Arch. Rational Mech. Anal. 159 (2001) 137–177
Digital Object Identifier (DOI) 10.1007/s002050100151

$W^{2,p}$ *Estimates for the Parabolic Monge-Ampère Equation*

CRISTIAN E. GUTIÉRREZ & QINGBO HUANG

In memory of Eugene B. Fabes

Communicated by L. C. EVANS

Abstract

When u is a solution to the equation $-u_t \det D_x^2 u = f$ with f positive, continuous, and f_t satisfying certain growth conditions, we establish estimates in L^∞ for u_t and show that $D_x^2 u$ satisfies uniform interior estimates in L^p for $0 < p < \infty$.

1. Introduction

The parabolic Monge-Ampère operator considered in this paper is

$$\mathcal{M}u = -u_t \det D^2 u, \tag{1.1}$$

where $u = u(x, t)$ is convex in x and nonincreasing in t, $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, and $D^2 u = D_x^2 u$ denotes the Hessian of u with respect to the variable x. This operator is relevant in the study of deformation of surfaces by Gauss-Kronecker curvature [Fir74, Tso85a], and in a maximum principle for parabolic equations [Tso85b]. Together with (1.1), KRYLOV [Kry76] introduced other parabolic versions of the elliptic Monge-Ampère operator, see [Lie96, pp. 406–416] for a complete description and related results.

Our purpose in this paper is to establish that solutions u to $\mathcal{M}u = f$ with f positive, continuous, and f_t satisfying certain growth conditions, have second derivatives in L^p for $0 < p < \infty$. This is the main result in this paper and is precisely stated in Section 2, Theorem 2.1. These types of interior estimates have been recently established by CAFFARELLI [Caf90a] for the elliptic Monge-Ampère equation $\det D^2 u = f$, and therefore we extend Caffarelli's result to the parabolic case. The origin of these estimates goes back to POGORELOV [Pog71], who proved that convex solutions to $\det D^2 u = 1$ on a bounded convex domain Ω with $u = 0$ on $\partial\Omega$ satisfy the L^∞ estimate

$$C_1(\Omega', \Omega)\, I \leqq D^2 u(x) \leqq C_2(\Omega', \Omega)\, I, \tag{1.2}$$

for $x \in \Omega'$, where Ω' is a convex domain with closure contained in Ω, I is the identity matrix, and C_i are positive constants depending only on the domains. The estimate (1.2) plays an important role in the fundamental estimates proved by Caffarelli, and the crucial estimate that leads to (1.2) is that one can bound the Hessian of u by means of its gradient, [Pog71, Theorem 2]. In [GH98], the parabolic analogue of [Pog71, Theorem 2] was used to establish a generalization of a celebrated Theorem by CALABI [Cal58]. Such extension plays an important role in the present paper, see Theorem 5.2 below. All these results use the recent theory for cross-sections of solutions to the Monge-Ampère equation developed in references [Caf90a, Caf91, CG97, CG96, GH00, Hua99, Gut]. One of the aims of this paper is to extend several results of this theory to the parabolic setting and the main difficulty for this extension is due to the presence of the time derivative in the definition of $\mathcal{M}$. However, under some conditions on the right-hand side, f, we prove that u_t is bounded away from zero and $-\infty$. This permit us to introduce an appropriate notion of parabolic cross-section, defined by (4.2), that has properties which lead to the desired result.

We mention that $C^{2+\alpha,1+\alpha/2}$ estimates for solutions to $\mathcal{M}u = f$ were obtained in [WW92] when f is Lipschitz continuous in x and t.

Throughout the paper we work with classical solutions but all the estimates are independent of the smoothness and depend only on the structural constants.

Each section in the paper contains results that are interesting in themselves. The organization is as follows. We begin in Section 2 introducing some notation, definitions and the statement of the main result. In Section 3, we show that under certain conditions on f, we can bound u_t in the interior of the domain by the bounds for the data on the parabolic boundary. This holds, for example, if f_t is of bounded mean oscillation. Section 4 contains the proofs of the properties of the parabolic cross-sections needed in Section 6. Section 5 contains the proof of an approximation theorem crucial for the proof of the $W^{2,p}$ estimates. Section 6 contains the proof of the $W^{2,p}$ estimates. Finally, the appendix, Section 7, contains the regularity properties of the parabolic convex envelope.

2. Notation, definitions and statement of the main result

If $Q \subset \mathbb{R}^{n+1}$ and $t \in \mathbb{R}$, then we define

$$Q(t) = \{x : (x,t) \in Q\}. \tag{2.1}$$

Let $Q \subset \mathbb{R}^{n+1}$ be a bounded set and $t_0 = \inf\{t : Q(t) \neq \emptyset\}$. The *parabolic boundary* of the bounded domain Q is defined by

$$\partial_p Q = \left(\overline{Q}(t_0) \times \{t_0\}\right) \cup \bigcup_{t\in\mathbb{R}} \left(\partial Q(t) \times \{t\}\right),$$

where $\overline{Q}$ denotes the closure of Q and $\partial Q(t)$ denotes the boundary of $Q(t)$. We say that the set $Q \subset \mathbb{R}^{n+1}$ is a *bowl-shaped domain* if $Q(t)$ is convex for each t and $Q(t_1) \subset Q(t_2)$ for $t_1 \leqq t_2$.

Let Q be a bowl-shaped domain in $\mathbb{R}^{n+1}$, and $u \in C(\overline{Q})$. A function $u(x,t)$ is *parabolically convex* in Q or *p-convex* if it is convex in x and nonincreasing in t.

Given $z_0 = (x_0, t_0) \in Q$, $\ell_{z_0}(x)$ is a *supporting affine function, or supporting hyperplane* for $u(\cdot, t_0)$ at $x = x_0$, if $\ell_{z_0}(x) = u(x_0, t_0) + p \cdot (x - x_0)$ and $u(x, t_0) \geqq \ell_{z_0}(x)$ for all $x \in Q(t_0)$. When u is regular we have $p = Du(x_0, t_0)$.

Given $h > 0$, we define

$$Q_h(z_0) = Q_h(u; z_0) = \{(x,t) : u(x,t) \leqq \ell_{z_0}(x) + h \text{ and } t \leqq t_0\}, \tag{2.2}$$

and

$$S_h(x_0|t_0) = S_h(u; x_0|t_0) = \{x : u(x, t_0) \leqq \ell_{z_0}(x) + h\}. \tag{2.3}$$

If $Q \subset \mathbb{R}^{n+1}$ is an open bounded bowl-shaped domain and $u : Q \to \mathbb{R}$ is continuous, then the *parabolic normal mapping of* u is the set-valued function $\mathcal{P}_u : Q \to \{E : E \subset \mathbb{R}^{n+1}\}$ defined by

$$\begin{aligned} \mathcal{P}_u(x_0, t_0) = \{(p, h) : u(x,t) \geqq u(x_0, t_0) + p \cdot (x - x_0), \\ \forall x \in Q(t), \text{ with } t \leqq t_0,\ h = p \cdot x_0 - u(x_0, t_0)\}, \end{aligned}$$

where $Q(t) = \{x : (x,t) \in Q\}$. If $E \subset Q$, then $\mathcal{P}_u(E) = \cup_{(x,t)\in E} \mathcal{P}_u(x,t)$.

Given a bounded convex domain $\Omega \subset \mathbb{R}^n$ with non-empty interior, let E be the ellipsoid of minimum volume containing Ω with center at the center of mass of Ω. Then there exists an affine transformation T such that $B_{\alpha_n}(0) \subset T(\Omega) \subset B_1(0)$ with $\alpha_n = n^{-3/2}$; see [Pog78, p. 90].

The main results in this paper can be summarized in the following theorem. The proof of conclusion (A) is given in Section 3, Theorem 3.1; and the proof of (B) is given at the end of Section 6.

Theorem 2.1. *Let u be a parabolically convex solution to $\mathcal{M}u = f$ in the cylinder $Q = \Omega \times (0, T]$ with $u = \phi$ on $\partial_p Q$. Suppose that*

(1) $B_{\alpha_n}(0) \subset \Omega \subset B_1(0)$ *convex*, $\partial\Omega \in C^{1,\alpha}$ *with* $\alpha > 1 - \frac{2}{n}$;
(2) $0 < \lambda \leqq f \leqq \Lambda$, $f \in C(\bar{Q})$, $f_t \in L^{n+1}(Q)$, *and* $\exp(A(-f_t)^+) \in L^1(Q)$ *for some* $A > 0$; *and*
(3) $\phi \in C^{2,1}(\bar{Q})$ *satisfying* $-c_2 \leqq \phi_t \leqq -c_1$ *and* $C_1 I \leqq D^2\phi \leqq C_2 I$ *in* Q, *where* c_i *and* C_i *are positive constants*, $i = 1, 2$.

Then:

(A) *There exist positive constants M_1 and M_2, depending only on the constants above and $\|f_t\|_{L^{n+1}(Q)}$, such that*

$$-M_1 \leqq u_t \leqq -M_2, \qquad \text{in } Q.$$

(B) *For each* $0 < p < \infty$ *and* $h > 0$ *we have*

$$\iint_{\Omega_h \times (h,T]} \|D_x^2 u(x,t)\|^p \, dx\, dt \leqq C,$$

where $\Omega_h = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > h\}$, *and C is a constant that depends only on p, h, T, and the parameters in* (1), (2), *and* (3).

3. Propagation of the bounds for u_t from the boundary to the interior

Let Ω be a bounded convex domain in $\mathbb{R}^n$, $Q = \Omega\times(0,T)$, and u a parabolically convex function solution to the problem:

$$-u_t \det D^2u = f \qquad \text{in } Q, \tag{3.1}$$

$$u = \phi \qquad \text{on } \partial_p Q. \tag{3.2}$$

The main result of this section is Theorem 3.1. We first show that if ϕ_t is bounded on $\partial_p Q$ then the same is true for u_t. This implies, together with our assumptions (3.5) and (3.6) on f_t, that u_t is bounded away from zero and $-\infty$ in the interior of Q.

Lemma 3.1. *Let ϕ be a function defined on $\bar{Q}$ such that there exist negative constants m_1 and m_2 so that*

$$m_1 \leqq \phi_t \leqq m_2 \qquad \text{on } \partial\Omega\times[0,T]. \tag{3.3}$$

Assume in addition that $\phi(x,0)$ is strictly convex in Ω, and $0<\lambda\leqq f(x,t)\leqq\Lambda$ in $\bar{Q}$. Then there exist negative constants m_1' and m_2' depending only on m_1, m_2, λ and Λ such that, if u solves (3.1) *and* (3.2)*, then*

$$m_1' \leqq u_t \leqq m_2' \qquad \text{on } \partial_p Q. \tag{3.4}$$

Proof. We have $u_t = \phi_t$ on $\partial\Omega\times[0,T]$. Also $u(x,0) = \phi(x,0)$ on $\bar{\Omega}\times\{0\}$. Hence $\det D^2u(x,0) = \det D^2\phi(x,0)$. So $\lambda_1 \leqq \det D^2u(x,0) \leqq \lambda_2$ and, using (3.1), we can complete the proof.

The following Theorem shows global bounds for u_t in Q. Notice that if, for example, $f_t \in BMO$, then conditions (3.5) and (3.6) hold.

Theorem 3.1. *Assume the hypotheses of Lemma 3.1 and that there exist positive constants A and B such that*

$$f_t \in L^{n+1}(Q) \tag{3.5}$$

$$\iint_Q e^{A(-f_t)^+}\,dx\,dt \leqq B \qquad (a^+ = \max\{a,0\}). \tag{3.6}$$

Then there exist negative constants M_1 and M_2, depending only on the constants above and $\|f_t\|_{L^{n+1}(Q)}$, such that

$$M_1 \leqq u_t \leqq M_2 \qquad \text{in } Q. \tag{3.7}$$

Proof. These inequalities will be proved by using auxiliary functions and the following Aleksandrov-Bakelman-Pucci type maximum principle proved by Tso [Tso85b]: if u is a smooth function defined on the cylinder Q, then

$$\sup_Q u \leqq \sup_{\partial_p Q} u + C\left(\iint_{\Gamma(u)} |u_t \det D^2u|\,dxdt\right)^{1/(n+1)}, \tag{3.8}$$

where C is a constant depending only on n, T and the diameter of Ω; and $\Gamma(u)$ is the set $\Gamma(u) = \{(x,t) \in Q : u_t(x,t) \geqq 0, \text{ and } D^2u(x,t) \leqq 0\}$. We consider the linearized parabolic Monge-Ampère operator associated with u and defined by

$$L(v) = -\frac{1}{u_t}\, v_t - \operatorname{trace}\left(\left(D^2u\right)^{-1}\, D^2v\right).$$

Differentiating (3.1) with respect to t yields

$$-(u_t)_t \det D^2u - u_t \operatorname{trace}\left((D^2u)^{-1} (\det D^2u)\, D^2(u_t)\right) = f_t.$$

Consequently, $L(u_t) = -\dfrac{f_t}{f}$, and if we let

$$v(x,t) = (t+M)^k\, u_t(x,t), \tag{3.9}$$

with $M > 0$, we then have

$$L(v) = \left(-\frac{k}{M+t} - \frac{f_t}{f}\right)(t+M)^k. \tag{3.10}$$

We first estimate the $\inf_Q u_t$. Applying (3.8) to $-u_t$ and noting that $(u_t)_t \leqq 0$ and $D^2u_t \geqq 0$ on $\Gamma(-u_t)$, we get

$$\begin{aligned}
&\inf_Q u_t - \inf_{\partial_p Q} u_t \\
&\geqq -C\left(\iint_{\Gamma(-u_t)} -(u_t)_t \det D^2(u_t)\, dx\, dt\right)^{1/(n+1)} \\
&= -C\left(\iint_{\Gamma(-u_t)} \frac{-(u_t)_t}{-u_t} (\det D^2u)^{-1} \det D^2(u_t)\, f\, dx\, dt\right)^{1/(n+1)} \quad \text{by (3.1)} \\
&\geqq -C\left(\iint_Q \left|\frac{(u_t)_t}{u_t} + \operatorname{trace}((D^2u)^{-1}D^2(u_t))\right|^{n+1} f\, dx\, dt\right)^{1/(n+1)} \quad \text{by (5.11)} \\
&= -C\left(\iint_Q |-L(u_t)|^{n+1} f\, dx\, dt\right)^{1/(n+1)} \\
&= -C\left(\iint_Q \left|\frac{f_t}{f}\right|^{n+1} f\, dx\, dt\right)^{1/(n+1)} \geqq -C\,\|f_t\|_{L^{n+1}(Q)},
\end{aligned}$$

and then from (3.4) the first inequality in (3.7) follows.

We now estimate $\sup_Q u_t$. Applying (3.8) to v defined in (3.9), we get

$$\begin{aligned}
&\sup_Q v - \sup_{\partial_p Q} v \\
&\qquad \leqq C \left(\iint_{\Gamma(v)} -(-v)_t \,\det D^2(-v)\, dx\, dt \right)^{1/(n+1)} \\
&\qquad = C \left(\iint_{\Gamma(v)} -\frac{(-v)_t}{-u_t} \frac{\det D^2(-v)}{\det D^2 u} \, f\, dx\, dt \right)^{1/(n+1)} \\
&\qquad \leqq C \left(\iint_{\Gamma(v)} \left(\frac{v_t}{-u_t} + \operatorname{trace}((D^2u)^{-1} D^2(-v)) \right)^{n+1} f\, dx\, dt \right)^{1/(n+1)} \\
&\qquad \leqq C \left(\iint_{Q} \left((Lv)^+\right)^{n+1} f\, dx\, dt \right)^{1/(n+1)},
\end{aligned}$$

since $\Gamma(v) \subset \{(x,t) : L(v)(x,t) \geqq 0\}$. Hence from (3.10) we obtain

$$\begin{aligned}
\sup_Q v \leqq &\sup_{\partial_p Q} v \\
&+ C \left(\iint_Q \left(\left(-\frac{k}{M+t} - \frac{f_t}{f} \right)^+ \right)^{n+1} (t+M)^{k(n+1)} f\, dx\, dt \right)^{1/(n+1)} \\
\leqq &\sup_{\partial_p Q} v + C \left(\iint_Q \left(\left(-\frac{\lambda\, k}{M+T} - f_t \right)^+ \right)^{n+1} dx\, dt \right)^{1/(n+1)} (T+M)^k .
\end{aligned}$$

We have

$$\begin{aligned}
&\iint_Q \left(\left(-\frac{\lambda\, k}{M+T} - f_t \right)^+ \right)^{n+1} dxdt \\
&\quad = \sum_{j=0}^{\infty} \iint_{(j+1)\frac{\lambda k}{M+T} \leqq -f_t < (j+2)\frac{\lambda k}{M+T}} \left(\left(-\frac{\lambda\, k}{M+T} - f_t \right)^+ \right)^{n+1} dxdt \\
&\quad \leqq \sum_{j=0}^{\infty} \left((j+1) \frac{\lambda\, k}{M+T} \right)^{n+1} \left| \left\{ (x,t) \in Q : -f_t(x,t) > (j+1)\frac{\lambda k}{M+T} \right\} \right| \\
&\quad \leqq \sum_{j=0}^{\infty} \left((j+1) \frac{\lambda\, k}{M+T} \right)^{n+1} \frac{B}{e^{A(j+1)\frac{\lambda\, k}{M+T}}} \qquad \text{by (3.6)} \\
&\quad \leqq C\, e^{-A\frac{\lambda\, k}{M+T}} \left(\frac{\lambda k}{M+T} \right)^{n+1}
\end{aligned}$$

if $\dfrac{A\lambda k}{M+T} \geqq 1$ with $C = \alpha\, B$, where α is a universal constant. From (3.4) and (3.9) it then follows that

$$\sup_Q u_t \leqq -C_0 \left(1 + \frac{T}{M} \right)^{-k} + C\, \frac{\lambda k}{M+T}\, e^{-k\frac{A\lambda}{(M+T)(n+1)}}$$

for $k \geqq \frac{M+T}{A\lambda}$ with $C_0 > 0$, and with C depending only on n, diam(Ω), λ, Λ, A, and B. Now let $M = 1$. If $T \leqq T_0 = \min\{1, \frac{A\lambda}{4(n+1)}\}$, we then have

$$\begin{aligned}\sup_Q u_t &\leqq -C_0\,(1+T)^{-k}\left(1 - \frac{C}{C_0}\,\frac{\lambda k}{(1+T)}\,e^{-k\frac{A\lambda}{(1+T)(n+1)}+k\ln(1+T)}\right) \qquad (3.11)\\ &\leqq -C_0\,(1+T)^{-k}\left(1 - \frac{C\lambda k}{C_0(1+T)}\,e^{-k\frac{A\lambda}{(1+T)(n+1)}+kT}\right)\\ &\leqq -C_0\,(1+T)^{-k}\left(1 - \frac{C\lambda k}{C_0}\,e^{-k\frac{A\lambda}{4(n+1)}}\right) \leqq -C_1\end{aligned}$$

for k large depending only on C, C_0, A, λ, n, and T_0.

