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Existence of Stationary Supersonic Flows Past a Pointed Body

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Abstract

In this paper we study the mathematical aspects of the stationary supersonic flow past a non-axisymmetric curved pointed body. The flow is described by a steady potential flow equation, which is a quasilinear hyperbolic equation of second order. We prove the local existence of the solution to this problem with a pointed shock attached at the tip of the pointed body, provided the pointed body is a perturbation of a circular cone, and the vertex angle of the approximate cone of the pointed body is less than a critical value. The solution is smooth in between the shock and the surface of the body. Consequently, such a structure of flow near the tip of the pointed body and its stability is verified mathematically.

1. Introduction

1.1. Background

Supersonic flow past a given body is a fundamental problem in gas dynamics, and has been well studied both computationally and experimentally (see [2, 5, 23, 27, 28]). The goal of the present article is to study the problem analytically. Physical observation shows that, when a projectile moves in the air with supersonic speed, a shock front will generally appear ahead of it. Depending on the shape of the projectile, the shock can be detached or attached. Namely, if the body has a blunt head, the shock in front of it is detached; otherwise, if the head of the body is sharp, then the shock will be attached to the head. In both cases the main task of the problem is to determine the location of the shock and the flow field in between the shock and the surface of the body.

Supersonic flow can be described mathematically by a quasilinear hyperbolic system of conservation laws. However, in the problem of supersonic flow past a blunt body, the flow behind the shock can be subsonic and then governed by an elliptic equation. Therefore, the whole problem will involve a nonlinear mixed-type equation with free boundary, and the mathematical theory for such problems is completely open. On the other hand, when the body placed in a supersonic flow has a sharp head, the flow behind the shock may possibly be totally supersonic. In this case the problem is reduced to a nonlinear boundary value problem for a hyperbolic system. Some progress in this case has been obtained in the past decades. Basically, two kinds of sharp bodies are considered – wings and pointed bodies. If the body is shaped like a wing, then under the assumption that the angle between the surface and the direction of the incoming flow is less than a critical value, the existence of a local solution near the edge of the wing has been proved. For instance, the conclusion is obtained in [5] for a plane wedge, in [9, 13–15, 25, 26] for wings with constant sections, and in [6] for wings with variable sections. On the other hand, when the body is conical, the tip of the body will often causes a strong singularity, and then the problem will be more complicated. If the body is a circular cone, and the incoming flow is parallel to the axis of the cone, the problem is discussed in [5]. If the body is assumed to have some symmetry, for instance, to be axisymmetric or conical with straight generators, the problems are discussed in [7, 8, 16, 24]. In this paper we are going to discuss the supersonic flow past a generic curved pointed body without these restrictions. Namely, we will verify the above-mentioned structure of flow field near the head of a pointed body in supersonic flow; in other words, we will prove the stability of such a structure in a local sense.

In this paper we always restrict ourselves to the case where the strength of the shock is small. Since the increase of the entropy of the flow is a small quantity of third order compared with the strength of the shock, it will be neglected in our discussion. Besides, according to the assumption on the shape of the pointed body, the shock front attached at its tip is also expected to be a perturbed cone. Therefore, the flow behind the shock can be assumed to be approximately isentropic and irrotational. It turns out that we can introduce a potential to simplify the system describing the flow. The potential flow equation is a second order quasilinear equation, which has been used and strongly recommended in many bibliography (see [17, 22]).

It may be convenient for readers if we specify all assumptions for our main result in advance. In the whole paper we only consider the polytropic gas with the state equation $p = A\rho^\gamma$. Therefore, the only data in our problem are the equation describing the surface of the body Γ and the parameters of the incoming flow: velocity q_∞, pressure p_∞ and density ρ_∞.

Our first assumption is

$$q_\infty > \left(\frac{\gamma p_\infty}{\rho_\infty}\right)^{\frac{1}{2}}, \tag{H$_1$}$$

which means the incoming flow is supersonic.

The equation of Γ in the cylindrical coordinates (z, R, θ) is $R = B(z, \theta)$. By introducing $r = R/z$, it can also be written as $r = b(z, \theta)$. The tangential cone Γ at the origin is $r = b(0, \theta)$, which is also the equation of the section of the tangential cone with the plane $z = 1$. To describe the fact that the pointed body Γ is a small perturbation of a circular cone $r = b_0$ (i.e., $R = b_0 z$), we assume that

$$\|b(0,\theta) - b_0\|_{C^{k_1}} \leqq \varepsilon_0, \tag{H$_2$}$$

$$\partial_z^k b(0,\theta) = 0 \quad \text{for} \quad 1 \leqq k \leqq k_2. \tag{H$_3$}$$

Here the condition (H$_2$) means that the perturbation is small in the θ direction, while the condition (H$_3$) means that the tangential cone is close to the pointed body with high order of tangency, i.e., the perturbation is small in a radial direction.

Our other assumption is on the sharpness of the pointed body. As it is known that the problem of supersonic flow past a symmetric cone is determined by the apple curve defined in [5] on the phase plane (u, v), which plays a similar role to shock polar in the discussion on the reflection of oblique plane shocks. The apple curve is symmetric with respect to the u axis and has the point $(q_\infty, 0)$ as its double point. The process of determining the weak entropy solution of the problem via the apple curve can be found in [5]. The conclusion is that if the vertex angle arctan b_0 is less than a critical value determined by the parameters of the incoming flow, then the problem admits a solution with an attached shock at the tip. Otherwise, the shock in front of the cone will be detached. Moreover, there is a constant $b_* < 1$, such that for the cone $r = b_0$ satisfying $b_0 < b_*$, the velocity u behind the shock is also supersonic. Therefore, to ensure that the equation governing the flow behind the shock is hyperbolic we assume that

$$\max_{z<z_0, 0\leqq\theta\leqq 2\pi} b(\theta) < b_*. \tag{H$_4$}$$

In the next subsection we will give a precise mathematical formulation of our problem. To prove the existence of the solution of the corresponding boundary value problem, we introduce several approximate solutions at different level. Assuming that the pointed body is a perturbation of the circular cone Γ, we call the solution of the incoming uniform supersonic flow past Γ a background solution, or approximate solution of level zero. In Section 2 we use a finite power expansion of z to look for an approximate solution which satisfies the equation and all boundary conditions with error $O(z^N)$. The first term of the finite power series is called an approximate solution of level one, and is nothing but the solution of the same uniform supersonic flow past the tangential cone of Γ. The determination of this term itself is also an independent problem, which we will discuss in Section 3 especially. All other terms in the finite expansion satisfy linear boundary value problems of elliptic equations with the same principal part. Under the assumption (H$_3$) we can determine all terms of the expansion up to N-th order. Here N can be as large as we want. The whole expansion is called an approximate solution of level two. Since this approximate solution satisfies the equation and the boundary conditions with error $O(z^N)$ for z near to zero, it is then possible to modify once more the approximate solution in Sobolev space with weight z^{-N}. Therefore, we can introduce Newton's iterative procedure in Section 4 to improve the approximation, and set up a new sequence of approximate solutions, which finally convergence to the precise solutions of the original problem.

Since the shock front is a free boundary to be determined with the unknown functions, then the boundary of the domain will be moving in the standard approximation procedure. To avoid this trouble we employ the partial hodograph transformation now and again in this paper. Here the main idea is to introduce a transformation which replaces an unknown function by one of the independent variables. Then a moving boundary will become fixed because the potential is given on this moving boundary. A disadvantage of the classical partial hodograph transformation is that the transformation may let a fixed boundary become a new moving boundary. To overcome this difficulty we combine it with the method of domain decomposition in Section 3. Meanwhile, in the last section the partial hodograph transformation is combined with two transformations of unknown functions to avoid the appearance of any new moving boundary. The detailed analysis will be given in Sections 3 and 4.

1.2. Formulation and result

The compressible Euler system describing conservation laws of mass, momentum, and total energy in the multidimensional case is given by

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \mathrm{div}(\boldsymbol{m}) &= 0, \\ \frac{\partial \boldsymbol{m}}{\partial t} + \mathrm{div}\left(\frac{\boldsymbol{m}\otimes\boldsymbol{m}}{\rho}\right) + \nabla p(\rho, S) &= 0, \\ \frac{\partial E}{\partial t} + \mathrm{div}\left(\boldsymbol{m}\left(\frac{E}{\rho}+\frac{p}{\rho}\right)\right) &= 0, \end{aligned} \tag{1.2.1}$$

where ρ is density, $\boldsymbol{m} = \rho\boldsymbol{v}$ is momentum, p is pressure, S is entropy, E is total energy; and $p = p(\rho, S)$, $E = E(\rho, S)$ are given functions. The last equation in (1.2.1) can also be replaced by

$$\frac{\partial S}{\partial t} + \boldsymbol{v}\cdot\nabla S = 0. \tag{1.2.2}$$

When the flow is isentropic and irrotational, (1.2.2) is satisfied automatically. Moreover, we can introduce a potential ϕ such that

$$\boldsymbol{v} = \nabla\phi. \tag{1.2.3}$$

Meanwhile, the momentum equation yields

$$\boldsymbol{v}_t + \tfrac{1}{2}\nabla(q^2) + \nabla h = 0, \tag{1.2.4}$$

where $q = |\boldsymbol{v}|$, $h(\rho)$ is the specific enthalpy determined within a constant by the thermodynamic equation of state and satisfies

$$h'(\rho) = \frac{1}{\rho}\frac{dp}{d\rho}(\rho, S_0).$$

For a polytropic gas,

$$p = A\rho^\gamma, \quad h = \frac{\gamma p}{(\gamma - 1)\rho} = \frac{A\gamma}{\gamma - 1}\rho^{\gamma-1}.$$

For stationary flow, all parameters of flow are independent of t. Then (1.2.4) leads to the Bernoulli's relation

$$\tfrac{1}{2}q^2 + h(\rho) = C_0, \tag{1.2.5}$$

where C_0 is a constant determined by the uniform incoming flow. Therefore, defining

$$H(\nabla\phi) = h^{-1}(C_0 - \tfrac{1}{2}|\nabla\phi|^2), \tag{1.2.6}$$

we obtain a steady potential flow equation

$$\sum \partial_{x_i}(\phi_{x_i} H(\nabla\phi)) = 0. \tag{1.2.7}$$

Notice that $H/H' = a^2$ with a being sonic speed, (1.2.7) is equivalent to

$$\left(\frac{v_1^2}{a^2} - 1\right)\phi_{x_1x_1} + \left(\frac{v_2^2}{a^2} - 1\right)\phi_{x_2x_2} + \left(\frac{v_3^2}{a^2} - 1\right)\phi_{x_3x_3} + \frac{2v_1v_2}{a^2}\phi_{x_1x_2} + \frac{2v_1v_3}{a^2}\phi_{x_1x_3} + \frac{2v_2v_3}{a^2}\phi_{x_2x_3} = 0 \tag{1.2.8}$$

where $v_i = \phi_{x_i}$, $(i = 1, 2, 3)$. The characteristic form of the right-hand side of (1.2.8) is

$$\left(\frac{v_1^2}{a^2} - 1\right)\xi_1^2 + \left(\frac{v_2^2}{a^2} - 1\right)\xi_2^2 + \left(\frac{v_3^2}{a^2} - 1\right)\xi_3^2 + \frac{2v_1v_2}{a^2}\xi_1\xi_2 + \frac{2v_1v_3}{a^2}\xi_1\xi_2 + \frac{2v_2v_3}{a^2}\xi_2\xi_3.$$

If $v_3 > a$, the quadratic form has two real roots ξ_3 for any ξ_1, ξ_2. Namely, (1.2.8) is strictly hyperbolic with respect to x_3. For the uniform supersonic incoming flow with $v_1 = v_2 = 0$, $v_3 = q_\infty$, the corresponding potential of the flow ahead of the possible shock is $\phi_0 = q_\infty x_3$.

Consider the problem of supersonic flow past a pointed body. As we mentioned above, ahead of the pointed body there will appear a shock front attached at the tip of the pointed body, provided the head is sharp in some sense which we will describe precisely later. Let the surface of the pointed body be given by $m(x_1, x_2, x_3) = 0$, and the corresponding shock front be given by $\mu(x_1, x_2, x_3) = 0$. Then on the surface $m(x_1, x_2, x_3) = 0$, the velocity of the fluid is tangent to the surface. Namely, we have

$$m_{x_1}\phi_{x_1} + m_{x_2}\phi_{x_2} + m_{x_3}\phi_{x_3} = 0. \tag{1.2.9}$$

On the shock front $\mu(x_1, x_2, x_3) = 0$ the potential ϕ is continuous across the shock front, i.e.,

$$\phi = \phi_-(= q_\infty x_3); \tag{1.2.10}$$

and the derivatives of ϕ must satisfy the Rankine-Hugoniot condition

$$(\mu_{x_1}\phi_{x_1} + \mu_{x_2}\phi_{x_2} + \mu_{x_3}\phi_{x_3})H = \mu_{x_3} q_\infty \rho_\infty. \tag{1.2.11}$$

The purpose of this paper is to prove the existence of the solution to the boundary value problem (1.2.8)–(1.2.11) in a neighbourhood of the origin. Here the function $\mu(x_1, x_2, x_3)$ is also unknown; it should be determined together with ϕ.

Since the directional derivative of ϕ normal to the shock front equals the normal component of velocity, which is never zero behind the shock, we can use $\phi(x_1, x_2, x_3) = q_\infty x_3$ to describe the shock front. Then we can omit an unknown function $\mu(x_1, x_2, x_3)$, and the Rankine-Hugoniot condition (1.2.11) can also be written as

$$\left(\phi_{x_1}^2 + \phi_{x_2}^2 + \phi_{x_3}(\phi_{x_3} - q_\infty)\right) H = (\phi_{x_3} - q_\infty) q_\infty \rho_\infty. \tag{1.2.12}$$

It is often convenient to discuss the problem (1.2.8)–(1.2.11) in a cylindrical coordinate system. Defining

$$R = \sqrt{x_1^2 + x_2^2}, \quad \theta = \arctan \frac{x_2}{x_1}, \quad z = x_3,$$

$r = R/z$, the components of velocity can be expressed as

$$\begin{aligned}
v_1 &= \phi_{x_1} = \frac{\cos\theta}{z}\phi_r - \frac{\sin\theta}{zr}\phi_\theta, \\
v_2 &= \phi_{x_2} = \frac{\sin\theta}{z}\phi_r + \frac{\cos\theta}{zr}\phi_\theta, \\
v_3 &= \phi_{x_3} = \phi_z - \frac{r}{z}\phi_r, \\
v_r &= v_1 \cos\theta + v_2 \sin\theta = \frac{\phi_r}{z}, \\
v_\theta &= v_2 \cos\theta - v_1 \sin\theta = \frac{\phi_\theta}{zr}.
\end{aligned}$$

The equation (1.2.8) can be reduced to the form

$$\begin{aligned}
z^2 a_{00}\phi_{zz} + a_{11}\phi_{rr} + a_{22}\phi_{\theta\theta} + 2z a_{01}\phi_{zr} + 2z a_{02}\phi_{z\theta} + 2a_{12}\phi_{r\theta} \\
+ a_1\phi_r + a_2\phi_\theta = 0,
\end{aligned} \tag{1.2.13}$$

where

$$\begin{aligned}
&a_{00} = \left(\frac{v_3}{a}\right)^2 - 1, \quad a_{11} = \frac{(v_r - r v_3)^2}{a^2} - (1 + r^2), \quad a_{22} = \frac{1}{r^2}\left(\frac{v_\theta^2}{a^2} - 1\right), \\
&a_{01} = \frac{v_3 v_r}{a^2} - r\left(\frac{v_3^2}{a^2} - 1\right), \quad a_{02} = \frac{v_3 v_\theta}{a^2 r}, \quad a_{12} = \frac{v_r v_\theta}{a^2 r} - \frac{v_\theta v_3}{a^2}, \\
&a_1 = \frac{v_\theta^2}{a^2 r} - \frac{1}{r} + 2r\left(\frac{v_3^2}{a^2} - 1\right) - \frac{2 v_3 v_r}{a^2}, \quad a_2 = \frac{2 v_r v_\theta}{a^2 r^2}.
\end{aligned} \tag{1.2.14}$$

Correspondingly, if we describe the surface of the pointed body by $r = b(z, \theta)$, then in the cylindrical coordinate system the boundary condition on it takes the form

$$(b + zb_z)\phi_z + \frac{b_\theta}{r^2}\left(\frac{\phi_\theta}{z}\right) - (1 + b(b + zb_z))\left(\frac{\phi_r}{z}\right) = 0. \tag{1.2.15}$$

Meanwhile, the boundary condition (1.2.12) becomes

$$\left(\left(\frac{\phi_r}{z}\right)^2 + \frac{1}{r^2}\left(\frac{\phi_\theta}{z}\right)^2 + \left(\phi_z - \frac{r\phi_r}{z} - q_\infty\right)\left(\phi_z - \frac{r\phi_r}{z}\right)\right) H \\ - \left(\phi_z - \frac{r\phi_r}{z} - q_\infty\right) q_\infty \rho_\infty = 0. \tag{1.2.16}$$

Obviously, when we solve the function ϕ from the above boundary value problem, all parameters of the flow including velocity, pressure and density can be obtained immediately.

If the pointed body placed in the supersonic flow is a circular cone with its axis parallel to the direction of the velocity of the flow, then the function $b(z, \theta)$ becomes a constant b_0. Correspondingly, all parameters of the flow also depend only on r, and are independent of θ and z. Therefore, the potential ϕ is a function independent of θ and homogeneous of degree one with respect to z. If we write $\phi = z\psi$, ψ is also independent of θ and z, and $v_r = \psi_r$, $v_3 = \psi - r(\partial\psi/\partial r)$. In this case the boundary value problem (1.2.8)–(1.2.11) becomes

$$a^2\left((1 + r^2)\psi_{rr} + \frac{1}{r}\psi_r\right) - ((1 + r^2)\psi_r - r\psi)^2\psi_{rr} = 0, \tag{1.2.17}$$

$$(1 + b_0^2)\psi_r - b_0\psi = 0 \qquad \text{on} \quad r = b_0, \tag{1.2.18}$$

$$\psi = q_\infty, \quad ((1 + s_0^2)\psi_r - s_0\psi)H + s_0 q_\infty \rho_\infty = 0 \quad \text{on} \quad r = s_0, \tag{1.2.19}$$

where s_0 is to be determined together with ψ.

