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Asymptotic Stability of Solitons for Subcritical Generalized KdV Equations

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Abstract

We prove in this paper the asymptotic completeness of the family of solitons in the energy space for generalized Korteweg-de Vries equations in the subcritical case (this includes in particular the KdV equation and the modified KdV equation). This result is obtained as a consequence of a rigidity theorem on the flow close to a soliton up to a scaling and a translation, which has its own interest. The proofs use some tools introduced in a previous paper to prove similar results in the case of critical generalized KdV equation.

1. Introduction

In this paper, we consider generalized Korteweg-de Vries equations in the subcritical case

$$\begin{aligned} &u_t + (u_{xx} + u^p)_x = 0, \qquad (t, x) \in \mathbf{R} \times \mathbf{R}, \\ &u(0, x) = u_0(x), \qquad x \in \mathbf{R}, \end{aligned} \tag{1}$$

for $p = 2, 3, 4$ and $u_0 \in H^1(\mathbf{R})$. These models appear in the study of waves on shallow water (see KORTEWEG & DE VRIES [11]), as well as in other areas of physics (see, e.g., LAMB [12]). These equations, together with nonlinear Schrödinger equations, are considered as universal models for Hamiltonian systems in infinite dimensions. From the Hamiltonian structure, we have formally the following two conservation laws in time:

$$\int u^2(t) = \int u_0^2, \tag{2}$$

and

$$\frac{1}{2}\int u_x^2(t) - \frac{1}{p+1}\int u^{p+1}(t) = \frac{1}{2}\int u_{0x}^2 - \frac{1}{p+1}\int u_0^{p+1}. \tag{3}$$

From these conservation laws, H^1 appears as an energy space, so that it is a natural space in which to study the solutions.

In equation (1), $p = 2$, corresponding to the KdV equation, is a special case in the theory. Indeed, from the integrability (see LAX [13] and MIURA [17]), we have for suitable u_0 (u_0 and its derivatives with exponential decay at infinity) an infinite number of conservation laws. In this situation, the problem can be transformed through nonlinear estimates into a linear problem. Unfortunately, these techniques, in order to be relevant, require very regular solutions with fast decay at infinity. The results that we will present are new in $H^1(\mathbf{R})$, even for $p = 2$.

Recall that the well-posedness of (1) in the energy space is now well understood: for any $p = 2, 3, 4$, it follows from the results of KENIG, PONCE & VEGA [9] that, for all $u_0 \in H^1(\mathbf{R})$, there exists a unique solution $u \in C(\mathbf{R}, H^1(\mathbf{R}))$ of (1) satisfying (2) and (3) for all $t \in \mathbf{R}$ (see Corollaries 2.2, 2.5 and 2.7 in [9]). The proof of this result involves oscillatory-integrals techniques and sharp space-time estimates. Throughout this paper we work in this framework, though in order to obtain compactness results in H^1, we also use the fact that the Cauchy problem is well posed in H^{s^*} for some $0 < s^* < 1$ ([9]).

We refer to KATO [8], GINIBRE & TSUTSUMI [6], for previous results on the H^s theory ($s > \frac{3}{2}$) and to BOURGAIN [3] for the case of periodic initial data.

In the subcritical case $p = 2, 3, 4$, the global existence of all solutions in H^1, as well as uniform bound in H^1, follow from the Gagliardo-Nirenberg inequality,

$$\forall\, v \in H^1(\mathbf{R}), \quad \int |v|^{p+1} \leqq C(p) \left(\int v^2\right)^{\frac{p+3}{4}} \left(\int v_x^2\right)^{\frac{p-1}{4}},$$

and relations (2), (3), which imply $\forall\, t \in \mathbf{R}$, $\int u^2(t) = \int u_0^2$ and $\int |u_x(t)|^2 \leqq C(|u_0|_{H^1})$.

This is in contrast with the critical and supercritical cases (resp., $p = 5$ and $p > 5$, p integer), where local well-posedness in H^1 is known from [9], but the existence of singularity in finite or infinite time for some solutions of (1) (i.e., $|u(t)|_{H^1} \to +\infty$ as $t \uparrow T$, for some $T > 0$) is conjectured. Note however that it is still an open problem.

In fact, few results on the qualitative properties of the solutions are known for these equations, for any $p > 1$.

Equation (1) has traveling-wave solutions (called solitons), of the form

$$u(t, x) = R_c(x - ct),$$

where $c > 0$ and $R_c > 0$ satisfies the equation

$$R_c \in H^1(\mathbf{R}), \quad R_{cxx} + R_c^p = cR_c,$$

or equivalently, by integration,

$$R_{cx}^2 + \frac{2}{p+1} R_c^{p+1} = cR_c^2 \quad \text{and} \quad R_c(x) = \left(\frac{c(p+1)}{2\mathrm{ch}^2\left(\frac{p-1}{2}\sqrt{c}\, x\right)}\right)^{\frac{1}{p-1}}. \tag{4}$$

The study of the flow around these solitons is crucial for the understanding of the generic behavior of solutions of (1). Indeed, these Hamiltonian systems in infinite dimension are known to have complicated dynamics even for short time (involving time oscillations). Therefore, any result of classification of solutions is relevant. Thanks to the structure of the equation near the solitons, a geometrical approach related to a refined control on the dispersion will reduce the complexity of the equation and allows us to look for an asymptotic regime for large time.

The notions of stability and asymptotic stability of the solitons are crucial. We say that for $c > 0$ the soliton $R_c(x - ct)$ is stable in H^1 if

$$\forall\, \delta_0 > 0,\ \exists\, \alpha_0 > 0\ /\ |u_0 - R_c|_{H^1} \leqq \alpha_0 \ \Rightarrow \quad \forall\, t \geqq 0,\ \exists\, x(t)\ /\ |u(t) - R_c(. - x(t))|_{H^1} \leqq \delta_0. \tag{5}$$

We say that the family of solitons $\{R_c(x - x_0 - ct),\ c > 0,\ x_0 \in \mathbf{R}\}$ is asymptotically stable if

$$\exists\, \alpha_0 > 0\ /\ |u_0 - R_c|_{H^1} \leqq \alpha_0 \ \Rightarrow \quad \forall\, t \geqq 0,\ \exists\, c(t), x(t)\ /\ u(t, . + x(t)) - R_{c(t)} \underset{t\to+\infty}{\rightharpoonup} 0 \text{ in } H^1. \tag{6}$$

We recall previously known results concerning the stability of the solitons and the asymptotic stability of the family of solitons, respectively in the subcritical, supercritical and critical cases:

- In the subcritical case, $p = 2, 3, 4$, it follows from energetic arguments that the solitons are H^1 stable (see Cazenave & Lions [5], Weinstein [22], and Bona, Souganidis & Strauss [2]). Moreover, Pego & Weinstein [19] proved that the family of solitons is asymptotically stable in the case $p = 2$ (KdV equation) and $p = 3$ (modified KdV equation), for initial data with exponential decay as $x \to +\infty$. In [19], the case of initial data in the energy space is not treated, for any value of p.
- In the supercritical case $p > 5$, numerical simulations (see Bona *et al.* [1] and references therein) suggest that blow-up in finite time occurs for some initial data. However, no rigorous proof of existence of such singular solutions exists. For $p > 5$, Bona, Souganidis & Strauss [2] proved, using Grillakis, Shatah & Strauss [7] type arguments, the H^1 instability of solitons. Their proof does not apply to the case $p = 5$. For example, the proof uses the vector $y = \frac{R_1}{2} + xR_{1x}$, which satisfies $(\mathcal{L}_c y, y) = 0$ for $p = 5$ (see top of page 406 in [2]). In fact, the existence of a suitable y in their proof is based on the condition $d''(c) < 0$ or equivalently $p > 5$, see Theorem 3.1 in [2].
- In the critical case, $p = 5$,

$$\begin{aligned} u_t + (u_{xx} + u^5)_x = 0, &\qquad (t, x) \in \mathbf{R}^+ \times \mathbf{R}, \\ u(0, x) = u_0(x), &\qquad x \in \mathbf{R}, \end{aligned} \tag{7}$$

  the problem becomes degenerated. Indeed, while in the subcritical and supercritical cases, for a given L^2 norm, there is one and only one soliton (up to translation), in the critical case, we have $\forall\, c > 0$, $|R_c|_{L^2} = |R_1|_{L^2}$ and moreover

$E(R_c) = 0$. In particular, all the solitons lie on the same level set of these two quantities. This property will have a consequent effect on the structure of the linearized operator around the solitons (see Section 4).

Note that the variational structure of the critical generalized KdV equation presents a lot of similarities with the one of the critical nonlinear Schrödinger equation in one space dimension. Indeed, on the one hand, it follows from the invariants (2) and (3) and a sharp Gagliardo-Nirenberg inequality that, for initial data satisfying $|u_0|_{L^2} < |R_1|_{L^2}$, the solution of (7) is global and bounded in H^1.

On the other hand, it is conjectured that there exist blow-up solutions of (7) such that $|u_0|_{L^2} \geqq |R_1|_{L^2}$ (see numerical simulations in BONA *et al.* [1]), though no rigorous proof of existence of a blow-up solution is available at this moment.

A first result in the direction of blow-up for the critical generalized KdV equation is established in MARTEL & MERLE [14], by showing that $u(t,x) = R_1(x-t)$ is unstable in H^1 (by scaling argument, all solitons are unstable). This result is proved in a qualitative way, finding the interior of a parabola as the instability region.

Next, in MARTEL & MERLE [15], we established some crucial properties in the direction of understanding the structure of the equation close to R_1: in a neighborhood of

$$R^*_{c_1,c_2} = \{R_{c_0}(x-x_0)\ ,\ c_0 \in (c_1,c_2),\ x_0 \in \mathbf{R}\}, \quad (0 < c_1 < c_2).$$

In particular, we proved a rigidity theorem on (7) close to $R^*_{c_1,c_2}$ in the energy space (i.e., a characterization of the soliton) related to the notion of dispersion. In some sense, this Liouville theorem says that if a solution defined for all $t \in \mathbf{R}$ is close at some point to a soliton and does not disperse, then it is exactly a soliton.

This Liouville theorem and the monotonicity in time of a quantity related to the dispersive effect of the linear Airy equation imply the asymptotic completeness of the solitons. Nevertheless, the proof is based on some specific algebraic properties due to the special structure of the critical case, and on an extensive use of the local well-posedness of the Cauchy problem in L^2, which is not valid in the subcritical case. See [15] for precise statements in the critical case $p = 5$.

In this paper, we prove similar results in the subcritical case. First, we state the result of asymptotic completeness of the family of solitons in the energy space.

Theorem 1 (Asymptotic stability for $p = 2, 3, 4$**).** *Let $p = 2, 3$ or 4, and let $c_0 > 0$. Let $u_0 \in H^1(\mathbf{R})$, and let $u(t)$ be the solution of* (1) *on $\mathbf{R}^+ \times \mathbf{R}$. There exists $\alpha_0 > 0$ such that if $|u_0 - R_{c_0}|_{H^1} < \alpha_0$, then there exist $c_{+\infty} > 0$ and a function $x(t)$ such that*

$$u(t, . + x(t)) \rightharpoonup R_{c_{+\infty}} \quad \text{in } H^1(\mathbf{R}) \text{ as } t \to +\infty.$$

Remark 1. The conclusion of Theorem 1 is also true for some $c_{-\infty}$ as $t \to -\infty$, by changing $x \to -x$, $t \to -t$.

Note that the result implies convergence in $L^2(\mathbf{R}, e^{-|x|}dx)$ and $L^\infty_{\text{loc}}(\mathbf{R})$. A result of strong convergence in $L^2(\mathbf{R})$ is false. For example, for $p = 2$, using integrability, we can construct a solution such that for a small $\epsilon > 0$,

$$|u(t, . + x(t)) - R_1 - R_\epsilon(. - x_\epsilon(t))|_{L^\infty} \underset{t\to+\infty}{\to} 0, \quad \text{where } x_\epsilon(t) \to -\infty.$$

Moreover, for $p = 2, 3, 4$, convergence in L^2 strong as $t \to +\infty$ in Theorem 1 implies that the solution is exactly a soliton by using the characterization of the soliton given in Theorem 2, see below (indeed, in this case, the solution is L^2 compact).

Note that, from the proof of Theorem 1, the L^2 dispersion occurs only at the left of the soliton.

Remark 2. For other results of this type for subcritical generalized KdV equations, we refer to PEGO & WEINSTEIN [19] (see also MIZUMACHI [18]). Their approach is based on linear theory around R_1 which allows only initial data with fast decay at $+\infty$ (exponential or polynomial decay). Therefore, the results are obtained in spaces different from the energy space of the nonlinear problem H^1. Moreover, this approach requires in some sense that the interaction between the linear dynamics and the nonlinear dynamics is decoupled enough. See also the result of BOURGAIN & WANG [4] for the critical nonlinear Schrödinger equation where decoupling is essential and requires the introduction of weighted spaces for the function and some of its derivatives as well as some additional orthogonality conditions on the initial data. Our approach does not rely on decoupling and thus we do not need this kind of assumption.

Note that due to the lack of control on the size of the tail of the solution in Theorem 1, we do not have precise control of the convergence rate to the soliton as in [19].

Remark 3. From the proof of Theorem 1, we also have $x_t(t) \to c_{+\infty}$ as $t \to +\infty$.

In the critical case, the result of asymptotic stability is stated as follows: $\exists\, c(t), x(t)$ such that

$$u(t, . + x(t)) - R_{c(t)} \rightharpoonup 0 \quad \text{in } H^1(\mathbf{R}) \text{ as } t \to +\infty.$$

(see Theorem 2 in [15]). This can be explained as follows. In both the critical and subcritical cases, the L^2_{loc} norm of the solution is monotone in time in some sense. In the subcritical case, the limit as $t \to +\infty$ of the L^2_{loc} norm of $u(t)$ selects the asymptotic soliton since, for a given L^2 norm, there is only one soliton up to translation; this is crucial for obtaining the limit of $c(t)$. In the critical case, the monotonicity of the L^2_{loc} norm does not prevent oscillations of $c(t)$ as $t \to +\infty$ since all solitons have same L^2 norm; this is an illustration of degeneracy. From the possible existence of blow-up solutions close to $R_{c(t)}$ for $c(t) \to +\infty$, we think that, in the critical case, there do exist solutions for which $c(t)$ oscillates between two values.