For general T, we cut $Q = \Omega \times (0, T)$ into a stack of thin slices $Q = \bigcup_{i=0}^{N} \Omega \times (iT_0, (i+1)T_0]$ with $N \leqq T/T_0 \leqq N+1$, and apply (3.11) on each slice. Indeed, applying (3.11) to $u(x, t+T_0)$ yields $\sup_{\Omega\times(T_0,2T_0]} u_t \leqq -C_2$. Continuing in this way, $\sup_{\Omega\times((N-1)T_0,NT_0]} u_t \leqq -C_N$. Therefore the estimate for u_t in Q follows.

To establish the L^p estimates of $D_x^2 u$ from now onwards we may assume, by Theorem 3.1, that $m_1 < -u_t < m_2$.

4. Properties of the parabolic sections

Our purpose here is to define a notion of parabolic section that is suitable for establishing the $W^{2,p}$ estimates. We could attempt to take as a notion of parabolic section of u the one given by (2.2). But the problem with the sets $Q_h(z_0)$ is that they do not satisfy the engulfing property of Lemma 4.3 and the shrinking property given in Lemma 4.4, and therefore, the type of decomposition given by Theorem 4.1 might fail to hold in terms of those sets. This can be fixed by introducing a new definition of parabolic section given by (4.2). These new sections are monotone in h, and satisfy the geometric properties needed to establish a Calderón-Zygmund type decomposition, Theorem 4.1, that will be crucial in Section 6 when we prove the power decay given by Proposition 6.2.

Let $\delta > 0$ be a small number that will be chosen in a moment. Let us consider the time $t_0 + \delta\, h$. Since $u(x, t)$ is nonincreasing in t we have $u(x, t_0 + \delta\, h) \leqq u(x, t_0)$ for all x. Let us look at the set

$$S = \{x : u(x, t_0 + \delta\, h) \leqq \ell_{z_0}(x)\}.$$

This set is non-empty because $x_0 \in S$. Consider

$$\Delta = \min_x \{u(x, t_0 + \delta\, h) - \ell_{z_0}(x)\}.$$

Notice that $\Delta \leqq 0$. Let $(x_0)^h_{\min}$ be the point where the minimum is attained, that is,

$$\Delta = u((x_0)^h_{\min}, t_0 + \delta\, h) - \ell_{z_0}((x_0)^h_{\min}),$$

and therefore

$$u(x, t_0 + \delta h) \geqq \ell_{z_0}(x) + u((x_0)^h_{\min}, t_0 + \delta h) - \ell_{z_0}((x_0)^h_{\min}) := \ell^*_{z_0}(x) \quad (4.1)$$

for all x. Since $u((x_0)^h_{\min}, t_0 + \delta h) = \ell^*_{z_0}((x_0)^h_{\min})$, it follows that $\ell^*_{z_0}(x)$ is a supporting affine function for $u(\cdot, t_0 + \delta h)$ at $x = (x_0)^h_{\min}$.

We define the section

$$Q^*_h(z_0) = \{(x, t) : u(x, t) \leqq \ell^*_{z_0}(x) + h \text{ and } t \leqq t_0 + \delta h\}, \quad (4.2)$$

and notice that

$$Q^*_h(z_0) = Q_h((x_0)^h_{\min}, t_0 + \delta h), \quad (4.3)$$

that is, each Q^*_h is a Q_h given by (2.2) at another point with t coordinate slightly larger than t_0. In the case where $t_0 + \delta h > T$, we replace $t_0 + \delta h$ by T.

Remark 4.1 *(*Location of $(x_0)^h_{\min}$*)*. We have that $((x_0)^h_{\min}, t_0) \in Q_{m_1 \delta h}(x_0, t_0)$, actually $(x_0)^h_{\min} \in S_{m_1 \delta h}(x_0|t_0)$, where $-u_t \leqq m_1$.

Indeed, we write

$$\begin{aligned} &u((x_0)^h_{\min}, t_0) - \ell_{z_0}((x_0)^h_{\min}) \\ &\quad = u((x_0)^h_{\min}, t_0) - u((x_0)^h_{\min}, t_0 + \delta h) + u((x_0)^h_{\min}, t_0 + \delta h) - \ell_{z_0}((x_0)^h_{\min}) \\ &\quad \leqq u((x_0)^h_{\min}, t_0) - u((x_0)^h_{\min}, t_0 + \delta h) \\ &\quad = u_t((x_0)^h_{\min}, \tau)(-\delta h) \leqq m_1 \delta h. \end{aligned}$$

We now recall the notion of normalization of the section $Q_h(x_0, t_0)$. Consider $S_h(x_0|t_0)$ given by (2.3), and let T be the affine transformation that normalizes $S_h(x_0|t_0)$, that is

$$B_{\alpha_n}(0) \subset T\,(S_h(x_0|t_0)) \subset B_1(0),$$

and define the transformation

$$T_p(x, t) = \left(Tx, \frac{t - t_0}{h}\right),$$

with its corresponding inverse

$$T_p^{-1}(y, s) = \left(T^{-1}y, t_0 + s\,h\right).$$

We let

$$v(y, s) = u(T_p^{-1}(y, s)) = u(T^{-1}y, t_0 + s\,h).$$

If $\bar{\ell}_{(Tx_0,0)}(y)$ is a supporting affine function for $v(\cdot, 0)$ at $y = Tx_0$, then we have

$$\bar{\ell}_{(Tx_0,0)}(y) = v(Tx_0, 0) + Dv(Tx_0, 0) \cdot (y - Tx_0).$$

This follows from the fact that $Dv(y, s) = (T^{-1})^t (Du)(T^{-1}y, t_0 + s\,h)$. If we define

$$Q_h(u; (x_0, t_0)) = \{(x, t) : u(x, t) \leqq \ell_{z_0}(x) + h \text{ and } t \leqq t_0\},$$

then we have the following formula:

$$T_p\,(Q_h(u; (x_0, t_0))) = Q_h(v; (Tx_0, 0)).$$

Lemma 4.1. *There exists $\delta > 0$ sufficiently small depending only on m_1, the lower bound of u_t, that is $u_t \geqq -m_1$, such that*

(1) *if $h \leqq H$, then $Q_h^*(z_0) \subset Q_H^*(z_0)$;*
(2) $Q_{h/2}(z_0) \subset Q_h^*(z_0)$.

Proof. We begin with (1). Let $(x,t) \in Q_h^*(z_0)$. Then $u(x,t) \leqq \ell_{z_0}^*(x) + h = \ell_{z_0}(x) + \Delta + h$ and $t \leqq t_0 + \delta h$. Hence

$$\begin{aligned} & u(x,t) - \ell_{z_0}(x) \\ & \quad = u((x_0)_{\min}^h, t_0 + \delta h) - \ell_{z_0}((x_0)_{\min}^h) + h \\ & \quad \leqq u((x_0)_{\min}^H, t_0 + \delta h) - \ell_{z_0}((x_0)_{\min}^H) + h \\ & \quad = u_t((x_0)_{\min}^H, \tau)\,(h-H)\,\delta + h + u((x_0)_{\min}^H, t_0 + \delta H) - \ell_{z_0}((x_0)_{\min}^H). \end{aligned}$$

Since $-m_1 \leqq u_t$ and $h - H < 0$, it follows that

$$u(x,t) - \ell_{z_0}(x) \leqq -m_1\,(h-H)\,\delta + h + u((x_0)_{\min}^H, t_0 + \delta H) - \ell_{z_0}((x_0)_{\min}^H).$$

If $m_1\,\delta \leqq 1$, then we obtain

$$u(x,t) - \ell_{z_0}(x) \leqq u((x_0)_{\min}^H, t_0 + \delta H) - \ell_{z_0}((x_0)_{\min}^H) + H.$$

Therefore $u(x,t) \leqq \ell_{z_0}(x) + u((x_0)_{\min}^H, t_0 + \delta H) - \ell_{z_0}((x_0)_{\min}^H) + H = \ell_{z_0}^{**}(x) + H$, that is, $(x,t) \in Q_H^*(z_0)$.

We now prove (2). Let $(x,t) \in Q_{h/2}(z_0)$. We have $t \leqq t_0$ and

$$\begin{aligned} & u(x,t) - \ell_{z_0}(x) \leqq h/2 \\ & \quad = (u - \ell_{z_0})((x_0)_{\min}^h, t_0 + \delta h) - (u - \ell_{z_0})((x_0)_{\min}^h, t_0 + \delta h) + \\ & \qquad\qquad (u - \ell_{z_0})((x_0)_{\min}^h, t_0) - (u - \ell_{z_0})((x_0)_{\min}^h, t_0) + \frac{h}{2} \\ & \quad \leqq (u - \ell_{z_0})((x_0)_{\min}^h, t_0 + \delta h) \\ & \qquad\qquad - \left[(u - \ell_{z_0})((x_0)_{\min}^h, t_0 + \delta h) - (u - \ell_{z_0})((x_0)_{\min}^h, t_0)\right] + \frac{h}{2} \\ & \quad \leqq (u - \ell_{z_0})((x_0)_{\min}^h, t_0 + \delta h) + m_1\,\delta\,h + \frac{h}{2}. \end{aligned}$$

If we now choose δ so that $m_1\,\delta < \frac{1}{2}$, we are done.

4.1. Engulfing property at different times

Consider $S_{2h}(x_0|t_0)$ and let T be the affine transformation normalizing $S_{2h}(x_0|t_0)$, and

$$T_p(x,t) = \left(Tx, \frac{t - t_0}{h}\right).$$

Let $\ell_{z_0}(x) = u(x_0,t_0) + Du(x_0,t_0)\cdot(x - x_0)$ and the function

$$v(y,s) = \frac{1}{h}(u - \ell_{z_0})(T^{-1}y, t_0 + s\,h).$$

Then $T_p\,(Q_{2h}(u;z_0)) = Q_2(v;(Tx_0,0))$ is normalized. We have $\min_{Q_2} v = v(Tx_0,0) = 0$, $v = 2$ on $\partial_p Q$ and $C^{-1} \leqq |v_t| \leqq C$.

Let $z_1 = (x_1,t_1) \in Q_1(v;(Tx_0,0))$, then

$$|Dv(x_1,t_1)| \leqq \frac{2}{\operatorname{dist}((x_1,t_1),\partial Q_2(t_1))} \leqq C$$

by [GH98, Theorem 2.1]. We claim that

$$S_1(v;x_1|t_1) \subset S_\theta(v;x_0|t_0).$$

Indeed, if $x \in S_1(v;x_1|t_1)$, then $v(x,t_1) \leqq v(x_1,t_1) + Dv(x_1,t_1)\cdot(x-x_1)+1 \leqq 2+C = \theta$. Since $t_1 < t_0$ and $v(x,\cdot)$ is monotonic in t, we have $v(x,t_0) \leqq \theta$ and hence $x \in S_\theta(v;x_0|t_0)$ because the supporting hyperplane defining $S_\theta(v;x_0|t_0)$ equals zero. Conversely, we show that

$$S_1(v;x_0|t_0) \subset S_\theta(v;x_1|t_1).$$

Because, if $x \in S_1(v;x_0|t_0)$ then $v(x,t_0) < 1$, and since $C^{-1} \leqq |v_t| \leqq C$, we have $v(x,t_1) < C$. Now $|\ell_{z_1}(x)| = |v(x_1,t_1) + Dv(x_1,t_1)\cdot(x-x_1)| \leqq C_1$. Hence $(u-\ell_{z_1})(x,t_1) \leqq C + C_1 = \theta$ and so $x \in S_\theta(v;x_1|t_1)$.

Lemma 4.2. *Let u be such that $0 < m_1 \leqq -u_t \leqq m_2$ and $0 < \lambda \leqq -u_t \det D^2 u \leqq \Lambda$. Suppose that $(x_1,t_1), (x_2,t_2) \in Q_h(u;z_0)$. Then there exists θ depending only on the parameters such that*

(1) $S_h(x_1|t_1) \subset S_{\theta h}(x_2|t_2)$,
(2) $S_h(x_2|t_2) \subset S_{\theta h}(x_1|t_1)$.

Proof. It is enough to prove the lemma when $(x_2,t_2) = (x_0,t_0)$. Normalize the section $Q_{2h}(u;z_0)$ as before. We have

$$T_p(x,t) = \left(Tx, \frac{t-t_0}{h}\right) = (x^*,t^*);$$
$$u^*(x^*,t^*) = \frac{1}{h}(u-\ell_{z_0})(T_p^{-1}(x^*,t^*));$$

$Q_2^*(z_0^*) = T_p Q_{2h}(z_0)$, and $Q_1^*(z_0^*) = T_p Q_h(z_0)$. We have $T S_h(x_1|t_1) = S_1^*(x_1^*|t_1^*)$. By the inclusions previously proved we have $S_1^*(x_1^*|t_1^*) \subset S_\theta^*(x_0^*|t_0^*)$ and $S_1^*(x_0^*|t_0^*) \subset S_\theta^*(x_1^*|t_1^*)$. Taking T^{-1} in these inclusions yields the lemma.

4.2. Engulfing property for parabolic sections

We now prove the engulfing property for the sections $Q_h^*(z_0)$.

Lemma 4.3 (Engulfing property**).** *There exists a constant $\vartheta > 1$ such that for each $z_1 \in Q_h^*(z_0)$, $Q_h^*(z_0) \subset Q_{\vartheta h}^*(z_1)$.*

Proof. Let $z_1 = (x_1, t_1) \in Q_h^*(z_0)$, with $z_0 = (x_0, t_0)$, and recall (4.1), (4.2), and (4.3). Let T be an invertible affine transformation to be chosen later,

$$T_p(x,t) = \left(Tx, \frac{t - t_0}{h}\right),$$

and

$$v(y,s) = \frac{1}{h}(u - \ell_{z_0}^*)(T^{-1}y, t_0 + sh).$$

We have

$$T_p(Q_h^*(z_0)) = \{(y,s) : v(y,s) \leqq 1 \text{ and } s \leqq \delta\}.$$

Consider $\ell_{z_1}(x)$ a supporting hyperplane for $u(\cdot, t_1)$ at $x = x_1$, and $(x_1)_{\min}^{\vartheta h}$ the point at which the minimum of $u(x, t_1 + \delta\,\vartheta\, h) - \ell_{z_1}(x)$ is attained, where ϑ will be chosen. We let

$$\ell_{z_1}^*(x) = \ell_{z_1}(x) + u((x_1)_{\min}^{\vartheta h}, t_1 + \delta\,\vartheta\, h) - \ell_{z_1}((x_1)_{\min}^{\vartheta h}).$$

We have

$$u(x,t) - \ell_{z_1}^*(x) = h\, v\left(Tx, \frac{t - t_0}{h}\right) + \ell_{z_0}^*(x) - \ell_{z_1}^*(x).$$

By definition of $Q_{\vartheta h}^*$ (see (4.2)) we have

$$\begin{aligned} T_p(Q_{\vartheta h}^*(x_1, t_1)) = \Big\{(y,s) : v(y,s) + \frac{1}{h}\{\ell_{z_0}^*(x) - \ell_{z_1}^*(x)\} \leqq \vartheta \\ \text{and } s \leqq \frac{t_1 - t_0}{h} + \delta h\Big\}. \end{aligned}$$

The inclusion in the lemma is equivalent to

$$T_p(Q_h^*(z_0)) \subset T_p(Q_{\vartheta h}^*(z_1)), \tag{4.4}$$

with $T_p(z_1) \in T_p(Q_h^*(z_0))$. To show (4.4), let $(y,s) \in T_p(Q_h^*(z_0))$. We have

$$v(y,s) + \frac{1}{h}\{\ell_{z_0}^*(T^{-1}y) - \ell_{z_1}^*(T^{-1}y)\} \leqq 1 + \frac{1}{h}\{\ell_{z_0}^*(T^{-1}y) - \ell_{z_1}^*(T^{-1}y)\},$$

and we shall prove that the right-hand side of the last inequality is less than ϑ.

We write

$$\begin{aligned} &\ell_{z_0}^*(T^{-1}y) - \ell_{z_1}^*(T^{-1}y) \\ &\quad = \ell_{z_0}(T^{-1}y) - \ell_{z_1}(T^{-1}y) + u((x_0)_{\min}^h, t_0 + \delta\, h) - \ell_{z_0}((x_0)_{\min}^h) \\ &\qquad - \{u((x_1)_{\min}^{\vartheta h}, t_1 + \delta\,\vartheta\, h) - \ell_{z_1}((x_1)_{\min}^{\vartheta h})\}. \end{aligned}$$

Since $u_t \geqq -m_1$, we obtain

$$u((x_1)_{\min}^{\vartheta h}, t_1 + \delta\,\vartheta\, h) - \ell_{z_1}((x_1)_{\min}^{\vartheta h}) \geqq -m_1\,\delta\,\vartheta\, h + u((x_1)_{\min}^{\vartheta h}, t_1) - \ell_{z_1}((x_1)_{\min}^{\vartheta h}).$$

Therefore

$$\begin{aligned}
&\ell^*_{z_0}(T^{-1}y) - \ell^*_{z_1}(T^{-1}y)\\
&\quad\leqq \ell_{z_0}(T^{-1}y) - \ell_{z_1}(T^{-1}y) + u((x_0)^h_{\min}, t_0 + \delta\, h) - \ell_{z_0}((x_0)^h_{\min})\\
&\qquad\qquad + m_1\, \delta\, \vartheta\, h - u((x_1)^{\vartheta h}_{\min}, t_1) + \ell_{z_1}((x_1)^{\vartheta h}_{\min})\\
&\quad= \ell_{z_0}(T^{-1}y) - \ell_{z_0}((x_0)^h_{\min}) + \ell_{z_1}((x_1)^{\vartheta h}_{\min}) - \ell_{z_1}(T^{-1}y)\\
&\qquad\qquad + u((x_0)^h_{\min}, t_0 + \delta\, h) - u((x_1)^{\vartheta h}_{\min}, t_1) + m_1\, \delta\, \vartheta\, h\\
&\quad\leqq \ell_{z_0}(T^{-1}y) - \ell_{z_0}((x_0)^h_{\min}) + \ell_{z_1}((x_1)^{\vartheta h}_{\min}) - \ell_{z_1}(T^{-1}y)\\
&\qquad\qquad + u((x_0)^h_{\min}, t_1) - u((x_1)^{\vartheta h}_{\min}, t_1) + m_1\, \delta\, \vartheta\, h\\
&\quad= A + B + C + m_1\, \delta\, \vartheta\, h,
\end{aligned}$$

since $u((x_0)^h_{\min}, t_0 + \delta\, h) \leqq u((x_0)^h_{\min}, t_1)$ because $t_1 \leqq t_0 + \delta\, h$.