The boundary problem (1.2.17)–(1.2.19) is essentially the same as the problem discussed in [5]. It can be solved by using the apple curve as mentioned in Section 1.1. The weak entropy solution of the problem (1.2.17)–(1.2.19) is called the background solution and is denoted by $\psi_B(r)$. Correspondingly, $\phi(z, r, \theta) = z\psi_B(r)$ is called the background solution of the problem (1.2.13), (1.2.15), (1.2.16) or, in this case, the approximate solution of level zero.

Now let us indicate some properties of the solution ψ of (1.2.17)–(1.2.19). These facts will be used in the discussion afterwards. If we define $\beta = \arctan r$, and

$$v_n = -v_r \cos\beta + v_3 \sin\beta = \frac{r\psi - (1 + r^2)\psi_r}{\sqrt{1 + r^2}},$$

then v_n is the component of velocity along the normal direction of the shock, satisfying $v_n < a$. Besides, in the whole region we have $v_r > 0$ behind the shock.

Moreover,

$$\psi_{rr} = \frac{a^2}{r}\psi_r\Big(\big((1+r^2)\psi_r - r\psi\big)^2 - a^2(1+r^2)\Big)^{-1}$$
$$= \frac{a^2}{r}\psi_r(1+r^2)(v_n^2 - a^2) < 0,$$

so v_r is monotonically decreasing with respect to r.

Our main result in this paper is

Theorem 1. *Assume that the conditions* (H_1)–(H_4) *are satisfied for sufficiently small* ε_0, *then we can find a number* $z_0 > 0$, *such that there is a* C^2 *function* $\phi(z,r,\theta)$ *defined in* $0 \leqq z \leqq z_0$, *satisfying the following conditions:*

(1) $\phi(0,r,\theta) = 0$, $\phi_r > 0$ *for* $z > 0$, *and then the equation* $\phi(z,r,\theta) = q_\infty z$ *defines a surface* $r = s(z,\theta)$*;*
(2) $\phi(z,r,\theta)$ *satisfies* (1.2.13) *in* $b(z,\theta) < r < s(z,\theta)$, $0 < z < z_0$, $0 \leqq \theta \leqq 2\pi$*;* (1.2.15) *on* $r = b(z,\theta)$*;* (1.2.16) *on* $r = s(z,\theta)$*;*
(3) $\left|\left(\frac{\phi_r}{z}\right)^2 + \frac{1}{r^2}\left(\frac{\phi_\theta}{z}\right)^2 + \left(\phi_z - \frac{r\phi_r}{z} - q_\infty\right)\left(\phi_z - \frac{r\phi_r}{z}\right)\right| < \left|\phi_z - \frac{r\phi_r}{z} - q_\infty\right| q_\infty$ *on* $r = s(z,\theta)$.

In a word, the problem (1.2.13), (1.2.15), (1.2.16) *admits a weak entropy solution with a pointed shock front attached at the origin, provided* ε_0 *is small enough.*

Remark 1. Since $b(z,\theta)$ is a C^∞ smooth function, for any integer p the local solution $\phi(z,r,\theta)$ can be in C^p. But the constants ε_0, k_1, k_2 and z_0 will depend on p.

Remark 2. The theorem indicates that the attached shock front and the flow behind the shock is stable under perturbation of the surface of the pointed body.

In the end of this section we would like to indicate that there have been many works on multidimensional conservation laws (see [1, 3, 4, 12, 17, 18, 20–22]). Among them, let us particularly mention two elegant works closely related to this paper. The first one is the study of the unsteady potential flow equation [22], which has the form in self-similar coordinates (ξ, η)

$$(a^2-(\psi_\xi-\xi)^2)\psi_{\xi\xi} - 2(\psi_\xi-\xi)(\psi_\eta-\eta)\psi_{\xi\eta} + (a^2-(\psi_\eta-\eta)^2)\psi_{\eta\eta} = 0. \quad (1.2.22)$$

The second one is on the study of the unsteady transonic small disturbance (UTSD) equation [3]

$$\begin{aligned} u_t + uu_x + v_y &= 0 \\ u_y - v_x &= 0, \end{aligned} \quad (1.2.23)$$

which leads to an equation in self-similar coordinates

$$\begin{aligned} (u-\xi)u_\xi - \eta u_\eta + v_\eta &= 0 \\ u_\eta - v_\xi &= 0. \end{aligned} \quad (1.2.24)$$

The authors in [4, 22] employed (1.2.22) and (1.2.24) to study the problem "reflection of shock by a ramp", which is a most important prototype problem in multidimensional hyperbolic conservation laws. Since on the (ξ, η) plane (1.2.22) or (1.2.24) may change type behind the shock, it turns out that a discussion of a boundary value problem for a mixed type equation or a degenerate elliptic equation is necessary. Returning to the problem of a steady potential equation in this paper, if the generators of the cone are straight lines, then the equation behind the shock in self-similar coordinates is elliptic due to the fact that the normal component of velocity behind the shock is subsonic. That is why the method in our paper is quite different from that in [4, 22]. Besides, another essential difference is that in our paper we have also to deal with the perturbation of the flow field in a radial direction.

2. Approximation of level one

2.1. Sub-boundary value problems

As the first step towards finding the approximate solution of the problem (1.2.8)–(1.2.11), we assume that the potential ϕ has a form of finite power expansion as

$$\phi(z, r, \theta) = \sum_{n=0}^{N} z^{n+1}\phi_n(r, \theta) + O(z^{N+2}). \tag{2.1.1}$$

By a suitable choice of ϕ_n, the equation and boundary conditions can be satisfied with error $O(z^N)$. Therefore, in a neighbourhood of the origin the form of the asymptotic series (2.1.1) offers a good approximation for large N.

According to (2.1.1),

$$\begin{aligned} v_r &= \frac{1}{z}\phi_r &&= \sum_{n=0}^{N} z^n \phi_{nr} + O(z^{N+1}),\\ v_\theta &= \frac{\phi_\theta}{zr} &&= \sum_{n=0}^{N} \frac{z^n}{r}\phi_{n\theta} + O(z^{N+1}),\\ v_z &= \phi_z - \frac{r}{z}\phi_r &&= \sum_{n=0}^{N} z^n((n+1)\phi_n - r\phi_{nr}) + O(z^{N+1}). \end{aligned}$$

For any smooth function f of v_z, v_r, v_θ we have the following expansion:

$$\begin{aligned} f(v_z, v_r, v_\theta) = f\Big(&\phi_0 - r\phi_{0r}, \phi_{0r}, \frac{1}{r}\phi_{0\theta}\Big)\\ &+ f_1(*)(2\phi_1 - r\phi_{1r})z + f_2(*)\phi_{1r}z + f_3(*)\phi_{1\theta}\frac{z}{r} + \cdots\\ &+ f_1(*)((n+1)\phi_n - r\phi_{nr})z^n + f_2(*)\phi_{nr}z^n\\ &+ f_3(*)\phi_{n\theta}\frac{z^n}{r} + F_n z^n, \end{aligned} \tag{2.1.2}$$

where $f_i(*)$ stands for the value of the derivative of f with respect to its i-th variables at $(\phi_0 - r\phi_{0r}, \phi_{0r}, \frac{1}{r}\phi_{0\theta})$, and F_n stands for a function depending on $\phi_\ell, \phi_{\ell r}, \phi_{\ell\theta}$ with $\ell < n$. More precisely, F_n can be written as

$$\sum_{i_{11}+\cdots+i_{3k_3}=n, i_{st}<n} g_{i_{11}+\cdots i_{3k_3}} \prod_{s=1}^{k_1} \phi_{i_{1s}} \prod_{s=1}^{k_2} (\phi_{i_{2s}})_r \prod_{s=1}^{k_3} (\phi_{i_{3s}})_\theta .$$

Substituting the expressions of v_z, v_r, v_θ into (1.2.13) leads to

$$\begin{aligned}&\sum_{n=2}^{N} n(n+1)a_{00}\phi_n z^{n+1} + \sum_{n=1}^{N}(n+1)(2a_{01}\phi_{nr} + 2a_{02}\phi_{n\theta})z^{n+1}\\ &\quad+ \sum_{n=0}^{N}(a_{11}\phi_{nrr} + a_{22}\phi_{n\theta\theta} + 2a_{12}\phi_{nr\theta} + a_1\phi_{nr} + a_2\phi_{n\theta})z^{n+1} = O(z^{N+2}).\end{aligned} \tag{2.1.3}$$

Expanding a_{ij} by means of (2.1.2) and comparing the terms with same power of z, we obtain

$$a_{11}(*)\phi_{0rr} + a_{22}(*)\phi_{0\theta\theta} + 2a_{12}(*)\phi_{0r\theta} + A(\phi_0, \phi_{0r}, \phi_{0\theta}) = 0, \tag{2.1.4}$$

and

$$\begin{aligned}a_{11}(*)\phi_{nrr} + a_{12}(*)\phi_{n\theta\theta} + 2a_{12}(*)\phi_{nr\theta} + B_r\phi_{nr} + B_\theta\phi_{n\theta} + C\phi_n\\ = G_n(\phi_\ell, \nabla\phi_\ell, \nabla^2\phi_\ell)_{\ell<n}, \qquad n \geqq 1,\end{aligned} \tag{2.1.5}$$

where

$$A(\phi_0, \phi_{0r}, \phi_{0\theta}) = -\frac{1}{r}\phi_{0r} + \frac{a^2}{r^2}\phi_{0r}^2\phi_{0\theta}^2 - \frac{2a^2}{r^3}(\phi_{0r}(1+r^2) - r\phi_0)\phi_{0\theta}^2,$$

G_n are given functions depending on ϕ_ℓ and its derivatives of first and second order, $G_n = 0$ if $\phi_\ell = 0$ for all $\ell < n$; $a_{ij}(*)$ stands for the value of a_{ij} at $(\phi_0 - r\phi_{0r}, \phi_{0r}, \frac{1}{r}\phi_{0\theta})$, while B_r, B_θ, C are also functions of ϕ_0 and its derivatives of first and second order.

We notice that the equation (2.1.4) and all equations in (2.1.5) have same principal symbol

$$Q(\xi,\eta) = \left(\frac{(rv_z - v_r)^2}{a^2} - (1+r^2)\right)\xi^2 + 2\left(\frac{v_r v_\theta}{a^2 r} - \frac{v_\theta v_z}{a^2}\right)\xi\eta + \frac{1}{r^2}\left(\frac{v_\theta^2}{a^2} - 1\right)\eta^2.$$

For the background solution $\phi_\theta = 0$, $v_\omega = (rv_z - v_r)/\sqrt{1+r^2} < a$, the symbol $Q(\xi,\eta)$ equals

$$(1+r^2)\left(\frac{v_\omega^2}{a^2} - 1\right)\xi^2 - \frac{1}{r^2}\eta^2,$$

which is definitely negative, hence all these equations are elliptic.

The solvability of boundary value problems of (2.1.5) relies on the maximum principle, whose validity inside a domain depends on the sign of the coefficient C. From the expansion of each term in (2.1.3) the coefficient C is

$$n(n+1)a_{00}(*) + O(n) + O(\phi_{0\theta}, \phi_{0\theta\theta}, \phi_{0r\theta}).$$

According to (1.2.14) we know $a_{00}(*) > 0$ and $a_{11}(*) < 0$, because the background solution is supersonic in the x_3 direction and the component v_ω of the velocity is subsonic. Therefore, we can find a constant k_2, which is determined by the parameters of the supersonic flow and the pointed body, such that the sign of coefficient C satisfies the requirement of the maximum principle for $n > k_2$.

Turn to the boundary condition (1.2.15) on the surface; $b(z,\theta)$ can also be written as a form of finite expansion

$$b(z,\theta) = \sum_{n=0}^{N} b_n(\theta) z^n + O\big(z^{N+1}\big). \tag{2.1.6}$$

Denote by $\underline{\phi}_n$ the expression $\phi_n(b_0(\theta),\theta)$ and by $g_n(\underline{\phi}, b)$ any expression depending on $\underline{\phi}_\ell, \underline{\phi}_{\ell r}, \underline{\phi}_{\ell\theta}, b_\ell, b_{\ell\theta}$ with $\ell < n$, where g_n may be different in different equalities. Substituting (2.1.6) into (2.1.1), we have

$$\begin{aligned}
\phi &= \sum_{n=0}^{N} z^{n+1}\phi_n\left(\sum_{\ell=0}^{N} b_\ell(\theta) z^\ell, \theta\right) + O(z^{N+2}) \\
&= \underline{\phi}_0 z + \sum_{n=1}^{N} (\underline{\phi}_n + \underline{\phi}_{0r} b_n + g_n(\underline{\phi}, b)) z^{n+1} + O(z^{N+2}).
\end{aligned}$$

Correspondingly,

$$\begin{aligned}
\phi_r &= \underline{\phi}_{0r} z + \sum_{n=1}^{N} (\underline{\phi}_{nr} + \underline{\phi}_{0rr} b_n + g_n(\underline{\phi}, b)) z^{n+1} + O(z^{N+2}), \\
\phi_\theta &= \underline{\phi}_{0\theta} z + \sum_{n=1}^{N} (\underline{\phi}_{n\theta} + \underline{\phi}_{0r\theta} b_n + g_n(\underline{\phi}, b)) z^{n+1} + O(z^{N+2}), \\
\phi_z &= \underline{\phi}_0 + \sum_{n=1}^{N} (n+1)(\underline{\phi}_n + \underline{\phi}_{0r} b_n + g_n(\underline{\phi}, b)) z^{n} + O(z^{N+1}).
\end{aligned}$$

Obviously, we have $g_1 = 0$, and $g_n = 0$ if $\underline{\phi}_\ell = 0$, $b_\ell = 0$ for all $\ell < n$.

Substituting these expressions into (1.2.15) and comparing the terms with the same power of z we obtain

$$b_0 \underline{\phi}_0 + \frac{1}{b_0^2} b_{0\theta} \underline{\phi}_{0\theta} - (1 + b_0^2)\underline{\phi}_{0r} = 0 \tag{2.1.7}$$

$$\begin{aligned}&(n+1)(\underline{\phi}_0 b_n + b_0(\underline{\phi}_n + \underline{\phi}_{0r} b_n))\\&\quad + \frac{1}{b_0^2}\left(b_{0\theta}(\underline{\phi}_{n\theta} + \underline{\phi}_{0r\theta} b_n) + \underline{\phi}_{0\theta} b_{n\theta} - 2\frac{b_n}{b_0} b_{0\theta}\underline{\phi}_{0\theta}\right)\\&\quad - (1+b_0^2)(\underline{\phi}_{nr} + \underline{\phi}_{0rr} b_n) - (n+2)\underline{\phi}_{0r} b_0 b_n = g_n(\underline{\phi}, b) \qquad (n>0). \quad (2.1.8)\end{aligned}$$

Meanwhile, because all coefficients b_n are given, $g_n(\underline{\phi}, b)$ can be simply written as $g_n(\underline{\phi})$, and (2.1.8) becomes

$$(n+1)b_0\underline{\phi}_n + \frac{1}{b_0^2} b_{0\theta}\underline{\phi}_{n\theta} - (1+b_0^2)\underline{\phi}_{nr} = g_n(\underline{\phi}) \qquad (n>0). \tag{2.1.9}$$

Notice that the differential operator acting on $\underline{\phi}_n$ $(n \geqq 0)$ in (2.1.7), (2.1.9) is $b_{0\theta} b_0^{-2}\partial_\theta - (1+b_0^2)\partial_r$, which is outward on the boundary $r = b(z,\theta)$. Moreover, $(n+1)b_0 > 0$ implies that ϕ_n cannot attain its non-negative maximum or non-positive minimun on the boundary $r = b_0(\theta)$. Namely, the boundary condition (2.1.8) satisfies the requirement of the maximum principle.