The proof of Theorem 1 is based on the following Liouville Theorem.

Theorem 2 (Liouville property close to R_{c_0} for $p = 2, 3, 4$**).** *Let* $p = 2, 3$ *or* 4, *and let* $c_0 > 0$. *Let* $u_0 \in H^1(\mathbf{R})$, *and let* $u(t)$ *be the solution of* (1) *for all time* $t \in \mathbf{R}$. *There exists* $\alpha_0 > 0$ *such that if* $|u_0 - R_{c_0}|_{H^1} < \alpha_0$, *and if there exists* $x(t)$ *such that* $v(t, x) = u(t, x + x(t))$ *satisfies*

$$\forall\, \delta_0 > 0, \exists\, A_0 > 0, \forall\, t \in \mathbf{R}, \quad \int_{|x|>A_0} v^2(t, x)dx \leqq \delta_0, \quad (L^2 \textit{ compactness}), \tag{8}$$

then there exists $c_1 > 0$, $x_1 \in \mathbf{R}$ *such that*

$$\forall\, t \in \mathbf{R}, \forall\, x \in \mathbf{R}, \quad u(t,x) = R_{c_1}(x - x_1 - c_1 t).$$

Remark 4. From [22], we know that R_{c_0} is stable, in the sense that (5) is true for R_{c_0}. In the above theorems, $u(t)$ is defined for all $t \in \mathbf{R}$ since we consider subcritical p's, whereas the further property

$$\forall\, t \in \mathbf{R}, \quad C_1 \leqq |u(t)|_{H^1} \leqq C_2 \tag{9}$$

is implied by the H^1 stability of R_{c_0} and the assumption that $|u_0 - R_{c_0}|_{H^1}$ is small.

In the critical case, (9) is not necessarily true under the same assumptions because of the possible existence of blow-up solutions close to the family of solitons. This is the reason why it is given as an additional assumption in the results of [15].

Remark 5. For a class of nonlinear parabolic equation

$$u_t = \Delta u + |u|^{p-1} u,$$

where $u : \mathbf{R}^N \to \mathbf{R}^M$, and $1 < p < \frac{N+2}{N-2}$, a Liouville theorem related to blow-up solutions has been established by MERLE & ZAAG [16]. Of course, the structure of the problem and the proof for the KdV equations are completely different.

Remark 6. It might be expected that the Liouville property would still true without the smallness of $|u_0 - R_{c_0}|_{H^1}$. However, at least for $p = 3$, a counter example (see references in LAMB [12] and KENIG, PONCE & VEGA [10]) proves that the smallness condition is necessary. This question for other values of p is open. Indeed, for $p = 3$, the following solutions of (1) (called breather solutions)

$$-\frac{2\sqrt{6}\,\omega}{\mathrm{ch}(\omega x + \gamma t)} \left(\frac{\cos(Nx + \delta t) - (\omega/N)\sin(Nx + \delta t)\tanh(\omega x + \gamma t)}{1 + (\omega/N)^2 \sin^2(Nx + \delta t)/\mathrm{ch}(\omega x + \gamma t)} \right),$$

where $\delta = N(N^2 - 3\omega^2)$ and $\gamma = \omega(3N^2 - \omega^2)$, are example of global in time solutions, without dispersion, which are not of the type $R_c(x - ct)$.

Remark 7. Note that apart from being a crucial result for proving the asymptotic completeness of the family of solitons, the Liouville property has its own interest. Indeed, it gives a characterization of the solitons: it says that, locally in the space H^1, the solitons are the only solutions which do not disperse. Extension of this classification result to all H^1 would give the generic behavior of the solutions of (1). Recall that in the case $p = 3$ there is another family of nondispersive solutions (see preceding remark).

In the proof of Theorem 1, the L^2 localization condition (8) is satisfied by an asymptotic object as $t \to +\infty$, and therefore it is a natural assumption in this problem. Moreover, from the proof of Theorem 2, we will see that the condition of L^2 localization uniform in time around the center of mass implies, surprisingly, a far more precise result of uniform pointwise exponential decay (see Section 3). In conclusion, an asymptotic object such as the one constructed in the proof of Theorem 1 has uniform exponential decay in space.

Now, we give some notation and an outline of the methods used in the proofs. The method used in the proof will be close to the one used in the critical case (see [15]); however various technical differences will be pointed out. We consider $p = 2, 3, 4$.

By scaling, in Theorems 1 and 2, we can restrict ourselves to the case where $c_0 = 1$ (if $u(t, x)$ is solution then for $\lambda_0 > 0$, $\lambda_0^{\frac{2}{p-1}} u(\lambda_0^3 t, \lambda_0 x)$ is also a solution). We note that

$$Q(x) = R_1(x) = \left(\frac{p+1}{2\mathrm{ch}^2\left(\frac{p-1}{2} x\right)} \right)^{\frac{1}{p-1}},$$

so that Q satisfies $Q_{xx} = Q - Q^p$.

Throughout this paper, we consider a solution $u(t)$ of (1) with $|u_0 - Q|_{H^1} \leqq \alpha_0$, where α_0, to be chosen later, is small. By the stability of Q in H^1, we have, for some $y(t)$,

$$\forall t, \quad |u(t) - Q(x - y(t))|_{H^1} \leqq \epsilon(\alpha_0) \tag{10}$$

where $\epsilon(\alpha_0) \to 0$ as $\alpha_0 \to 0$. From modulation theory and the invariances of the equation (if $u(t, x)$ is a solution then for $\lambda_0 > 0$ and $x_0 \in \mathbf{R}$, $\lambda_0^{\frac{2}{p-1}} u(\lambda_0^3 t, \lambda_0 x + x_0)$ is also a solution), it is natural to set

$$v(t, y) = \lambda^{\frac{2}{p-1}}(t) u(t, \lambda(t) y + x(t)) \quad \text{and} \quad \varepsilon(t, y) = v(t, y) - Q(y),$$

where the geometrical parameters $\lambda(t)$, $x(t)$ are chosen so that

$$\forall t \in \mathbf{R}, \quad (\varepsilon(t), Q) = (\varepsilon(t), Q_y) = 0$$

(see [14], [15]). Recall that the orthogonality with respect to Q_y is related to a choice of center of mass $x(t)$, while $\lambda(t)$, the scaling of the approximate soliton, is chosen through orthogonality with respect to Q. There are many other possibilities (small perturbations of the functions above for example).

Note that in the subcritical case, such a choice is possible since

$$\begin{aligned} \left(\frac{d}{d\lambda} \int \lambda^{\frac{2}{p-1}} Q(\lambda x) Q(x)\, dx \right)_{\lambda=1} &= \int \left(\frac{2Q}{p-1} + x Q_x \right) Q \\ &= \frac{5-p}{2(p-1)} \int Q^2 \neq 0 \end{aligned} \tag{11}$$

and

$$\left(\frac{d}{dx'} \int Q(x + x') Q_x(x)\, dx \right)_{x'=0} = \int Q_x^2 \neq 0. \tag{12}$$

(See Proposition 1 in [14].) In this paper the orthogonality with respect to Q is strongly used. The choice of the orthogonality condition $(\varepsilon(t), Q) = 0$ was not possible in the critical case since $\left(\left(\frac{1}{2} Q + x Q_x\right), Q\right) = 0$ in this case. However, from an algebraic property of the linearized operator in the critical case, this condition was automatically recovered for the asymptotic linear problem considered in Section 5 (see [15], Part B).

If we change the time variable as follows,

$$s = \int_0^{t'} \frac{dt}{\lambda^3(t')} \quad \text{or equivalently} \quad \frac{ds}{dt} = \frac{1}{\lambda^3},$$

then ε satisfies, by direct calculations, for $s \in \mathbf{R}$, $y \in \mathbf{R}$,

$$\varepsilon_s = (L\varepsilon)_y + \frac{\lambda_s}{\lambda}\left(\frac{2Q}{p-1} + yQ_y\right) + \left(\frac{x_s}{\lambda} - 1\right) Q_y + \frac{\lambda_s}{\lambda}\left(\frac{2\varepsilon}{p-1} + y\varepsilon_y\right)$$
$$+ \left(\frac{x_s}{\lambda} - 1\right)\varepsilon_y - ((Q+\varepsilon)^p - (Q^p + pQ^{p-1}\varepsilon))_y, \tag{13}$$

where

$$L\varepsilon = L_p\varepsilon = -\varepsilon_{xx} + \varepsilon - pQ^{p-1}\varepsilon. \tag{14}$$

Note that $|(Q+\varepsilon)^p - (Q^p + pQ^{p-1}\varepsilon)| \leqq C\varepsilon^2$. By (10), there exists $\epsilon_1(\alpha_0) \to 0$ as $\alpha_0 \to 0$ such that we have

$$\forall s, \quad |\varepsilon(s)|_{H^1} + |\lambda(s) - 1| \leqq C\epsilon_1(\alpha_0). \tag{15}$$

Finally, note that with the above choice of orthogonality conditions on ε, using (11), the properties $LQ_y = 0$, $LQ_{yy} = p(p-1)Q^{p-2}Q_y^2$, and parity properties (see Lemma 4 in [14] for similar calculations), we have the following relations between $\frac{\lambda_s}{\lambda}$ and $\frac{x_s}{\lambda} - 1$:

$$\frac{\lambda_s}{\lambda}\left(\frac{5-p}{2(p-1)}\int Q^2 - \int yQ_y\varepsilon\right) + \int Q_y R(\varepsilon) = 0, \tag{16}$$

$$-\frac{\lambda_s}{\lambda}\int yQ_{yy}\varepsilon + \left(\frac{x_s}{\lambda} - 1\right)\left(\frac{1}{2}\int Q^2 - \int Q_{yy}\varepsilon\right)$$
$$-p(p-1)\int Q^{p-2}Q_y^2\varepsilon + \int Q_{yy}R(\varepsilon) = 0. \tag{17}$$

In particular, this gives smallness of $\frac{\lambda_s}{\lambda}$ and $\frac{x_s}{\lambda} - 1$.

As in the critical case, Theorem 2 is equivalent to the following proposition in terms of ε.

Proposition 1 (Liouville Theorem for ε**).** *There exists* $a_1 > 0$ *such that if* $\varepsilon \in C(\mathbf{R}, H^1(\mathbf{R})) \cap L^\infty(\mathbf{R}, H^1(\mathbf{R}))$ *is a solution of* (13) *on* $\mathbf{R} \times \mathbf{R}$ *satisfying* $|\varepsilon(0)|_{H^1} \leqq a_1$ *and*

(H1) *Orthogonality conditions:*

$$\forall s \in \mathbf{R}, \quad (\varepsilon(s), Q) = (\varepsilon(s), Q_y) = 0,$$

(H3) L^2 *compactness:* $\forall\, \delta_0 > 0,\ \exists\, A_0(\delta_0) > 0$, *such that*

$$\forall\, s \in \mathbf{R}, \quad |\varepsilon(s)|_{L^2(|y|>A_0)} \leqq \delta_0,$$

then $\varepsilon \equiv 0$ *on* $\mathbf{R} \times \mathbf{R}$.

Theorem 1 will be a consequence of the following proposition.

Proposition 2 (Asymptotic behavior of ε**).** *Let*

$$\varepsilon \in C(\mathbf{R}^+, H^1(\mathbf{R})) \cap L^\infty(\mathbf{R}^+, H^1(\mathbf{R}))$$

be a solution of (13) *on* $\mathbf{R}^+ \times \mathbf{R}$. *There exists* $a_2 > 0$ *such that if* $|\varepsilon(0)|_{H^1} \leqq a_2$, *then* $\varepsilon(s) \rightharpoonup 0$ *in* $H^1(\mathbf{R})$ *as* $s \to +\infty$.

Remark 8. Recall that, from the stability of Q in the subcritical case and the decomposition, we have under the assumptions of Propositions 1, 2:

(H2) H^1 *bounds:* there exists $\lambda_1, \lambda_2 > 0$ such that $\forall\, s \in \mathbf{R}$, $\lambda_1 \leqq \lambda(s) \leqq \lambda_2$.

In the critical case [15], (H2) is an additional assumption, which is necessary due to the possible existence of blow-up solutions.

The rest of this paper is organized as follows. Section 2 is devoted to basic estimates on (1). In Section 3, we show that the proof of Theorem 1 (asymptotic stability) can be reduced to the proof of the Liouville property, Theorem 2. This is the analogue of Part C in [15]. The main difference comes from the spaces in which the Cauchy problem is solved (see [9]). It introduces some technical modifications. As in [15], we use a quantity which measures the mass of the solution at the right of the soliton, and we control the variation in time of this quantity. The conclusion is that as $t \to +\infty$, the solution has to remain L^2 compact, which reduces the rest of the proof to a nonlinear Liouville property (Theorem 2). To obtain the convergence of the scaling parameter $\lambda(t)$ as $t \to +\infty$, we use a monotony property in L^2 and the subcriticality of p (i.e., there is a unique R_c (up to translation) of given L^2 norm), see Remark 3.

In Section 4, we show that the nonlinear Liouville property close to Q is equivalent to a Liouville property on a linear problem. To do this, as in Part A of [15], we introduce a sequence $w_n = \frac{\varepsilon_n}{a_n}$ of renormalized solutions of (13) ($a_n = \sup_{s\in\mathbf{R}} |\varepsilon_n(s)|_{H^1}$). We show that the L^2 compactness leads to the following properties:

$$\forall\, s \in \mathbf{R},\ \forall\, y \in \mathbf{R}, \quad |w_n(s, y)| \leqq Ce^{-c_2|y|}, \quad \text{and} \quad |w_n(s)|_{L^2} \geqq c > 0, \tag{18}$$

which implies the equivalence of all norms of w_n. The proof of (18) is based on the nonlinear decomposition of ε_n in a nonlinear part, which decays in time, and a localized part, which decays in space on the right. The decay in time of the nonlinear part is obtained through a monotonicity property of small solutions of (7) proved in Section 2. Note that the same technique applies in the critical case, even if in [15] we have shown this property in a different way.

