Arch. Rational Mech. Anal. 150 (1999) 281–305. © Springer-Verlag 1999

# *A Discrete Convolution Model for Phase Transitions*

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*Communicated by* P. J. HOLMES

**Abstract**

We study a discrete convolution model for Ising-like phase transitions. This nonlocal model is derived as an $l_2$-gradient flow for a Helmholtz free energy functional with general long range interactions. We construct traveling waves and stationary solutions, and study their uniqueness and stability. In particular, we find some criteria for "propagation" and "pinning", and compare our results with those for a previously studied continuum convolution model.

## 1. Introduction

We study the following infinite system of coupled semilinear evolution equations

$$\dot{u}_n = (J * u)_n - u_n - \lambda f(u_n),\ n \in \mathbb{Z}, \tag{1.1}$$

where $(J * u)_n \equiv \sum_{i\in\mathbb{Z}\setminus\{0\}} J(i)u_{n-i}$, $\sum_{i\in\mathbb{Z}\setminus\{0\}} J(i) = 1$, $f$ is bistable and $\lambda > 0$. The values of the kernel $J$ can change sign but some of our results require $J$ to be nonnegative and even. Equation (1.1) can be obtained using a microscopic viewpoint, from the following argument.

Consider a lattice $\Lambda$ whose sites are occupied by blocks, each consisting of many "atoms" arranged in a finer lattice. Each atom can exist in one of two states, A and B (such as spin up or spin down). We take the point of view that each arrangement of atoms has a potential energy of interaction consisting of (i) interactions between atoms within each block, and (ii) interactions between blocks. Furthermore, we will account for random (thermal) fluctuations within each block to give an entropy of mixing but we will not consider random fluctuations between blocks of $\Lambda$.

The sites of $\Lambda$ will be denoted by $r$ but also by $l(r)$, the latter signifying the smaller lattice which is the block at position $r \in \Lambda$. We will use $\sigma(s)$ to denote the occupancy of an A-atom at site $s \in l(r)$, i.e., $\sigma(s) = 1$ if an A-atom occupies site $s$ and is 0 otherwise. We will use $a(r)$ to denote the average A-occupancy of $l(r)$.

The Helmholtz free energy of our system with an arrangement $\{a(r)\}_{r\in\Lambda}$ is given by

$$E = H - TS, \tag{1.2}$$

where $H$ is the internal interaction energy, $T$ the absolute temperature and $S$ the entropy.

As postulated above, the interaction energy has the form

$$H(a) = I_1(a) + I_2(a),$$

where $I_1$ is the total energy of interaction within the blocks which make up $\Lambda$ and $I_2$ is that between blocks. We may write

$$\begin{aligned} I_1(a) = -\tfrac{1}{2}\sum_{r\in\Lambda}\sum_{s,s'\in l(r)} \Big[ & J^{AA}(s-s')\,\sigma(s)\,\sigma(s') + J^{BB}(s-s')(1-\sigma(s)) \\ & \times(1-\sigma(s')) + J^{AB}(s-s')(\sigma(s)(1-\sigma(s')) + (1-\sigma(s))\sigma(s'))\Big], \end{aligned}$$

where $J^{\alpha\beta}(s-s')$ is the energy of interaction between atoms of types $\alpha$ and $\beta$ at sites $s$ and $s'$ (see [21]). In this form $I_1$ actually depends upon the microscopic configuration given by the $\sigma$'s but a simplifying assumption below resolves this issue.

The interaction energy between blocks is given by

$$\begin{aligned} I_2(a) = -\tfrac{1}{2}\sum_{r,r'\in\Lambda} \Big[ & j^{AA}(r-r')a(r)a(r') + j^{BB}(r-r')(1-a(r))(1-a(r')) \\ & + j^{AB}(r-r')(a(r)(1-a(r')) + (1-a(r))a(r'))\Big], \end{aligned}$$

where $j^{\alpha\beta}(r-r')$ denotes the interaction energy between atoms of types $\alpha$ and $\beta$ at sites $r$ and $r'$, which we assume are symmetric in $(r-r')$.

By completing the square, we may write this as

$$\begin{aligned} I_2(a) = & \tfrac{1}{4}\sum_{r,r'\in\Lambda} j(r-r')(a(r)-a(r'))^2 - \tfrac{1}{2}\sum_{r\in\Lambda}\bar{j}(a^2(r)-a(r)) \\ & -\tfrac{1}{2}\sum_{r\in\Lambda}\bar{d}a(r) - c, \end{aligned}$$

where $j(r) = j^{AA}(r) + j^{BB}(r) - 2j^{AB}(r)$, $\bar{d} = \sum_{r\in\Lambda}(j^{AA}(r) - j^{BB}(r))$, $\bar{j} = \sum_{r\in\Lambda} j(r)$ and $c = \sum_{r,r'\in\Lambda} j^{BB}(r-r')$.

Since we will be considering the gradient flow of the free energy, there is no loss in dropping $c$. Also, we are considering an extremely large lattice $\Lambda$ and will ignore boundary effects, so we take the $j's$ to be translation-invariant, which is why we get constants $\bar{j}$ and $\bar{d}$, above.

On the other hand, we will consider each identical $l(r)$ to be spatially of such small size that the $J's$ are independent of position. Then, if each small lattice has $N$ vertices, we may write $I_1$ as

$$I_1(a) = -\tfrac{1}{2}\sum_{r\in\Lambda} N^2\bar{J}(a^2(r) - a(r)) - \tfrac{1}{2}\sum_{r\in\Lambda} N^2 Da(r) - C,$$

where $\bar{J} = J^{AA} + J^{BB} - 2J^{AB}$, $D = J^{AA} - J^{BB}$, and $C = \frac{1}{2}J^{BB}N^2|\Lambda|$.

The entropy, $S(a)$, is given by

$$S(a) = \sum_{r\in\Lambda} s(a(r)),$$

where, for a block of $N$ vertices with A-occupancy fraction $\alpha = p/N$, $s(\alpha)$ is given by

$$\exp(Ns(\alpha)/k) = \frac{N!}{p!(N-p)!}.$$

The expression on the right side is the number of arrangements of $p$ atoms among $N$ sites, and $k$ is Boltzman's constant.