Now

$$\begin{aligned}
A &= \ell_{z_0}(T^{-1}y) - \ell_{z_0}((x_0)^h_{\min}) = Du(x_0, t_0) \cdot (T^{-1}y - (x_0)^h_{\min}),\\
B &= \ell_{z_1}((x_1)^{\vartheta h}_{\min}) - \ell_{z_1}(T^{-1}y) = Du(x_1, t_1) \cdot ((x_1)^{\vartheta h}_{\min} - T^{-1}y),\\
C &= u((x_0)^h_{\min}, t_1) - u((x_1)^{\vartheta h}_{\min}, t_1) = Du(\xi, t_1) \cdot ((x_0)^h_{\min} - (x_1)^{\vartheta h}_{\min}).
\end{aligned}$$

If at the beginning of the proof we subtract from u the supporting hyperplane $\ell_{z_0}(x)$, then we may assume that $Du(x_0, t_0) = 0$. By definition of v it follows that

$$Du(x, t) = h\, T^t\, Dv\Big(Tx, \frac{t - t_0}{h}\Big).$$

We have

$$\begin{aligned}
A &= 0,\\
B &= (T^{-1})^t\, Du(x_1, t_1) \cdot (T((x_1)^{\vartheta h}_{\min}) - y)\\
&= h\, Dv(Tx_1, \frac{t_1 - t_0}{h}) \cdot (T((x_1)^{\vartheta h}_{\min}) - y),\\
C &= h\, Dv(T\xi, \frac{t_1 - t_0}{h}) \cdot (T((x_0)^h_{\min}) - T((x_1)^{\vartheta h}_{\min})).
\end{aligned}$$

Let x_{t_1} be the point where the minimum of $u(\cdot, t_1)$ is attained. Then applying repeatedly Lemma 4.2 we get

$$S_{Mh}(x_0|t_0) \subset S_{\theta Mh}(x_{t_1}|t_1) \subset S_{(\theta)^2 Mh}(x_0|t_0)$$

for $M \geqq 1$, (θ is the constant in Lemma 4.2). Now choosing $M = C\,\vartheta$ and T an affine transformation normalizing $S_{Mh}(x_0|t_0)$ we can bound Dv, and we find that $B \leqq L\, h$ and $C \leqq L\, h$. Therefore we get

$$\ell^*_{z_0}(T^{-1}y) - \ell^*_{z_1}(T^{-1}y) \leqq 2L\, h + m_1\, \delta\, \vartheta\, h.$$

If we choose δ so that $m_1\, \delta \leqq 1/2$ and then ϑ such that $1 + 2L + \frac{1}{2}\vartheta \leqq \vartheta$, we obtain (4.4) and the proof of the engulfing property is complete.

4.3. Shrinking property of parabolic sections

Proposition 4.1. *Let Q be a normalized bowl-shaped domain in $\mathbb{R}^{n+1}$, and u a parabolically convex function in Q satisfying: $\lambda \leqq -u_t \det D^2 u \leqq \Lambda$ in Q, $u \geqq 0$, $\min_Q u = 0$, $-m_1 \leqq u_t \leqq -m_2$ in Q, and $u = 1$ on $\partial_p Q$. Then*

(1) *If $u(z_0) < 1-\varepsilon$, then $\operatorname{dist}(z_0, \partial_p Q) \geqq C\,\varepsilon^{n+1}$, where C is a constant depending only on n, m_1 and Λ.*
(2) *If $\operatorname{dist}(z_0, \partial_p Q) \geqq \varepsilon$, then $u(z_0) \leqq 1 - m_2\,\varepsilon$.*

Proof. We begin with the proof of (1). Let $v(x,t) = u(x,t) - 1$. We have $\lambda \leqq -v_t \det D^2 v \leqq \Lambda$, $\min_Q v = -1$ and $v = 0$ on $\partial_p Q$. By application of [GH98, Theorem 2.1] to the function v we get

$$|v(x_0, t_0)|^{n+1} \leqq C(n, \Lambda) \operatorname{dist}(x_0, \partial(Q(t_0))),$$

and since $v(x_0, t_0) \leqq -\varepsilon$, we obtain $\operatorname{dist}(x_0, \partial(Q(t_0))) \geqq C(n, \Lambda)\,\varepsilon^{n+1}$. Since Q is bowl-shaped, we have $Q(t_0) \subset Q(t)$ for $t \geqq t_0$, and consequently

$$\operatorname{dist}(x_0, \partial(Q(t))) \geqq C(n, \Lambda)\,\varepsilon^{n+1} \qquad \text{for } \ t \geqq t_0.$$

For $0 \leqq t_0 - t \leqq c_1\,\varepsilon$, we get

$$u(x_0, t) \leqq u(x_0, t_0) + m_1 c_1\,\varepsilon \leqq 1 - \varepsilon + m_1 c_1\,\varepsilon = 1 - \frac{\varepsilon}{2},$$

by choosing $c_1 = 1/(2\,m_1)$. Again from [GH98, Theorem 2.1] we obtain

$$\operatorname{dist}(x_0, \partial(Q(t))) \geqq C(n, \Lambda)\,\varepsilon^{n+1} \qquad \text{for } t_0 - c_1\,\varepsilon \leqq t \leqq t_0.$$

Now let $z_1 = (x_1, t_1) \in \partial_p Q$ be such that $\operatorname{dist}(z_0, \partial_p Q) = |z_0 - z_1|$. If $t_1 > t_0 - c_1\,\varepsilon$, then

$$|x_0 - x_1| \geqq \operatorname{dist}(x_0, \partial(Q(t_1))) \geqq C\,\varepsilon^{n+1}.$$

If $t_1 < t_0 - c_1\,\varepsilon$, then $\operatorname{dist}(z_0, \partial_p Q) \geqq t_0 - t_1 \geqq c_1\,\varepsilon$. Therefore in either case we obtain the inequality $\operatorname{dist}(z_0, \partial_p Q) \geqq C(n, m_1, \Lambda)\,\varepsilon^{n+1}$.

We now prove (2). Let t_{bdy} be the time such that $(x_0, t_{\text{bdy}}) \in \partial_p Q$. We have $t_0 - t_{\text{bdy}} \geqq \operatorname{dist}(z_0, \partial_p Q)$. Hence

$$u(x_0, t_0) \leqq u(x_0, t_{\text{bdy}}) - m_2 \operatorname{dist}(z_0, \partial_p Q) = 1 - m_2 \operatorname{dist}(z_0, \partial_p Q) \leqq 1 - m_2\,\varepsilon.$$

Lemma 4.4. *Let $z \notin Q_h^*(z_0)$, and let T_p be a parabolically affine transformation normalizing $Q_h^*(z_0)$. Then there exist structural positive constants C and ν such that*

$$T_p(Q^*_{(1-\varepsilon)h}(z_0)) \cap K(T_p(z), C\,\varepsilon^{\nu}) = \emptyset$$

for $0 < \varepsilon < 1$, where $K(z, R)$ is the standard parabolic cylinder given by $K(z, R) = B_R(x) \times (t - R^2, t + R^2]$; $z = (x, t)$.

Proof. With the notation of Lemma 4.3 and similarly to the proof of Lemma 4.1 we have

$$\begin{aligned}
&(u-\ell_{z_0})((x_0)^{(1-\varepsilon)h}_{\min}, t_0+\delta(1-\varepsilon)h)+(1-\varepsilon)h\\
&\quad\leqq (u-\ell_{z_0})((x_0)^{h}_{\min}, t_0+\delta(1-\varepsilon)h)+(1-\varepsilon)h\\
&\quad\leqq (u-\ell_{z_0})((x_0)^{h}_{\min}, t_0+\delta h)+\sup|u_t|\,\delta\varepsilon h+(1-\varepsilon)h\\
&\quad\leqq (u-\ell_{z_0})((x_0)^{h}_{\min}, t_0+\delta h)+\left(1-\frac{\varepsilon}{2}\right)h.
\end{aligned}$$

Hence

$$Q^*_{(1-\varepsilon)h}(z_0)\subset Q_{(1-\frac{\varepsilon}{2})h}((x_0)^h_{\min}, t_0+\delta h)\cap\{t\leqq t_0+\delta(1-\varepsilon)h\}.$$

Let

$$T_p(x,t)=\left(Tx,\,\frac{t-(t_0+\delta h)}{h}\right)$$

normalizing $Q_h(z_h)$ with $z_h=((x_0)^h_{\min}, t_0+\delta h)$. Set

$$T_p(Q_h(z_h))=\hat{Q}_1(0);\qquad T_p(Q_{(1-\frac{\varepsilon}{2})h}(z_h))=\hat{Q}_{1-\frac{\varepsilon}{2}}(0);$$

and

$$\hat{u}(\hat{x},\hat{t})=\frac{1}{h}\{u(T_p^{-1}(\hat{x},\hat{t})-\ell_{z_h}(T^{-1}\hat{x})\}.$$

To complete the proof of the lemma it is sufficient to show that

$$\operatorname{dist}(\partial\hat{Q}_1(0),\,\hat{Q}_{1-\frac{\varepsilon}{2}}(0)\cap\{t\leqq-\varepsilon\,\delta\})\geqq C_\delta\,\varepsilon^\nu.$$

This follows from Proposition 4.1, item (1), because

$$\operatorname{dist}(\partial_p\hat{Q}_1(0),\,\hat{Q}_{1-\frac{\varepsilon}{2}}(0))\geqq C\,\varepsilon^{n+1},\quad\text{and}\quad\operatorname{dist}(\{t=0\},\{t\leqq-\varepsilon\,\delta\})=\varepsilon\,\delta.$$

4.4. Size of parabolic sections

Lemma 4.5. *Let $Q^*_{h_0}(z_0)$ be a section and T_p a transformation that normalizes it. If $h\leqq h_0$ and $Q^*_h(z')\cap Q^*_{h_0}(z_0)\neq\emptyset$, then $|T_p(z')|\leqq M$ and*

$$K\left(T_p(z'),\,C_1\left(\frac{h}{h_0}\right)^{\varepsilon_1}\right)\subset T_p(Q^*_h(z'))\subset K\left(T_p(z'),\,C_2\left(\frac{h}{h_0}\right)^{\varepsilon_2}\right),$$

with M, C_1, C_2, ε_1 and ε_2 positive constants depending only on the structure; and $K(z,R)$ denotes the parabolic cylinder defined in Lemma 4.4.

Proof. We can assume $h_0 = 1$, $Q_1^*(z_0)$ is already normalized and T_p =identity. Applying Lemma 4.3 three times we get $Q_1^*(z') \subset Q_{\vartheta^2}^*(z_0)$. Since $Q_1^*(z_0)$ is normalized, it follows from Lemma 4.2 that $Q_{2\vartheta^2}^*(z_0)$ is also normalized. We have $|z'| \leqq M$ and

$$\begin{aligned} Q_h^*(z') &= Q_h((x')_{\min}^h, t' + \delta h) \subset S_h((x')_{\min}^h | t' + \delta h) \times (t' - ch, t' + \delta h] \\ &\subset B((x')_{\min}^h, C_{2,\vartheta}\, h^{\varepsilon_2}) \times (t' - ch, t' + \delta h] \qquad \text{by [GH00, Theorem 2.3]} \\ &\subset B(x', 2C\, h^{\varepsilon_2}) \times (t' - ch, t' + \delta h]. \end{aligned}$$

On the other hand,

$$S_{ch}(\hat{x}|t' - \delta h) \times (t' - \delta h, t' + \delta h) \subset Q_h^*(z'),$$

where $\hat{x}$ is the minimum point of $u(\cdot, t' - \delta h)$. (Assume $\ell_{((x')_{\min}^h, t' + \delta h)} = 0$; $c \geqq 3/4$ and δ small.) Since x' is minimum point of $u(\cdot, t')$, we have

$$u(x', t' - \delta h) \leqq u(x', t') + \hat{c}\delta h \leqq u(\hat{x}, t') + \hat{c}\delta h \leqq u(\hat{x}, t' - \delta h) + \hat{c}\delta h.$$

Hence $x' \in S_{ch/2}(\hat{x}|t' - \delta h)$. From [GH00, Theorem 2.4(i)] we get $S_{\eta h}(x'|t' - \delta h) \subset S_{ch}(\hat{x}|t' - \delta h)$. Therefore from [GH00, Theorem 2.3] we obtain

$$\begin{aligned} B(x', Ch^{\varepsilon_1}) \times (t' - \delta h, t' + \delta h] &\subset S_{\eta h}(x'|t' - \delta h) \times (t' - \delta h, t' + \delta h] \\ &\subset Q_h^*(z'). \end{aligned}$$

Remark 4.2. More precisely, the first inclusion in Lemma 4.5 can be written as

$$K\left(T_p(z'), C_1 \left(\frac{h}{h_0}\right)^{\varepsilon_1}\right) \cap \{t \leqq T\} \subset T_p(Q_h^*(z')).$$

4.5. *Second size property of sections*

Lemma 4.6. *Let $Q_1(z_0)$ be a normalized section. There exist positive constants C and p such that, if $0 < r < s < 1$ and $z' \in Q_r(z_0)$, then $Q_h(z') \subset Q_s(z_0)$ for $h \leqq C\,(s - r)^p$.*

Proof. By Lemma 4.1(2), and Lemma 4.5 we get

$$Q_h(z') \subset Q_{2h}^*(z') \subset K(z', C\, h^{\varepsilon_1}).$$

From [GH98, Theorem 2.1] we have

$$|Du(z')| \leqq \frac{C}{(1 - r)^{n+1}}.$$

If $z \in Q_h(z')$ then

$$\begin{aligned} u(z) &\leqq u(z') + Du(z') \cdot (x - x') + h \\ &\leqq r + \frac{C}{(1 - r)^{n+1}}\, C\, h^{\varepsilon_1} + h \leqq s \qquad (\text{assume } \ell_{z_0} = 0), \end{aligned}$$

when $h \leqq \frac{1}{2}(s - r)$ and $h^{\varepsilon_1} \leqq \eta\,(s - r)^{n+2}$. That is, $Q_h(z') \subset Q_s(z_0)$.

4.6. Besicovitch type covering lemma

Lemma 4.7. *Let Q be a parabolic section and $\mathcal{O} \subset Q$. Suppose that for each $z \in \mathcal{O}$ a section $Q_r^*(z)$ is given so that $r \leqq M$. Assume that the engulfing and size properties hold (Lemmas 4.3, 4.4 and 4.5). Then we can choose a countable subfamily $\{Q_{r_k}^*(z_k)\}_{k=1}^{\infty}$ with the following properties:*

(1) $\mathcal{O} \subset \cup_{k=1}^{\infty} Q_{r_k}^*(z_k)$;
(2) $z_k \notin \cup_{j<k} Q_{r_j}^*(z_j)$ *for* $k \geqq 2$;
(3) $\sum_{k=1}^{\infty} \chi_{Q^*_{(1-\varepsilon)r_k}(z_k)}(z) \leqq C_0 \log \dfrac{1}{\varepsilon}$;

where χ_E denotes the characteristic function of the set E.

Proof. It is the same as the one given in [CG96] and [Hua99].

4.7. A Calderón-Zygmund type decomposition

We now give a proposition needed in the proof of the Calderón-Zygmund decomposition.

Proposition 4.2. *There exists a positive constant C depending only on the structure such that*

$$|Q_h^*(z_0) \setminus Q^*_{(1-\varepsilon)h}(z_0)| \leqq C\sqrt{\varepsilon}\,|Q_h^*(z_0)|,$$

with $0 < \varepsilon < 1$.

Proof. The inequality is affine invariant. We assume $h = 1$ and $Q_1^*(z_0)$ normalized. Let us also assume that $z_0 = 0$, $\ell_{((x_0)^1_{\min},\delta)} = 0$ and let $t_{\min}$ denote the minimum t in $Q_1^*(0)$ and $(\hat{x}, t_{\min}) \in \partial Q_1^*(0)$. Also let $m(t) = \min_x u(x,t)$ and $x^t_{\min}$ be the point where the minimum of $u(\cdot, t)$ is attained. We write

$$\begin{aligned}
|Q_1^*(0) \setminus Q^*_{1-\varepsilon}(0)| \leqq{}& \int_{\delta(1-\varepsilon)}^{\delta} |S_{1-m(t)}(x^t_{\min}|t)|\,dt \\
&+ \int_{t_{\min}+\sqrt{\varepsilon}}^{\delta(1-\varepsilon)} |S_{1-m(t)}(x^t_{\min}|t) \setminus S_{1-\varepsilon-m(t)+\alpha}(x^t_{\min}|t)|\,dt \\
&+ \int_{t_{\min}}^{t_{\min}+\sqrt{\varepsilon}} |S_{1-m(t)}(x^t_{\min}|t)|\,dt = \mathrm{I} + \mathrm{II} + \mathrm{III},
\end{aligned}$$

where

$$\alpha = u(x^{1-\varepsilon}_{\min}, (1-\varepsilon)\delta) \leqq u(x^1_{\min}, (1-\varepsilon)\delta) \leqq \sup|u_t|\,\varepsilon\delta \leqq \frac{\varepsilon}{2}.$$

We have

$$\mathrm{I} \leqq \delta\,\varepsilon |S_1(x^1_{\min}|1)| \leqq C\,\varepsilon \leqq \varepsilon\,|Q_1^*(0)|.$$

Also

$$\mathrm{III} \leqq \sqrt{\varepsilon}\,|S_{1-m(t')}(x^{t'}_{\min}|t')| \leqq C\sqrt{\varepsilon},$$

with $t' = t_{\min} + \sqrt{\varepsilon}$. For $t \geqq t_{\min} + \sqrt{\varepsilon}$ we have $m(t) \leqq u(\hat{x}, t) \leqq u(\hat{x}, t_{\min}) + u_t\,(t - t_{\min}) \leqq 1 - C\sqrt{\varepsilon}$. Hence by [CG97, Lemma 5.20] we get

$$\begin{aligned}
\mathrm{II} &\leqq \int_{t_{\min}+\sqrt{\varepsilon}}^{\delta(1-\varepsilon)} |S_{1-m(t)}(x^t_{\min}|t) \setminus S_{\left(1-\frac{\varepsilon-\alpha}{1-m(t)}\right)(1-m(t))}(x^t_{\min}|t)|\,dt \\
&\leqq \int_{t_{\min}+\sqrt{\varepsilon}}^{\delta(1-\varepsilon)} n\,\frac{\varepsilon-\alpha}{1-m(t)}|S_{1-m(t)}(x^t_{\min}|t)|\,dt \\
&\leqq C\,\frac{\varepsilon}{c\sqrt{\varepsilon}} = C\sqrt{\varepsilon} \leqq C\sqrt{\varepsilon}\,|Q_1^*(0)|.
\end{aligned}$$

We conclude this section with the decomposition needed in the proof of the $W^{2,p}$ estimates.