Finally let us turn to the shock boundary. The condition of continuity of the potential on the shock boundary is (1.2.10). Write the equation of the boundary as

$$r = s(z,\theta) = \sum_{n=0}^{N} s_n(\theta) z^n + O(z^{N+1}). \tag{2.1.10}$$

Then the potential on $r = s(z,\theta)$ is

$$\phi(z,r,\theta) = \sum_{n=0}^{N} z^{n+1}\phi_n\left(\sum_{\ell=0}^{N} s_\ell(\theta) z^\ell, \theta\right) + O(z^{N+2}).$$

Denote by $\bar{\phi}_n$ the function $\phi_n(s_0(\theta),\theta)$, by $g_n(\bar{\phi}, s)$ any expression depending on $\bar{\phi}_\ell, \bar{\phi}_{\ell r}, \bar{\phi}_{\ell\theta}, s_\ell, s_\theta$ with $\ell < n$. Then for $r = s_0(\theta)$ we have

$$\begin{aligned}\phi_r &= \bar{\phi}_{0r} z + \sum_{n=1}^{N} (\bar{\phi}_{nr} + \bar{\phi}_{0rr} s_n + g_n(\bar{\phi}, s)) z^{n+1} + O(z^{N+2}),\\ \phi_\theta &= \bar{\phi}_{0\theta} z + \sum_{n=1}^{N} (\bar{\phi}_{n\theta} + \bar{\phi}_{0r\theta} s_n + g_n(\bar{\phi}, s)) z^{n+1} + O(z^{N+2}),\\ \phi_z &= \bar{\phi}_0 + \sum_{n=1}^{N} (n+1)(\bar{\phi}_n + \bar{\phi}_{0r} s_n + g_n(\bar{\phi}, s)) z^n + O(z^{N+1}).\end{aligned}$$

The equality (1.2.10) implies

$$\bar{\phi}_0 + \sum_{n=1}^{N} (\bar{\phi}_n + \bar{\phi}_{0r} s_n(\theta) + g_n(\bar{\phi}, s)) z^n + O(z^{N+1}) = q_\infty, \tag{2.1.11}$$

which leads to

$$\bar{\phi}_0 = q_\infty, \tag{2.2.12}$$

$$s_n(\theta) = -\frac{\bar{\phi}_n}{\bar{\phi}_{0r}} + g_n(\bar{\phi}, s), \qquad n \geqq 1. \tag{2.1.13}$$

For the Rankine-Hugoniot condition (1.2.16), we first rewrite each term in the form of a finite power series:

$$\begin{aligned}
&\phi_z - \frac{r\phi_r}{z} - q_\infty \\
&\quad = (\phi_0 - r\phi_{0r} - q_\infty) + \sum_{n=1}^{N}((n+1)\phi_n - r\phi_{nr} + g_n(\phi))z^n + O(z^{N+1}) \\
&\quad = (\bar{\phi}_0 - r\bar{\phi}_{0r} - q_\infty) + \sum_{n=1}^{N}((n+1)\bar{\phi}_n - r\bar{\phi}_{nr} + ((n+1)\bar{\phi}_{0r} - r\bar{\phi}_{0rr})s_n \\
&\qquad + g_n(\bar{\phi}))z^n + O(z^{N+1}) \\
&\quad = (\bar{\phi}_0 - r\bar{\phi}_{0r} - q_\infty) + \sum_{n=1}^{N}(-r\bar{\phi}_{nr} + r\bar{\phi}_{0rr}\bar{\phi}_{0r}^{-1}\bar{\phi}_n + g_n(\bar{\phi}))z^n + O(z^{N+1}),
\end{aligned} \tag{2.1.14}$$

$$\begin{aligned}
&\left(\frac{\phi_r}{z}\right)^2 + \frac{1}{r^2}\left(\frac{\phi_\theta}{z}\right)^2 + \left(\phi_z - \frac{r\phi_r}{z}\right)^2 \\
&\quad = \bar{\phi}_{0r}^2 + \frac{1}{r^2}\bar{\phi}_{0\theta}^2 + (\bar{\phi}_0 - r\bar{\phi}_{0r})^2 + 2\sum_{n=1}^{N}(\bar{\phi}_{0r}(\bar{\phi}_{nr} - \bar{\phi}_{0rr}\bar{\phi}_{0r}^{-1}\bar{\phi}_n) \\
&\qquad + \frac{1}{r^2}\bar{\phi}_{0\theta}(\bar{\phi}_{n\theta} - \bar{\phi}_{0r\theta}\bar{\phi}_{0r}^{-1}\bar{\phi}_n) + (\bar{\phi}_0 - r\bar{\phi}_{0r}) \\
&\qquad \cdot (-r\bar{\phi}_{nr} + r\bar{\phi}_{0rr}\bar{\phi}_{0r}^{-1}\bar{\phi}_n + g_n(\bar{\phi}))z^n + O(z^{N+1}).
\end{aligned} \tag{2.1.15}$$

Also,

$$\begin{aligned}
&H\left(C_0 - \frac{1}{2}|\nabla\phi|^2\right) \\
&\quad = H_0 - H_0'\sum_{n=1}^{N}\Bigg(\bar{\phi}_{0r}(\bar{\phi}_{nr} - \bar{\phi}_{0rr}\bar{\phi}_{0r}^{-1}\bar{\phi}_n) + \frac{1}{r^2}\bar{\phi}_{0\theta}(\bar{\phi}_{n\theta} - \bar{\phi}_{0r\theta}\bar{\phi}_{0r}^{-1}\bar{\phi}_n) \\
&\qquad + (\bar{\phi}_0 - r\bar{\phi}_{0r})(-r\bar{\phi}_{nr} + r\bar{\phi}_{0rr}\bar{\phi}_{0r}^{-1}\bar{\phi}_n) + g_n(\bar{\phi})\Bigg)z^n + O(z^{N+1}),
\end{aligned}$$

where H_0, H_0' take their value at $C_0 - \frac{1}{2}\left(\bar{\phi}_{0r}^2 + \frac{1}{r^2}\bar{\phi}_{0\theta}^2 + (\bar{\phi}_0 - r\bar{\phi}_{0r})^2\right)$, and $H_0 = \rho_0$, $H_0/H_0' = a_0^2$.

Defining

$$D_0 = \bar\phi_{0r}^2 + \frac{1}{r^2}\bar\phi_{0\theta}^2 + (\bar\phi_0 - r\bar\phi_{0r})(\bar\phi_0 - r\bar\phi_{0r} - q_\infty)$$
$$= \bar\phi_{0r}^2 + \frac{1}{r^2}\bar\phi_{0\theta}^2 - (\bar\phi_0 - r\bar\phi_{0r})r\bar\phi_{0r}$$

and substituting all the above expressions into (1.2.16), we have

$$D_0\rho_0 = -rho\bar\phi_{0r}q_\infty\rho_\infty, \tag{2.1.16}$$

$$\begin{aligned}
&-\frac{D_0}{a_0^2}\Big(\bar\phi_{0r}(\bar\phi_{nr} - \bar\phi_{0rr}\bar\phi_{0r}^{-1}\bar\phi_n) + \frac{1}{r^2}\bar\phi_{0\theta}(\bar\phi_{n\theta} - \bar\phi_{0r\theta}\bar\phi_{0r}^{-1}\bar\phi_n)\\
&+ (\bar\phi_0 - r\bar\phi_{0r})(-r\bar\phi_{nr} + r\bar\phi_{0rr}\bar\phi_{0r}^{-1}\bar\phi_n)\Big)\\
&+ 2\bar\phi_{0r}(\bar\phi_{nr} - \bar\phi_{0rr}\bar\phi_{0r}^{-1}\bar\phi_n) + \frac{2}{r^2}\bar\phi_{0\theta}(\bar\phi_{n\theta} - \bar\phi_{0r\theta}\bar\phi_{0r}^{-1}\bar\phi_n)\\
&+ (2\bar\phi_0 - 2r\bar\phi_{0r} - q_\infty)(-r\bar\phi_{nr} + r\bar\phi_{0rr}\bar\phi_{0r}^{-1}\bar\phi_n)\\
&- (-r\bar\phi_{nr} + r\bar\phi_{0rr}\bar\phi_{0r}^{-1}\bar\phi_n)\frac{q_\infty\rho_\infty}{\rho_0} = g_n(\bar\phi, s), \qquad n \geqq 1.
\end{aligned} \tag{2.1.17}$$

The equality can also be rewritten as

$$\gamma_1\bar\phi_{nr} + \gamma_2\bar\phi_{n\theta} + \gamma_3\bar\phi_n = g_n(\bar\phi), \tag{2.1.18}$$

where $g_n(\bar\phi, s)$ has been rewritten as $g_n(\bar\phi)$ by inductively applying (2.1.13) with index $i < n$. Direct computation implies $g_1 = 0$, and $g_n = 0$ if $\bar\phi_\ell = 0$ for all $\ell < n$. Moreover, the coefficients in (2.1.18) are

$$\gamma_1 = -\frac{D_0}{a_0^2}(-r\bar\phi_0 + (1 + r^2)\bar\phi_{0r})) + 2\bar\phi_{0r}(1 + r^2) - 2r\bar\phi_0 + \frac{rq_\infty\rho_\infty}{\rho_0},$$
$$\gamma_2 = -\frac{D_0}{a_0^2}\cdot\frac{\bar\phi_{0\theta}}{r^2} + \frac{2}{r^2}\bar\phi_{0\theta},$$
$$\gamma_3 = -\gamma_1\frac{\bar\phi_{0rr}}{\bar\phi_{0r}} - \gamma_2\frac{\bar\phi_{0r\theta}}{\bar\phi_{0r}}.$$

Now let us observe the sign of the coefficients in (2.1.18). To simplify calculation we may neglect all derivatives with respect to θ because our problem is a small perturbation of the symmetric case, so that all these derivatives are small. Recalling the physical meaning of relevant quantities, we have the following relations on the boundary $r = s_0(\theta)$:

$$\phi_{0r} = v_{0r}, \quad \phi_0 - r\phi_{0r} = v_{0z},$$
$$r\phi_0 - (1 + r^2)\phi_{0r} = rv_{0z} - v_{0r} = \Big(\sqrt{1 + r^2}\Big)v_{0n},$$

where the subscript 0 stands for the state behind the shock, v_{0n} is the inner normal component of the velocity and satisfies $0 < v_{0n} < \infty$. Furthermore,

$$D_0 = -\frac{r\phi_{0r}q_\infty\rho_\infty}{\rho_0} = -\frac{\rho_\infty q_{\infty n}v_{0r}\sqrt{1+r^2}}{\rho_0} = -v_{0n}v_{0r}\sqrt{1+r^2},$$

which leads to

$$\begin{aligned}\gamma_1 &= -(1+r^2)v_{0r}\frac{v_{0n}^2}{a_0^2} + v_{0r}(1+r^2) - \sqrt{1+r^2}v_{0n} + \frac{rq_\infty\rho_\infty}{\rho_0}\\ &= v_{0r}(1+r^2)\left(1-\frac{v_{0n}^2}{a_0^2}\right) > 0.\end{aligned}$$

Besides, $\gamma_2 \sim 0$, and $\gamma_3 > 0$ due to $\bar\phi_{0r} > 0$ and $\bar\phi_{0rr} < 0$. Hence the boundary condition (2.1.18) also satisfies the requirement of the maximum principle.

Now (2.1.4), (2.1.7), (2.1.12), (2.1.16) form a nonlinear boundary value problem for $\phi_0(r,\theta)$ with a free boundary $r = s_0(\theta)$, which is to be determined together with the solution $\phi_0(r,\theta)$. On the other hand, once $s_0(\theta)$ is obtained, the boundary value problem for (2.1.5) with boundary condition (2.1.8) on $r = b_0(\theta)$ and condition (2.1.17) on $r = s_0(\theta)$ for each $n > 0$ is a linear boundary value problem in a fixed domain. In this way we have derived a set of sub-boundary value problems from the original problem (1.2.8)–(1.2.11). When these boundary value problems are solved, the approximate solution $\tilde\phi(z,r,\theta) = \sum z^{n+1}\phi_n(r,\theta)$ satisfying (1.2.13), (1.2.15), (1.2.16) with error $O(z^{N+2})$ is also obtained.

2.2. Existence and uniqueness for sub-boundary value problems

If $b(z,\theta)$ is independent of z, the surface of the pointed body is a cone with straight generators. In this case the expected attached shock is also a cone with straight generators, and all parameters of the flow behind the shock are constant on each ray starting from the origin. Noticing that in this case $\phi_z = \phi_0$, $b_z = 0$, we see that (1.2.13), (1.2.15), (1.2.16) is automatically reduced to (2.1.4), (2.1.7), (2.1.16). For the latter we can establish the following theorem, which is a special case of Theorem 1 in fact.

Theorem 2. *Assume that the surface of a pointed body is given by $r = b(\theta)$, the conditions* (H_1), (H_2), (H_4) *are satisfied for a suitable integer k_1, and a sufficiently small ε_0, then there is a C^2 function $\phi_0(z,r,\theta)$, satisfying the following conditions:*

(1) $\phi_{0r} > 0$, *and then the equation $\phi_0(r,\theta) = q_\infty$ defines a surface $r = s_0(\theta)$;*
(2) $\phi_0(r,\theta)$ *satisfies* (2.1.4) *in* $b_0(\theta) < r < s_0(\theta)$, $0 \leqq \theta \leqq 2\pi$; (2.1.7) *on* $r = b_0(z)$; (2.1.16) *on* $r = s_0(\theta)$;
(3) $|\phi_{0r}^2 + \frac{1}{r^2}\phi_{0\theta}^2 - (\phi_{0r} - r\phi_{0r})r\phi_{0r}| < r\phi_{0r}$ *on* $r = s_0(\theta)$.

In a word, the problem (2.1.4), (2.1.7), (2.1.16) *admits a weak entropy solution with a pointed shock front attached at the origin, provided ε_0 is small enough.*

Theorem 2 can be reduced to the solvability of a free boundary value problem of a nonlinear elliptic equation. We leave the nonlinear problem to the next section due to its complexity. Instead, let us first consider the problem (2.1.5), (2.1.8), (2.1.17).

Notice that the domain where (2.1.5) is defined has fixed boundary $r = b_0(\theta)$ and $r = s_0(\theta)$, and both (2.1.5) and the boundary conditions (2.1.8), (2.1.17) are linear. Besides, for the linear boundary value problem all requirements of the maximum principle on boundary conditions are satisfied, so that $\phi_n(r,\theta)$ cannot attain its non-negative maximum and non-positive minimum on

$$r = b_0(\theta), \quad r = s_0(\theta).$$

Moreover, as we indicated above, for a suitable integer k_1, the coefficient C in the equation (2.1.5) is negative. Therefore, the boundary value problem (2.1.5), (2.1.8), (2.1.17) for each $n > k_2$ is uniquely solvable.

According to the assumption (1.2.20), all coefficients $b_n(\theta)$ in (2.1.5) vanish for $1 \leqq n \leqq k_2$. Therefore, $\phi_n(r,\theta) = 0$ with $1 \leqq n \leqq k_2$ satisfies the linear boundary value problem (2.1.5), (2.1.8), (2.1.17). Meanwhile, $s_n(\theta)$ also vanishes for $1 \leqq n \leqq k_2$. Combining this with the solvability of this boundary value problem for $n > k_2$, we can determine all solutions $\phi_n(r,\theta)$ with $1 \leqq n \leqq N$, where N can be as large as we want. Therefore, according to the process of deriving all conditions satisfied by $\phi_n(r,\theta)$ we confirm that the finite expansion $\tilde{\phi}(z,r,\theta) = \sum_{n=0}^{N} z^{n+1}\phi_n(r,\theta)$ satisfies (1.2.8)–(1.2.11) with error $O(z^N)$. This means that a required approximate solution with error $O(z^N)$ for any large N can be obtained once Theorem 2 is proved.

3. The approximate solution of level one

3.1. Decomposition of nonlinear problems

In this section we are going to prove Theorem 2. The solution confirmed by this theorem will be chosen as the approximate solution of level one for the problem (1.2.13), (1.2.15), (1.2.16), and then will be employed to seek $\phi_n(r,\theta)$ defined in Section 2. To simplify the notation we replace $\phi_0, b_0(\theta), s_0(\theta)$ by $\psi, b(\theta), s(\theta)$ respectively, and write the problem (2.1.4), (2.1.7), (2.1.16) as

$$a_{11}\psi_{rr} + a_{22}\psi_{\theta\theta} + 2a_{12}\psi_{r\theta} + A(\psi,\psi_r,\psi_\theta) = 0, \tag{3.1.1}$$

$$b\psi + \frac{1}{b^2} b_\theta\psi_\theta - (1+b^2)\psi_r = 0 \quad \text{on} \quad r = b(\theta), \tag{3.1.2}$$

$$\left(\psi_r^2 + \frac{1}{r^2}\psi_\theta^2 + (\psi - r\psi_r)r\psi_r\right)\rho_0 = -r\psi_r q_\infty\rho_\infty \quad \text{when} \quad \psi = q_\infty. \tag{3.1.3}$$

where $b(\theta)$ satisfies (2.2.1). We emphasize here that the solution of (3.1.1)–(3.1.3) solves the problem of supersonic flow past a conical body with the surface $b = b(\theta)$. Certainly, this solution is also a perturbation of the background solution of (1.3.1)–(1.3.3).

In order to fix the free boundary we introduce a partial hodograph transformation (see [11, 18, 19, 21]). Notice that $\psi_r > a_0 > 0$ holds for background solutions; the inequality will also be true for perturbed solutions. Therefore, we can take ψ as the new coordinate p, and perform a partial hodograph transformation $T\,:\ (r,\theta) \mapsto (p,\sigma)$:

$$\begin{aligned} \sigma &= \theta, \\ p &= \psi(r,\theta). \end{aligned} \tag{3.1.4}$$

Its inverse transform is T^{-1} :

$$\begin{aligned} \theta &= \sigma, \\ r &= u(p,\sigma). \end{aligned} \tag{3.1.5}$$

In the new coordinates, the shock front becomes a fixed boundary $p = \psi_0$, and $u(p,\sigma)$ becomes a new unknown function, which satisfies $u(\psi_0,\sigma) = s(\sigma)$ on the shock front.