Following Bragg & Williams [8] we can use Stirling's formula to approximate the factorials and discard small terms ($N$ is very large even though each $l(r)$ is spatially very small) to obtain the approximation

$$s(\alpha) = -k[\alpha\log\alpha + (1-\alpha)\log(1-\alpha)]$$

and so

$$S(a) = -k\sum_{r\in\Lambda}[a(r)\log(a(r)) + (1-a(r))\log(1-a(r))].$$

Rearranging the terms which comprise (1.2) and dropping additive constants, we get

$$E(a) = \tfrac{1}{4}\sum_{r,r'\in\Lambda} j(r-r')(a(r)-a(r'))^2 + \sum_r \Big\{da(r) + Tk[a(r)\log(a(r)) + (1-a(r))\log(1-a(r))] - q(a^2(r)-a(r))\Big\}, \quad (1.3)$$

where

$$d = -\tfrac{1}{2}(\bar{d} + N^2D) \text{ and } q = \tfrac{1}{2}(\bar{j} + N^2\bar{J}).$$

We assume that $q > 0$, which is typical for ferromagnetic-like materials. Note that for $|d| < q$ and $Tk$ small enough, the last summand in (1.3) has two minima, which are of equal depth if $d = 0$.

Fix $T > 0$ sufficiently small so that this summand has two minima, at $a = a_1$ and $a = a_2$, say. Change variables, letting $u = -1 + 2(a - a_1)/(a_2 - a_1)$ so that (1.3) becomes

$$E(u) = \tfrac{1}{4}\sum_{r,r'\in\Lambda} j(r-r')(u(r)-u(r'))^2 + \sum_{r\in\Lambda} W(u(r)), \quad (1.4)$$

where the factor $(a_2-a_1)^2/4$ has been absorbed by redefining $j$ and $W$ has minima at $u=\pm 1$, not necessarily of equal depth.

Observe that even though we assume $q>0$ we do not always assume $j$ is everywhere positive. However, in much of our analysis, that will be the case.

From now on, for simplicity we assume that $\Lambda=\mathbb{Z}$. The $l_2$-gradient $\nabla E$ of $E$ is

$$\nabla E(u)_n = -(j*u)_n + \Big(\sum_m j(m)\Big)u_n + W'(u_n), \quad n\in\mathbb{Z}.$$

We assume that solutions evolve along curves of 'steepest descent' for $E$. These curves, at each point, lie in the direction of the negative gradient $-\nabla E(u)$. Thus the evolution can be represented by a gradient flow (see [13])

$$\dot{u}_n = -\nabla E(u)_n, \quad n\in\mathbb{Z}. \tag{1.5}$$

Set $J=\frac{j}{\sum j(m)}$, $f(u)=W'(u)$ and $\lambda=\frac{1}{\sum j(m)}$. With this normalization, the 'strength' of the interaction kernel $j$ is contained in the constant $\lambda$. Dividing by $\sum j(m)$ and rescaling time brings (1.5) into the form of (1.1). In general, the various physical forces which contribute to $J$ result in $J$ being not necessarily monotone or even positive on the set of positive integers.

One may observe that in the above derivation, the function

$$g(u) = u + \lambda f(u)$$

is monotone if we ignore short range interaction and set $\bar{J}=0$. However, in the analysis of (1.1) that follows, we do not require $g$ to be monotone, and in fact some results rely upon the nonmonotonicity of $g$.

We remark that in the continuum mean field approximation, (1.4) becomes

$$E(u) = \frac{1}{4}\int_{\mathbb{R}}\int_{\mathbb{R}} j(x-y)(u(x)-u(y))^2\,dx\,dy + \int_{\mathbb{R}} W(u(x))\,dx, \tag{1.6}$$

as was derived in [3]. An $L^2$-gradient flow of (1.6) is then

$$u_t = J*u - u - \lambda f(u), \tag{1.7}$$

where $J=\frac{j}{\int j}$ and $\lambda=\frac{1}{\int j}$. This equation has been studied recently in [5,15,9, 10] and [3].

Going one step further, one can change variables in (1.6), using $\eta=\frac{x-y}{2}$, $\xi=\frac{x+y}{2}$, and expand $u(x)=u(\xi+\eta)$ and $u(y)=u(\xi-\eta)$ about $\xi$, to get the formal expression

$$2\int_{\mathbb{R}}\int_{\mathbb{R}} J(2\eta)\Big(\sum_{k=0}^{\infty}\frac{D^{2k+1}u(\xi)\eta^{2k+1}}{(2k+1)!}\Big)^2\,d\xi\,d\eta + \int_{\mathbb{R}} W(u(x))\,dx. \tag{1.8}$$

Truncating the summation in (1.8) at $k=0$ gives, for $c=\int J(2\eta)\eta^2 d\eta$, an energy

$$c\int_{\mathbb{R}}|u'(x)|^2\,dx + \int_{\mathbb{R}} W(u(x))\,dx,$$

whose $L^2$-gradient flow is

$$u_t = cu_{xx} - f(u). \tag{1.9}$$

This is the familiar Allen-Cahn equation [2], often referred to as the bistable equation (e.g., [14]). Some results for higher order truncations have been obtained in [7] and [4].

A version of (1.1) with nearest-neighbor interactions, i.e.,

$$\dot{u}_n = \tfrac{1}{2}u_{n+1} + \tfrac{1}{2}u_{n-1} - u_n - \lambda f(u_n), \quad n \in \mathbb{Z}, \tag{1.10}$$

has been studied in other contexts. It is often referred to as a discrete Allen-Cahn or Nagumo equation. Many results specific to this equation may be found in [18, 22,23], [17,19,16,11,12] and [20]. In these papers, the authors investigate both traveling waves and so-called 'pinning' ('lack of propagation'), which occurs for large $\lambda$ even when the wells of $W$ are of unequal depth.

Pinning also occurs in the continuum convolution equation (1.7), as was discovered in [5,9,10] and [3]. However, it does not occur in the local bistable equation (1.9).