Finally, in Section 5, we prove the linear Liouville property. This is where the main differences with the critical case $p = 5$ appear. Indeed, in the subcritical case, the linear operator has less structure, see Remark 13. However, in the three cases $p = 2, 3, 4$, we can reduce the proof of the linear Liouville property to the study of a quadratic form. As in the critical case, it is crucial that the linear operators are classical ones, so we can do explicit calculations.

2. Preliminaries

In the critical case, we have established a monotonicity property for (1). This property says that in some sense, the mass of the solution cannot travel to the right of the solitons. We recall two main results in this direction.

For $K > 0$ to be chosen later, define

$$\forall\, x \in \mathbf{R}, \quad \phi(x) = \phi_K(x) = cQ\left(\frac{x}{K}\right), \qquad \psi(x) = \psi_K(x) = \int_{-\infty}^{x} \phi(y)\, dy,$$

where

$$c = \frac{K}{\displaystyle\int_{-\infty}^{+\infty} Q(y)\ dy},$$

so that $\psi(x) \to 0$ as $x \to -\infty$ and $\psi(x) \to 1$ as $x \to +\infty$.

Let z be a solution of (1), and define, for $\sigma > 0$,

$$\forall\, t \geqq 0, \quad \mathcal{I}(t) = \mathcal{I}_\sigma(t) = \int z^2(t,x)\psi(x - \sigma t)\ dx.$$

Lemma 1 (Monotonicity of $\mathcal{I}$ for small solutions of (1)**).** *For any $\sigma > 0$, if $K \geqq \sqrt{\frac{2}{\sigma}}$, and*

$$\sup_{t \geqq 0} |z(t)|_{L^\infty} \leqq d_0 = \left(\frac{(p+1)\sigma}{8p}\right)^{\frac{1}{p-1}}, \tag{19}$$

then the function $\mathcal{I}$ is nonincreasing in t.

Proof. The proof is very similar to the proof for the critical case, see Lemma 16 in [15]. Recall that it makes use of a Virial-type identity for (1): for every C^3 function φ, we have

$$\frac{d}{dt}\int z^2(t)\varphi = -3\int z_x^2(t)\varphi' + \int z^2(t)\varphi^{(3)} + \frac{2p}{p+1}\int z^{p+1}(t)\varphi'. \tag{20}$$

Let $u(t)$ be a solution of (1) such that (10) is satisfied. For $x_0 \in \mathbf{R}$, let

$$\mathcal{I}_{x_0}(t) = \int u^2(t,x)\psi(x - x(0) - \sigma t - x_0)\, dx.$$

With the decomposition of $u(t)$ in terms of $\varepsilon(t)$, $\lambda(t)$, $x(t)$ (se the introduction) we have the following corollary of the monotonicity lemma.

Corollary 1 (Almost monotonicity for a solution close to the soliton**).** *Let* $\sigma > 0$, $K > 0$ *such that*

$$\sigma \leqq \frac{1}{4\lambda_2^2}, \quad K \geqq \sqrt{\frac{2}{\sigma}}.$$

There exists $a_0 = a_0(\sigma)$ *such that if* $\sup_{t \geqq 0} |\varepsilon(t)|_{H^1} \leqq a_0$, *then for* $C = C(\sigma, K)$, $\forall\, x_0 \leqq 0, \forall\, t \geqq 0, \quad \mathcal{I}_{x_0}(t) - \mathcal{I}_{x_0}(0) \leqq C e^{\frac{x_0}{K}}$.

The proof of Corollary 1 is omitted since it is exactly the same as in the critical case (see Lemma 20 in [15]).

Moreover, let us recall a Virial-type identity for $\varepsilon(s)$, solution of (13). See also [14], Lemma 5, for similar calculations.

Define

$$I(s) = \tfrac{1}{2} \int y\varepsilon^2(s).$$

Then, we have, by direct calculations:

$$\begin{aligned} &I'(s) + 2\frac{p-3}{p-1}\frac{\lambda_s}{\lambda} I(s) \\ &= \frac{\lambda_s}{\lambda}\int y\left(\frac{2Q}{p-1} + yQ_y\right)\varepsilon + \left(\frac{x_s}{\lambda} - 1\right)\left(\int yQ_y\varepsilon - \frac{1}{2}\int \varepsilon^2\right) \\ &\quad - \frac{3}{2}(L\varepsilon, \varepsilon) + \int \varepsilon^2 - \frac{p(p-1)}{2}\int Q^{p-2}\left(\frac{2Q}{p-1} + yQ_y\right)\varepsilon^2 \\ &\quad + \frac{p}{p+1}\int \varepsilon^{p+1} - \int ((Q+\varepsilon)^p - Q^p - pQ^{p-1}\varepsilon - \varepsilon^p)_y y\varepsilon. \end{aligned} \tag{21}$$

3. Asymptotic behavior of $\varepsilon(s)$ and $\lambda(s)$ as $s \to +\infty$

First, we prove Proposition 2, and then we conclude the proof of Theorem 1 by proving the convergence of $\lambda(t)$ using monotonicity properties in L^2.

3.1. Asymptotic property of $\varepsilon(s)$

In this subsection, we show that the Liouville theorem for ε implies the asymptotic behavior result on ε. Proposition 1 is proved in Sections 4 and 5.

Proof of Proposition 2, assuming Proposition 1. The proof is by contradiction. Assume that for some sequence $s_n \to +\infty$, we have

$$\varepsilon(s_n) \not\rightharpoonup 0 \quad \text{in } H^1 \text{ as } n \to +\infty.$$

Note that by the H^1 stability of Q (5), since $|\varepsilon(0)|_{H^1} \leqq a_2$, we have $a = \sup_{s\geqq 0} |\varepsilon(s)|_{H^1}$ as small as we want provided that a_2 is small enough.

Since $|\varepsilon(s_n)|_{H^1} \leqq C$ and $\lambda_1 \leqq \lambda(s_n) \leqq \lambda_2$, there exists a subsequence of (s_n), which we still denote by (s_n), $\widehat{\varepsilon}_0 \in H^1(\mathbf{R})$ and $\widehat{\lambda}_0 > 0$ such that

$$\widehat{\varepsilon}_0 \not\equiv 0, \quad \varepsilon(s_n) \rightharpoonup \widehat{\varepsilon}_0 \quad \text{in } H^1, \quad \text{and} \quad \lambda(s_n) \to \widehat{\lambda}_0 \quad \text{as } n \to \infty. \tag{22}$$

Note that $|\widehat{\varepsilon}_0|_{H^1} \leqq a$.

Denote by $\widehat{\varepsilon}(s)$ the solution of (13) for all $s \in \mathbf{R}$, with $\widehat{\varepsilon}(0) = \widehat{\varepsilon}_0$ and $(\widehat{\lambda}, \widehat{x})$ such that $\widehat{\varepsilon}$ satisfies $(\widehat{\varepsilon}, Q) = (\widehat{\varepsilon}, Q_y) = 0$. Set $v(t, y) = Q(y) + \varepsilon(t, y) = \lambda^{\frac{2}{p-1}}(t)u(t, \lambda(t)y + x(t))$, and $\widehat{v} = Q + \widehat{\varepsilon}$.

We have the following lemma, relating ε and $\widehat{\varepsilon}$.

Lemma 2 (Stability of weak convergence with respect to time**).** *For all $s \in \mathbf{R}$,*

$$\varepsilon(s_n + s) \rightharpoonup \widehat{\varepsilon}(s) \quad \textit{in } H^1(\mathbf{R}) \textit{ as } n \to +\infty\,. \tag{23}$$

We claim the following property.

Lemma 3 (L^2 compactness of $\widehat{\varepsilon}$**).** *The function $\widehat{\varepsilon}$ is L^2 compact, i.e.,*

$$\forall\, \delta_0 > 0,\ \exists\, A_0 = A_0(\delta_0) > 0,\ \textit{such that}\ \forall\, s \in \mathbf{R},\ \int_{|y|>A_0} \widehat{\varepsilon}^2(s) < \delta_0. \tag{24}$$

Assuming (24), the conclusion of the proof of Proposition 2 follows from Proposition 1. Indeed, from Lemma 2, note that $\forall\, s \in \mathbf{R}$, $|\widehat{\varepsilon}(s)|_{H^1} \leqq a$, and $\widehat{\varepsilon}$ is a solution of (13) satisfying (H1) and (H3). Therefore, for a_2 small enough, the Liouville property implies that

$$\widehat{\varepsilon} \equiv 0 \quad \text{on} \quad \mathbf{R} \times \mathbf{R}.$$

In particular, $\widehat{\varepsilon}_0 \equiv 0$, which is a contradiction. This concludes the proof of Proposition 2.

We can now prove Lemma 2 and Lemma 3.

Proof of Lemma 2. Arguing as in [15], Lemma 17 and Appendix D, and using the orthogonality conditions and (16), (17), Lemma 2 is equivalent to the following property on u.

Claim. Assume that there exists a sequence $t_n \to +\infty$ and $\widehat{u}_0 \in H^1(\mathbf{R})$ such that

$$u(t_n, x(t_n) + .) \rightharpoonup \widehat{u}_0 \quad \text{in } H^1(\mathbf{R}).$$

Then, if $\widehat{u}$ is the solution of (1) with initial value $\widehat{u}(0) = \widehat{u}_0$, we have

$$\forall\, t \in \mathbf{R}, \quad u(t_n + t, x(t_n) + .) \rightharpoonup \widehat{u}(t, .) \quad \text{in } H^1(\mathbf{R}) \text{ as } n \to +\infty, \tag{25}$$

$$\forall\, t \in \mathbf{R}, \quad u(t_n + ., x(t_n) + .) \to \widehat{u} \quad \text{in } C([-t, t], L^2_{\text{loc}}(\mathbf{R})) \text{ as } n \to +\infty. \tag{26}$$

To prove this property, we have to work in the spaces introduced in [9] to solve in a sharp way the Cauchy problem for (1). From the structure of the norms, the proof is rather different from the one in the critical case. The keys of the proof are a Virial identity which gives a smallness property, and the well-posedness of the Cauchy problem in H^1 and in H^s, for $s \in (3/4, 1)$ if $p = 2$, $s \in [1/4, 1)$ if $p = 3$ and $s \in [1/12, 1)$ if $p = 4$, for the generalized KdV equation. In fact, we just need a local Cauchy theory in some H^{s^*}, $0 < s^* < 1$ to prove the claim. Since the cases $p = 2, 3, 4$ are similar (the problems are solved in H^s, for some $0 < s < 1$), we will concentrate on the case $p = 2$.

Set $p = 2$. Let M be such that

$$\forall\, t \in \mathbf{R}, \quad |u(t)|_{H^1} \leqq M.$$

Note that it suffices to prove (25) on an interval $[-t_0, t_0]$, with $t_0 = t_0(M) > 0$, then the claim is obtained by iteration in time.

Since $t_n \to +\infty$, we may assume that $\forall\, n \in \mathbf{N}$, $t_n \geqq 1$. For $t \in [-1, 1]$, we set,

$$\forall\, x \in \mathbf{R}, \quad x_n = x(t_n), \quad u_n(t, x) = u(t_n + t, x_n + x).$$

We first decompose $u_n(t)$ into compact and noncompact parts.

Since

$$\int u_n^2(0) \leqq M^2, \quad u_n(0, .) = u(t_n, x(t_n) + .) \to \widehat{u}_0 \quad \text{in } L^2_{\text{loc}}(\mathbf{R}),$$

we can write

$$u_n(0) = u_{1,n}(0) + u_{2,n}(0),$$

where

$$u_{1,n}(0) \to \widehat{u}_0 \quad \text{in } L^2 \text{ as } n \to +\infty, \quad \left| \int (u_{1,n}(0))^2 - \int \widehat{u}_0^2 \right| \leqq \frac{1}{n},$$
$$u_{2,n}(0, x) = 0, \quad \text{if } |x| \leqq 2\rho_n, \quad \text{with } \rho_n \to +\infty \text{ as } n \to +\infty.$$

Next, we set $z_n(0) = u_{1,n}(0) - \widehat{u}_0$; we have $u_n(0) = \widehat{u}_0 + z_n(0) + u_{2,n}(0)$, with

$$\int z_n^2(0) \leqq \frac{c}{n}, \quad |\widehat{u}_0|_{H^1},\ |u_{1,n}(0)|_{H^1},\ |z_n(0)|_{H^1},\ |u_{2,n}(0)|_{H^1} \leqq K_0. \tag{27}$$

We then consider the solutions $\widehat{u}(t)$, $z_n(t)$, $u_{2,n}(t)$ of (1), with respective initial values $\widehat{u}_0$, $z_n(0)$, $u_{2,n}(0)$. Finally, we define the interaction term $R_n(t) = u_n(t) - (\widehat{u}(t) + z_n(t) + u_{2,n}(t))$.

We show the stability in time of the properties of $z_n(t)$, $u_{2,n}(t)$ and $\widehat{u}(t)$.

Recall that the Cauchy problem for (1) can be solved in $H^s(\mathbf{R})$ for all $s > 3/4$. Fix $s \in (3/4, 1)$. The idea is to use the norms related to H^1 to control the size of the norms related to H^s.