The function $u(p,\sigma)$ satisfies a second order differential equation, which can be deduced from (3.1.1). By the chain the rule we have

$$\begin{aligned} \partial_r &= \frac{1}{u_p}\partial_p, & \partial_\theta &= \partial_\sigma - \frac{u_\sigma}{u_p}\partial_p, \\ \psi_r &= \frac{1}{u_p}, & \psi_\theta &= -\frac{u_\sigma}{u_p}, \\ \psi_{rr} &= -\frac{1}{u_p^3}u_{pp}, & \psi_{r\theta} &= \frac{u_\sigma}{u_p^3}u_{pp} - \frac{1}{u_p^2}u_{p\sigma}, \\ \psi_{\theta\theta} &= -\frac{1}{u_p}u_{\sigma\sigma} + \frac{2u_\sigma}{u_p^2}u_{p\sigma} - \frac{u_\sigma^2}{u_p^3}u_{pp}, \end{aligned}$$

and

$$|\nabla\phi|^2 = \frac{1}{u_p^2}\Big(1 + \frac{u_\sigma^2}{u^2} + (pu_p - u)^2\Big). \tag{3.1.6}$$

Therefore, in the new coordinates (3.1.1) becomes

$$\begin{aligned} & a^2\Big(-\frac{u_{pp}}{u_p^3}(1+u^2) + \frac{1}{uu_p} + \frac{2u_\sigma}{u^2u_p^2}u_{p\sigma} - \frac{1}{u^2u_p}u_{\sigma\sigma} - \frac{u_\sigma^2}{u^2u_p^3}u_{\sigma\sigma}\Big) \\ & \quad + \Big(\frac{1+u^2}{u_p} - up\Big)^2\frac{u_{pp}}{u_p^3} + \frac{2}{u^2}\Big(\frac{1+u^2}{u_p} - up\Big)\frac{u_\sigma}{u_p}\Big(\frac{u_\sigma}{u_p^3}u_{pp} - \frac{u_{p\sigma}}{u_p^2}\Big) \\ & \quad - \frac{1}{u^4}\frac{u_\sigma^2}{u_p^2}\Big(-\frac{u_{\sigma\sigma}}{u_p} + \frac{2u_\sigma}{u_p^2}u_{p\sigma} - \frac{u_\sigma^2}{u_p^3}u_{pp}\Big) - \frac{u_\sigma^2}{u^3u_p^3} + \frac{2}{u^3}\Big(\frac{1+u^2}{u_p} - up\Big)\frac{u_\sigma^2}{u_p^2} = 0. \end{aligned} \tag{3.1.7}$$

The boundary conditions will also have a new form. First, on the shock front the variable p takes constant ψ_0, hence the boundary becomes fixed, and the boundary condition is

$$\left((1+u^2)+\frac{u_\sigma^2}{u^2}-upu_p\right)H+upu_p\rho_\infty=0. \tag{3.1.8}$$

However, since the potential ψ is not known on the surface of the body, then the corresponding boundary in the (p,σ) coordinate system becomes unknown. Therefore, the boundary conditions on it should be described by two equations:

$$u=b(\sigma), \tag{3.1.9}$$

$$(1+b^2(\sigma))+\frac{b'(\sigma)}{b^2(\sigma)}u_\sigma-b(\sigma)pu_p=0. \tag{3.1.10}$$

In what follows we will call the problem (3.1.1)–(3.1.3) (NL), and call the problem (3.1.7)–(3.1.10) (NL)*. Evidently, these two problems are equivalent. If one of them is solved, then the solution of the other one is also obtained.

The problem (NL) has a fixed boundary $r=b(\theta)$ and a free boundary $r=s(\theta)$. Conversely, the problem (NL)* had a fixed boundary at $p=\psi_0$ and a free boundary $p=g(\sigma)$. Motivated by Schwarz alternating iteration we will also use the domain decomposition method to decompose the problems (NL) and (NL)* into a set of auxiliary nonlinear boundary value problems with fixed boundaries, so that the combination of successive solutions to these problems leads to the solutions to (NL) and (NL)*. Namely, for small $\delta>0$ we introduce constants r_1, r_2 and two monotonically increasing sequences $\{\alpha_\ell\}$, $\{\beta_\ell\}$ with $1\leqq\ell\leqq k$, satisfying

$$\begin{aligned}
&b_0<r_2<r_1<b_0+\delta,\\
&\alpha_1=\psi_B(r_2),\\
&\alpha_\ell<\beta_{\ell-1}<\alpha_{\ell+1}<\beta_\ell, \qquad 1<\ell<k\\
&\beta_k=\psi_0.
\end{aligned} \tag{3.1.11}$$

Denoting the interior annulus $b(\theta)<r<r_1, 0\leqq\theta\leqq 2\pi$ on the (r,θ) plane by Ω_i, and the exterior annulus $\alpha_\ell<p<\beta_\ell, 0\leqq\sigma\leqq 2\pi$ on (p,σ) plane by Ω_{e_ℓ}, we can write the $k+1$ auxiliary boundary value problems as follows.

$$(\mathrm{NL})^{(i)}:\begin{cases}\text{equation (3.1.1)} & \text{in}\quad \Omega_i,\\ \text{boundary condition (3.1.2)} & \text{on}\quad r=b(\theta),\\ \psi=d(\theta) & \text{on}\quad r=r_1,\end{cases} \tag{3.1.12}$$

$$(\mathrm{NL})^{(e_\ell)}:\begin{cases}\text{equation (3.1.7)} & \text{in}\quad \Omega_{e_\ell},\\ u=q_{1\ell}(\sigma) & \text{on}\quad p=\alpha_\ell,\\ u=q_{2\ell}(\sigma) & \text{on}\quad p=\beta_\ell,\end{cases} \tag{3.1.13}$$

where $1\leqq\ell\leqq k-1$.

$$(\mathrm{NL})^{(e_k)}:\begin{cases}\text{(3.1.7)} & \text{in}\quad \Omega_{e_k},\\ u=q_1(\sigma) & \text{on}\quad p=\alpha_k,\\ \text{boundary condition (3.1.8)} & \text{on}\quad p=\psi_0.\end{cases} \tag{3.1.14}$$

Remark 3. Here and later we assume that $\delta' = \max_{1\leqq \ell \leqq k} \alpha_\ell$ are small, so that the comparison principle holds for these problems, and assume that $\min_{1\leqq \ell\leqq k}(\beta_\ell - \alpha_{\ell+1}, \alpha_{\ell+1} - \beta_{\ell-1}) > \frac{\delta'}{10}$ without loss of generality. We also assume that the constant ε_0, which dominates the perturbation, is usually much smaller than δ, δ'. When the notation $O(\varepsilon_0)$ is applied, the quantities δ, δ' are regarded as fixed.

The solvability of problems $(\mathrm{NL})^{(i)}$, $(\mathrm{NL})^{(e_\ell)}$ $(1 \leqq \ell \leqq k)$ and corresponding estimates of their solutions will be given in the next subsection. To emphasize the dependence on the corresponding data given on boundaries we also denote the solution of the problems (3.1.12), (3.1.13), (3.1.14) by $(\mathrm{NL})^{(i)}\{b(\theta), d(\theta)\}$, $(\mathrm{NL})^{(e_\ell)}\{q_{1\ell}(\sigma), q_{2\ell}(\sigma)\}$ and $(\mathrm{NL})^{(e_k)}\{q_{1k}(\sigma)\}$ respectively. Similarly, we will also use $(\mathrm{NL})\{b(\theta)\}$ to denote the solution of (1.2.13), (1.2.15), (1.2.16).

3.2. The problem in the interior annulus

For the problem $(\mathrm{NL})^{(i)}$, we first use the transformation

$$\begin{aligned} \tilde\theta &= \theta \\ \frac{\tilde r - b_0}{r_1 - b_0} &= \frac{r - b(\theta)}{r_1 - b(\theta)} \end{aligned} \tag{3.2.1}$$

to change the boundary $r = b(\theta)$ into $\tilde r = b_0$, then $(\mathrm{NL})^{(i)}$ becomes a new boundary problem defined on $b_0 \leqq \tilde r \leqq r_1, 0 \leqq \tilde\theta \leqq 2\pi$. Consider the linearization of this at $\psi = \psi_B(r), b(\theta) = b_0, d(\theta) = \psi_{10}$. Since at the the background solution, $\tilde r_r = 1, \tilde r_\theta = 0$, and $(\psi_B)_\theta = (\psi_B)_{r\theta} = (\psi_B)_{\theta\theta} = 0$, we obtain the linearization of (3.1.1) for the perturbation $\dot\psi$:

$$L^{(i)}\dot\psi = A_{11}\dot\psi_{rr} + A_{22}\dot\psi_{\theta\theta} + B_1\dot\psi_r + C\dot\psi = f, \tag{3.2.2}$$

where

$$\begin{aligned} A_{11} &= a^2(1+r^2) - ((1+r^2)\psi_r - r\psi)^2, \quad A_{22} = \frac{a^2}{r^2}, \\ B_1 &= \frac{a^2}{r} - \Big((\gamma-1)\Big((1+r^2)\psi_{rr} + \frac{\psi_r}{r}\Big) + 2(1+r^2)\psi_{rr}\Big) \\ &\quad \cdot ((1+r^2)\psi_r - r\psi), \\ C &= 2r((1+r^2)\psi_r - r\psi)\psi_{rr} - ((1+r^2)\psi_{rr} + \frac{1}{r}\psi_r)(\gamma-1)(\psi - r\psi_r), \end{aligned}$$

and $\tilde r, \tilde\theta$ are denoted by r, θ again. Correspondingly, the boundary conditions for the linearized problem are

$$(1+b_0^2)\dot\psi_r - b_0\dot\psi = g \qquad \text{on} \quad r = b_0, \tag{3.2.3}$$

$$\dot\psi = h \qquad \text{on} \quad r = r_1. \tag{3.2.4}$$

The linearized problem (3.2.2)–(3.2.4) is denoted by $(\mathrm{L})^{(i)}$, which is a linear elliptic boundary value problem because $A_{11} > 0$ and $A_{22} > 0$.

Lemma 1. *There is $\delta > 0$, such that the solution of* $(\mathrm{L})^{(\mathrm{i})}$ *uniquely exists, and*

$$\|\dot{\psi}\|_{C^{2+\alpha}[b_0,r_1;0,2\pi]} \leqq C_1\big(\|f\|_{C^{\alpha}[b_0,r_1;0,2\pi]} + \|g\|_{C^{1+\alpha}(0,2\pi)} + \|h\|_{C^{2+\alpha}(0,2\pi)}\big), \quad (3.2.5)$$

$$\|\dot{\psi}\|_{C^{2+\alpha}[b_0,r_1-\frac{\delta}{10};0,2\pi]} \leqq C_2\big(\|f\|_{C^{\alpha}[b_0,r_1;0,2\pi]} + \|g\|_{C^{1+\alpha}(0,2\pi)} + \|h\|_{C^{0}(0,2\pi)}\big) \quad (3.2.6)$$

provided $|r_1 - b_0| < \delta$.

Proof. First, let us show that the solution $\dot{\psi}$ of the linearized problem (3.2.2)–(3.2.4) monotonically depends on its boundary value on $r = r_1$, provided f and g vanish. Namely, $h_1 \geqq h_2$ on $r = r_1$ implies $\dot{\psi}_1 \geqq \dot{\psi}_2$ inside the domain. In fact, making a transformation of an unknown function $v = e^{K(r-b_0)^2}\dot{\psi}$ for the problem $(\mathrm{L})^{(i)}$, then

$$L^{(i)}\dot{\psi} = L^{(i)}(e^{-K(r-b_0)^2}v) = e^{-K(r-b_0)^2}L_K^{(i)}v,$$

where

$$\begin{aligned} L_K^{(i)}v = L^{(i)}v - 4K(r-b_0)A_{11}v_r + ((4K^2(r-b_0)^2 \\ - 2K)A_{11} - 2K(r-b_0)B_1)v. \end{aligned} \quad (3.2.7)$$

Obviously, v satisfies the elliptic equation

$$L_K^{(i)}v = 0,$$

provided $\dot{\psi}$ satisfies $L^{(i)}\dot{\psi} = 0$. When δ is sufficiently small and $K = \delta^{-1}$, the coefficient of v in (3.2.7) is

$$(4K^2(r-b_0)^2 - 2K)A_{11} - 2K(r-b_0)B_1 + C < (4-2\delta^{-1})A_{11} + 2|B_1| + |C| < 0.$$

On the other hand, the boundary condition (3.2.3) implies that v satisfies

$$\gamma_1\frac{\partial v}{\partial n} + \gamma_2 v = 0$$

on $r = b_0$, where γ_1, γ_2 are both positive. By the maximum principle, the solution of $L_K^{(i)}v = 0$ can not take negative value on $r = b_0$ and inside the domain $b_0 < r < r_1$, $0 \leqq \theta \leqq 2\pi$, if it is non-negative on $r = r_1$. This fact implies that v depends on its boundary value on $r = r_1$ monotonically. Hence it is also true for the solution $\dot{\psi}$ of the problem $(\mathrm{L})^{(i)}$.

The above argument indicates that the elliptic operator $L^{(i)}$ under homogeneous boundary conditions corresponding to (3.2.3), (3.2.4) does not have a non-negative eigenvalue, provided δ is sufficiently small. Namely, the problem (3.2.2)–(3.2.4) is uniquely solvable. Besides, (3.2.5), (3.2.6) are just the generalized global and interior Schauder estimates.

Remark 4. Returning to the original (r, θ) coordinate system, the estimates obtained in the lemma can be written as

$$\|\dot{\psi}\|_{C^{2+\alpha}(\Omega_i)} \leqq C_1(\|f\|_{C^\alpha(\Omega_i)} + \|g\|_{C^{1+\alpha}(0,2\pi)} + \|h\|_{C^{2+\alpha}(0,2\pi)}),$$
$$\|\dot{\psi}\|_{C^{2+\alpha}(\Omega_i^-)} \leqq C_2(\|f\|_{C^\alpha(\Omega_i)} + \|g\|_{C^{1+\alpha}(0,2\pi)} + \|h\|_{C^0(0,2\pi)}),$$

where $\Omega_i = \{(r, \theta); b(\theta) \leqq r \leqq r_1, 0 \leqq \theta \leqq 2\pi\}$, $\Omega_i^- = \{(r, \theta); b(\theta) \leqq r \leqq r_1 - \frac{\delta}{10}, 0 \leqq \theta \leqq 2\pi\}$. Moreover, denote the coefficients of (3.2.2) by $A_{11}(\psi)$, $A_{22}(\psi)$ etc. If ψ is replaced by ψ_1, satisfying $\|\psi - \psi_1\|_{C^{2+\alpha}} \leqq C$ uniformly, then the estimates (3.2.5), (3.2.6) hold with uniform constants C_1, C_2.

Lemma 2. *Assume that δ_1, ε are sufficiently small in the sense of Remark 3, $|r_1 - b_0| < \delta_1$, $\|b(\theta) - b_0\|_{C^{2+\alpha}(0,2\pi)} < \varepsilon$, $\|d(\theta) - \psi_{10}\|_{C^{2+\alpha}(0,2\pi)} < \varepsilon$ with $\psi_{10} = \psi_B(r_1)$, then the problem* $(\mathrm{NL})^{(i)}\{b(\theta), d(\theta)\}$ *has a unique solution $\psi(r, \theta)$. Moreover,*

$$\|\psi(r, \theta) - \psi_B(r)\|_{C^{2+\alpha}(\Omega_a)} \to 0 \quad \textit{when} \quad \varepsilon \to 0. \tag{3.2.8}$$

Proof. When $b(\theta) = b_0$, the function $\psi_B(r)$ is the solution of the nonlinear problem $(\mathrm{NL})^{(i)}\{b_0, \psi_{10}\}$. For $|r_1 - b_0| < \delta_1$, the linearization $(\mathrm{L})^{(i)}$ of the nonlinear problem $(\mathrm{NL})^{(i)}$ at $b(\theta) = b_0$, $\psi = \psi_{10}$ has estimate (3.2.5), where the constant C_1 is uniform with respect to $b(\theta)$. Then the implicit function theorem implies that the problem $(\mathrm{NL})^{(i)}\{b(\theta), d(\theta)\}$ has a unique solution, which is a small perturbation of $\psi = \psi_B(r)$, provided $\|b(\theta) - b_0\|_{C^{2+\alpha}} < \varepsilon$, $\|d(\theta) - \psi_{10}\|_{C^{2+\alpha}} < \varepsilon$ for sufficiently small ε. Furthermore, (3.2.8) follows from the conclusion of the implicit function theorem.

Lemma 3. *Assume that δ_1, ε are sufficiently small, $|r_1 - b_0| < \delta_1$, $\|b(\theta) - b_0\|_{C^{2+\alpha}(0,2\pi)} < \varepsilon$, $\|d_j(\theta) - \psi_{10}\|_{C^{2+\alpha}} < \varepsilon$ $(j = 1, 2)$, and $\psi_j(r, \theta)$ is the solution of the problem* $(\mathrm{NL})^{(i)}\{b(\theta), d_j(\theta)\}$. *Then the comparison principle is valid, i.e., $d_2 \geqq d_1$ implies $\psi_2 \geqq \psi_1$.*

Proof. For the nonlinear problem $(\mathrm{NL})^{(i)}$, we assume that $\psi_j(r, \theta)$ is the solution of problem $(\mathrm{NL})^{(i)}\{b(\theta), d_j(\theta)\}$ with $d_2(\theta) > d_1(\theta)$. Subtracting the equation satisfied by ψ_2 and ψ_1, we find that $\dot{\psi} = \psi_2 - \psi_1$ satisfies

$$A_{11}(\psi_1)\dot{\psi}_{rr} + D_0\dot{\psi}_{r\theta} + A_{22}(\psi_1)\dot{\psi}_{\theta\theta} + (B_1(\psi_1) + D_1)\dot{\psi}_r + D_2\dot{\psi}_\theta + (C(\psi_1) + D_3)\dot{\psi} = 0, \tag{3.2.9}$$

where $A_{11}(\psi_1)$, $A_{22}(\psi_1)$, $B_1(\psi_1)$, $C(\psi_1)$ are the coefficients in (3.2.2) with ψ replaced by ψ_1, $D_j\,(j = 0, 1, 2, 3)$ are small quantities, dominated by $\sum_{|\alpha|\leqq 2} |\partial^\alpha \dot{\psi}| + |\partial_\theta^\alpha \psi_{1,2}|$. Correspondingly, the boundary conditions for $\dot{\psi}$ are

$$(1 + b^2)\dot{\psi}_r - \frac{b'}{b^2}\dot{\psi}_\theta - b\dot{\psi} = 0 \qquad \text{on} \quad r = b(\theta), \tag{3.2.10}$$
$$\dot{\psi} = d_2 - d_1 \quad \text{on} \quad r = r_1. \tag{3.2.11}$$

In view of the estimates given in Lemma 1, the coefficients of the equation (3.2.9) are small perturbations of the corresponding coefficients of (3.2.2), provided that

$\|d_j(\theta)-\psi_{10}\|_{C^{2+\alpha}}$ and $\|b(\theta)-b_0\|_{C^{2+\alpha}}$ are sufficiently small. Therefore, $d_2-d_1 \geqq 0$ on $r=r_1$ implies $\dot{\psi} \geqq 0$ in Ω_i.