The paper is organized as follows:

In Section 2, under the assumption that $J$ is even and nonnegative, we construct traveling wave solutions. Our method is different than those in [22,17] and [11] which rely on the finite-range coupling in equation (1.10). Our approach is to consider a sequence of traveling wave solutions of (1.7) with varying $J$'s, which are smooth approximations of sums of delta functions. We show that these solutions have a limit, which is a traveling wave of (1.1). We then attempt to relate its speed to the general nonlinearity $f$. However, a 'propagation / no-propagation criterion' seems to be much harder to establish than that for the equation (1.7) [5]. One of the reasons is that a traveling wave of (1.7) is always a solution of an integrodifferential equation. However, a traveling wave of (1.1) is a solution of either a functional equation, or an infinite system of coupled ODE's, depending on whether its speed is nonzero or zero. Our 'criterion' is thus not complete, but nonetheless it is stronger than the corresponding results in [18,22] and [19]. In particular, we find that (1.1) admits 'less propagation' than (1.7). Next, we show uniqueness of traveling waves with nonzero speed, using a certain 'squeezing' technique. This method also enables us to obtain a stability result.

In Section 3, we adapt a technique from [3] to prove that for large $\lambda$, equation (1.1) admits pinning. We construct all stationary solutions and give precise criteria for their stability. We note that this result holds for multi-dimensional lattices and general $J$'s, including those which change sign.

## 2. Monotone traveling waves, in the case $J \geqq 0$

In this section, we study monotone traveling wave solutions of (1.1).

Define $J_\delta(x) = \sum_{|i|\geqq 1} J(i)\delta(x-i)$, where $J(i) = J(-i) \geqq 0$ for all $i \in \mathbb{Z}$, $\sum_{|i|\geqq 1} J(i) = 1$ and $\sum_{|i|\geqq 1} |i|J(i) < \infty$. We also assume that the support of

$J$ contains either $i = 1$ or two relatively prime integers, $i = p$ and $i = q$. We consider the equation

$$cu'(x) + (J_\delta * u)(x) - u(x) - \lambda f(u(x)) = 0, \quad x \in \mathbb{R}, \tag{2.1}$$

together with the boundary conditions

$$u(-\infty) = -1, \quad u(+\infty) = 1. \tag{2.2}$$

Here $f$ is bistable, with $f \in C^r(\mathbb{R})$, $r \geqq 1$, and $f$ has only three zeros, at $-1$, $1$ and $a \in (-1, 1)$. Clearly, solutions $(u(x), c)$ of (2.1), subject to (2.2), give traveling waves for (1.1), by setting $u_n(t) = u(n - ct)$.

Define $g(u) \equiv u + \lambda f(u)$. For simplicity, we make the following assumptions about $g$:

$g$ has at most three intervals of monotonicity, $[-1, \beta)$, $[\beta, \gamma]$ and $(\gamma, 1]$, for some $\beta \leqq \gamma$. Moreover,

$$g' > 0 \text{ on } [-1, \beta) \cup (\gamma, 1], \quad g' \leqq 0 \text{ on } (\beta, \gamma).$$

In the case $\beta < \gamma$, for any number

$$k \in K \equiv \{g(u) : u \in [-1, \beta]\} \cap \{g(u) : u \in [\gamma, 1]\}$$

define $g_k(u)$ to be the continuous nondecreasing function obtained by modifying $g$ to be the constant value $k$ between the ascending branches of $g$.

In the case $\beta = \gamma$, $k$ can be chosen to be any number in $[-1, 1]$, and $g_k(u) = g(u)$ for all $u$.

**Theorem 2.1 (**Existence of monotone traveling waves**).** *There exists a strictly monotone traveling wave solution* $u_n(t) = u_\delta(n - c_\delta t)$ *of* (1.1), *such that* $u_\delta(-\infty) = -1$ *and* $u_\delta(+\infty) = 1$. *Moreover:*

*A.* $\operatorname{sgn} c_\delta = \operatorname{sgn} \int_{-1}^{1} f(u)\,du$, *if* $c_\delta \neq 0$.

*B.* $c_\delta = 0$ if there exists $k$ such that $\int_{-1}^{1} g_k(u)\,du = 0$.

*C.* $c_\delta \neq 0$ *if* $\int_{-1}^{1} f(u)\,du \neq 0$ *and* $\lambda \leqq \lambda(f)$, *where* $\lambda(f)$ *is small enough.*

*D. In the case* $g$ *monotone,* $c_\delta < 0$ $(c_\delta > 0)$ *if there exists* $u^* \in (-1, 1)$ *such that* $\lambda f(u^*) < -1$ $(\lambda f(u^*) > 1)$.

*E. In the case* $g$ *nonmonotone,* $c_\delta < 0$ $(c_\delta > 0)$ *if there exists* $u^* \in (-1, \beta)$ $((\gamma, 1))$ *such that* $\lambda f(u^*) < -1$ $(\lambda f(u^*) > 1)$, *or if there exists* $u^* \in (\gamma, 1)$ $((-1, \beta))$ *such that* $\lambda f(u^*) < -1$ $(\lambda f(u^*) > 1)$ *and* $g(\beta) \leqq g(u^*)$ $(g(\gamma) \geqq g(u^*))$.

**Proof.** The idea is to 'approximate' (2.1) by

$$cu' + J_m * u - u - \lambda f(u) = 0, \tag{2.3}$$

where $\{J_m\}$ is a family of sums of delta sequences, such that

$$J_m * \phi \to J_\delta * \phi \text{ as } m \to \infty \tag{2.4}$$

uniformly on compact sets for each $\phi \in C_0^\infty(\mathbb{R})$. The kernel $J_m$ can be defined as follows:

Let $\psi$ be any smooth function, such that $\psi \geqq 0$ and $\int_\mathbb{R} \psi(x)dx = 1$. Then $\delta_m(x) \equiv m\psi(mx)$ is a delta sequence, i.e., $(\delta_m * \phi)(x) \to \phi(x)$ as $m \to \infty$, for $\phi \in C_0^\infty(\mathbb{R})$. For simplicity, we assume that $\psi$ is even and has compact support. Let

$$J_m(x) \equiv \sum_{1 \leqq |i| \leqq m} \frac{1}{w_m} J(i)\delta_m(x-i), \tag{2.5}$$

where $w_m \equiv \sum_{1\leqq|k|\leqq m} J(k)$. To show (2.4), let $\phi \in C_0^\infty(\mathbb{R})$. For any fixed $x$, there exists some $M_x$, such that $\delta_m(x-y-i) = 0$ and $\phi(x-i) = 0$ for all $m$, $|i| \geqq M_x$ and $y \in \operatorname{supp}\phi$, so that

$$\begin{aligned} J_m * \phi(x) &= \sum_{1\leqq|i|\leqq M} \frac{1}{w_m} J(i) \int_\mathbb{R} \delta_m(x-y-i)\phi(y)\,dy \\ &\to \sum_{1\leqq|i|\leqq M} J(i)\phi(x-i) = J_\delta * \phi(x) \quad \text{as } m \to \infty, \end{aligned}$$

where $M \equiv \min\{M_x, m\}$. Since $\phi'$ is bounded, it is easily seen that convergence is uniform on compact sets for each $\phi \in C_0^\infty(\mathbb{R})$.