Let

$$|f|_{H^s} = |D_x^s f|_{L^2} + |f|_{L^2}, \quad |\zeta|_{L^p_x L^q_T} = \left(\int_{-\infty}^{\infty} \left(\int_{-T}^{T} |\zeta(t, x)|^q dt \right)^{p/q} dx \right)^{1/p},$$
$$|\zeta|_{L^q_T L^p_x} = \left(\int_{-T}^{T} \left(\int_{-\infty}^{\infty} |\zeta(t, x)|^p dx \right)^{q/p} dt \right)^{1/q}.$$

To solve the Cauchy problem in $H^s(\mathbf{R})$, we consider, for $\zeta : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$, and $T > 0$,

$$\lambda_1^T(\zeta) = \sup_{t\in[-T,T]} |\zeta(t)|_{H^s}, \quad \lambda_2^T(\zeta) = |\zeta|_{L_T^2 L_x^\infty}, \quad \lambda_3^T(\zeta) = |D_x^s \zeta_x|_{L_x^\infty L_T^2},$$

$$\lambda_4^T(\zeta) = (1+T)^{-1}|\zeta|_{L_x^2 L_T^\infty}, \quad \Lambda^T(\zeta) = \max_{j=1,\dots,4} \lambda_j^T(\zeta).$$

To solve the Cauchy problem in $H^1(\mathbf{R})$, we consider

$$\widetilde{\Lambda}^T(\zeta) = \max\left\{ \sup_{t\in[-T,T]} |\zeta(t)|_{H^1}, \lambda_2^T(\zeta), |\zeta_{xx}|_{L_x^\infty L_T^2}, \lambda_4^T(\zeta) \right\}.$$

Let $S(t)$ represent the convolution with $(3t)^{-1/3}\mathrm{Ai}(x(3t)^{-1/3})$. From [9], proof of Theorem 2.1, for $T > 0$ and $F, G : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$, we have,

$$\begin{aligned} \Lambda^T\left(\int_0^t S(t-s)F(s)G_x(s)ds\right) &\leqq C|FG_x|_{L_x^2 L_T^2} + C|D_x^s(FG_x)|_{L_x^2 L_T^2} \\ &\leqq C(1+T)\Lambda^T(F)\widetilde{\Lambda}^T(G). \end{aligned} \tag{28}$$

By the global well-posedness result in H^1 (which is a consequence of (28) and global energy bounds – see Theorem 2.2 in [9]), there exists $K_1 > 0$ such that, if $z(t)$ is a solution of (1) and satisfies $|z(0)|_{H^1} \leqq K_0$, then $\forall\, t \in \mathbf{R}$, $\widetilde{\Lambda}^t(z) \leqq K_1$.

In particular, we have

$$\forall\, t \in \mathbf{R}, \quad \widetilde{\Lambda}^t(\widehat{u}),\ \widetilde{\Lambda}^t(u_n),\ \widetilde{\Lambda}^t(z_n),\ \widetilde{\Lambda}^t(u_{2,n}) \leqq K_1. \tag{29}$$

Remark 9. To solve the Cauchy problem in H^s, KENIG, PONCE & VEGA have used $\lambda_2^{\prime T}(\zeta) = |\zeta|_{L_T^4 L_x^\infty}$ instead of λ_2^T, see page 580 in [9]. Let us remark, as regards their proof, that the inhomogeneous term in the fixed point is estimated by Λ^T (see (4.10) just below (4.9)), and that $\lambda_2^T \leqq CT^{1/4}\lambda_2^{\prime T}$. It seems that if one uses $\lambda_2^{\prime T}$, one is not able to prove the claim.

We claim the following property.

Claim. There exists $n_0 \in \mathbf{N}$, $t_1 > 0$, such that $\forall\, n \geqq n_0$,

(i) $\left(\Lambda^{t_1}(z_n)\right)^2 \leqq \dfrac{C}{n^{1-s}}$,

(ii) $\forall\, t \in [-t_1, t_1]$, $\displaystyle\int_{|x|<\rho_n} (u_{2,n}(t,x))^2 dx \leqq \frac{C}{\rho_n}$,

(iii)
$$\lim_{n\to\infty} \sup_{t\in(-t_1,t_1)} \left(\int_{|x|\leqq\rho_n} \left(D_x^s u_{2,n}\right)^2\right) + \int_{t\in(-t_1,t_1)} \left(\sup_{|x|\leqq\rho_n^{1/4}} (u_{2,n})_x\right)^2$$
$$+ \sup_{|x|\leqq\rho_n^{1/2}} \left(\int_{t\in(-t_1,t_1)} (D_x^s u_{2,nx})^2\right) + \int_{|x|\leqq\rho_n^{1/4}} \left(\sup_{t\in(-t_1,t_1)} |u_{2,n}|\right)^2 = 0.$$

Using properties (i)–(iii), (28) and the technique of Step 3 of the proof of Lemma 30 in [15], it is easy to finish the proof of the claim, by a fixed point argument using the Λ^T norm.

Proof of (i). Since $|z_n(0)|_{H^s} \leqq C|z_n(0)|_{H^1}^s |z_n(0)|_{L^2}^{1-s} \leqq \frac{C}{n^{1-s}}$ by interpolation, the result follows from the proof of Theorem 2.1 in [9], for some $t_1 > 0$ (local existence in H^s). Now, the value of t_1 is fixed.

Proof of (ii). Consider $\gamma : [0, +\infty) \to [0, 1]$ a smooth function satisfying

$$\gamma(r) = 1 \ \text{ for } 0 \leqq r \leqq 1, \quad \gamma(r) = 0 \ \text{ for } r \geqq 2.$$

By formula (20), we have, for $t \in [-t_1, t_1]$,

$$\begin{aligned} \frac{d}{dt}\int \gamma\left(\frac{|x|}{\rho_n}\right)(u_{2,n}(t,x))^2\,dx = & -\frac{3}{\rho_n}\int \gamma'\left(\frac{|x|}{\rho_n}\right)(u_{2,n}(t,x))_x^2\,dx \\ & + \frac{1}{\rho_n^3}\int \gamma^{(3)}\left(\frac{|x|}{\rho_n}\right)(u_{2,n}(t,x))^2\,dx \\ & + \frac{2p}{(p+1)\rho_n}\int \gamma'\left(\frac{|x|}{\rho_n}\right)(u_{2,n}(t,x))^{p+1}\,dx. \end{aligned}$$

For n large enough so that $\rho_n \geqq 1$, and using the Gagliardo-Nirenberg inequality, we obtain, $\forall\, t \in [-t_1, t_1]$,

$$\begin{aligned} &\left|\frac{d}{dt}\int \gamma\left(\frac{|x|}{\rho_n}\right)(u_{2,n}(t,x))^2\,dx\right| \\ &\quad \leqq \frac{C}{\rho_n}\Big(|\gamma'|_{L^\infty}|(u_{2,n})_x|_{L^2}^2 \\ &\qquad + |\gamma^{(3)}|_{L^\infty}|u_{2,n}|_{L^2}^2 + |\gamma'|_{L^\infty}|(u_{2,n})_x|_{L^2}^2|u_{2,n}|_{L^2}^{p-1}\Big) \\ &\quad \leqq \frac{C}{\rho_n}. \end{aligned}$$

Since $\int \gamma\left(\frac{|x|}{\rho_n}\right)(u_{2,n}(0,x))^2 = 0$, (ii) follows.

Proof of (iii). The first term in (iii) is small, as can be shown by using the interpolation inequality $|w|_{H^s} \leqq C|w|_{H^1}^s|w|_{L^2}^{1-s}$ for $w \in H^1(\mathbf{R})$, and $\forall\, t \in (-t_1, t_1)$,

$$\int_{|x|\leqq\rho_n/2}(D_x^s u_{2,n})^2 \leqq C\left(\int_{|x|\leqq\rho_n}(D_x u_{2,n})^2 + (u_{2,n})^2\right)^s\left(\int_{|x|\leqq\rho_n}(u_{2,n})^2\right)^{1-s}.$$

Then (ii) and the bound $|u_{2,n}(t)|_{H^1} \leqq K_1$ imply the result.

For the second term, we observe that by the Sobolev inequality (Gagliardo-Nirenberg inequality and cut-off), we have $\forall\, t \in (-t_1, t_1)$, for $\sigma > 0$,

$$\begin{aligned}
&\sup_{|x| \leqq \rho_n^{1/4}} (u_{2,nx}(t,x))^2 \\
&\quad \leqq C \left(\int_{|x| \leqq 2\rho_n^{1/4}} (D_x^2 u_{2,n})^2 \right)^{3/4} \left(\int_{|x| \leqq 2\rho_n^{1/4}} u_{2,n}^2 \right)^{1/4} \\
&\qquad + C \left(\int_{|x| \leqq 2\rho_n^{1/4}} u_{2,n}^2 \right) \\
&\quad \leqq C\sigma_n \left(\int_{|x| \leqq 2\rho_n^{1/4}} (D_x^2 u_{2,n})^2 \right) + \frac{C}{\sigma_n} \left(\int_{|x| \leqq 2\rho_n^{1/4}} u_{2,n}^2 \right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_{t \in (-t_1, t_1)} \sup_{|x| \leqq \rho_n^{1/4}} (u_{2,nx}(x))^2 \\
&\quad \leqq C\sigma_n \int_{t \in (-t_1, t_1)} \int_{|x| \leqq 2\rho_n^{1/4}} (D_x^2 u_{2,n})^2 + \frac{C}{\sigma_n} \int_{t \in (-t_1, t_1)} \int_{|x| \leqq 2\rho_n^{1/4}} u_{2,n}^2 \\
&\quad \leqq C\sigma_n \rho_n^{1/4} \sup_{x \in \mathbf{R}} \int_{t \in (-t_1, t_1)} (D_x^2 u_{2,n})^2 + \frac{C}{\sigma_n} \sup_{t \in (-t_1, t_1)} \left(\int_{|x| \leqq \rho_n} u_{2,n}^2 \right).
\end{aligned}$$

By (29) and (ii), we obtain the result for $\sigma_n = \rho_n^{-1/2}$.

For the third term in (iii), we set $F(x) = \int_{t \in (-t_1, t_1)} (D_x^{1+s} u_{2,n})^2$. By the Sobolev inequality, and (29),

$$\begin{aligned}
&\int_{|x| \leqq 2\rho_n^{1/2}} F(x)\, dx \\
&\quad \leqq C\sigma_n \int_{t \in (-t_1, t_1)} \int_{|x| \leqq 4\rho_n^{1/2}} |D_x^2 u_{2,n}|^2 + \frac{C}{\sigma_n} \int_{t \in (-t_1, t_1)} \int_{|x| \leqq 4\rho_n^{1/2}} |u_{2,n}|^2 \\
&\quad \leqq C\sigma_n \rho_n^{1/2} \sup_{x \in \mathbf{R}} \int_{t \in (-t_1, t_1)} |D_x^2 u_{2,n}|^2 + \frac{C}{\sigma_n} \sup_{t \in (-t_1, t_1)} \int_{|x| \leqq 4\rho_n} |u_{2,n}|^2 \\
&\quad \leqq CK_1 \sigma_n \rho_n^{1/2} + \frac{C}{\sigma_n \rho_n}.
\end{aligned}$$

Taking $\sigma_n = \rho_n^{-3/4}$, we obtain $\int_{|x| \leqq 2\rho_n^{1/2}} F(x)\, dx \leqq C\rho_n^{-1/4}$.

Let $q \in \mathbf{N}$, $q \geqq 1$. Then $D_x^{1-s} F(x) = \int_{t \in (-t_1, t_1)} D^{1-s} (D^{1+s} u_{2,n}(x))^2$. Therefore,

$$|D_x^{1-s} F|_{L^q(|x| \leqq \rho_n)} \leqq \int_{t \in (-t_1, t_1)} |D^{1-s} (D^{1+s} u_{2,n})^2|_{L^q(|x| \leqq \rho_n)}.$$

From (A.6) in [9] (chain rule for fractional derivatives), we have

$$|D_x^{1-s} (D_x^{1+s} u_{2,n})^2|_{L^q} \leqq C_q |D_x^{1+s} u_{2,n}|_{L^{2q}} |D_x^2 u_{2,n}|_{L^{2q}},$$

and by localization arguments

$$|D_x^{1-s}(D_x^{1+s}u_{2,n})^2|_{L^q(|x|\leqq\rho_n/2)}$$
$$\leqq C_q|D_x^{1+s}u_{2,n}|_{L^{2q}(|x|\leqq\rho_n)}\left(|D_x^2u_{2,n}|_{L^{2q}(|x|\leqq\rho_n)}+|D_x^{1+s}u_{2,n}|_{L^{2q}(|x|\leqq\rho_n)}\right),$$

Hence,

$$\begin{aligned}
|D_x^{1-s}F|_{L^q(|x|\leqq\rho_n)}
&\leqq C_q\int_{t\in(-t_1,t_1)}|D_x^{1+s}u_{2,n}|_{L^{2q}(|x|\leqq\rho_n)}|D_x^2u_{2,n}|_{L^{2q}(|x|\leqq\rho_n)}\\
&\leqq C_q\rho_n^{1/q}\int_{t\in(-t_1,t_1)}\left(\sup_{x\in\mathbf{R}}|D_x^{1+s}u_{2,n}(x)|\ \sup_{x\in\mathbf{R}}|D_x^2u_{2,n}(x)|\right)\\
&\leqq C_q\rho_n^{1/q}\left(\int_{t\in(-t_1,t_1)}\sup_{x\in\mathbf{R}}|D_x^{1+s}u_{2,n}(x)|^2\right)^{1/2}\\
&\quad\left(\int_{t\in(-t_1,t_1)}\sup_{x\in\mathbf{R}}|D_x^2u_{2,n}(x)|^2\right)^{1/2}\\
&\leqq C_qK_1\rho_n^{1/q}.
\end{aligned}$$

Similarly $|F|_{L^q(|x|\leqq\rho_n)}\leqq C_qK_1\rho_n^{1/q}$. From the Gagliardo-Nirenberg inequality, for some $\theta=\theta_q\in(0,1)$, we have

$$\begin{aligned}
\sup_{|x|\leqq\rho_n^{1/2}}|F(x)|&\leqq C\left(\int_{|x|\leqq2\rho_n^{1/2}}F\right)^{\theta}\left(|D_x^{1-s}F|_{L^q(|x|\leqq\rho_n)}+|F|_{L^q(|x|\leqq\rho_n)}\right)^{1-\theta}\\
&\leqq C_q\rho_n^{-\frac{\theta}{4}+\frac{1-\theta}{q}}.
\end{aligned}$$

Now, observe that as $q\to+\infty$, $\theta\to\theta_0\in(0,1)$. Therefore, for q large enough, we obtain

$$\sup_{|x|\leqq\rho_n^{1/2}}|F(x)|\leqq C\rho_n^{\frac{-\theta_0}{8}}.$$

For the last term, we write

$$\begin{aligned}
\int_{|x|\leqq\rho_n^{1/4}}\sup_{t\in(-t_1,t_1)}|u_{2,n}(t,x)|^2
&\leqq 2\rho_n^{1/4}\sup_{\substack{t\in(-t_1,t_1)\\ x\in(-\rho_n,\rho_n)}}|u_{2,n}(t,x)|^2\\
&\leqq 2\rho_n^{1/4}\sup_{t\in(-t_1,t_1)}|u_{2,n}(t)|_{H^1}\sup_{t\in(-t_1,t_1)}\left(\int_{|x|\leqq2\rho_n^{1/4}}u_{2,n}^2\right)^{1/2}\\
&\leqq C\rho_n^{-1/4}.
\end{aligned}$$

This concludes the proof of Lemma 2.