Theorem 4.1. *Let $\mathcal{O}$ be an open subset of a section Q, $0 < \delta < 1$ small, and $\gamma > 0$. Suppose that for each $z \in \mathcal{O}$ a section $Q^*_{h_z}(z)$ is given with $h_z \leqq \gamma$, and*

$$\frac{|Q^*_{h_z}(z) \cap \mathcal{O}|}{|Q^*_{h_z}(z)|} = \delta.$$

Then there exists a family of parabolic sections $\{Q^*_{h_k}(z_k)\}_{k=1}^{\infty}$ with the following properties:

(1) *$z_k \in \mathcal{O}$ and $h_k \leqq \gamma$ for all $k \in \mathbb{N}$;*
(2) *$\mathcal{O} \subset \cup_{k=1}^{\infty} Q^*_{h_k}(z_k)$;*
(3) *$\dfrac{|Q^*_{h_k}(z_k) \cap \mathcal{O}|}{|Q^*_{h_k}(z_k)|} = \delta$;*
(4) *$|\mathcal{O}| \leqq \sqrt{\delta}\,|\cup_{k=1}^{\infty} Q^*_{h_k}(z_k)|$.*

Proof. The theorem can be proved by combining Lemma 4.7 with Proposition 4.2 and using the technique in [CG96], see [Gut].

5. Approximation Theorem

Let $(x_0, t_0) \in Q$, $\sigma > 0$, and

$$P_\sigma(x,t) = \sigma(|x - x_0|^2 - (t - t_0)) + p \cdot (x - x_0) + u(x_0, t_0).$$

If $(x_0, t_0) \in S \subset Q$, then we say that u is touched from below by P_σ in S if $u(x,t) \geqq P_\sigma(x,t)$ for all $(x,t) \in S$. Notice that if $S = Q \cap \{t = t_0\}$, then $p = D_x u(x_0, t_0)$.

Let us define the following sets:

$$A_\sigma(u) = \{(x_0, t_0) : u \text{ is touched from below by } P_\sigma \text{ in } Q \cap \{t \leqq t_0\}\},$$

and

$$A^*_\sigma(u) = \{(x_0, t_0) : u(x, t_0) \text{ is touched from below by } P_\sigma(x, t_0) \text{ in } Q \cap \{t = t_0\}\}. \tag{5.1}$$

Notice that $A_\sigma \subset A^*_\sigma$, and if $u_t \leqq -\sigma$ in Q, then we also have $A^*_\sigma \subset A_\sigma$.

Theorem 5.1 (Approximation Theorem**).** *Let Q be a bowl-shaped domain in $\mathbb{R}^{n+1}$ such that*

$$B_\delta(0)\times(-\delta^2,0]\subset Q\subset B_1(0)\times(-1,0],$$

where $Q=\{(x,t):\Phi(x,t)<0, t\leqq 0\}$ with Φ parabolically convex.[1] *Suppose that $0<\varepsilon<1/2$, and u is a parabolically convex function in Q, a classical solution of*

$$(1-\varepsilon)^{n+1}\leqq \mathcal{M}u\leqq(1+\varepsilon)^{n+1}\qquad \text{in } Q, \tag{5.2}$$

$$u=0\qquad \text{on } \partial_p Q, \tag{5.3}$$

and

$$-m_1\leqq u_t\leqq -m_2\qquad \text{in } Q, \tag{5.4}$$

where m_1 and m_2 are positive constants.

Let $0<\alpha<1$ and set

$$Q_\alpha=\left\{(x,t)\in Q : u(x,t)<(1-\alpha)\min_Q u\right\}. \tag{5.5}$$

Then there exist positive constants $\sigma>0$ and C_n depending only on the dimension n and α, and both independent of Q (depending only on the bounds for the time derivatives of the function Φ defining Q), u and ε such that

$$|Q_\alpha\setminus A_\sigma|\leqq C_n\,\varepsilon\,|Q_\alpha|.$$

Proof. By [GH98, Lemma 2.1] we have $C_1\leqq -\min_Q u\leqq C_2$. Let w be a parabolically convex solution of

$$-w_t\det D^2 w=1\qquad \text{in } Q, \tag{5.6}$$

$$w=0\qquad \text{on } \partial_p Q.$$

We have $w\in C(\overline{Q})\cap C^\infty(Q)$. We use the following Comparison Principle: if $\mathcal{M}v\geqq\mathcal{M}u$ in Q, then

$$u(x,t)-v(x,t)\geqq\min_{\partial_p Q}\{u(x,t)-v(x,t)\}\qquad \text{for all } (x,t)\in Q,$$

see [WW93, Proposition 2.3]. In our case, $\mathcal{M}((1+\varepsilon)w)\geqq\mathcal{M}u\geqq\mathcal{M}((1-\varepsilon)w))$, and so

$$(1+\varepsilon)w\leqq u\leqq(1-\varepsilon)w\qquad \text{in } Q. \tag{5.7}$$

Thus

$$\left(\frac{1}{2}+\varepsilon\right)w\leqq u-\frac{w}{2}\leqq\left(\frac{1}{2}-\varepsilon\right)w\qquad \text{in } Q. \tag{5.8}$$

[1] It is assumed here that Φ_t is bounded away from 0 and $-\infty$; see [WW92, Definition 3.1, p. 428 and Lemma 3.1]

We have $w < 0$ in Q, so

$$\left(\frac{1}{2}+\varepsilon\right)(-w) \geqq \left|u-\frac{w}{2}\right| \geqq \left(\frac{1}{2}-\varepsilon\right)(-w) \qquad \text{in } Q.$$

Since Q is normalized, it follows again by [GH98, Lemma 2.1] that $C_1 \leqq |\min_Q w| \leqq C_2$ and consequently $C_1' \leqq \max_Q |u - \frac{1}{2}w| \leqq C_2'$.

Let $\Gamma(x,t)$ be the parabolic convex envelope of $u - \frac{1}{2}w$ in Q, see definition (7.1).

Claim 1. For all $(x,t) \in Q$ we have the inequality

$$\left|\frac{w(x,t)}{2} - \Gamma(x,t)\right| \leqq C_n \varepsilon.$$

Indeed, by (5.8) and since w is parabolically convex, we have

$$\left(\frac{1}{2}+\varepsilon\right) w(x,t) \leqq \Gamma(x,t) \leqq \left(\frac{1}{2}-\varepsilon\right) w(x,t) \qquad \text{in } Q,$$

and the claim follows.

Claim 2. Let D be a bowl-shaped open and bounded domain, $u, v \in C(\bar{D})$ parabolically convex, $u = v$ on $\partial_p D$, and $v \leqq u$ in D. Then

$$\mathcal{P}_u(D) \subset \mathcal{P}_v(D) \qquad \text{a.e.},$$

that is, $\mathcal{P}_u(D) \setminus E \subset \mathcal{P}_v(D)$ for some $|E| = 0$. Indeed, let $(p,h) \in \mathcal{P}_u(D)$. Then $\ell(x) = u(x_0,t_0) + p \cdot (x - x_0) \leqq u(x,t)$ for all $t \leqq t_0$ and $x \in D(t)$, with $(x_0,t_0) \in D$. Slide ℓ in a parallel fashion in the direction of t negative until it touches for the last time the graph of v. Say ℓ touches v at (x_1,t_1), $t_1 \leqq t_0$. Then $\ell(x_1) = v(x_1,t_1) = u(x_0,t_0) + p \cdot (x_1 - x_0)$ and so $p \cdot x_1 - v(x_1,t_1) = p \cdot x_0 - u(x_0,t_0) = h$. If $(x_1,t_1) \in \partial_p D$, then $v(x_1,t_1) = u(x_1,t_1) = p \cdot x_1 - h$. Since $u(x,t) \geqq u(x_1,t_1) + p \cdot (x - x_1)$ for all $t \leqq t_0$ and $x \in D(t)$, it follows that $(p,h) \in P_u(x_1,t_1)$. That is $(p,h) \in P_u(x_1,t_1) \cap P_u(x_0,t_0)$, but if $(x_0,t_0) \neq (x_1,t_1)$, then this set of (p,h) has measure zero and the claim follows.

Since $w = 0$ on $\partial_p Q$, it follows that

$$\mathcal{P}_{(\frac{1}{2}-\varepsilon)w}(Q) \subset \mathcal{P}_\Gamma(Q) \subset \mathcal{P}_{(\frac{1}{2}+\varepsilon)w}(Q) \qquad \text{a.e.}$$

By the results in Section 7, Corollary 7.1,

$$\mathcal{M}\Gamma \leqq \mathcal{M}\left(u - \frac{w}{2}\right)\chi_{\mathcal{C}}, \tag{5.9}$$

where

$$\mathcal{C} = \left\{(x,t) \in Q : \Gamma(x,t) = u(x,t) - \frac{w(x,t)}{2}\right\}.$$

If $(x_0, t_0) \in \mathcal{C}$, then $u - \frac{1}{2}w - \Gamma$ attains its minimum 0 at the point (x_0, t_0) and hence

$$D_x^2\left(u - \frac{w}{2}\right)(x_0, t_0) \geqq 0, \qquad \text{and} \qquad \left(u - \frac{w}{2}\right)_t (x_0, t_0) \leqq 0. \tag{5.10}$$

On the other hand, if $a \geqq 0$ and $A \geqq 0$ is an $n \times n$ symmetric matrix then

$$\begin{aligned}&(a \ \det A)^{1/(n+1)} \\ &= \frac{1}{n+1} \ \inf\{\mathrm{trace}(B\,A) + b\,a : b > 0,\ B > 0 \text{ symmetric with } b \det B = 1\}.\end{aligned} \tag{5.11}$$

Thus

$$((a_1 + a_2)\det(A_1 + A_2))^{1/(n+1)} \geqq (a_1 \det A_1)^{1/(n+1)} + (a_2 \det A_2)^{1/(n+1)}$$

for $a_i \geqq 0$ and $A_i \geqq 0$ symmetric, $i = 1, 2$. Since $\frac{1}{2}w$ is parabolically convex, it follows using the previous inequality and (5.10) that

$$\{\mathcal{M}u(x,t)\}^{1/(n+1)} \geqq \left\{\mathcal{M}\left(u - \frac{w}{2}\right)(x,t)\right\}^{1/(n+1)} + \left\{\mathcal{M}\left(\frac{w}{2}\right)(x,t)\right\}^{1/(n+1)}$$

for $(x, t) \in \mathcal{C}$. Consequently

$$\begin{aligned}\left\{\mathcal{M}\left(u - \frac{w}{2}\right)(x_0, t_0)\right\}^{1/(n+1)} &\leqq \{\mathcal{M}u(x_0, t_0)\}^{1/(n+1)} - \left\{\mathcal{M}\left(\frac{w}{2}\right)(x_0, t_0)\right\}^{1/(n+1)} \\ &\leqq (1+\varepsilon) - \frac{1}{2} = \frac{1}{2} + \varepsilon,\end{aligned} \tag{5.12}$$

an inequality valid for a.e. point in $\mathcal{C}$. By Claim 2, $|\mathcal{P}_{(\frac{1}{2}-\varepsilon)w}(Q)| \leqq |\mathcal{P}_\Gamma(Q)|$, and from (5.9) and (5.12) we get

$$\left(\frac{1}{2} - \varepsilon\right)^{n+1} \int_Q \mathcal{M}w(x,t)\,dx\,dt \leqq \int_{\mathcal{C}} \mathcal{M}\Gamma(x,t)\,dx\,dt \leqq \left(\frac{1}{2} + \varepsilon\right)^{n+1} |\mathcal{C}|.$$

This yields the estimate

$$|\mathcal{C}| \geqq \left(\frac{\frac{1}{2} - \varepsilon}{\frac{1}{2} + \varepsilon}\right)^n |Q| \geqq (1 - C_n\varepsilon)|Q|,$$

which implies

$$|Q \setminus \mathcal{C}| \leqq C_n\,\varepsilon\,|Q|. \tag{5.13}$$

We now prove that there exists a universal constant $\sigma > 0$ so that

$$Q_\alpha \cap \mathcal{C} \subset A_\sigma.$$

We recall the following variant of a theorem due to Pogorelov, see [GH98, Theorem 2.2].

Theorem 5.2. *Let $D \subset \mathbb{R}^{n+1}$ be a bounded open bowl-shaped domain and $v \in C(\overline{D})$ such that v is parabolically convex in D. Suppose that v is a smooth solution of*

$$-v_t \det D^2 v = 1 \qquad \text{in } \overline{D} \setminus \partial_p D, \tag{5.14}$$

$$v(x,t) = 0 \qquad \text{for } (x,t) \in \partial_p D. \tag{5.15}$$

Let $\alpha \in \mathbb{R}^n$, $|\alpha| = 1$,

$$w(x,t) = |v(x,t)| D_{\alpha\alpha} v(x,t) e^{\frac{1}{2}(D_\alpha v(x,t))^2},$$

and $M = \max_{\bar{D}} w(x,t)$. Then there exists $P \in \overline{D} \setminus \partial_p D$ where the maximum M is attained and the following inequality holds:

$$M \leqq C_n \ (1 + |D_\alpha v(P)|)\, e^{\frac{1}{2}(D_\alpha v(P))^2},$$

with C_n a positive constant depending only on the dimension n.

We apply this theorem to w. Let $\delta > 0$ and

$$Q_\delta(w) = \{(x,t) \in Q : w(x,t) < -\delta\}.$$

Notice that since $0 < \varepsilon < \frac{1}{2}$, it follows from (5.7) that $\frac{3}{2}\, w \leqq u \leqq \frac{1}{2}\, w$ and consequently

$$Q_\delta(u) \subset Q_{2\delta/3}(w) \qquad \text{and} \qquad Q_\delta(w) \subset Q_{\delta/2}(u). \tag{5.16}$$

Arguing as in the proof of (3-7) and (3-8) of [GH98], we obtain

$$|D_x w(x,t)| \leqq C(\delta) \qquad \text{for } (x,t) \in Q_\delta(w),$$

which implies that

$$|D_{\alpha\alpha} w(x,t)| \leqq C'(\delta)$$

on the same set and for all $|\alpha| = 1$. Thus

$$D_x^2 w(x,t) \leqq M_\delta I \qquad \forall\, (x,t) \in Q_\delta(w). \tag{5.17}$$

This estimate used together with the equation yields the following upper bound for the time derivative of w:

$$w_t(x,t) \leqq -C(\delta) \tag{5.18}$$

for $(x,t) \in Q_\delta(w)$, with $C(\delta) > 0$. To obtain the lower estimate of w_t we invoke [WW92, Lemma 3.3]. (Notice that this estimate depends on the time derivative of the defining function Φ.) Thus from (5.17) and (5.6) we obtain

$$D_x^2 w(x,t) \geqq M'_\delta I \qquad \forall\, (x,t) \in Q_\delta(w). \tag{5.19}$$

Consequently, if $(x_0, t_0) \in Q_\delta(w)$ by the convexity of w, we then obtain the estimate

$$w(x, t_0) \geqq w(x_0, t_0) + D_x w(x_0, t_0) \cdot (x - x_0) + m\, |x - x_0|^2$$

for all $(x, t_0) \in Q$ with m a positive constant depending only on n and δ. (Here we use the Taylor polynomial of second order of $w(\cdot, t_0)$ with the remainder written in integral form and the convexity of $w(., t_0)$ together with (5.19) to obtain the inequality valid in all $Q(t_0)$.)

Recall that $\Gamma(x, t) \leqq u(x, t) - \frac{1}{2}w(x, t)$ for all $(x, t) \in Q$. Since $\Gamma(x, t_0)$ is convex, let ℓ_{x_0} be a supporting hyperplane for $\Gamma(x, t_0)$ at $x = x_0$. Then

$$\begin{aligned} u(x, t_0) &\geqq \ell_{x_0}(x) + \frac{w(x, t_0)}{2} \\ &\geqq \ell_{x_0}(x) + \frac{1}{2}\left(w(x_0, t_0) + D_x w(x_0, t_0) \cdot (x - x_0) + m\,|x - x_0|^2\right) \\ &= P_{m/2}(x, t_0) \end{aligned}$$

for all $x \in Q(t_0)$. For $(x_0, t_0) \in \mathcal{C}$, we know that $P_{m/2}(x, t_0)$ touches $u(x, t_0)$ from below in $Q(t_0)$. Since $u_t \leqq -m_2$ in Q, if we take $\sigma = \min\{m_2, \frac{1}{2}m\}$ then $u_t \leqq -\sigma$. Hence $(x_0, t_0) \in A^*_\sigma \subset A_\sigma$, that is, $Q_\delta(w) \cap \mathcal{C} \subset A_\sigma$.

Now taking into account (5.16) and choosing δ so that $\frac{3}{2}\delta$ is close to $-(1 - \alpha)\min_Q u$ we obtain

$$Q_\alpha \setminus A_\sigma \subset Q \setminus \mathcal{C}.$$

Then by (5.13), and since $|Q_\alpha|$ and $|Q|$ are comparable, we obtain the theorem.

6. $W^{2,p}$ estimates

We prove here L^p estimates of the second derivatives in x of is solutions to $\mathcal{M}u = f$. This done in several steps. We first establish a strict convexity result, Lemma 6.1. Second, we prove a density result, Proposition 6.1, for which the approximation Theorem 5.1 is used. This density result, combined with the decomposition Theorem 4.1 and once more the approximation Theorem 5.1, yields the power decay, Proposition 6.2. Once this is done, we obtain $W^{2,p}$ estimates on parabolic sections, Theorem 6.1, that is with zero boundary data. Next, we use a strict convexity result due to Caffarelli, Theorem 6.2, which coupled with Theorem 3.1 yields a the strict convexity result in the parabolic case, Theorem 6.3. This last result and Theorem 6.1 yield by means of a covering argument the main result in the paper, Theorem 2.1.