Remark 5. Let $\tilde{b}_0$ be a constant in $(b_0, b_0+\varepsilon_0)$ and $\tilde{\psi}_B(r)$ be the solution of problem (1.2.17)–(1.2.19) with b_0 replaced by $\tilde{b}_0$. Then we know from [3] that $\tilde{\psi}_B(r)$ is well defined on $(\tilde{b}_0, \tilde{s}_0)$ with $\tilde{s}_0 > s_0$, and $\tilde{\psi}_B(r) < \psi_B(r)$ holds on $\tilde{b}_0 \leqq r \leqq s_0$. Furthermore, if we extend $\tilde{\psi}_B(r)$ to $(b_0, \tilde{b}_0)$ by using (1.2.17) and its value in $r \geqq \tilde{b}_0$, the relation $\tilde{\psi}_B(r) < \psi_B(r)$ still holds in $(b_0, \tilde{b}_0)$. In what follows the functions $\psi_B(r)$ and $\tilde{\psi}_B(r)$ will be applied to dominate the solution of (NL) from above and below.

Lemma 4. *Assume that* $b_0 \leqq b(\theta) \leqq \tilde{b}_0$, $\|b(\theta)-b_0\|_{C^{2+\alpha}(0,2\pi)} < \varepsilon$, $|b_0-\tilde{b}_0| < \varepsilon$, $\psi_B(r) = (\mathrm{NL})(b_0)$, $\tilde{\psi}_B(r) = (\mathrm{NL})(\tilde{b}_0)$, $\psi(r,\theta) = (\mathrm{NL})^{(i)}\{b(\theta), d(\theta)\}$, $\tilde{\psi}_B(r_1) \leqq d(\theta) \leqq \psi_B(r_1)(=\psi_{10})$, *then* $\tilde{\psi}_B(r) \leqq \psi(r,\theta) \leqq \psi_B(r)$ *holds in* Ω_i. *Besides,*

$$\|\psi(r,\theta)-\psi_B(r)\|_{C^{2+\alpha}(\Omega_i^-)} \leqq C(\|d(\theta)-\psi_{10}\|_{C^{(}0,2\pi)}+\varepsilon), \tag{3.2.12}$$

where $\psi_{10} = \psi_B(r_1)$.

Proof. The background solution ψ_B satisfies

$$r\psi_B - (1+r^2)(\psi_B)_r = 0 \qquad \text{on} \quad r = b_0,$$

hence

$$\frac{(\psi_B)_r}{\psi_B} \leqq \frac{b_0}{1+b_0^2}.$$

If ε is small enough such that $b_0+\varepsilon < 1$, the relation $b_0 \leqq b(\theta)$ implies

$$\frac{(\psi_B)_r}{\psi_B} \leqq \frac{b(\theta)}{1+b(\theta)^2},$$

which leads to

$$b(\theta)\psi_B - (1+b^2(\theta))(\psi_B)_r \geqq 0. \tag{3.2.13}$$

Set $\Delta_1\psi = \psi - \psi_B$. It satisfies

$$A_{11}(\psi_B)(\Delta_1\psi)_{rr} + A_{22}(\psi_B)(\Delta_1\psi)_{\theta\theta} + \tilde{D}_0(\Delta_1\psi)_{r\theta} + (B_1(\psi_B)+\tilde{D}_1)(\Delta_1\psi)_r + \tilde{D}_2(\Delta_1\psi)_\theta + (C(\psi_B)+\tilde{D}_3)\Delta_1\psi = 0 \tag{3.2.14}$$

and corresponding boundary conditions

$$(1+b^2)(\Delta_1\psi)_r - \frac{b'}{b^2}(\Delta_1\psi)_\theta - b\Delta_1\psi \geqq 0 \qquad \text{on} \ r = b(\theta), \tag{3.2.15}$$

$$\Delta_1\psi = d(\theta) - \psi_{10} \quad \text{on} \ r = r_1. \tag{3.2.16}$$

Again noticing that the coefficients of (3.2.14) are small perturbation of the corresponding coefficients of (3.2.2), and using an argument similar to that in Lemma 3.3, we may confirm that $\Delta_1\psi$ cannot attain its positive maximum inside the domain and on the boundary. Namely, we have $\psi(r,\theta) \leqq \psi_B(r)$.

On the other hand, $\tilde{\psi}_B(r)$ satisfies

$$r\tilde{\psi}_B - (1+r^2)(\tilde{\psi}_B)_r = 0 \quad \text{on } r = \tilde{b}_0.$$

Then $b(\theta) \leqq \tilde{b}_0$ implies

$$(1+b(\theta)^2)(\tilde{\psi}_B)_r - b(\theta)\tilde{\psi}_B \geqq 0 \quad \text{on } r = b(\theta). \tag{3.2.17}$$

Then we can derive an elliptic equation with the form (3.2.14) for $\Delta_2\psi = \psi - \tilde{\psi}_B$, which also satisfies boundary conditions

$$(1+b^2)(\Delta_2\psi)_r - \frac{b'}{b^2}(\Delta_2\psi)_\theta - b\Delta_2\psi \leqq 0 \quad \text{on } r = b(\theta), \tag{3.2.18}$$

$$\Delta_2\psi \geqq 0 \quad \text{on } r = r_1. \tag{3.2.19}$$

Hence $\psi \geqq \tilde{\psi}_B$ can be obtained as above.

To prove the estimate of the $C^{2+\alpha}$ norm of ψ , we use the transformation (3.2.1). Denote by ψ_B^* the inverse image of ψ_B; then ψ_B^* is defined in Ω_i. Since ψ_B satisfies (1.2.17), we have

$$a^2((1+\tilde{r}^2)e^2(\psi_B^*)_{rr} + \frac{1}{\tilde{r}}e(\psi_B^*)_r) - ((1+\tilde{r}^2)e(\psi_B^*)_r - \tilde{r}\psi_B^*)^2 e^2(\psi_B^*)_{rr} = 0, \tag{3.2.20}$$

where

$$e = \frac{\partial r}{\partial \tilde{r}} = \frac{r_1 - b(\theta)}{r_1 - b_0} = 1 + \frac{b_0 - b(\theta)}{r_1 - b_0} = 1 + O(\varepsilon).$$

Set $\Delta\psi^* = \psi(r,\theta) - \psi_B^*$. This satisfies

$$\begin{aligned} &A_{11}(\psi_B)(\Delta\psi^*)_{rr} + A_{22}(\psi_B)(\Delta\psi^*)_{\theta\theta} + D_0^*(\Delta\psi^*)_{r\theta} \\ &+ (B_1(\psi_B) + D_1^*)(\Delta\psi^*)_r + D_2^*(\Delta\psi^*)_\theta + (C(\psi_B) + D_3^*)\Delta\psi^* = f \end{aligned} \tag{3.2.21}$$

and corresponding boundary conditions

$$(1+b^2)(\Delta\psi^*)_r - \frac{b'}{b^2}(\Delta\psi^*)_\theta - b\Delta\psi^* = g \qquad \text{on } r = b(\theta), \tag{3.2.22}$$

$$\Delta\psi^* = d(\theta) - \psi_{10} \quad \text{on } r = r_1, \tag{3.2.23}$$

where $\|f\|_{C^{2+\alpha}}, \|g\|_{C^{2+\alpha}}$ are quantities $O(\varepsilon)$. Therefore, we have

$$\|\Delta\psi^*\|_{C^{2+\alpha}(\Omega_i^-)} \leqq C(\|d(\theta) - \psi_{10}\|_{C^0(0,2\pi)} + \varepsilon),$$

which leads to (3.2.12) directly.

3.3. The problems in exterior annuli

Next we consider the nonlinear problem $(\mathrm{NL})^{(e_\ell)}$ in the exterior annulus Ω_{e_ℓ} with fixed boundary $p=\alpha_\ell$ and $p=\beta_\ell$. The image of $\psi_B(r)$ under the inverse partial hodograph transformation T^{-1} will be called the background solution for $(\mathrm{NL})^*$, and will be denoted by $u_B(p)$. To linearize $(\mathrm{NL})^{(e_\ell)}$, we multiply (3.2.4) by u_p^3 and then linearize the background solution $u=u_B(p)$. In view of

$$\frac{\partial u_B}{\partial \sigma}=0,\quad a^2=\frac{\gamma-1}{2}(C-u_p^{-2}-(p-uu_p^{-1})^2),$$

we obtain the linearized equation for $\dot u$ as follows.

$$\begin{aligned}
L^{(e)}\dot u &= (N_1^2-(1+u^2)a^2)\dot u_{pp}-\frac{a^2u_p^2}{u^2}\dot u_{\sigma\sigma}\\
&\quad+\left(\frac{2a^2u_p}{u}-2u_{pp}N_1\frac{1+u^2}{u_p^2}+N_2\left(\frac{1+u^2}{u_p^3}-\frac{up}{u_p^2}\right)\right)\dot u_p \qquad (3.3.1)\\
&\quad+\left(\frac{2N_1u_{pp}(2u-pu_p)}{u_p}-\frac{a^2}{u^2}u_p^2-2a^2uu_{pp}+N_2\left(\frac{p}{u_p}-\frac{u}{u_p^2}\right)\right)\dot u\\
&= f,
\end{aligned}$$

where $N_1=(1+u^2)/u_p-up$, $N_2=(\gamma-1)(u_p^2/u-(1+u^2)u_{pp})$.

The boundary conditions on $p=\alpha_1,\cdots,\alpha_k,\beta_1,\cdots,\beta_{k-1}$ are taken as

$$\dot u=q. \qquad (3.3.2)$$

Meanwhile, the boundary condition of $\dot u$ on $p=\psi_0(=\beta_k)$ is the linearization of (3.1.10), that is,

$$\begin{aligned}
&2u\dot u\cdot H-(H-\rho_\infty)p(u\dot u_p+u_p\dot u)\\
&\qquad+(1+u^2-upu_p)\cdot H'\cdot\left(\frac{\dot u_p}{u_p^3}+(p-\frac{u}{u_p})(\frac{\dot u}{u_p}-\frac{u\dot u_p}{u_p^2})\right)=g_1.
\end{aligned}$$

In view of $H=\rho$, $H'=\rho/a^2$, the condition can be rewritten as

$$\begin{aligned}
&\left(\left(1-\frac{\rho_\infty}{\rho}\right)pu+\frac{1+u^2-upu_p}{a^2}\left(\left(p-\frac{u}{u_p}\right)\frac{u}{u_p^2}-\frac{1}{u_p^3}\right)\right)\dot u_p\\
&\qquad+\left(-2u+\left(1-\frac{\rho_\infty}{\rho}\right)pu_p-\frac{1+u^2-upu_p}{a^2}\left(p-\frac{u}{u_p}\right)\frac{1}{u_p}\right)\dot u=g,
\end{aligned}$$

which will be denoted by

$$\gamma_3\dot u_p+\gamma_4\dot u=g. \qquad (3.3.3)$$

Therefore, we can establish the linearized boundary value problems

$$(\mathrm{L})^{(e_\ell)} : \begin{cases} \text{equation (3.3.1)} & \text{in} \quad \Omega_{e_\ell}, \\ \dot{u} = q_{1\ell} & \text{on} \quad p = \alpha_\ell, \\ \dot{u} = q_{2\ell} & \text{on} \quad p = \beta_\ell, \end{cases} \tag{3.3.4}$$

for $1 \leqq \ell \leqq k-1$, and the problem

$$(\mathrm{L})^{(e_k)} : \begin{cases} \text{equation (3.3.1)} & \text{in} \quad \Omega_{e_k}, \\ \dot{u} = q_{1k} & \text{on} \quad p = \alpha_k, \\ \text{condition(3.3.3)} & \text{on} \quad p = \psi_0. \end{cases} \tag{3.3.5}$$

Notice that all $|\beta_\ell - \alpha_\ell|$ are chosen small, so that the coefficients of (3.3.1) satisfy the requirement of the comparison principle. Then the principle is available for solutions of (3.3.4) or (3.3.5), provided γ_3, γ_4 in (3.3.3) take the same sign. To verify the last fact we notice that on the boundary $p = \psi_0$

$$p - \frac{u}{u_p} = \psi - r\psi_r = v_z,$$

$$1 + u^2 - upu_p = \frac{1}{\psi_r}((1+r^2)\psi_r - r\psi) = -\frac{v_n}{v_r}\sqrt{1+r^2} < 0,$$

$$\left(p - \frac{u}{u_p}\right)\frac{u}{u_p^2} - \frac{1}{u_p^3} = -\frac{1}{u_p^3}(1 + u^2 - upu_p) = \sqrt{1+r^2}\cdot v_r^2 \cdot v_n > 0.$$

According to the Rankine-Hugoniot condition (3.1.8) we have $(1+u^2)\rho = puu_p(\rho - \rho_\infty)$ for the background solution. This implies $\left(1 - \frac{\rho_\infty}{\rho}\right)pu_p = \frac{1+u^2}{u}$, then

$$\begin{aligned} \gamma_3 &= \frac{1+r^2}{u_p} - \frac{1}{a^2}\frac{v_n}{v_r}\sqrt{1+r^2}\sqrt{1+r^2}v_r^2 v_n \\ &= v_r(1+r^2)\left(1 - \frac{v_n^2}{a^2}\right) > 0, \end{aligned}$$

$$\begin{aligned} \gamma_4 &= -2r + \frac{1+r^2}{r} - \frac{1}{a^2}\sqrt{1+r^2}\frac{v_n}{v_r}v_z v_r \\ &= \frac{1}{r} - r - \frac{1}{a^2}\sqrt{1+r^2}v_z v_n \\ &\geqq \frac{1}{r}\left(1 + \frac{v_n^2}{a^2}\right) - r\left(1 - \frac{v_n^2}{a^2}\right). \end{aligned}$$

Under the assumption of weak shock the qualtity $1 - \frac{v_n^2}{a^2}$ is small, hence $\gamma_4 > 0$. Since $\frac{\partial}{\partial p}$ points in the outward normal direction of the boundary $p = \psi_0$ of Ω_{e_k}, then the sign of coefficients in the linearized boundary condition (3.3.3) satisfies the requirement of the maximum principle for elliptic boundary value problems. Therefore, we can use a similar method to that in Lemma 1 to establish the following proposition.

Lemma 5. *There is $\delta' > 0$, such that the solution $\dot{u}_\ell$ of* $(\mathrm{L})^{(e_\ell)}$ *uniquely exists, and satisfies the comparison principle, that is,*

$$\begin{aligned} f &= 0, \qquad q_{1\ell} \geqq 0, \quad q_{2\ell} \geqq 0 \Longrightarrow \dot{u}_\ell \geqq 0 \quad \text{in} \quad \Omega_{e_\ell}, \\ f &= g = 0, \qquad q_{1k} \geqq 0 \Longrightarrow \dot{u}_\ell \geqq 0 \quad \text{in} \quad \Omega_{e_k}. \end{aligned}$$

Meanwhile, the following estimates hold:

$$\|\dot{u}_\ell\|_{C^{2+\alpha}(\Omega_{e_\ell})} \leqq C_1\big(\|f\|_{C^\alpha(\Omega_{e_\ell})} + \|q_{1\ell}\|_{C^{2+\alpha}(0,2\pi)} + \|q_{2\ell}\|_{C^{2+\alpha}(0,2\pi)}\big), \tag{3.3.6}$$

$$\|\dot{u}_\ell\|_{C^{2+\alpha}(\Omega^-_{e_\ell})} \leqq C_2\big(\|f\|_{C^\alpha(\Omega_{e_\ell})} + \|q_{1\ell}\|_{C^0(0,2\pi)} + \|q_{2\ell}\|_{C^0(0,2\pi)}\big), \tag{3.3.7}$$

for $1 \leqq \ell \leqq k-1$, and

$$\|\dot{u}_k\|_{C^{2+\alpha}(\Omega_{e_k})} \leqq C_1\big(\|f\|_{C^\alpha(\Omega_{e_k})} + \|g\|_{C^{1+\alpha}(0,2\pi)} + \|q_{1k}\|_{C^{2+\alpha}(0,2\pi)}\big), \tag{3.3.8}$$

$$\|\dot{u}_k\|_{C^{2+\alpha}(\Omega^-_{e_k})} \leqq C_2\big(\|f\|_{C^\alpha(\Omega_{e_k})} + \|g\|_{C^{1+\alpha}(0,2\pi)} + \|q_{1k}\|_{C^0(0,2\pi)}\big), \tag{3.3.9}$$

where $\Omega^-_{e_\ell} = [\alpha_\ell + \frac{1}{10}\delta_1, \beta_\ell - \frac{1}{10}\delta_1; 0, 2\pi]$ for $1 \leqq \ell \leqq k-1$ and $\Omega^-_{e_k} = [\alpha_k + \frac{1}{10}\delta_1, \beta_k; 0, 2\pi]$.

Furthermore, similar to Lemmas 2 and 3, we can establish the solvability of nonlinear problems $(\mathrm{NL})^{(e_\ell)}$ for $1 \leqq \ell \leqq k$ and the corresponding comparison principle.

Lemma 6. (a) *If $\|q_{1\ell}(\sigma) - u_B(\alpha_\ell)\|_{C^{2+\alpha}(0,2\pi)} < \varepsilon$, $\|q_{2\ell}(\sigma) - u_B(\beta_\ell)\|_{C^{2+\alpha}(0,2\pi)} < \varepsilon (1 \leqq \ell \leqq k-1)$ hold for sufficiently small $\varepsilon > 0$, then $(\mathrm{NL})^{(e_\ell)}\{q_{1\ell}(\sigma), q_{2\ell}(\sigma)\}$ has a unique solution. Moreover,*

$$\|u(p,\sigma) - u_B(p)\|_{C^{2+\alpha}(\Omega_{e_\ell})} \to 0 \quad \textit{when} \quad \varepsilon \to 0. \tag{3.3.10}$$

(b) *If $\|q_{1k}(\sigma) - u_B(\alpha_2)\|_{C^{2+\alpha}(0,2\pi)} < \varepsilon$ holds for sufficiently small $\varepsilon > 0$, then $(\mathrm{NL})^{(e_k)}\{q_{1k}(\sigma)\}$ has a unique solution. Moreover,*

$$\|u(p,\sigma) - u_B(p)\|_{C^{2+\alpha}(\Omega_{e_k})} \to 0 \quad \textit{when} \quad \varepsilon \to 0. \tag{3.3.11}$$

Proof. The proof for cases (a) and (b) are similar, so we will only prove case (b). For the problem $(\mathrm{NL})^{(e_k)}$ the three equations in (3.1.14) can be regarded as a map from $C^{2+\alpha}(\Omega_{e_k})$ to $C^\alpha(\Omega_{e_k}) \times C^{2+\alpha}(0,2\pi) \times C^{1+\alpha}(0,2\pi)$. It has been shown that for $(0, u_B(\alpha_k), 0) \in C^\alpha \times C^{2+\alpha} \times C^{1+\alpha}$, the nonlinear problem has a solution $u_B(p)$, which is the inverse of $\psi_B(r)$. Besides, the estimate (3.3.6) holds for the linearized problem. According to the implicit function theorem, there is an $\varepsilon > 0$ such that $(\mathrm{NL})^{(e_k)}$ has a unique solution, provided $\|q_{1k}(\sigma) - u_B(\alpha_2)\|_{C^{2+\alpha}} < \varepsilon$. Finally, (3.3.11) also follows from the implicit function theorem.

