:

$$D(\mathrm{Lap}^\varepsilon) = \big\{u \in H^2(\mathcal{O}(\varepsilon)) : \partial u/\partial n = 0 \text{ on } \partial\mathcal{O}(\varepsilon)\big\}, \quad \mathrm{Lap}^\varepsilon u = -\Delta u.$$

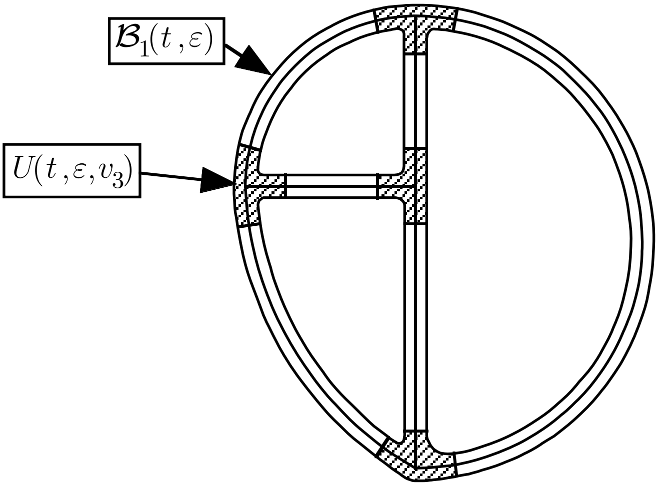


**Fig. 6.** Junctions and fat branches; compare with Fig. 1.

## 5. The plumber's shop I: from $H^1(M)$ to $H^1(\mathcal{O}(\varepsilon))$

In this section, we define a mapping $Q^\varepsilon$ from $H^1(M)$ to $H^1(\mathcal{O}(\varepsilon))$ and we prove some precise comparison results on the $L^2$ norms and the energies. The title of this section and the next two comes from the fact that $\mathcal{O}(\varepsilon)$ looks like a set of pipes, which are soldered at the vertices, and the soldered parts, as can be expected, are slightly fatter than the pipes themselves. We recall that each soldered part has $|J(x)| \geqq 2$ arms. We go from $H^1(M)$ to $H^1(\mathcal{O}(\varepsilon))$ basically by extending a function $f$ on $M$ as a constant along the normals to the curves $M_j$; this works well if $f$ is constant close to $\mathcal{V}$. Thus we replace $f$ by a $\tilde{f}$ which has about the same $H^1(M)$ norm as $f$ and which is constant close to $\mathcal{V}$.

As a consequence of the study performed in Section 4, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \leqq \varepsilon_0$, $\Psi_j$ is a diffeomorphism from $(a_j + t_1, b_j - t_1) \times (-\varepsilon, \varepsilon)$ to its image, $\mathcal{B}(t_1, \varepsilon)$. Therefore, we can express the Euclidean metric on $\mathcal{B}_j(t, \varepsilon)$ in the $(\theta, s)$ coordinates in terms of the metric tensor; it is given by

$$G_{\theta\theta} = |\psi_j'(\theta) + s\rho\psi_j''(\theta)|^2, \quad G_{s\theta} = G_{\theta s} = 0, \quad G_{ss} = 1, \tag{16}$$

and it is used to measure the length of vector fields. To comply with standard notation, we write the volume element in these local coordinates $\sqrt{G}\, d\theta\, ds$; $G$ is simply equal to $G_{\theta\theta}$.

Let us define now a mapping $Q^\varepsilon$ from $H^1(M)$ to $H^1(\mathcal{O}(\varepsilon))$. Let $f$ belong to $H^1(M)$; recall that $f_j = f \circ \psi_j$; define a piecewise affine change of variable on $[a_j, b_j]$ by

$$T_j(\theta) = \begin{cases} a_j, & \text{if } a_j \leqq \theta \leqq a_j + \varepsilon R, \\ b_j, & \text{if } b_j - \varepsilon R \leqq \theta \leqq b_j, \\ a_j + \dfrac{\theta - a_j - \varepsilon R}{b_j - a_j - 2\varepsilon R}(b_j - a_j), & \text{otherwise.} \end{cases} \tag{17}$$

Observe that the dependence of $T_j$ upon $\varepsilon$ is understated, in order to simplify the notation. The reciprocal function of $T_j$ is $\Theta_j$, which we define by

$$\Theta_j(t) = a_j + t_1 + \frac{t - a_j}{b_j - a_j}(b_j - a_j - 2t_1), \quad t \in [a_j, b_j]. \tag{18}$$

With this notation, we let $\tilde{f}_j(\theta) = f_j(T_j(\theta))$, and we define $Q^\varepsilon f$ by

$$(Q^\varepsilon f)(y) = \begin{cases} \tilde{f}_j(\theta) & \text{if } y = \Psi_j(\theta, s) \in \mathcal{B}_j(t_1, \varepsilon), \\ f(v) & \text{if } y \in \overline{U(t_1, \varepsilon, v)}. \end{cases} \tag{19}$$

Before we state the first estimate, let us give another definition: the bilinear energy form on $H^1(\mathcal{O}(\varepsilon))$ is

$$E^\varepsilon(u, w) = \int_{\mathcal{O}(\varepsilon)} \operatorname{grad} u(x) \operatorname{grad} w(x)\, dx.$$

**Theorem 1.** *There exists a constant $C$ such that for all $f$ in $H^1(M)$ the following inequalities hold:*

$$2\varepsilon(1 + C\varepsilon)|f|^2_{L^2(M)} + C\varepsilon^2 E(f, f) \geqq \left|Q^\varepsilon f\right|^2_{L^2(\mathcal{O}(\varepsilon))} \geqq 2\varepsilon(1 - C\varepsilon)|f|^2_{L^2(M)}, \tag{20}$$

$$2\varepsilon(1 + C\varepsilon)E(f, f) \geqq E^\varepsilon(Q^\varepsilon f, Q^\varepsilon f). \tag{21}$$

**Proof.** Let us calculate the $L^2$ norm of $Q^\varepsilon f$ on $\mathcal{B}_j(t_1, \varepsilon)$: we recall that the Euclidean volume element in $\theta$ and $s$ coordinates is $\sqrt{G}\, d\theta\, ds$; therefore

$$\int_{\mathcal{B}_j(t_1,\varepsilon)} |Q^\varepsilon f|^2\, dx = \int_{a_j+t_1}^{b_j-t_1} \int_{-\varepsilon}^{\varepsilon} |\tilde{f}_j(\theta)|^2 \sqrt{G}\, ds\, d\theta,$$

and with the help of the change of variable (18) this expression can be rewritten

$$\int_{\mathcal{B}_j(t_1,\varepsilon)} |Q^\varepsilon f|^2\, dx = \int_{-\varepsilon}^{\varepsilon} \int_{a_j}^{b_j} \left|f_j(t)\right|^2 \sqrt{G(\Theta_j(t), s)}\, dt ds \frac{b_j - a_j - 2t_1}{b_j - a_j}.$$

We observe that $G(\Theta_j(t), s) = 1 + O(\varepsilon)$ uniformly in $\varepsilon$, according to our regularity assumptions over $\psi_j$. Therefore, we can see that there exists a constant $C$ such that, for all $j$ and all sufficiently small $\varepsilon$,

$$\int_{\mathcal{B}_j(t_1,\varepsilon)} |Q^\varepsilon f|^2\, dx \geqq 2\varepsilon\big(1 - C\varepsilon\big) \int_{a_j}^{b_j} |f_j(\theta)|^2\, d\theta.$$

Summing these inequalities with respect to $j$ proves the second inequality in (20).

In order to prove the first of inequalities (20), we observe that, as we are in dimension 1,

$$\max\{|f(v)| : v \in \mathcal{V}\} \leqq C|f|_{H^1(M)}.$$

The area of $U(t_1,\varepsilon,v)$ is of order $O(\varepsilon^2)$; thus the contribution of $U(t_1,\varepsilon,v)$ to $|Q^\varepsilon f|^2_{L^2}$ is bounded by $C\varepsilon^2|f|^2_{H^1(M)}$; the contribution of $\mathcal{B}_j(t_1,\varepsilon)$ to $|Q^\varepsilon f|^2_{L^2}$ is estimated from above by

$$2\varepsilon\bigl(1+C\varepsilon\bigr)\int_{a_j}^{b_j}|f_j(\theta)|^2\,d\theta.$$

This proves the desired assertion, by summing over the branches and the junctions.

Let us address now the energy estimates: there is no contribution to the energy on the junctions $U(t_1,\varepsilon,v)$; therefore, we are left with

$$E^\varepsilon(Q^\varepsilon f,Q^\varepsilon f)=\sum_j\int_{\mathcal{B}_j(t_1,\varepsilon)}\bigl|\mathrm{grad}\,Q^\varepsilon f\bigr|^2\,dx.$$

In $(\theta,s)$ coordinates, the square of the gradient of $Q^\varepsilon f_j$ becomes

$$G^{-1}\left|\frac{\partial Q^\varepsilon f_j}{\partial\theta}\right|^2,$$

since $Q^\varepsilon f$ is constant in the normal direction on $\mathcal{B}_j(t_1,\varepsilon)$. Thus,

$$\int_{\mathcal{B}_j(t_1,\varepsilon)}\bigl|\mathrm{grad}\,Q^\varepsilon f\bigr|^2\,dx=\int_{-\varepsilon}^{\varepsilon}\int_{a_j+t_1}^{b_j-t_1}G^{-1}\left|\frac{d\tilde f_j}{d\theta}\right|^2\sqrt{G}\,d\theta\,ds.$$

After the change of variables (18), we find that

$$\int_{\mathcal{B}_j(t_1,\varepsilon)}\bigl|\mathrm{grad}\,Q^\varepsilon f\bigr|^2\,dx$$
$$=\int_{-\varepsilon}^{\varepsilon}\int_{a_j}^{b_j}\bigl(G(\Theta_j(t),s)\bigr)^{-1/2}\frac{b_j-a_j}{b_j-a_j-2t_1}|(f_j)'(t)|^2\,dt\,ds.$$

The same argument as above yields (21).

## 6. The plumber's shop II: the subpipes

Going from $H^1\bigl(\mathcal{O}(\varepsilon)\bigr)$ to $H^1(M)$ is more complicated than the inverse operation: the essential idea is to take normal averages on the curves $M_j$; but what should we do at junctions? We have to compare normal averages close to them.

For each junction $U(t,\varepsilon,v)$, $t\geqq t_1$ we pick a pair of arms indexed by $\kappa$ and $\lambda$; a subpipe $W_{\kappa\lambda}(t,\varepsilon)$ will be made out of two curved rectangles, one in each of the branches $\kappa$ and $\lambda$, and a circular sector. We also define a function $h_{\kappa\lambda}$ on $W_{\kappa\lambda}(t,\varepsilon)$ whose gradient is tangent to the boundary on the segments $\Psi_\kappa(\{t\}\times(-\varepsilon,\varepsilon))$ and $\Psi_\lambda(\{t\}\times(-\varepsilon,\varepsilon))$, and normal to it everywhere else. Moreover, the gradient of $h_{\kappa\lambda}$ is of norm 1 almost everywhere. A Stokes formula allows us to estimate the difference of the averages of $u$ on the end segments in terms of $\varepsilon$, $t$ and the energy norm of $u$ on $W_{\kappa\lambda}$.