Proof of Lemma 3. The proof of Lemma 3 is exactly the same as in [15] (see the proof of Proposition 6 in [14] for more details.)

The main idea is to use the following two properties.

Lemma 4. (i) Irreversibility of the loss of mass on the left: *There exists* $a_3 > 0$ *such that if* $0 < a < a_3$ *then, for all* $\delta_0 > 0$, $\varepsilon_0 \in (0, 1)$, *there exists* $A_1 = A_1(\delta_0, \varepsilon_0) > 0$ *such that for all* $y_0 > A_1$ *and* $t_0 \geqq 0$,

$$\forall\, t \geqq t_0, \quad \int_{y<\frac{-\lambda_1 y_0}{2\lambda_2}} v^2(t, y)\, dy \geqq (1 - \varepsilon_0) \int_{y<-y_0} v^2(t_0, y)\, dy - \delta_0.$$

(ii) L^2 compactness of v on the right: *There exists* $a_4 > 0$ *such that if* $0 < a < a_4$, *then we have the following property:* $\forall\, \delta_0 > 0$, $\exists\, R_2 = R_2(\delta_0) > 0$ *such that*

$$\forall\, t \geqq 0, \quad \int_{y>R_2} v^2(t, y)\, dy \leqq \delta_0. \tag{30}$$

Proof of Lemma 4. The proof of the irreversibility (property (i)) follows as in [15], Lemma 19, from the almost-monotonicity property given in Corollary 1. The idea is to use the almost-monotonicity on the following quantity

$$I_{x_0}(t) = \int u^2(t, x)\psi(x - x(0) - \sigma t - x_0).$$

The proof of the L^2 compactness on the right is based on two procedures: first, a decomposition of the solution close to the soliton; second, use of a monotonicity property of the mass on the right-hand side of the soliton.

Equation (13) can be rewritten

$$\varepsilon_s + \varepsilon_{yyy} - \frac{x_s}{\lambda}\varepsilon_y = \frac{\lambda_s}{\lambda}\left(\frac{2\varepsilon}{p-1} + y\varepsilon_y\right) + f_1 + f_{2y} - (\varepsilon^p)_y,$$

with

$$f_1(s, y) = \frac{\lambda_s}{\lambda}\left(\frac{2Q}{p-1} + yQ_y\right),$$
$$f_2(s, y) = \left(\frac{x_s}{\lambda} - 1\right) Q - ((Q + \varepsilon)^p - Q^p - \varepsilon^p).$$

We introduce

$$\eta(s, x) = \lambda^{-\frac{2}{p-1}}(s)\varepsilon(s, \lambda^{-1}(s)x).$$

We verify that

$$\lambda^{\frac{2}{p-1}}\eta_s + \lambda^{\frac{3p-1}{p-1}}\eta_{xxx} - \lambda^{\frac{2}{p-1}}x_s\eta_x = f_1(s, \lambda^{-1}x) + f_{2y}(s, \lambda^{-1}x) - (\varepsilon^p)_y. \tag{31}$$

Changing the time variable $s \to t$ by the formula

$$s = \int_0^t \frac{dt'}{\lambda^3(t')}, \quad \text{or equivalently,} \quad \frac{ds}{dt} = \frac{1}{\lambda^3}, \tag{32}$$

we obtain

$$\eta_t + \eta_{xxx} - x_t \eta_x = g_1 + g_{2x} - (\eta^p)_x, \tag{33}$$

where

$$g_1(t,x) = \lambda^{-\frac{3p-1}{p-1}} f_1(t, \lambda^{-1}x), \tag{34}$$

$$g_2(t,x) = \lambda^{-\frac{2}{p-1}} \left(\frac{x_s}{\lambda} - 1\right) Q(\lambda^{-1}x) + \left(\lambda^{-\frac{2}{p-1}} Q(\lambda^{-1}x) + \eta\right)^p - \left(\lambda^{-\frac{2}{p-1}} Q(\lambda^{-1}x)\right)^p - \eta^p. \tag{35}$$

We can split η into two parts:

$$\eta(t,x) = \eta_{\mathrm{I}}(t,x) + \eta_{\mathrm{II}}(t,x),$$

where η_{I} satisfies the purely nonlinear equation

$$(\eta_{\mathrm{I}})_t + (\eta_{\mathrm{I}})_{xxx} - x_t(t)(\eta_{\mathrm{I}})_x = -(\eta_I^p)_x,$$
$$\eta_{\mathrm{I}}(0) = \eta(0),$$

and η_{II} satisfies

$$(\eta_{\mathrm{II}})_t + (\eta_{\mathrm{II}})_{xxx} - x_t(t)(\eta_{\mathrm{II}})_x = g_1(t) + g_{2x}(t) - (\eta^p - (\eta_{\mathrm{I}})^p)_x,$$
$$\eta_{\mathrm{II}}(0) = 0.$$

Using the monotonicity property for small solutions of (1) (i.e., Lemma 1), the quantity

$$I(t) = \int \overline{\eta}_{\mathrm{I}}^2(t,x)\psi(x - \sigma t - x_0)\, dx,$$

where $\overline{\eta}_{\mathrm{I}}(t,x) = \eta_{\mathrm{I}}(t, x - x(t) + x(0))$, is monotone in time. The L^2 compactness of η_{I} then follows from this property for x_0 large. This is proved in the same way as in the critical case.

We claim the following lemma, proving that η_{II} satisfies an L^2 compactness property.

Lemma 5 (Exponential estimate for $x > 0$**).** *There exists* $a_0' > 0$ *and* $\theta_1, \theta_2 > 0$ *such that if* $0 < a < a_0'$*, then*

$$\forall t \geqq 0, \forall x \geqq 0, \quad |\eta_{\mathrm{II}}(t,x)| \leqq \sqrt{ab}\, \theta_1 e^{-\theta_2 x}. \tag{36}$$

Proof of Lemma 5. The proof is exactly the same as for the critical case (see Lemma 2 in [15]). It is based on pointwise estimates, and then does not depend on the value of p. Let us just recall that we use an estimate of the solutions of a shifted nonhomogeneous linear Airy equation, with exponentially decaying second member, see Lemma 5 in [15].

This concludes the proof of Lemma 4 and Proposition 2.

3.2. Convergence of $\lambda(s)$

Here, we use the monotonicity in time of the L^2_{loc} norm of $u(t)$, and the fact that the problem in subcritical (there is a unique soliton of given L^2 norm), so that the limit L^2_{loc} selects the asymptotic soliton.

Proposition 3 (Convergence of $\lambda(t)$**).** *Let* $p = 2, 3, 4$. *Under the assumptions of Proposition 2, there exists* $\lambda_{+\infty} > 0$ *such that*

$$\lambda(t) \to \lambda_{+\infty} \quad \text{as } t \to +\infty.$$

Remark 10. The conclusion of Theorem 1 follows easily from this result. Note that it also gives convergence of $x_t(t)$. Indeed, $|\lambda^2(t)x_t(t) - 1| \leqq C \int e^{-c|x|}|\varepsilon(s)|$ for some $C, c > 0$ (by (16), (17)), and thus $x_t(t) \to 1/\lambda^2_{+\infty} = c_{+\infty}$ as $t \to +\infty$.

Recall that by Proposition 2, we have

$$v(t,.) = \lambda^{\frac{2}{p-1}}(t)u(t, \lambda(t). + x(t)) \rightharpoonup Q \quad \text{in } H^1(\mathbf{R}) \text{ as } t \to +\infty.$$

The convergence of the scaling parameter $\lambda(t)$ as $t \to +\infty$ relies on a careful description of behavior of the local L^2 mass of $v(t)$ as $t \to +\infty$. Indeed, we have in some sense $|v(t)|_{L^2_{\text{loc}}} \sim \lambda^{\frac{p-5}{p-1}}(t) \int Q^2$, as $t \to +\infty$. Moreover, by using the L^2 compactness of $v(t)$ on the right (Lemma 4(ii)), and Corollary 1 which says that the mass of $u(t)$ which is at some time at the left of the soliton will never return to the soliton, $|v(t)|_{L^2_{\text{loc}}}$ is almost decreasing in time. This gives monotonicity and thus convergence of $\lambda(t)$.

Proof. Recall that

$$\forall\, t \geqq 0, \quad \lambda_1 \leqq \lambda(t) \leqq \lambda_2. \tag{37}$$

We consider ψ as in Section 2 and $\sigma = \frac{1}{4\lambda_2^2}$, $K = \frac{2}{\sigma}$. Let $\delta > 0$ arbitrary and $x_0 < 0$ be such that $Ce^{x_0/K} < \delta$, where C appears in Corollary 1. From Corollary 1, since $x(t) \geqq x(t') + \sigma(t - t')$ by the choice of σ, we have $\forall\, t \geqq t' \geqq 0$,

$$\int u^2(t,x)\psi(x - x(t) - x_0)\, dx \leqq \int u^2(t',x)\psi(x - x(t') - x_0)\, dx + \delta.$$

Therefore,

$$\lambda^{\frac{p-5}{p-1}}(t) \int v^2(t,y)\psi(\lambda(t)y - x_0)\, dy \leqq \lambda^{\frac{p-5}{p-1}}(t') \int v^2(t',y)\psi(\lambda(t')y - x_0)\, dy + \delta.$$

By compactness of $v(t)$, on the right-hand side, uniform in time (Lemma 4(ii)), weak convergence $v(t) \rightharpoonup Q$ in $H^1(\mathbf{R})$ as $t \to +\infty$, and (37), there exists $T = T(\delta)$ and $x_0 = x_0(\delta) < 0$ ($|x_0|$ large enough) such that $\forall\, t > T$,

$$\left| \int v^2(t,y)\psi(\lambda(t)y - x_0)\, dy - \int Q^2 \right| \leqq \delta.$$

(Recall that $\psi(y)$ decays exponentially to 0 as $y \to -\infty$ and goes to 1 as $y \to +\infty$.) Thus, $\forall\, \delta > 0$, there exists $T > 0$ such that $\forall\, t \geqq t' \geqq T$,

$$\lambda^{\frac{p-5}{p-1}}(t) \int Q^2 \leqq \lambda^{\frac{p-5}{p-1}}(t') \int Q^2 + \delta + 2\lambda_1^{\frac{p-5}{p-1}} \delta.$$

It follows that $\lambda^{\frac{p-5}{p-1}}(t)$ and $\lambda(t)$ have a limit when $t \to +\infty$.

This completes the proof of Proposition 3 and the proof of Theorem 1.

4. Passage from a nonlinear Liouville property to a linear Liouville property

This is the first step of the proof of Theorem 2 (or Proposition 1). We claim that the nonlinear Liouville property for small ε is equivalent to a linear Liouville property.

We want to show that for $|\varepsilon|_{H^1}$ small and satisfying the assumptions in Proposition 1 (orthogonality condition, L^2 localization, smallness in H^1), we necessarily have $\varepsilon \equiv 0$. For the sake of contradiction we assume that there exists a sequence $\varepsilon_n \not\equiv 0$ of solutions of (13) satisfying $|\varepsilon_n(0)|_{H^1} \to 0$ as $n \to +\infty$. By the strict convexity of the functional $E(v) + \frac{c}{2}\int v^2$, we have $a_n = \sup_{s\in\mathbf{R}} |\varepsilon_n(s)|_{H^1} \to 0$ as $n \to \infty$ (see the proof of the stability result in [5]). We claim the following convergence result for a sequence of renormalizations of the (ε_n).

Proposition 4 (Convergence to a linear problem**).** *Consider a sequence* $\varepsilon_n \in C(\mathbf{R}, H^1(\mathbf{R})) \cap L^\infty(\mathbf{R}, H^1(\mathbf{R}))$ *of solutions of* (13) *satisfying* (H1) *and* (H3) (*without any uniformity in n for* (H3))*. Assume that*

$$a_n = \sup_{s\in\mathbf{R}} |\varepsilon_n(s)|_{H^1} \to 0 \quad \textit{as } n \to +\infty.$$

Then:

(i) *There exist a sequence* $(s_n) \in \mathbf{R}$ *and a subsequence* $(\varepsilon_{n'})$ *such that*

$$\frac{\varepsilon_{n'}(s_{n'} + s)}{a_{n'}} \to w(s) \quad \textit{in } L^\infty_{\mathrm{loc}}(\mathbf{R}, L^2(\mathbf{R})),$$

where $w \in C(\mathbf{R}, H^1(\mathbf{R})) \cap L^\infty(\mathbf{R}, H^1(\mathbf{R}))$ *satisfies*

$$w \not\equiv 0,$$
$$w_s - (Lw)_y = \alpha(s)\left(\frac{2Q}{p-1} + yQ_y\right) + \beta(s)Q_y, \quad (s, y) \in \mathbf{R}^2 \tag{38}$$

for some continuous functions α *and* β.

(ii) *Moreover, there exist* $C > 0$ *and* $\theta_2 > 0$ *such that* w *satisfies*

(H1′) $\quad \forall\, s \in \mathbf{R}, \quad (w(s), Q) = 0, \quad (w(s), Q_y) = 0,$

(H2′) $\quad \forall\, s \in \mathbf{R},\ \forall\, y \in \mathbf{R}, \quad |w(s, y)| \leqq Ce^{-\theta_2|x|}.$

In Section 5, we show that the solution w built in Proposition 4 does not exists, which concludes the contradiction argument and the proof of Theorem 2.

Proof. We proceed in three steps.

Step 1. We claim the following lemma.

Lemma 6 (Uniform exponential decay**).** *Let* $\varepsilon \in C(\mathbf{R}, H^1(\mathbf{R})) \cap L^\infty(\mathbf{R}, H^1(\mathbf{R}))$ *be a solution of equation* (13) *satisfying* (H1) *and* (H3). *Let a and b be defined by*

$$a = \sup_{s\in\mathbf{R}} |\varepsilon(s)|_{H^1}, \quad b = \sup_{s\in\mathbf{R}} |\varepsilon(s)|_{L^2}. \tag{39}$$

There exists $a_0 > 0$ *and two constants* $\theta_1, \theta_2 > 0$, *such that if* $a < a_0$, *then*

$$\forall\, s \in \mathbf{R},\ \forall\, y \in \mathbf{R}, \quad |\varepsilon(s, y)| \leqq \theta_1 \sqrt{ab}\, e^{-\theta_2 |y|}. \tag{40}$$

Remark 11. From this result, we obtain *a posteriori* an explicit $A(\delta_0)$ in the L^2 compactness assumption (H3) of Proposition 1, $A(\delta_0) \sim \log(1/\delta_0)$.