Let $u(x, t)$ be a parabolically convex function in the bowl-shaped domain Q with $u = 0$ on $\partial_p Q$. Given $0 < \alpha \leqq 1$, recall the definition (5.5). We have $Q_\alpha \subset Q_\beta$ for $0 < \alpha \leqq \beta \leqq 1$. Given $z_0 = (x_0, t_0) \in Q$, let us keep in mind definitions (2.1)–(2.3), (4.2), and (5.1).

We have the following strict convexity result which states that sections with base points in Q_α are contained in Q for sufficiently small values of the parameter h and independently of the base point.

Lemma 6.1. *Let Q be a normalized bowl-shaped domain and u a solution to $\lambda \leqq \mathcal{M}u \leqq \Lambda$ in Q with $u = 0$ on $\partial_p Q$ and $0 < m_2 \leqq -u_t \leqq m_1$ in Q. Given $0 < \alpha \leqq \alpha_0 < 1$ there exists $\eta_\alpha > 0$ such that, if $h \leqq \eta_\alpha$ and $(x_0, t_0) \in Q_\alpha$, then*

$$Q_h(x_0, t_0) \subset Q_{(\alpha_0+1)/2}.$$

Proof. It is by contradiction. Suppose there exist $z_j = (x_j, t_j)$, u_j, Q^j such that $z_j \in Q^j_\alpha$, $Q_{1/j}(z_j) \not\subset Q^j_{\alpha_0}$, Q^j is normalized, $\lambda \leqq \mathcal{M}u_j \leqq \Lambda$ in Q^j, $0 < m_2 \leqq -(u_j)_t \leqq m_1$ in Q^j, and $u_j = 0$ on $\partial_p Q^j$. Let $(y_j, s_j) \in Q_{1/j}(z_j) \setminus Q^j_{\alpha_0}$. Then

$$(1-\alpha) \min_{Q^j} u_j \leqq u_j(y_j, s_j) \leqq \ell_j(y_j) + \frac{1}{j},$$

where ℓ_j is a supporting hyperplane for u_j at z_j. Letting $j \to \infty$ by compactness we obtain a function u_∞ and a normalized convex domain Q^∞ such that $\lambda \leqq \mathcal{M}u_\infty \leqq \Lambda$ in Q^∞, $0 < m_2 \leqq -(u_\infty)_t \leqq m_1$ in Q^∞, and $u_\infty = 0$ on $\partial_p Q^\infty$. Moreover,

$$(1-\alpha) \min_{Q^\infty} u_\infty \leqq u_\infty(y_\infty, s_\infty) = \ell_\infty(y_\infty),$$

where ℓ_∞ is a supporting hyperplane for u_∞ and $z_\infty = (x_\infty, s_\infty) \in Q^\infty_\alpha$. Therefore $(y_\infty, s_\infty), z_\infty \in \{u_\infty = \ell_\infty\}$. If $s_\infty \neq t_\infty$, i.e., $s_\infty < t_\infty$, then since u is decreasing in t we have $u_\infty(y_\infty, t) = \ell_\infty(y_\infty)$, for $s_\infty \leqq t \leqq t_\infty$. This contradicts the fact that $m_1 \leqq -u_t \leqq m_2$. Since $c^{-1} \leqq \det D^2_x u_\infty(\cdot, t_\infty) \leqq c$ in $Q^\infty \cap \{t = t_\infty\}$, it follows from CAFFARELLI's result, [Caf90b, Theorem 1], that no extremal points of $\{u_\infty = \ell_\infty\}$ can be inside the interior of the domain and therefore all must be outside $Q^\infty_{\alpha_0} \cap \{t = t_\infty\}$. The point x_∞ must be a convex combination of extremal points, i.e., $x_\infty = \sum_{i=1}^k \lambda_i P_i$ with $\lambda_i \geqq 0$ and $\sum_{i=1}^k \lambda_i = 1$, and $u_\infty(P_i) \geqq (1-\alpha_0) \min_{Q^\infty} u_\infty$. Thus

$$(1-\alpha) \min_{Q^\infty} u_\infty \geqq u_\infty(x_\infty) = \ell_\infty(x_\infty) = \sum_{i=1}^k \lambda_i\, u_\infty(P_i) \geqq (1-\alpha_0) \min_{Q^\infty} u_\infty,$$

a contradiction.

For $\lambda > 0$ and $0 < \alpha \leqq \alpha_0 < 1$, we define the set

$$D^\alpha_\lambda = \{(x_0, t_0) \in Q_\alpha : S_h(x_0|t_0) \subset B_{\lambda\sqrt{h}}(x_0), \quad \text{for all } h \leqq \eta_0\},$$

where $\eta_0 = \eta_{\alpha_0}$ is the number in Lemma 6.1 corresponding to $\alpha = \alpha_0$.

Lemma 6.2. *Let u be a parabolically convex function on a bounded bowl-shaped domain Q and $0 < \alpha_0 < 1$. There exists a constant $C_1 > 0$ depending only on α_0 and $\max_t\{diam(Q(t))\}$ such that*

$$D^\alpha_\lambda = Q_\alpha \cap A^*_{1/\lambda^2}(u)$$

for all $\lambda \geqq C_1$ and $0 < \alpha \leqq \alpha_0 < 1$.

Proof. Suppose $z_0 = (x_0, t_0) \in D^\alpha_\lambda$. Let $\ell_{z_0}(x)$ be a supporting hyperplane to $u(\cdot, t_0)$ at $x = x_0$, i.e., $u(x, t_0) \geqq \ell_{z_0}(x)$ for all $x \in Q(t_0)$. Let $x \in Q(t_0)$ and $\mu = u(x, t_0) - \ell_{z_0}(x)$, so $x \in S_\mu(x_0|t_0)$. If $\mu < \eta_0$, then $S_\mu(x_0|t_0) \subset B_{\lambda\sqrt{\mu}}(x_0)$ and

$$\frac{1}{\lambda^2}|x - x_0|^2 + \ell_{z_0}(x) \leqq u(x, t_0).$$

On the other hand, if $\mu \geqq \eta_0$, then

$$u(x,t_0) - \ell_{z_0}(x) = \mu \geqq \frac{\eta_0}{\operatorname{diam}(Q(t_0))^2}\,|x-x_0|^2.$$

If

$$\frac{1}{\lambda^2} \leqq \frac{\eta_0}{\max_t\{\operatorname{diam}(Q(t))^2\}},$$

then

$$u(x,t_0) - \ell_{z_0}(x) \geqq \frac{1}{\lambda^2}\,|x-x_0|^2$$

and the inclusion follows, taking

$$C_1 = \frac{\max_t\{\operatorname{diam}(Q(t))\}}{\sqrt{\eta_0}}.$$

If $(x_0,t_0) \in Q_\alpha \cap A^*_{1/\lambda^2}(u)$, then

$$u(x,t_0) \geqq \frac{1}{\lambda^2}\,|x-x_0|^2 + \ell_{z_0}(x)$$

for all $x \in Q(t_0)$. Let $x \in S_h(x_0|t_0)$ and $h < \eta_0$, then $S_h(x_0|t_0) \subset Q(t_0)$ and

$$\frac{1}{\lambda^2}\,|x-x_0|^2 + \ell_{z_0}(x) \leqq u(x,t_0) < \ell_{z_0}(x) + h$$

and so $x \in B_{\lambda\sqrt{h}}(x_0)$.

Proposition 6.1. *Let* $0 < \varepsilon < 1/2$*, and* u *be a solution of* (5.2) *in the normalized bowl-shaped domain* Q *satisfying* (5.4)*. There exists a constant* $c_0 > 0$ *depending only on* n *and* σ *in Theorem 5.1 such that, if* $z_0 = (x_0,t_0) \in Q_{\alpha_0}$ *and* $h \leqq \eta_0/2$*, then*

$$\frac{|Q_h(z_0) \setminus A^*_{c_0\,h}(u)|}{|Q_h(z_0)|} \leqq C_n\,\varepsilon.$$

Moreover, if $\lambda \geqq \dfrac{2}{c_0\,\eta_0}$*, then*

$$\frac{|Q_h(z_0) \setminus A^*_{1/\lambda}(u)|}{|Q_h(z_0)|} \leqq C_n\,\varepsilon$$

for $h \geqq \dfrac{1}{c_0\,\lambda}$*; (*$C_n$ *is the constant in the approximation Theorem 5.1).*

Proof. The idea of the proof is to normalize u and then apply Theorem 5.1. Consider the elliptic section $S_h(x_0|t_0)$, $h < \eta_0$, and let T be the affine transformation that normalizes $S_h(x_0|t_0)$. That is, $B_{\alpha_n}(0) \subset T\,(S_h(x_0|t_0)) \subset B_1(0)$. We define

$$T_p(x,t) = \left(Tx,\, \frac{t-t_0}{h}\right) = (y,s).$$

Then from the estimates for u_t, see [GH98, Lemma 3.1], we have

$$(-\varepsilon_1, 0] \times B_{\alpha_n}(0) \subset T_p\left(Q_h(z_0)\right) \subset (-\varepsilon_2, 0] \times B_1(0),$$

where ε_1 and ε_2 are constants. Set

$$Q_h^*(z_0) = T_p\left(Q_h(z_0)\right).$$

We have $T_p^{-1}(y, s) = \left(T^{-1}y, t_0 + h\,s\right)$. Let $\ell_{z_0}(x)$ be the affine function defining $Q_h(z_0)$. We define

$$v(x, t) = \frac{C}{h}\left(u(x, t) - \ell_{z_0}(x) - h\right),$$

where C is a constant that will be determined in a moment. Let

$$u^*(y, s) = v(T_p^{-1}(y, s)) = v(T^{-1}y, t_0 + h\,s).$$

We have $u_s^*(y, s) = \dfrac{C}{h}\, h\, u_t(T_p^{-1}(y, s))$, and

$$D^2u^*(y, s) = \frac{C}{h}\,(T^{-1})^t (D^2u)(T_p^{-1}(y, s))(T^{-1}).$$

Hence

$$\det D^2u^*(y, s) = \left(\frac{C}{h}\right)^n |\det T^{-1}|^2 \det D^2u(T_p^{-1}(y, s)).$$

Consequently,

$$\begin{aligned}
&-u_s^*(y, s)\ \det D^2u^*(y, s)\\
&\qquad = \frac{C^{n+1}}{h^n}\,|\det T^{-1}|^2 \left(-u_t(T_p^{-1}(y, s))\ \det D^2u(T_p^{-1}(y, s))\right).
\end{aligned}$$

We now choose C such that

$$\frac{C^{n+1}}{h^n}\,|\det T^{-1}|^2 = 1. \tag{6.1}$$

Since u satisfies (5.2), it follows that u^* satisfies

$$(1-\varepsilon)^{n+1} \leqq -u_t^*\ \det D^2u^* \leqq (1+\varepsilon)^{n+1} \qquad \text{in } Q_h^*(z_0) \tag{6.2}$$

$$u^* = 0 \qquad \text{on } \partial Q_h^*(z_0). \tag{6.3}$$

By the definition of u^*, we have $\min_{Q_h^*(z_0)} u^* = -C$. By properties of the elliptic sections, see [GH00, Proposition 1.1], we have $c^{-1}h^{n/2} \leqq |S_h(x_0|t_0)| \leqq ch^{n/2}$, and hence $c^{-1}h^{-n/2} \leqq |\det T| \leqq ch^{-n/2}$. Therefore C in (6.1) depends only on n. Applying Theorem 5.1 with $Q \to Q_h^*(z_0)$, $\alpha \to \beta$, and $u \to u^*$, we obtain

$$\frac{|Q_{\beta\,h}^*(z_0) \setminus A_\sigma^*|}{|Q_{\beta\,h}^*(z_0)|} \leqq C_n\,\varepsilon, \qquad \text{with } T_p(Q_{\beta\,h}(z_0)) = Q_{\beta\,h}^*(z_0),$$

where $A^*_\sigma = A^*_\sigma(u^*)$ (notice that $(Q^*_h(z_0))_\beta = T_p(Q_{\beta h}(z_0)) = Q^*_{\beta h}(z_0)$ and $A_\sigma(u^*) \subset A^*_\sigma(u^*)$). We now show that there exist universal constants $0 < \beta < 1$ and $c_0 > 0$ such that

$$T_p^{-1}\left(Q^*_{\beta h}(z_0) \cap A^*_\sigma\right) \subset Q_{\beta h}(z_0) \cap A^*_{c_0 h}(u). \tag{6.4}$$

Let $z_1^* = (x_1^*, t_1^*) \in Q^*_{\beta h}(z_0) \cap A^*_\sigma$ and $z_1 = (x_1, t_1) = T_p^{-1} z_1^* \in Q_{\beta h}(z_0)$. Since $z_1^* \in A^*_\sigma$, we have

$$u^*(x^*, t_1^*) - \ell^*(x^*) \geqq \sigma\, |x^* - x_1^*|^2$$

for all $x^* \in Q^*_h \cap \{t_1^*\}$ with $\ell^*(x^*) = \ell_{z_1}(T^{-1}x^*)$ where ℓ_{z_1} is a supporting hyperplane for $u(\cdot, t_1)$ at $x = x_1$. Hence

$$\frac{C(u(x,t_1) - \ell_{z_0}(x) - h)}{h} - \frac{C(\ell_{z_1}(x) - \ell_{z_0}(x) - h)}{h} \geqq \sigma\, |Tx - Tx_1|^2,$$

for $x \in Q_h(z_0) \cap \{t_1\}$. Therefore

$$u(x,t_1) - \ell_{z_1}(x) \geqq \frac{1}{C}\,\sigma\, h\, |Tx - Tx_1|^2 \qquad \text{in } Q_h(z_0) \cap \{t_1\}.$$

By rotating the coordinates, we may assume that the ellipsoid of minimum volume containing $S_h(x_0|t_0)$ with center at x_h, the center of mass of $s_h(x_0/t_0)$ has axes on the coordinate axes. That is,

$$Tx = \Big(\frac{x_1 - x_h^1}{\mu_1}, \cdots, \frac{x_n - x_h^n}{\mu_n}\Big),$$

where μ_i are the axes of the ellipsoid. Since Q is bounded, we have $\mu_i \leqq$ const, and so $\mu_i^{-1} \geqq$ const. Therefore $|Tx - Tx_1| \geqq C'\,|x - x_1|$. Consequently,

$$u(x,t_1) - \ell_{z_1}(x) \geqq C''\,\sigma\, h\, |x - x_1|^2 \qquad \text{in } Q_h(z_0) \cap \{t_1\}. \tag{6.5}$$

We now want to show that a similar inequality holds in $Q \cap \{t_1\}$. Since $z_1 \in Q_{\beta h}(z_0)$, by the engulfing property Lemma 4.3, we have $Q_{\beta h}(z_0) \subset Q_{\theta \beta h}(z_1)$. Again by the engulfing property, $Q_{\theta\beta h}(z_1) \subset Q_{\theta^2 \beta h}(z_0)$, so taking $\beta = 1/\theta^2$ yields $Q_{h/\theta}(z_1) \subset Q_h(z_0)$ and consequently (6.5) holds in $Q_{h/\theta}(z_1) \cap \{t_1\}$. Now, if $x \notin Q_{h/\theta}(z_1) \cap \{t_1\}$, then $u(x,t_1) - \ell_{z_1}(x) \geqq h/\theta$, and since Q is normalized, $h \geqq h\, C'''\,\sigma\,|x - x_1|^2$. Therefore $(x_1, t_1) \in A^*_{\bar{C}\sigma h}(u)$, and letting $c_0 = \bar{C}\sigma$ we obtain (6.4) with $\beta = 1/\theta^2$.

Therefore (6.4) implies that

$$Q_{\beta h}(z_0) \setminus A^*_{c_0 h}(u) \subset T_p^{-1}(Q^*_{\beta h}(z_0) \setminus A^*_\sigma),$$

and consequently

$$\frac{|Q_{\beta h}(z_0) \setminus A^*_{c_0 h}(u)|}{|Q_{\beta h}(z_0)|} \leqq \frac{|T_p^{-1}(Q^*_{\beta h}(z_0) \setminus A^*_\sigma)|}{|T_p^{-1}(Q^*_{\beta h}(z_0))|} = \frac{|Q^*_{\beta h}(z_0) \setminus A^*_\sigma|}{|Q^*_{\beta h}(z_0)|} \leqq C_n\, \varepsilon$$

for $h < \eta_0$, which yields the first conclusion of the proposition.

To prove the second conclusion, notice that if $\sigma \geqq \mu$, then $A^*_\sigma \subset A^*_\mu$. Hence $A^*_{c_0 h}(u) \subset A^*_{1/\lambda}(u)$ for $1/\lambda \leqq c_0\, h$.

We recall the definition of D_λ^α,

$$D_\lambda^\alpha = \{(x_0, t_0) \in Q_\alpha : S_h(x_0|t_0) \subset B_{\lambda\sqrt{h}}(x_0) \quad \text{for all } h \leqq \eta_0\};$$

here $0 < \alpha \leqq \alpha_0 < 1$, $\lambda > 0$, and $\eta_0 > 0$ is from Lemma 6.1 so that $Q_h(x_0, t_0) \subset Q_{(\alpha_0+1)/2}$ for $h \leqq \eta_0$.

The following proposition gives the power decay needed for the proof of the $W^{2,p}$ estimates.

Proposition 6.2 (Power decay). *Let $0 < \varepsilon < 1/2$ and u be a solution satisfying the hypotheses of Theorem 5.1. Set*

$$(D_\lambda^\alpha)^c = Q_\alpha \setminus D_\lambda^\alpha; \qquad (D_{M\lambda}^\tau)^c = Q_\tau \setminus D_{M\lambda}^\tau, \quad 0 < \tau < \alpha \leqq \alpha_0.$$

There exist positive constants M, p_0 and C_2 such that

$$|(D_{M\lambda}^\tau)^c| \leqq \sqrt{C_n\, \varepsilon}\, |(D_\lambda^\alpha)^c| \tag{6.6}$$

for all $\lambda \geqq C_2$ and $\alpha - \tau = (M\lambda)^{-p_0}$.