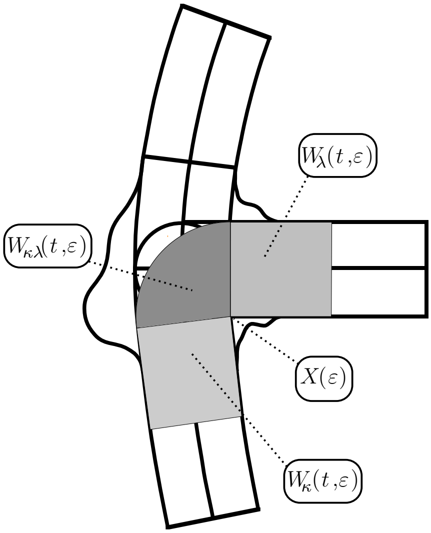


**Fig. 7.** A pipe in a junction.

For the detailed statement of this result and the proof, it is convenient to use the language of differential forms, which enables us to apply exterior differentiation in a coordinate independent fashion, and to make use of the Stokes formula under its most convenient form; the reader is referred to classical works in differential geometry (e.g., [2, 12, 8]) for a description of this language.

**Lemma 2.** *There exists a constant $C$ such that for all $v$ in $\mathcal{V}$, for all distinct $\kappa$ and $\lambda$ in $J(v)$, for all $t \in [t_1, \ell/2]$ and all sufficiently small $\varepsilon > 0$, there exists an open set $W_{\kappa\lambda}(t,\varepsilon)$ and a piecewise $C^1$ function $h_{\kappa\lambda}$ on $W_{\kappa\lambda}(t,\varepsilon)$ whose differential is also piecewise $C^1$ such that the following conditions hold:*

(1) *The open set $W_{\kappa\lambda}(t,\varepsilon)$ is the union of*

$$\Psi_\kappa\big((\theta_\kappa(\varepsilon), t) \times (-\varepsilon, \varepsilon)\big), \quad \Psi_\lambda\big((\theta_\lambda(\varepsilon), t) \times (-\varepsilon, \varepsilon)\big),$$

*and the circular sector of the disk of center*

$$X_{\kappa\lambda}(\varepsilon) = \Psi_\lambda(\theta_\lambda(\varepsilon), -\varepsilon) = \Psi_\kappa(\theta_\kappa(\varepsilon), \varepsilon)$$

*and of radius $2\varepsilon$ which fills the angle between the first two pieces.*

(2) *The differential $dh_{\kappa\lambda}$ vanishes on the tangent vectors to $\partial W_{\kappa\lambda}(t,\varepsilon)$ except on the parts $\Psi_\kappa(\{t\} \times (-\varepsilon, \varepsilon))$ and $\Psi_\lambda(\{t\} \times (-\varepsilon, \varepsilon))$.*

(3) *Since $\rho$ is the rotation of angle $\pi/2$, it follows that*

$$\begin{aligned} dh_{\kappa\lambda}\big(\rho\psi_\kappa'(t)\big) &= -1 \text{ on } \Psi_\kappa\big(t \times (-\varepsilon, \varepsilon)\big),\\ dh_{\kappa\lambda}\big(\rho\psi_\lambda'(t)\big) &= 1 \quad \text{on } \Psi_\lambda\big(t \times (-\varepsilon, \varepsilon)\big). \end{aligned} \tag{22}$$

(4) *For almost every $y$ in $W_{\kappa\lambda}(t,\varepsilon)$,*

$$|dh_{\kappa\lambda}| = 1. \tag{23}$$

(5) *The differential $dh_{\kappa\lambda}$ is piecewise $C^1$ and its differential vanishes in the sense of distributions.*

(6) *The area of $W_{\kappa\lambda}(t,\varepsilon)$ is bounded as follows:*

$$|W_{\kappa\lambda}(t,\varepsilon)| \leqq 4t\varepsilon + C\varepsilon^2 t. \tag{24}$$

**Proof.** We use the notation of Lemma 1: without loss of generality, we assume that $v = 0$, and we use the notation (8). Here, $\arg\hat{e} = \beta$ belongs to $(0, \pi/2]$ and we consider separately the cases $\beta = \pi/2$ and $\beta < \pi/2$. We assume henceforth that $\varepsilon$ is at most equal to $\varepsilon_0$ defined in Lemma 1.

Assume first that $\beta = \pi/2$, and define the function

$$\tilde{\Psi}(\theta,s) = \begin{cases} \hat{\Psi}(\theta,s) & \text{if } (\theta,s) \in [0,\ell]\times[-\varepsilon,\varepsilon], \\ \Psi(-\theta,-s) & \text{if } (\theta,s) \in [-\ell,0]\times[-\varepsilon,\varepsilon]. \end{cases}$$

This definition makes sense, since $\hat{\Psi}(0,s)$ and $\Psi(0,-s)$ coincide for $s \in [-\varepsilon,\varepsilon]$. Moreover, $\Psi$ and $\hat{\Psi}$ are of class $C^1$. It is convenient to write $\tilde{\Psi}(\theta,s) = \tilde{\psi}(\theta) + s\rho\tilde{\psi}'(\theta)$. Therefore, we let

$$\tilde{W}(t,\varepsilon) = \tilde{\Psi}((-t,t)\times(-\varepsilon,\varepsilon))$$

and $h(y) = s$ if $y = \tilde{\Psi}(\theta,s) \in W(t,\varepsilon)$. We calculate immediately $dh$: in $(s,\theta)$ coordinates, $dh = ds$, and therefore it is easy to see that $dh$ vanishes on the tangent vectors to $\tilde{\Psi}\big((-t,t)\times\{\pm\varepsilon\}\big)$; the identities (22) are clear. We can see immediately that $|dh| = 1$ everywhere. Finally, the fact that $dh$ is piecewise $C^1$ is obvious; the vanishing of its differential in the sense of distributions is due to the fact that $dh$ does not jump across the segment $\Psi\big(\{0\}\times(-\varepsilon,\varepsilon)\big)$. Finally the area of $W(t,\varepsilon)$ is equal to

$$\int_{-t}^{t}\int_{-\varepsilon}^{\varepsilon} \left|\tilde{\psi}'(\theta) + s(\nu\circ\psi)'(\theta)\right| \, ds\, d\theta \leqq 4t\varepsilon(1 + C\varepsilon).$$

Consider now the case $\beta \in (0,\pi/2)$; we let

$$\begin{aligned} W(t,\varepsilon) &= \Psi\big((\theta(\varepsilon),t)\times(-\varepsilon,\varepsilon)\big), \\ \hat{W}(t,\varepsilon) &= \hat{\Psi}\big((\hat{\theta}(\varepsilon),t)\times(-\varepsilon,\varepsilon)\big), \\ W_0(t,\varepsilon) &= \big\{y : |y - X(\varepsilon)| < 2\varepsilon \text{ and } (y - X(\varepsilon))\cdot\psi'(\theta(\varepsilon)) \leqq 0 \\ &\qquad \text{and } (y - X(\varepsilon))\cdot\psi'(\hat{\theta}(\varepsilon)) \leqq 0\big\}, \\ \tilde{W}(t,\varepsilon) &= W(t,\varepsilon) \cup W_0(t,\varepsilon) \cup \hat{W}(t,\varepsilon). \end{aligned}$$

Then, $h$ is defined as follows:

$$h(y) = \begin{cases} \varepsilon - s & \text{if } y = \Psi(\theta, s) \in W(t, \varepsilon), \\ -|y - X(\varepsilon)| & \text{if } y \in W_0(t, \varepsilon), \\ s - \varepsilon & \text{if } y = \hat{\Psi}(\theta, s) \in \hat{W}(t, \varepsilon). \end{cases}$$

The verification of properties (1) to (3) of Lemma 2 is is left to the reader. The calculation of $|dh|$ on $W(t, \varepsilon)$ and on $\hat{W}(t, \varepsilon)$ is performed as in the previous situation; on the circular sector, the calculation of $|dh|$ is immediate, and it is clear that there is no jump of $dh$ across the common boundaries of $W(t, \varepsilon)$ and $W_0(t, \varepsilon)$ or of $\hat{W}(t, \varepsilon)$ and $W_0(t, \varepsilon)$. The area of $\tilde{W}(t, \varepsilon)$ is equal to

$$\begin{aligned} &\int_{\theta(\varepsilon)}^{t} \int_{-\varepsilon}^{\varepsilon} \left|\psi'(\theta) + s(\nu \circ \psi)'(\theta)\right| \, ds \, d\theta \\ &\qquad + \int_{\hat{\theta}(\varepsilon)}^{t} \int_{-\varepsilon}^{\varepsilon} \left|\hat{\psi}'(\theta) + s(\nu \circ \hat{\psi})'(\theta)\right| \, ds \, d\theta \\ &\qquad + 2\varepsilon^2 \arccos \psi'(\theta(\varepsilon)) \cdot \hat{\psi}'(\hat{\theta}(\varepsilon)). \\ &\quad \leqq 2\varepsilon\big(t - \theta(\varepsilon)\big)(1 + C\varepsilon) + 2\varepsilon\big(t - \hat{\theta}(\varepsilon)\big)(1 + C\varepsilon) + 2\pi\varepsilon^2. \end{aligned}$$

The conclusion is now clear.

The following immediate consequence of Lemma 2 will be used in the proof of Theorem 2:

**Corollary 1.** *There exists a constant $C$ such that, for all $u$ in $H^1(W_{\kappa\lambda}(\varepsilon, t))$,*

$$\begin{aligned} &\left| \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} u(\Psi_\kappa(t, s)) \, ds - \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} u(\Psi_\lambda(t, s)) \, ds \right| \\ &\qquad \leqq C \left(\frac{t}{\varepsilon}\right)^{1/2} \left(\int_{W_{\kappa\lambda}(\varepsilon, t)} |\operatorname{grad} u|^2 \, dy\right)^{1/2}. \end{aligned} \tag{25}$$

**Proof.** It is enough to observe that if $u$ is of class $C^1$ on $W_{\kappa\lambda}(\varepsilon, t)$, we apply Stokes' theorem as follows

$$\int_{\partial W_{\kappa\lambda}(\varepsilon,t)} u dh_{\kappa\lambda} = \int_{W_{\kappa\lambda}(\varepsilon,t)} d\big(u dh_{\kappa\lambda}\big) = \int_{-\varepsilon}^{\varepsilon} u(\Psi_\lambda(t, s)) \, ds - \int_{-\varepsilon}^{\varepsilon} u(\Psi_\kappa(t, s)).$$

But $d\big(u dh_{\kappa\lambda}\big) = du \wedge dh_{\kappa\lambda}$. Thanks to estimates (23) and (24), the conclusion follows immediately.