We recall the result in [5], that for each $J_m$, there exists a (strictly) monotone traveling wave solution $u(x,t) = u_m(x - c_m t)$ of (1.7), such that $u_m(-\infty) = -1$, $u_m(+\infty) = 1$. Note that here $u_m$ is a function defined on $\mathbb{R}$ and should not be confused with the sequence $\{u_n\}$ appearing in (1.1) and elsewhere.

**Proposition 2.1.** *There exists a solution* $(u_m, c_m)$ *of* (2.3)*, such that* $u_m$ *is (strictly) monotone,* $u_m(-\infty) = -1$ *and* $u_m(+\infty) = 1$. *Moreover,*

$$c_m = 0 \text{ if, and only if, there exists } k \text{ such that } \int_{-1}^1 g_k(u)\,du = 0, \tag{2.6}$$

*and otherwise,* $\operatorname{sgn} c_m = \operatorname{sgn} \int_{-1}^1 f(u)\,du$.

For proof of this theorem, see [5].

The solutions $(u_m, c_m)$ are of course also weak solutions of (2.3), i.e., for any $\phi \in C_0^\infty(\mathbb{R})$ they satisfy

$$-c\int_\mathbb{R} u\phi' + \int_\mathbb{R} [J_m * u - u - \lambda f(u)]\phi = 0. \tag{2.7}$$

Consider first the case $c_m \geqq 0$. Take $\alpha \in (a, 1)$ and translate each $u_m$ so that $u_m(0) = \alpha$. By Helly's Theorem, there exists a subsequence of $u_m$, which we still denote by $u_m$, converging pointwise to a monotone function $u_\delta$ as $m \to \infty$. Moreover, the $c_m$'s are uniformly bounded, as can be seen from the following argument.

Assume to the contrary, that there is a sequence $c_m \to \infty$ as $m \to \infty$. From (2.3) we see that $|c_m u_m'|_\infty \leqq$ const, from which we get $|u_m'|_\infty \to 0$ as $m \to \infty$. This implies $u_\delta \equiv \alpha$, which gives a contradiction, since

$$\begin{aligned} 0 > \lambda f(\alpha) &= \lim_{m\to\infty} \lambda f(u_m) \geqq \lim_{m\to\infty} (J_m * u_m - u_m) \\ &= \lim_{m\to\infty} \int_{\mathbb{R}} J_m(x-y)(y-x)u_m'(p(x,y))\,dy = 0, \end{aligned}$$

for some function $p$, where the last limit follows from $|u_m'|_\infty \to 0$ as $m \to \infty$ (we can pass to the limit under the integral sign since $\int_{\mathbb{R}} |x| J_m(x)\,dx$ are uniformly bounded).

Thus by passing to another subsequence, we also have $c_m \to c_\delta$, for some $c_\delta$, as $m \to \infty$. We now show that $u_\delta$ solves (2.1) and that $u_\delta(\pm\infty) = \pm 1$. By passing to the limit $m \to \infty$ in (2.7), we note that $u_\delta$ is a weak solution of (2.1), i.e., it satisfies

$$-c_\delta \int_{\mathbb{R}} u\phi' + \int_{\mathbb{R}} [J_\delta * u - u - \lambda f(u)]\phi = 0 \tag{2.8}$$

for $\phi \in C_0^\infty(\mathbb{R})$. This follows from Lebesgue's Dominated Convergence Theorem and the limit

$$\int_{\mathbb{R}} (J_m * u_m)\phi = \int_{\mathbb{R}} (J_m * \phi)u_m \to \int_{\mathbb{R}} (J_\delta * \phi)u_\delta = \int_{\mathbb{R}} (J_\delta * u_\delta)\phi. \tag{2.9}$$

as $m \to \infty$. The first equality is obvious from Fubini's Theorem (recall that each $J_m$ is defined to be a finite sum). The limit follows from Lebesgue's Dominated Convergence Theorem, since $|u_m| \leqq 1$ and $\phi$ has compact support. To show the last equality we need the following lemma.

**Lemma 2.1 (**Dominated Convergence Theorem**).** *Let* $\{f_{i,k}\}$ *be a double sequence of summable functions (i.e.,* $\sum_{|i|\geqq 1} f_{i,k} < \infty$*), such that* $f_{i,k} \to f_i$ *as* $k \to \infty$ *for all* $|i| \geqq 1$. *If there exists a summable sequence* $\{g_i\}$ *such that* $|f_{i,k}| \leqq g_i$ *for all* $i, k$*'s, then*

$$\sum_{|i|\geqq 1} f_{i,k} \to \sum_{|i|\geqq 1} f_i \quad \text{as } k \to \infty.$$

The proof is similar to that of Lebesgue's Dominated Convergence Theorem.

Note that

$$\begin{aligned} \int_{\mathbb{R}} (J_\delta * \phi)u_\delta &= \lim_{k\to\infty} \int_{-k}^{k} \sum_{|i|\geqq 1} J(i)\phi(x+i)u_\delta(x)\,dx \\ &= \lim_{k\to\infty} \sum_{|i|\geqq 1} J(i) \int_{-k}^{k} \phi(x+i)u_\delta(x)\,dx, \end{aligned}$$

where the last equality follows because $\phi$ has compact support and thus the sum is finite. Since

$$\left| \int_{-k}^{k} \phi(x+i) u_\delta(x)\, dx \right| \leqq \int_{\mathbb{R}} |\phi(x+i)|\, dx = const,$$

we can use Lemma 2.1, the evenness of $J$ and Lebesgue's Dominated Convergence Theorem to conclude that

$$\begin{aligned} \int_{\mathbb{R}} (J_\delta * \phi) u_\delta &= \sum_{|i| \geqq 1} J(i) \int_{\mathbb{R}} \phi(x+i) u_\delta(x)\, dx \\ &= \sum_{|i| \geqq 1} J(i) \int_{\mathbb{R}} u_\delta(x+i) \phi(x)\, dx = \int_{\mathbb{R}} (J_\delta * u_\delta)\phi, \end{aligned}$$

which shows the last equality in (2.9).

