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Heteroclinic Orbits and Bernoulli Shift for the Elliptic Collision Restricted Three-Body Problem

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Abstract

We consider two mass points of masses $m_1 = m_2 = \frac{1}{2}$ moving under Newton's law of gravitational attraction in a collision elliptic orbit while their centre of mass is at rest. A third mass point of mass $m_3 \approx 0$, moves on the straight line L, perpendicular to the line of motion of the first two mass points and passing through their centre of mass. Since $m_3 \approx 0$, the motion of the masses m_1 and m_2 is not affected by the third mass, and from the symmetry of the motion it is clear that m_3 will remain on the line L. So the three masses form an isosceles triangle whose size changes with the time. The elliptic collision restricted isosceles three-body problem consists in describing the motion of m_3.

In this paper we show the existence of a Bernoulli shift as a subsystem of the Poincaré map defined near a loop formed by two heteroclinic solutions associated with two periodic orbits at infinity. Symbolic dynamics techniques are used to show the existence of a large class of different motions for the infinitesimal body.

1. Introduction

Sitnikov [16] showed the possibility of the existence of oscillatory motions for the elliptic non-collision restricted isosceles three-body problem. This problem is now called the *Sitnikov problem*. Alekseev [1,2] (see also Moser [13]) proved the existence of such a motion by using some homoclinic or heteroclinic orbits. We extended all the dynamics found in the elliptic non-collision restricted isosceles three-body problem to the collision problem. These two problems are very different due to the existence of the triple collision in the second problem.

We have two mass points with equal masses, $m_1 = m_2$ (called *primaries*), moving under Newton's law of gravitational attraction in a collision elliptic orbit while their centre of mass is at rest. We consider a third mass point of mass $m_3 \approx 0$, moving on the straight line L (z-axis) perpendicular to the line of motion (x-axis) of the first two mass points and passing through their centre of mass. Since $m_3 \approx 0$, the motion of the masses m_1 and m_2 is not affected by the third mass, and from the symmetry of the motion it is clear that m_3 will remain on the line L. So the three masses form an isosceles triangle whose size changes with the time. The origin of the coordinates (x, z) is at the centre of mass. The *elliptic collision restricted isosceles three-body problem* consists in describing the motion of m_3. In what follows we call it simply *the restricted isosceles three-body problem*.

Using MCGEHEE's blow up at infinity [11] we study a neighbourhood of both infinities ($z = \pm\infty$). This allows us to show that there are two special periodic orbits at infinity denoted by Γ^+ and Γ^-. Each of these periodic orbits has one stable and one unstable invariant manifold formed by parabolic orbits, (i.e., orbits which start and end at infinity with zero radial velocity). We denote by $P^{\pm,u}$ and $P^{\pm,s}$ the unstable and stable invariant manifolds of $\Gamma^\pm$. These manifolds are analytic cylinders, their expressions are computed in Appendix B.

The intersections of $P^{+,u} \cap P^{-,s}$ and $P^{-,u} \cap P^{+,s}$ define two transversal heteroclinic orbits ξ_1 and ξ_2. Since the restricted isosceles three-body problem is far from an integrable one (see the end of Section 8), we use numerical computations in order to obtain the transversality of the heteroclinic orbits ξ_1 and ξ_2. The loop formed by ξ_1 and ξ_2 provides the necessary recurrent motion for showing the existence of rich dynamics in its neighbourhood, and for studying all final evolutions of m_3 without taking into account triple collisions. For a study of this triple collision see [3]. For analysing the flow near the loop we use the Poincaré map F defined on the transversal section $z = 0$ near the intersection with the loop.

ALEKSEEV [1] and MOSER [13] characterize the orbits of the Sitnikov problem by using symbolic dynamics. This technique has been used later on to study many different three-body problems, see for instance CORS & LLIBRE [4–6], DEVANEY [7], LLIBRE, MARTÍNEZ & SIMÓ [8], LLIBRE & SIMÓ [9,10], and MEYER & WANG [12]. For the restricted isosceles three-body problem we shall define a Bernoulli shift on the set of sequences of symbols $\{a_n\}$ determined by the number of binary collisions of m_1 and m_2 between two consecutive passings of m_3 through the transversal section $z = 0$. We prove that this shift appears as a subsystem of the Poincaré map F, see Theorem 8.2. The dynamics described by this theorem are one of the main results of this paper. An immediate consequence is the existence of periodic orbits and the non-existence of real analytic integrals different from the total energy of the system.

Finally, we give the topology of the set of initial conditions for the capture and escape orbits (i.e., orbits which start or end parabolically or hyperbolically after crossing $z = 0$ n times).

The rest of the paper is organized as follows. First, in Section 2 we deduce for the restricted isosceles three-body problem its equations of motion, its final evolutions, study its three symmetries, and define the Poincaré map. The flow in a neighbourhood of infinity is analysed in Section 3. In Section 4 we deal with the invariant manifolds formed by the parabolic orbits, and compute numerically their transversal intersections in Section 5. The Bernoulli shift is defined in Section 6, and the conditions for appearing as a subsystem of a map, in Section 7. In Section 8 we prove that the Bernoulli shift is a subsystem of the Poincaré map of the restricted isosceles three-body problem. Finally, in Section 9 we study the capture and escape orbits.

2. Formulation of the problem

We have taken the units of length, mass and time in such way that $m_1 = m_2 = \frac{1}{2}$, the time between two consecutive binary collision of the primaries equals 2π, and the gravitational constant equals 1.

Let x be the distance between the centre of mass and m_1, and let z be the distance of m_3 from the centre of mass. So, $(x, 0)$ denotes the position of m_1, $(-x, 0)$ that of m_2, and $(0, \pm z)$ the position of m_3. Then the motion of the two primaries is given by

$$x(E) = \tfrac{1}{2}(1 - \cos E), \qquad t = E - \sin E$$

where we have chosen $x(0) = t(0) = 0$, t the time and E the eccentric anomaly; see for more details [15].

Note that binary collisions between m_1 and m_2 are only possible when $E = 0$ (mod 2π). Of course, we have $x(E) = x(E + 2\pi)$ for all $E \in \mathbb{R}$.

From Newton's laws of mechanics, the equation of motion of m_3 is

$$\frac{d^2 z}{dt^2} = -\frac{z}{(x^2(t) + z^2)^{3/2}}, \tag{2.1}$$

where $x(t) = x(E(t))$.

We denote by (ω_-, ω_+) the *maximal interval* where a solution $z(t)$ of the differential equation (2.1) is defined.

Let $z(t)$ be a solution of system (2.1) defined in the maximal interval $(-\infty, +\infty)$. A *final evolution* of this solution $z(t)$ of the restricted isosceles three-body problem describes its asymptotic behaviour when $t \to -\infty$ or $t \to +\infty$. We have the following possible final evolutions for a solution $z(t)$ of system (2.1).

When $\omega_+ = +\infty$ we say that $z(t)$ has a final evolution of *parabolic* type $(\mathcal{P}^+)$, if $\lim_{t\to+\infty} z(t) = \pm\infty$ and $\lim_{t\to+\infty} \dot{z}(t) = 0$; of *hyperbolic* type $(\mathcal{H}^+)$, if $\lim_{t\to+\infty} z(t) = \pm\infty$ and $\lim_{t\to+\infty} |\dot{z}(t)| > 0$; and of *elliptic* type $(\mathcal{E}^+)$, if m_3 intersects infinitely many times the axes $z = 0$ when $t \to +\infty$. Inside the class $\mathcal{E}^+$ there are the *oscillatory* (O^+) final evolutions which are characterized by $\lim_{t\to+\infty} \sup |z(t)| = +\infty$ and $\lim_{t\to+\infty} \inf |z(t)| = 0$, and the periodic orbits.

In a similar way when $\omega_- = -\infty$ we define the final evolutions $\mathcal{P}^-$, $\mathcal{H}^-$, $\mathcal{E}^-$ and $\mathcal{O}^-$.

We describe a solution $z(t) \not\equiv 0$ of (2.1) by giving its velocity $\dot{z}_0$ and time t_0 when $z(t_0) = 0$. We note that such a zero of $z(t)$ always exists, and usually it is not unique. If $z(t_0) = 0$ we are interested in the solutions such that $\dot{z}(t_0) \neq 0$, otherwise by the uniqueness theorem of the solutions of a differential equation we would get the trivial solutions $z(t) \equiv 0$. Thus we can describe an arbitrary orbit $z(t)$ of system (2.1) by giving $t_0 \pmod{2\pi}$ and $\dot{z}(t_0)$. The values $t_0 \pmod{2\pi}$ and $\dot{z}(t_0) = 0$ correspond to Euler's collinear solution, $z(t) \equiv 0$ (see for instance [15]). In short, we have the following result.

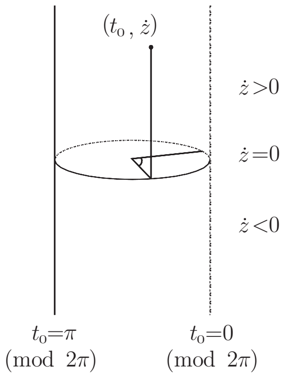

Fig. 2.1. The cylinder $z = 0$.

Proposition 2.1. *Orbits of the mass m_3 are determined by points of a cylinder $z = 0$, parametrized by $t_0 \pmod{2\pi}$ and $\dot{z}(t_0)$, without the generatrix $t_0 = 0$ which corresponds to triple collision orbits (see Fig. 2.1).*

We note that the differential equation (2.1) is invariant under the symmetries

$$\begin{aligned}\mathcal{S}_0 &: (z, \dot{z}, t) \to (-z, -\dot{z}, t),\\ \mathcal{S}_1 &: (z, \dot{z}, t) \to (z, -\dot{z}, -t),\\ \mathcal{S}_2 &: (z, \dot{z}, t) \to (-z, \dot{z}, -t).\end{aligned}$$

We define a Poincaré map f_+ (or f_-) from the cylinder $z = 0$ into itself by following the solution with initial conditions $z(t_0) = 0$, $\dot{z}_0 = \dot{z}(t_0) > 0$ (or $\dot{z}(t_0) < 0$) to its next z zero, and denote by t_1 the smallest $t > t_0$ for which $z(t_1) = 0$ and $\dot{z}(t_1) < 0$ (respectively, $\dot{z}(t_1) > 0$), if it exists. Then, if $\dot{z}_1 = \dot{z}(t_1)$, $f_+(\dot{z}_0, t_0) = (\dot{z}_1, t_1)$ (respectively, $f_-(\dot{z}_0, t_0) = (\dot{z}_1, t_1)$).

If $\mathcal{E}_0^{+,s}$ ($\mathcal{E}_0^{-,s}$) denotes the domain of definition of the map $f_+(f_-)$, then $\mathcal{E}_0^{+,s}$ ($\mathcal{E}_0^{-,s}$) represents the initial values of the orbits which return to $z = 0$ in forward time. From the continuity of the solutions of (2.1) with respect to initial conditions, we have that $\mathcal{E}_0^{+,s}$ and $\mathcal{E}_0^{-,s}$ are open sets.

Now we study the complement of $\mathcal{E}_0^{+,s} \cup \mathcal{E}_0^{-,s}$. First, let $z(t)$ be a solution of (2.1) satisfying $z(t_0) = 0$, $\dot{z}(t_0) > 0$ and $t_1 = +\infty$; then $\dot{z}(t) > 0$ for all $t > t_0$. Thus $z(t)$ is monotonically increasing for $t > t_0$ and, $z(t) \to +\infty$ when $t \to +\infty$. Since by (2.1) $\ddot{z}(t) < 0$ for all $t > t_0$, the function $\dot{z} > 0$ is monotonically decreasing and

$$\dot{z}(\infty) = \lim_{t\to\infty} z(t) \geqq 0$$

exists. We have shown that $z(t)$ is a *parabolic* (or *hyperbolic*) orbit if $\dot{z}(\infty) = 0$ (respectively, $\dot{z}(\infty) > 0$). A similar study can be made for an orbit with initial conditions in $\mathcal{E}_0^{-,s}$.

In fact, the complement of the set $\mathcal{E}_0^{+,s} \cup \mathcal{E}_0^{-,s}$ in $C = \{z = 0, \dot{z} \neq 0, t \neq 0 \pmod{2\pi}\}$ corresponds to initial conditions for parabolic or hyperbolic orbits when $t \to +\infty$. The set of initial conditions in C of parabolic (or hyperbolic) orbits for $t \to +\infty$ is denoted by $P_0^{+,s} \cup P_0^{-,s}$ (respectively, $H_0^{+,s} \cup H_0^{-,s}$), with $P_0^{+,s} \cup H_0^{+,s} \subset (C \cap \{\dot z > 0\})$ and $P_0^{-,s} \cup H_0^{-,s} \subset (C \cap \{\dot z < 0\})$. We note that C is contained in a cylinder, but it is not a cylinder because the generatrix $\{z = 0, \dot z, t = 0 \pmod{2\pi}\}$ which corresponds to initial conditions of triple collision is omitted. However, roughly speaking, it can called a cylinder, and in what follows we will talk about the cylinder C.

Now, we define the inverse Poincaré map. Let $(\dot z_0, t_0)$ be one point on the cylinder C, and $z(t)$ be the orbit of m_3 such that $z(t_0) = 0$ and $\dot z(t_0) = \dot z_0$. We denote by t_{-1} the largest $t < t_0$ for which $z(t_{-1}) = 0$, if it exists. If $\dot z_{-1} = \dot z(t_{-1})$, then we define $f_+^{-1}(\dot z_0, t_0) = (\dot z_{-1}, t_{-1})$ if $\dot z_{-1} > 0$, and $f_-^{-1}(\dot z_0, t_0) = (\dot z_{-1}, t_{-1})$ if $\dot z_{-1} < 0$. We note that f_+^{-1} (or f_-^{-1}) is the inverse Poincaré map of f_+ (respectively, f_-).

If $\mathcal{E}_0^{+,u}$ and $\mathcal{E}_0^{-,u}$ denote respectively the domains of definition f_+^{-1} and f_-^{-1}, then $\mathcal{E}_0^{+,u}$ and $\mathcal{E}_0^{-,u}$ are the set of initial conditions for the orbits which go back to $z = 0$ at least once for backward time. From the continuity of the solutions of (2.1) with respect to initial conditions, we have that $\mathcal{E}_0^{+,u}$ and $\mathcal{E}_0^{-,u}$ are open sets.

The orbits $z(t)$ defined on the complement of $\mathcal{E}_0^{-,u} \cup \mathcal{E}_0^{+,u}$ in C satisfy the conditions that $z(t_0) = 0$, $t_{-1} = -\infty$, $|z(t)| \to +\infty$ when $t \to -\infty$, and $|\dot z(-\infty)| \geqq 0$. Therefore the points in C in the complement of $\mathcal{E}_0^{+,u} \cup \mathcal{E}_0^{-,u}$ correspond to parabolic or hyperbolic orbits when $t \to -\infty$. We denoted by $P_0^{+,u} \cup P_0^{-,u}$ (or $H_0^{+,u} \cup H_0^{-,u}$), with $P_0^{+,u} \cup H_0^{+,u} \subset (C \cap \{\dot z < 0\})$ and $P_0^{-,u} \cup H_0^{-,u} \subset (C \cap \{\dot z > 0\})$ the set of initial conditions in C for parabolic (respectively, hyperbolic) orbits when $t \to -\infty$.

In order to find the domain of definition of f_+^{-1} (f_-^{-1}) we use the symmetry $\mathcal{S}_1$. We remark that on the cylinder C the symmetry $\mathcal{S}_1$ is given by $\mathcal{S}_1(\dot z_0, t_0) = (-\dot z_0, 2\pi - t_0)$ if $t_0 \in (0, 2\pi)$; so $f_+^{-1} = \mathcal{S}_1^{-1} \circ f_+ \circ \mathcal{S}_1$ and $f_-^{-1} = \mathcal{S}_1^{-1} \circ f_- \circ \mathcal{S}_1$. Consequently we have the following result.

Proposition 2.2. *The following equalities hold:*

$$\mathcal{E}_0^{+,u} = \mathcal{S}_1(\mathcal{E}_0^{+,s}),\ H_0^{+,u} = \mathcal{S}_1(H_0^{+,s}),\ P_0^{+,u} = \mathcal{S}_1(P_0^{+,s}),$$
$$\mathcal{E}_0^{-,u} = \mathcal{S}_1(\mathcal{E}_0^{-,s}),\ H_0^{-,u} = \mathcal{S}_1(H_0^{-,s}),\ P_0^{-,u} = \mathcal{S}_1(P_0^{-,s}).$$

We remark that $\mathcal{E}^+ \subsetneqq \mathcal{E}_0^{\pm,s}$ and $\mathcal{E}^- \subsetneqq \mathcal{E}_0^{\pm,u}$. This means that not all orbits with initial conditions in $\mathcal{E}_0^{\pm,u(s)}$ have elliptic final evolution. In a similar way, we have that $P_0^{\pm,s} \subsetneqq \mathcal{P}^+$, $H_0^{\pm,s} \subsetneqq \mathcal{H}^+$, $P_0^{\pm,u} \subsetneqq \mathcal{P}^-$ and $H_0^{\pm,u} \subsetneqq \mathcal{H}^-$.