## 7. The plumber's shop III: from $H^1\big(\mathcal{O}(\varepsilon)\big)$ to $H^1(M)$

Recall that $t_1 = (R + 1)\varepsilon$. For $\varepsilon \leqq \varepsilon_0$, for all edge indices $j \in \{1, \ldots, j\}$, for all $z \in H^1\big(\mathcal{O}(\varepsilon)\big)$, and for all $\theta \in [a_j + t_1, b_j - t_1]$ we define a normal average of $z$ at $x = \psi_j(\theta)$ as follows:

$$(N_j z)(\theta) = \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} z(\Psi_j(\theta, s)) \, ds. \tag{26}$$

This normal average is well defined, since traces of functions belonging to $H^1\big(\mathcal{O}(\varepsilon)\big)$ over smooth curves $\mathcal{C}$ are well defined, and belong to $H^{1/2}(\mathcal{C})$, which is a space of locally integrable functions. Provided that $t_1 \leqq \theta \leqq \ell$, we define for all $v \in \mathcal{V}$ and all $z$ in $H^1\big(\mathcal{O}(\varepsilon)\big)$, the normal average with a $\kappa$ indexation

$$N_\kappa z(\theta) = \big(N_{\kappa[1]}\big)\big(\kappa[2] + \kappa[3]\theta\big),$$

and the average of normal averages around $v$

$$c(\theta, v, z) = \frac{1}{|J(v)|} \sum_{\kappa \in J(v)} N_\kappa z(\theta).$$

We use henceforth the following notation:

$$\begin{aligned}
&t_2 \text{ is an element, to be defined later, of } [t_1, \ell],\\
&\mathcal{U}_1 = U(t_1, \varepsilon, v), \quad \mathcal{U}_2 = U(t_2, \varepsilon, v), \quad \mathcal{O}_\kappa = \Psi_\kappa\big((t_1, t_2) \times (-\varepsilon, \varepsilon)\big),\\
&\eta_1 = \int_{\mathcal{U}_1} |\text{grad}\, u|^2\, dx, \quad \eta_2 = \int_{\mathcal{U}_2} |\text{grad}\, u|^2\, dx,\\
&\zeta_1 = \int_{\mathcal{U}_1} u^2\, dx, \qquad\quad \zeta_2 = \int_{\mathcal{U}_2} u^2\, dx,
\end{aligned} \tag{27}$$

Given $u$ in $H^1\big(\mathcal{O}(\varepsilon)\big)$, we define $P^\varepsilon u$ as follows:

$$(P^\varepsilon u)(x) = \begin{cases} (N_j u)(\theta) & \text{if } x = \psi_j(\theta) \in M_j(t_2),\\[1ex] (N_\kappa u)(\theta) + \dfrac{(t_2 - \theta)\big(c(t_1, v, u) - (N_\kappa u)(t_1)\big)}{t_2 - t_1} & \\ \qquad\qquad \text{if } x = \psi_\kappa(\theta), t_1 \leqq \theta \leqq t_2, & \\[1ex] c(t_1, v, u) & \text{if } x = \psi_\kappa(\theta), 0 \leqq \theta \leqq t_1. \end{cases} \tag{28}$$

The value of $t_2$ is not yet determined; it will be given in (51). The definition of $P^\varepsilon u$ is motivated as follows: we approximate the restriction of $u$ to $\mathcal{U}_2$ by a function $w$ on $\mathcal{U}_2$ which is constant on $\mathcal{U}_1$ and depends only on $\theta$ on $\mathcal{U}_2 \setminus \mathcal{U}_1$. A natural way of doing that is to minimize

$$\begin{aligned}
&|\text{grad}(u - w)|^2_{L^2(\mathcal{U}_2)}\\
&= \sum_{\kappa \in J(v)} \int_{t_1}^{t_2} \int_{-\varepsilon}^{\varepsilon} \left[ G^{-1} \left| \frac{\partial (u - w) \circ \Psi_\kappa}{\partial \theta} \right|^2 + \left| \frac{\partial (u - w) \circ \Psi_\kappa}{\partial s} \right|^2 \right] \sqrt{G}\, ds\, d\theta;
\end{aligned}$$

however, the terms $G^{-1/2}$ and $G^{1/2}$ are equal to $1 + 0(\varepsilon)$ and we will make only a small error if we minimize instead the simpler expression

$$\sum_{\kappa \in J(v)} \int_{t_1}^{t_2} \int_{-\varepsilon}^{\varepsilon} \left[ \left| \frac{\partial (u - w) \circ \Psi_\kappa}{\partial \theta} \right|^2 + \left| \frac{\partial (u - w) \circ \Psi_\kappa}{\partial s} \right|^2 \right] ds\, d\theta. \tag{29}$$

We impose the boundary condition $N_\kappa w(t_2) = N_\kappa u(t_2)$ for all $\kappa \in J(v)$. We find by standard arguments that

$$w \circ \psi_\kappa(\theta) = N_\kappa u(\theta) + \alpha_\kappa(t_2 - \theta).$$

We substitute this relation into (29), which becomes

$$\sum_{\kappa \in J(v)} \int_{t_1}^{t_2} \int_{-\varepsilon}^{\varepsilon} \left[ \left| \frac{\partial(u \circ \Psi_\kappa - N_\kappa u)}{\partial \theta} \right|^2 + \left| \frac{\partial u \circ \Psi_\kappa}{\partial s} \right|^2 + \alpha_\kappa^2 \right] ds\, d\theta,$$

since the integral

$$\int_{-\varepsilon}^{\varepsilon} \frac{\partial(u \circ \Psi_\kappa - N_\kappa u)}{\partial \theta}\, ds$$

vanishes in $[t_1, t_2]$. Denote by $\hat{c}$ the value of $w$ over $\mathcal{U}_1$. By continuity, we must have, for all $\kappa \in J(v)$,

$$\alpha_\kappa(t_2 - t_1) + N_\kappa u(t_1) = \hat{c}.$$

Therefore, if we express $\alpha_\kappa$ in terms of $\hat{c}$, we find that we have to minimize

$$\sum_{\kappa \in J(v)} \left(\hat{c} - N_\kappa u(t_1)\right)^2,$$

and therefore

$$\hat{c} = \frac{1}{|J(v)|} \sum_{\kappa \in J(v)} N_\kappa u(t_1),$$

which is exactly equal to $c(t_1, v, u)$; $\alpha_\kappa$ is equal to $(\hat{c} - N_\kappa u(t_1))/(t_2 - t_1)$.

Our purpose now is to prove the following theorem:

**Theorem 2.** *There exists a constant C such that for all u in $H^1\big(\mathcal{O}(\varepsilon)\big)$ the following estimates hold:*

$$E(P^\varepsilon u, P^\varepsilon u) \leqq \frac{1 + C\sqrt{\varepsilon}}{2\varepsilon} E^\varepsilon(u, u), \tag{30}$$

$$\left|P^\varepsilon u\right|^2_{L^2(M)} \geqq \frac{1}{2\varepsilon} \Big( (1 - C\sqrt{\varepsilon})\, |u|^2_{L^2(\mathcal{O}(\varepsilon))} - C\sqrt{\varepsilon} E^\varepsilon(u, u) \Big). \tag{31}$$

Lemma 3, Theorem 4 and Corollary 3 are the technical results which enable us to ignore the details of the smoothing at the junctions. In particular, they enable us to estimate $\zeta_1$ in terms of $\eta_2$ and $\eta_2 - \zeta_1$ (see (32)). This is necessary for inequality (31), since $P^\varepsilon u$ does not contain any explicit information on $\zeta_1$. Estimate (32) would be obtained simply by a scaling argument if the open set $(U(t_1, \varepsilon, v) - v)/\varepsilon$ were equal to a constant set. However, this relation can hold only if all the edges of the graph are straight lines, at least in a neighborhood of the vertices, and if we choose a special smoothing which scales with $\varepsilon$. Since we want to be unencumbered by the details of the smoothing and of the shape of the edges, we prove estimates which enable us to forget about them. The main idea of the proof is to construct a sequence of Lipschitz-continuous bijections whose inverse is also a bijection, which map $\phi(\mathbb{R}, \varepsilon_n)$ into a fixed closed curve, while mapping the region enclosed by $\phi(\mathbb{R}, \varepsilon_n)$ into a fixed region.

The result that we will use for our estimates is the following:

**Theorem 3.** *There exists a constant $C$ such that for all small enough $\varepsilon$ and all $z \in H^1(\mathcal{U}_1)$ the following inequality holds:*

$$\int_{\mathcal{U}_1} |z|^2 \, dx \leqq C\varepsilon^2 \biggl( \int_{\mathcal{U}_1} |\operatorname{grad} z|^2 \, dx + \Bigl( \sum_{\kappa \in J(v)} N_\kappa z(t_1) \Bigr)^2 \biggr). \tag{32}$$

The proof of this result relies on several technical lemmas and theorems. The main idea is the following: if the junctions could be blown up to a fixed set by an appropriate linear transformation, relation (32) would be immediate. Therefore, we look for an approximate blow-up, and we have to impose conditions on the boundary of the junction which will ensure enough compactness; the conditions we chose to impose are precisely conditions (15). While they may not be necessary, they are reasonable, and one can construct examples of nasty behavior when they are not satisfied.

We define the approximate blow-ups in a slightly more general setting than the one described by (11)–(14) and (37), and this slightly more general formulation simplifies the proof somewhat.

If we start from a family of functions $\phi(\cdot, \varepsilon)$ as in (15), we define

$$\phi_n(s) = \frac{\phi(s\varepsilon_n, \varepsilon_n) - v}{\varepsilon_n} \tag{33}$$

for any sequence $\varepsilon_n$ decreasing to 0. More generally, suppose we are given a sequence of functions $\phi_n$ from $\mathbb{R}$ to $\mathbb{R}^2$ which have the following properties: there exists a number $L$ such that

$$\text{for all } n \in \mathbb{N},\ \phi_n \text{ is continuous and periodic of period } L. \tag{34}$$

We are given $p$ real numbers $l_1, \dots, l_p$ belonging to $(0, L]$, and we define, for all $k \in \mathbb{Z}$,

$$l_{j+kp} = l_j + pL. \tag{35}$$

We assume that the sequence $\phi_n$ is of class $C^2$ over each interval $(l_j, l_{j+1})$ and that there exists a number $M$ such that

$$\sup_n \sup_j \sup_{x \in (l_j, l_{j+1})} \bigl( |\phi_n| + |\phi_n'| + |\phi_n''| \bigr) \leqq M. \tag{36}$$

Assumption (36) implies that at each point $l_j$, $\phi_n'(x)$ has limits on the left and on the right. We suppose that there exists a number $m > 0$ such that

$$\forall n \in \mathbb{N}, \forall x, x' \in \mathbb{R}, \ \bigl|\phi_n(x) - \phi_n(x')\bigr| \geqq m \min\bigl(m, \min_{k \in \mathbb{Z}} |x - x' - kL|\bigr). \tag{37}$$

This condition implies that for each $x \notin \{l_j, j \in \mathbb{Z}\}$, $\left|\phi_n'(x)\right| \geqq m$. Moreover, at each $l_j$, the right and left limits of $\phi_n'$ satisfy the same inequality. Finally the image of $\phi_n$ is homeomorphic to a circle. Thanks to Jordan's theorem, $\mathbb{R}^2 \setminus \phi_n(\mathbb{R})$ has two connected components; the bounded component is denoted by $\Omega_n$. Finally, we assume that for all $j \in \mathbb{Z}$,

$$\inf_{n,j} \frac{\phi_n'(l_j - 0) \cdot \phi_n'(l_j + 0)}{\left|\phi_n'(l_j - 0)\right| \left|\phi_n'(l_j + 0)\right|} \geqq -1 + m. \tag{38}$$

Under these assumptions, it is possible to find a subsequence, still denoted by $\phi_n$, and a function $\phi_\infty$ such that $\phi_n$ converges to $\phi_\infty$ in $C^0(\mathbb{R})$, and such that on every interval $[l_j, l_{j+1}]$, $\phi_n$ converges to $\phi$ in $C^{1,\alpha}([l_j, l_{j+1}])$. The derivative of $\phi_\infty$ is Lipschitz continuous, and $\phi_\infty$ satisfies estimates analogous to (36) and (37). In particular, the image of $\phi_\infty$ is a closed curve; we denote by $\Omega_\infty$ the connected component of $\mathbb{R}^2 \setminus \phi_\infty(\mathbb{R})$.

**Lemma 3.** *Under assumptions* (34) *and* (36)–(38) *there exists for all large enough $n$ a Lipschitz continuous mapping $\Phi_n$ from $\mathbb{R}^2$ to itself which maps $\Omega_n$ to $\Omega_\infty$ and such that*

$$\Phi_n \circ \phi_\infty = \phi_n. \tag{39}$$

*Moreover, $\Phi_n$ and $\Phi_n^{-1}$ converge uniformly to the identity mapping as $n$ tends to infinity; $D\Phi_n$ and $D\Phi_n^{-1}$ converge uniformly to the constant mapping taking the value identity as $n$ tends to infinity.*

**Proof.** We construct the mapping $\Phi_n$ in two steps. The first step amounts to fixing the angular points $\phi_n(l_j)$. Denote by $B_r(x)$ the open Euclidean ball of radius $r$ about $x$. Let $r$ be a strictly positive number such that the intersection of the balls $B_{2r}(\phi_\infty(l_j))$ and $B_{2r}(\phi_\infty(l_i))$ is empty for $1 \leqq i < j \leqq p$.