If $c_\delta \neq 0$, then (2.8) implies that $u_\delta \in W^{1,\infty}(\mathbb{R})$. A bootstrap argument then shows that $u_\delta$ is $C^1(\mathbb{R})$ (and actually, $C^{r+1}(\mathbb{R})$) and thus a traveling wave solution of (1.1).

If $c_\delta = 0$, then $u_\delta$ need not be continuous, so $J_\delta * u_\delta(n)$ need not equal $\sum_{|i| \geqq 1} J(i) u_\delta(n-i)$. However, $u_\delta$ is monotone, and so the set of jump discontinuities is at most countable. Thus we can find a sequence $\{s_k\}$ such that $s_k \searrow 0$ as $k \to \infty$ and $u_\delta$ is continuous at $n + s_k$ for all $n \in \mathbb{Z}$ and $k > 0$. Equation (2.8) implies that

$$\begin{aligned} &J_\delta * u_\delta(n+s_k) - u_\delta(n+s_k) - \lambda f(u_\delta(n+s_k)) \\ &= \sum_{|i| \geqq 1} J(i) u_\delta(n + s_k - i) - u_\delta(n+s_k) - \lambda f(u_\delta(n+s_k)) = 0 \end{aligned}$$

for all $n \in \mathbb{Z}$ and $k > 0$. It is then easily seen that the sequence $u_\delta$ defined by

$$u_{\delta n} \equiv \lim_{k \to \infty} u_\delta(n + s_k), \quad n \in \mathbb{Z},$$

satisfies

$$\sum_{|i| \geqq 1} J(i) u_{\delta n-i} = u_{\delta n} - \lambda f(u_{\delta n}),$$

so is a stationary wave solution of (1.1).

We now show that $u_\delta(-\infty) = -1$ and $u_\delta(+\infty) = 1$. From the monotonicity of $u_\delta$ we easily see that $f(u_\delta(\pm\infty)) = 0$. Since $u_\delta(0) = \alpha$, we have $u_\delta(+\infty) = 1$ and $u_\delta(-\infty) \in \{a, -1\}$. If $u_\delta(-\infty) = -1$, we are done. So assume otherwise, that $u_\delta(-\infty) = a$. Then $f(u_\delta(x)) < 0$ on $\mathbb{R}$.

Consider first the case $c_\delta > 0$. Integrate equation (2.1) over $(-N, N)$ to get

$$c_\delta \int_{-N}^{N} u_\delta'(x)\, dx + \int_{-N}^{N} (J_\delta * u_\delta(x) - u_\delta(x))\, dx = \lambda \int_{-N}^{N} f(u_\delta(x))\, dx < 0.$$

To obtain a contradiction, we show that $\lim_{N\to\infty}\int_{-N}^{N}(J_\delta * u_\delta - u_\delta) = 0$. We have

$$\begin{aligned}
\int_{-N}^{N}(J_\delta * u_\delta - u_\delta) &= \int_{-N}^{N}\sum_{|i|\geqq 1}[J(i)(u_\delta(x-i)-u_\delta(x))]\,dx \\
&= -\sum_{|i|\geqq 1} J(i)\int_{-N}^{N}\int_0^1 u'_\delta(x-ti)i\,dt\,dx \\
&= -\sum_{|i|\geqq 1} iJ(i)\int_0^1 (u_\delta(N-ti)-u_\delta(-N-ti))\,dt \\
&\to -(1-a)\sum_{|i|\geqq 1} iJ(i) = 0 \text{ as } N\to\infty.
\end{aligned}$$

In the above calculation we used Lebesgue's Dominated Convergence Theorem, Fubini's Theorem, Lemma 2.1 and the evenness of $J$.

Next, assume that $c_\delta = 0$. Then, using an argument similar to the above, the sequence $\{u_\delta\}$ defined by $u_{\delta n} \equiv \lim_{k\to\infty} u_\delta(n+s_k)$, $n\in\mathbb{Z}$, is a stationary solution of (1.1), i.e.,

$$(J * u_\delta)_n - u_{\delta n} - \lambda f(u_{\delta n}) = 0,\ n\in\mathbb{Z}.$$

However, it is easily seen that

$$0 = \sum_{n\in\mathbb{Z}}((J*u_\delta)_n - u_{\delta n}) = \sum_{n\in\mathbb{Z}}\lambda f(u_{\delta n}) < 0,$$

a contradiction again.

Finally, in the case $c_m \leqq 0$, a similar argument is used taking $\alpha \in (-1, a)$.

To show strict monotonicity, we consider first the case $c_\delta = 0$. For simplicity of notation here, we drop the subscript $\delta$, i.e., we let $u \equiv u_\delta$.

We argue by contradiction. Assume that $u_{n_0+1} = u_{n_0}$ for some $n_0 \in \mathbb{Z}$. We then have

$$\sum_{|i|\geqq 1} J(i)(u_{n_0+1-i} - u_{n_0-i}) = 0,$$

i.e., $u_{n_0+1-i} = u_{n_0-i}$ for $i \in \operatorname{supp} J$. Since either $J(\pm 1) > 0$ or $J(p),\ J(q) > 0$ for some relatively prime integers $p$ and $q$, by induction it then follows that $u \equiv$ const, a contradiction.

Let $c \neq 0$. Suppose $u'(x_0) = 0$ for some $x_0$. Since $u'(x) \geqq 0$ for all $x\in\mathbb{R}$, $u''(x_0) = 0$. Therefore

$$0 = -cu''(x_0) = \sum_{|i|\geqq 1} J(i)u'(x_0 - i)$$

and $u'(x_0 - i) = 0$ for all $i \in \text{supp } J$. Since either $J(\pm 1) > 0$ or $J(p) > 0$ and $J(q) > 0$ with $p$ and $q$ relatively prime, we conclude that $u'(x_0 + n) = 0$ for all $n \in \mathbb{Z}$. Define $u_n(t) = u(n - ct)$, then $u_n$ satisfies the initial value problem

$$\dot{w}_n = \sum_{|i| \geqq 1} J(i) w_{n-i} - w_n - \lambda f(w_n)$$
$$w_n(-\frac{x_0}{c}) = u(n + x_0).$$

Since $u'(n + x_0) = 0$ for all $n \in \mathbb{Z}$, the constant $w_n(t) \equiv u(n + x_0)$ also solves (2), contradicting the uniqueness of solutions to (2).