3. The flow near infinity

An orbit *escapes* at (or *comes* from) infinity if $z(t)$ tends to $\pm\infty$ when t tends to $+\infty$ (respectively, $-\infty$). To study the flow in a neighbourhood of the infinity we use the transformation introduced by McGehee [11], that is,

$$z = \operatorname{sign}(z)\frac{2}{q^2}, \quad \dot z = -\operatorname{sign}(z)p, \quad dt = 4q^{-3}ds,$$

with $0 < q < +\infty$. So $q \to 0$ corresponds to $z \to \pm\infty$. Then, (2.1) becomes

$$\begin{aligned} \frac{dq}{ds} &= p, \\ \frac{dp}{ds} &= q\left(1 + \frac{q^4}{4}x^2(t)\right)^{-3/2}, \end{aligned} \tag{3.1}$$

defined in the phase space $\{(q, p, t) \in \mathbb{R}^3 : q > 0\}$. We note that with system (3.1) we can study both neighbourhoods of the infinity $q = 0$, i.e., the neighbourhood of $z = -\infty$ and the neighbourhood of $z = +\infty$.

Since the function $x(t) = \frac{1}{2}(1 - \cos E)$ is 2π-periodic in E and $t = E - \sin E$, we have that in $\{(q, p, t) \in \mathbb{R}^3 : q \geqq 0\}$ the orbit $q = p = 0$ is 2π-periodic for the system

$$\begin{aligned} \frac{dq}{dt} &= \frac{q^3 p}{4}, \\ \frac{dp}{dt} &= \frac{q^4}{4}\left(1 + \frac{q^4}{4}x^2(t)\right)^{-3/2}, \\ \frac{dt}{dt} &= 1. \end{aligned} \tag{3.2}$$

We write the system as follows

$$\begin{aligned} \frac{dq}{dt} &= -\frac{1}{4}q^3(-p), \\ \frac{d(-p)}{dt} &= -\frac{1}{4}q^4(1 + g_1(q, x(t))), \\ \frac{dt}{dt} &= 1, \end{aligned} \tag{3.3}$$

where $g_1(q, x(t))$ is a power series expression in q starting with order 4 and 2π-periodic in t.

System (3.3) has a 2π-periodic solution at $(q, p) = (0, 0)$ (which corresponds to Γ^+ and Γ^- for $z = +\infty$ and $z = -\infty$ respectively).

Proposition 3.1. *The Poincaré map in a neighbourhood of the periodic orbit $q = 0$, $p = 0$ is given by*

$$\mathcal{P}(q, -p) = \left(q + \frac{\pi}{2}q^3(p + r_1(q, p)), -p - \frac{\pi}{2}q^3(p + r_2(p, q))\right),$$

where r_1 and r_2 are real analytical functions of third order in q and p.

Proof. See Appendix A. □

For every open neighbourhood U of the origin of $\mathbb{R}^2$ we define $A^+(\mathcal{P}, U)$ as $\{a \in U : \mathcal{P}^k(a) \in U$ for all $k > 0$, and $\mathcal{P}^k(a) \to 0$ when $k \to +\infty\}$. Clearly, the parabolic orbits for $t \to +\infty$ contained in U are exactly $A^+(\mathcal{P}, U)$. McGehee proved that there exists U such that $A^+(\mathcal{P}, U)$ is an analytic curve in $q > 0$. Using $\mathcal{P}^{-1}$ we define $A^-(\mathcal{P}, U)$ which is the set of parabolic orbits for $t \to -\infty$.

The flow near (q, p) is obtained by rotating Fig. 3.1 around the p axis. The point $(q, p) = (0, 0)$ can be seen as a fixed point for the Poincaré map having one 1-dimensional stable invariant manifold P^s and one 1-dimensional invariant manifold P^u which are analytic in a neighbourhood of $(0, 0)$ with $q > 0$. Of course, $P^s\ |_U = \mathcal{A}^+(\mathcal{P}, U)$ and $P^u\ |_U = \mathcal{A}^-(\mathcal{P}, U)$.

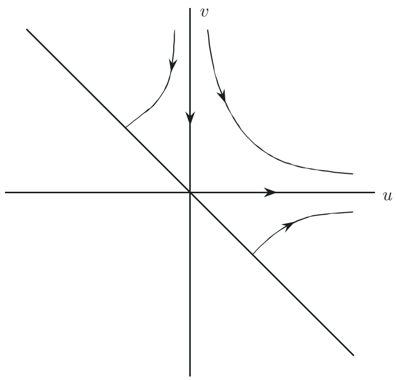

Fig. 3.1. The flow at infinity $z = +\infty$.

System (3.2) is invariant by the symmetry $(q, p, t) \to (q, -p, -t)$. Therefore, if the local stable invariant manifold $A^+(\mathcal{P}, U)$ is given by the equation $q = F(-p, -t)$, then $q = F(p, t)$ is the equation of the local unstable invariant manifold $A^-(\mathcal{P}, U)$. We just point out that the expansion $q = F(p, t) = \sum_{n \geqq 0} a_n(t) p^n$ can be computed by comparing coefficients, after substituting this expansion in (3.3).

Proposition 3.2. *The points (q, p, t) of the local unstable invariant manifold $A^-(\mathcal{P}, U)$ of the periodic orbit $q = p = 0$ at infinity satisfy*

$$q = F(p, t) = p + a_5 p^5 + a_8(t) p^8 + \cdots ,$$

and the points of the local stable invariant manifold $A^+(\mathcal{P}, U)$ verify that $q = F(-p, -t)$, where F is 2π-periodic in t and analytic in $p \in (0, b)$ for some b.

Proof. See Appendix B. □

When we go back to the variables $(z, \dot{z}, t)$, the invariant manifolds P^u and P^s duplicate to $P^{\pm,u}$ and $P^{\pm,s}$, where $P^{+,u}$ and $P^{+,s}$ are near $z = +\infty$, and $P^{-,u}$ and $P^{-,s}$ are near $z = -\infty$.

4. Manifolds of parabolic orbits

We transform the invariant manifolds in the coordinate planes through the change of variables

$$\begin{aligned} u &= \tfrac{1}{4}(q - F(-p,-t)) = \tfrac{1}{4}(q+p) + O_5, \\ v &= \tfrac{1}{4}(q - F(p,t)) = \tfrac{1}{4}(q-p) + O_5, \end{aligned} \tag{4.1}$$

and system (3.1) goes over to

$$\begin{aligned} \frac{du}{ds} &= u + O_5, \\ \frac{dv}{ds} &= -v + O_5, \\ \frac{dt}{ds} &= \{2(u+v)^3 + O_7\}^{-1} = 4q^{-3}. \end{aligned} \tag{4.2}$$

This differential system is restricted to the domain $\{(u,v,t) : q = 2(u+v)+O_5 > 0\}$. Here O_n stands for a C^∞ function $f = f(u,v,t)$ of period 2π in t and such that $\lambda^{-n} f(\lambda u, \lambda v, t)$ is bounded uniformly in t when $\lambda \to 0$ running through positive values.

In order to study the orbits near $u = v = 0$ we eliminate the variable s and write

$$\begin{aligned} \frac{du}{dt} &= 2u(u+v)^3 + O_7, \\ \frac{dv}{dt} &= -2v(u+v)^3 + O_7. \end{aligned} \tag{4.3}$$

We have the following lemma; its proof is easy.

Lemma 4.1. *The differential system*

$$\begin{aligned} \dot{u} &= 2u(u+v)^3, \\ \dot{v} &= -2v(u+v)^3, \end{aligned}$$

in the domain $\mathcal{R} = \{(u,v) \in \mathbb{R}^2 : u+v > 0\}$ *verifies:*

(a) *The family of curves* $uv = c$ *in* $\mathcal{R}$ *are solutions of the system.*
(b) *Its flow is topologically equivalent to the one described in Fig. 4.1.*
(c) *The flow on the second (fourth) quadrant tends to the boundary of* $\mathcal{R}$ *when* $t \to +\infty$ $(-\infty)$.

From this lemma we can discuss the local behaviour of the solutions of differential system (4.3) in a sufficiently small neighbourhood of $u = v = 0$.

The set $\{(u,v) : u < 0, v > 0, q > 0\}$ is contained in the second quadrant, its solutions approach $u + v = 2q = 0$ when $t \to +\infty$. On the other hand, if $u = \frac{1}{4}(q+p) + O_5 < 0$ then p tends to a negative number or zero when $t \to +\infty$. Since $z = \operatorname{sign}(z)\frac{2}{q^2}$ and $\dot{z} = -\operatorname{sign}(z)p$, we can observe that $z(+\infty) = +\infty$ with $\dot{z}(+\infty) > 0$, and $z(+\infty) = -\infty$ with $\dot{z}(+\infty) < 0$. Hence these points

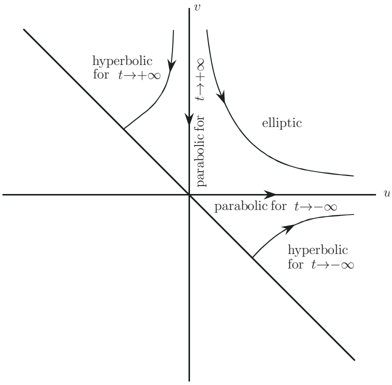

Fig. 4.1. Orbits of m_3 near the infinity.

correspond to orbits which escape with positive or negative velocity at infinity, that is, hyperbolic orbits. The points $\{(u, v) : u = 0, v > 0, q > 0\}$ are such that $z(+\infty) = \pm\infty$ with $\dot{z}(+\infty) = 0$. Thus they correspond to orbits which escape with zero velocity at infinity, that is, parabolic orbits for $t \to +\infty$. The set $\{(u, v) : u > 0, v > 0, q > 0\}$ corresponds to orbits which pass close to infinity and then return.

In a similar way we find that $\{(u, v) : u > 0, v < 0, q > 0\}$ and $\{(u, v) : u > 0, v = 0, q > 0\}$ correspond to hyperbolic and parabolic orbits respectively for $t \to -\infty$.

We will study the topology of the invariant manifolds of parabolic orbits, and their first intersection with the cylinder C. We have denoted by $P^{+,u}$ and $P^{+,s}$ the unstable and stable invariant manifold associated with the periodic orbit $q = p = 0$ at $z = +\infty$. In a similar way we have $P^{-,u}$ and $P^{-,s}$ at $z = -\infty$.

We define $P_0^{+,s}$ (and $P_0^{-,s}$) as the first intersection of $P^{+,s}$ (respectively, $P^{-,s}$) in backward time with $z = 0$. In the same way, $P_0^{+,u}$ (or $P_0^{-,u}$) is the first intersection in forward time of $P^{+,u}$ (respectively, $P^{-,u}$) with $z = 0$.

Now we are interested in the first intersections of the parabolic orbits with the cylinder C, that is in the study of the curves $P_0^{\pm,u}$ and $P_0^{\pm,s}$. Later on we see numerically that the curves $P_0^{+,u}$ and $P_0^{-,s}$ intersect transversally.

The curve $P_0^{+,u}$ has been computed numerically, solving in forward time the system

$$\begin{aligned} \frac{dz}{dE} &= (1 - \cos E)v, \\ \frac{dv}{dE} &= -(1 - \cos E)\frac{z}{(x^2(E) + z^2)^{3/2}}, \end{aligned} \tag{4.4}$$

starting on the unstable manifold $P^{+,u}$ near $z = +\infty$ and ending in the first intersection with $z = 0$. The initial conditions on $P^{+,u}$ can be obtained from the series expansion of $P^{+,u}$ given in Proposition 3.2. Working in this way we obtain the curve $P_0^{+,u}$, and the curves $P_0^{+,s}$, $P_0^{-,u}$ and $P_0^{-,s}$ can be obtained by using the

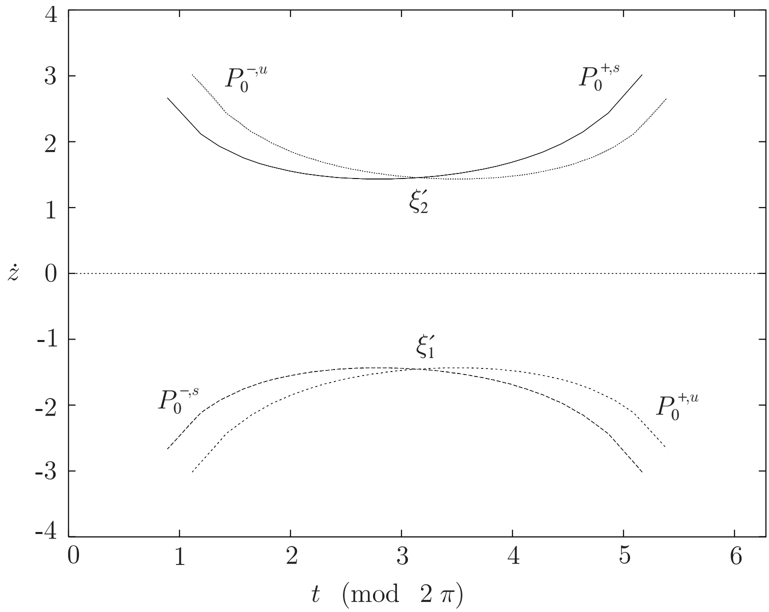

Fig. 4.2. Parabolic orbits on the cylinder $(z = 0, \dot{z}, t \pmod{2\pi})$.

symmetries $\mathcal{S}_1$, $\mathcal{S}_0$ and $\mathcal{S}_2$ respectively (see Fig. 4.2). See Appendix C for more details about the numerical computation.

The numerical results have been drawn on the cylinder$(z = 0, \dot{z}, t \pmod{2\pi})$. Since in the triple collision the velocity $|\ \dot{z}\ | = +\infty$, from Fig. 4.2 there is numerical evidence that there exists at least one intersection point of the manifolds $P^{\pm,u(s)}$ with the generatrix $t = 0 \pmod{2\pi}$ of the cylinder $z = 0$ at infinity (i.e., $|\ \dot{z}\ | = +\infty$). In other words, there is at least a parabolic orbit starting at $z = +\infty$ which ends at collision, and the symmetric ones. Later on we will give topological arguments for proving the existence of a such solution.

Now we consider system (3.2) at infinity $(q = 0)$,

$$\frac{dp}{dt} = 0, \quad \frac{dt}{dt} = 1.$$

Since $t \pmod{2\pi}$ is an angular variable and $p \in \mathbb{R}$, the invariant manifold at infinity is the cylinder

$$\{(q, p, t) : q = 0, p \in \mathbb{R}, t \in S^1\} \cong \mathbb{R} \times S^1,$$

foliated by the periodic orbits $p =$ constant. Therefore the α-limit and ω-limit sets of the hyperbolic and parabolic orbits are periodic orbits. In fact, the infinity manifold is formed by two cylinders, associated with $z = -\infty$ and $z = +\infty$ respectively.

Theorem 4.2. *The set $P_0^{+,s}$ (or $P_0^{-,s}$) of the initial conditions for the parabolic orbits which tend to $z = +\infty$ (respectively, $z = -\infty$) when $t \to +\infty$ is homeomorphic to a simple closed curve with at least one point on $|\ \dot{z}\ | = +\infty$ when*

$t_0 = 0 \pmod{2\pi}$. *The curve* $P_0^{+,s}$ (*or* $P_0^{-,s}$) *divides the cylinder* $z = 0$ *with* $\dot{z} > 0$ (*respectively,* $\dot{z} < 0$) *into two components. One component is formed by the initial conditions* $\mathcal{E}_0^{+,s}$ (*or* $\mathcal{E}_0^{-,s}$)*, and the other by the initial conditions* $H_0^{+,s}$ (*respectively,* $H_0^{-,s}$).

Proof. We have seen that the parabolic orbits which tend to $z = +\infty$ (or $z = -\infty$) when $t \to +\infty$ correspond in a small neighbourhood of infinity with $\{(u, v, t) : u = 0, v > 0\}$ and t arbitrary. Since t is an angular variable, we have that the intersection of this set with the plane $v = v_0 > 0$ for v_0 sufficiently small is a simple closed curve (see Fig. 4.3). This intersection is transversal, otherwise the parabolic orbit would not reach the infinity.

Since any orbit different to $z \equiv 0$ intersects the cylinder C transversally, it is clear that the mapping $(u = 0, v_0, t) \to (z = 0, \dot{z}_0, t_0)$ obtained by following in backward time the solutions from $v = v_0$ to C is well defined. By the uniqueness theorem of the solutions of a differential equation this map is a diffeomorphism. Thus the parabolic orbits which tend to $z = +\infty$ (or $z = -\infty$) when $t \to +\infty$ is a cylinder such that its first intersection with the cylinder C is a simple curve $P_0^{+,s}$ (respectively, $P_0^{-,s}$) with at least one point in $|\dot{z}| = +\infty$ with $t_0 = 0 \pmod{2\pi}$. The curve $P_0^{+,s}$ (or $P_0^{-,s}$) is the boundary of $\mathcal{E}_0^{+,s}$ denoted by $\partial\mathcal{E}_0^{+,s}$ (respectively, $\mathcal{E}_0^{-,s}$ and $\partial\mathcal{E}_0^{-,s}$). The set $\mathcal{E}_0^{+,s}$ (or $\mathcal{E}_0^{-,s}$) corresponds to initial values for the orbits returning to C at least once in forward (respectively, backward) time. The orbits with initial values in $\partial\mathcal{E}_0^{+,s}$ (or $\partial\mathcal{E}_0^{-,s}$) correspond to parabolic orbits for $z = +\infty$ (respectively, $z = -\infty$) when $t \to +\infty$. Clearly, we also have $P_0^{+,s} = \partial H_0^{+,s}$ and $P_0^{-,s} = \partial H_0^{-,s}$, where $H_0^{+,s}$ (or $H_0^{-,s}$) correspond to the orbits which escape hyperbolically to $z = +\infty$ (respectively, $z = -\infty$) when $t \to +\infty$, and do not intersect in forward time the cylinder C. □

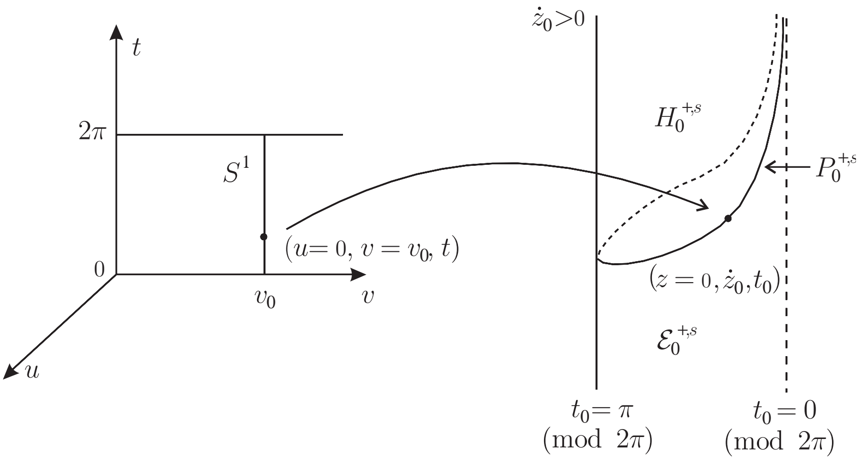

Fig. 4.3. Parabolic orbits on $u = 0, v = v_0, t \pmod{2\pi}$ and on the cylinder $(z = 0, \dot{z} > 0, t \pmod{2\pi})$.