Proof of Lemma 6. The method used in the proof of Lemma 6 is similar to the one used in Proposition 1 in [15]. It involves a nonlinear decomposition of η, as in the proof of Lemma 4. However, the treatment of the purely nonlinear part of the solution is different and relies on the monotonicity lemma for small solutions of (1), see Lemma 1. This difference is due to the fact that no scattering result is available for $p < 5$ in L^2 nor H^s, $s \in (0, 1]$ (see [9], Introduction). Indeed, in general, the monotonicity result can be seen as a more robust property implying convergence to zero in a weak sense on compact sets for small solutions in the case where there is no scattering.

Let (t_n) be a sequence such that $t_n \to -\infty$. For $n \in \mathbf{N}$, define η_n by

$$\eta_n(t, x) = \eta(t + t_n, x).$$

Then η_n satisfies

$$(\eta_n)_s + (\eta_n)_{xxx} - x_t(t + t_n)(\eta_n)_x = g_1(t + t_n) + g_{2x}(t + t_n) - (\eta_n^p)_x,$$

where g_1, g_2 are defined in (34), (35), and

$$\eta_n(0, x) = \eta(t_n, x),$$

as in the proof of Lemma 4.

As before, we split η_n into two parts: this is a “nonlinear decomposition of η”. We set

$$\eta_n(t, x) = \eta_{\mathrm{I},n}(t, x) + \eta_{\mathrm{II},n}(t, x), \tag{41}$$

where $\eta_{\mathrm{I},n}$ is solution of the purely nonlinear equation

$$\begin{aligned} &(\eta_{\mathrm{I},n})_t + (\eta_{\mathrm{I},n})_{xxx} - x_t(t + t_n)(\eta_{\mathrm{I},n})_x = -(\eta_{\mathrm{I},n}^p)_x, \\ &\eta_{\mathrm{I},n}(0, x) = \eta(t_n, x), \quad x \in \mathbf{R}, \end{aligned} \tag{42}$$

and $\eta_{\mathrm{II},n}$ is solution of

$$(\eta_{\mathrm{II},n})_t + (\eta_{\mathrm{II},n})_{xxx} - x_t(t+t_n)(\eta_{\mathrm{II},n})_x = g_1(t+t_n) + g_{2x}(t+t_n) \\ - (\eta_n^p - (\eta_{\mathrm{I},n})^p)_x, \tag{43}$$

$$\eta_{\mathrm{II},n}(0,x) = 0, \quad x \in \mathbf{R}.$$

We claim the following:

Claim. (a) Exponential estimate of $\eta_{\mathrm{II},n}$:

$$\forall\, t \in \mathbf{R},\ \forall\, x \geqq 0, \quad |\eta_{\mathrm{II},n}(t,x)| \leqq \sqrt{ab}\,\theta_1 e^{-\theta_2 x}. \tag{44}$$

(b) Asymptotic behavior of $\eta_{\mathrm{I},n}$. Let $t_0 \in \mathbf{R}$. For any sequence $t_n \to -\infty$, we have

$$\eta_{\mathrm{I},n}(t_0 - t_n) \to 0 \quad \text{in } L^\infty_{\mathrm{loc}}(\mathbf{R}) \text{ as } n \to +\infty. \tag{45}$$

Property (a) follows from Lemma 5 applied to $\eta_{\mathrm{II},n}$.

Proof of (b). Let $A > 0$ and $t_0 \in \mathbf{R}$ fixed. First, we want to show that

$$\int_{x>-A} \eta^2_{\mathrm{I},n}(t_0 - t_n, x)\, dx \to 0 \quad \text{as } n \to \infty.$$

The function ψ being defined as in Section 2, it is sufficient to show that

$$\int \eta^2_{\mathrm{I},n}(t_0 - t_n, x)\psi(x+A)dx \to 0 \quad \text{as } n \to \infty.$$

By applying Lemma 1 to $\overline{\eta_{\mathrm{I},n}}(t, x+x_0)$, where $\overline{\eta_{\mathrm{I},n}}(t,x) = \eta_{\mathrm{I},n}(t, x - x(t + t_n) + x(t_n))$, for a small enough, and with $\sigma = \frac{1}{2\lambda_2^2}$, we obtain

$$\int \eta^2_{\mathrm{I},n}(t_0 - t_n, x)\psi(x + x(t_0) - x(t_n) - \sigma(t_0 - t_n) - x_0) \\ \leqq \int \eta^2_{\mathrm{I},n}(0,x)\psi(x - x_0). \tag{46}$$

Let $x_0 = -A - \sigma(t_0 - t_n) + (x(t_0) - x(t_n))$. Since $\left|\lambda^2 x_t - 1\right| \leqq Ca$, for a small enough, we have $x_t \geqq \frac{3\sigma}{2}$. Therefore, $-\sigma(t_0 - t_n) + x(t_0) - x(t_n) \geqq \frac{\sigma}{2}(t_0 - t_n)$, and so $x_0 \geqq -A + \frac{\sigma}{2}(t_0 - t_n)$.

We obtain from (46), and the fact that ψ is nondecreasing,

$$\int \eta^2_{\mathrm{I},n}(t_0 - t_n, x)\psi(x+A)dx \leqq \int \eta^2_{\mathrm{I},n}(0,x)\psi(x + A - \frac{\sigma}{2}(t_0 - t_n)) \\ \leqq \int \eta^2(t_n, x)\psi(x + A - \frac{\sigma}{2}(t_0 - t_n)).$$

From the compactness in L^2 of $\eta(t,x)$ and the fact $\sigma(t_0 - t_n) \to +\infty$ as $n \to +\infty$, we obtain

$$\int_{x>-A} \eta^2_{\mathrm{I},n}(t_0 - t_n, x)\, dx \to 0, \quad \text{as } n \to \infty.$$

Therefore, $\eta_{\mathrm{I},n}(t_0 - t_n) \to 0$ in L^2_{loc}. Since $\sup_{t\in\mathbf{R}} |\eta_{\mathrm{I},n}(t)|_{H^1} \leqq C$, we have $\eta_{\mathrm{I},n}(t_0 - t_n) \to 0$ in L^∞_{loc}. Thus (b) is proved.

Using (a) and (b), we now conclude the proof of Lemma 6. Fix $t \in \mathbf{R}$ and $x \geqq 0$. Recall that, $\forall\, n \in \mathbf{N}$, we have

$$\eta(t) = \eta_{\mathrm{I},n}(t - t_n) + \eta_{\mathrm{II},n}(t - t_n).$$

Property (b) yields

$$\eta_{\mathrm{I},n}(t - t_n, x) \to 0 \quad \text{as } n \to \infty.$$

Therefore,

$$\eta_{\mathrm{II},n}(t - t_n, x) \to \eta(t, x) \quad \text{as } n \to \infty.$$

It follows from (44) that $\forall\, n \in \mathbf{N}$,

$$|\eta_{\mathrm{II},n}(t - t_n, x)| \leqq \sqrt{ab}\,\theta_1 e^{-\theta_2 x}.$$

Hence, we obtain, letting n go to ∞,

$$|\eta(t, x)| \leqq \sqrt{ab}\,\theta_1 e^{-\theta_2 x}.$$

Thus,

$$\forall\, t \in \mathbf{R},\ \forall\, x \geqq 0, \quad |\eta(t, x)| \leqq \sqrt{ab}\,\theta_1 e^{-\theta_2 x}. \tag{47}$$

Let us now prove the result for $x \leqq 0$. We use the symmetry of (1) under the following transformation: if $u(t, x)$ is a solution of (1) then $\widetilde{u}(t, x) = u(-t, -x)$ is also a solution of (1) satisfying (H1) and (H3).

Therefore, the same argument yields

$$\forall\, t \in \mathbf{R},\ \forall\, x \geqq 0, \quad |\widetilde{\eta}(t, x)| \leqq \sqrt{ab}\,\theta_1 e^{-\theta_2 x}. \tag{48}$$

By (47) and (48), we obtain

$$\forall\, t \in \mathbf{R},\ \forall\, x \in \mathbf{R}, \quad |\eta(t, x)| \leqq \sqrt{ab}\,\theta_1 e^{-\theta_2 |x|}.$$

Since

$$\varepsilon(s, y) = \lambda^{-1/2}(s)\eta(s, \lambda(s)y),$$

by (H2) (control of $\lambda(s)$, see remark after Proposition 2), we have

$$|\varepsilon(s, y)| \leqq \sqrt{ab}\,\theta_1' e^{-\theta_2' |x|},$$

which concludes the proof of Lemma 6.

Step 2. We claim the following lemma.

Lemma 7 (Comparison between L^2 and H^1 norms of ε**).** *Under the assumptions of Lemma 6, there exist* $a_1 > 0$ *and* $C > 0$ *such that, if* $a < a_1$ *then,*

$$b \leqq a \leqq Cb,$$

where

$$a = \sup_{s\in\mathbf{R}} |\varepsilon(s)|_{H^1}, \quad b = \sup_{s\in\mathbf{R}} |\varepsilon(s)|_{L^2}.$$

Proof. The proof is based on a Virial-type identity (see proof of Proposition 2 in [15] for more details). Recall that in Section 2 we set

$$I(s) = \tfrac{1}{2}\int y\varepsilon^2(s).$$

From (21), we claim that

$$\frac{d}{ds}(\lambda^{2\frac{p-3}{p-1}}I)(s) \leqq Cb^2 - \frac{3\lambda_1^{2\frac{p-3}{p-1}}}{2}|\varepsilon(s)|_{H^1}^2. \tag{49}$$

Indeed, note that in (21), by integration by parts, the last term can be written as the sum of scalar products of ε^i, for $i \geqq 3$, with rapidly decaying functions (see [14], Lemma 14). Thus (49) follows from $(L\varepsilon, \varepsilon) \geqq |\varepsilon|_{H^1}^2 - C|\varepsilon|_{L^2}^2$, and

$$\left|\frac{\lambda_s}{\lambda}\right| + \left|\frac{x_s}{\lambda} - 1\right| \leqq Cb,$$

which is a consequence of (16) and (17).

Next, we claim that there exists $\sigma > 0$ and $c_0 > 0$ such that if a is small enough, then

$$|\varepsilon(s_0)|_{H^1} \geqq \frac{a}{2} \text{ implies } \forall\, s \in (s_0, s_0+\sigma),\ |\varepsilon(s)|_{H^1} \geqq c_0 a. \tag{50}$$

This follows from the arguments of [9] to show that the Cauchy problem for (1) is well posed in H^1 (see also Appendix B1 in [15]).

By definition of a, there exists $s_0 \in \mathbf{R}$, such that $\frac{a}{2} \leqq |\varepsilon(s_0)|_{H^1} \leqq a$. We deduce from (50) that

$$\forall\, s \in (s_0, s_0+\sigma), \quad |\varepsilon(s)|_{H^1} \geqq c_0 a,$$

and then

$$\forall\, s \in (s_0, s_0+\sigma), \quad (\lambda^{2\frac{p-3}{p-1}}I)(s_0+\sigma) - (\lambda^{2\frac{p-3}{p-1}}I)(s_0) \leqq \sigma(Cb^2 - C'a^2).$$

Now, by Lemma 6, we have $\forall\, s \in \mathbf{R}$, $\lambda^{2\frac{p-3}{p-1}}(s)I(s) \leqq Cab$. Therefore, for some $C > 0$, we have $a^2 \leqq C(b^2+ab)$ and thus $Ca \leqq b$.

This proves Lemma 7.

Step 3. Conclusion of the proof of Proposition 4. We define

$$w_n(s,y) = \frac{\varepsilon_n(s+s_n, y)}{b_n},$$

where s_n is such that

$$|\varepsilon_n(s_n)|_{L^2} \geqq \frac{b_n}{2} = \frac{1}{2}\sup_{s\in\mathbf{R}} |\varepsilon_n(s)|_{L^2}.$$

Using Steps 1 and 2, exactly as in the proof of Proposition 3 in [15], we have, for a subsequence,

$$w_n \to w \quad \text{in } L^\infty_{\text{loc}}(\mathbf{R}, L^2(\mathbf{R})),$$

where $w \in C(\mathbf{R}, H^1(\mathbf{R})) \cap L^\infty(\mathbf{R}, H^1(\mathbf{R}))$ satisfies

$$w \not\equiv 0$$

(as a consequence of Step 2), and equation (38).

This concludes the proof of Proposition 4.

5. Linear Liouville property

The objective of this section is to show that the solution w obtained in Proposition 4 does not exist, which concludes the proof of Proposition 1 (and Theorem 2).

Throughout this section, let $w \in C(\mathbf{R}, H^1(\mathbf{R})) \cap L^\infty(\mathbf{R}, H^1(\mathbf{R}))$ be a solution of

$$w_s - (Lw)_y = \alpha(s)\left(\frac{2Q}{p-1} + yQ_y\right) + \beta(s)Q_y, \quad (s, y) \in \mathbf{R} \times \mathbf{R}, \tag{51}$$

where α and β are continuous functions of s and w satisfies

(H1′) Orthogonality conditions:

$$\forall\, s \in \mathbf{R}, \quad (w(s), Q) = (w(s), Q_y) = 0.$$

(H2′) Exponential decay condition:

$$\forall\,(s, y) \in \mathbf{R} \times \mathbf{R}, \quad |w(s, y)| \leqq Ce^{-c|y|}.$$

We claim the following result, which implies Theorem 1.

Proposition 5 (A linear Liouville theorem for equation (51)**).** *Let*

$$w \in C(\mathbf{R}, H^1(\mathbf{R})) \cap L^\infty(\mathbf{R}, \quad H^1(\mathbf{R}))$$

be a solution of (51) *satisfying* (H1′) *and* (H2′). *Then*

$$w \equiv 0 \quad \textit{on } \mathbf{R} \times \mathbf{R}.$$

Remark 12. In fact, we obtain by Proposition 5 a characterization of the stationary solutions of $w_s - (Lw)_y = 0$. If w is a solution of this equation satisfying (H2′), then $w \equiv \delta_0 Q_y$ (see [15] for details).