**Remark.** It is only here where we use the assumption that either $1 \in \text{supp } J$ or $p, q \in \text{supp } J$. An interesting open question is then how to relax these restrictions on $J$ to preserve strict monotonicity and $u' > 0$ (for waves with nonzero speed).

To complete the proof, we now show A–E.

Suppose that $c_\delta \neq 0$. Multiply equation (2.1) by $u'_\delta(x)$ and integrate over $\mathbb{R}$. We get

$$c_\delta \int_{\mathbb{R}} u'_\delta(x)^2 \, dx + \int_{\mathbb{R}} (J_\delta * u_\delta - u_\delta) u'_\delta = \int_{-1}^{1} \lambda f(u) \, du. \qquad (2.10)$$

However,

$$\int_{\mathbb{R}} (J_\delta * u_\delta - u_\delta) u'_\delta = \sum_{|i| \geqq 1} J(i) \int_{\mathbb{R}} (u_\delta(x - i) - u_\delta(x)) u'_\delta(x) \, dx = 0,$$

where we used Lebesgue's Dominated Convergence Theorem, the evenness of $J$, and the equalities

$$\int_{\mathbb{R}} u_\delta(x - i) u'_\delta(x) \, dx = -\int_{\mathbb{R}} u_\delta(x + i) u'_\delta(x) \, dx$$

and $\int_{\mathbb{R}} u_\delta(x) u'_\delta(x) \, dx = 0$. From (2.10) we thus get $\text{sgn } c_\delta = \text{sgn } \int_{-1}^{1} f(u) \, du$ (it also follows that $\int_{-1}^{1} f(u) \, du = 0$ implies $c_\delta = 0$). This proves A.

If $c_\delta = 0$, then $u_\delta$ is not necessarily $C^1(\mathbb{R})$ (in fact, we conjecture that it may be a step function, constant on half-closed intervals of unit length), so this argument cannot be used anymore. However, from our construction of $(u_\delta, c_\delta)$ and Proposition 2.1, B easily follows.

To prove C, we argue by contradiction. Assume that $\int_{-1}^{1} f(u) \, du \neq 0$ and $c_\delta = 0$. Let $u^\lambda \equiv u_\delta$ be the stationary solution to (1.1), i.e.,

$$(J * u^\lambda)_n - u_n^\lambda = \lambda f(u_n^\lambda), \quad n \in \mathbb{Z}. \qquad (2.11)$$

Multiply (2.11) by $u^{\lambda}_{n+1} - u^{\lambda}_{n-1}$ and sum over $n \in \mathbb{Z}$, to get

$$\frac{1}{\lambda}\sum_{n\in\mathbb{Z}}\Big(\sum_{|i|\geqq 1} J(i)u^{\lambda}_{n-i} - u^{\lambda}_{n}\Big)(u^{\lambda}_{n+1} - u^{\lambda}_{n-1}) = \sum_{n\in\mathbb{Z}}[f(u^{\lambda}_{n})(u^{\lambda}_{n+1} - u^{\lambda}_{n}) + f(u^{\lambda}_{n})(u^{\lambda}_{n} - u^{\lambda}_{n-1})]. \tag{2.12}$$

First note that

$$\max_{n\in\mathbb{Z}}(u^{\lambda}_{n+1} - u^{\lambda}_{n}) \to 0 \text{ as } \lambda \to 0. \tag{2.13}$$

This follows from the following argument by contradiction. Assume there exists $\varepsilon > 0$ and a sequence $\{\lambda_k\}$ converging to 0 such that

$$\max_{n\in\mathbb{Z}}(u^{\lambda_k}_{n+1} - u^{\lambda_k}_{n}) = u^{\lambda_k}_{n_k+1} - u^{\lambda_k}_{n_k} \geqq \varepsilon \text{ as } k \to \infty.$$

Through translation, we can take $n_k = 0$. Since every solution $u^{\lambda_k}$ is monotone, by Helly's Theorem there exists a subsequence of $u^{\lambda_k}$ which converges to some monotone $u^0$. It is easily seen from (2.11) that $u^0$ satisfies

$$\sum_{|i|\geqq 1} J(i)(u^0_{n-i} - u^0_n) = 0 \tag{2.14}$$

for all $n \in \mathbb{Z}$ and that $u^0_1 - u^0_0 \geqq \varepsilon$. Let $J(0) = 0$, then (2.14) becomes

$$\sum_{i\in\mathbb{Z}} J(n-i)(u^0_i - u^0_n) = \sum_{i\in\mathbb{Z}} J(n-i)(u^0_{i+1} - u^0_{n+1}) = 0$$

for all $n \in \mathbb{Z}$. After subtraction, we get

$$\sum_{i\in\mathbb{Z}} J(n-i)(u^0_{i+1} - u^0_i) = u^0_{n+1} - u^0_n \tag{2.15}$$

for all $n \in \mathbb{Z}$. Since $-1 \leqq u^0_n \leqq 1$ for all $n \in \mathbb{Z}$, the number of integers $n$ for which $u^0_{n+1} - u^0_n \geqq \varepsilon$ is finite, which gives us a contradiction to (2.15).

Statement(2.13) implies that the right side of (2.12) is a Riemann sum, which tends to $2\int_{-1}^{1} f(u)\,du \neq 0$ as $\lambda \to 0$. However, the left side of (2.12) is 0 for all $\lambda$'s, as is seen from the following calculation. Let $J(0) = 0$. Since

$$\sum_{n\in\mathbb{Z}}\sum_{i\in\mathbb{Z}} |J(i)(u_{n-i} - u_n)(u_{n+1} - u_{n-1})| \leqq 2\sum_{n\in\mathbb{Z}}\sum_{i\in\mathbb{Z}} J(i)(u_{n+1} - u_{n-1}) = 8,$$

the double series $\sum_{n\in\mathbb{Z}}\sum_{i\in\mathbb{Z}} J(i)(u_{n-i} - u_n)(u_{n+1} - u_{n-1})$ is summable. By an equivalent of Fubini's Theorem for series, we can rearrange this series to get

$$\begin{aligned}\sum_{n\in\mathbb{Z}}\sum_{i\in\mathbb{Z}} J(i)(u_{n-i} - u_n)(u_{n+1} - u_{n-1}) &= \sum_{n\in\mathbb{Z}}\sum_{i\in\mathbb{Z}} J(n-i)(u_i - u_n)(u_{n+1} - u_{n-1}) \\ &= \sum_{i\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} J(n-i)(u_i - u_n)(u_{n+1} - u_{n-1}) \equiv S.\end{aligned}$$