Applying the symmetry $\mathcal{S}_1$ to the Theorem 4.2 we obtain:

Theorem 4.3. *The set $P_0^{+,u}$ (or $P_0^{-,u}$) of the initial conditions for the parabolic orbits which tends to $z = +\infty$ (respectively, $z = -\infty$) when $t \to -\infty$ is homeomorphic to a simple closed curve with at least one point, with $|\dot{z}| = +\infty$ when $t_0 = 0 \pmod{2\pi}$. The curve $P_0^{+,u}$ (or $P_0^{-,u}$) divides the cylinder $z = 0$ with $\dot{z} < 0$ (respectively, $\dot{z} > 0$) into two components. One component is formed by the initial conditions $\mathcal{E}_0^{+,u}$ (respectively, $\mathcal{E}_0^{-,u}$), and the other by the initial conditions $H_0^{+,u}$ (respectively, $H_0^{-,u}$).*

Figure 4.4 shows the curves $P_0^{\pm,u(s)}$ dividing the set of initial conditions in the cylinder C.

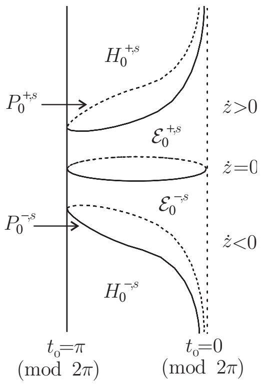

Fig. 4.4. Curves which divide the cylinder C.

5. Transversality of the parabolic manifolds

The numerical computations of the curves $P_0^{\pm,u(s)}$ in the coordinates ($z = 0, \dot{z}, t \pmod{2\pi}$), show that the intersection of $P_0^{-,u}$ (or $P_0^{+,u}$) with $P_0^{+,s}$ (respectively, $P_0^{-,s}$) is nontangential at $t = \pi \pmod{2\pi}$, i.e., these intersections are transversal, see Fig. 4.2.

Consequently we have the following result.

Proposition 5.1. *For the restricted isosceles three-body problem the curves $P_0^{+,u}$ (or $P_0^{-,u}$) and $P_0^{-,s}$ (respectively, $P_0^{+,s}$) intersect transversally on the generatrix $t_0 = \pi \pmod{2\pi}$. That is, there exists one orbit ξ_1 (respectively, ξ_2) beginning parabolically at $z = +\infty$ (respectively, $z = -\infty$), ending parabolically at $z = -\infty$ (respectively, $z = +\infty$), and crossing only once the surface $z = 0$.*

We defined by $\xi_1' = P_0^{+,u} \cap P_0^{-,s}$ and $\xi_2' = P_0^{-,u} \cap P_0^{+,s}$. Note that the orbits ξ_1 and ξ_2 which intersect $z = 0$ at the points ξ_1' and ξ_2' are heteroclinic orbits to the periodic orbits $q = p = 0$ at the infinity $z = +\infty$ and $z = -\infty$. These two heteroclinic orbits form a loop (see Fig. 6.2) which will provide a rich recurrent motion.

6. The Bernoulli shift

In this section we introduced two cylinders $\{z = 0, \dot{z} \neq 0, t \pmod{2\pi}\}$. One is associated with initial coordinates defining the sets of hyperbolic orbits $H_0^{\pm,s}$, parabolic orbits $P_0^{\pm,s}$ and orbits $\mathcal{E}_0^{\pm,s}$ for forward time, see Fig. 6.1. The other cylinder is the image by $\mathcal{S}_1$ (see Section 2) of the cylinder of Fig. 6.1, and it is associated with initial conditions defining the sets of hyperbolic orbits $H_0^{\pm,u}$, parabolic orbits $P_0^{\pm,u}$ and orbits $\mathcal{E}_0^{\pm,u}$ for backward time.

The final evolutions for $t \to +\infty$ of the orbits $H_0^{\pm,s}$ and $P_0^{\pm,s}$, and for $t \to -\infty$ of the orbits $H_0^{\pm,u}$ and $P_0^{\pm,u}$ are clear. Now we want to study the final evolutions of the orbits having initial conditions on $\mathcal{E}_0^{\pm,s}$ and $\mathcal{E}_0^{\pm,u}$. More specifically, the objective of the following sections is to study the regions $\mathcal{E}_0^{-,u} \cap \mathcal{E}_0^{+,s}$ and $\mathcal{E}_0^{+,u} \cap \mathcal{E}_0^{-,s}$ in a neighbourhood of the transversal heteroclinic points ξ_1' and ξ_2'. Remember that the heteroclinic orbit ξ_1 (or ξ_2) starts parabolically in the periodic orbit $q = p = 0$ at $z = +\infty$ (respectively, $z = -\infty$) crosses $z = 0$ once, and ends parabolically in the periodic orbit $q = p = 0$ at $z = -\infty$ (respectively, $z = +\infty$).

We will show that in a neighbourhood of the heteroclinic loop formed by the union of the two heteroclinic orbits ξ_1 and ξ_2 there exist oscillatory motions, as a consequence of which the Poincaré map will have a Bernoulli shift as a subsystem (see Fig. 6.2).

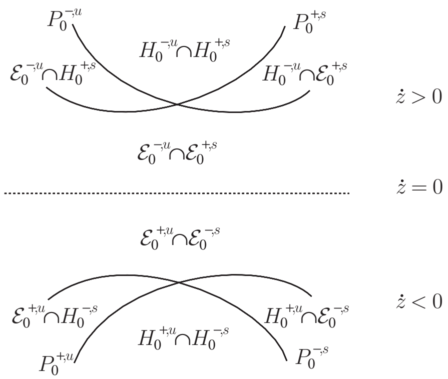

Fig. 6.1. Initial conditions set on cylinder $z = 0$.

We introduce a sequence of integers related to the orbit of m_3 in the following way: given the time t_0 and the position $z(t_0) = 0$, we consider the sequence of all t_n such that $z(t_n) = 0$, ordered in such a way that $t_n < t_{n+1}$. There are four possibilities:

(a) There exist t_n for all $n \in \mathbb{Z}$.

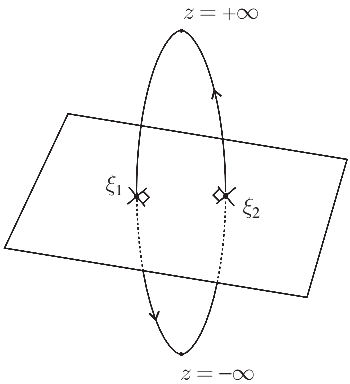

Fig. 6.2. Neighbourhood of the heteroclinic loop formed by ξ_1 and ξ_2.

(b) There exist t_n for all $n \in \mathbb{Z}$ with $n > 0$, but there is some integer k such that it is the first negative integer, starting at -1, for which t_n does not exist. Then we take $t_k = -\infty$.
(c) For all $n \in \mathbb{Z}$ with $n < 0$ there exists t_n, but there is some integer l such that it is the first positive integer for which t_n does not exist. We take $t_l = +\infty$.
(d) The only indices n for which t_n exists satisfy $k < n < l$ with $k < 0$ and $l > 0$. In this case we set $t_k = -\infty$ and $t_l = +\infty$.

We can define the integers

$$a_n = \left[\frac{t_n - t_{n-1}}{2\pi} \right],$$

and let $[y]$ denote the integer part of y, if $y \in \mathbb{R}$. Then $a_n \in \mathbb{N} \cup \{\infty\}$ measure the number of binary collisions of the primaries between two zeros of $z(t)$, that is, between two consecutives crossings of the axis $z = 0$ by m_3.

In this way, we can associate a double infinite sequence with each orbit of m_3, where a is the set of the natural numbers. The sequences are of four types:

(α) $$(\cdots, a_{-2}, a_{-1}, a_0, a_1, a_2, \cdots)$$

with $a_n \in \mathbb{N}$ for all $n \in \mathbb{Z}$. These sequences describe orbits with m_3 cutting $z = 0$ an infinite number of times in positive time, and therefore they never escape to infinity, similarly for negative time. As a consequence orbits of this kind never escape and are never captured by infinity.

(β) $$(\infty, a_{k+1}, a_{k+2}, \cdots)$$

with $k \leqq 0$, and $a_n \in \mathbb{N}$, for all $n \in \mathbb{Z}$ such that $n > k$. These sequences describe orbits which come from infinity and remain cutting $z = 0$ for all time.

(γ) $$(\cdots, a_{l-2}, a_{l-1}, \infty)$$

with $l \geqq 1$ and $a_n \in \mathbb{N}$, for all $n \in \mathbb{Z}$ such that $n < l$. These sequences describe orbits for which m_3 for negative times cuts $z = 0$ an infinite number of times, but at some positive time m_3 is captured by the infinity.

The orbits	define sequences of type
P_0^u y H_0^u	$(\infty, a_1, a_2, \cdots)$
	$(\infty, a_1, \cdots, a_{l-1}, \infty)$ con $l \geqq 1$
P_0^s y H_0^s	$(\cdots, a_{-1}, a_0, \infty)$
	$(\infty, a_{k+1}, \cdots, a_0, \infty)$ con $k \leqq 0$
$P_0^u \cap P_0^s$, $P_0^u \cap H_0^s$	$(a_0, a_1) = (\infty, \infty)$
$H_0^u \cap P_0^s$, $H_0^u \cap H_0^s$	

Table 6.1.

Sequences types	Final evolutions			
α	$\mathcal{E}^- \cap \mathcal{E}^+$,	$\mathcal{E}^- \cap \mathcal{O}^+$,	$\mathcal{O}^- \cap \mathcal{E}^+$,	$\mathcal{O}^- \cap \mathcal{O}^+$
β	$\mathcal{H}^- \cap \mathcal{E}^+$,	$\mathcal{H}^- \cap \mathcal{O}^+$,	$\mathcal{P}^- \cap \mathcal{E}^+$,	$\mathcal{P}^- \cap \mathcal{O}^+$
γ	$\mathcal{E}^- \cap \mathcal{H}^+$,	$\mathcal{E}^- \cap \mathcal{P}^+$,	$\mathcal{O}^- \cap \mathcal{H}^+$,	$\mathcal{O}^- \cap \mathcal{P}^+$
δ	$\mathcal{H}^- \cap \mathcal{H}^+$,	$\mathcal{H}^- \cap \mathcal{P}^+$,	$\mathcal{P}^- \cap \mathcal{H}^+$,	$\mathcal{P}^- \cap \mathcal{P}^+$

Table 6.2.

(δ) $\qquad (\infty, a_{k-1}, \cdots, a_{l-1}, \infty)$

with $k \leqq 0$, $l \geqq 1$ and $a_n \in \mathbb{N}$, for all $n \in \mathbb{Z}$ such that $k < n < l$. These sequences describe orbits of capture and escape.

In order that sequences are well defined it is sufficient that $t_n \neq 0 \pmod{2\pi}$ for all $t_n \in \mathbb{R}$ such that $z(t_n) = 0$. Otherwise there would be a triple collision.

In Table 6.1 we show the types of sequences which are defined by the orbits with initial conditions P_0^u, P_0^s, H_0^u and H_0^s, and their intersection.

The main result in this paper is the following theorem, which describes the dynamics of m_3 in a neighbourhood of the heteroclinic loop formed by the orbits ξ_1 and ξ_2.

Theorem 6.1. *There exists an integer $b > 0$ such that for every sequence of integers $\{b_n\}$ of one of the previous types, providing that $b_n \geqq b$ (for all $n \in \mathbb{Z}$ such that b_n is defined), there is an orbit of m_3 such that its associated sequence is $\{b_n\}$.*

Theorem 6.1 shows the existence of all possible final evolutions (without triple collision) for the restricted isosceles three-body problem, which are given in Table 6.2. To prove this theorem we shall introduce some notions.

7. The Bernoulli shift as a subsystem of a map

As usual $\mathbb{Z}$ will denote the set of integer numbers, and $\mathbb{N}$ will denote the set of non-negative integers numbers.

Let A be the set $\mathbb{N}\bigcup\{\infty\}$, where ∞ is an arbitrary element. We provide A with a total ordering with the usual ordering of naturals extended by $a < \infty$ for all $a \in \mathbb{N}$.

Let S be the set of sequences of elements of A of the followings types:

(a) $(\cdots, a_{-2}, a_{-1}, a_0, a_1, a_2, \cdots)$ with $a_n \neq \infty$ for all $n \in \mathbb{Z}$;
(b) $(a_k, a_{k+1}, a_{k+2}, \cdots)$ with $k \leqq 0$, $a_k = \infty$, and $a_n \neq \infty$ for all $n \in \mathbb{Z}$ such that $n > k$;
(c) $(\cdots, a_{l-2}, a_{l-1}, a_l)$ with $l \geqq 1$, $a_l = \infty$, and $a_n \neq \infty$ for all $n \in \mathbb{Z}$ such that $n < l$;
(d) $(a_k, a_{k+1}, \cdots, a_{l-1}, a_l)$ with $k \leqq 0$, $l \geqq 1$, con $a_k = a_l = \infty$, and $a_n \neq \infty$ for all $n \in \mathbb{Z}$ such that $k < n < l$.

We introduce a topology in S as follows. For each element $a \in S$ there exists a base of neighbourhoods $\{U_j(a)\}$, $j = 1, 2, 3, \ldots$ defined by

$$
\begin{aligned}
U_j(a) &= \{a' \in S : a'_n = a_n \text{ if } \mid n \mid < j\},\\
U_j(a) &= \{a' \in S : a'_n = a_n \text{ if } k < n \leqq j,\ a'_k \geqq j\},\\
U_j(a) &= \{a' \in S : a'_n = a_n \text{ if } -j \leqq n < l,\ a'_l \geqq j\},\\
U_j(a) &= \{a' \in S : a'_n = a_n \text{ if } k < n < l,\ a'_k, a'_l \geqq j\},
\end{aligned}
$$

where a is of type (a), (b), (c) and (d) respectively. For all $a \in S$ the neighbourhoods $\{U_j(a)\}$ satisfy the following.

(1) Each $U_j(a)$ is not empty.
(2) For all j, $a \in U_j(a)$.
(3) Given two arbitrary neighbourhoods $U_i(a)$ and $U_j(a)$, there exists another neighbourhood $U_m(a)$ such that $U_m(a) \subset U_i(a) \cap U_j(a)$. It is sufficient to take $m \geqq \max\{i,j\}$.
(4) For all $U_j(a)$ there exists a subset U of $U_j(a)$ such that $a \in U$, and for all $b \in U$ there exists some $U_i(b)$ contained in U.

We remark that if $U = U_i(a)$ and $i \geqq j+1$, then statement (4) is satisfied. In short, the neighbourhood basis $\{U_j(a)\}$ provides S with the structure of a topological space.

Now, we will prove the following proposition by using the technique of continued fractions.

Proposition 7.1. *With the topology induced by the neighbourhood basis $\{U_j(a)\}$ the topological space S is compact.*

Proof. In order to show that S is compact we shall prove the existence of a homeomorphism between S and the square $C = [0, 1] \times [0, 1]$. Since C is compact with the Euclidean topology of $\mathbb{R} \times \mathbb{R}$, it follows that S is compact.

A sequence $m_0, m_1, m_2, \cdots, m_j$ of arbitrary real numbers, all positive except perhaps m_0, defines the continued fraction

$$[m_0, m_1, m_2, \cdots, m_j] = m_0 + \cfrac{1}{m_1 + \cfrac{1}{m_2 + \cfrac{\ddots}{+\cfrac{1}{m_{j-1} + \cfrac{1}{m_j}}}}}.$$

A continued fraction is *simple* if all $m_i \in \mathbb{N}$.

Let $m_1, m_2, \cdots, m_j$ a finite sequence of numbers in $\mathbb{N}$, $m_0 \in \mathbb{Z}$ and $[m_0, m_1, m_2, \cdots m_j]$ be the continued fraction associated with them. From Theorem 7.2 of [14] we get that any finite simple continued fraction represent a rational number.

Now, we consider the infinite sequence $m_1, m_2, \cdots$ of numbers in $\mathbb{N}$ and $m_0 \in \mathbb{Z}$. The value of any infinite simple continued fraction $[m_0, m_1, m_2 \cdots]$ is define as $\lim_{n\to\infty}[m_0, m_1, m_2 \cdots, m_n]$. By Theorem 7.7 of [14] we have that this limit converges to an irrational number.