Let $\chi$ be a function of class $C^2$ over $\mathbb{R}^2$ takings its values in [0, 1], identically equal to 1 on $B_{r/2}(0)$ and identically equal to 0 outside $B_r(0)$. Write

$$\chi_j(x) = \chi(x - \phi_\infty(l_j)),$$

and define a mapping $\tilde{\Phi}_n$ by

$$\tilde{\Phi}_n(x) = \left(1 - \sum_{j=1}^{p} \chi_j(x)\right)x + \sum_{j=1}^{p} \chi_j(x)\big(x + \phi_n(l_j) - \phi_\infty(l_j)\big).$$

The function $\tilde{\Phi}_n$ is of class $C^2$ and maps $\phi_\infty(l_j)$ to $\phi_n(l_j)$ by construction. Let us prove that for $n$ large enough, it has an inverse of class $C^2$. There exists $n_r$ such that for all $n \geqq n_r$ and for all $j$, $\left|\phi_n(l_j) - \phi_\infty(l_j)\right| \leqq r$. Then the closure of the ball $B_{2r}(\phi_\infty(l_j))$ is invariant by $\tilde{\Phi}_n$: if $\left|x - \phi_\infty(l_j)\right| \leqq r$, then

$$\left|\tilde{\Phi}_n(x) - \phi_\infty(l_j)\right| \leqq \left|x - \phi_\infty(l_j)\right| + \left|\chi_j(x)\right| \left|\phi_\infty(l_j) - \phi_n(l_j)\right| \leqq 2r,$$

if $n \geqq n_r$. If $r \leqq \left|x - \phi_\infty(l_j)\right| \leqq 2r$, then $\tilde\Phi_n(x) = x$, and thus we have proved the invariance with respect to $\tilde\Phi_n$ of the ball of radius $2r$ about $\phi_\infty(l_j)$. Therefore, in order to solve the equation

$$\tilde\Phi_n(x) = y \tag{40}$$

we have to consider two different situations: either $\min_j \left|y - \phi_\infty(l_j)\right|$ is strictly larger than $2r$ and then $x = y$ solves equation (40), or there exists $j$ such that $\left|y - \phi_\infty(l_j)\right|$ is at most equal to $2r$ and we seek a solution $y$ under the form $x = y - z$. Thus, we have to solve the problem

$$z = \chi_j\big(y - z\big)\big(\phi_n(l_j) - \phi_\infty(l_j)\big). \tag{41}$$

If we choose $n_r$ so large that for all $n \geqq n_r$,

$$\left|\phi_n(l_j) - \phi_\infty(l_j)\right| \max |D\chi(x)| \leqq 1/2,$$

then (41) possesses a unique solution whose dependence over $y$ is of class $C^2$. Therefore, we have proved that the inverse of $\tilde\Phi_n$ exists and is of class $C^2$.

For the next step of the construction of $\Phi_n$ we define

$$\tilde\phi_n = \tilde\Phi_n^{-1} \circ \phi_n.$$

The reader will check that $\tilde\phi_n$ has properties (34) and (36)–(38), possibly with different constants, and only for $n$ larger than $n_r$.

The idea of the proof now is to define a bijection which will map $[0, L) \times (-\tau, \tau)$ to a tube around the boundary of $\Omega_\infty$. If $\partial\Omega_\infty$ were a smooth curve, this would be a standard tube construction. We modify this construction so as to retain its main properties; however, the bijection that we shall define is only Lipschitz continuous, as well as its inverse.

Without loss of generality, assume that $\phi_\infty$ is parametrized so that $\partial\Omega_\infty$ is positively oriented, and denote by $\nu_\infty(s)$ the exterior normal to $\Omega_\infty$ at $\phi_\infty(s)$. For all $s \notin \{l_j, j_{\in}\mathbb{Z}\}$, $\nu_\infty(s)$ is obtained from $\phi'_\infty(s)$ by a normalization and a rotation of $-\pi/2$. At $l_j$, the left and right limits of $\nu$ are defined, and (38) implies that

$$\nu_\infty(l_j - 0) \cdot \nu_\infty(l_j + 0) \neq -1. \tag{42}$$

It is possible to find a strictly positive number $m_0$ and an $L$-periodic $C^2$ function $\mu$ from $\mathbb{R}$ to $\mathbb{R}^2$ such that, for all $s \notin \{l_j, j \in \mathbb{Z}\}$,

$$\mu(s) \cdot \nu_\infty(s) \geqq 2m_0.$$

Denote by $\nu_n$ the exterior normal to $\Omega_n$ at $\tilde\phi_n(s)$. There exists $n_0$ such that for all $n \geqq n_0$, and all $s \notin \{l_j, j \in \mathbb{Z}\}$

$$\mu(s) \cdot \nu_n(s) \geqq 3m_0/2. \tag{43}$$

Consider now the transformations

$$\hat\Phi_n(s, \tau) = \tilde\phi_n(s) + \tau\mu(s), \quad n \in \mathbb{N} \cup \{\infty\}.$$

Relations (42) and (43) imply that there exists a strictly positive number $\tau_0$ such that for all $s \notin \{l_j, j \in \mathbb{Z}\}$ and all $\tau \in [-\tau_0, \tau_0]$ the mappings $D\hat{\Phi}_n$ for $n \geqq n_0$ and $D\hat{\Phi}_\infty$ are invertible, and that their inverses are uniformly bounded. Moreover, the standard argument on tubular neighborhoods of a submanifold extends here and we can choose $\tau_0$ so small that $\hat{\Phi}_n$ is a Lipschitz-continuous bijection with Lipschitz-continuous inverse from $\mathbb{R}/L\mathbb{Z} \times [-\tau_0, \tau_0]$ for all $n = n_0, \dots, \infty$. It will be convenient henceforth to denote

$$S(t) = \hat{\Phi}\big(\mathbb{R} \times [-\tau, \tau]\big).$$

Let $\chi_0$ be a $C^2$ function from $\mathbb{R}$ to $[0, 1]$ such that $\chi_0$ vanishes outside $[-2\tau_0/3, 2\tau_0/3]$ and is equal to 1 on $[-\tau_0/3, \tau_0/3]$. Define

$$\chi_1(x) = \begin{cases} \chi_0(\tau) & \text{if } x \in S(\tau_0) \text{ and } x = \hat{\Phi}_\infty(s, t), \\ 0 & \text{if } x \notin S(\tau_0). \end{cases}$$

If we perform a convolution of $\chi_1$ with an adequate smoothing kernel we can obtain a $C^2$ function $\chi_2$ which is equal to 1 on $S(\tau_0/4)$ and which vanishes outside $S(3\tau_0/4)$. Define a mapping $\bar{\Phi}_n$ by

$$\bar{\Phi}_n(x) = \begin{cases} \chi_2(x)\, \hat{\Phi}_n \circ \hat{\Phi}_\infty^{-1}(x) + (1 - \chi_2(x))\, x & \text{if } x \in S(\tau_0), \\ x & \text{otherwise.} \end{cases}$$

We observe that by continuity, there exists an $r_0 > 0$ such that any ball of radius $r_0$ about a point of $S(3\tau_0/4)$ is included in $S(\tau_0)$. There exists $n_1$ such that for all $n \geqq n_1$, $\sup_s \left|\tilde{\phi}_n(s) - \phi_\infty(s)\right| \leqq r_0$. This implies that for $n \geqq n_1$, $\bar{\Phi}_n^{-1}$ maps $S(\tau_0)$ into itself: if $x$ belongs to $S(\tau_0) \setminus S(3\tau_0/4)$, $\hat{\Phi}_n \circ \hat{\Phi}_\infty^{-1}(x)$ is equal to $x$. If $x = \Phi_\infty(s, \tau)$ belongs to $S(3\tau_0/4)$, then

$$\bar{\Phi}_n(x) - x = \chi_2(x)\big(\tilde{\phi}_n(s) - \phi_\infty(s)\big).$$

Therefore, for $n \geqq n_2$, $\bar{\Phi}_n(x)$ belongs to $S(\tau_0)$. For large enough $n$, $\bar{\Phi}_n$ is invertible over $\mathbb{R}^2$: the resolution of the equation

$$\bar{\Phi}_n(x) = y$$

is equivalent to one of the two problems

$$y \in \mathbb{R}^2 \setminus S(\tau_0), \qquad x = y, \tag{44}$$

$$y \in S(\tau_0), \qquad \chi_2(x)\hat{\Phi}_n \circ \hat{\Phi}_\infty^{-1}(x) + (1 - \chi_2(x))x = y. \tag{45}$$

Problem (44) is trivial. Problem (45) can be solved thanks to the strict contraction principle: we seek a solution of (45) of the form $x = y - z$, so that it is equivalent to solving

$$z = \chi_2(y - z)\,\big(\hat{\Phi}_n - \Phi_\infty\big) \circ \Phi_\infty^{-1}(y - z).$$

For large enough $n$, the right-hand side is Lipschitz continuous with respect to $z$ with Lipschitz ratio at most $1/2$. The same argument shows that $\bar{\Phi}_n$ tends uniformly to the identity mapping and that $D\bar{\Phi}_n$ converges uniformly to the constant mapping taking the value identity as $n$ tends to infinity.

Thus, we have constructed a Lipschitz-continuous function $\bar{\Phi}_n$ whose inverse is Lipschitz continuous, such that $\tilde{\phi}_n = \bar{\Phi}_n \circ \phi_\infty$.

For $n$ large enough, we can define $\Phi_n = \tilde{\Phi}_n \circ \bar{\Phi}_n$; it is invertible. The previous study shows that $\Phi_n$ and $\Phi_n^{-1}$ converge uniformly to the identity mapping and that $D\Phi_n$ and $D\Phi_n^{-1}$ converge uniformly to the constant mapping taking the value identity as $n$ tends to infinity. By construction, $\Phi_n$ maps $\Omega_\infty$ to $\Omega_n$ and (39) holds.

We are now able to prove a Poincaré-type estimate with a constant independent of $n$ on $\Omega_n$: let $I$ be a subset of positive measure of $(0, L]$; for all $n$, let $C_n$ be defined by

$$C_n = \inf_{u \in H^1(\Omega_n)\setminus\{0\}} \frac{\int_{\Omega_n} |u|^2 \, dx}{\int_{\Omega_n} |\operatorname{grad} u|^2 \, dx + \left(\int_{\phi_n(I)} u \, d\Gamma\right)^2}.$$

The existence of $C_n$ is an exercise.

We prove the following theorem, which is essential in our estimates:

**Theorem 4.** *Under assumptions* (34) *and* (36)–(38), *the sequence $C_n$ is bounded.*

**Proof.** Assume $C_n$ unbounded; possibly extracting a subsequence, we may assume that $C_n$ goes to infinity as $n$ goes to infinity. It is not difficult to see that $C_n$ is attained for a certain $u_n$. We can normalize $u_n$ so that

$$\int_{\Omega_n} |u_n|^2 \, dx = 1.$$

Then, thanks to the definition of $C_n$,

$$\lim_{n\to\infty} \int_{\Omega_n} |\operatorname{grad} u_n|^2 \, dx + \left(\int_{\phi_n(I)} u_n \, d\Gamma\right)^2 = 0.$$

Define now

$$w_n = u_n \circ \Phi_n.$$

The function $w_n$ belongs to $H^1(\Omega_\infty)$; we can see that

$$\int_{\Omega_\infty} |w_n|^2 \, dx = \int_{\Omega_n} |u_n|^2 \left|\det \Phi_n^{-1}\right| \, dy.$$

Therefore, the limit of the $L^2$ norm of $w_n$ as $n$ tends to infinity is equal to 1. Similarly,

$$\int_{\Omega_\infty} |\nabla w_n|^2 \, dx = \int_{\Omega_n} \left|Du_n \, D\Phi_n \circ \Phi_n^{-1}\right|^2 \left|\det \Phi_n^{-1}\right| \, dy,$$

and $\nabla w_n$ tends to 0 in $L^2(\Omega_\infty)$. Therefore, after possibly multiplying some terms of the sequence $u_n$ by $-1$, we can see that $w_n$ converges strongly in $H^1(\Omega_\infty)$ to the constant function $u_\infty = |\Omega_\infty|^{-1/2}$.