In S (the last series) we interchange $i$ with $n$ and use the evenness of $J$ to get

$$S = \sum_{i\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} J(n-i)(u_n - u_i)(u_{i+1} - u_{i-1}).$$

Summing the last two series, we get

$$\begin{aligned} 2S = &- \sum_{i\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} J(n-i)(u_n - u_i)(u_{n+1} - u_{i+1}) \\ &+ \sum_{i\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} J(n-i)(u_n - u_i)(u_{n-1} - u_{i-1}) = 0, \end{aligned}$$

which gives us the desired contradiction.

To show D, we assume without loss of generality that there exists $u^* \in (-1, 1)$ such that $\lambda f(u^*) < -1$. Choose a small $\varepsilon > 0$, and define a monotone function $g_\varepsilon$ as follows:

$$g_\varepsilon(u) \begin{cases} = g(u), & \text{for } u \geqq u^*, \\ = g(u^*), & \text{for } -1+\varepsilon \leqq u \leqq u^*, \\ \geqq g(u) \text{ and is increasing,} & \text{for } -1 \leqq u \leqq -1+\varepsilon. \end{cases}$$

We first prove that the equation

$$\dot{u}_n = (J * u)_n - g_\varepsilon(u_n), \quad n \in \mathbb{Z}, \tag{2.16}$$

has a solution $u_\varepsilon(n - c_\varepsilon t)$, with $c_\varepsilon < 0$, and then show that this implies that $c_\delta < 0$ as well.

From the existence part, it clearly follows that (2.16) has a solution $u_\varepsilon(n - c_\varepsilon t)$. To show that $c_\varepsilon < 0$, we argue by contradiction. If $c_\varepsilon \neq 0$, then sgn $c_\varepsilon =$ sgn $\int_{-1}^{1} f_\varepsilon(u)\,du$, where $\lambda f_\varepsilon(u) \equiv g_\varepsilon(u) - u$. However, from the facts that there exists $u^* \in (-1, 1)$ such that $\lambda f_\varepsilon(u^*) < -1$ and $g_\varepsilon$ is monotone, we have $\int_{-1}^{1} f_\varepsilon(u)\,du < 0$, so $c_\varepsilon \neq 0$ implies that $c_\varepsilon < 0$. Now assume that $c_\varepsilon = 0$. Let $u_\varepsilon$ be the stationary solution of (2.16). For convenience, let us drop the subscript $\varepsilon$, i.e., we let $u \equiv u_\varepsilon$. Since $u_n$ is strictly increasing, so is the sequence $(J * u)_n$. Thus, $u_n \notin [-1+\varepsilon, u^*]$ for any $n \in \mathbb{Z}$. To see this, assume otherwise and let $n_0$ be such that $u_{n_0} \in [-1+\varepsilon, u^*]$. Then since $g_\varepsilon$ is constant on $[-1+\varepsilon, u^*]$,

$$g_\varepsilon(u^*) = g_\varepsilon(u_{n_0}) > \sum_{i\leqq -1} J(i)u^* + \sum_{i\geqq 1} J(i)(-1) = \tfrac{1}{2}u^* - \tfrac{1}{2}.$$

Recalling that $\lambda f_\varepsilon(u^*) < -1$, we then conclude $u^* > 1$, a contradiction.

Since $u$ is translationally invariant, we can assume that $u_0 < -1+\varepsilon$ and $u_1 > u^*$. This implies that

$$g_\varepsilon(u_0) < g_\varepsilon(u^*)$$

and

$$g_\varepsilon(u_1) > \sum_{i\leqq -1} J(i)u^* + \sum_{i\geqq 1} J(i)(-1) = \tfrac{1}{2}u^* - \tfrac{1}{2},$$

from which we get

$$\frac{\varepsilon}{2} + \frac{1}{2}(1-u^*) > \sum_{|i| \geqq 1} J(i)(u_{1-i} - u_{-i}) = g_\varepsilon(u_1) - g_\varepsilon(u_0)$$
$$> \frac{1}{2}u^* - g_\varepsilon(u^*) - \frac{1}{2} = -\lambda f_\varepsilon(u^*) - \frac{1}{2} - \frac{1}{2}u^*.$$

Hence,

$$\lambda f_\varepsilon(u^*) > -1 - \frac{\varepsilon}{2},$$

which, for small enough $\varepsilon$, contradicts the assumption $\lambda f(u^*) < -1$.

It is obvious from the above strict inequalities that $g_\varepsilon$ can be made $C^r(\mathbb{R})$, with $c_\varepsilon < 0$ preserved. In the following, we assume that $g_\varepsilon$ is $C^r(\mathbb{R})$.

To show that $c_\varepsilon < 0$ implies $c_\delta < 0$, we argue by contradiction. Assume that $c_\delta = 0$. Define

$$w_n^\varepsilon(t) = -u_\varepsilon(n - z - c_\varepsilon t - B(1 - e^{-\alpha t})) + \mu e^{-\alpha t} + u_{\delta n}$$

where $z$ is chosen so that

$$w_n^\varepsilon(0) = -u_\varepsilon(n+z) + \mu + u_{\delta n} > 0$$

and all other constants are chosen as in (2.18)-(2.22) below. Since $g_\varepsilon \geqq g$, a calculation similar to (2.24) below shows that $w_n^\varepsilon(t) > 0$ for all $n \in \mathbb{Z}$ and $t \geqq 0$. However, since $c_\varepsilon < 0$, we obtain a contradiction.

The proof of E is analogous to that of D, and we leave the details to the reader. □

**Remark.** Conclusion B of Theorem 2.1 is only a one-way implication while (2.6) gives an equivalence in the continuous case. Also, conclusions C-E do not exclude the possibility that B should also be an equivalence. Thus, an interesting question arises as to whether (2.6) holds for traveling wave solutions of (1.1) as well, i.e., does (1.1) admit the same 'amount' of propagation as (1.7)? The answer is no, as the following simple example shows.