Proof. By Lemma 6.2 we have

$$D_{M\lambda}^\tau = Q_\tau \cap A^*_{1/(M\lambda)^2}(u) \qquad \text{for } M\,\lambda \geqq C_1 \text{ and } \tau < \alpha_0.$$

Since $D_{M\lambda}^\tau$ is closed, $\mathcal{O} = (D_{M\lambda}^\tau)^c$ is open and we obtain

$$\mathcal{O} = Q_\tau \setminus D_{M\lambda}^\tau = Q_\tau \cap (A^*_{1/(M\lambda)^2}(u))^c$$

for $M\,\lambda \geqq C_1$ and $\tau < \alpha_0$. Consequently

$$Q_h(z_0) \cap \mathcal{O} \subset Q_h(z_0) \cap Q_\tau \cap (A^*_{1/(M\lambda)^2}(u))^c \subset Q_h(z_0) \setminus A^*_{1/(M\lambda)^2}(u).$$

Therefore by Proposition 6.1 we obtain

$$\frac{|Q_h(z_0) \cap \mathcal{O}|}{|Q_h(z_0)|} \leqq \frac{|Q_h(z_0) \setminus A^*_{1/(M\lambda)^2}(u)|}{|Q_h(z_0)|} \leqq C_n\, \varepsilon$$

for

$$M\,\lambda \geqq \max\left\{C_1, \sqrt{\frac{2}{c_0\eta_0}}\right\}, \quad \tau < \alpha_0, \quad \frac{1}{(\lambda M)^2} \leqq h \leqq \eta_0/2, \quad z_0 \in Q_{\alpha_0}. \tag{6.7}$$

Let us now consider the sections $Q_h^*(x_0, t_0)$ defined by (4.2), and keep in mind (4.3). Since the set $\mathcal{O}$ is open, we have

$$\lim_{h\to 0} \frac{|Q_h^*(z_0) \cap \mathcal{O}|}{|Q_h^*(z_0)|} = 1, \qquad z_0 \in \mathcal{O}. \tag{6.8}$$

By Proposition 6.1 we have

$$\frac{|Q_h^*(x_0, t_0) \cap \mathcal{O}|}{|Q_h^*(x_0, t_0)|} = \frac{|Q_h((x_0)^h_{\min}, t_0 + \delta\, h) \cap \mathcal{O}|}{|Q_h((x_0)^h_{\min}, t_0 + \delta\, h)|} \leqq C_n\, \varepsilon,$$

with h satisfying (6.7), since $(x_0, t_0) \in Q_\alpha$ implies that $((x_0)^h_{\min}, t_0 + \delta h) \in Q_{(\alpha+1)/2}$, see Remark 4.1, and $m_1 \delta < \eta_0/2$.

If $\mathcal{O} = Q_\tau \setminus D^\tau_{M\lambda}$, then for $z \in \mathcal{O}$ we choose h_z, the largest h such that

$$\frac{|Q^*_h(z) \cap \mathcal{O}|}{|Q^*_h(z)|} \geqq C_n\, \varepsilon.$$

Then by (6.7) and (6.8) we get $h_z \leqq 1/(M\lambda)^2$. Applying Theorem 4.1 to $\mathcal{O}$ with $\gamma = 1/(M\lambda)^2$, and $\delta = C_n\,\varepsilon$, we obtain a family of sections $\{Q^*_{h_k}(z_k)\}_{k=1}^\infty$, $z_k = (x_k, t_k)$, with $h_k \leqq 1/(M\lambda)^2$.

We shall prove that

$$Q^*_{h_k}(x_k, t_k) \subset (D^\alpha_\lambda)^c = Q_\alpha \setminus D^\alpha_\lambda. \tag{6.9}$$

By Remark 4.1 and Lemma 4.6 we know that if $(x_k, t_k) \in Q_\tau$, then

$$((x_k)^h_{\min}, t_k) \in Q_{m_1 \delta h}(x_k, t_k) \subset Q_{\tau + c(\delta h)^{1/p}} \subset Q_{\tau + (M\lambda)^{-p_0}}.$$

That is, $Q^*_{h_k}(x_k, t_k) \subset Q_{\tau + 2(M\lambda)^{-p_0}}$. For $\tau = \alpha - (M\lambda)^{-p_0}$ and since $(x_k, t_k + \delta h_k) \in Q_\tau$, it follows that $Q_{h_k}((x_k)^h_{\min}, t^h_k) \subset Q_\alpha$ where $t^h_k = t_k + \delta h_k$.

To complete the proof of (6.9) we proceed by contradiction. Suppose there exists $z_0 = (x_0, t_0) \in Q_{h_k}((x_k)^h_{\min}, t^h_k) \cap D^\alpha_\lambda$. By the engulfing property of elliptic sections at different times, Lemma 4.2, we have

$$S_{h_k}((x_k)^h_{\min} | t^h_k) \subset S_{\theta h_k}(x_0 | t_0) \subset B_{\lambda\sqrt{\theta h_k}}(x_0),$$

with $z_0 \in D^\alpha_\lambda$ ($\theta\, h_k \leqq \eta_0$ by choosing M large). As in the proof of Proposition 6.1 we normalize the section $S_{2h_k}((x_k)^h_{\min} | t^h_k)$. That is, $B_{\alpha_n}(0) \subset T(S_{2h_k}((x_k)^h_{\min} | t^h_k)) \subset B_1(0)$, and let $Q^*_k = T_p\left(Q_{2h_k}((x_k)^h_{\min}, t^h_k)\right)$. We set

$$u^* = \frac{c}{2h_k}(u - \ell - 2h_k)(T_p^{-1}(x^*, t^*)),$$

and we have $(1-\varepsilon)^{n+1} \leqq Mu^* \leqq (1+\varepsilon)^{n+1}$ in Q^*_k. By the approximation Theorem 5.1 we then have

$$|(Q^*_k)_{1/2} \setminus A^*_\sigma| < C_n\, \varepsilon\, |(Q^*_k)_{1/2}|, \tag{6.10}$$

(notice that $(Q^*_k)_{1/2} = T_p\left(Q_{h_k}((x_k)^h_{\min}, t^h_k)\right)$). We now claim that

$$T_p^{-1}((Q^*_k)_{1/2} \cap A^*_\sigma) \subset D^\alpha_{M\lambda} \qquad \text{for } M \text{ large.} \tag{6.11}$$

Let $z^*_1 = (x^*_1, t^*_1) \in (Q^*_k)_{1/2} \cap A^*_\sigma$ and $z_1 = T_p^{-1} z^*_1 = (x_1, t_1) \in Q_{h_k}\left((x_k)^h_{\min}, t^h_k\right)$. Since $(x^*_1, t^*_1) \in A^*_\sigma$, we have $u^*(x^*, t^*_1) - \ell^*(x^*) \geqq \sigma\, |x^* - x^*_1|^2$ and hence $S^*_h(x^*_1 | t^*_1) \subset B(x^*_1, \sqrt{h/\sigma})$. Therefore $T^{-1}\big(S^*_h(x^*_1 | t^*_1)\big) \subset T^{-1}\big(B(x^*_1, \sqrt{h/\sigma})\big)$ for $h \leqq$ const, and consequently

$$S_{c\, h_k\, h}(x_1 | t_1) \subset T^{-1}\big(B(x^*_1, \sqrt{h/\sigma})\big) \subset B\big(x_1, \lambda\sqrt{\theta 2h_k}\sqrt{h/\sigma}\big),$$

because T dilates at least $(\lambda\sqrt{\theta 2h_k})^{-1}$ and T^{-1} contracts at least $\lambda\sqrt{\theta 2h_k}$. Then $S_h(x_1|t_1) \subset B(x_1, \lambda\sqrt{ch/\sigma})$ for $h \leqq \text{const}\, h_k$. If $h_k \leqq h \leqq \eta_0$, then (x_0, t_0), $(x_1, t_1) \in Q_h((x_k)^h_{\min}, t^h_k)$. By the engulfing property at different times $S_h(x_1|t_1) \subset S_{\theta h}(x_k|t^h_k) \subset S_{\theta^2 h}(x_0|t_0) \subset B_{\lambda\sqrt{\theta^2 h}}(x_0)$, since $z_0 \in D^\alpha_\lambda$. Therefore $(x_1, t_1) \in Q_\alpha \cap D^\alpha_{M\lambda}$ for some M large, and the proof of (6.11) is complete.

Therefore by Lemma 6.2,

$$\begin{aligned} Q_{h_k}((x_k)^h_{\min}, t^h_k) \cap \mathcal{O} &= Q_{h_k}((x_k)^h_{\min}, t^h_k) \cap (Q_\tau \setminus D^\tau_{M\lambda}) \\ &\subset Q_{h_k}((x_k)^h_{\min}, t^h_k) \setminus D^\alpha_{M\lambda} \subset T_p^{-1}((Q^*_k)_{1/2} \setminus A^*_\sigma), \end{aligned}$$

and by (6.10) we obtain

$$\frac{|Q^*_{h_k}(x_k, t_k) \cap \mathcal{O}|}{|Q^*_{h_k}(x_k, t_k)|} \leqq \frac{|(Q^*_k)_{1/2} \setminus A^*_\sigma|}{|(Q^*_k)_{1/2}|} < C_n\, \varepsilon,$$

which contradicts (3) in Theorem 4.1. This completes the proof of the power decay.

Theorem 6.1. *Let Q be a normalized bowl-shaped bounded domain and let u satisfy the hypothesis of Theorem 5.1. Then, given $0 < p < \infty$ and $0 < \tau < \alpha \leqq \alpha_0$, there exists $\varepsilon(p, \tau) > 0$ such that*

$$\iint_{Q_\tau} D_{ee}u(x, t)^p\, dx\, dt \leqq C$$

for all $|e| = 1$ and $0 < \varepsilon < \varepsilon(p, \tau)$ with C a constant depending only on the structure.

Proof. We iterate the inequality in Proposition 6.2. Notice that we can choose M large so that the statement of Proposition 6.2 holds for all $\lambda \geqq M$. We begin the iteration with $\lambda = M$ and therefore $(\tau =)\alpha_1 = \alpha - (M^2)^{-p_0}$ and we get

$$|Q_{\alpha_1} \setminus D^{\alpha_1}_{M^2}| \leqq \sqrt{C_n\, \varepsilon}\, |Q_\alpha \setminus D^\alpha_M|.$$

Continuing in this way, we let $\lambda = M^k$ and $\alpha_k = \alpha - \sum_{j=1}^k M^{-p_0(j+1)}$, obtaining

$$|Q_{\alpha_k} \setminus D^{\alpha_k}_{M^{k+1}}| \leqq C\left(\sqrt{C_n\, \varepsilon}\right)^k \qquad \text{for } k = 1, 2, \cdots.$$

We fix $\tau < \alpha$ and choose M large so that $\alpha_k \geqq \alpha - \sum_{j=1}^\infty M^{-(j+1)p_0} \geqq \tau$. We claim that if $(x_0, t_0) \in A^*_\sigma(u)$, then $u(x, t_0) \leqq C(n)\, \sigma^{-n+1}\, |x - x_0|^2 + \ell_{z_0}(x)$ for all x sufficiently close to x_0. Indeed, we have $S_h(x_0|t_0) \subset B_{\sqrt{h/\sigma}}(x_0)$ and, by properties of the elliptic sections, $c^{-1}h^{n/2} \leqq |S_h(x_0|t_0)| \leqq ch^{n/2}$. Applying Aleksandrov's maximum principle to the convex function $u(x, t_0) - \ell_{z_0}(x) - h$ on the set $S_h(x_0|t_0)$ yields $\text{dist}(x_0, \partial S_h(x_0|t_0)) \geqq \sigma^{n/2}\sqrt{h/\sigma}$, and the claim follows. Therefore, if $(x_0, t_0) \in A^*_\sigma(u)$, then $D_{ee}u(x_0, t_0) \leqq 2\, C(n)\, \sigma^{-n+1}$ for any $|e| = 1$. By Lemma 6.2, if $(x_0, t_0) \in D^{\alpha_i}_{M^{i+1}}$, then $(x_0, t_0) \in Q_\alpha \cap A^*_{1/M^{2(i+1)}}(u)$ and so $D_{ee}u(x_0, t_0) \leqq 2\, C(n)\, M^{2(n-1)(i+1)}$. Therefore

$$D^{\alpha_i}_{M^{i+1}} \subset \{(x, t) \in Q_{\alpha_i} : D_{ee}u(x, t) \leqq 2\, C(n)\, M^{2(n-1)(i+1)}\}.$$

Thus

$$\begin{aligned}
&\|D_{ee}u\|^p_{L^p(Q_\tau)} \\
&\quad \leqq M^{2(n-1)p}\,|Q_\tau| \\
&\qquad + \sum_{i=0}^{\infty} \int_{\{(x,t)\in Q_\tau : M^{2(n-1)(i+1)} < D_{ee}u(x,t) \leqq M^{2(n-1)(i+2)}\}} D_{ee}u(x,t)^p\,dxdt \\
&\quad \leqq C(M,n,\alpha,\tau,p) + \sum_{i=0}^{\infty} |Q_{\alpha_i} \setminus D^{\alpha_i}_{M^{i+1}}|\, M^{2(n-1)(i+2)p} \\
&\quad \leqq C(M,n,\alpha,\tau,p) + C(n) \sum_{i=0}^{\infty} (\sqrt{C_n\varepsilon})^{i+1}\, M^{2(n-1)(i+2)p} < \infty
\end{aligned}$$

for ε sufficiently small.

To complete the proof of the $W^{2,p}$ estimates we need the following result due to CAFFARELLI, see [Caf90b].

Theorem 6.2. *Let u be a convex solution to*

$$\lambda \leqq \det D^2 u \leqq \Lambda \qquad \text{in } \Omega, \tag{6.12}$$

$$u = f \qquad \text{on } \partial\Omega, \tag{6.13}$$

where $\Omega \subset \mathbb{R}^n$ is a $C^{1,\alpha}$ normalized convex domain and $f \in C^{1,\alpha}$, with $\alpha > 1 - \frac{2}{n}$. Then for each $h > 0$ there exists $\delta > 0$ such that for $x_0 \in \Omega_h = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > h\}$ we have

$$S(x_0,\delta) = \{x : u(x) < \ell_{x_0}(x) + \delta\} \subset \Omega_{h/2},$$

where δ depends only on $h, \lambda, \Lambda, n, \alpha$ and the $C^{1,\alpha}$ norms of f and Ω.

Now we are ready to prove the following result for the parabolic case.

Theorem 6.3. *Let u be a solution to $\mathcal{M}u = f$ in the cylinder $Q = \Omega \times (0,T]$ with $u = \phi$ on $\partial_p Q$. Suppose that*

(1) $B_{\alpha_n}(0) \subset \Omega \subset B_1(0)$, $\partial\Omega \in C^{1,\alpha}$ *with* $\alpha > 1 - \frac{2}{n}$;
(2) $0 < \lambda \leqq f \leqq \Lambda$, $f \in C(\bar{Q})$, $f_t \in L^{n+1}(Q)$ *and* $\exp(A(-f_t)^+) \in L^1(Q)$ *for some* $A > 0$;
(3) $\phi \in C^{2,1}(\bar{Q})$ *satisfying* $-c_2 \leqq \phi_t \leqq -c_1$ *and* $C_1 I \leqq D^2\phi \leqq C_2 I$ *in* Q.

Then for each $h > 0$ there exists $\delta > 0$ such that for $(x_0,t_0) \in \Omega_h \times (h,T]$, $\Omega_h = \{x \in \Omega : \operatorname{dist}(x,\partial\Omega) > h\}$, we have

$$Q_\delta(x_0,t_0) = \{(x,t) \in Q : u(x,t) < \ell_{x_0}(x) + \delta, \quad t \leqq t_0\} \subset \Omega_{h/2} \times (h/2, T],$$

where δ depends only on h and the parameters.

Proof. By Theorem 3.1 we get $-m_1 \leqq u_t \leqq -m_2$ in Q. Therefore $u_0(\cdot) = u(\cdot, t_0)$ satisfies (6.12) and by Theorem 6.2 there exists δ such that, if $x_0 \in \Omega_h$, then $S_\delta(x_0|t_0) = \{x : u(x, t_0) < \ell_{x_0}(x) + \delta\} \subset \Omega_{h/2}$. Since $-m_1 \leqq u_t \leqq -m_2$, it follows that $Q_\delta(x_0, t_0) \subset S_\delta(x_0|t_0) \times (t_0 - c\,\delta, t_0] \subset \Omega_{h/2} \times (h/2, T]$.

We are now in a position to complete the proof of the main result in the paper.

Proof of Theorem 2.1(B). The proof will follow, combining Theorems 6.1 and 6.3. Let $z_0 = (x_0, t_0) \in \Omega_h \times (h, T]$ and suppose that we have a section $Q^\delta = Q_u(z_0, \delta) \subset \Omega_{h/2} \times (h/2, T]$ such that $|f(z_0) - f(z)| \leqq \varepsilon$ for each $z = (x, t) \in Q_u(z_0, \delta)$. Taking δ sufficiently small, by Theorem 6.3 we may assume that $Q_u(z_0, \delta) \subset \Omega_{h/2} \times (h/2, T]$. Notice that since Q is normalized we have from the size property of sections, Lemma 4.5, that

$$K(z_0, K_1\,\delta^{\varepsilon_1}) \subset Q_u(z_0, \delta) \subset K(z_0, K_2\,\delta^{\varepsilon_2}), \tag{6.14}$$

with K_i, ε_i being positive constants depending only on λ, Λ and n, and $K(z, R)$ is the standard parabolic cylinder defined in the statement of Lemma 4.4. Let T be an affine transformation normalizing $S_\delta(x_0|t_0)$,

$$T_p(x, t) = \left(Tx, \frac{t - t_0}{\delta}\right)$$

as in the comment following remark (4.1), and consider the function

$$v(x, t) = \frac{C}{\delta}\,\left(u(T_p^{-1}(x, t)) - \ell_y(T_p^{-1}(x, t)) - \delta\right),$$

where ℓ_{x_0} is the supporting hyperplane for $u(\cdot, t_0)$ at x_0, and C is a constant that will be determined in a moment. We look at v on the set $T_p(Q_u(z_0, \delta))$, and we have $v = 0$ on $\partial T_p(Q_u(z_0, \delta))$,

$$\begin{aligned} D_x^2 v(x, t) &= \frac{C}{\delta}\,\left\{(T^{-1})^t\,(D_x^2 u)(T_p^{-1}(x, t))\,T^{-1}\right\}, \qquad \text{and} \\ v_t(x, t) &= C\,u_t(T_p^{-1}(x, t)). \end{aligned}$$

Hence

$$\mathcal{M}v(x, t) = \frac{C^{n+1}}{\delta^n}\,|\det T|^{-2}\,f(T_p^{-1}(x, t)) = \frac{f(T_p^{-1}(x, t))}{f(z_0)}$$

for $C = \dfrac{\delta^{n/(n+1)}\,|\det T|^{2/(n+1)}}{f(z_0)^{1/(n+1)}}$. Now $f(z_0) - \varepsilon \leqq f(z) \leqq f(z_0) + \varepsilon$ for $z \in Q^\delta$, and so

$$1 - \frac{\varepsilon}{f(z_0)} \leqq \frac{f(T_p^{-1}z)}{f(z_0)} \leqq 1 + \frac{\varepsilon}{f(z_0)}$$

for $z \in T_p(Q^\delta)$. Since $f(z_0) \geqq \lambda$, it follows that

$$1 - \frac{\varepsilon}{\lambda} \leqq \frac{f(T_p^{-1}z)}{f(z_0)} \leqq 1 + \frac{\varepsilon}{\lambda} \qquad \text{for } z \in T_p(Q^\delta).$$

Then applying our result on the set $T_p(Q^\delta)$ to the function v, we get

$$\int_{(T_p(Q^\delta))_h} D_{ee}v(x,t)^p\,dx\,dt \leqq C(n,h,p)$$

for each unit vector e and $\varepsilon \leqq \varepsilon(p,h)$.