On the other hand, if $m_1, m_2, \cdots$ is a periodic sequence, that is, there is a $n \in \mathbb{N}$ such that $m_r = m_{n+r}$ for all $r \in \mathbb{N}$ sufficiently larger, $m_0 \in \mathbb{Z}$. Then the associated continued fraction is periodic, and by Theorem 7.19 of [14] we have that it converges to an irrational number.

From the sequences of S we define the following continued fractions:

$$x_{a_0 a_{-1} a_{-2} \cdots} = \cfrac{1}{a_0 + \cfrac{1}{a_{-1} + \cfrac{1}{a_{-2} + \ddots}}},$$

$$y_{a_1 a_2 a_3 \cdots} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}},$$

$$x_{a_0 a_{-1} \cdots a_{k+1}} = \cfrac{1}{a_0 + \cfrac{1}{a_{-1} + \cfrac{\ddots}{+\cfrac{1}{a_{k+1}}}}} \quad \text{if} \quad k < 0,$$

$$y_{a_1 a_2 \cdots a_{l-1}} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{\ddots}{+\cfrac{1}{a_{l-1}}}}} \quad \text{if} \quad l > 1.$$

The map f_a that assigns to each sequence of type (a) the point of C with coordinates $(x_{a_0a_{-1}a_{-2}\cdots}, y_{a_1a_2a_3\cdots})$ is a homeomorphism between the set of sequences of type (a) and the subset of C formed by points having irrational coordinates.

The map f_b that assigns to each sequence of type (b) the pair $(x_{a_0a_{-1}\cdots a_{k+1}}, y_{a_1a_2a_3\cdots})$ if $k < 0$, and $(0, y_{a_1a_2a_3\cdots})$ if $k = 0$, is a homeomorphism between the set of sequences of type (b) and the subset of C formed by the points having a rational number as first coordinate and as second coordinate an irrational number.

The map f_c that assigns to each sequence of type (c) the pair $(x_{a_0a_{-1}a_{-2}\cdots}, y_{a_1a_2\cdots a_{l-1}})$ if $l > 1$, or $(x_{a_0a_{-1}a_{-2}\cdots}, 0)$ if $l = 1$, is a homemomorphism between the set of type (c) sequences and the subset of C formed by the points with irrational first coordinate and rational second coordinate.

The map f_d that assigns to each sequence of type (d) the pair $(x_{a_0a_{-1}\cdots a_{k+1}}, y_{a_1a_2\cdots a_{l-1}})$ if $k < 0$ and $l > 1$, $(x_{a_0a_{-1}\cdots a_{k+1}}, 0)$ if $k < 0$ and $l = 1$; $(0, y_{a_1a_2\cdots a_{l-1}})$ if $k = 0$ and $l > 1$, and $(0, 0)$ if $k = 0$ and $l = 1$, is a homeomorphism from the set of type (d) sequences onto the subset of points of C having rational coordinates.

In short, the application $f : S \to C$ restricted to the sets of sequences of type (a), (b), (c) and (d) is f_a, f_b, f_c and f_d respectively; it is a homeomorphism between S and C. □

The mapping $\sigma : S \to S$ defined by $(\sigma(a))_n = a_{n+1}$, with $a \in S$ is known as *the Bernoulli shift* of S. The domain of definition of σ is $D(\sigma) = \{a \in S : a_0 \neq \infty\}$, and its image is $\operatorname{Im} \sigma = \{a \in S : a_1 \neq \infty\}$.

The version of the Bernoulli shift as a subsystem of a convenient Poincaré map used here was inspired by MOSER [13]. If f is a continuous map from X into itself, and g is another continuous map from Y into itself, then we say that g is a *subsystem* of f if there is a homeomorphism h from Y to $h(Y) \subset X$ such that the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & X \\ {\scriptstyle h}\uparrow & & {\scriptstyle h}\uparrow \\ Y & \xrightarrow{\ g\ } & Y \end{array}$$

Let $\mathcal{U}^+$ and $\mathcal{U}^-$ be two copies of the square $[0, 1] \times [0, 1]$. Now, from the geometry of a convenient map $\mathcal{F}$ from $\mathcal{U} = \mathcal{U}^+ \cup \mathcal{U}^-$ into itself, we will show that there are two copies of the Bernoulli shift σ as a subsystem of $\mathcal{F}$. We need some preliminary definitions.

Let $\mu \in (0, 1)$. We define a *vertical curve* in $\mathcal{U}^+$ as $x = v(y)$ if $0 \leqq v(y) \leqq 1$ for all $0 \leqq y \leqq 1$, and

$$| v(y_1) - v(y_2) | \leqq \mu \mid y_1 - y_2 \mid \quad \text{for all} \quad 0 \leqq y_1 \leqq y_2 \leqq 1.$$

If $v_1(y)$ and $v_2(y)$ define two vertical curves, and if $0 \leqq v_1(y) < v_2(y) \leqq 1$ for all $0 \leqq y \leqq 1$, we call the set $V = \{(x, y) \in \mathcal{U}^+ : v_1(y) \leqq x \leqq v_2(y)\}$ a *vertical strip* in $\mathcal{U}^+$.

We define the *diameter* of a vertical strip V as

$$d(V) = \max_{0 \leqq y \leqq 1} (v_2(y) - v_1(y)).$$

Similarly, we describe a *horizontal curve* in $\mathcal{U}^+$ as $y = h(x)$, if $0 \leqq h(x) \leqq 1$ for all $0 \leqq x \leqq 1$, and

$$| \, h(x_1) - h(x_2) \, | \leqq \mu \, | \, x_1 - x_2 \, | \quad \text{for all} \quad 0 \leqq x_1 \leqq x_2 \leqq 1.$$

If $h_1(y)$ and $h_2(y)$ define two horizontal curves, and if $0 \leqq h_1(x) < h_2(x) \leqq 1$ for all $0 \leqq x \leqq 1$, we call the set $H = \{(x, y) \in \mathcal{U}^+ : h_1(x) \leqq y \leqq h_2(x)\}$ a *horizontal strip* in $\mathcal{U}^+$.

We define the *diameter* of a horizontal strip H as

$$d(H) = \max_{0 \leqq x \leqq 1} (h_2(x) - h_1(x)).$$

Similarly, we define the *vertical curves* and *strips* in $\mathcal{U}^-$.

Proposition 7.2 ([13], Lemma 1, p. 70**).** *Let $V_1 \supset V_2 \supset \cdots$ be a sequence of vertical strips (or horizontal strips). If $d(V_k) \to 0$ as $k \to \infty$, then $\bigcap_{k=1}^{\infty} V_k$ defines a vertical curve (respectively, horizontal curve).*

Proposition 7.3 ([13], Lemma 2, p. 70**).** *A vertical curve and a horizontal curve in $\mathcal{U}^+$ or $\mathcal{U}^-$ intersect at exactly one point.*

We set the following conditions:

(a) Let $\mathcal{F}$ be a homeomorphism from $\mathcal{U}$ to $\mathcal{F}(\mathcal{U}) \subset \mathbb{R}^2$ such that there are two families of pairwise disjoint vertical strips V_n^+ and V_n^- with $n \in \mathbb{N}$, and two families of pairwise disjoint horizontal strips H_n^+ and H_n^- with $n \in \mathbb{N}$, and $\mathcal{F}(V_n^+) = H_n^-$, $\mathcal{F}(V_n^-) = H_n^+$ for all $n \in \mathbb{N}$. The vertical boundaries ∂V_n^+ and ∂V_n^- of the vertical strips V_n^+ and V_n^- under $\mathcal{F}$ are the boundaries of the horizontal strips preserving the order; that is, $\mathcal{F}(\partial V_n^+) = \partial H_n^-$ and $\mathcal{F}(\partial V_n^-) = \partial H_n^+$ (see Fig. 7.1).

Furthermore the strips $V_n^{\pm}$ and $H_n^{\pm}$ are ordered as in Fig. 7.2. The set of vertical strips is completed by adding $V_\infty^+ = \{(x, y) \in \mathcal{U}^+ \mid x = 1\}$ and $V_\infty^- = \{(x, y) \in \mathcal{U}^- \mid x = 1\}$. We need that $V_n^+ \to V_\infty^+$ and $V_n^- \to V_\infty^-$ when $n \to \infty$. The limit set H_∞^+ and H_∞^- are defined in a similar way.

(b) Let $V \subset \cup_{n=1}^{+\infty} V_n^+$ be a vertical strip in $\mathcal{U}^+$. Then $V_n' = V_n^- \cap \mathcal{F}^{-1}(V)$ is a vertical strip in $\mathcal{U}^-$ for all n, and for some $r \in (0, 1)$ we have that $d(V_n') \leqq r \cdot d(V)$. Let $H \subset \cup_{n=1}^{+\infty} H_n^+$ be a horizontal strip in $\mathcal{U}^+$. Then $H_n' = H_n^- \cap \mathcal{F}(H)$ is a horizontal strip in $\mathcal{U}^-$ for all n and we have also $d(H_n') \leqq r \cdot d(H)$. A similar assertion is true for the families of vertical and horizontal strips in $\mathcal{U}^-$.

We let S^+ and S^- be two copies of the set S, and we define $\widetilde{S} = S^+ \cup S^-$. We consider a homeomorphism $h : \widetilde{S} \to \mathcal{U}$, such that $h_+ = h \mid_{S^+} : S^+ \to \mathcal{U}^+$ and $h_- = h \mid_{S^-} : S^- \to \mathcal{U}^-$. On other hand, we define the map $\widetilde{\sigma} : \widetilde{S} \to \widetilde{S}$ such that $\widetilde{\sigma} \mid_{S^+} : S^+ \to S^-$ and $\widetilde{\sigma} \mid_{S^-} : S^- \to S^+$ are copies of the Bernoulli shift σ. Then the domain of definition of $\widetilde{\sigma}$ is $D(\widetilde{\sigma}) = \{a \in \widetilde{S} : a_0 \neq \infty\}$ and its image is $\operatorname{Im} \widetilde{\sigma} = \{a \in \widetilde{S} : a_1 \neq \infty\}$. Let $\mathcal{F}$ be a continuous map of $\mathcal{U}$ into itself, such that $\mathcal{F}_+ = \mathcal{F} \mid_{\mathcal{U}^+} : \mathcal{U}^+ \to \mathcal{U}^-$ and $\mathcal{F}_- = \mathcal{F} \mid_{\mathcal{U}^-} : \mathcal{U}^- \to \mathcal{U}^+$. If the following diagram

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\mathcal{F}} & \mathbb{R}^2 \\ {\scriptstyle h}\uparrow & & \uparrow{\scriptstyle h} \\ \widetilde{S} & \xrightarrow{\widetilde{\sigma}} & \widetilde{S} \end{array}$$

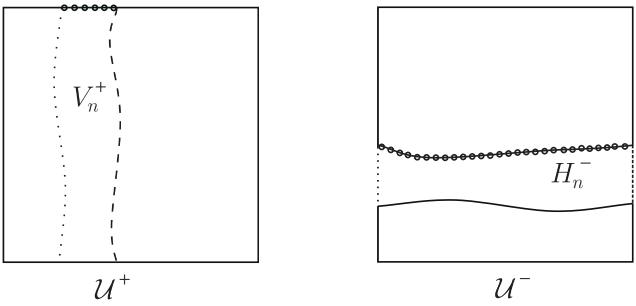

Fig. 7.1. Application of the vertical strips V_n^+ into the horizontal strips H_n^-.

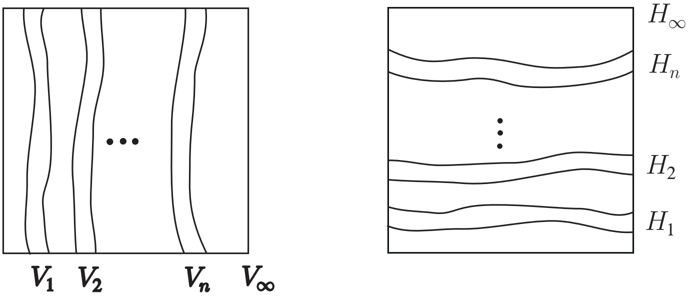

Fig. 7.2. The ordering of the vertical and horizontal strips.

is commutative, we say that $\mathcal{F}$ has *two copies of Bernoulli shift* σ as a subsystem.

Theorem 7.4. *If* $\mathcal{F} : \mathcal{U} \to \mathbb{R}^2$ *is a map satisfying conditions* (a) *and* (b). *Then* $\mathcal{F}$ *has two copies of the Bernoulli shift* σ *as a subsystem.*

Proof. The set $\widetilde{S}$ has four types of elements α, β, γ and δ. We give explicitly the images for elements of types α and β under h, the ones for types γ and δ are similar.

From statements (a) and (b), we can verify that $\mathcal{F}$ (or $\mathcal{F}^{-1}$) maps each of the squares $\mathcal{U}^+$ and $\mathcal{U}^-$ into horizontal (respectively, vertical) strips contained in $\mathcal{U}^-$ and $\mathcal{U}^+$ respectively.

Let $a = (\cdots, a_{-1}, a_0, a_1, \cdots)$ be a sequence of type α. We define recursively for $n \geqq 1$

$$V^+_{a_0 a_{-1}\cdots a_{-n}} = V^+_{a_0} \cap \mathcal{F}^{-1}(V^-_{a_{-1}\cdots a_{-n}}),$$

and

$$V^-_{a_0 a_{-1}\cdots a_{-n}} = V^-_{a_0} \cap \mathcal{F}^{-1}(V^+_{a_{-1}\cdots a_{-n}}).$$

Using condition (b) we have that $V^+_{a_0 a_{-1}\cdots a_{-n}}$ is a vertical strip contained in $\mathcal{U}^+$ and $V^-_{a_0 a_{-1}\cdots a_{-n}}$ is a vertical strip contained in $\mathcal{U}^-$. The same condition assures us that the diameters of $V^\pm_{a_0 a_{-1}\cdots a_{-n}}$ satisfy

$$d(V^\pm_{a_0 a_{-1}\cdots a_{-n}}) \leqq r \cdot d(V^\pm_{a_{-1}\cdots a_{-n}}) \leqq r^n \cdot d(V^\pm_{a_{-n}}) \leqq r^n,$$

in consequence, the diameters tend to zero when $n \to \infty$.

From the definition of vertical strip we have

$$V^+_{a_0a_{-1}\cdots a_{-n+1}} = V^+_{a_0} \cap \mathcal{F}^{-1}(V^-_{a_{-1}} \cap \mathcal{F}^{-1}(V^+_{a_{-2}} \cap \mathcal{F}^{-1}(\cdots \mathcal{F}^{-1}(V^{\operatorname{sign}(-1)^{n-1}}_{a_{-n+1}}) \cdots)))$$

and

$$V^+_{a_0a_{-1}\cdots a_{-n}} = V^+_{a_0} \cap \mathcal{F}^{-1}(V^-_{a_{-1}} \cap \mathcal{F}^{-1}(V^+_{a_{-2}} \cap \mathcal{F}^{-1}(\cdots \mathcal{F}^{-1}(V^{\operatorname{sign}(-1)^{n}}_{a_{-n}}) \cdots))).$$

Then we find that

$$V^+_{a_0a_{-1}\cdots a_{-n}} \subset V^+_{a_0a_{-1}\cdots a_{-n+1}}$$

is satisfied, and therefore using Proposition 7.2,

$$V^+(a) = \bigcap_{n=0}^{+\infty} V^+_{a_0a_{-1}\cdots a_{-n}}$$

is a vertical curve in $\mathcal{U}^+$. Equivalent properties hold for the vertical strips $V^-_{a_0a_{-1}\cdots a_{-n}}$ in $\mathcal{U}^-$. Thus we have the vertical curves $V^+(a)$ in $\mathcal{U}^+$ and $V^-(a)$ contained in $\mathcal{U}^-$,

$$\begin{aligned} V^+(a) = \{p \in \mathcal{U}^+ : \mathcal{F}^{-m}(p) &\in V^+_{a_m} \quad \text{for } m = 0, -2, -4, \ldots \text{ and} \\ \mathcal{F}^{-m}(p) &\in V^-_{a_m} \quad \text{for } m = -1, -3, \ldots\}, \end{aligned}$$

$$\begin{aligned} V^-(a) = \{p \in \mathcal{U}^- : \mathcal{F}^{-m}(p) &\in V^-_{a_m} \quad \text{for } m = 0, -2, -4, \ldots \text{ and} \\ \mathcal{F}^{-m}(p) &\in V^+_{a_m} \quad \text{for } m = -1, -3, \ldots\}. \end{aligned}$$

For $n \geqq 2$ we define recursively

$$H^+_{a_1a_2\ldots a_n} = H^+_{a_1} \cap \mathcal{F}(H^-_{a_2\ldots a_n}),$$

and

$$H^-_{a_1a_2\ldots a_n} = H^-_{a_1} \cap \mathcal{F}(H^+_{a_2\ldots a_n}).$$

Condition (b) assures us that $H^+_{a_1a_2\ldots a_n}$ is a horizontal strip in $\mathcal{U}^+$, and $H^-_{a_1a_2\ldots a_n}$ is a horizontal strip in $\mathcal{U}^-$. As above, the diameters of the horizontal strips satisfy

$$d(H^{\pm}_{a_1a_2\cdots a_n}) \leqq r \cdot d(H^{\pm}_{a_2\cdots a_n}) \leqq r^{n-1} \cdot d(H^{\pm}_{a_n}) \leqq r^{n-1}.$$

Therefore, the limit of the diameters tend to zero when $n \to \infty$.