Remark 13. The proof of this result will differ from the one of the critical case. The first change is the fact that in some sense the geometry around Q is simpler (no degeneracy). Only considerations on the Virial identity will give monotonicity (this is the key of the proof). The second change is that, to prove the monotonicity, we have to use spectral estimates on some operators. It turns out that these estimates are less accurate than in the critical case, and the calculations are more delicate, due to the fact that we have only two orthogonality conditions on w, while in the critical case an additional orthogonality condition on w is obtained by the structure of the operator.

Proof of Proposition 5. We introduce a new function using an explicit solution of (51):

$$\overline{w} = w + \gamma(s) Q_y, \tag{52}$$

where $\gamma(s)$ is the bounded, continuous function defined by

$$\gamma(s) = -\frac{1}{\int Q_y y Q} \int y Q w(s) = \frac{2}{\int Q^2} \int y Q w(s).$$

Thus, we obtain

$$(\overline{w}, Q) = (\overline{w}, yQ) = 0, \quad \forall s \in \mathbf{R},\ \forall y \in \mathbf{R},\ |\overline{w}(s, y)| \leqq Ce^{-c|y|}. \tag{53}$$

Since $L(Q_y) = 0$, $\overline{w}$ satisfies (51), for $\overline{\alpha}(s) = \alpha(s)$, $\overline{\beta}(s) = \beta(s) + \gamma'(s)$.

Now, from the orthogonality conditions, we claim that

$$\overline{\alpha}(s) = 0, \quad \overline{\beta}(s) = \frac{2}{\int Q^2} \int \overline{w}(s)(2Q + (p-3)Q^p). \tag{54}$$

Indeed, on the one hand, multiplying equation (51) for $\overline{w}$ by Q and integrating by parts, we have

$$0 = \frac{d}{ds} \int Q\overline{w} = -\int (L\overline{w}) Q_y + \overline{\alpha}(s) \int \left(\tfrac{2Q}{p-1} + yQ_y\right) Q + \overline{\beta}(s) \int Q_y Q.$$

From $LQ_y = 0$, $\int QQ_y = 0$, and (11), it follows that $\overline{\alpha}(s) \equiv 0$.

On the other hand, multiplying (51) by yQ, we have

$$0 = \frac{d}{ds} \int \overline{y} Q w = -\int (L\overline{w}) \left(Q + yQ_y\right) + \overline{\beta}(s) \int Q_y y Q.$$

Since

$$\begin{aligned} L(Q + yQ_y) &= -Q_{yy} - 2Q_{yy} - yQ_{yyy} + Q + yQ_y - pQ^p + pyQ^{p-1}Q_y \\ &= -3Q_{yy} - y(Q - Q^p)_y + Q + yQ_y - pQ^p + pyQ^{p-1}Q_y \\ &= -2Q - (p-3)Q^p, \end{aligned}$$

we obtain

$$-\int \overline{w}(s)(2Q + (p-3)Q^p) + \beta(s)\frac{\int Q^2}{2} = 0.$$

The equation of $\overline{w}(s)$ then reduces to

$$\overline{w}_s = (L\overline{w})_y + \frac{2Q_y}{\int Q^2} \int \overline{w}(s)(2Q + (p-3)Q^p). \tag{55}$$

We have the following relations for (55). Let

$$I(s) = \tfrac{1}{2} \int y\overline{w}^2(s, y)\, dy.$$

Lemma 8 (Identities for equation(55)**).** *For all* $s \in \mathbf{R}$,

(i) $(L\overline{w}(s), \overline{w}(s)) = (L\overline{w}(0), \overline{w}(0)) = (Lw(0), w(0))$,

(ii) $\dfrac{d}{ds} I(s) = -H^*(\overline{w}(s), \overline{w}(s))$, *where*

$$H^*(\overline{w}, \overline{w}) = H(\overline{w}, \overline{w}) - \frac{2}{\int Q^2} \left(\int \overline{w} y Q_y \right) \left(\int \overline{w} (2Q + (p-3)Q^p) \right),$$

$$\begin{aligned} H(\overline{w}, \overline{w}) &= -((L\overline{w})_y, y\overline{w}) = (L_1 \overline{w}, \overline{w}) \\ &= \frac{3}{2}(L\overline{w}, \overline{w}) - (\overline{w}, \overline{w}) + p \int Q^{p-2} \left(Q + \frac{p-1}{2} y Q_y \right) \overline{w}^2, \end{aligned}$$

with

$$L_1 \overline{w} = -\frac{3}{2} \overline{w}_{yy} + \frac{1}{2} \overline{w} - \frac{p}{2} Q^{p-1} \overline{w} + \frac{p(p-1)}{2} y Q_y Q^{p-2} \overline{w}.$$

Proof. Since $L(Q_y) = 0$, when we take the scalar product of (51) and $L\overline{w}$, we obtain $\frac{d}{ds}(L\overline{w}, \overline{w}) = 0$. Moreover, since $\overline{w} = w + \gamma(s) Q_y$, we have $(L\overline{w}(0), \overline{w}(0)) = (Lw(0), w(0))$.

Property (ii) is proved following similar calculations as for the nonlinear problem (see Section 2). Note that by (H2′), $y\overline{w}^2 \in L^1(\mathbf{R})$.

In order to complete the proof of Proposition 5, as in the critical case, we claim the following properties, which will be proved later.

Proposition 6 (Positivity properties of H^***).** *Let* $p = 2, 3$ *or* 4. *There exists* $\sigma_0 > 0$ *such that if* $\overline{w} \in H^1(\mathbf{R})$ *satisfies* $(\overline{w}, Q) = (\overline{w}, yQ) = 0$, *then*

(i) *if* $\overline{w} \not\equiv 0$, *then* $(L\overline{w}, \overline{w}) > 0$,

(ii) $H^*(\overline{w}, \overline{w}) \geqq \sigma_0 (L\overline{w}, \overline{w})$.

Remark 14. Several difficulties prevent us from following the same procedure for the choice of the orthogonality conditions as in the critical case. First, we are not able (except for $p = 3$) to consider orthogonality conditions such that $H^*(w, w)$ does not contain a scalar product part. Second, an orthogonality condition with Q_y is not enough to guarantee the positivity in the three cases.

End of the proof of Proposition 5. Assume, for contradiction, that $\overline{w} \not\equiv 0$. We have $\forall s \in \mathbf{R}$,

$$\frac{d}{ds} I(s) = -H^*(\overline{w}(s), \overline{w}(s)) \leqq -\sigma_0 (L\overline{w}(s), \overline{w}(s)),$$

and, since $\overline{w}(0) \not\equiv 0$ and $(\overline{w}(0), Q) = (\overline{w}(0), yQ) = 0)$,

$$(L\overline{w}(s), \overline{w}(s)) = (L\overline{w}(0), \overline{w}(0)) > 0.$$

Therefore, $\forall s \in \mathbf{R}$, $\frac{d}{ds} I(s) \leqq -\sigma_0' < 0$, which contradicts the fact that I is uniformly bounded from (H2′):

$$\forall s \in \mathbf{R}, \quad |I(s)| \leqq C \int |y| e^{-c|y|} dy \leqq C.$$

Thus $\overline{w}(0) \equiv 0$, and similarly, $\forall s \in \mathbf{R}$, $\overline{w}(s) \equiv 0$. From the orthogonality conditions on $w(s)$, we have the conclusion.

Proof of Proposition 6. Property (i) follows from Proposition 2.9 in [21].

Now, we prove (ii). Let $p = 2, 3$ or 4. The arguments of the proof are similar to the ones of the proof of Proposition 4 in [15], although there are technical differences.

We give some notation. Let B be a bilinear form on a vector space V. Let us define the index of B on V by

$$\begin{aligned}\mathrm{ind}_V(B) = \max\Big\{&k \in \mathbf{N}/ \text{ there exists a sub-space } P \text{ of codimension } k \\ &\text{such that } B_{|P} \text{ is positive definite}\Big\}.\end{aligned}$$

Let H_e^1 (respectively, H_o^1) denote the sub-space of even (respectively, odd) H^1 functions. Assume that H_e^1 is B-orthogonal to H_o^1. We say that B defined on H^1 has index $i + j$ if $\mathrm{ind}_{H_e^1} = i$ and $\mathrm{ind}_{H_o^1} = j$.

First, we check that we have a result similar to Lemma 23 in [15], giving a lower bound on H in terms of a quadratic form with explicit index $2 + 1$, related to a classical operator. All eigenelements of this operator are described in terms of hypergeometric functions. See for example Titchmarsh [20] for its complete description.

Then, by considering first the case where $w \in H_e^1(\mathbf{R})$ and $(w, Q) = 0$, and next, the case where $w \in H_o^1(\mathbf{R})$ and $(w, yQ) = 0$, we prove $H^*(w, w) \geqq 0$. Parity considerations imply Proposition 6.

Note that we can restrict ourselves to proving (ii) with $\sigma_0 = 0$, i.e., $H^*(w, w) \geqq 0$ whenever w satisfies $(w, Q) = (w, yQ) = 0$. Indeed, calculations in this proof being not optimal, by a continuity argument, it follows that (ii) is true for some small $\sigma_0 > 0$.

(a) Upper bound on the index of H. Here, we use that fact that L_1 can be compared to an operator of the type

$$\widetilde{L}_{a,b}u = -u_{yy} + au - b\frac{u}{\mathrm{ch}^2\,(y)},$$

where $a, b \in \mathbf{R}$. Note that this class of operators is well known (see, for example, Titchmarsh [20]).

We have

$$\begin{aligned}L_1 u = &-\frac{3}{2}u_{yy} + \frac{1}{2}u - \frac{p(p+1)}{4}\frac{u}{\mathrm{ch}^2\left(\frac{p-1}{2}y\right)} \\ &- \frac{p(p-1)(p+1)}{4}\,y\,\frac{\mathrm{sh}\left(\frac{p-1}{2}y\right)}{\mathrm{ch}^3\left(\frac{p-1}{2}y\right)}u.\end{aligned}$$

Recall that, from (174) in [15], we have the following inequality

$$\forall a \in \mathbf{R}, \quad a\frac{\operatorname{sh}(a)}{\operatorname{ch}^3(a)} \leqq \frac{1}{50}\left(1+\frac{92}{\operatorname{ch}^2(a)}\right), \tag{56}$$

and so

$$(L_1 u, u) \geqq \frac{3}{2}\left[\int u_y^2 + \frac{50-p(p+1)}{150}\int u^2 - \frac{39p(p+1)}{50}\int \frac{u^2}{\operatorname{ch}^2\left(\frac{p-1}{2}y\right)}\right]. \tag{57}$$

Let us separate the cases $p = 2, 3, 4$.

Case $p = 2$**:**

$$\begin{aligned}(L_1 u, u) &\geqq \frac{3}{2}\left[\int u_y^2 + \frac{44}{150}\int u^2 - \frac{117}{25}\int \frac{u^2}{\operatorname{ch}^2\left(\frac{y}{2}\right)}\right] \\ &\geqq \frac{3}{2}\left[\int u_y^2 + \frac{44}{150}\int u^2 - 5\int \frac{u^2}{\operatorname{ch}^2\left(\frac{y}{2}\right)}\right].\end{aligned}$$

Let

$$\widetilde{L}u = -u_{yy} + \frac{44}{150}u - 5\frac{u}{\operatorname{ch}^2\left(\frac{y}{2}\right)}.$$

As in the book by TITCHMARSH [20], the operator $\widetilde{L}$ has three nonpositive eigenvalues. The first and third eigenvalues correspond to the following even eigenfunctions:

$$\chi_1(y) = \operatorname{ch}^{-4}\left(\frac{y}{2}\right), \quad \chi_2(y) = \frac{6}{7}\operatorname{ch}^{-2}\left(\frac{y}{2}\right) - \operatorname{ch}^{-4}\left(\frac{y}{2}\right).$$

Case $p = 3$**:**

$$\begin{aligned}(L_1 u, u) &\geqq \frac{3}{2}\left[\int u_y^2 + \frac{19}{75}\int u^2 - \frac{234}{25}\int \frac{u^2}{\operatorname{ch}^2(y)}\right] \\ &\geqq \frac{3}{2}\left[\int u_y^2 + \frac{19}{75}\int u^2 - 12\int \frac{u^2}{\operatorname{ch}^2(y)}\right].\end{aligned}$$

Let

$$\widetilde{L}u = -u_{yy} + \frac{19}{75}u - 12\frac{u}{\operatorname{ch}^2(y)}.$$

The operator $\widetilde{L}$ has three nonpositive eigenvalues. The first and third eigenvalues correspond to the following even eigenfunctions:

$$\chi_1(y) = \operatorname{ch}^{-3}(y), \quad \chi_2(y) = \frac{4}{5}\operatorname{ch}^{-1}(y) - \operatorname{ch}^{-3}(y).$$

Case $p = 4$**:**

$$(\overline{L}_1 u, u) \geqq \frac{3}{2}\left[\int u_y^2 + 5\int u^2 - \frac{78}{5}\int \frac{u^2}{\mathrm{ch}^2\left(\frac{3y}{2}\right)}\right]$$
$$\geqq \frac{3}{2}\left[\int u_y^2 + 5\int u^2 - 22\int \frac{u^2}{\mathrm{ch}^2\left(\frac{3y}{2}\right)}\right].$$

Let

$$\widetilde{L}u = -u_{yy} + 5u - 22\frac{u}{\mathrm{ch}^2\left(\frac{3y}{2}\right)}.$$

The operator $\widetilde{L}$ has three nonpositive eigenvalues. The first and third eigenvalues correspond to the following even eigenfunctions:

$$\chi_1(y) = \mathrm{ch}^{-8/3}\left(\frac{3y}{2}\right), \quad \chi_2(y) = \frac{10}{13}\mathrm{ch}^{-2/3}\left(\frac{3y}{2}\right) - \mathrm{ch}^{-8/3}\left(\frac{3y}{2}\right).$$

Let us remark that the second eigenvalue is associated with an odd eigenfunction. Note also that in the three cases, we have the helpful property

$$\mathrm{span}(\chi_1, \chi_2) = \mathrm{span}(Q, Q^p). \tag{58}$$

We now claim from these estimates and algebraic relations the following properties on L_1.