By the definition of v, we have

$$D_x^2u(x,t) = \frac{\delta}{C}\,T^t\,(D_x^2v)(T_p(x,t))\,T,$$

and consequently

$$\begin{aligned} D_{ee}u(x,t) &= \langle D_x^2u(x,t)\,e,\ e\rangle \\ &= \frac{\delta}{C}\,|Te|^2\,\langle (D_x^2v)(T_p(x,t))\,e', e'\rangle = \frac{\delta}{C}\,|Te|^2\,(D_{e'e'}v)(T_p(x,t)), \end{aligned}$$

with $e' = \dfrac{Te}{|Te|}$. We have $(T_p(Q^\delta))_h = T_p((Q^\delta)_h)$. Therefore

$$\begin{aligned} \int_{(Q^\delta)_h} D_{ee}u(x,t)^p\,dx\,dt &= \left(\frac{\delta}{C}\right)^p\,|Te|^{2p}\int_{(T(Q^\delta))_h}(D_{e'e'}v)(z)^p\,|\det T|^{-1}\,\delta\,dz \\ &\leqq f(z_0)^{p/(n+1)}\left(\frac{\delta^{\frac{1}{p}+\frac{1}{n+1}}\,|Te|^2}{|\det T|^{\frac{2}{n+1}+\frac{1}{p}}}\right)^p\,C(h,n,p). \end{aligned}$$

To estimate the term between parentheses, let E be the ellipsoid of minimum volume containing $S_\delta(x_0|t_0)$, and let $\mu_1,\cdots,\mu_n$ be the axes of E. If δ is small, then by properties of elliptic sections we have $c^{-1}\delta^{n/2} \leqq |S_\delta(x_0|t_0)| \leqq c\delta^{n/2}$, see [Gut]. The affine transformation that normalizes $S_\delta(x_0|t_0)$ has the form

$$Tx = \left(\frac{x_1 - x_1^0}{\mu_1},\ldots,\frac{x_n - x_n^0}{\mu_n}\right),$$

where $(x_1^0,\cdots,x_n^0)$ is the center of the ellipsoid E (the center of mass of $S_\delta(x_0|t_0)$). We have $c^{-1}\delta^{-n/2} \leqq |\det T| \leqq c\delta^{-n/2}$, and from (6.14) it follows that $\mu_i \geqq K_1\,\delta^{\varepsilon_1}$. Hence

$$\frac{\delta^{\frac{1}{p}+\frac{1}{n+1}}\,|Te|^2}{|\det T|^{\frac{2}{n+1}+\frac{1}{p}}} \leqq C\,|Te|^2\,\delta^{1+\frac{1}{p}+\frac{n}{2p}} \leqq C\,\delta^{1+\frac{1}{p}+\frac{n}{2p}-2\varepsilon_1},$$

and consequently

$$\int_{(Q^\delta)_h} D_{ee}u(x,t)^p\,dx\,dt \leqq C(\lambda,\Lambda,n,h,p)\,\delta^{p+1+\frac{n}{2}-2p\varepsilon_1}. \tag{6.15}$$

We now choose δ small depending only on the parameters λ, Λ, h and the modulus of continuity of f, so that $|f(z_0)-f(z)| \leqq \varepsilon$ in $K(z_0, K_2\,\delta^{\varepsilon_2})$, $z_0 \in \Omega_h \times (h,T]$, and next select a finite covering of $\Omega_h \times (h,T]$ by standard parabolic cylinders $\{K(z_j, K_1\,\delta^{\varepsilon_1})\}_{j=1}^N$ with $z_j \in \Omega_h \times (h,T]$. The desired inequality then follows by adding (6.15) over $(Q(z_j,\delta))_h$.

7. The parabolic convex envelope on a bowl-shaped domain

Let Q be a bowl-shaped domain in $\mathbb{R}^{n+1}$, and $u \in C(\overline{Q})$. We define *the parabolic convex envelopes* Γ_u and Γ_u^p as follows. Given $(x_0, t_0) \in Q$ we let

$$\begin{aligned}
\Gamma_u(x_0, t_0) &= \sup\{v(x_0, t_0) : v \leqq u \text{ in } Q \text{ with } v \in C(Q) \text{ and } p\text{-convex in } Q\};\\
\Gamma_u^p(x_0, t_0) &= \sup\{v(x_0, t_0) : v \leqq u, \text{ in } Q \cap \{t \leqq t_0\} \\
&\qquad \text{with } v \text{ continuous and } p\text{-convex in } Q \cap \{t \leqq t_0\}\}.
\end{aligned} \tag{7.1}$$

The set $\mathcal{C}$ *of contact points, or contact set,* is given by

$$\mathcal{C} = \{(x, t) \in \overline{Q} : u(x, t) = \Gamma_u(x, t)\}.$$

Lemma 7.1. *The following equality holds:*

$$\Gamma_u = \Gamma_u^p \qquad \textit{in } Q. \tag{7.2}$$

Proof. We obviously have $\Gamma_u \leqq \Gamma_u^p$ in Q. Given $(x_0, t_0) \in Q$ and $\varepsilon > 0$, let v be continuous and p-convex in $Q \cap \{t \leqq t_0\}$ such that $v \leqq u$ in $Q \cap \{t \leqq t_0\}$ and

$$v(x_0, t_0) \geqq \Gamma_u^p(x_0, t_0) - \varepsilon.$$

Since v is p-convex there exists a supporting hyperplane $\ell_{x_0}(x)$ such that

$$\ell_{x_0}(x) \leqq v(x, t) \qquad \text{in } Q \cap \{t \leqq t_0\}, \text{ and} \qquad \ell_{x_0}(x_0) = v(x_0, t_0).$$

By continuity of ℓ_{x_0} and u, there exists $\delta > 0$ so that $\ell_{x_0}(x) - \varepsilon \leqq u(x, t)$ in $Q \cap \{t \leqq t_0 + \delta\}$. Let $0 \leqq \alpha(t) \leqq 1$ be a continuous and nonincreasing function on $(0, t_0 + \delta)$ with $\alpha(t) = 1$ on $(0, t_0)$ and $\alpha(t_0 + \delta) = 0$. Set

$$w(x, t) = \alpha(t)(\ell_{x_0}(x) - \varepsilon) + (1 - \alpha(t))K,$$

where $K = \min\{\min_Q(\ell_{x_0} - \varepsilon), \min_Q u\}$. It is easy to see that w is continuous and p-convex in Q, and satisfies

$$\begin{aligned}
w \leqq \ell_{x_0} - \varepsilon \leqq u \quad &\text{in } Q \cap \{t \leqq t_0 + \delta\},\\
w = K \leqq u \quad &\text{in } Q \cap \{t > t_0 + \delta\}.
\end{aligned}$$

Hence $w \leqq u$ in Q. Therefore

$$\Gamma_u(x_0, t_0) \geqq w(x_0, t_0) = \ell_{x_0}(x_0) - \varepsilon = v(x_0, t_0) - \varepsilon \geqq \Gamma_u^p(x_0, t_0) - 2\varepsilon,$$

and (7.2) follows by letting $\varepsilon \to 0$.

Lemma 7.2. *Let $u \in C^{2,1}(\bar{Q})$. If $(x_0, t_0) \in \mathcal{C} \cap Q$, then there exist $\varepsilon_0 > 0$, $M > 0$, and $p = D_x u(x_0, t_0)$, depending only on u (bounded by the $C^{2,1}$ norm of u in $\bar{Q}$), such that*

$$\Gamma_u(x, t) \leqq \Gamma_u(x_0, t_0) + p \cdot (x - x_0) + M\left(|x - x_0|^2 + t_0 - t\right) \tag{7.3}$$

for all $(x, t) \in B_{\sqrt{\varepsilon_0}}(x_0) \times (t_0 - \varepsilon_0, t_0] \cap Q$.

Proof. By the Taylor expansion,

$$\begin{aligned} u(x,t) \leqq u(x_0,t_0) + u_t(x_0,t_0)(t-t_0) + Du(x_0,t_0)\cdot(x-x_0) \\ + \tfrac{1}{2}\langle D_x^2 u(x_0,t_0)(x-x_0), x-x_0\rangle + \varepsilon(|x-x_0|^2 + t_0 - t) \end{aligned}$$

as $x \to x_0$ and $t \to t_0^-$, and for ε small. Since $\Gamma_u(x,t) \leqq u(x,t)$ and $(x_0,t_0) \in \mathcal{C} \cap Q$, the lemma follows.

Lemma 7.3. *Assume $u \in C^{2,1}(\bar{Q})$. Let $(x_0,t_0) \in Q \setminus \mathcal{C}$ and let $L(x) = \alpha + p\cdot x$ be a supporting hyperplane for $\Gamma_u(\cdot,t_0)$ at $x = x_0$. Then there exist at most $n+1$ points $(x_i,t_i) \in \mathcal{C}$ such that*

$$x_0 = \sum_{i=1}^{n+1} \lambda_i x_i, \tag{7.4}$$

where $\lambda_i \geqq 0$, $\sum_{i=1}^{n+1} \lambda_i = 1$, $t_i \leqq t_0$, $L(x_i) = \Gamma_u(x_i,t_i) = u(x_i,t_i)$ and $p = D_x u(x_i,t_i)$, $i = 1,\cdots,n+1$.

Proof. We have $\Gamma_u(x_0,t_0) < u(x_0,t_0)$. Since $L(x)$ is a supporting hyperplane for $\Gamma_u(x,t_0)$ at x_0, then $\Gamma_u(x,t_0) \geqq L(x)$ for all $x \in Q \cap \{t = t_0\}$ and $\Gamma_u(x_0,t_0) = L(x_0)$. We have $\Gamma_u(x,t) \geqq \Gamma_u(x,t_0)$ for all $(x,t) \in Q \cap \{t \leqq t_0\}$. Since $u(x,t) \geqq \Gamma_u(x,t)$, it follows that

$$u(x,t) \geqq L(x) \qquad \text{for all } (x,t) \in Q \cap \{t \leqq t_0\}. \tag{7.5}$$

Let

$$H = \{x : \text{there exists } t \text{ such that } (x,t) \in \overline{Q} \cap \{t \leqq t_0\} \text{ and } u(x,t) = L(x)\}.$$

We have $H \neq \emptyset$. Otherwise, by (7.5), $u(x,t) > L(x)$ in $\overline{Q} \cap \{t \leqq t_0\}$ and by compactness $u(x,t) - L(x) \geqq \delta > 0$ on the same set and for some $\delta > 0$. Hence $\Gamma_u^p(x,t_0) \geqq L(x) + \delta$. Using (7.2) and letting $x = x_0$ we get a contradiction. It is clear that the set H is closed.

Let $z \in H$ and $s \leqq t_0$ such that $u(z,s) = L(z)$. Then $(z,s) \in \mathcal{C}$. Indeed,

$$u(x,t) \geqq \Gamma_u(x,t) \geqq \Gamma_u(x,t_0) \geqq L(x) \qquad \text{for all } (x,t) \in Q \cap \{t \leqq t_0\},$$

and letting $x = z$ and $t = s$ we obtain $u(z,s) = \Gamma_u(z,s)$.

Let $\mathrm{Con}(H)$ be the convex hull of H. We claim that $x_0 \in \mathrm{Con}(H)$. Assume by contradiction that $x_0 \notin \mathrm{Con}(H)$ and let N be a neighborhood of $\mathrm{Con}(H)$ and $\ell(x)$ an affine function such that $\ell(x_0) > 0$ and $\ell(x) < 0$ in N. We have

$$\min\{u(x,t) - L(x) : (x,t) \in Q \cap \{t \leqq t_0\} \setminus N \times [a,t_0]\} \geqq \delta > 0,$$

with a lower bound for t when $(x,t) \in Q$. Hence, there exists $\varepsilon > 0$ such that $u(x,t) - L(x) \geqq \varepsilon\ell(x)$ for all $x \notin N$ and $t \leqq t_0$. Therefore, by (7.5), $u(x,t) \geqq L(x) + \varepsilon\ell(x)$ for all $(x,t) \in Q \cap \{t \leqq t_0\}$ and consequently $\Gamma_u(x,t) \geqq L(x) + \varepsilon\ell(x)$ on the same set. Since $\Gamma_u(x_0,t_0) = L(x_0)$, we obtain a contradiction.

Therefore by Carathéodory's theorem, see [Sch93, Theorem 1.1.3, p.3]

$$x_0 = \sum_{i=1}^{n+1} \lambda_i\, x_i, \tag{7.6}$$

where $\lambda_i \geqq 0$, $\sum_{i=1}^{n+1} \lambda_i = 1$, and $x_i \in H$. Let $t_i \leqq t_0$ be the t's corresponding to x_i's such that $(x_i, t_i) \in Q \cap \{t \leqq t_0\}$ and $u(x_i, t_i) = L(x_i)$. We have $u(x_i, t_i) \geqq \Gamma_u(x_i, t_i) \geqq \Gamma_u(x_i, t_0) \geqq L(x_i) = u(x_i, t_i)$, and so $u(x_i, t_i) = \Gamma_u(x_i, t_i) = L(x_i)$. We also know that L is a supporting hyperplane for $u(\cdot, t_i)$ at $x = x_i$ for $i = 1, \cdots, n+1$. Since u is regular,

$$L(x) \leqq u(x, t_i) = u(x_i, t_i) + Du(x_i, t_i) \cdot (x - x_i) + o(|x - x_i|^2)$$

as $x \to x_i$. Since $L(x_i) = \alpha + p \cdot x_i = u(x_i, t_i)$, we get $L(x) = u(x_i, t_i) + p \cdot (x - x_i)$ and so

$$p \cdot (x - x_i) \leqq Du(x_i, t_i) \cdot (x - x_i) + o(|x - x_i|^2), \tag{7.7}$$

and the lemma follows.

Lemma 7.4. *If $u \in C(\overline{Q})$, then $\Gamma_u \in C(Q)$.*

Proof. We know that Γ_u is p-convex in Q. We claim that

$$\lim_{t \downarrow t_0} \Gamma_u(x_0, t) = \Gamma_u(x_0, t_0), \qquad (x_0, t_0) \in Q. \tag{7.8}$$

By monotonicity $\Gamma_u(x_0, t) \leqq \Gamma_u(x_0, t_0)$ for $t \geqq t_0$. Hence

$$\lim_{t \downarrow t_0} \Gamma_u(x_0, t) \leqq \Gamma_u(x_0, t_0).$$

To show the opposite inequality, given $\varepsilon > 0$ there exists $v \in C(Q)$, p-convex, so that $v \leqq u$ in Q and $v(x_0, t_0) + \varepsilon \geqq \Gamma_u(x_0, t_0)$. Since $v(x_0, t)$ is continuous and nonincreasing in t, there exists $\delta > 0$ so that $0 \leqq v(x_0, t_0) - v(x_0, t) < \varepsilon$ for $t_0 \leqq t \leqq t_0 + \delta$. Hence $\Gamma_u(x_0, t_0) \leqq v(x_0, t) + 2\varepsilon$, for $t_0 \leqq t \leqq t_0 + \delta$, and taking the limit as $t \downarrow t_0$ yields

$$\Gamma_u(x_0, t_0) \leqq \liminf_{t \downarrow t_0} \Gamma_u(x_0, t) + 2\varepsilon.$$

Letting $\varepsilon \to 0$ we obtain (7.8).

Let $(x_0, t_0) \in \{z \in Q : u(z) = \Gamma_u(z)\}$. We claim that Γ_u is continuous at (x_0, t_0). Notice that, by monotonicity, if $t \leqq t_0$, then $\Gamma_u(x_0, t) \geqq \Gamma_u(x_0, t_0)$ and, since $\Gamma_u(x_0, t)$ is nonincreasing, we get

$$\lim_{t \uparrow t_0} \Gamma_u(x_0, t) \geqq \Gamma_u(x_0, t_0). \tag{7.9}$$

Since $u \in C(Q)$ and $(x_0, t_0) \in \mathcal{C}$, it follows that

$$\lim_{t \uparrow t_0} \Gamma_u(x_0, t) \leqq \liminf_{t \uparrow t_0} u(x_0, t) = u(x_0, t_0) = \Gamma_u(x_0, t_0).$$

By (7.9) we then have

$$\lim_{t\uparrow t_0} \Gamma_u(x_0,t) = \Gamma_u(x_0,t_0). \tag{7.10}$$

This combined with (7.8) yields

$$\lim_{t\to t_0} \Gamma_u(x_0,t) = \Gamma_u(x_0,t_0). \tag{7.11}$$

On the other hand, since Γ_u is bounded in Q and convex in x, it follows by [GH00, Lemma 1.1] that $|\Gamma_u(x_1,t) - \Gamma_u(x_2,t)| \leqq C\,|x_1 - x_2|$ for (x_1,t), (x_2,t) in a neighborhood of (x_0,t_0). Therefore

$$\begin{aligned}|\Gamma_u(x,t) - \Gamma_u(x_0,t_0)| &\leqq |\Gamma_u(x,t) - \Gamma_u(x_0,t)| + |\Gamma_u(x_0,t) - \Gamma_u(x_0,t_0)| \\ &\leqq C\,|x - x_0| + |\Gamma_u(x_0,t) - \Gamma_u(x_0,t_0)| \to 0\end{aligned}$$

as $(x,t) \to (x_0,t_0)$ by (7.11).