On the other hand,

$$\int_{\phi_n(I)} u_n \, d\Gamma = \int_I u_n \circ \phi_n \left|\phi_n'\right| \, ds = \int_I w_n \circ \phi_\infty |\phi_n'| \, ds.$$

Therefore, the limit as $n$ tends to infinity of $\int_{\phi_n(I)} u_n \, d\Gamma$ is equal to the measure of $\phi_\infty(I)$ multiplied by $u_\infty$, which leads to a contradiction.

We can prove now the almost scaled Poincaré-type inequality stated previously:

**Proof of Theorem 3.** This is just a scaled version of Theorem 4, under the scaling defined by (33).

We also need the following easy lemma:

**Lemma 4.** *There exists a constant $C$ such that, for all small enough $\varepsilon$ and all $z \in H^1(\mathcal{O}_\kappa)$, the following inequality holds:*

$$|N_\kappa z(t_1)| \leqq C\big((t_2/\varepsilon)^{1/2} \, |\text{grad}\, z|_{L^2(\mathcal{O}_\kappa)} + (t_2\varepsilon)^{-1/2} \, |z|_{L^2(\mathcal{O}_\kappa)}\big). \tag{46}$$

**Proof.** Let $Q = (0,1)^2$. There exists a constant $C$ such that for all $w \in H^1(Q)$ we have the inequality

$$\left|\int_0^1 w(0, y_2) \, dy_2\right| \leqq C \, |w|_{H^1(Q)} \, . \tag{47}$$

For all $z \in H^1(\mathcal{O}_\kappa)$ define

$$w(y) = z\big(\Psi_\kappa(t_1 + y_1(t_2 - t_1), \varepsilon(2y_2 - 1))\big).$$

We observe that

$$\begin{aligned} \int_Q |w|^2 \, dy &= \int_{t_1}^{t_2} \int_{-\varepsilon}^{\varepsilon} |z \circ \Psi_\kappa(\theta, s)|^2 \, \frac{d\theta \, ds}{2\varepsilon\sqrt{G}(t_2 - t_1)} \\ &= \frac{1 + O(\varepsilon)}{2\varepsilon(t_2 - t_1)} \int_{\mathcal{O}_\kappa} |z|^2 \, dx. \end{aligned} \tag{48}$$

Similarly,

$$\int_Q |\text{grad}\, w|^2 \, dy$$
$$= \int_{-\varepsilon}^{\varepsilon} \int_t^{t_2} \left( (t_2 - t_1)^2 \left| \frac{\partial(z \circ \Psi_\kappa)}{\partial\theta} \right|^2 + 4\varepsilon^2 \left| \frac{\partial(z \circ \Psi_\kappa)}{\partial s} \right|^2 \right) \frac{d\theta \, ds}{2\varepsilon(t_2 - t_1)}.$$

If we assume that $\varepsilon$ is small enough, we can see that

$$\int_Q |\text{grad}\, w|^2 \, dy \leqq \frac{(t_2 - t_1)(1 + O(\varepsilon))}{2\varepsilon} \int_{\mathcal{O}_\kappa} |\text{grad}\, z|^2 \, dx. \tag{49}$$

Finally,

$$\int_0^1 w(0, y_2)\, dy_2 = N_\kappa z(t_1).$$

Then

$$|N_\kappa z(t_1)| \leqq C\Big(\Big(\frac{t_2 - t_1}{2\varepsilon}\Big)^{1/2} |\mathrm{grad}\, z|_{L^2(\mathcal{O}_\kappa)} + \frac{1}{(2\varepsilon(t_2 - t_1))^{1/2}} |z|_{L^2(\mathcal{O}_\kappa)}\Big).$$

This immediately implies relation (46), for $\varepsilon$ small enough.

**End of the proof of Theorem 2.** Let us prove first inequality (30). A straightforward calculation shows that

$$\begin{aligned}\int_{a_j+t_2}^{b_j-t_2} \left|\frac{\partial N_j u(\theta)}{\partial\theta}\right|^2 d\theta &\leqq \frac{1}{2\varepsilon}\int_{a_j+t_2}^{b_j-t_2}\int_{-\varepsilon}^{\varepsilon}\left|\frac{\partial(u\circ\Psi_j)}{\partial\theta}\right|^2 ds\, d\theta \\ &\leqq \frac{1+C\varepsilon}{2\varepsilon}\int_{\mathcal{B}_j(t_2,\varepsilon)} |\mathrm{grad}\, u|^2\, dx.\end{aligned} \tag{50}$$

The contribution of $\cap_{\kappa\in J(v)}\psi_\kappa(0, t_2)$ to the energy of $P^\varepsilon u$ is

$$\sum_{\kappa\in J(v)}\int_{t_1}^{t_2}\left|\frac{\partial N_\kappa u}{\partial\theta} - \frac{\hat{c} - N_\kappa u(t_1)}{t_2 - t_1}\right|^2 d\theta.$$

We observe that

$$\sum_{\kappa\in J(v)}\frac{(\hat{c} - N_\kappa u(t_1))^2}{t_2 - t_1}$$

can be estimated by $C\eta_1/t_2$ according to (25). Henceforth,

$$t_2 \text{ is any finite strictly positive number at most equal to } \ell/2, \tag{51}$$

and $\varepsilon$ must be no larger than $\min(\varepsilon_0, t_2/R)$. Therefore,

$$\begin{aligned}&\sum_{\kappa\in J(v)}\int_{t_1}^{t_2}\left|\frac{\partial N_\kappa u}{\partial\theta} - \frac{\hat{c} - N_\kappa u(t_1)}{t_2 - t_1}\right|^2 d\theta \\ &\quad\leqq (1+\sqrt{\varepsilon})\sum_{\kappa\in J(v)}\int_{t_1}^{t_2}\left|\frac{\partial N_\kappa u}{\partial\theta}\right| d\theta + (1 + 1/\sqrt{\varepsilon})\sum_{\kappa\in J(v)}\frac{(\hat{c} - N_\kappa u(t_1))^2}{t_2 - t_1} \\ &\quad\leqq \frac{1+\sqrt{\varepsilon}}{2\varepsilon}\int_{\mathcal{O}_\kappa}|\mathrm{grad}\, u|^2\, dx + \frac{C}{\sqrt{\varepsilon}}\int_{\mathcal{U}_1}|\mathrm{grad}\, u|^2\, dx.\end{aligned}$$

This relation, together with (50), implies (30).

Let us turn now to the proof of (31). Consider first the contribution of $M_j(t_2)$ to $|P^\varepsilon u|_{L^2(M)}$: it is easy to see that

$$|u\circ\Psi_\kappa(\theta, s) - N_\kappa u(\theta)| \leqq 2\varepsilon\Big(\int_{-\varepsilon}^{\varepsilon}\left|\frac{\partial(u\circ\Psi_\kappa)}{\partial s}\right|^2 ds\Big)^{1/2}.$$

Therefore we have the inequality

$$\int_{t_1}^{t_2}\int_{-\varepsilon}^{\varepsilon} |u \circ \Psi_\kappa(\theta, s) - N_\kappa u(\theta)|^2 \, ds \, d\theta \leqq 2\varepsilon(1 + C\varepsilon) \int_{\mathcal{B}_j(t_2,\varepsilon)} |\operatorname{grad} u|^2 \, dx.$$

We apply the inequality $(a + b)^2 \geqq (1 - \sqrt{\varepsilon})a^2 - b^2/\sqrt{\varepsilon}$ with $a = u \circ \Psi_\kappa(\theta, s)$ and $b = N_\kappa u(\theta) - u \circ \Psi_\kappa(\theta, s)$, and we can see that

$$\int_{a_j+t_2}^{b_j-t_2} |N_\kappa u(\theta)|^2 \, d\theta \geqq \frac{1}{2\varepsilon}\Big[(1 - C\sqrt{\varepsilon})|u|^2_{L^2(\mathcal{B}_j(t_2,\varepsilon))} - C\sqrt{\varepsilon}|\operatorname{grad} u|^2_{L^2(\mathcal{B}_j(t_2,\varepsilon))}\Big]. \quad (52)$$

The contribution of $\cup_{\kappa\in J(v)}\psi_\kappa([0, t_2])$ to $|P^\varepsilon u|^2_{L^2(M)}$ is

$$t_1|J(v)|\hat{c}^2 + \sum_{\kappa\in J(v)} \int_{t_1}^{t_2} \big(N_\kappa u(\theta) + (t_2 - \theta)(\hat{c} - N_\kappa u(t_1))/(t_2 - t_1)\big)^2 \, d\theta.$$

The same type of argument as above yields

$$\int_{t_1}^{t_2} \big[N_\kappa u(\theta) + (t_2 - \theta)(\hat{c} - N_\kappa u(t_1))/(t_2 - t_1)\big]^2 \, d\theta \geqq \frac{1}{2\varepsilon}\big[(1 - C\sqrt{\varepsilon})(\eta_2 - \eta_1) - C\sqrt{\varepsilon}\eta_2\big]. \quad (53)$$

It remains for us to show that $\eta_2 - \zeta_1$ can be conveniently estimated from below. We infer from Theorem 3 that

$$\zeta_1 \leqq C\varepsilon^2\Big(\eta_1 + \sum_\kappa N_\kappa u(t_1)^2\Big). \quad (54)$$

We substitute estimate (46) into (54) and we can see that

$$\zeta_1 \leqq C\varepsilon\eta_2 + C\varepsilon(\eta_2 - \zeta_1).$$

Therefore,

$$\eta_2 - \zeta_1 \geqq (1 - C\varepsilon)\eta_2 - C\varepsilon\eta_2. \quad (55)$$

Therefore, if we put together (52), (53) and (55), we have obtained inequality (31).

**Remark 1.** We may now compare our the approximation process to that developed by EVANS & SAITO [9] and SAITO [20]. We take for simplicity a cross-shaped domain $\mathcal{O}(\varepsilon)$ of width $2\varepsilon$ defined by

$$\mathcal{O}(\varepsilon) = \{(x_1, x_2) : \max(x_1, x_2) \leqq 1, \quad \min(x_1, x_2) \leqq \varepsilon\},$$

and calculate explicitly the operator $L^\varepsilon = (Q^\varepsilon)^* \operatorname{Lap}^\varepsilon Q^\varepsilon$; also for simplicity, we require Dirichlet conditions at the end-points of the cross. We obtain

$$D(L^\varepsilon) = \Big\{ f \in H^1(\Gamma) : f_j \in H^2(0,1),$$
$$\sum_{j=1}^{4} (1-\varepsilon) f_j'(0) + \varepsilon^2 f(0) = 0, \quad f_j(1) = 0 \Big\}$$
$$(L^\varepsilon f)_j = -(1-\varepsilon)^2 f_j'';$$

Evans and Saito construct a mapping $\tilde{Q}^\varepsilon$ given by

$$(\tilde{Q}^\varepsilon f)(x_1, x_2) = f(x_1), \text{ if } 0 < x_1 < 1\,;$$

with analogous expressions on the three other branches. Then a direct calculation gives for $\tilde{L}^\varepsilon = (\tilde{Q}^\varepsilon)^* \operatorname{Lap}^\varepsilon \tilde{Q}^\varepsilon$:

$$D(\tilde{L}^\varepsilon) = \Big\{ f = (f_j)_{1 \leqq j \leqq 4} : \min(x_j, \varepsilon) f_j' \in L^2(0,1),$$
$$\frac{(\min(x_j, \varepsilon) f_j')'}{\min(x_j, \varepsilon)} \in L^2(0,1) \Big\}$$
$$(\tilde{L}^\varepsilon f_j) = -\frac{1}{\min(x_j, \varepsilon)} \frac{d}{dx} (\min(x_j, \varepsilon) f_j).$$

Therefore, the approximation of Lap by $L^\varepsilon$ is smoother than the approximation by $\tilde{L}^\varepsilon$. An explicit calculation in this particular case shows that, in general, $(\tilde{L}^\varepsilon)^{-1} g$ does not belong to $L^2(\Gamma)$, even with very smooth data $g$, reflecting thus the singular character of the approximation.