Now, we use the definition of horizontal strips to obtain

$$\begin{aligned} H^+_{a_1a_2\ldots a_n} &= H^+_{a_1} \cap \mathcal{F}(H^-_{a_2} \cap \mathcal{F}(H^+_{a_3} \cap \mathcal{F}(\cdots \mathcal{F}(H^{\operatorname{sign}(-1)^{n-1}}_{a_n}) \cdots))), \\ H^+_{a_1a_2\ldots a_{n+1}} &= H^+_{a_1} \cap \mathcal{F}(H^-_{a_2} \cap \mathcal{F}(H^+_{a_3} \cap \mathcal{F}(\cdots \mathcal{F}(H^{\operatorname{sign}(-1)^{n}}_{a_{n+1}}) \cdots))), \end{aligned}$$

and from both equalities we get

$$H^+_{a_1a_2\ldots a_{n+1}} \subset H^+_{a_1a_2\ldots a_n}.$$

Finally, from Proposition 7.2, $H^+(a) = \cap_{n=1}^{+\infty} H^+_{a_1a_2\ldots a_n}$ is horizontal curve in $\mathcal{U}^+$. Equivalent properties hold for the horizontal strips $H^-_{a_1a_2\ldots a_n}$ in $\mathcal{U}^-$. Hence, we get the following horizontal curves in $\mathcal{U}^+$ and $\mathcal{U}^-$ respectively,

$$\begin{aligned} H^+(a) = \{p \in \mathcal{U}^+ : \mathcal{F}^{-m}(p) &\in H^+_{a_{m+1}} \quad \text{for } m = 0, 2, 4, \ldots \text{ and} \\ \mathcal{F}^{-m}(p) &\in H^-_{a_{m+1}} \quad \text{for } m = 1, 3, \ldots \}, \end{aligned}$$

and

$$\begin{aligned} H^-(a) = \{p \in \mathcal{U}^- : \mathcal{F}^{-m}(p) &\in H^-_{a_{m+1}} \quad \text{for } m = 0, 2, 4, \ldots \text{ and} \\ \mathcal{F}^{-m}(p) &\in H^+_{a_{m+1}} \quad \text{for } m = 1, 3, \ldots \}. \end{aligned}$$

From Proposition 7.3 we obtain the fact that $V^+(a) \cap H^+(a)$ (or $V^-(a) \cap H^-(a)$) is a unique point $p \in \mathcal{U}^+$ (respectively, $\mathcal{U}^-$).

Now, we shall consider a sequence of type β, that is, $b = (a_k, a_{k+1}, \cdots)$ where $k \leqq 0$. In this case we have that the sequence $a_1, a_2, \ldots$ defines horizontal curves

$$\begin{aligned} H^+(b) = \{p \in \mathcal{U}^+ : \mathcal{F}^{-m}(p) &\in H^+_{a_{m+1}} \quad \text{for } m = 0, 2, 4, \ldots \text{ and} \\ \mathcal{F}^{-m}(p) &\in H^-_{a_{m+1}} \quad \text{for } m = 1, 3, \ldots \}, \end{aligned}$$

and

$$\begin{aligned} H^-(b) = \{p \in \mathcal{U}^- : \mathcal{F}^{-m}(p) &\in H^-_{a_{m+1}} \quad \text{for } m = 0, 2, 4, \ldots \text{ and} \\ \mathcal{F}^{-m}(p) &\in H^+_{a_{m+1}} \quad \text{for } m = 1, 3, \ldots \}. \end{aligned}$$

Proposition 7.2 tell us that $V^+_n \cap \mathcal{F}^{-1}(V^-_\infty)$ (or $V^-_n \cap \mathcal{F}^{-1}(V^+_\infty)$) is a vertical curve in $\mathcal{U}^+$ (respectively, $\mathcal{U}^-$) for all $n \in \mathbb{N}$. Moreover, the inverse image by $\mathcal{F}$ of a vertical curve of $\mathcal{U}^+$ ($\mathcal{U}^-$) is a vertical curve inside each strip V^-_n (respectively, V^+_n). Then

$$\begin{aligned} V^+(b) &= V^+_{a_0} \cap \mathcal{F}^{-1}(V^-_{a_{-1}} \cap \mathcal{F}^{-1}(V^+_{a_{-2}} \cap \cdots \\ &\qquad \cdots \cap \mathcal{F}^{-1}(V^{\text{sign}(-1)^{-k+1}}_{a_{k+1}} \cap \mathcal{F}^{-1}(V^{\text{sign}(-1)^{-k}}_\infty)))) \\ &= \{p \in \mathcal{U}^+ : \mathcal{F}^{-m}(p) \in V^+_{a_m} \quad \text{for } m = 0, -2, -4, \ldots \text{ and} \\ &\qquad \mathcal{F}^{-m}(p) \in V^-_{a_m} \quad \text{for } m = -1, -3, \ldots \text{ with } 0 \leqq m \leqq k\} \end{aligned}$$

and

$$\begin{aligned} V^-(b) &= V^-_{a_0} \cap \mathcal{F}^{-1}(V^+_{a_{-1}} \cap \mathcal{F}^{-1}(V^-_{a_{-2}} \cap \cdots \\ &\qquad \cdots \cap \mathcal{F}^{-1}(V^{\text{sign}(-1)^{-k+1}}_{a_{k+1}} \cap \mathcal{F}^{-1}(V^{\text{sign}(-1)^{-k}}_\infty)))) \\ &= \{p \in \mathcal{U}^- : \mathcal{F}^{-m}(p) \in V^-_{a_m} \quad \text{for } m = 0, -2, -4, \ldots \text{ and} \\ &\qquad \mathcal{F}^{-m}(p) \in V^+_{a_m} \quad \text{for } m = -1, -3, \cdots \text{ with } 0 \leqq m \leqq k\} \end{aligned}$$

are vertical curves in $\mathcal{U}^+$ and $\mathcal{U}^-$ respectively. Consequently, from Proposition 7.3 we get that $V^+(b) \cap H^+(b)$ (or $V^-(b) \cap H^-(b)$) is a unique point $p \in \mathcal{U}^+$ (respectively, $\mathcal{U}^-$).

Note that for the sequences a of type α all the succesive images and inverse images of $h(a)$ under $\mathcal{F}$ exist. However, for a sequence a of type β, $a = (a_k, a_{k+1}, \cdots)$ all the preimages exist, but only the first $|k|$ images. That is, $\mathcal{F}^{|k|}(h_+(a))$ belongs to a vertical curve with index ∞.

Let $p \in \mathcal{U}$ be the intersection point of the vertical curve with a horizontal curve, associated with one of the sequences α, β, γ or δ. Then, we define $h_+(a) = V^+(a) \cap H^+(a)$ if $a \in S^+$, and $h_-(a) = V^-(a) \cap H^-(a)$ if $a \in S^-$.

The continuity of h is obtained as follows. We take two sequences in $\widetilde{S}$; if they are of type α, many terms coincide, and if they are of type β then an element at a_k is large enough. Then points p and p' belong to the same vertical and horizontal strips with small diameters if the elements are close enough; that is, if a large number of terms of the sequence coincide.

The map h is injective because the strips of the same class having different subindices are disjoint. Since $\widetilde{S}$ is compact and h is continuous, we have that $h(\widetilde{S})$ is a compact set. Since h is injective, $h^{-1}\mid_{h(\widetilde{S})}$ is a continuous map. Hence, h is homeomorphism between $\widetilde{S}$ and $h(\widetilde{S})$.

Due to the construction, we have that the Bernoulli shift $\widetilde{\sigma}$ is a subsystem of $\mathcal{F}$, that is, $h \circ \widetilde{\sigma} = \mathcal{F} \circ h \mid_{D(\widetilde{\sigma})}$. □

When the mapping $\mathcal{F}$ is $\mathcal{C}^1$, condition (b) is usually replaced by another condition which is easier to verify. We will replace it by the following:

(c) When $R \in (0, 1)$ we can define in the tangent bundle the sector

$$\Sigma^+_{(x,y)} = \{(u, v) \in T_{(x,y)}\mathcal{U} : |v| \leqq R|u|\},$$

on the set of points (x, y) belonging to vertical strips (where $T_{(x,y)}\mathcal{U}$ is the tangent space for $\mathcal{U}$ at (x, y)), so that:

(I) The bundle Σ^+ is mapped into itself under the differential $D\mathcal{F}$, that is, $D\mathcal{F}_{(x,y)}(\Sigma^+_{(x,y)}) \subset \Sigma^+_{\mathcal{F}(x,y)}$.

(II) If $(u_0, v_0) \in \Sigma^+_{(x,y)}$ and $(u_1, v_1) = D\mathcal{F}_{(x,y)}(u_0, v_0) \in T_{\mathcal{F}(x,y)}\mathcal{U}$, then $|u_0| \geqq R^{-1}|u_1|$.

In an analogous way, if Σ^- is the bundle of sectors defined over horizontal strips by $|u| \leqq R|v|$, it is mapped into itself under $D\mathcal{F}^{-1}$ and $|v_0| \geqq R^{-1}|v_1|$.

Proposition 7.5. *If $\mathcal{F} : \mathcal{U} \to \mathbb{R}^2$ is a map which is continuously differentiable and satisfies conditions* (a) *and* (c) *with $0 < R < 1/2$, then condition* (b) *holds with $r = R(1 - R)^{-1}$.*

Proof. The proof is similar of the proof of Theorem 3.2 in [13] p. 77. □

Theorem 7.6. *Let $\mathcal{F} : \mathcal{U} \to \mathbb{R}^2$ be a map which satisfies conditions* (a) *and* (c). *Then $\mathcal{F}$ has two copies of the Bernoulli shift σ as a subsystem.*

Proof. The proof follows from Theorem 7.4 and Proposition 7.5. □

We have seen that in the neighbourhood of $q = 0$, $p = 0$ the flow of the restricted isosceles three-body problem is very close to the Sitnikov problem flow, so the proofs of Lemmas 4 and 5 ([13], pp. 167–181) follow also for the restricted isosceles three-body problem. This is a consequence of the fact that in our problem, as in the Sitnikov one, when the third body arrives parabolically at infinity there is no distinction between the case where the primaries are moving in an elliptic collision orbit and that where they are in an elliptic orbit.

Remember that Poincaré maps f_+ and f_- defined in Section 2 are given by $f_\pm(\dot z_0, t_0) = (\dot z_1, t_1)$, such that $\mathcal{E}_0^{+,u} = f_+(\mathcal{E}_0^{+,s})$ and $\mathcal{E}_0^{-,u} = f_-(\mathcal{E}_0^{-,s})$.

Lemma 7.7 ([13] , Lemma 4, p. 89**).** *Let $\gamma = \{(\dot z_0, t_0) \mid \dot z_0 = \dot z_0(\lambda), t_0 = t_0(\lambda)$ with $0 \leqq \lambda \leqq 1\}$ be a $\mathcal{C}^1$ arc contained in $\mathcal{E}_0^{+,s}$ (or $\mathcal{E}_0^{-,s}$) such that γ meets $P_0^{+,s}$ (respectively, $P_0^{-,s}$) in the endpoint corresponding to $\lambda = 0$. Just at this point the curves γ and $P_0^{+,s}$ (respectively, $P_0^{-,s}$) are nontangential. Then the image curve $f_+(\gamma) = \{(\dot z_1, t_1) \mid \dot z_1 = \dot z_1(\lambda), t_1 = t_1(\lambda)$ with $0 \leqq \lambda \leqq 1\}$ (or $f_-(\gamma)$) approaches $P_0^{+,u}$ (respectively, $P_0^{-,u}$) spiralling, that is, $t_1(\lambda) \to +\infty$ when $\lambda \to 0$ (see Fig. 7.3).*

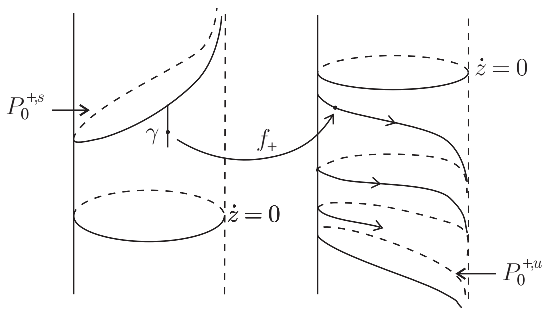

Fig. 7.3. Showing Lemma 7.7.

For $\varepsilon > 0$ sufficiently small, we define $\mathcal{E}_0^{\pm,s}(\varepsilon)$ as the set of points in $\mathcal{E}_0^{\pm,s}$ whose distance from $P_0^{\pm,s}$ is less ε. Since the curve $P_0^{\pm,s}$ is continuously differentiable (actually analytic), we can associate with any point $\zeta \in \mathcal{E}_0^{\pm,s}(\varepsilon)$ a unique point $\xi \in P_0^{\pm,s}$ such that $d(\zeta, P_0^{\pm,s}) = d(\zeta, \xi)$. Here d is the distance over cylinder $z = 0$ induced by Euclidean distance of $\mathbb{R}^3$.

In $\mathcal{E}_0^{+,s}(\varepsilon)$ (and $\mathcal{E}_0^{-,s}(\varepsilon)$) we define two bundles of sectors: The bundle $\Sigma_0 = \Sigma_0(\varepsilon^{1/3})$ assigns to every point $\zeta \in \mathcal{E}_0^{+,s}(\varepsilon)$ (respectively, $\mathcal{E}_0^{-,s}(\varepsilon)$) the set of lines of the tangent plane to the cylinder $z = 0$ at this point, which form an angle less than or equal to $\varepsilon^{1/3}$ with the line through ξ_2' (respectively, ξ_1') parallel to the tangent to the curve $P_0^{+,s}$ (respectively, $P_0^{-,s}$) at ζ. The bundle Σ_0' assigns to every point the set of lines complementary to that of Σ_0 (see Fig. 7.4). Similarly, Σ_1 and Σ_1' are the corresponding bundles of sectors over $\mathcal{E}_0^{+,u}$ (respectively $\mathcal{E}_0^{-,u}$), obtained for example applying the symmetry $\mathcal{S}_1$ to Σ_0 and Σ_0'. Remember that $\mathcal{S}_1(\dot z, t) = (-\dot z, -t)$.

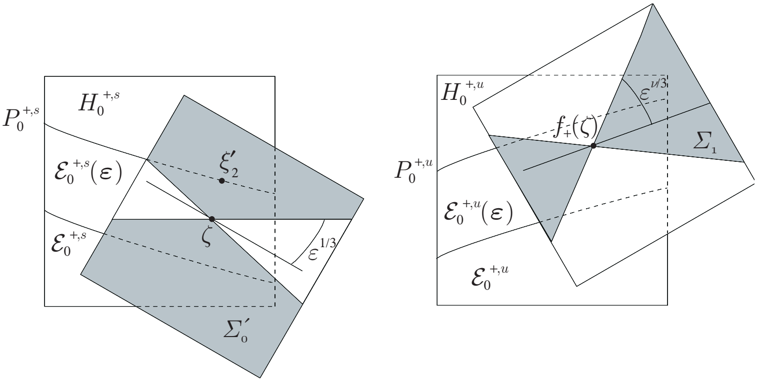

Fig. 7.4. Bundles of sectors.

Lemma 7.8 ([13] , Lemma 5, p. 91**).** *There exists a ν in $0 < \nu < 1$, such that for sufficiently small ε the map f_+ (or f_-) takes $\mathcal{E}_0^{+,s}(\varepsilon)$ (respectively, $\mathcal{E}_0^{-,s}(\varepsilon)$) into $\mathcal{E}_0^{+,u}(\varepsilon^\nu)$ (respectively, $\mathcal{E}_0^{-,u}(\varepsilon^\nu)$), and the tangent map Df_+ (or Df_-) takes the bundle $\Sigma_0' = \Sigma_0'(\varepsilon^{1/3})$ into $\Sigma_1 = \Sigma_1(\varepsilon^{\nu/3})$. Moreover, if $w_0 \in \Sigma_0'$, $w_1 = Df_+(w_0)$ (or $Df_-(w_0)$) and u_0 is the orthogonal projection of w_0 into the centre line of Σ_0' and u_1 that of w_1 into the centre line of Σ_1, then $|u_1| \leqq \varepsilon^{-1/3}|u_0|$.*

The situation of Lemma 7.8 is depicted in Fig. 7.4. Now, we take a curve γ as in Lemma 7.7. This one is differentiable ($\mathcal{C}^1$) with startpoint in $P_0^{+,s}$ (or $P_0^{-,s}$), and it will lie in Σ_0' for sufficiently small ε. Hence $Df_+(\gamma)$ (or $Df_-(\gamma)$) will lie in Σ_1; that is, the direction of $Df_+(\gamma)$ (resprectively, $Df_-(\gamma)$) deviates from the nearest tangent at most by an angle $\varepsilon^{\nu/3}$. This shows that $f_+(\gamma)$ (or $f_-(\gamma)$) approaches $P_0^{+,u}$ (respectively, $P_0^{-,u}$) also in its tangent direction.

8. The Bernoulli shift as a subsystem of the Poincaré map

Sitnikov has shown the possibility of existence of oscillatory motions for a special restricted three-body problem [16]. Alekseev proved the existence of such a motion by using the existence of homoclinic or heteroclinc orbits, showing how to embed the shift of infinite elements in two-dimensional diffeomorphisms [1], [2]. A simplified geometrical version of this type of statements was provided by Moser [13].