It remains to show that Γ_u is continuous when $(x_0,t_0) \notin \mathcal{C}$. By subtracting L from u we may assume that $u(x_i,t_i) = 0$, and therefore $u(x_i,t_i) = \Gamma_u(x_i,t_i) = 0$. (Notice that this implies that $\Gamma_u(x_0,t_0) = 0$.) Since (7.8) holds by reviewing the previous argument, we notice that to prove the continuity of Γ_u at (x_0,t_0) it is enough to establish (7.10). Actually it is enough to show that

$$\lim_{t\uparrow t_0} \Gamma_u(x_0,t) \leqq \Gamma_u(x_0,t_0).$$

Case 1. Suppose $(x_i,t_i) \in Q$, $t_i < t_0$. Then

$$\begin{aligned}\lim_{\Delta t\to 0^+} \Gamma_u(x_i, t_0 - \Delta t) &\leqq \lim_{\Delta t\to 0^+} \Gamma_u(x_i, t_i - \Delta t) \\ &\leqq \lim_{\Delta t\to 0^+} u(x_i, t_i - \Delta t) = u(x_i,t_i) = 0.\end{aligned}$$

Case 2. Suppose $(x_i,t_i) \in \partial_p Q$, $t_i < t_0$. For each $\varepsilon > 0$ there exist Δx and h so that $|\Delta x| < \varepsilon$, $|h| < \varepsilon$ and such that $(x_i + \Delta x, t_i + h) \in Q$ and $u(x_i + \Delta x, t_i + h) < u(x_i,t_i) + \varepsilon = \varepsilon$. Therefore

$$\lim_{\Delta t\to 0^+} \Gamma_u(x_i + \Delta x, t_0 - \Delta t) \leqq \Gamma_u(x_i + \Delta x, t_i + h) \leqq u(x_i + \Delta x, t_i + h) \leqq \varepsilon.$$

Case 3. Suppose $(x_i,t_i) \in \partial_p Q$, $t_i = t_0$. For any ε there exists $|\Delta x_i| < \varepsilon$ such that $(x_i + \Delta x_i, t_0) \in Q$ and $u(x_i + \Delta x_i, t_0) < u(x_i,t_i) + \varepsilon$. Hence

$$\begin{aligned}\lim_{\Delta t\to 0^+} \Gamma_u(x_i + \Delta x_i, t_0 - \Delta t) &\leqq \lim_{\Delta t\to 0^+} u(x_i + \Delta x_i, t_0 - \Delta t) \\ &= u(x_i + \Delta x_i, t_0) < \varepsilon.\end{aligned}$$

Summing up: if $x_0 = \sum_{i=1}^{n+1} \lambda_i x_i$, then for each (x_i,t_i) we have

$$\lim_{\Delta t\to 0^+} \Gamma_u(x_i + \Delta x_i, t_0 - \Delta t) \leqq \varepsilon,$$

with some Δx_i possibly equal to zero. Therefore

$$\lim_{\Delta t\to 0^+} \Gamma_u\Big(\sum \lambda_i (x_i+\Delta x_i), t_0-\Delta t\Big) \leqq \sum \lambda_i \lim_{\Delta t\to 0^+} \Gamma_u(x_i+\Delta x_i, t_0-\Delta t) \leqq \varepsilon.$$

Since $(x_0,t_0)\in Q$, and all Δx_i, Δt are small, by convexity of Γ_u it follows from [GH00, Lemma 1.1] that Γ_u is locally Lipschitz in x with a Lipschitz constant uniform in t. Hence

$$\begin{aligned}\lim_{\Delta t\to 0^+} \Gamma_u(x_0,t_0-\Delta t) &\leqq K\Big|\sum \lambda_i \Delta x_i\Big| + \lim_{\Delta t\to 0^+} \Gamma_u\Big(x_0+\sum \lambda_i\Delta x_i, t_0-\Delta t\Big)\\ &\leqq (K+1)\,\varepsilon.\end{aligned}$$

That is, $\lim_{\Delta t\to 0^+} \Gamma_u(x_0,t_0-\Delta t)=0$, and hence Γ_u is continuous at (x_0,t_0).

Proposition 7.1 (Regularity of Γ_u**).** *Let $u\in C^{2,1}\left(\overline{Q}\right)$, where Q is a bowl-shaped domain, $u=0$ on $\partial_p Q$, and $u<0$ in Q. Assume in addition that Q is defined by $Q=\{(x,t):\Phi(x,t)<0, t<T\}$ where Φ is p-convex, and that if $\Phi(x_0,t_0)=0$, then there exist $c>0$ so that $\Phi(x_0+\Delta x, t_0-\Delta t)\leqq 0$ for $|\Delta x|\leqq c\Delta t$.*[2] *Then Γ_u is locally in $W^{2,1}_\infty(Q)$ and $\mathcal{M}\Gamma_u \leqq \chi_{\mathcal{C}}\ \mathcal{M}u$, where $\chi_{\mathcal{C}}$ denotes the characteristic function of the contact set $\mathcal{C}$.*

Proof. If $(x_0,t_0)\in\mathcal{C}\cap Q$, then the proposition follows from Lemma 7.2. Suppose that $(x_0,t_0)\notin\mathcal{C}$. Let $K\Subset Q$ be compact such that $(x_0,t_0)\in K$, and L be a supporting hyperplane as in Lemma 7.3 and (x_i,t_i) the corresponding points.

Step 1. There exist a compact $K_0\Subset Q$ and a constant $C>0$, both depending only on K and u, and at least one (x_i,t_i), say (x_1,t_1), such that $(x_1,t_1)\in K_0$ with $\lambda_1\geqq C$. Indeed, let $-\delta_0=\max_K u<0$, and take $K_0\Subset Q$ such that

$$u>-\frac{\delta_0}{n+1} \qquad \text{in } \overline{Q}\setminus K_0.$$

Since $L(x_0)=L\left(\sum_{i=1}^{n+1}\lambda_i x_i\right)=\sum_{i=1}^{n+1}\lambda_i L(x_i)$, we get $-\delta_0\geqq u(x_0,t_0)\geqq L(x_0)=\sum_{i=1}^{n+1}\lambda_i\,u(x_i,t_i)$. Hence $\delta_0\leqq (n+1)\ \max_i \lambda_i|u(x_i,t_i)|$, and assuming the maximum is attained when $i=1$, we get $\delta_0\leqq (n+1)\,\lambda_1|u(x_1,t_1)|$. If $\lambda_1\leqq 1$ then

$$u(x_1,t_1)\leqq -\frac{\delta_0}{n+1},$$

that is $(x_1,t_1)\in K_0$ and consequently

$$\lambda_1\geqq \frac{\delta_0}{(n+1)\ \max_Q |u|}.$$

[2] This holds if for example Φ is Lipschitz in (x,t).

Step 2. The function $\Gamma_u(x, t_0)$ is $C^{1,1}$ in x.

Let $\Delta x < \operatorname{dist}(K, \partial_p Q)$. By (7.4), we write

$$\begin{aligned}
\Gamma_u(x_0 + \Delta x, t_0) &= \Gamma_u\left(\sum_{i>1} \lambda_i\, x_i + \lambda_1 \left(x_1 + \frac{\Delta x}{\lambda_1}\right), t_0\right) \\
&\leqq \sum_{i>1} \lambda_i\, \Gamma_u(x_i, t_0) + \lambda_1\, \Gamma_u\left(\left(x_1 + \frac{\Delta x}{\lambda_1}\right), t_1\right) \\
&\leqq \sum_{i>1} \lambda_i\, L(x_i) + \lambda_1 \left(L\left(x_1 + \frac{\Delta x}{\lambda_1}\right) + M\left|\frac{\Delta x}{\lambda_1}\right|^2\right) \qquad \text{by Lemma 7.2} \\
&= L(x_0 + \Delta x) + \frac{M}{\lambda_1}|\Delta x|^2.
\end{aligned}$$

Step 3. The function $\Gamma_u(x_0, t)$ is Lipschitz in t, $t \leqq t_0$.

By assumption, $(x_i + \Delta x_i, t_0 - \Delta t) \in Q$ with $|\Delta x| < C\Delta t$. From (7.4), we have

$$\begin{aligned}
\Gamma_u\left(x_0 + \sum_i \lambda_i \Delta x_i, t_0 - \Delta t\right) &= \Gamma_u\left(\sum_i \lambda_i (x_i + \Delta x_i), t_0 - \Delta t\right) \\
&\leqq \sum_i \lambda_i\, \Gamma_u((x_i + \Delta x_i), t_0 - \Delta t) \\
&\leqq \sum_i \lambda_i\, \Gamma_u((x_i + \Delta x_i), t_i - \Delta t) \\
&\leqq \sum_i \lambda_i\, \left(L(x_i + \Delta x_i) + M(|\Delta x_i|^2 + \Delta t)\right) \\
&\leqq L\left(x_0 + \sum_i \lambda_i \Delta x_i\right) + C\, M\, \Delta t.
\end{aligned}$$

On the other hand, since Γ_u is bounded in Q and convex in x by [GH00, Lemma 1.1] we have $|\Gamma_u(x_1, t) - \Gamma_u(x_2, t)| \leqq C\, |x_1 - x_2|$ for (x_1, t), (x_2, t) in a neighborhood of (x_0, t_0). Therefore,

$$\begin{aligned}
\Gamma_u(x_0, t_0 - \Delta t) &\leqq C\, |\sum_i \lambda_i \Delta x_i| + L(x_0 + \sum_i \lambda_i \Delta x_i) + CM\Delta t \\
&\leqq L(x_0 + \sum_i \lambda_i \Delta x_i) + 2\, C\, M\, \Delta t \leqq L(x_0) + C'\, M\, \Delta t,
\end{aligned}$$

and Step 3 is proved.

Step 4. The function $\Gamma_u(x, t_0)$ is affine in the simplex generated by $\{x_i\}$. In fact, let $x = \sum \mu_i x_i$ with $\mu_i \geqq 0$ and $\sum \mu_i = 1$. Since $\Gamma_u(x_i, t_i) = L(x_i)$ and$\Gamma_u(x, t) \geqq$

$\Gamma_u(x, t_0) \geqq L(x)$ for all x and $t \leqq t_0$, we get

$$L(x) \leqq \Gamma_u(\sum \mu_i x_i, t_0) \leqq \sum \mu_i \Gamma_u(x_i, t_i) = L(x),$$

and so $\Gamma_u(\sum \mu_i x_i, t_0) = L(\sum \mu_i x_i)$ which proves Step 4.

Consequently, $\det D_x^2 \Gamma_u(x, t_0) = 0$ for x in the simplex generated by $\{x_i\}$ and in particular for $x = x_0$. This completes the proof of the proposition.

Remark 7.1. Let Q be as in Proposition 7.1, so $\partial_p Q = \{(x, t) : \Phi(x, t) = 0\}$. Then Γ_u is continuous up to the boundary of Q and $\Gamma_u = 0$ on $\partial_p Q$. Let $(x_0, t_0) \in \partial_p Q$. Let $\Delta t > 0$ be small and $\ell(x) = D_x \Phi(x_{\Delta t}, t_0 + \Delta t) \cdot (x - x_{\Delta t})$be a supporting hyperplane for $Q \cap \{t = t_0 + \Delta t\}$ with $(x_{\Delta x}, t_0 + \Delta t) \in \partial_p Q$. Choose K very negative and $\varepsilon > 0$ small so that $K\ell(x) - \varepsilon \leqq u(x, t)$ in $Q \cap \{t \leqq t_0 + \Delta t\}$. Hence $K\ell(x) - \varepsilon \leqq \Gamma_u(x, t) \leqq u(x, t)$ in $Q \cap \{t \leqq t_0 + \Delta t\}$. Fixing for a moment Δt and $x_{\Delta t}$, since Φ is Lipschitz we get

$$-K\,C\,|x - x_{\Delta t}| - \varepsilon \leqq \Gamma_u(x, t) \leqq u(x, t),$$

and now letting $(x, t) \to (x_0, t_0)$ yields

$$-K\,C\,|x_0 - x_{\Delta t}| - \varepsilon \leqq \liminf_{(x,t)\to(x_0,t_0)} \Gamma_u(x, t) \leqq 0.$$

Letting $\Delta t \to 0$ we get $x_{\Delta t} \to x_0$ and consequently

$$-\varepsilon \leqq \liminf_{(x,t)\to(x_0,t_0)} \Gamma_u(x, t) \leqq 0,$$

and so $\Gamma_u(x_0, t_0) = 0$.

Corollary 7.1. *Let $u \in C(\bar{Q}) \cap C^{2,1}(Q)$ with $u = 0$ on $\partial_p Q$, $u < 0$ in Q, a bowl-shaped domain whose defining function is Lipschitz in x. Then $\Gamma_u \in C(\bar{Q})$ and $\Gamma_u = 0$ on $\partial_p Q$ and*

$$\mathcal{M}\Gamma_u \leqq \chi_{\mathcal{C}}\; \mathcal{M}u,$$

where $\chi_{\mathcal{C}}$ denotes the characteristic function of the contact set $\mathcal{C}$.

Proof. The first part follows from the previous remark.

Let ϕ be a mollifier in $\mathbb{R}$ and

$$f_\varepsilon(x) = \int_{|y|\leqq 1} \phi(y)\, g_\varepsilon\Big(x - \frac{\varepsilon}{3} y\Big)\, dy,$$

where

$$g_\varepsilon(x) = \begin{cases} 0 & \text{for } x > -4\varepsilon/3, \\ 5\left(x + \dfrac{4\varepsilon}{3}\right) & \text{for } -5\varepsilon/3 < x < -4\varepsilon/3, \\ x & \text{for } x < -5\varepsilon/3. \end{cases}$$

Then $f_\varepsilon \in C^\infty$ and

$$f_\varepsilon(x) = \begin{cases} 0 & \text{for } x > -\varepsilon, \\ \uparrow & \text{for } -2\varepsilon \leqq x \leqq -\varepsilon, \\ x & \text{for } x < -2\varepsilon. \end{cases}$$

Let $u_\varepsilon = f_\varepsilon(u) \to u$ in $C(\bar{Q})$. Take $Q_\varepsilon \uparrow Q$, where Q_ε is a smooth bowl-shaped domain such that $u_\varepsilon \leqq 0$ in a small neighborhood of $\partial_p Q_\varepsilon$. Then $u_\varepsilon \in C^{2,1}(\bar{Q})$, and applying Proposition 7.1 to u_ε yields

$$\mathcal{M}\Gamma_{u_\varepsilon, Q_\varepsilon} \leqq \mathcal{M}u_\varepsilon \ \chi_{\{u_\varepsilon = \Gamma_{u_\varepsilon, Q_\varepsilon}\}}.$$

Since $\Gamma_{u_\varepsilon, Q_\varepsilon} \to \Gamma_{u,Q}$ and $\mathcal{M}u_\varepsilon = \mathcal{M}u$ for $K \Subset Q$ compact, we obtain $\mathcal{M}\Gamma_{u,Q} \leqq \mathcal{M}u \ \chi_{\{u=\Gamma_{u,Q}\}}$.

Acknowledgements. CRISTIAN GUTIÉRREZ was partially supported by the NSF. The work was carried out while QINGBO HUANG was at the University of Texas at Austin.

References

[Caf90a] L. A. CAFFARELLI. Interior $W^{2,p}$ estimates for solutions of the Monge-Ampère equation. *Ann. of Math.*, **131**, 135–150, 1990.

[Caf90b] L. A. CAFFARELLI. A localization property of viscosity solutions to the Monge-Ampère equation and their strict convexity. *Ann. of Math.*, **131**, 129–134, 1990.

[Caf91] L. A. CAFFARELLI. Some regularity properties of solutions of Monge-Ampère equation. *Comm. Pure Appl. Math.*, **44**, 965–969, 1991.

[Cal58] E. CALABI. Improper affine hyperspheres of convex type and a generalization of a theorem by K. Jörgens. *Michigan Math. J.*, **5**, 105–126, 1958.

[CG96] L. A. CAFFARELLI & C. E. GUTIÉRREZ. Real analysis related to the Monge-Ampère equation. *Trans. A.M.S.*, **348**, 1075–1092, 1996.

[CG97] L. A. CAFFARELLI & C. E. GUTIÉRREZ. Properties of the solutions of the linearized Monge-Ampère equation. *Amer. J. Math.*, **119**, 423–465, 1997.

[Fir74] W. J. FIREY. Shapes of worn stones. *Mathematika*, **21**, 1–11, 1974.

[GH98] C. E. GUTIÉRREZ & QINGBO HUANG. A generalization of a theorem by Calabi to the parabolic Monge-Ampère equation. *Indiana Univ. Math. J.*, **47**, 1459–1480, 1998.

[GH00] C. E. GUTIÉRREZ & QINGBO HUANG. Geometric properties of the sections of solutions to the Monge-Ampère equation. *Trans. A.M.S.*, **352**, 4381–4396, 2000.

[Gut] C. E. GUTIÉRREZ. *The Monge-Ampère equation*. Book Progress in Nonlinear Differential equations and their applications, Volume **44**, Birkhäuser, Boston, 2001.

[Hua99] QINGBO HUANG. Harnack inequality for the linearized parabolic Monge-Ampère equation. *Trans. A.M.S.*, **351**, 2025–2054, 1999.

[Kry76] N. V. KRYLOV. Sequences of convex functions and estimates of the maximum of the solution of a parabolic equation. *Siberian Math. J.*, **17**, 226–236, 1976.

[Lie96] G. LIEBERMAN. *Second order parabolic differential equations*. World Scientific Publishing Co., Singapore, 1996.

[Pog71] A. V. POGORELOV. On the regularity of generalized solutions of the equation $\det(\frac{\partial^2 u}{\partial x^i \partial x^j}) = \phi(x^1, \cdots, x^n) > 0$. *Soviet Math. Dokl.*, **12**, 1436–1440, 1971.

[Pog78] A. V. POGORELOV. *The Minkowski Multidimensional Problem.* John Wiley & Sons, Washington, D. C., 1978.

[Sch93] R. SCHNEIDER. *Convex bodies: the Brunn-Minkowski theory*, volume 44 of *Encyclopedia of Math. and its Appl.* Cambridge U. Press, Cambridge, UK, 1993.

[Tso85a] KAISING TSO. Deforming a hypersurface by its Gauss-Kronecker curvature. *Comm. Pure Appl. Math.*, **38**, 867–882, 1985.

[Tso85b] KAISING TSO. On an Aleksandrov-Bakelman type maximum principle for second-order parabolic equations. *Comm. Partial Differential Equations*, **10**, 543–553, 1985.

[WW92] ROUHUAI WANG & GUANGLIE WANG. On existence, uniqueness and regularity of viscosity solutions for the first initial boundary value problem to parabolic Monge-Ampère equation. *Northeastern Math. J.*, **88**, 417–446, 1992.

[WW93] ROUHUAI WANG & GUANGLIE WANG. The geometric measure theoretical characterization of viscosity solutions to parabolic Monge-Ampère type equation. *J. Partial Diff. Eqs.*, **6**, 237–254, 1993.

Department of Mathematics
Temple University
Philadelphia, PA 19122
e-mail: gutierrez@math.temple.edu

and

Department of Mathematics
Wright State University
Dayton, Ohio 45435
e-mail: qhuang@math.wright.edu

(*Accepted January 23, 2001*)