Lemma 7. (a) *If* $\|q_{1\ell}^{(j)}(\sigma) - u_B(\alpha_1)\|_{C^{2+\alpha}} < \varepsilon$, $\|q_{2\ell}^{(j)}(\sigma) - u_B(\beta_1)\|_{C^{2+\alpha}} < \varepsilon$ *and* $u_\ell^{(j)}(p,\sigma) = (\mathrm{NL})^{(e_\ell)}\{q_{1\ell}^{(j)}(\sigma), q_{2\ell}^{(j)}(\sigma)\}$ $(1 \leqq \ell \leqq k-1,\ j = 1,2)$ *hold for sufficiently small* $\varepsilon > 0$, *then*

$$q_{1\ell}^{(2)}(\sigma) \geqq q_{1\ell}^{(1)}(\sigma), q_{2\ell}^{(2)}(\sigma) \geqq q_{2\ell}^{(1)}(\sigma) \Longrightarrow u_\ell^{(2)}(p,\sigma) \geqq u_\ell^{(1)}(p,\sigma). \tag{3.3.12}$$

Besides,

$$\|u_\ell^{(j)}(p,\sigma) - u_B(p)\|_{C^{2+\alpha}(\Omega_{e_\ell}^-)} \leqq C(\|q_{1\ell}^{(j)}(\sigma) - u_B(\alpha_\ell)\|_{C^0(0,2\pi)} + \|q_{2\ell}^{(j)}(\sigma) - u_B(\beta_\ell)\|_{C^0(0,2\pi)}). \tag{3.3.13}$$

(b) *If* $\|q_{1k}^{(j)}(\sigma) - u_B(\alpha_k)\|_{C^{2+\alpha}} < \varepsilon$ *holds for sufficiently small* $\varepsilon > 0$, $u_k^{(j)}(p,\sigma) = (\mathrm{NL})^{(e_k)}\{q_{1k}^{(j)}\}$, *then*

$$q_{1k}^{(2)}(\sigma) \geqq q_{1k}^{(1)}(\sigma) \Longrightarrow u_k^{(2)}(p,\sigma) \geqq u_k^{(1)}(p,\sigma). \tag{3.3.14}$$

Moreover,

$$\|u_k^{(j)}(p,\sigma) - u_B(p)\|_{C^{2+\alpha}(\Omega_{e_k}^-)} \leqq C\|q_{1k}^{(j)}(\sigma) - u_B(\alpha_k)\|_{C^0(0,2\pi)}. \tag{3.3.15}$$

Proof. We only prove the conclusion (b). Take $\dot q_{1\ell} = q_{1\ell}^{(2)} - q_{1\ell}^{(1)}$, $\dot q_{2\ell} = q_{2\ell}^{(2)} - q_{2\ell}^{(1)}$, $\dot u_\ell = u_\ell^{(2)} - u_\ell^{(1)}$, $(1 \leqq \ell \leqq k)$, then $\dot u_\ell$ satisfies

$$\begin{aligned}&(N_1^2 - (1+u^2)a^2)\dot u_{pp} - \frac{a^2u_p^2}{u^2}\dot u_{\sigma\sigma} + D_0\dot u_{p\sigma}\\ &+\left(\frac{2a^2u_p}{u} - 2u_{pp}N_1\frac{1+u^2}{u_p^2} + N_2\left(\frac{1+u^2}{u_p^3} - \frac{up}{u_p^2}\right) + D_1\right)\dot u_p + D_2\dot u_\sigma\\ &+\left(-\frac{a^2u_p^2}{u^2} + \frac{2N_1u_{pp}(2u - pu_p)}{u_p} - 2a^2uu_{pp} + N_2\left(\frac{p}{u_p} - \frac{u}{u_p^2}\right) + D_3\right)\dot u = 0,\end{aligned} \tag{3.3.16}$$

where N_1, N_2 have the same expression as in (3.3.1) and $u, \dot u$ stand for $u_\ell^{(1)}, \dot u_\ell$ to simplify the notation. According to the assumptions of the lemma, $\dot u_\ell$ is non-negative on the boundary $p = \alpha_1, \beta_1$, and $\dot u_k$ is non-negative on $p = \alpha_k$. Moreover $\dot u_k$ satisfies

$$\begin{aligned}&\left(\left(1 - \frac{\rho_\infty}{\rho}\right)pu + \frac{1+u^2 - upu_p}{a^2}\left(-\frac{1}{u_p^3} + (p - \frac{u}{u_p})\frac{u}{u_p^2}\right) + E_1\right)\dot u_p + E_2\dot u_\sigma\\ &+\left(-2u + \left(1 - \frac{\rho_\infty}{\rho}\right)pu_p - \frac{1+u^2 - upu_p}{a^2}\left(p - \frac{u}{u_p}\right)\frac{1}{u_p} + E_3\right)\dot u = 0\end{aligned} \tag{3.3.17}$$

on the boundary $p = \psi_0$. Since in (3.3.16), (3.3.17) all D_j, E_j are small quantities of order $O(\varepsilon)$, then the coefficients in (3.3.16), (3.3.17) are perturbations of the corresponding coefficients in (3.3.1), (3.3.2). Using Lemma 3.5 we obtain (3.3.14) and (3.3.15).

3.4. Monotone alternating approximation

Based on the discussion of $(\mathrm{NL})^{(i)}$ and $(\mathrm{NL})^{(e_\ell)}$, we are able to construct the solution of (NL) or $(\mathrm{NL})^*$ now. Using the value of the background solution on the boundary $r = r_1$ as data, we first solve $(\mathrm{NL})^{(i)}$in Ω_i. Then by alternatively solving $(\mathrm{NL})^{(e_\ell)}$ in Ω_{e_ℓ} and $(\mathrm{NL})^{(i)}$ in Ω_i, we will establish sequences $\{\psi^{(n)}\}$ and $\{u_\ell^{(n)}\}$ of solutions for $(\mathrm{NL})^{(i)}$ and $(\mathrm{NL})^{(e_\ell)}$. It will be shown that these sequences are convergent, the limits of $u_\ell^{(n)}$ and $u_{\ell+1}^{(n)}$ coincide in $\Omega_\ell \cap \Omega_{\ell+1}$ for $1 \leqq \ell \leqq k-1$, and the limits of $u_1^{(n)}$ and $\psi^{(n)}$ are the inverse of each other in their overlapped domain. Hence we are led to the solution of (NL).

First let us describe the method of choosing $r_1, \alpha_\ell, \beta_\ell (0 \leqq \ell \leqq k)$ precisely. Take $r_1 > \tilde{b}_0 = \max b(\theta)$, so that $|r_1 - b(\theta)| < \delta$ and the problem $(\mathrm{NL})^{(i)}$ is uniquely solvable in the domain $\Omega_i : b(\theta) \leqq r \leqq r_1$. According to Lemmas 2 to 4 the solution ψ of $(\mathrm{NL})^{(i)}(b(\theta), \psi_B(r_1))$ satisfies $\psi \leqq \psi_B$ and the value of ψ in Ω_i depends monotonically on the boundary value on $r = r_1$. Applying Lemma 6 and Lemma 7, we can find $\{\alpha_\ell\}$, $\{\beta_\ell\}$ $(1 \leqq \ell \leqq k)$ satisfying (3.1.11), so that the problem $(\mathrm{NL})^{(e_\ell)}$ is uniquely solvable in the domain Ω_{e_ℓ} and satisfies the comparison principle with respect to the data on boundaries.

Next, to simplify the notation, we assume $k = 2$ in the following discussion without loss of generality. Besides, in what follows, r as a function of p, θ is denoted by $\psi^{-1}(p,\theta)$, provided $p = \psi(r,\theta)$, while p as a function of r, σ is denoted by $u^{-1}(r,\sigma)$, provided $r = u(p,\sigma)$.

The approximate sequences will be established as follows. Setting $\psi^{(0)} = \psi_B$, $u_1^{(0)} = u_{1*}^{(0)} = u_2^{(0)} = u_B$, we choose $\psi^{(1)}(r,\theta)$ to be the solution of the problem $(\mathrm{NL})^{(i)}\{b(\theta), \psi_B(r_1)\}$. For $n \geqq 1$ we take

$$
\begin{aligned}
u_1^{(n)}(p,\sigma) &= (\mathrm{NL})^{(e_1)}\{(\psi^{(n)})^{-1}(\alpha_1), (u_{1*}^{(n-1)})^{-1}(\beta_1)\},\\
u_2^{(n)}(p,\sigma) &= (\mathrm{NL})^{(e_2)}\{u_1^{(n)}(\alpha_2)\},\\
u_{1*}^{(n)}(p,\sigma) &= (\mathrm{NL})^{(e_1)}\{u_1^{(n)}(\alpha_1), u_2^{(n)}(\beta_1)\},\\
\psi^{(n+1)}(r,\theta) &= (\mathrm{NL})^{(i)}\{b(\theta), (u_{1*}^{(n)})^{-1}(r_1,\theta)\}
\end{aligned}
$$

inductively.

Lemma 8. *If $\|b(\theta) - b_0\|_{C^{2+\alpha}} < \varepsilon$ with ε being sufficiently small, then the sequences $\{\psi^{(n)}\}, \{u_1^{(n)}\}, \{u_2^{(n)}\}, \{u_{1*}^{(n)}\}$ are well defined. The first sequence is monotone, decreasing with respect to n, and other three sequences are monotone, increasing with respect to n. Furthermore, we have*

$$
\begin{aligned}
\tilde{\psi}_B(r) \leqq \psi^{(n)}(r,\theta) &\leqq \psi_B(r) \quad \text{in} \quad \Omega_i^-,\\
\|\psi^{(n)}(r,\theta) - \psi_B(r)\|_{C^{2+\alpha}(\Omega_i^-)} &\leqq C\varepsilon,
\end{aligned}
\tag{3.4.1}
$$

$$\begin{gathered}\tilde{u}_B(p) \geqq u_1^{(n)}(p,\sigma), u_{1*}^{(n)}(p,\sigma) \geqq u_B(p) \quad \text{in} \quad \Omega_{e_1}^-,\\ \tilde{u}_B(p) \geqq u_2^{(n)}(p,\sigma) \geqq u_B(p) \quad \text{in} \quad \Omega_{e_2}^-,\\ \|u_1^{(n)}(p,\sigma) - u_B(p)\|_{C^{2+\alpha}(\Omega_{e_1}^-)} \leqq C\varepsilon,\\ \|u_{1*}^{(n)}(p,\sigma) - u_B(p)\|_{C^{2+\alpha}(\Omega_{e_1}^-)} \leqq C\varepsilon,\\ \|u_2^{(n)}(p,\sigma) - u_B(p)\|_{C^{2+\alpha}(\Omega_{e_2}^-)} \leqq C\varepsilon,\end{gathered} \tag{3.4.2}$$

where all constants are independent of n.

Proof. Lemma 4 indicates that the solution $\psi^{(1)}(r,\theta)$ of the nonlinear problem $(\mathrm{NL})^{(i)}\{b(\theta), \psi_B(r_1)\}$ exists and satisfies $\tilde{\psi}_B \leqq \psi^{(1)} \leqq \psi_B$, where $\tilde{\psi}_B = (\mathrm{NL})\{\tilde{b}_0\}$. In view of $|b_0 - \tilde{b}_0| < \varepsilon$, we have $\|\tilde{\psi}_B - \psi_B\|_{C^0} < C_3\varepsilon$, and then

$$\|\psi^{(1)} - \psi_B\|_{C^0(\Omega_i)} \leqq C_3\varepsilon, \tag{3.4.3}$$

$$\|\psi^{(1)} - \psi_B\|_{C^{2+\alpha}(\Omega_i^-)} \leqq C_4\varepsilon. \tag{3.4.4}$$

Because $\psi^{(1)}$ is a small perturbation of ψ_B, then $\psi_r^{(1)} \geqq \zeta > 0$, where ζ is a constant independent of n. Hence $(\psi^{(1)})^{-1}$ is well defined and $(\psi^{(1)})^{-1} \geqq (\psi_B)^{-1} = u_B$. Moreover, from (3.4.3) and $\psi_r^{(1)} > \zeta > 0$, we have

$$\|(\psi^{(1)})^{-1}(\alpha_1,\sigma) - u_B(\alpha_1)\|_{C^0(0,2\pi)} < \frac{C_3\varepsilon}{\zeta}. \tag{3.4.5}$$

Taking $(\psi^{(1)})^{-1}(\alpha_1,\sigma)$ and $u_B(\beta_1)$ as the data for the nonlinear problem $(\mathrm{NL})^{(e_1)}$ on the boundary $p = \alpha_1$ and $p = \beta_1$ respectively, we obtain the solution $u_1^{(1)}(p,\sigma)$ of $(\mathrm{NL})^{(e_1)}$ by Lemma 6, while Lemma 7 implies the following estimates:

$$\begin{aligned}\|u_1^{(1)} - u_B\|_{C^{2+\alpha}(\Omega_{e_1}^-)} &\leqq C_2\|(\psi^{(1)})^{-1}(\alpha_1,\sigma) - u_B(\alpha_1)\|_{C^0(0,2\pi)}\\ &\leqq \frac{C_2C_3\varepsilon}{\zeta}.\end{aligned} \tag{3.4.6}$$

$$\tilde{u}_B(\alpha_1) \geqq u_1^{(1)}(\alpha_1,\sigma) \geqq u_B(\alpha_1) \quad \text{in} \quad \Omega_{e_1}. \tag{3.4.7}$$

Hence

$$\tilde{u}_B(p) \geqq u_1^{(1)}(p,\sigma) \geqq u_B(p) \tag{3.4.8}$$

holds on $p = \alpha_2$.

Using Lemmas 6 and 7 we can solve the problem $(\mathrm{NL})^{(e_2)}\{u_1^{(1)}(\alpha_2)\}$. Moreover,

$$\begin{gathered}\tilde{u}_B(p) \geqq u_1^{(1)}(p,\sigma) \geqq u_B(p) \quad \text{in} \quad \Omega_{e_2}^-,\\ \|u_2^{(1)}(p,\sigma) - u_B(p)\|_{C^{2+\alpha}(\Omega_{e_2}^-)} \leqq C\varepsilon\end{gathered} \tag{3.4.9}$$

hold.

Returning to the domain Ω_{e_1}, we solve the Dirichlet problem $(\mathrm{NL})^{(e_1)}$ once more, and obtain

$$u_{1*}^{(1)}(p,\sigma) = (\mathrm{NL})^{(e_1)}\{u_1^{(1)}(\alpha_1,\sigma), u_2^{(1)}(\beta_1,\sigma)\},$$

which also satisfies (3.4.6) and (3.4.7). From $(u_{1*}^{(1)})_p > 0$, we know $(u_{1*}^{(1)})^{-1} \leqq \psi_B$ on $r = r_1$ and

$$\|(u_{1*}^{(1)})^{-1}(r_1,\theta) - \psi_B(r_1)\|_{C^0(0,2\pi)} \leqq |\tilde{\psi}_B(r_1) - \psi_B(r_1)| < \varepsilon. \tag{3.4.10}$$

In addition, we can solve the problem $(\mathrm{NL})^{(i)}\{b(\theta), (u_{1*}^{(1)})^{-1}(r_1,\theta)\}$ in the interior annulus $\Omega^{(i)}$ by Lemma 2. Its solution $\psi^{(2)}(r,\theta)$ satisfies $\psi^{(2)} \leqq \psi^{(1)}$ according to Lemma 3 and satisfies $\psi^{(2)} \geqq \tilde{\psi}_B$ according to Lemma 4. Therefore, we have

$$\|\psi^{(2)}(r_2,\theta) - \psi_B(r_2)\|_{C^0(0,2\pi)} < \varepsilon, \tag{3.4.11}$$

$$\begin{aligned}\|\psi^{(2)}(r,\theta) - \psi_B(r)\|_{C^{2+\alpha}(\Omega_i^-)} &\leqq C_2\|(u^{(1)})^{-1}(r_1,\theta) - \psi_B(r_1)\|_{C^0(0,2\pi)} \\ &\leqq C_2\varepsilon. \end{aligned}\tag{3.4.12}$$

By the same procedure, we obtain $u_1^{(2)}(p,\sigma), u_2^{(2)}(p,\sigma), u_{1*}^{(2)}(p,\sigma), \psi^{(3)}(r,\theta)$ and so on. Then (3.4.1) and (3.4.2) can be proved by induction.