In [8] Llibre, Martínez & Simó studied the restricted circular three-body problem for values of the Jacobi constant C, near the value C_2, associated with the Euler critical point L_2. A Lyapunov family of periodic orbits near L_2, the so-called family (c), is born at $C = C_2$ and exists for values of C less than C_2. These periodic orbits are hyperbolic. The corresponding invariant manifolds meet tranversally along homoclinic orbits (see Fig. 8.1). Symbolic dynamic techniques are used to show the existence of orbits passing in a random way (in a given sense) from the region near one primary to the region near the other.

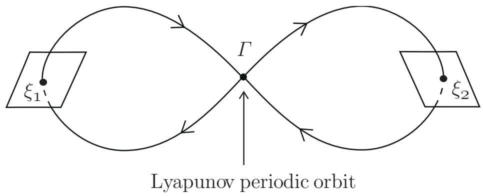

Fig. 8.1. Loop formed by two homoclinic orbits ξ_1 and ξ_2 to the Γ Lyapunov periodic orbit around the critical point L_2 for restricted circular three-body problem.

We have introduced two periodic orbits Γ_- and Γ_+ in $z = -\infty$ and $z = +\infty$, respectively, for the restricted three-body problem. The unstable and stable manifolds are analytic (2-dimensional cylinders), and they are formed by parabolic orbits. Besides, in the plane $z = 0$, they intersect each other transversally at heteroclinic points ξ_1' and ξ_2', which correspond to two heteroclinic orbits ξ_1 and ξ_2 (see Fig. 8.2). We shall show the existence of solutions with final evolution of oscillatory type in the restricted isosceles three-body problem. In the case of the Sitnikov problem, these orbits exist in a neighbourhood of the homoclinic loop. In fact, if we consider a convenient identification of phase space using the symmetry of the restricted isosceles three-body problem, we can reduce the heteroclinic loop to a homoclinic one, in the same way that Moser did it for the Sitnikov problem. Here, we do not consider such an identification in order that the description of the dynamics of the flow for the restricted isosceles three-body problem be clearer and more direct. To understand the dynamics, around the loop of Fig. 8.2, we will apply Theorem 7.6.

For $\varepsilon > 0$ sufficiently small, we define $\mathcal{E}_0^{+,s}(\varepsilon) = \{\zeta \in \mathcal{E}_0^{+,s} : d(\zeta, P_0^{+,s}) \leqq \varepsilon\}$. $\mathcal{E}_0^{+,u}(\varepsilon)$ as the symmetric region $\mathcal{E}_0^{+,u}(\varepsilon) = \mathcal{S}_1(\mathcal{E}_0^{+,s}(\varepsilon))$. In a similar way, we define $\mathcal{E}_0^{-,s}(\varepsilon) = \{\zeta \in \mathcal{E}_0^{-,s} : d(\zeta, P_0^{-,s}) \leqq \varepsilon\}$. $\mathcal{E}_0^{-,u}(\varepsilon)$ as the symmetric region $\mathcal{E}_0^{-,u}(\varepsilon) = \mathcal{S}_1(\mathcal{E}_0^{-,s}(\varepsilon))$.

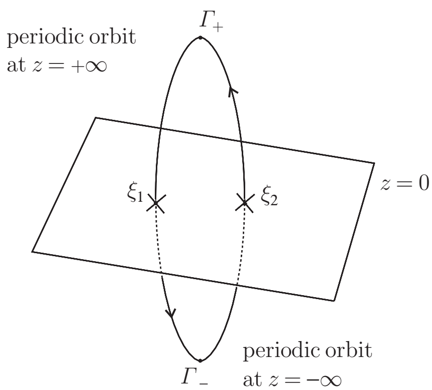

Fig. 8.2. Loop formed by the two heteroclinic orbits ξ_1 and ξ_2 to the periodic orbits Γ_- and Γ_+ at infinity, for the elliptic collision restricted isosceles three-body problem.

Let U^+ be the connected component of $\mathcal{E}_0^{-,u}(\varepsilon) \cap \mathcal{E}_0^{+,s}(\varepsilon)$ which contains the heteroclinic point ξ_2'; it is clear from Proposition 5.1 that U^+ contains $\xi_2' \in P_0^{-,u} \cap P_0^{+,s}$. In a similar way let U^- be the connected component of $\mathcal{E}_0^{+,u}(\varepsilon) \cap \mathcal{E}_0^{-,s}(\varepsilon)$ which contains the heteroclinic point ξ_1'. Note that $\mathcal{S}_1(U^+) = U^-$.

For sufficiently small ε the boundaries $\partial U^{\pm}$ are four arcs of class $\mathcal{C}^1$: $C_1^{\pm}$, $C_2^{\pm}$, $C_3^{\pm}$ and $C_4^{\pm}$ respectively (see Fig. 8.3).

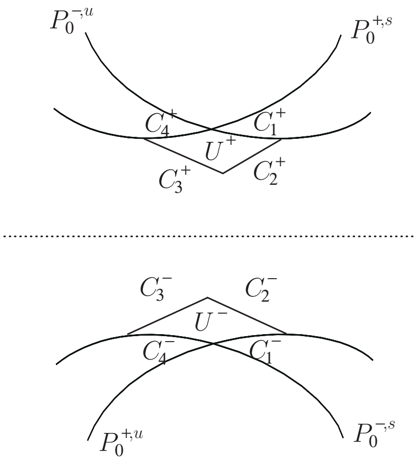

Fig. 8.3. Arcs $C_1^{\pm}$, $C_2^{\pm}$, $C_3^{\pm}$ and $C_4^{\pm}$.

Let U be the union of rectangles U^+ and U^-. From now on, U^+ and U^- will play the role of $\mathcal{U}^+$ and $\mathcal{U}^-$ (respectively) in Theorem 7.6.

We define the mapping $F : U \to f_+(\mathcal{E}_0^{+,s}) \cup f_-(\mathcal{E}_0^{-,s})$ in such a way that $F|_{U^+} = f_+$ and $F|_{U^-} = f_-$. The mapping F will play the role of $\mathcal{F}$ in Theorem 7.4.

From Proposition 5.1 the arcs C_1^+ (C_1^-) and C_3^+ (C_3^-) are curves ending on $P_0^{+,s}$ ($P_0^{-,s}$). Therefore, by Lemma 7.7, the image $f_+(U^+)$ spirals towards $P_0^{+,u}$ intersecting in infinitely many components to U^-, in such a way that the diameter of the strips tend to zero when the strips tends to $P_0^{+,u}$. So, $f_+(U^+) \cap U^-$ is the union of infinitely many horizontal strips (perhaps dropping finitely many components of $f_+(U^+) \cap U^-$ which are not horizontal strips, see Fig. 8.4). We denote these horizontal strips by H_1^-, H_2^-, ..., beginning with the strip nearest to $C_3^- \cap U^-$. Then $H_\infty^- = \lim_{n\to+\infty} H_n^- = C_1^- \cap U^-$.

In a similar way, we have that the strip $f_-(U^-)$ approaches $P_0^{-,u}$ spiralling (Lemma 7.7). Hence $f_-(U^-) \cap U^+$ is the disjoint union of infinitely many horizontal strips H_1^+, H_2^+, ..., beginning with the strip nearest to $C_3^+ \cap U^+$ (perhaps dropping finitely many of the components of $f_-(U^-) \cap U^+$). In consequence, $H_\infty^+ = \lim_{n\to+\infty} H_n^+ = C_1^+ \cap U^+$.

We can make a similar study to analyse $f_-^{-1}(U^+)$. So, $f_-^{-1}(U^+)$ is a strip contained in $\mathcal{E}_0^{-,u}$ which cuts U^- infinitely many times, spiralling towards $P_0^{-,s}$, in

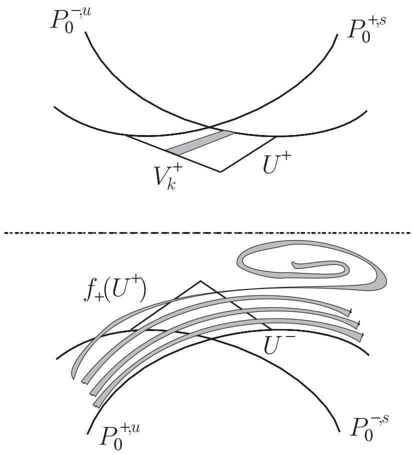

Fig. 8.4. $f_+(U^+) \cap U^-$.

such a way that the diameter of the strip decreases to zero when it approaches $P_0^{-,s}$. The intersection of this strip with U^- has infinitely many components, bounded by C_2^- and C_4^-. Each one of the components of $f_-^{-1}(U^+) \cap U^-$ will be called a vertical strip of U^-. Hence, we have defined a family of vertical strips $V_1^-, V_2^-, \ldots,$ contained in U^-, which are bounded by the curves C_2^- and C_4^-, in such a way that its diameter goes to zero when $n \to \infty$. We define $V_\infty^- = \lim_{n\to\infty} V_n^- = C_4^- \cap U^-$.

In the same manner, $f_+^{-1}(U^-)$ is a strip contained in $\mathcal{E}_0^{+,u}$, which cuts U^+ infinitely many times, spiralling to $P_0^{+,s}$. The components of $f_+^{-1}(U^-) \cap U^+$ are vertical strips $V_1^+, V_2^+, \ldots,$ bounded by the curves C_2^+ and C_4^+. Their diameter goes to zero when they approach C_4^+, and we define $V_\infty^+ = \lim_{n\to\infty} V_n^+ = C_4^+ \cap U^+$.

Observe that $V_n^+ = \mathcal{S}_1(H_n^-)$ and $V_n^- = \mathcal{S}_1(H_n^+)$.

Lemma 8.1. *The following relations hold:*

$$f_+(V_n^+) = H_n^-, \quad f_-(V_n^-) = H_n^+.$$

Proof. From Section 2 we have $f_\pm^{-1} = \mathcal{S}_1 \circ f_\pm \circ \mathcal{S}_1$. Therefore

$$V_n^+ \subset \mathcal{S}_1(f_+(U^+)) = f_+^{-1}\mathcal{S}_1 U^+,$$

and hence

$$f_+(V_n^+) \subset \mathcal{S}_1 U^+ = \bigcup_{k=1}^{\infty} H_k^-. \tag{8.1}$$

In a similar way we have

$$f_+^{-1}(H_n^-) = \mathcal{S}_1 f_+ \mathcal{S}_1(H_n^-) = \mathcal{S}_1 f_+(V_n^+) \subset \mathcal{S}_1\left(\bigcup_{k=1}^{\infty} H_k^-\right),$$

and hence

$$H_n^- \subset f_+ \mathcal{S}_1 \left(\bigcup_{k=1}^{\infty} H_k^- \right) = f_+ \left(\bigcup_{k=1}^{\infty} V_k^+ \right). \tag{8.2}$$

Then, from (8.1) and (8.2) it follows that $\bigcup_{n=1}^{\infty} f_+(V_n^+) = \bigcup_{n=1}^{\infty} H_n^-$. In a similar way the equality $f_-(V_m^-) = H_n^+$ can be proved. We have strip V_m^+ and H_n^- pairwise disjoint; each vertical strip V_m^+ must be sent by f_+ onto a horizontal strip H_n^-. It remains to show that $m = n$.

Let $\hat{\gamma}$ be the symmetry line of U^+, that is, the diagonal of U^+ through ξ_1'. Then $\hat{\gamma}$ has a point in $P_0^{+,s}$, and therefore by Lemma 7.7 the image $f_+(\hat{\gamma})$ spirals towards $P_0^{+,u}$. Hence $f_+(\hat{\gamma})$ and $\mathcal{S}_1(\hat{\gamma})$ have infinitely many points p_k of intersection. We order these points so that p_1 is the first point of intersection with $\mathcal{S}_1(\hat{\gamma})$, which is an arc with endpoint at $\mathcal{E}_0^{-,u} \cap \mathcal{E}_0^{+,s}$, p_2 is the second point, etc. So, $\bigcup_{k=1}^{\infty} p_k = \mathcal{S}_1(\hat{\gamma}) \cap f_+^{-1}(\hat{\gamma})$ and

$$f_+ \mathcal{S}_1(\hat{\gamma}) \cap f_+^{-1}(\hat{\gamma}) = f_+ \mathcal{S}_1(\hat{\gamma}) \cap \hat{\gamma} = \mathcal{S}_1 f_+^{-1}(\hat{\gamma}) \cap \hat{\gamma}.$$

Considering the ordering of intersection, it follows that $f_+(p_k) = \mathcal{S}_1 p_k$. As p_k belongs to a vertical strip V_m^+ and $\mathcal{S}_1(p_k)$ belongs to a horizontal strip H_n^- we get that one is the image of the other, that is, $m = n$. □

Now, conditions (a) and (c) of Theorem 7.6 follow essentially from Lemmas 7.8 and 8.1. In short, we have proved the following result (see the notation of Section 8).

Theorem 8.2. *The Poincaré map F defined in U has two copies of the Bernoulli shift $\widetilde{\sigma}$ defined in $D(\widetilde{\sigma}) \subset \widetilde{S}$ as a subsystem. That is, there exists a homeomorphism h of $D(\widetilde{\sigma})$ onto $h(D(\widetilde{\sigma})) \subset U$ such that $F \circ h = h \circ \widetilde{\sigma}$.*

Now, we see that Theorem 6.1 is a consequence of Theorem 8.2.

Proof of Theorem 6.1. From the sequence $\{b_n\}$ of Theorem 6.1 we construct a new sequence $\{a_n\}$ such that $a_n = b_n - b$. The points in V_k^- (V_k^+) are initial conditions for the orbits of m_3, such that the time up to the next collision between m_1 and m_2 is $2\pi(k + b + \vartheta)$, where b is related with the integer number of turns given by $F(U)$ around the cylinder $z = 0$ before intersecting U though the side $C_4^+ \cap U^+$ ($C_4^- \cap U^-$), and $\vartheta \in [0, 1)$.

The sequences $\{a_n\}$ belong to $D(\widetilde{\sigma})$, and by Theorem 7.6 we can associate with each of these sequences a unique point $\zeta \in U^+ \cup U^-$. By Theorem 7.6 we know that $F^{-n}(\zeta) \in V_{a_n}^+ \cup V_{a_n}^-$ for all a_n of the sequence $\{a_n\}$. Therefore for the orbit defined by ζ the integers b_n measure the number of binary collisions of m_1 and m_2 between two consecutives crossings by $z = 0$. □

From Theorem 6.1 we can derive some consequences of the final evolutions of our problem. From Theorems 4.2 and 4.3 we know that in a neighbourhood of the heteroclinic orbits defined by ξ_1' and ξ_2', there exist orbits with final evolutions of types $\mathcal{H}^- \cap \mathcal{H}^+$, $\mathcal{P}^- \cap \mathcal{H}^+$ and $\mathcal{H}^- \cap \mathcal{P}^+$; while the orbits associated with ξ_1 and ξ_2 have final evolutions of types $\mathcal{P}^- \cap \mathcal{P}^+$.

Using Theorem 6.1 we have that in $h(D(\widetilde{\sigma}))$ there are points with associated orbits of the other types of final evolution described in Table 6.2. Furthermore we can establish the existence of infinitely many periodic orbits. Let $\{b_n\}$ be a m-periodic orbit sequence with $b_n \geqq b$. Then Theorem 6.1 assures the existence of a point $\zeta = h(\{b_n\})$ in $U^+ \cup U^-$ such that the associated orbit of m_3 is periodic of period $2\pi m$, because

$$F^m(\zeta) = h\widetilde{\sigma}^m h^{-1}(\zeta) = h\widetilde{\sigma}^m(\{b_n\}) = h(\{b_n\}) = \zeta.$$

As the Poincaré map F has the Bernoulli shift as subsystem, from Theorem 3.10 of [13], p. 107 we have the following proposition.

Proposition 8.3. *The elliptic collision restricted isosceles three-body problem does not have a real analytic integral.*

9. Capture or escape orbits

In Section 8 we have seen that there are 16 types of final evolutions for the restricted isosceles three-body problem (see Table 6.2), without taking into consideration those beginning or ending in triple collisions.

Let $p = (\dot{z}_0, t_0 \pmod{2\pi})$ be a point of the cylinder $z = 0$ with $t_0 \pmod{2\pi} \neq 0$. We suppose that $\dot{z}_0 > 0$. We say that p defines a *parabolic orbit of type* $P_{2n}^{+,s}$ for $n \in \mathbb{N}$ and write $p \in P_{2n}^{+,s}$ if the orbit through p when time is zero crosses $2n$ times the line $z = 0$ before escaping parabolically to $z = +\infty$ when $t \to +\infty$; or equivalently $(f_- \circ f_+)^n(p) \in P_0^{+,s}$.

We say that p defines a *parabolic orbit of type* $P_{2n+1}^{-,s}$ for $n \in \mathbb{N} \cup \{0\}$, and we write $p \in P_{2n+1}^{-,s}$ if the orbit through p when time is zero crosses $2n+1$ times the line $z = 0$ before escaping parabolically to $z = -\infty$ when $t \to +\infty$, or equivalently $f_+ \circ (f_- \circ f_+)^n(p) \in P_0^{-,s}$.

Now, we suppose that $\dot{z} < 0$. We say that p defines a *parabolic orbit of type* $P_{2n}^{-,s}$ for $n \in \mathbb{N}$, and write $p \in P_{2n}^{-,s}$ if $(f_+ \circ f_-)^n(p) \in P_0^{-,s}$.

We say that p defines a *parabolic orbit of type* $P_{2n+1}^{+,s}$ for $n \in \mathbb{N} \cup \{0\}$, and write $p \in P_{2n+1}^{+,s}$ if $f_- \circ (f_+ \circ f_-)^n(p) \in P_0^{+,s}$.