## 8. Comparison of the eigenvalues

We start this section by deriving a Hilbertian lemma on the comparison of the eigenvalues of two self-adjoint operators which are bounded from below and which operate in different Hilbert spaces. This result will enable us to relate the eigenvalues of the Laplace operator on $\mathcal{O}(\varepsilon)$ and the eigenvalues of Lap on $M$.

Let $H$ be a Hilbert space, whose scalar product is denoted by $(\,,\,)$ and whose norm is denoted by $|\;|$. Recall that if $A$ is a self-adjoint operator in $H$, the Rayleigh quotient associated with $A$ is the expression

$$\mathcal{R}(u) = \frac{(Au, u)}{|u|^2}.$$

It is defined for all $u$ in the domain of $A$. The bilinear form $a$ associated with $A$ is defined *a priori* on the domain of $A$ by

$$a(u, v) = (Au, v).$$

This bilinear form can be classically extended as follows if $A$ is bounded from below, i.e., if there exists a real number $\alpha$ such that

$$(Au, u) + \alpha |u|^2 \geqq 0. \tag{56}$$

The completion of $D(A)$ for this norm is a dense subspace $V$ of $H$ which is often called the domain of the maximal extension of the quadratic form $a$.

The expression

$$\lambda_n(A) = \inf\big\{\{\sup \mathcal{R}(v) : v \in W\}, \dim W = n\big\} \tag{57}$$

is *a priori* an element of $\mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$. If $A$ is bounded from below, i.e., if (56) holds, the minimax theorem (see for instance [17], Theorem XIII.1) states that for each $n$,

(1) either there are at least $n$ eigenvalues (counting degenerate eigenvalues a number of times equal to their multiplicity) below the bottom of the essential spectrum and $\lambda_n(A)$ is the $n$-th eigenvalue of $A$ counting multiplicity,
(2) or $\lambda_n(A)$ is the bottom of the essential spectrum of $A$, and in that case $\lambda_n(A) = \lambda_{n+1}(A) = \dots$ and there are at most $n - 1$ eigenvalues (counting multiplicity) below $n$.

We prove now the comparison theorem for the eigenvalues:

**Lemma 5.** *Let $H_0$ and $H_1$ be Hilbert spaces equipped with scalar products denoted by $(\ ,\ )_j$ for $j = 0, 1$; let $A_j$ be a self-adjoint operator in $H_j$, which is bounded from below. Denote by $V_j$ the domain of the maximal quadratic form associated with $A_j$ and by $\mathcal{R}_j$ the Rayleigh quotient associated with $A_j$. Suppose that there exists a continuous operator $S$ mapping $V_1$ to $V_0$ and an increasing function $\phi$ from $\mathbb{R}$ to $\mathbb{R} \cup \{+\infty\}$ such that* $\exp(-\phi)$ *is continuous over $\mathbb{R}$ and such that*

$$\forall u \in V_1 \setminus \ker S, \quad \mathcal{R}_0(Su) \leqq \phi(\mathcal{R}_1(u)). \tag{58}$$

*Assume that for a given $n$*

$$\mu = \inf\{\mathcal{R}_1(v) : v \in V_1 \cap \ker S\} > \lambda_n(A_1). \tag{59}$$

*Then,*

$$\lambda_n(A_0) \leqq \phi(\lambda_n(A_1)). \tag{60}$$

*In particular, if $S$ is one-to-one, relation* (60) *holds for all integer $n$.*

**Proof.** Assume first that $\lambda_n(A_1)$ is an eigenvalue of $A_1$. Let $u_1, \dots u_n$ be the eigenvectors of $A_1$ relative to the eigenvalues $\lambda_1(A), \dots \lambda_n(A)$, spanning the space $W$; condition (59) implies that the restriction of $S$ to $W$ is one-to-one. Indeed, if

$$u = \sum_{j=1}^{n} \xi_j u_j$$

belongs to the kernel of $S$, we have the relation

$$\sum_{j=1}^{n} \lambda_j |\xi_j|^2 \geqq \sum_{j=1}^{n} \mu |\xi_j|^2$$

which implies immediately that $u$ vanishes. Condition (60) implies that $Su_j$ belongs to $V_0$ for all $j = 1, \ldots, n$, and the argument we just made implies that the space spanned by $Su_1, \ldots Su_n$ is of dimension $n$; therefore, the expression (57) implies

$$\begin{aligned} \lambda_n(A_0) &\leqq \sup\bigl\{\mathcal{R}_0(z) : z \in SW \setminus \{0\}\bigr\} \\ &= \sup\bigl\{\mathcal{R}_0(Sw) : w \in W \setminus \{0\}\bigr\} \\ &\leqq \sup\bigl\{\phi(\mathcal{R}_1(w)) : w \in W \setminus \{0\}\bigr\} \\ &= \phi(\lambda_n(A_1)). \end{aligned}$$

If $\lambda_n(A_1)$ is the lower bound of the essential spectrum of $A_1$, we use Weyl's criterion: there exists an orthonormal family $(u_j)_{j\geqq 1}$ such that $\mathcal{R}(u_n)$ converges to $\lambda_n(A)$. If $W_j$ is the space spanned by $u_j, \ldots, u_{j+n-1}$, then for $j$ large enough, the intersection of $W_j$ and $\ker S$ is reduced to 0, and the above argument shows that

$$\lambda_n(A_0) \leqq \sup\{\phi(\mathcal{R}_1(w)) : w \in W_j\};$$

the construction of $W_j$ implies that

$$\lim_{k\to\infty} \sup\{\mathcal{R}_1(w) : w \in W_j\}$$

is equal to $\lambda_j(A_1)$, and the theorem is proved.

As a corollary of Lemma 5, we obtain the following result on the comparison between the eigenvalues of $\mathrm{Lap}^\varepsilon$ and Lap:

**Theorem 5.** *There exists a constant $C$ such that*

$$\lambda_p(\mathrm{Lap}^\varepsilon) \leqq (1 + C\varepsilon)\lambda_p(\mathrm{Lap}), \tag{61}$$

*and for all $\varepsilon \leqq \varepsilon(p)$,*

$$\lambda_p(\mathrm{Lap}) \leqq \frac{\lambda_p(\mathrm{Lap}^\varepsilon)(1 + C\sqrt{\varepsilon})}{1 - C\sqrt{\varepsilon} - C\lambda_p(\mathrm{Lap}^\varepsilon)\sqrt{\varepsilon}}. \tag{62}$$

**Proof.** Denote by $\mathcal{R}$ the Rayleigh quotient associated with the operator Lap, and by $\mathcal{R}^\varepsilon$ the Rayleigh quotient associated with the operator $\mathrm{Lap}^\varepsilon$. The mapping $Q_\varepsilon$ is clearly one-to-one; relations (20) and (21) imply the relation

$$\mathcal{R}_\varepsilon(Q^\varepsilon f) \leqq (1 + C\varepsilon)\mathcal{R}(f),$$

for all $f \in H^1(M)$, and for all small enough $\varepsilon$. Lemma 5 implies then that for all small enough $\varepsilon$ and all $p$, (61) holds.

For the converse, we have to estimate $\mathcal{R}^\varepsilon(u)$ when $P^\varepsilon u$ vanishes. Thanks to the definition (28) of $P^\varepsilon$, $u$ belongs to the kernel of $P^\varepsilon$ if and only if

$$\forall\, j \in \{1, \dots, |\mathcal{E}|\}, \quad \forall\, \theta \in M_j(t_1), \quad N_j u(\theta) = 0.$$

An elementary computation shows that for all $v \in H^1(-\varepsilon, \varepsilon)$ whose mean vanishes,

$$\int_{-\varepsilon}^{\varepsilon} v'^2\, ds \geqq \frac{\pi^2}{4\varepsilon^2} \int_{-\varepsilon}^{\varepsilon} v^2\, ds.$$

Therefore, if $u$ belongs to the kernel of $P^\varepsilon$, we can see that

$$\begin{aligned}
&\int_{\mathcal{B}_j(t_1,\varepsilon)} |\mathrm{grad}\, u|^2 \; d\theta\, ds \\
&\qquad \geqq \int_{a_j+t_1}^{b_j-t_1} \int_{-\varepsilon}^{\varepsilon} \left| \frac{\partial u \circ \Psi_j}{\partial s} \right|^2 \frac{1}{\sqrt{G}}\, d\theta\, ds \\
&\qquad \geqq \frac{(1-C\varepsilon)\pi^2}{4\varepsilon^2} \int_{a_j+t_1}^{b_j-t_1} \int_{-\varepsilon}^{\varepsilon} \left| u \circ \Psi_j \right|^2 \sqrt{G}\, d\theta\, ds \\
&\qquad \geqq \frac{(1-C\varepsilon)\pi^2}{4\varepsilon^2} \int_{\mathcal{B}_j(t_1,\varepsilon)} |u|^2 \; d\theta\, ds
\end{aligned}$$

On the junctions, we can use Theorem 3, which can be made uniform with respect to the vertices of the graph $M$. Therefore, we have shown that there exists a constant $C$ such that for all small enough $\varepsilon$,

$$\inf\left\{\mathcal{R}^\varepsilon(u) : u \in \ker P^\varepsilon\right\} \geqq \frac{C}{\varepsilon^2}. \tag{63}$$

Define

$$\phi^\varepsilon(r) = \begin{cases} r\bigl(1 + C\sqrt{\varepsilon}\bigr)/\bigl(1 - C\sqrt{\varepsilon} - rC\sqrt{\varepsilon}\bigr) & \text{if } r \leqq (1 - C\sqrt{\varepsilon})/\sqrt{\varepsilon}, \\ +\infty & \text{otherwise.} \end{cases}$$

Then relations (30) and (31) imply, for all $u \notin \ker P^\varepsilon$, the relation

$$\mathcal{R}(P^\varepsilon u) \leqq \phi\bigl(\mathcal{R}^\varepsilon(u)\bigr).$$

Moreover, for all $p$, there exists $\varepsilon(p)$ such that, if $\varepsilon \leqq \varepsilon(p)$, then thanks to relations (61) and (63), we have

$$\inf\left\{\mathcal{R}^\varepsilon(u) : u \in \ker P^\varepsilon\right\} \geqq \lambda_p(\mathrm{Lap}^\varepsilon).$$

Thus, we may apply again Lemma 5, and we find that

$$\lambda_p(\mathrm{Lap}) \leqq \phi^\varepsilon(\lambda_p(\mathrm{Lap}^\varepsilon)),$$

which is exactly (62).