To prove the monotonicity of the sequences $\{\psi^{(n)}\}$ and $\{u^{(n)}\}$, we are going to verify the following inequalities by induction

$$\begin{aligned}\psi^{(n)}(r,\theta) &\leqq \psi^{(n-1)}(r,\theta),\\ u_1^{(n)}(p,\sigma) &\geqq u_1^{(n-1)}(p,\sigma),\\ u_2^{(n)}(p,\sigma) &\geqq u_2^{(n-1)}(p,\sigma),\\ u_{1*}^{(n)}(p,\sigma) &\geqq u_{1*}^{(n-1)}(p,\sigma).\end{aligned}\tag{3.4.13}$$

According to the process of establishing these sequences, we have $\psi^{(1)}(r,\theta) \leqq \psi_B(r)$ from Lemma 4. Furthermore, $u_1^{(1)}(p,\sigma) \geqq u_B(p)$, $u_{1*}^{(1)}(p,\sigma) \geqq u_B(p)$, $u_2^{(1)}(p,\sigma) \geqq u_B(p)$ follow from Lemma 7. Hence (3.4.13) holds for $n = 1$. Now assume that (3.4.13) with index n is valid. Then Lemma 4 implies

$$\psi^{(n+1)}(r,\theta) \leqq \psi^{(n)}(r,\theta), \tag{3.4.14}$$

because their boundary value on $r = r_1$ satisfies this inequality by the assumption of induction. Equation (3.4.14) implies $u_1^{(n+1)}(\alpha_1,\sigma) \geqq u_1^{(n)}(\alpha_1,\sigma)$. Then, combining this with the boundary condition $u_{1*}^{(n+1)}(\beta_1,\sigma) \geqq u_{1*}^{(n)}(\beta_1,\sigma)$, we have

$$u_1^{(n+1)}(p,\sigma) \geqq u_1^{(n)}(p,\sigma) \tag{3.4.15}$$

according to part (a) of Lemma 7. Finally, by using (3.4.15) on the line $p = \alpha_2$ and the boundary condition on $p = \beta_2 (= \psi_0)$, the inequality

$$u_2^{(n+1)}(p,\sigma) \geqq u_2^{(n)}(p,\sigma) \tag{3.4.16}$$

is also valid according to part (b) of Lemma 7. Furthermore, (3.4.16) on the line $p = \beta_1$ and (3.4.15) on the line $p = \alpha_1$ lead us to

$$u_{1*}^{(n+1)}(p, \sigma) \geqq u_{1*}^{(n)}(p, \sigma). \tag{3.4.17}$$

Now (3.4.17) yields

$$(u_{1*}^{(n+1)})^{-1}(r_1, \theta) \leqq (u_{1*}^{(n)})^{-1}(r_1, \theta). \tag{3.4.18}$$

Therefore, we come back to (3.4.14) with index $n + 1$:

$$\psi^{(n+2)}(r, \theta) \leqq \psi^{(n+1)}(r, \theta).$$

Hence the monotonicity as shown in (3.4.13) is proved by induction.

Finally, we prove the following lemma, which leads to the conclusion of Theorem 2 directly.

Lemma 9. *If* $\|b(\theta) - b_0\|_{C^{2+\alpha}} \leqq \varepsilon$ *with* ε *being sufficiently small, then the problems* (NL) *and* (NL)* *are solvable.*

Proof. As we proved in the above lemma, the sequences $\{u_1^{(n)}(p, \sigma)\}$, $\{u_2^{(n)}(p, \sigma)\}$, $\{u_{1*}^{(n)}(p, \sigma)\}$ and $\{\psi^{(n)}(r, \theta)\}$ are bounded and monotone with respect to n, so these sequences are convergent. Let us denote their limits by $u_1(p, \sigma)$, $u_2(p, \sigma)$, $u_{1*}(p, \sigma)$, $\psi(r, \theta)$ respectively. Notice that the $C^{2+\alpha}$ norm of $\psi^{(n)}$ on $r = u_B(\alpha_1)$ is dominated by its C^0 norm on $r = r_1$, the $C^{2+\alpha}$ norm of $u_1^{(n)}$, $u_{1*}^{(n)}$ in $\alpha_1 + \frac{1}{10}\delta \leqq p \leqq \beta_1 - \frac{1}{10}\delta$ is dominated by their C^0 norm on $p = \alpha_1$, $p = \beta_1$, and the $C^{2+\alpha}$ norm of $u_2^{(n)}$ in $\alpha_2 + \frac{1}{10}\delta \leqq p \leqq \psi_0$ is dominated by its C^0 norm on $p = \alpha_2$, so the $C^{2+\alpha}$ norms of $\psi^{(n)}$, $u_1^{(n)}$, $u_{1*}^{(n)}$, $u_2^{(n)}$ are uniformly bounded. This fact implies that ψ, u_1, u_{1*}, u_2 are $C^{2+\alpha}$ functions in their respective domains.

On the other hand, ψ satisfy (3.1.1), (3.1.2) and $\psi(r_1, \theta) = (u_1)^{-1}(r_1, \theta)$; u_1, u_{1*} satisfy (3.1.7) and $u_1 = u_{1*}$ on the boundary $p = \alpha_1$ and $p = \beta_1$; u_2 satisfies (3.1.7), (3.1.8) and $u_1(\alpha_2, \sigma) = u_2(\alpha_2, \sigma)$. Notice that (3.1.7) is the equation for the function whose inverse satisfies (3.1.1), and vice versa. Hence both $\psi(r, \theta)$ and $(u_1)^{-1}(r, \theta)$ satisfy (3.1.1) in the overlapped domain $\Omega_i \cap T^{-1}(\Omega_{e_1})$. Besides, these two functions coincide on the boundaries $r = r_1$ and $r = u(\alpha_1, \theta)$ (equivalently $p = \psi(r_1, \sigma)$ and $p = \alpha_1$). Since the domain Ω_i is chosen so small, the linearized operator $L^{(i)}$ does not have a non-negative eigenvalue in Ω_i. Therefore, there is no non-negative eigenvalue in its subdomain $\Omega_i \cap T^{-1}(\Omega_{e_1})$ either. By the uniqueness of the Dirichlet problem for the nonlinear elliptic equation (3.1.1), the functions $\psi(r, \theta)$ and $u_1^{-1}(r, \theta)$ coincide on the whole domain $\Omega_i \cap T^{(-1)}(\Omega_{e_1})$. Moreover, $u_1(p, \sigma)$, $u_{1*}(p, \sigma)$, $u_2(p, \sigma)$ coincide on the corresponding overlapped domain. Therefore, viewing functions $u_1^{-1}(r, \theta)$, $u_2^{-1}(r, \theta)$ as extensions of $\psi(r, \theta)$, we obtain the solution of (NL) in the whole domain $b(\theta) < r < \psi^{-1}(\psi_0, \theta)$. Correspondingly, the problem (NL)* is also solved simultaneously.

It is obvious that the solution of (NL) or (NL)* satisfies the entropy condition on the shock front, because the solution is a small perturbation of the background solution, which already satisfies the entropy condition on the corresponding shock front.

4. Approximation on level two

4.1. Generalized hodograph transformation

We have obtained the approximate solution of level one in Section 3. Combining this with the solutions $\phi_1(r,\theta),\cdots,\phi_N(r,\theta)$ of the problem (2.1.5), (2.1.8), (2.1.17), we obtain $\tilde{\phi}(z,r,\theta)$ – the approximate solution satisfying the boundary value problem with error $O(z^N)$. Starting from this approximate solution we will use the Newton iterative procedure to modify it and finally obtain the precise solution of our problem. Let us simply call $\tilde{\phi}$ the approximate solution of level two, and call the iterative process starting from $\tilde{\phi}$ an approximation on level two. Writing $\phi(z,r,\theta)$ as $z\chi(z,r,\theta)$, we consider the problem for χ instead. To further simplify the notation and remove the formal degeneracy at $z=0$, we introduce a transformation

$$t=\ln z, \qquad z=e^t, \tag{4.1.1}$$

which maps the point $z=0$ to $t=-\infty$. By using coordinates (t,θ,r) the problem (1.2.15), (1.2.17), (1.2.18) for the new function χ becomes

$$\begin{aligned} L(\chi) \equiv\ & a_{00}(\chi_t+\chi_{tt})+a_{11}\chi_{rr}+a_{22}\chi_{\theta\theta}+2a_{01}\chi_{tr}+2a_{02}\chi_{t\theta}+2a_{12}\chi_{r\theta}\\ &+(a_1+2a_{01})\chi_r+(a_2+2a_{02})\chi_\theta=0, \end{aligned}\tag{4.1.2}$$

$$E(\chi)\equiv(b+b_t)(\chi+\chi_t)+\frac{b_\theta}{r^2}\chi_\theta-(1+r(b+b_t))\,\chi_r=0 \tag{4.1.3}$$

$$\begin{aligned} G(\chi) \equiv\ & \Big(\chi_r^2+r^{-2}\chi_\theta^2+(\chi+\chi_t-r\chi_r)(\chi+\chi_t-r\chi_r-q_\infty)\Big)\,H\\ &-(\chi+\chi_t-r\chi_r-q_\infty)\,q_\infty\rho_\infty\\ =\ & 0,\\ \chi =\ & q_\infty, \end{aligned}\tag{4.1.4}$$

where (4.1.3) is given on the fixed boundary $r=b(t,\theta)$, while (4.1.4) is given on the free boundary $r=s(t,\theta)$.

The rest of this paper is devoted to proving the existence of a solution for (4.1.2)–(4.1.4) near $t=-\infty$, which will obviously imply Theorem 2. Since the function (2.1.1) $\tilde{\phi}(z,r,\theta)$ satisfies (1.2.13), (1.2.15), (1.2.16) with error $O(z^N)$, then the corresponding $\tilde{\chi}(t,r,\theta)$ satisfies (4.1.2)–(4.1.4) with error $O(e^{-N|t|})$.

Since the shock front is a free boundary, we will also use a transformation like the partial hodograph transformation in Section 3 to fix the boundary. In view of the hyperbolicity of (4.1.2) in the domain, we prefer to make a modification to the partial hodograph transformation so that both the boundary $r=b(t,\theta)$ and $r=s(t,\theta)$ are transformed into fixed boundaries. The transformation is

$$y_0=t,\quad y_1=\theta,\quad y_2=\frac{r-b}{r-b+q_\infty-\chi}, \tag{4.1.5}$$

which transforms the boundary $r=b$ and $r=s$ into $y_2=0$ and $y_2=1$ respectively.

We notice that here $q_\infty-\chi>0$ away from the shock front, and $r-b>0$ away from the surface of the pointed body. Therefore, we always have $d(t,\theta,r)=$

$r - b(t, \theta) + u_\infty - \chi(t, \theta, r) > 0$. Besides, in view of $\chi_r = v_r$, which is positive for the background solution, we also have

$$\partial_r y_2 = d^{-2}(\chi_r(r-b) + (q_\infty - \chi)) > 0 \tag{4.1.6}$$

in the whole domain $b(t, \theta) < r < s(t, \theta)$; so the transformation (4.1.5) is uniformly nonsingular.

The differential relations of first order for (4.1.5) are

$$\begin{aligned} &dy_0 = dt, \qquad dy_1 = d\theta, \\ &dy_2 = \frac{-db_t - d_t(r-b)}{d^2}dt + \frac{-db_\theta - d_\theta(r-b)}{d^2}d\theta + \frac{d - d_r(r-b)}{d^2}dr, \end{aligned} \tag{4.1.7}$$

where $d_t = -b_t - \chi_t$, $d_\theta = -b_\theta - \chi_\theta$, $d_r = 1 - \chi_r$. Consequently,

$$dr = \frac{db_t + d_t(r-b)}{d_1}dy_0 + \frac{db_\theta + d_\theta(r-b)}{d_1}dy_1 + \frac{d^2}{d_1}dy_2, \tag{4.1.8}$$

where $d_1 = d - d_r(r-b) = \chi_r(r-b) + q_\infty - \chi > 0$. Hence we have

$$r_{y_0} = \frac{db_t + d_t(r-b)}{d_1}, \quad r_{y_1} = \frac{db_\theta + d_\theta(r-b)}{d_1}, \quad r_{y_2} = \frac{d^2}{d_1}.$$

To derive a terse form of nonlinear problem (4.1.2)–(4.1.4) in coordinates (y_0, y_1, y_2), we choose

$$\omega(y_0, y_1, y_2) = d(y_0, y_1, r(y_0, y_1, y_2))$$

as the unknown function, where $r(y_0, y_1, y_2)$ is the inverse function determined by the last equality of (4.1.5). We notice that the choice of the independent variables and the new unknown function is equivalent to the combination of the following transformations:

(1) Transformation of unknown function

$$H(t, \theta, r) = \frac{r-b}{r - b + q_\infty - \chi(t, r, \theta)}, \tag{4.1.9}$$

(2) Partial hodograph transformation

$$y_0 = t, \; y_1 = \theta, \; y_2 = H(t, r, \theta), \tag{4.1.10}$$

(3) Transformation of unknown function

$$\omega(y_0, y_1, y_2) = (r - b(y_0, y_1))/y_2. \tag{4.1.11}$$

Such a combination is called a generalized hodograph transformation.

By the chain rule $\omega_{y_2} = d_r r_{y_2} = d^2 d_1{}^{-1} d_r$. Meanwhile, from $d_t = \omega_{y_0} + \omega_{y_2} y_{2t}$, $d_\theta = \omega_{y_1} + \omega_{y_2} y_{2\theta}$, we have

$$\omega_{y_0} = d_t + \frac{db_t + d_t(r-b)}{d_1} d_r, \quad \omega_{y_1} = d_\theta + \frac{db_\theta + d_\theta(r-b)}{d_1} d_r.$$

Therefore, denoting

$$(t, \theta, r) \quad \text{by } (\xi_0, \xi_1, \xi_2),$$

$$\frac{\partial(\xi_0, \xi_1, \xi_2)}{\partial(y_0, y_1, y_2)}) \quad \text{by } J, \quad \text{and} \quad \frac{\partial(\omega_{y_0}, \omega_{y_1}, \omega_{y_2})}{\partial(d_{\xi_0}, d_{\xi_1}, d_{\xi_2})} \quad \text{by } \tilde{J},$$

we have

$$\tilde{J} = \begin{pmatrix} \dfrac{d}{d_1} & 0 & \dfrac{(db_t + d_t(r-b))d}{d_1^2} \\ 0 & \dfrac{d}{d_1} & \dfrac{(db_\theta + d_\theta(r-b))d}{d_1^2} \\ 0 & 0 & \dfrac{d^3}{d_1^2} \end{pmatrix} = \frac{d}{d_1} J^T. \tag{4.1.12}$$

In view of the expression of d, the second derivatives χ_{ij} equal $-d_{ij}$, adding a given function involving the second derivatives of $b(t, \theta)$. Therefore, the principle part of (4.1.2) is

$$\begin{aligned} \sum a_{ij} \frac{\partial^2 \chi}{\partial \xi_i \partial \xi_j} &= \sum a_{ij} \frac{\partial^2 d}{\partial \xi_i \partial \xi_j} + \text{lower order} \\ &= \sum a_{ij} \frac{\partial d_i}{\partial \omega_\ell} \frac{\partial^2 \omega}{\partial y_\ell \partial y_k} \frac{\partial y_k}{\partial \xi_j} + \text{lower order} \\ &= \sum \alpha_{\ell k} \frac{\partial^2 \omega}{\partial y_\ell \partial y_k} + \text{lower order}, \end{aligned}$$

where

$$\alpha_{\ell k} = \sum \frac{\partial d_i}{\partial \omega_\ell} (a_{\ell k}) \frac{\partial y_k}{\partial \xi_j}.$$

Therefore,

$$(a_{ij}) = dd_1{}^{-1} J^T (\alpha_{ij}) J, \tag{4.1.13}$$

and ω satisfies a quasilinear equation

$$\mathbf{P}\omega \equiv \sum_{i,j=0,1,2} \alpha_{ij}(\omega, \nabla\omega) \omega_{y_i y_j} + R(\omega, \nabla\omega) = 0 \tag{4.1.14}$$

in the domain $(-\infty, -T) \times [0, 2\pi] \times (0, 1)$.

Turn to the boundary conditions. On the boundary $y_2 = 0$ corresponding to $r = b$, we have $d_1 = d$ and

$$\begin{pmatrix} \omega_{y_0} \\ \omega_{y_1} \\ \omega_{y_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & b_t \\ 0 & 1 & b_\theta \\ 0 & 0 & d \end{pmatrix} \begin{pmatrix} d_t \\ d_\theta \\ d_r \end{pmatrix} = \tilde{J} \begin{pmatrix} d_t \\ d_\theta \\ d_r \end{pmatrix}.$$

Hence the boundary condition for ω can be written as

$$\mathbf{Q_b}\omega \equiv \tilde{M}\nabla_y\omega + R_b(\omega) = 0 \quad \text{on} \quad y_2 = 0, \tag{4.1.15}$$

where $\tilde{M} = M(J^T)^{-1}$ and M is the vector formed by the coefficients of $\nabla_\xi \chi$ in (4.1.3).

Finally, according to the relation among $\nabla_y\omega, \nabla_\xi d$ and $\nabla_\xi \chi$ the condition (4.1.4) is transformed to

$$\mathbf{Q_s}\omega \equiv \tilde{G}(\omega, \omega_{y_0}, \omega_{y_1}, \omega_{y_2}) = 0 \quad \text{on} \quad y_2 = 1. \tag{4.1.16}$$

We notice here that the relation

$$\nabla_q \tilde{G} = \nabla_p G \tilde{J}^{-1} \tag{4.1.17}$$

holds, where q, p stand for the argument $\nabla_y\omega, \nabla_\xi \chi$ in the nonlinear functions $\tilde{G}, G_1$ respectively.

When χ is replaced by the approximate solution $\tilde{\chi}$ of level two, the right-hand side of (4.1.2) should be replaced by $O(e^{Nt})$. Therefore, the corresponding approximate solution $\tilde{\omega}(y_0, y_1, y_2)$ obtained from $\tilde{\chi}(t, \theta, r)$ by the above generalized hodograph transformation satisfies (4.1.14)–(4.1.16) with error $O(e^{Ny_0})$, because $\chi(t, \theta, r)$ satisfies (4.1.2)–(4.1.4) with error $O(e^{Nt})$. Obviously, our main theorem will be proved if we obtain the existence and uniqueness of the precise solution to the boundary value problem (4.1.14)–(4.1.16). In the remaining part of this paper we will use Newton's iteration to prove the conclusion:

There exists $T \gg 0$, such that (4.1.14)–(4.1.16) has a unique classical solution $\omega(y_0, y_1, y_2)$ in $(-\infty, -T)$.