We say that p defines a *parabolic orbit of type* $P_{2n}^{-,u}$ for $n \in \mathbb{N}$ and write $p \in P_{2n}^{-,u}$ if the orbit through p when time is zero crosses $2n$ times the line $z = 0$ before being captured parabolically to $z = -\infty$ when $t \to -\infty$, in other words $(f_+ \circ f_-)^{-n}(p) \in P_0^{-,u}$.

We say that p defines a *parabolic orbit of type* $P_{2n+1}^{+,u}$ for $n \in \mathbb{N} \cup \{0\}$, and write $p \in P_{2n+1}^{+,u}$, if $f_+^{-1} \circ (f_+ \circ f_-)^{-n}(p) \in P_0^{+,u}$.

We consider $\dot{z} > 0$ again. We say that p defines a *parabolic orbit of type* $P_{2n}^{+,u}$ for $n \in \mathbb{N}$, and write $p \in P_{2n}^{+,u}$, if $(f_- \circ f_+)^{-n}(p) \in P_0^{+,u}$.

In a similar way, p defines a *parabolic orbit of type* $P_{2n+1}^{-,u}$ for $n \in \mathbb{N} \cup \{0\}$, and we write $p \in P_{2n+1}^{-,u}$ if $f_-^{-1} \circ (f_- \circ f_+)^{-n}(p) \in P_0^{-,u}$.

We say that p defines a *hyperbolic orbit of type* $H_{2n}^{+,s}$ for $n \in \mathbb{N}$, and write $p \in H_{2n}^{+,s}$, if the orbit through p when time is zero crosses $2n$ times the line $z = 0$, before escaping parabolically to $z = +\infty$ when $t \to +\infty$; that is, $(f_- \circ f_+)^n(p) \in H_0^{+,s}$.

We say that p defines a *hyperbolic orbit of type* $H_{2n+1}^{-,s}$ for $n \in \mathbb{N} \cup \{0\}$, and write $p \in H_{2n+1}^{-,s}$, if $f_+ \circ (f_- \circ f_+)^n(p) \in H_0^{-,s}$. Let $\dot{z} < 0$. We say that p defines a *hyperbolic orbit of type* $H_{2n}^{-,s}$ for $n \in \mathbb{N}$ if $(f_+ \circ f_-)^n(p) \in H_0^{-,s}$.

Similarly p defines a *hyperbolic orbit of type* $H_{2n+1}^{+,s}$ for $n \in \mathbb{N} \cup \{0\}$, and we write $f_- \circ (f_+ \circ f_-)^n(p) \in H_0^{+,s}$.

Let $\dot{z} > 0$. We say that p defines a *hyperbolic orbit of type* $H_{2n}^{+,u}$ for $n \in \mathbb{N}$, and write $p \in H_{2n}^{+,u}$, if the orbit through p when time is zero crosses $2n$ times the line $z = 0$ before being captured by $z = -\infty$ when $t \to -\infty$; that is, $(f_- \circ f_+)^{-n}(p) \in H_0^{+,u}$.

We say that p defines a *hyperbolic orbit of type* $H_{2n+1}^{+,u}$ for $n \in \mathbb{N} \cup \{0\}$, and write $p \in H_{2n+1}^{+,u}$, if $f_-^{-1} \circ (f_- \circ f_+)^{-n}(p) \in H_0^{+,u}$.

Let $\dot{z} < 0$. We say p defines a *hyperbolic orbit of type* $H_{2n}^{-,u}$ for $n \in \mathbb{N}$, and write $p \in H_{2n}^{-,u}$, if $(f_+ \circ f_-)^{-n}(p) \in H_0^{-,u}$.

We say that p defines a *hyperbolic orbit of type* $H_{2n+1}^{-,u}$ for $n \in \mathbb{N} \cup \{0\}$, and write $p \in H_{2n+1}^{-,u}$, if $f_+^{-1} \circ (f_+ \circ f_-)^{-n}(p) \in H_0^{-,u}$.

Let $\mathcal{E}_{2n}^{+,s} \subset \{\dot{z} > 0\}$ and $\mathcal{E}_{2n}^{-,s} \subset \{\dot{z} < 0\}$ be the domains of definition of $f_+ \circ (f_- \circ f_+)^n$ and $f_- \circ (f_+ \circ f_-)^n$ for $n \in \mathbb{N} \cup \{0\}$ respectively; and define $\mathcal{E}_{2n-1}^{+,s} \subset \{\dot{z} > 0\}$ and $\mathcal{E}_{2n-1}^{-,s} \subset \{\dot{z} < 0\}$ with $n \in \mathbb{N}$ as the domains of $(f_- \circ f_+)^n$ and $(f_+ \circ f_-)^n$, respectively.

Also, we define $\mathcal{E}_{2n}^{+,u} \subset \{\dot{z} < 0\}$ and $\mathcal{E}_{2n}^{-,u} \subset \{\dot{z} > 0\}$ as the domains of definition for $f_+ \circ (f_- \circ f_+)^{-n}$ and $f_- \circ (f_+ \circ f_-)^{-n}$ for $n \in \mathbb{N} \cup \{0\}$, respectively; and in a similar way $\mathcal{E}_{2n-1}^{+,u} \subset \{\dot{z} < 0\}$ and $\mathcal{E}_{2n-1}^{-,u} \subset \{\dot{z} > 0\}$ with $n \in \mathbb{N}$ are defined as the domains of $(f_- \circ f_+)^{-n}$ and $(f_+ \circ f_-)^{-n}$, respectively.

We observe that $\mathcal{E}_0^{+,s}$, $\mathcal{E}_0^{-,s}$, $\mathcal{E}_0^{+,u}$ and $\mathcal{E}_0^{-,u}$ are the domains of the maps f_+, f_-, f_+^{-1} and f_-^{-1} respectively, that we obtained in Section 2.

We will study the final evolution of the points in $\mathcal{E}_0^{+,s}$ for the points in $\mathcal{E}_0^{-,s}$ a similar study can be made.

We consider γ, a $\mathcal{C}^1$ arc, contained in $\mathcal{E}_0^{+,s}$ with endpoint at $P_0^{+,s}$ (see Fig. 9.1). Now, we use Lemma 7.7 to obtain the domain of map $f_- \circ f_+$ restricted to γ, which will be denoted by $\gamma_1^{+,s}$. Observe that $\mathcal{E}_1^{+,s} = f_+^{-1}(\mathcal{E}_0^{-,s} \cap f_+(\gamma \cap \mathcal{E}_0^{+,s}))$, then we omit from γ infinitely many closed intervals I_i such that $f_+(I_i) \not\subseteq \mathcal{E}_0^{-,s}$. It is clear that such intervals accumulate to $P_0^{+,s}$. By continuity, the domain $\mathcal{E}_1^{+,s}$ is $\mathcal{E}_0^{+,s}$ minus the strips $B_1, B_2, \ldots$, which intersect the arc γ in the closed intervals $I_1, I_2, \ldots$ respectively. These strips are the preimage of $f_+(\mathcal{E}_0^{+,s}) \cap H_0^{-,s}$ by f_+.

Each one of these strips B_i has two curves as boundary, one of these curves corresponds to the points of $P_1^{-,s}$ and the other one corresponds to orbits which end in triple collision. The interior of the strip B_i is formed by the points of $H_1^{-,s}$. Using the symmetry $\mathcal{S}_1$ we obtain the sets $\mathcal{E}_1^{+,u}$, $P_1^{+,u}$ and $H_1^{+,u}$.

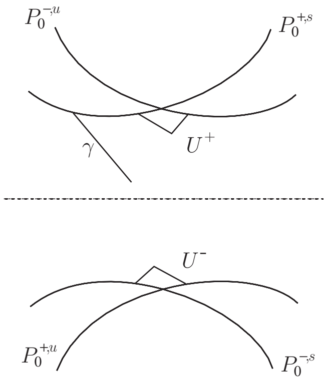

Fig. 9.1. The arc γ in $\mathcal{E}_0^{+,s}$.

The domain of definition of $f_+ \circ f_- \circ f_+$ restricted to γ is

$$\gamma_2^{+,s} = f_+^{-1}(f_-^{-1}(\mathcal{E}_0^{+,s} \cap f_-(\mathcal{E}_0^{-,s} \cap f_+(\gamma \cap \mathcal{E}_0^{+,s})))).$$

From Lemma 7.7, we know that $f_+(\gamma \cap \mathcal{E}_0^{+,s})$ is a curve spiralling to $P_0^{+,u}$, therefore the components of $f_+(\gamma \cap \mathcal{E}_0^{+,s})$ which are close enough to $P_0^{+,u}$ are transversal to $P_0^{-,s}$ in such a way that $f_-(\mathcal{E}_0^{-,s} \cap f_+(\gamma \cap \mathcal{E}_0^{+,s}))$ is a curve which spirals, tending to $P_0^{-,u}$. Then, to obtain $\gamma_2^{+,s}$ we must omit from γ, not only $\bigcup_{i=1}^{+\infty} I_i$, but also $\bigcup_{i=1}^{+\infty} \bigcup_{j=1}^{+\infty} I_{ij}$ where each I_{ij} is a closed interval such that $f_-(f_+(I_{ij})) \not\subseteq \mathcal{E}_0^{+,s}$. We must observe that when varying j, the intervals I_{ij} accumulate to the interval I_i as we show in Fig. 9.2.

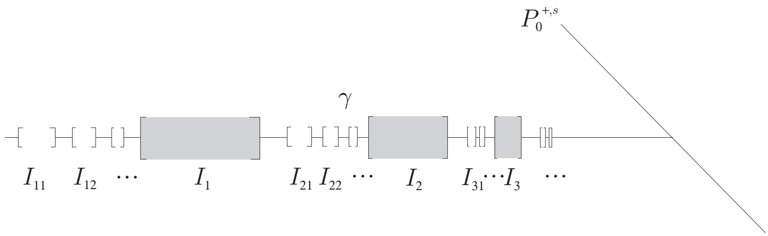

Fig. 9.2. Accumulation of the intervals I_{ij}.

By continuity, the domain $\mathcal{E}_2^{+,s}$ is $\mathcal{E}_1^{+,s}$ minus infinitely many strips B_{11}, B_{12}, $\dots$, B_{21}, B_{22}, $\dots$, B_{31}, B_{32}, $\dots$ which intersect the arc γ in the closed intervals $I_{11}, I_{12}, \dots, I_{21}, I_{22}, \dots, I_{31}, I_{32}, \cdots$ respectively.

By a recursive process we can obtain the domains $\mathcal{E}_3^{+,s}, \mathcal{E}_4^{+,s}, \dots$. Since $\mathcal{E}_n^{+,u} = \mathcal{S}_1 \mathcal{E}_n^{+,s}$ it is enough to obtain $\mathcal{E}_n^{+,s}$ for $n \geqq 0$. It is clear that $\gamma \bigcap \left(\bigcap_{n=0}^{+\infty} \mathcal{E}_n^{+,s} \right)$ is a Cantor set.

In consequence, the following theorem holds.

Theorem 9.1. *For the restricted isosceles three-body problem, the set of the initial conditions which define orbits of type $P_n^{\pm,u}$, $P_n^{\pm,s}$, $H_n^{\pm,u}$ and $P_n^{\pm,s}$ for $n \in \mathbb{N} \cup \{0\}$, on the cylinder $z = 0$, is the one shown in Table 9.1.*

The set of initial conditions of type	is homeomorphic to
$P_0^{\pm,u}$ or $P_0^{\pm,s}$	$S^1 \setminus$ {point}
$P_n^{\pm,u}$ or $P_n^{\pm,s}$, $n \geqq 1$	countable union of intervals open and disjoint
$H_n^{\pm,u}$ or $H_n^{\pm,s}$, $n \geqq 1$	set of positive Lebesgue measure
$P_n^{\pm,u} \cap P_m^{\pm,s}$	a countable set of points
$P_n^{\pm,u} \cap H_m^{\pm,s}$ or $H_n^{\pm,u} \cap P_m^{\pm,s}$	countable union of intervals open and disjoints
$H_n^{\pm,u} \cap H_m^{\pm,s}$	set of positive Lebesgue measure

Table 9.1.

Appendix A. Computations of the unstable manifold of the parabolic orbits

The unstable manifold of the periodic orbit $q = p = 0$ for the differential system:

$$\frac{dq}{dt} = \frac{q^3 p}{4},$$
$$\frac{dp}{dt} = \frac{q^4}{4}\left(1 + \frac{q^4}{4}x^2(t)\right)^{-3/2},$$
$$\frac{dt}{dt} = 1,$$

is an analytic curve in p and 2π-periodic in t. In order to obtain its expression

$$q = F(p,t) = \sum_{n=0}^{+\infty} a_n(t) p^n,$$

we expand the coefficients $a_n(t)$ in Fourier series. Derivating $q = F(p,t)$ with respect to t and substituting dq/dt and dp/dt as power series of p from (3.3), we obtain a system $da_n(t)/dt = g_n(t)$ that can be solved recursively. From (B.2) we get $a_0(t) = 0$ and $a_1(t) = 1$.

More specifically, we have

$$\dot{q} = \frac{dq}{dt} = \frac{\partial F}{\partial p}\frac{dp}{dt} + \frac{\partial F}{\partial t},$$

and substituting $\frac{dq}{dt}$ and $\frac{dp}{dt}$ from the differential system

$$\begin{aligned} \frac{dq}{dt} &= \frac{q^3 p}{4}, \\ \frac{dp}{dt} &= \frac{q^4}{4}\left(1 + \frac{q^4}{4}x^2(t)\right)^{-3/2}, \\ \frac{dt}{dt} &= 1, \end{aligned}$$

in the above expression, we have

$$\frac{q^3 p}{4} = \left(\sum_{n=0}^{+\infty} a_n(t) n p^{n-1}\right)\left(\frac{q^4}{4}\left(1 + \frac{q^4}{4}x^2(t)\right)^{-3/2}\right) + \sum_{n=0}^{+\infty} \dot{a}_n(t) p^n.$$

From the development in Taylor series

$$\left(1 + \frac{q^4}{4}x^2(t)\right)^{-3/2} = 1 - \frac{3}{8}q^4 x^2(t) + \frac{15}{128}q^8 x^4(t) + \mathcal{O}(q^{12}),$$

we can write

$$\frac{q^3 p}{4} = \left(\sum_{n=0}^{\infty} a_n(t) n p^{n-1}\right)\left(\frac{q^4}{4} - \frac{3}{32}q^8 x^2(t) + \mathcal{O}(q^{12})\right) + \sum_{n=0}^{\infty} \dot{a}_n(t) p^n.$$

Now replacing q by the series $\sum_{n=0}^{+\infty} a_n(t) p^n$, we obtain

$$\begin{aligned} \frac{1}{4}\left(\sum_{n=0}^{\infty} a_n(t) p^n\right)^3 p = \frac{1}{4}&\left(\sum_{n=0}^{\infty} a_n(t) n p^{n-1}\right)\left(\sum_{n=0}^{\infty} a_n(t) p^n\right)^4 \\ &- \frac{3}{32}\left(\sum_{n=0}^{\infty} a_n(t) n p^{n-1}\right)\left(\sum_{n=0}^{\infty} a_n(t) p^n\right)^8 x^2(t) \\ &+ \mathcal{O}(p^{12}) + \sum_{n=0}^{\infty} \dot{a}_n(t) p^n, \end{aligned}$$

or equivalently,

$$\begin{aligned} \textstyle\sum_{n=0}^{\infty} \dot{a}_n(t) p^n = \frac{1}{4}&\left(\textstyle\sum_{n=0}^{\infty} a_n(t) p^n\right)^3 p \\ &- \tfrac{1}{4}\left(\textstyle\sum_{n=0}^{\infty} a_n(t) n p^{n-1}\right)\left(\textstyle\sum_{n=0}^{\infty} a_n(t) p^n\right)^4 \qquad \text{(A.1)} \\ &+ \tfrac{3}{32}\left(\textstyle\sum_{n=0}^{\infty} a_n(t) n p^{n-1}\right)\left(\textstyle\sum_{n=0}^{\infty} a_n(t) p^n\right)^8 x^2(t) + \mathcal{O}(p^{12}). \end{aligned}$$

Now, by comparing the coefficients of the same powers of p on both sides of (A.1), we find a sequence of infinite differential equations

$$da_n(t)/dt = \dot{a}_n(t) = f_n(t), \tag{A.2}$$

that can be solved recursively.

We remark that the series on the right-hand side of (A.1) starts with p^4, hence from (A.2) we can see that $\dot{a}_2(t) = \dot{a}_3(t) = 0$, so $a_2 =$ constant and $a_3 =$ constant.

From (A.2) for $n = 4$, we have $\dot{a}_4(t) = a_1^3 - a_1^4 = 1 - 1 = 0$. Then $a_4 =$constant.

From (A.2) for $n = 5$, we have $\dot{a}_5(t) = 3a_1^2 a_2 - a_1^4 a_2 - \frac{1}{4} a_1^4 a_2 = \frac{7}{4} a_2$. Since a_2=constant and $a_5(t)$ is a periodic function, we have that $a_2 = 0$. Hence, using $\dot{a}_5(t) = a_2 = 0$, we obtain that $a_5(t)$=constant.

From (A.2) for $n = 6$, we have $\dot{a}_6(t) = \frac{11}{4} a_3$. The same argument used in the case $n = 5$ gives $a_3 = 0$ and $a_6 =$constant.

From (A.2) for $n = 7$, we obtain $\dot{a}_7(t) = -a_4(t)$. So $a_4 = 0$ and $a_7 =$constant.