**Remark 2.** The convergence of $\mathrm{Lap}^\varepsilon$ to Lap has (at least) two possible interpretations. In the first one, we consider the operator $L^\varepsilon$ defined by the quadratic form $E(Q^\varepsilon f, Q^\varepsilon f)/(2\varepsilon)$ on $H^1(\Gamma)$; it is then an exercise to show that for all $f \in H^1(\Gamma)$, $L^\varepsilon f$ tends to $\mathrm{Lap}^\varepsilon f$, and to infer that for all $z$ in the resolvant set of Lap, $(z-L^\varepsilon)^{-1}$ converges to $(z-\mathrm{Lap})^{-1}$. The second one would be the symmetric statement relative to the comparison of the operator $M^\varepsilon$ defined by the quadratic form $E^\varepsilon(P^\varepsilon u, P^\varepsilon u)$ with $\mathrm{Lap}^\varepsilon$. This is much more subtle, because the kernel of $M^\varepsilon$ is of infinite dimension; thus, we see that the statement of Theorem 5 is quite strong, since it tells us that the spectral approximation property holds, even though there is a very large kernel. The only relevant elements in the kernel are the constant functions since non-constant functions belonging to the kernel of $P^\varepsilon$ are highly oscillatory and therefore contribute only to the large eigenvalues of $\mathrm{Lap}^\varepsilon$.

## 9. Asymptotics for the spectrum of the magnetic Schrödinger operator

The energy levels of a quantum mechanical system under an external magnetic field are determined by the spectrum of the magnetic Schrödinger equation. In addition, this spectrum is important in superconductivity. There the dependence of the first eigenvalue on the magnetic field is equivalent to the curve $T_c(H_e)$ describing the phase transition temperature $T_c$ as a function of the applied magnetic field $H_e$.

To define the magnetic Schrödinger operator we introduce the magnetic vector potential related to the applied magnetic field. For this purpose we set $\mathbb{A} : \mathcal{O}(\varepsilon) \mapsto \mathbb{R}^2$ of class $C^1$. The magnetic energy functional associated with this field is given by

$$E^\varepsilon(u, u; \mathbb{A}) = \int_{\mathcal{O}(\varepsilon)} |(\mathrm{grad} + i\mathbb{A})u|^2 \, dx, \tag{64}$$

and it is defined for all $u \in H^1(\mathcal{O}(\varepsilon); \mathbb{C})$, the set of complex-valued and square-integrable functions over $\mathcal{O}(\varepsilon)$ whose first derivatives are also square integrable.

The Schrödinger operator $S^\varepsilon[\mathbb{A}]$ associated with the quadratic form (64) is defined by

$$D(S^\varepsilon[\mathbb{A}]) = \left\{u \in H^2(\mathcal{O}(\varepsilon); \mathbb{C}) : \left(\partial_\nu + i\mathbb{A}_\nu\right)u = 0 \text{ on } \partial\mathcal{O}(\varepsilon)\right\}, \tag{65}$$

and

$$\begin{aligned} S^\varepsilon[\mathbb{A}]u &= (\mathrm{grad} + i\mathbb{A})^*(\mathrm{grad} + i\mathbb{A})u \\ &= -\Delta u - i\mathbb{A} \cdot \mathrm{grad}\, u - i \,\mathrm{div}(\mathbb{A}u) + |\mathbb{A}|^2 u. \end{aligned} \tag{66}$$

In relation (65), $\partial_\nu$ is the normal derivative along the boundary $\partial\mathcal{O}(\varepsilon)$ of $\mathcal{O}(\varepsilon)$, and $\mathbb{A}_\nu = \mathbb{A} \cdot \nu$ is the normal component of $\mathbb{A}$.

In order to define an analogous one-dimensional magnetic energy on the graph $M$, we first identify $\mathbb{A}$ with a differential form of degree 1,

$$A = \mathbb{A}_1 dx^1 + \mathbb{A}_2 dx^2,$$

and then we express this form in local coordinates on each branch $\mathcal{B}_j(R\varepsilon, \varepsilon)$:

$$A = A_{j,\theta}\, d\theta + A_{j,s}\, ds,$$

where

$$A_{j,\theta}(\theta, s) = \big(\mathbb{A} \circ \Psi_j\big)(\theta, s) \cdot \big(\psi_j'(\theta) + s\rho\psi_j''(\theta)\big),$$
$$A_{j,s}(\theta, s) = \big(\mathbb{A} \circ \Psi_j\big)(\theta, s) \cdot \big(\rho\psi_j'(\theta)\big).$$

With this notation, the differential form $\mathbf{A}$ of degree 1 on $M$ is defined in local coordinates on $M_j$ by

$$A_j = A_{j,\theta}(\theta, 0),$$

and the magnetic energy on the graph $M$ is given by

$$E(f, f; \mathbf{A}) = \int_M |df + if\mathbf{A}|^2 \, d\mathrm{vol}_M = \sum_{j=1}^{|\mathcal{E}|} \int_{a_j}^{b_j} \left| f_j' + iA_j f_j \right|^2 d\theta.$$

The associated operator is $S[\mathbf{A}]$ defined by

$$D(S[\mathbf{A}]) = \Big\{ f \in H^1(M; \mathbb{C}) : \quad \forall\, j = 1, \dots, |\mathcal{E}|, \quad f_j \in H^2((a_j, b_j); \mathbb{C}), \sum_{\kappa \in J(v)} \kappa[3]\big(f'_{\kappa[1]} + if_{\kappa[1]}A_{\kappa[1]}\big)(\kappa[2]) = 0, \Big\} \tag{67}$$

$$\big(S[\mathbf{A}]f\big)_j(\theta) = -f_j'' - i\mathbf{A}_j f_j' - i(\mathbf{A}_j f_j)' + \left|\mathbf{A}_j\right|^2 f_j. \tag{68}$$

In this section, we shall prove estimates analogous to those of Sections 5 and 7, and we will apply them to compare the spectra of $S^\varepsilon[\mathbb{A}]$ and $S[\mathbf{A}]$.

We replace the transformation $Q^\varepsilon$ which has been defined in Section 5 by a new transformation which takes the vector potential into account. More precisely, for all $f \in H^1(M; \mathbb{C})$ we define

$$\big(\hat{Q}^\varepsilon f\big)(y) = \begin{cases} \tilde{f}_j(\theta) \exp\big(-i \int_0^s A_{j,s}(\theta, s')\, ds'\big) & \text{if } y = \Psi_j(\theta, s) \in \mathcal{B}_j(t_1, \varepsilon), \\ u_v(y) & \text{if } y \in \overline{\mathcal{U}_1}. \end{cases} \tag{69}$$

The function $u_v$ appearing in (69) is the minimizer of the Dirichlet integral over $\mathcal{U}_1 = \mathcal{U}(t_1, \varepsilon, v)$, assuming that the trace of $u_v$ coincides with the trace of $\hat{Q}^\varepsilon f$ on the common boundary of $\mathcal{U}_1$ and $\mathcal{B}_j(t_1, \varepsilon)$.

With this notation, we can now state our first magnetic estimates:

**Theorem 6.** *There exists a constant $C$, which is bounded if the $C^1$ norm of $\mathbb{A}$ is bounded, such that for all $f \in H^1(M; \mathbb{C})$ the following estimates hold:*

$$2\varepsilon(1 - C\varepsilon)|f|^2_{L^2(M)} \leqq |\hat{Q}^\varepsilon f|^2_{L^2(\mathcal{O}(\varepsilon))} \leqq 2\varepsilon(1 + C\varepsilon)|f|^2_{L^2(M)} + \varepsilon^2 E(f, f; \mathbf{A}), \tag{70}$$

$$E^\varepsilon\big(\hat{Q}^\varepsilon f, \hat{Q}^\varepsilon f; \mathbb{A}\big) \leqq 2\varepsilon\big(1 + C\varepsilon\big)E(f, f; \mathbf{A}) + C\varepsilon^2 |f|^2_{L^2(M)}. \tag{71}$$

**Proof.** The first step is to estimate the Dirichlet integral of $u_v$; we use a test function $u$ defined by

$$u(y) = \begin{cases} f(v) \qquad \text{if } y \in \mathcal{U}(R\varepsilon, \varepsilon, v), \\ \big[(R+1)\varepsilon - \theta + (\theta - R\varepsilon)\exp\big(-i\int_0^s A_{\kappa,s}(\theta, s')\, ds'\big)\big] f(v) \\ \qquad\qquad \text{if } y \in \Psi_\kappa((R\varepsilon, (R+1)\varepsilon) \times (-\varepsilon, \varepsilon). \end{cases}$$

Since $|\operatorname{grad} u|$ is bounded by $C\varepsilon|f(v)|$ where $C$ depends only on the $C^1$ norm of $\mathbb{A}$ and the geometry of $M$, we obtain

$$\int_{\mathcal{U}_1} |\operatorname{grad} u_v|^2\, dx \leqq C\varepsilon^4 |f(v)|^2. \tag{72}$$

It is immediately obvious that

$$\int_{\mathcal{B}_j(t_1,\varepsilon)} \left|\hat{Q}^\varepsilon f\right|^2 dx \in 2\varepsilon[1 - C\varepsilon, 1 + C\varepsilon] \int_{a_j}^{b_j} \left|f_j(\theta)\right|^2 d\theta.$$

According to Theorem 3 and to the definition of $\hat{Q}^\varepsilon f$, we can see that

$$\int_{\mathcal{U}_1} |\hat{Q}^\varepsilon f|^2\, dx \leqq C\varepsilon^2\, |f(v)|^2\,.$$

But we have the classical estimate

$$|f(v)|^2 \leqq C\, |f|^2_{H^1(M)}\,,$$

and we can see also that

$$E(f, f) \leqq 2E(f, f; \mathbf{A}) + 2\, \|\mathbf{A}\|^2_{C^0}\, |f|^2_{L^2(M)}\,. \tag{73}$$

These relations imply that there exists $C$ such that

$$|f(v)|^2 \leqq C\big(E(f, f; \mathbf{A}) + |f|^2_{L^2(M)}\big), \tag{74}$$

and the number $C$ is bounded when the $C^0$ norm of $\mathbb{A}$ is bounded. All these relations show that (70) holds.

Let us turn now to the energy estimates for $\hat{Q}^\varepsilon f$. An immediate computation gives the following relation, which is valid on $\mathcal{B}_j(t_1, \varepsilon, v)$:

$$\left|\operatorname{grad} \hat{Q}^\varepsilon f\right|^2 dx = G^{-1/2} \left| \frac{\partial \hat{Q}^\varepsilon f}{\partial\theta} + i A_{j,\theta} \hat{Q}^\varepsilon f \right|^2 d\theta\, ds.$$

The expression between vertical bars on the right-hand side of the above equation is equal to

$$\left(\tilde{f}_j'(\theta) + i A_{j,\theta}(T_j(\theta), 0)\tilde{f}_j(\theta) T_j'(\theta) + i\zeta_j \tilde{f}_j(\theta)\right) \exp\left(-i \int_0^s A_{j,s}, (\theta, s')\, ds'\right),$$

where the remainder $\zeta_j$ is defined by

$$\begin{aligned}\zeta_j(\theta, s) = A_{j,\theta}(\theta, s) - A_{j,\theta}(\theta, 0) + A_{j,\theta}(\theta, 0)(1 - T_j'(\theta)) \\ + T_j'(\theta)\big[A_j(\theta, 0) - A_j(T_j(\theta), 0)\big].\end{aligned}$$

Since $\mathbb{A}$ is of class $C^1$, we can see immediately that

$$\left|\zeta_j(\theta, s)\right| \leqq C\varepsilon.$$

On the other hand,

$$\tilde{f}_j' + T_j' A_{j,\theta}(T_j, 0)\tilde{f}_j = T_j'\big[f_j' + iA_j f_j\big] \circ T_j.$$

Thanks to the inequality $2\,|\alpha\beta| \leqq \varepsilon\,|\alpha|^2 + \varepsilon^{-1}\,|\beta|^2$, we can see that

$$\begin{aligned}&\left|\tilde{f}_j'(\theta) + iA_{j,\theta}(T_j(\theta), 0)\tilde{f}_j(\theta)T_j'(\theta) + i\zeta_j\tilde{f}_j(\theta)\right|^2 \\ &\qquad\leqq (1+\varepsilon)\left|T_j'(\theta)\big[f_j' + iA_j f_j\big] \circ T_j(\theta)\right|^2 + C\varepsilon\left|\tilde{f}_j\right|^2.\end{aligned}$$

Therefore, we have the estimate

$$\begin{aligned}&\int_{\mathcal{B}_j(t_1,\varepsilon)} |\mathrm{grad}\,\hat{Q}^\varepsilon f|^2\,dx \\ &\qquad\leqq 2\varepsilon\big(1 + C\varepsilon\big)\int_{a_j}^{b_j}\left|f_j' + iA_j f_j\right|^2\,d\theta + C\varepsilon^2\int_{a_j}^{b_j}\left|f_j\right|^2\,d\theta.\end{aligned}$$

Estimates (72) and (74) allow us to conclude that (71) holds.