Lemma 9. *The operator* L_1 *has the following properties:*

(i) $H(Q, Q) < 0, \quad H(Q_y, Q_y) = 0$*;* (60)
(ii) *the kernel of* L_1 *is* $\{0\}$*;*
(iii) $\forall\, \psi \in H^1(\mathbf{R})$*, there exists a unique* $\psi^* \in H^1(\mathbf{R})$ *such that* $L_1\psi^* = \psi$*.*

Proof. (i) We have

$$H(Q, Q) = -((LQ)_y, yQ) = -((-(p-1)Q^p)_y, yQ)$$
$$= \frac{p-1}{p+1}\int Q^{p+1} - (p-1)\int Q^{p+1} < 0,$$

and $H(Q_y, Q_y) = -\int (LQ_y)_y(yQ_y) = 0$, since $LQ_y = 0$.

(ii) Suppose that there exists $\chi \in H^1(\mathbf{R})$, such that $L_1\chi = 0$. Write $\chi = \chi_e + \chi_o$, where $\chi_e \in H_e^1$ and $\chi_o \in H_o^1$. We still have $L_1\chi_e = L_1\chi_o = 0$, since $L_1\chi_e$ is even (respectively, $L_1\chi_o$ is odd). We decompose $\chi_e = aQ + bQ^p + \chi_e^\perp$, where $(L_1 Q, \chi_e^\perp) = 0$ and $(L_1 Q^p, \chi_e^\perp) = 0$ (check directly that $\mathrm{span}(Q, Q^p)$ is not degenerate for H). Next, we have $0 = (L_1\chi_e, \chi_e^\perp) = H(\chi_e^\perp, \chi_e^\perp)$. Since $H(\chi_e^\perp, \chi_e^\perp) \geqq \frac{3}{2}(\widetilde{L}\chi_e^\perp, \chi_e^\perp)$, and $\chi_e^\perp$ is orthogonal is the L^2 sense to χ_1 and χ_2, we have from the spectral properties of $\widetilde{L}$ that $\chi_e^\perp = 0$. Since $L_1 Q$ and $L_1 Q^p$ are not colinear, we have $a = b = 0$, and so $\chi_e = 0$.

Observe now that we have $L_1 Q_y \neq 0$. Indeed,

$$L_1 Q_y = -\tfrac{1}{2}(y(L(Q_y))_y - LQ_y - L(yQ_{yy})) = \tfrac{1}{2}L(yQ_{yy}),$$

and $L(yQ_{yy}) \neq 0$ since the spectrum of L is exactly $\operatorname{span}(Q_y)$ (see [14], Lemma 2). Note that since $(L_1 Q_y, Q_y) = 0$, and $L_1 Q_y \neq 0$, there exists a negative eigenvalue of L_1 associated with an odd eigenfunction, denoted by ψ (this is a classical argument). Note that L_1 is coercive on $\operatorname{span}(\psi)^\perp$ by using the spectral properties of $\widetilde{L}$. Now, by a decomposition of χ_o in $a\psi + \chi_o^\perp$, we have the conclusion as before. Thus, we have proved that the kernel of L_1 is $\{0\}$.

(iii) We prove that L_1 is surjective in H^1.

Since $H(Q, Q) < 0$, there exists a first negative eigenvalue λ_1, associated with an even eigenvalue ψ_1. By the lower bound $H(w, w) \geqq (\widetilde{L}w, w)$, we know that there exists $\sigma > 0$ such that if w is orthogonal to Q and Q^p in the L^2 sense, then $H(w, w) \geqq \sigma(w, w)$.

Set

$$\lambda_2 = \inf_{\substack{(\psi_1, w) \\ |w|_{L^2}=1}} (L_1 w, w).$$

Two cases may occur.

If $\lambda_2 > 0$, then it follows that L_1 is coercive on $\operatorname{span}(\psi_1)^\perp$.

If $\lambda_2 \leqq 0$, then it follows from standard arguments that there exists $\psi_2 \neq 0$ such that $L_1\psi_2 = \lambda_2\psi_2 + \theta\psi_1$ and $(\psi_1, \psi_2) = 0$. By taking the scalar product with ψ_1, we find $\theta = 0$. Property (ii) implies that $\lambda_2 < 0$, and so ψ_2 is a even eigenfunction associated with a negative eigenvalue. From the comparison with $\widetilde{L}$, L_1 is coercive on $\operatorname{span}(\psi_1, \psi_2)^\perp$.

Now, we prove the surjectivity on H_e^1. Assume for example that we are in the second case. For any $\chi \in H_e^1$, we have $\chi = \chi_0 + a_1\psi_1 + \alpha_2\psi_2$, where $\chi_0 \in \operatorname{span}(\psi_1, \psi_2)^\perp$. Since L_1 is coercive on $\operatorname{span}(\psi_1, \psi_2)^\perp$, from the Lax-Milgram Theorem, there exists ϕ_0 such that $L_1\phi_0 = \chi_0$, and then $\phi = \phi_0 + \frac{a_1}{\lambda_1}\psi_1 + \frac{a_2}{\lambda_2}\psi_2$ is such that $L_1\phi = \chi$.

Surjectivity on H_o^1 follows from similar and simpler arguments (there is one negative eigenvalue associated with ψ_3 and L_1 is coercive on $\operatorname{span}(\psi_3)^\perp$). This concludes the proof of the lemma.

Let us introduce the following notation: $\forall\, u \in H^1(\mathbf{R}), \quad L_1 u^* = u.$

(b) Positivity property on H_e^1. We claim that if $w \in H_e^1(\mathbf{R})$ is such that $(w, Q) = 0$ then $H^*(w, w) \geqq 0$.

*Claim (*Numerical results*).* We have

$$(Q, Q^*) < 0, \quad \Delta_1 = (\chi_1, \chi_1^*) - \frac{(\chi_1^*, Q)^2}{(Q, Q^*)} + l_{11} > 0 \tag{60}$$

and

$$\Delta_2 = \left((\chi_1, \chi_1^*) - \frac{(\chi_1, Q)^2}{(Q, Q^*)} + l_{11}\right)\left((\chi_3, \chi_3^*) - \frac{(\chi_3^*, Q)^2}{(Q, Q^*)} + l_{33}\right) - \left((\chi_1, \chi_3^*) - \frac{(\chi_1^*, Q)(\chi_3^*, Q)}{(Q, Q^*)} + l_{13}\right)^2 > 0, \tag{61}$$

where

$$l_{ij} = -\frac{p-3}{\int Q^2}\left[(\chi_i^{\perp}, yQ_y)(\chi_j^{\perp}, Q^p) + (\chi_j^{\perp}, yQ_y)(\chi_i^{\perp}, Q^p)\right], \quad i, j = 1, 3.$$

Indeed, from numerical calculations, we have:

(i) For $p = 2$, $(Q, Q^*) \sim -5.6$, $\Delta_1 \sim 0.44$, $\Delta_2 \sim 0.038$.
(ii) For $p = 3$, $(Q, Q^*) \sim -1.9$, $\Delta_1 \sim 0.12$, $\Delta_2 \sim 0.0029$.
(iii) For $p = 4$, $(Q, Q^*) \sim -0.85$, $\Delta_1 \sim 0.037$, $\Delta_2 \sim 0.0017$.

To obtain these values, we have used the software MAPLE.

Claim. If $w \in H_e^1(\mathbf{R})$ is such that $(w, Q) = 0$, then $H^*(w, w) \geqq 0$.

Proof of the claim. Let us define

$$H^*(u, v) = H(u, v) - \frac{1}{\int Q^2}\left[(u, yQ_y)(v, 2Q + (p-3)Q^p) + (v, yQ_y)(u, 2Q + (p-3)Q^p)\right],$$

and $u^{\perp} = u^* - aQ^*$, where $a = \frac{(u^*, Q)}{(Q, Q^*)}$ is chosen so that $(u^{\perp}, Q) = 0$ ($(Q, Q^*) \neq 0$).

The proof is divided in several steps. First, we consider the space

$$E = \mathrm{span}(Q^*, (Q^p)^*, (yQ_y)^*)$$

(one additional dimension is needed compared with the case $p = 5$ to control the scalar product and property (58) reduces the size of E). We define $E^{\perp}$ the orthogonal space of E in $H_e^1(\mathbf{R})$ for the scalar product H (but not H^*). By a comparison argument, we prove that H^* is nonnegative on $E^{\perp}$. Then we conclude the proof from calculations in the space E.

We claim

$$H_e^1(\mathbf{R}) \cap \{w,\ (w, Q) = 0\} = E^{\perp} + \mathrm{span}((Q^p)^{\perp}, (yQ_y)^{\perp}). \tag{62}$$

Indeed, $\forall\, u \in E^{\perp}$, we have $(u, yQ_y) = H(u, (yQ_y)^*) = 0$, and so $H^*(u, u) = H(u, u) = (L_1 u, u) \geqq \frac{3}{2}(\widetilde{L}u, u)$. Since $(u, Q) = (u, Q^p) = 0$, from (58) and the spectral property of $\widetilde{L}$, H^* is coercive on $E^{\perp}$.

This implies that $E^{\perp} \cap E = \{0\}$, and thus E is not degenerated for H. In particular, $\forall\, u \in H_e^1(\mathbf{R})$, we have the following decomposition $u = u_1 + u_2$, $u_1 \in E^{\perp}$, $u_2 \in E$. If $(u, Q) = 0$, then $(u_1, Q) + (u_2, Q) = 0$. Since $(u_1, Q) = H(u_1, Q^*) = 0$, we deduce $(u_2, Q) = 0$. The conclusion then follows from

$$E \cap \{w,\ (w, Q) = 0\} = \mathrm{span}((Q^p)^{\perp}, (yQ_y)^{\perp}).$$

Thus, claim (62) is proved, and H^* on $E^{\perp}$ is nonnegative.

Finally, we claim that H^* restricted to $\mathrm{span}((Q^p)^{\perp}, (yQ_y)^{\perp})$ is nonnegative. This is equivalent to verifying that the two following properties are satisfied:

$$H^*((Q^p)^{\perp}, (Q^p)^{\perp}) > 0,$$

and

$$\begin{vmatrix} H^*((Q^p)^\perp, (Q^p)^\perp) & H^*((Q^p)^\perp, (yQ_y)^\perp) \\ H^*((Q^p)^\perp, (yQ_y)^\perp) & H^*((yQ_y)^\perp, (yQ_y)^\perp) \end{vmatrix} > 0.$$

Since $((Q^p)^\perp, Q) = ((yQ_y)^\perp, Q) = 0$, and since from the definition of $u^\perp$,

$$H(u^\perp, v^\perp) = (L_1(u^\perp), v^\perp) = (u, v^*) - \frac{(u^*, Q)(v^*, Q)}{(Q, Q^*)},$$

this is equivalent to the two inequalities (60) and (61). Thus H^* is nonnegative on $\operatorname{span}((Q^p)^\perp, (yQ_y)^\perp)$.

Therefore, $\forall\, w \in H_e^1(\mathbf{R})$, such that $(w, Q) = 0$, we have $w = w_1 + w_2$, where $w_1 \in E^\perp$ and $w_2 \in E$, with $(w_2, Q) = 0$, and

$$H^*(w, w) = H^*(w_1, w_1) + H^*(w_2, w_2) + 2H^*(w_1, w_2) \geqq 2H^*(w_1, w_2).$$

From the definition of E and $E^\perp$, we have $(w_1, yQ_y) = (w_1, Q) = (w_1, Q^p) = 0$, and so $H^*(w_1, w_2) = H(w_1, w_2) = 0$, which concludes the proof of the claim.

(c) Positivity property on H_o^1. We show that if $w \in H_o^1$ is such that $(w, yQ) = 0$, then $H^*(w, w) = H(w, w) \geqq 0$.

Claim (Numerical result).

$$((yQ)^*, yQ) < 0. \tag{63}$$

Indeed, for $p = 2$, we find numerically $((yQ)^*, yQ) \sim -23.90$; for $p = 3$, we find numerically $((yQ)^*, yQ) \sim -8.97$; for $p = 4$, we find numerically $((yQ)^*, yQ) \sim -5.15$.

Claim. If $w \in H_o^1(\mathbf{R})$ satisfies $(w, yQ) = 0$, then $H^*(w, w) \geqq 0$.

Proof of the claim. First, if we define $P_2 = \operatorname{span}(Q_y, (yQ)^*)$, then H is not degenerate on P_2 since

$$\begin{vmatrix} H(Q_y, Q_y) & H(Q_y, (yQ)^*) \\ H(Q_y, (yQ)^*) & H((yQ)^*, (yQ)^*) \end{vmatrix} = -(H(Q_y, (yQ)^*))^2 = -\tfrac{1}{4}(Q, Q)^2 \neq 0.$$

From (63) (or $H(Q_y, Q_y) = 0$), and the spectral property of $\widetilde{L}$, H is nonnegative on $P_2^\perp$, where $P_2^\perp$ is the orthogonal of P_2 in H_o^1, with respect to the quadratic form H.

Finally, if $w \in P_2$, $w \neq 0$ is such that $(w, yQ) = 0$,

$$w = \alpha Q_y + \beta (yQ)^*,$$

with $\beta \neq 0$, and

$$\frac{\alpha}{\beta} = -\frac{(yQ, (yQ)^*)}{(Q_y, yQ)} = -\frac{(yQ, (yQ)^*)}{H(Q_y, (yQ)^*)}.$$

It follows that

$$\frac{H(w,w)}{\beta^2} = \left(\frac{\alpha}{\beta}\right)^2 H(Q_y, Q_y) + 2\left(\frac{\alpha}{\beta}\right) H(Q_y, (yQ)^*) + H((yQ)^*, (yQ)^*)$$
$$= -H((yQ)^*, (yQ)^*) = -(yQ, (yQ)^*) > 0,$$

from (63) and $H(Q_y, Q_y) = 0$.

The conclusion follows from the fact that if $w \in H_o^1$, with $(w, yQ) = 0$, then $w = w_1 + w_2$, with $w_1 \in P_2^{\perp}$, and $w_2 \in P_2$, $(w_2, yQ) = 0$.

This together with parity considerations concludes the proof of Proposition 6(ii).

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