Since $\dot{a}_n(t)$ are 2π-periodic functions in t, they admit a development in Fourier series. The constant term of these expansion is

$$\dot{a}_n^0 = \frac{1}{2\pi} \int_0^{2\pi} \dot{a}_n(t)\, dt.$$

From (A.2) for $n = 8$ we obtain $\dot{a}_8(t) = 2a_5 + \frac{3}{32} x^2(t)$. Therefore

$$\begin{aligned}
\dot{a}_8^0(t) &= \frac{1}{2\pi} \int_0^{2\pi} \left(2a_5 + \frac{3}{32} x^2(t)\right) dt \\
&= \frac{a_5}{\pi} \int_0^{2\pi} dE + \frac{3}{256\pi} \int_0^{2\pi} (1 - \cos E)^3 dE \\
&= 2a_5 + \frac{15}{256}.
\end{aligned}$$

From $\dot{a}_8^0 = 0$, we have that $a_5 = -\frac{15}{512}$ and

$$a_8(t) = 2a_5 t + \frac{3}{128} \int_0^t (1 - \cos E)^3\, dt,$$

where $t = E - \sin E$.

From (A.2) for $n = 9$, we have $\dot{a}_9(t) = -\frac{3}{4} a_6$. Then $a_6 = 0$ and $a_9 =$constant.

From (A.2) for $n = 10$, we have $\dot{a}_{10} = -a_7$. Then $a_7 = 0$ and $a_{10} =$constant.

$\vdots$

Therefore, the expansion of the unstable manifold $F(p, t)$ up to order eight is

$$q = p - \frac{15}{512} p^5 + a_8(t) p^8 + \cdots \quad .$$

Appendix B. The Poincaré map near infinity

We consider the differential system $\dot{x} = f(x)$, where $x \in \mathcal{U}$ and $\mathcal{U} \subset \mathbb{R}^n$ is an open set. Let $\varphi(t, x) = \varphi_t(x)$ be the solution of the system $\dot{x} = f(x)$ satisfying the initial condition $\varphi(0, x) = \varphi_0(x) = x$. We have that, if f is of class $\mathcal{C}^r$, then $\varphi_t(x)$ is $\mathcal{C}^r$ too (see [17]). Hence,

$$\varphi_t(x+h) = \varphi_t(x) + \frac{\partial \varphi_t(x)}{\partial x} h + \frac{1}{2!}\frac{\partial^2 \varphi_t(x)}{\partial x^2} h^2 + \cdots$$

Since $\varphi_t(x)$ is of class $\mathcal{C}^r$, we can obtain the first variational equations of $\dot{x} = f(x)$ by finding the derivative of the relation $\dot{\varphi}_t(x) = f(\varphi_t(x))$ with respect to x, and then changing the derivatives with respect to t and x. In this way the first variational equations are

$$\frac{d}{dt}\frac{\partial \varphi_t(x)}{\partial x} = Df(\varphi_t(x))\frac{\partial \varphi_t(x)}{\partial x},$$

with initial conditions

$$\left.\frac{\partial \varphi_t(x)}{\partial x}\right|_{t=0} = I,$$

where Df is the Jacobian matrix and I is the identity matrix of $\mathbb{R}^n$.

The variational equations of second, third and higher order, can be obtained in a similar way:

$$\frac{d}{dt}\frac{\partial^2 \varphi_t(x)}{\partial x^2} = D^2 f(\varphi_t(x))\left(\frac{\partial \varphi_t(x)}{\partial x}\right)^2 + Df(\varphi_t(x))\frac{\partial^2 \varphi_t(x)}{\partial x^2},$$

with $\left.\frac{\partial^2 \varphi_t(x)}{\partial x^2}\right|_{t=0} = 0$;

$$\frac{d}{dt}\frac{\partial^3 \varphi_t(x)}{\partial x^3} = D^3 f(\varphi_t(x))\left(\frac{\partial \varphi_t(x)}{\partial x}\right)^3 + 3D^2 f(\varphi_t(x))\left(\frac{\partial^2 \varphi_t(x)}{\partial x^2}, \frac{\partial \varphi_t(x)}{\partial x}\right) + Df(\varphi_t(x))\frac{\partial^3 \varphi_t(x)}{\partial x^3},$$

with $\left.\frac{\partial^3 \varphi_t(x)}{\partial x^3}\right|_{t=0} = 0$;
etcetera.

To find the derivatives of the flow $\varphi(t, x)$ with respect to the initial conditions α_0 of a periodic orbit, we use the variational equations. In fact, we study the flow in a neighbourhood of the periodic orbit $q = p = 0$ by using the Poincaré map defined in a section transverse to this periodic orbit.

Let Σ be a section transverse to the periodic orbit $q = p = 0$, defined by the initial condition $\alpha_0 \in \Sigma$. Let $\sigma = \sigma(\alpha)$ be the time needed by the orbit that passes through $\alpha \in \Sigma$ to return the first time to Σ. Taking the transversal section $\Sigma' \subset \Sigma$ sufficiently small, it is clear from the theorem of continuous dependence with respect to initial conditions that $\sigma : \Sigma' \to \Sigma$ is defined for all $\alpha \in \Sigma'$. If we start with the initial condition $\alpha_1 \in \Sigma'$, then the Poincaré map $\mathcal{P} : \Sigma' \to \Sigma$ is given by

$$\mathcal{P}(\alpha_1) = \alpha_2 = \varphi(\sigma(\alpha_1), \alpha_1),$$

with $\alpha_2 \in \Sigma$. Hence we can write the map $\mathcal{P}$ into the form

$$\mathcal{P}(\alpha) = \varphi(\sigma(\alpha), \alpha).$$

Now we calculate the development in Taylor series of $\mathcal{P}$ around the fix point $\alpha_0 \in \Sigma$. Therefore we need the derivatives of $\mathcal{P}$ with respect to the initial conditions. For the first derivative we have

$$\frac{\partial \mathcal{P}}{\partial \alpha} = \frac{\partial \varphi}{\partial \sigma}\frac{\partial \sigma}{\partial \alpha} + \frac{\partial \varphi}{\partial \alpha} = f(\varphi_\sigma(\alpha))\frac{\partial \sigma}{\partial \alpha} + \frac{\partial \varphi}{\partial \alpha},$$

where $\frac{\partial \varphi}{\partial \alpha}$ is computed from the first variational equations. For the second derivatives

$$\frac{\partial^2 \mathcal{P}}{\partial \alpha^2} = \frac{\partial^2 \varphi}{\partial \alpha^2} + \frac{\partial^2 \varphi}{\partial \sigma \partial \alpha}\frac{\partial \sigma}{\partial \alpha} + \frac{\partial \varphi}{\partial \sigma}\frac{\partial^2 \sigma}{\partial \alpha^2},$$

we compute $\frac{\partial^2 \varphi}{\partial \alpha^2}$ and $\frac{\partial^2 \varphi}{\partial \sigma \partial \alpha}$ from the first and second variational equations, respectively. In a similar way we calculate the derivatives of $\mathcal{P}$ for higher orders. Thus the Poincaré map up to order two is:

$$\mathcal{P}(\alpha) = \alpha_0 + \left(\left.\frac{\partial \mathcal{P}}{\partial \alpha}\right|_{\alpha=\alpha_0} \right)(\alpha - \alpha_0) + \frac{1}{2!}\left(\left.\frac{\partial^2 \mathcal{P}}{\partial \alpha^2}\right|_{\alpha=\alpha_0} \right)(\alpha - \alpha_0)^2 + \mathcal{O}(3),$$

where $\mathcal{O}(3) = \mathcal{O}((\alpha - \alpha_0)^3)$.

We apply this method to compute the Poincaré map $\mathcal{P}$ defined at time 2π in a neighborhood of the periodic orbit $(q, p, t) = (0, 0, t)$ of the period 2π for differential system (3.2). More precisely $\mathcal{P} : \{(q, -p, t) \mid t = 0\} \to \{(q, -p, t) : t = 2\pi\}$ and $\mathcal{P}(0, 0, 0) = \mathcal{P}(0, 0, 2\pi)$. We consider the differential system

$$\begin{aligned} \frac{dq}{dt} &= -\frac{q^3(-p)}{4}, \\ \frac{d(-p)}{dt} &= -\frac{q^4}{4}\left(1 + \frac{q^4}{4}x^2(t)\right)^{-3/2}, \\ \frac{dt}{dt} &= 1, \end{aligned} \tag{B.1}$$

or equivalently,

$$\begin{aligned} \frac{dq}{dt} &= -\frac{q^3 P}{4}, \\ \frac{dP}{dt} &= -\frac{q^4}{4} + \frac{3}{32}q^8 x^2(t) + \mathcal{O}(q^{12}), \\ \frac{dt}{dt} &= 1, \end{aligned}$$

where $P = -p$. Hence the vector field on Σ is given by

$$f(q, P, t) = \begin{pmatrix} \frac{-q^3 P}{4} \\ -\frac{q^4}{4} + \frac{3}{32} q^8 x^2(t) + \mathcal{O}(q^{12}) \\ 1 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}.$$

We denote the components of the flow φ from $\dot{x} = f(x)$ by $(\varphi_1, \varphi_2, \varphi_3)$, here $x = (q, P, t)$. From (B.1) it is clear that $\varphi_3(t, \alpha) = t$; then $\sigma(\alpha) = 2\pi$. Therefore, since $\sigma(\alpha)$ is constant, we have $\frac{\partial^n \mathcal{P}}{\partial \alpha^n} = \frac{\partial^n \varphi(\sigma(\alpha), \alpha)}{\partial \alpha^n}$. In our case the Poincaré map $\mathcal{P}$ becomes

$$\begin{aligned} \mathcal{P}(q, P) = \alpha_0 &+ \frac{\partial \varphi}{\partial q}(\alpha_0, \sigma(\alpha_0)) q + \frac{\partial \varphi}{\partial P}(\alpha_0, \sigma(\alpha_0)) P \\ &+ \frac{1}{2!}\left(\frac{\partial^2 \varphi}{\partial q^2}(\alpha_0, \sigma(\alpha_0)) q^2 + 2 \frac{\partial^2 \varphi}{\partial q \partial P}(\alpha_0, \sigma(\alpha_0)) q P \right. \\ &\left. + \frac{\partial^2 \varphi}{\partial P^2}(\alpha_0, \sigma(\alpha_0)) P^2 \right) \\ &+ \frac{1}{3!}\left(\frac{\partial^3 \varphi}{\partial q^3}(\alpha_0, \sigma(\alpha_0)) q^3 + 3 \frac{\partial^3 \varphi}{\partial q^2 \partial P}(\alpha_0, \sigma(\alpha_0)) q^2 P \right. \\ &\left. + 3 \frac{\partial^3 \varphi}{\partial q \partial P^2}(\alpha_0, \sigma(\alpha_0)) q P^2 + \frac{\partial^3 \varphi}{\partial P^3}(\alpha_0, \sigma(\alpha_0)) P^3 \right) \\ &+ \frac{1}{4!}\left(\frac{\partial^4 \varphi}{\partial q^4}(\alpha_0, \sigma(\alpha_0)) q^4 + 4 \frac{\partial^4 \varphi}{\partial q^3 \partial P}(\alpha_0, \sigma(\alpha_0)) q^3 P \right. \\ &+ 6 \frac{\partial^4 \varphi}{\partial q^2 \partial P^2}(\alpha_0, \sigma(\alpha_0)) q^2 P^2 + 4 \frac{\partial^4 \varphi}{\partial q \partial P^3}(\alpha_0, \sigma(\alpha_0)) q P^3 \\ &\left. + \frac{\partial^4 \varphi}{\partial P^4}(\alpha_0, \sigma(\alpha_0)) P^4 \right) \\ &+ \mathcal{O}(5). \end{aligned}$$

We take the initial condition $\alpha_0 = (q_0, P_0) = (0, 0)$, which is a point on the periodic solution $(0, 0, t)$, and consequently it is a fixed point for $\mathcal{P}$. On the other hand, as the orbit is 2π-periodic we have that $\sigma(\alpha_0) = 2\pi$.

Since we have $\left.\frac{\partial \varphi}{\partial \alpha}\right|_{t=0} = I$, then $\frac{\partial \varphi_1}{\partial q} = 1$ and $\frac{\partial \varphi_2}{\partial P} = 1$.

If we integrate

$$\begin{aligned} &\frac{\partial^2 f_1}{\partial q^2} = \frac{3}{2} q P, && \frac{\partial^2 f_1}{\partial q \partial P} = -\frac{3}{4} q^2, && \frac{\partial^2 f_1}{\partial P^2} = 0, \\ &\frac{\partial^2 f_2}{\partial q^2} = -3q^2 + \frac{21}{4} q^6 x^2(t) + \mathcal{O}(q^{10}), && \frac{\partial^2 f_2}{\partial q \partial P} = 0, && \frac{\partial^2 f_2}{\partial P^2} = 0, \end{aligned}$$

on the periodic orbit, we get the solution of the second variational equations, and its value at $\alpha_0 = (0, 0)$ is zero.

As above, we integrate along the periodic orbit

$$
\begin{aligned}
&\frac{\partial^3 f_1}{\partial q^3} = \frac{3}{2}P, && \frac{\partial^3 f_1}{\partial q^2 \partial P} = -\frac{3}{2}q, && \frac{\partial^3 f_1}{\partial q \partial P^2} = 0,\\
&\frac{\partial^3 f_1}{\partial P^3} = 0, && \frac{\partial^3 f_1}{\partial P^2 \partial q} = 0, && \frac{\partial^3 f_1}{\partial P \partial q^2} = -\frac{1}{4}q,\\
&\frac{\partial^3 f_2}{\partial q^3} = -6q + \frac{63}{2}q^5 x^2(t) + \mathcal{O}(q^9), && \frac{\partial^3 f_2}{\partial q^2 \partial P} = 0, && \frac{\partial^3 f_2}{\partial q \partial P^2} = 0,\\
&\frac{\partial^3 f_2}{\partial P^3} = 0, && \frac{\partial^3 f_2}{\partial P^2 \partial q} = 0, && \frac{\partial^3 f_2}{\partial P \partial q^2} = 0,
\end{aligned}
$$

and we obtain the solution of the third variational equations, which vanishes at the initial conditions.

In the same way, we integrate

$$
\begin{aligned}
&\frac{\partial^4 f_1}{\partial q^4} = 0, && \frac{\partial^4 f_1}{\partial q^3 \partial P} = -\frac{3}{2}, && \frac{\partial^4 f_1}{\partial q^2 \partial P^2} = 0,\\
&\frac{\partial^4 f_1}{\partial q \partial P^3} = 0, && \frac{\partial^4 f_1}{\partial P^4} = 0, && \frac{\partial^4 f_1}{\partial P^4} = 0,\\
&\frac{\partial^4 f_1}{\partial P^3 \partial q} = 0, && \frac{\partial^4 f_1}{\partial P^2 \partial q^2} = 0, && \frac{\partial^4 f_1}{\partial P \partial q^3} = 0,\\
&\frac{\partial^4 f_2}{\partial q^4} = -6, && \frac{\partial^4 f_2}{\partial q^3 \partial P} = 0, && \frac{\partial^4 f_2}{\partial q^2 \partial P^2} = 0,\\
&\frac{\partial^4 f_2}{\partial q \partial P^3} = 0, && \frac{\partial^4 f_2}{\partial P^4} = 0, && \frac{\partial^4 f_2}{\partial P^3 \partial q} = 0,
\end{aligned}
$$

along the periodic orbit, obtaining the solution of the fourth variational equations, therefore $\frac{\partial^4 \varphi_1}{\partial q^3 \partial P} = 3\pi$ and $\frac{\partial^4 \varphi_2}{\partial q^4} = -12\pi$.

Joining all the preceding computations we have the expression for $\mathcal{P}$,

$$
\begin{aligned}
q &\to q - \frac{1}{4!}(4)(3\pi)q^3 P + \cdots\\
P &\to P + \frac{1}{4!}(-12\pi)q^4 + \cdots
\end{aligned}
$$

Therefore, changing the variable P by $-p$, we get that the Poincaré map up to fourth order is giving by

$$
\mathcal{P}(q, -p) = \left(q + \frac{\pi}{2}q^3 p + \mathcal{O}(5),\ -p - \frac{\pi}{2}q^4 + \mathcal{O}(5)\right). \qquad \text{(B.2)}
$$

Appendix C. Computation of parabolic orbits

We took the initial conditions on the unstable manifold of the parabolic orbits, given by the series $q = F(p, t)$, and we computed numerically the first intersection of $P^{+,u}$ with the cylinder $z = 0$ solving in backward time system (4.4).

In particular, we took $p = 0.01$ and $q = p - \frac{15}{512}p^5 + O(p^8)$, so we obtained $z = 20000.000011718750005149841310605 41$ and $\dot{z} = -0.01$. For the eccentric anomaly E we chose 60 equally spaced points on the interval $[0, 2\pi)$.

The principal program called ISOSCELES uses the subroutines RK78 and DERIV. We used the integration routine RK78 (i.e., Runge-Kutta-Felberg of order 7 and 8) with quadruple precision and tolerance of 10^{-18}, while the routine DERIV defines the vector field associated with system (4.4).

As $-1 \leqq \cos E \leqq 1$, in order to avoid errors we have replaced $1 - \cos E$ by $\sin^2 E/1 + \cos E$ when $E \in [0, \frac{\pi}{2}] \bigcup [\frac{3\pi}{2}, 2\pi]$, leaving the initial expression when $E \in [\frac{\pi}{2}, \frac{3\pi}{2}]$.

Taking $z = 0$ as the transversal section, once the orbits have crossed $z = 0$ we obtain the zero making a refinement using the Newton-Raphson method with tolerance of 10^{-20}.

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