For the estimates in the opposite direction, we do not need to redefine $P^\varepsilon$: we just extend the definition given in (28) to the complex case by linearity.

**Theorem 7.** *There exists a constant $C$ such that for all $u \in H^1(\mathcal{O}(\varepsilon); \mathbb{C})$ the following estimates hold:*

$$E(P^\varepsilon u, P^\varepsilon u; \mathbf{A}) \leqq \frac{1 + C\sqrt{\varepsilon}}{2\varepsilon}E(u, u; \mathbb{A}) + C\varepsilon^{-1/2}\,|u|^2_{L^2(\mathcal{O}(\varepsilon))}, \tag{75}$$

$$\left|P^\varepsilon u\right|^2_{L^2(M)} \geqq \frac{1}{2\varepsilon}\Big(\big(1 - C\sqrt{\varepsilon}\big)\,|u|^2_{L^2(\mathcal{O}(\varepsilon))} - C\sqrt{\varepsilon}E^\varepsilon(u, u; \mathbb{A})\Big). \tag{76}$$

**Proof.** Estimate (76) is a direct consequence of estimate (31) and of (73). Let us prove now relation (75). We observe that

$$\begin{aligned}\frac{\partial N_j u}{\partial\theta} + iA_j N_j u = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\Big[\frac{\partial u \circ \Psi_j}{\partial\theta} + A_{j,\theta}u \circ \Psi_j\Big](\theta, s)\,ds \\ + \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\big[A_{j,\theta}(\theta, 0) - A_{j,\theta}(\theta, s)\big]u \circ \Psi_j(\theta, s)\,ds.\end{aligned}$$

We infer from the assumption that $\mathbb{A}$ is of class $C^1$, that

$$\left|\frac{\partial N_j u}{\partial \theta}\right|^2 \leqq \frac{1+\sqrt{\varepsilon}}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \left|\frac{\partial u \circ \Psi_j}{\partial \theta} + A_{j,\theta} u \circ \Psi_j\right|^2 d\theta + C\sqrt{\varepsilon} \int_{-\varepsilon}^{\varepsilon} \left|u \circ \Psi_j\right|^2 d\theta,$$

so that

$$\int_{a_j+t_2}^{b_j-t_2} \left|\frac{\partial N_j u}{\partial \theta}\right|^2 d\theta \leqq \frac{1+C\sqrt{\varepsilon}}{2\varepsilon} \int_{\mathcal{B}_j(t_2,\varepsilon)} |(\operatorname{grad} + i\mathbb{A})u|^2 \, dx + C\sqrt{\varepsilon} \int_{\mathcal{B}_j(t_2,\varepsilon)} |u|^2 \, dx.$$

The next terms to estimate are

$$\sum_{\kappa \in J(v)} \int_{t_1}^{t_2} \left| \frac{\partial N_\kappa u}{\partial \theta} + i A_\kappa N_\kappa u - \frac{\hat{c} - N_\kappa u(t_1)}{t_2 - t_1} - i A_\kappa \frac{(t_2-\theta)\big(\hat{c} - N_\kappa u(t_1)\big)}{t_2 - t_1} \right|^2 d\theta. \quad (77)$$

We estimate the term $\partial N_\kappa u / \partial \theta + i A_\kappa N_\kappa u$ as above, and we use the classical inequality

$$|\alpha + \beta + \gamma|^2 \leqq (1 + C\sqrt{\varepsilon})|\alpha|^2 + C\big(|\beta|^2 + |\gamma|^2\big)/\sqrt{\varepsilon}.$$

We find that the expression (77) is estimated by

$$\sum_{\kappa \in J(v)} \frac{1+C\sqrt{\varepsilon}}{2\varepsilon} \int_{t_1}^{t_2} \int_{-\varepsilon}^{\varepsilon} \left|\frac{\partial u \circ \Psi_\kappa}{\partial \theta} + i A_{\kappa,\theta} u \circ \Psi_\kappa\right|^2 d\theta \, ds + C\sqrt{\varepsilon} \int_{\mathcal{O}_\kappa} |u|^2 \, dx + \frac{C}{\sqrt{\varepsilon}} \left|\hat{c} - N_\kappa u(t_1)\right|^2.$$

The last term to estimate is the contribution

$$\sum_{\kappa \in J(v)} \int_0^{t_1} \left|i A_\kappa \hat{c}\right|^2 d\theta \leqq C \left|\hat{c}\right|^2 \varepsilon.$$

We recall that, from (25),

$$\left|\hat{c} - N_\kappa u(t_1)\right|^2 \leqq C \int_{\mathcal{U}_1} |\operatorname{grad} u|^2 \, dx,$$

and from (46),

$$\left|\hat{c}\right|^2 \leqq C\varepsilon^{-1} \int_{\mathcal{O}_\kappa} \big(|\operatorname{grad} u|^2 + |u|^2\big) \, dx,$$

and we find that

$$\begin{aligned} &E(P^\varepsilon u, P^\varepsilon u; \mathbf{A}) \\ &\quad \leqq \frac{1+C\sqrt{\varepsilon}}{2\varepsilon} \sum_{j=1}^{|\mathcal{E}|} \int_{\mathcal{B}_j(t_1,\varepsilon)} |(\operatorname{grad} + i\mathbb{A})u|^2 \, dx \\ &\qquad + C\sqrt{\varepsilon} \sum_{j=1}^{|\mathcal{E}|} \int_{\mathcal{B}_j(t_1,\varepsilon)} |u|^2 \, dx + C\varepsilon^{-1/2} \sum_{v\in\mathcal{V}} \int_{\mathcal{U}(t_1,\varepsilon,v)} |\operatorname{grad} u|^2 \, dx \\ &\qquad + C \sum_{v\in\mathcal{V}} \sum_{\kappa\in J(v)} \int_{\mathcal{O}_\kappa} \left(|\operatorname{grad} u|^2 + |u|^2\right) dx. \end{aligned}$$

But

$$|\operatorname{grad} u|^2 \leqq 2\,|(\operatorname{grad} + i\mathbb{A})u|^2 + 2\,\|\mathbb{A}\|_{C^0}\, |u|^2 .$$

We conclude that (75) holds.

These theorems suffice to compare the spectra of $S^\varepsilon[\mathbb{A}]$ and $S[\mathbf{A}]$; we may even replace $\mathbb{A}$ by a vector potential $\mathbb{A}^\varepsilon$ depending on $\varepsilon$. If we define

$$\delta(\varepsilon) = \|\mathbf{A}^\varepsilon - \mathbf{A}^0\|_{C^0},$$

we can state the following result:

**Theorem 8.** *Assume that the norm of $\mathbb{A}^\varepsilon$ is bounded in $C^1(\mathcal{O}(\varepsilon))$ independently of $\varepsilon$; then, if $\delta(\varepsilon)$ tends to 0 as $\varepsilon$ tends to 0, the spectrum of $S^\varepsilon[\mathbb{A}^\varepsilon]$ tends to the spectrum of $S[\mathbf{A}^0]$ in the following sense. There exists $C$ such that for all integer $p$, the following estimate holds:*

$$\lambda_p(S^\varepsilon[\mathbb{A}^\varepsilon]) \leqq \frac{(1+C\varepsilon)\big[(1+C\delta(\varepsilon))\lambda_p(S[\mathbf{A}^0] + C\delta(\varepsilon)\big] + C\varepsilon}{1-C\varepsilon}; \tag{78}$$

*for all integer $p$, there exists $\varepsilon_p$ such that for all $\varepsilon \leqq \varepsilon_p$,*

$$\lambda_p(S[\mathbf{A}^0]) \leqq (1+C\delta(\varepsilon)) \frac{(1+C\sqrt{\varepsilon})\lambda_p(S^\varepsilon[\mathbb{A}^\varepsilon]) + C\sqrt{\varepsilon}}{1-C\sqrt{\varepsilon} - C\sqrt{\varepsilon}\lambda_p(S^\varepsilon[\mathbb{A}^\varepsilon])}. \tag{79}$$

**Proof.** The proof of this statement is straightforward, We compare first the spectra of $S[\mathbf{A}^\varepsilon]$ and $S[\mathbf{A}^0]$: we have the inequality

$$\begin{aligned} &\left| \int_M \left( \left|df + if\mathbf{A}^\varepsilon\right|^2 - \left|df + if\mathbf{A}^0\right|^2 \right) d\mathrm{vol}_M \right| \\ &\quad \leqq C\delta(\varepsilon) \int_M \left(|df|^2 + |f|^2\right) d\mathrm{vol}_M \\ &\quad \leqq C\delta(\varepsilon) \bigg( \int_M |f|^2 \, d\mathrm{vol}_M \\ &\qquad + \min\left( \int_M \left|df + i\mathbf{A}^0 f\right|^2 d\mathrm{vol}_M, \int_M \left|df + i\mathbf{A}^\varepsilon f\right|^2 d\mathrm{vol}_M \right) \bigg). \end{aligned}$$

In the statement of Lemma 5, the Hilbert spaces $H_0$ and $H_1$ are taken equal to $L^2(M)$, and $S$ will be the identity. We infer immediately that

$$\lambda_p(S[\mathbf{A}^\varepsilon]) \leqq (1 + C\delta(\varepsilon))\lambda_p(S[\mathbf{A}^0]) + C\delta(\varepsilon), \tag{80}$$

$$\lambda_p(S[\mathbf{A}^0]) \leqq (1 + C\delta(\varepsilon))\lambda_p(S[\mathbf{A}^\varepsilon]) + C\delta(\varepsilon). \tag{81}$$

We use now the argument of the proof of Theorem 5: we have thus the estimate

$$\lambda_p(S^\varepsilon[\mathbb{A}^\varepsilon]) \leqq \frac{(1 + C\varepsilon)\lambda_p(S[\mathbf{A}^\varepsilon]) + C\varepsilon}{1 - C\varepsilon},$$

which, with (80) implies (78).

Conversely, relation (63) still holds thanks to the second inequality of (70), and thus we infer from Lemma 5 and relations (75) and (76) that

$$\lambda_p(S[\mathbf{A}^\varepsilon]) \leqq \frac{(1 + C\sqrt{\varepsilon})\lambda_p(S^\varepsilon[\mathbb{A}^\varepsilon]) + C\sqrt{\varepsilon}}{1 - C\sqrt{\varepsilon} - C\sqrt{\varepsilon}\lambda_p(S^\varepsilon[\mathbb{A}^\varepsilon])},$$

and the conclusion (79) follows with the help of (81).

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