Remark 6. The above-mentioned generalized hodograph transformation retains the property of normal hyperbolicity of the equation with respect to an assigned direction. This means, the new form of the equation in (y_0, y_1, y_2) coordinates is still normal hyperbolic with respect to the image of the assigned direction under the transformation. Besides, for the nonlinear boundary conditions we can define linearized boundary operators and corresponding multidimensional linear stability conditions (see [18]), so that the property of satisfying the multidimensional linear stability condition also remains valid under the nonlinear hodograph transformation.

The above conclusion for the usual partial hodograph transformation is proved in [18]. The conclusion for generalized hodograph transformation introduced in this section is still valid because our transformation is a combination of the usual partial hodograph transformation with two transformations of the unknown function, as shown in (4.1.9)–(4.1.11). On the other hand, the conclusion can also be verified directly, because up to a constant factor the change of coefficients obey the same rule as that under the normal coordinate transformation:

$$\begin{aligned} M &= \tilde{M}\tilde{J}, \\ A &= (\frac{d_1}{d})\tilde{J}^T\tilde{A}\tilde{J}, \\ \nabla_p G_1 &= \nabla_q \tilde{G}\tilde{J}. \end{aligned} \tag{4.1.18}$$

4.2. Basic energy estimate for linearized problems

In order to obtain the precise solution of (4.1.14)–(4.1.16) by successively modifying $\tilde{\omega}$, we linearize them at $\hat{\omega}$ near $\tilde{\omega}$. The linearization is

$$\begin{aligned} P(\hat{\omega})\dot{\omega} &\equiv \sum_{i,j=0}^{2} \alpha_{ij}(\hat{\omega})\partial_{y_i}\partial_{y_j}\dot{\omega} + \sum_{j=0}^{2}\alpha_j(\hat{\omega})\partial_{y_j}\dot{\omega} + \alpha_3(\hat{\omega})\dot{\omega} \\ &= f \quad \text{in } \ 0 < y_2 < 1, \end{aligned} \tag{4.2.1}$$

$$Q_b(\hat{\omega})\dot{\omega} \equiv \sum_{j=0}^{2} \beta_{1j}\partial_{y_j}\dot{\omega} + \beta\dot{\omega} = g_1 \ \text{ on } \ y_2 = 0, \tag{4.2.2}$$

$$Q_s(\hat{\omega})\dot{\omega} \equiv \sum_{j=0}^{2} \gamma_{1j}\partial_{y_j}\dot{\omega} + \gamma_0\dot{\omega} = g_2 \ \text{ on } \ y_2 = 1, \tag{4.2.3}$$

where $\beta_0, \beta_{1j}, \gamma_0, \gamma_{1j}$ are derivatives of $\tilde{B}$, R_b, $\tilde{G}$ with respect to ω, ω_{y_j} respectively. As usual we will use the η-weighted Sobolev norm to derive energy estimates in our hyperbolic problem. For smooth functions ω which vanish near $y_0 = -\infty$, we define the following norms for a non-negative integer s.

$$|\omega|^2_{s,\eta,y_0} = \sum_{\tau_0+|\tau|=s} \int_0^1\int_0^{2\pi} e^{-2\eta y_0}\eta^{2\tau_0}|\nabla_y^\tau\omega(y_0,y_1,y_2)|^2\,dy_1dy_2, \tag{4.2.4}$$

$$||\omega||^2_{s,\eta,T} = \int_{-\infty}^{T} |\omega|^2_{s,\eta,y_0}\,dy_0. \tag{4.2.5}$$

On the boundary $y_2 = 0, 1$, we define the boundary norms

$$\begin{aligned} \langle\omega\rangle^2_{s,\eta,T} &= \int_{-\infty}^{T}\int_0^{2\pi} \sum_{\tau_0+|\tau|=s,\,y_2=0,1} e^{-2\eta y_0}\eta^{2\tau_0}|\nabla^\tau_{y_0,y_1}\omega(y_0,y_1,y_2)|^2\,dy_1dy_0, \\ \langle\langle\omega\rangle\rangle^2_{s,\eta,T} &= \int_{-\infty}^{T}\int_0^{2\pi} \sum_{\tau_0+|\tau|=s,\,y_2=0,1} e^{-2\eta y_0}\eta^{2\tau_0}|\nabla^\tau_{y}\omega(y_0,y_1,y_2)|^2\,dy_1dy_0. \end{aligned} \tag{4.2.6}$$

The latter includes the estimates of normal derivatives. Here we are able to introduce the norms involving the same regularity of derivatives in the normal direction and in the tangential direction because the boundary $y_2 = 0$ and $y_2 = 1$ is not characteristic. Finally, we define

$$|||\omega|||^2_{s,\eta,T} = \sup_{-\infty<y_0<T} |\omega|_{s,\eta,y_0} + \eta||\omega||^2_{s,\eta,T} + \langle\langle\omega\rangle\rangle^2_{s,\eta,T}. \tag{4.2.7}$$

The completion of the set of all smooth functions vanishing near $y_0 = -\infty$ with respect to the norm (4.2.7) is a Sobolev space and denoted by H_s. Since the dimension of the space of independent variables is 3, then for $s > 1 + n$, H_s functions have continuous classical derivatives up to order n.

For solutions of the linear boundary value problem (4.2.1)–(4.2.3), we have the following energy estimate:

Theorem 3. *Assume that $s > 2$ is an integer, $\eta \geqq \eta_0$ is sufficiently large, $\tilde{\omega}$ is the approximate solution of level two, and $\hat{\omega} \in H_{s+1}$ satisfying $|||\hat{\omega} - \tilde{\omega}|||_{s+1,\eta,T} \leqq \delta_0$. Then the solution $\dot{\omega}$ of* (4.2.1)–(4.2.3) *satisfies the energy estimate*

$$|||\dot{\omega}|||^2_{s+1,\eta,T} \leqq C_s \left(\frac{1}{\eta} ||f||^2_{s,\eta,T} + \langle g_1 \rangle^2_{s,\eta,T,} + \langle g_2 \rangle^2_{s,\eta,T,} \right), \tag{4.2.8}$$

where the constant C_s is independent of the choice of $\hat{\omega}$.

Proof. The general conclusion for boundary value problems of hyperbolic equation of second order in given in [18]. Applying the result obtained there we only need to verify that the boundary conditions (4.2.2), (4.2.3) satisfy the linear stability condition (L.S.C.). Writing the linearized boundary operators on the boundary $y_2 = 1$ as

$$\nu = \gamma_{10} \frac{\partial}{\partial y_0} + \gamma_{11} \frac{\partial}{\partial y_1} + \gamma_{12} \frac{\partial}{\partial y_2},$$

the requirement of L.S.C. is that the vector B intersects the boundary transversally, and $\frac{\alpha_{22}}{\gamma_{12}} \nu + N$ is a timelike direction, where $N = -\sum \alpha_{0j} \partial_{y_j}$. According to the analysis in Remark 6, we only need to verify L.S.C. for the boundary conditions on a physical space with coordinates x_1, x_2, x_3. Since our problem is a perturbation of the background solution, it is enough to verify L.S.C. for the latter.

Take a point O_1 on the shock front. Due to the symmetry of the background solution we assume $x_2 = 0$ at O_1 without loss of generality. Introduce new local coordinates z_1, z_2, z_3, such that $z_2 = x_2$, $O_1 z_1$ is tangential to the shock, and $O_1 z_3$ points in the normal direction. Then $\phi_{z_2} = 0$ at O_1 due to the axisymmetry. Meanwhile, in the coordinate system $O_1 z_1 z_2 z_3$, the equation of the tangential plane of the shock is

$$\psi - q_\infty z_1 \cos \beta - q_\infty z_3 \sin \beta = 0,$$

and the Rankine-Hugeniot condition becomes

$$\begin{aligned} &(\phi_{z_1}(\phi_{z_1} - q_\infty \cos \beta) + \phi_{z_2}^2 + \phi_{z_3}(\phi_{z_3} - q_\infty \sin \beta))H \\ &\qquad = q_\infty \rho_\infty ((\phi_{z_1} - q_\infty \cos \beta) \cos \beta + (\phi_{z_3} - q_\infty \sin \beta) \sin \beta) \end{aligned} \tag{4.2.9}$$

with $H = H(C - \frac{1}{2}(\phi_{z_1}^2 + \phi_{z_2}^2 + \phi_{z_3}^2))$. Regarding ϕ_{z_3} as an argument p_3, we have

$$\begin{aligned} \frac{\partial G}{\partial p_3} &= (2\phi_{z_3} - q_\infty \sin \beta) H - q_\infty \rho_\infty \sin \beta + (\phi_{z_1}(\phi_{z_1} - q_\infty \cos \beta) \\ &\quad + \phi_{z_3}(\phi_{z_3} - q_\infty \sin \beta)) H'(-\phi_{z_3}). \end{aligned}$$

Recalling the physical meaning of all quantities, $\phi_{z_3} = v_n, q_\infty \sin \beta = v_{n\infty}$, $q_\infty \cos \beta = v_{t\infty} = \phi_{z_1}$, we have

$$\begin{aligned} \frac{\partial G}{\partial p_3} &= (2v_n - v_{n\infty})\rho - \rho_\infty v_{n\infty} + v_n(v_n - v_{n\infty}) a^{-2} \rho(-v_n) \\ &= (v_n - v_{n\infty})\rho(1 - a^{-2} v_n^2) \neq 0. \end{aligned} \tag{4.2.10}$$

Besides, the second requirement of L.S.C. becomes trivial in view of $G_{p_2} = 0$. By the same method we confirm that L.S.C. is satisfied on the boundary corresponding to the surface of the body. Then (4.2.8) is valid according to [10] and [18]. Hence the conclusion in Theorem 3 is obtained.

4.3. Iterative scheme and convergence

Using the energy estimate (4.2.7) obtained in Theorem 3, we can employ Newton's iteration to establish the existence of a precise solution for (4.1.14)–(4.1.16). Write the precise solution as

$$\omega = \tilde{\omega} + \dot{\omega}, \tag{4.3.1}$$

and the problem is reduced to determining $\dot{\omega}$. Since $\tilde{\omega}$ is an approximate solution of (4.1.14)–(4.1.16) with error $O(e^{Ny_0})$, then for any $\eta > 0$, we can take $N \geqq \eta$, so that

$$\|\mathbf{P}\tilde{\omega}\|^2_{s,\eta,T}, \quad \langle \mathbf{Q_b}\tilde{\omega}\rangle^2_{s,\eta,T}, \quad \langle \mathbf{Q_s}\tilde{\omega}\rangle^2_{s,\eta,T}$$

are bounded. Therefore, for given $\kappa > 0$, we have

$$C_s(\|\mathbf{P}\tilde{\omega}\|^2_{s,\eta,T} + \langle \mathbf{Q_b}\tilde{\omega}\rangle^2_{s,\eta,T} + \langle \mathbf{Q_s}\tilde{\omega}\rangle^2_{s,\eta,T}) < \kappa \tag{4.3.2}$$

if $T < T_0 \ll 0$. Moreover, if δ_0 is sufficiently small and $\hat{\omega} \in H_{s+1}$ satisfies

$$|||\hat{\omega} - \tilde{\omega}|||_{s+1,\eta,T} \leqq \delta_0, \tag{4.3.3}$$

then (4.3.2) with $\tilde{\omega}$ replaced by $\hat{\omega}$ is uniformly valid. Namely, T_0 is independent of the choice of $\hat{\omega}$.

After linearizing (4.1.14)–(4.1.16) we perform the iteration scheme as follows. Take $\dot{\omega}_0 = 0$, and let $\dot{\omega}_{k+1}$ be defined as the solution of the linear boundary value problem:

$$\begin{aligned} P(\tilde{\omega}+\dot{\omega}_k)\dot{\omega}_{k+1} &\equiv \sum_{i,j=0,}^{2} \alpha_{ij}(\tilde{\omega}+\dot{\omega}_k)\partial_{y_i}\partial_{y_j}\dot{\omega}_{k+1} + \sum_{j=0}^{2}\alpha_j(\tilde{\omega}+\dot{\omega}_k)\partial_{y_j}\dot{\omega}_{k+1} \\ &\quad + \alpha(\tilde{\omega}+\dot{\omega}_k)\dot{\omega}_{k+1} \\ &= f_k \ \text{ in } \ 0 < y_2 < 1, \end{aligned} \tag{4.3.4}$$

$$\begin{aligned} Q_b(\tilde{\omega}+\dot{\omega}_k)\dot{\omega}_{k+1} &\equiv \sum_{j=0}^{2} \beta^{(k)}_{1j}\partial_{y_j}\dot{\omega}_{k+1} + \beta^{(k)}_0\dot{\omega}_{k+1} \\ &= g_{1k} \ \text{ on } \ y_2 = 0, \end{aligned} \tag{4.3.5}$$

$$\begin{aligned} Q_s(\tilde{\omega}+\dot{\omega}_k)\dot{\omega}_{k+1} &\equiv \sum_{j=0}^{2} \gamma^{(k)}_{1j}\partial_{y_j}\dot{\omega}_{k+1} + \gamma^{(k)}_0\dot{\omega}_{k+1} \\ &= g_{2k} \ \text{ on } \ y_2 = 1, \end{aligned} \tag{4.3.6}$$

where $f_k = -\mathbf{P}(\tilde{\omega}+\dot{\omega}_k) + P(\tilde{\omega}+\dot{\omega}_k)\dot{\omega}_k$, $g_{2k} = -\mathbf{Q_s}(\tilde{\omega}+\dot{\omega}_k) + Q_s(\tilde{\omega}+\dot{\omega}_k)\dot{\omega}_k$, $\gamma^{(k)}_{1j} = \gamma_{1j}(\tilde{\omega}+\dot{\omega}_k)$ etc. Next we are going to prove the boundedness in the high norm and the contraction in the low norm of the sequence $\{\dot{\omega}_k\}$.

In order to prove the boundedness in the high norm, for given $\kappa < \delta_0$ we take T sufficiently negative such that

$$C_s\left(\frac{1}{\eta}\|f_0\|^2_{s,\eta,T} + \langle g_{10}\rangle^2_{s,\eta,T} + \langle g_{20}\rangle^2_{s,\eta,T}\right) < \kappa. \tag{4.3.7}$$

Then the estimate

$$|||\dot{\omega}_k|||^2_{s+1,\eta,T} \leqq \kappa \tag{4.3.8}$$

can be proved by induction. Indeed (4.3.8) for $\kappa = 0$ is trivial. Now suppose (4.3.8) is valid for index k, then (4.3.3) holds, and (4.3.2) with $\tilde{\omega}$ replaced by $\tilde{\omega} + \dot{\omega}_k$. Therefore, Theorem 3 can be applied to the problem (4.3.4)–(4.3.6). By using energy estimate (4.2.8) we obtain

$$|||\dot{\omega}_{k+1}|||^2_{s+1,\eta,T} \leqq C_s \left(\frac{1}{\eta} \| f_k \|^2_{s,\eta,T} + \langle g_{1k} \rangle^2_{s,\eta,T} + \langle g_{2k} \rangle^2_{s,\eta,T} \right) < \kappa. \tag{4.3.9}$$

Hence (4.3.8) is valid for any k by induction.

To prove the convergence of the sequence $\{\dot{\omega}_k\}$, we take

$$\Delta_k = \dot{\omega}_{k+1} - \dot{\omega}_k$$

and obtain the following boundary value problem for Δ_k:

$$P(\tilde{\omega} + \dot{\omega}_k)\Delta_k = \tilde{f}_k \quad \text{in } 0 < y_2 < 1, \tag{4.3.10}$$

$$Q_b(\tilde{\omega} + \dot{\omega}_k)\Delta_k = -\tilde{g}_{1k} \quad \text{on } y_2 = 0, \tag{4.3.11}$$

$$Q_s(\tilde{\omega} + \dot{\omega}_k)\Delta_k = -\tilde{g}_{2k} \quad \text{on } y_2 = 1, \tag{4.3.12}$$

where $\tilde{g}_{2k} = -\mathbf{Q_s}(\tilde{\omega} + \dot{\omega}_k) + \mathbf{Q_s}(\tilde{\omega} + \dot{\omega}_{k-1}) + Q_s(\tilde{\omega} + \dot{\omega}_{k-1})\Delta_{k-1}$ etc. In view of Theorem 3, Δ_k satisfies

$$\begin{aligned} |||\Delta_k|||^2_{s-1,\eta,T} &\leqq C_{s-2} \left(\frac{1}{\eta} \| \tilde{f}_k \|^2_{s-2,\eta,T} + \| \tilde{g}_{1k} \|^2_{s-2,\eta,T} + \| \tilde{g}_{2k} \|^2_{s-2,\eta,T} \right) \\ &\leqq C' C_{s-2} |||\Delta_{k-1}|||^2_{s-1,\eta,T} \, |||\dot{\omega}_k|||^2_{s,\eta,T} \\ &\leqq C' C_{s-2} \kappa^2 |||\Delta_{k-1}|||^2_{s-1,\eta,T}. \end{aligned}$$

Therefore, $\{\dot{\omega}_k\}$ is contractive in H_{s-1} if κ satisfies $C' C_{s-1} \kappa^2 < \frac{1}{4}$. Namely, the sequence $\{\dot{\omega}_k\}$ is convergent. The limit of $\tilde{\omega} + \dot{\omega}_k$ solves the problem (4.1.14)–(4.1.16).

According to the analysis in the beginning of this section we also solve the problem (4.1.2)–(4.1.4). Correspondingly, we obtain a function $\phi(z, r, \theta)$ satisfying (1.2.13), (1.2.15), (1.2.16). Besides, since $\phi(z, r, \theta)$ is a small perturbation of the background solution ϕ_B, which satisfies the entropy condition, then $\phi(z, r, \theta)$ also satisfies the entropy condition on the shock front. Hence Theorem 1 is proved.

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