Assume $g$ is strictly monotone, $\int_{-1}^{1} g(u)\,du = 0$ and let $u$ be a stationary wave solution of (1.1) with this $g$, as constructed in Theorem 2.1. Since $u$ is a strictly monotone sequence, we can redefine $g$ on an interval which is not in the range of $u$, in such a way that the 'new' $\bar{g}$ is still strictly monotone and $\int_{-1}^{1} \bar{g}(u)\,du \neq 0$. Note that $u$ is still a stationary wave solution for (1.1) with $\bar{g}$, there are no traveling wave solutions with nonzero speed by Theorem 2.2 below, and that the traveling wave solution for (1.7) with $\bar{g}$ has nonzero speed by (2.6).

This example shows that equation (1.1) admits 'less propagation' than equation (1.7).

We now study uniqueness of our solutions.

**Theorem 2.2 (**Uniqueness of traveling waves with nonzero speed**).** *Let* $(u_\delta, c_\delta)$ *be a solution to* (2.1) *and* (2.2)*, as given in Theorem 2.1, such that* $c_\delta \neq 0$. *Let* $(\hat{u}_\delta, \hat{c}_\delta)$ *be another solution to* (2.1) *and* (2.2)*. Then* $\hat{c}_\delta = c_\delta$ *and, up to a translation,* $\hat{u}_\delta = u_\delta$.

**Proof.** We begin the proof with the observation that if $(\hat{u}, \hat{c})$ is a solution to (2.1) and (2.2), then

$$|\hat{u}|_\infty \leqq 1. \tag{2.17}$$

Suppose otherwise, i.e., let $x_0$ be such that $\max_{x\in\mathbb{R}} |\hat{u}(x)| = |\hat{u}(x_0)| > 1$. Without loss of generality, let us assume that $\hat{u}(x_0) > 1$. We obviously have $\hat{u}'(x_0) = 0$, and so

$$0 \geqq J_\delta * \hat{u}(x_0) - \hat{u}(x_0) = \lambda f(\hat{u}(x_0)) > 0$$

gives a contradiction.

First we show that $\hat{c}_\delta = c_\delta$. We use the 'squeezing' technique from [1]. For convenience, we drop the subscript $\delta$.

Suppose that $c \neq 0$. Choose $\alpha, d, M > 0$ such that

$$\lambda f'(z) > \alpha \text{ when } |z \pm 1| < d, \tag{2.18}$$

$$|u(x) - 1| < \frac{d}{2} \text{ when } x \geqq M, \tag{2.19}$$

$$|u(x) + 1| < \frac{d}{2} \text{ when } x \leqq -M, \tag{2.20}$$

$$u'(x) > d \text{ when } |x| \leqq M. \tag{2.21}$$

Note that (2.21) is possible since in the proof of Theorem 2.1 we showed that $u'(x) > 0$ for all $x \in \mathbb{R}$.

Let $\mu \in (0, \frac{d}{2})$ and define

$$B = \frac{\mu}{\alpha d}\Big[\alpha - \min_{z\in[-1,1]} \lambda f'(z)\Big]. \tag{2.22}$$

First, consider the case where $c > \hat{c}$ and define

$$w(x,t) = u(x + z + (\hat{c} - c)t + B(1 - e^{-\alpha t})) + \mu e^{-\alpha t} - \hat{u}(x), \tag{2.23}$$

where by (2.17) $z$ can be chosen so that

$$w(x, 0) = u(x + z) + \mu - \hat{u}(x) > 0.$$

We claim that $w(x,t) > 0$ for all $x \in \mathbb{R}$ and $t \geqq 0$. To see this, suppose that there exists $(x_0, t_0)$ such that

$$w(x_0, t_0) = 0 \leqq w(x,t) \text{ for } x \in \mathbb{R} \quad \text{and} \quad 0 \leqq t \leqq t_0.$$

From (2.23) we see that $w_t(x_0, t_0)$ exists and is nonpositive and that if $\hat{c} \neq 0$ (so that $\hat{u}$ is $C^{r+1}$), $w_x(x_0, t_0) = 0$. Furthermore, $J_\delta * w(x_0, t_0) \leqq 0$. Define

$$P(x,t) \equiv x + z + (\hat{c} - c)t + B(1 - e^{-\alpha t}).$$

Using (2.1), we have at $(x_0, t_0)$,

$$\begin{aligned} 0 \geqq\ & w_t - J_\delta * w \\ =\ & (\hat{c} - c + B\alpha e^{-\alpha t_0})u'(P) - \alpha\mu e^{-\alpha t_0} - J_\delta * u(P) - \mu e^{-\alpha t_0} + J_\delta * \hat{u}(x_0) \\ =\ & -u(P) - \lambda f(u(P)) + (\hat{c} + B\alpha e^{-\alpha t_0})u'(P) - \alpha\mu e^{-\alpha t_0} - \mu e^{-\alpha t_0} \\ & - \hat{c}\hat{u}'(x_0) + \hat{u}(x_0) + \lambda f(\hat{u}(x_0)) \\ =\ & -\lambda f(u(P)) + B\alpha e^{-\alpha t_0}u'(P) - \alpha\mu e^{-\alpha t_0} + \lambda f(u(P) + \mu e^{-\alpha t_0}) \\ =\ & B\alpha e^{-\alpha t_0}u'(P) - \alpha\mu e^{-\alpha t_0} + \lambda f'(z_0)\mu e^{-\alpha t_0} \end{aligned} \tag{2.24}$$

for some $z_0 \in (u(P), u(P) + \mu e^{-\alpha t_0}) \subset [-1, 1]$.

If $|P(x_0, t_0)| \leqq M$ then by (2.21) the right hand side of (2.24) is strictly greater than

$$e^{-\alpha t_0}[B\alpha d - \alpha\mu + \mu\lambda f'(z_0)]$$

which is nonnegative by (2.22), contradicting the inequality on the left side of (2.24).

If $|P(x_0, t_0)| \geqq M$, then $|u(P(x_0, t_0)) - 1| < \frac{d}{2}$ or $|u(P(x_0, t_0)) + 1| < \frac{d}{2}$ by (2.19) and (2.20), so the choice of $\mu$ implies

$$|z_0 - 1| < d \text{ or } |z_0 + 1| < d.$$

Therefore, $\lambda f'(z_0) > \alpha$ by (2.18). Since $u'(P) > 0$ we see that the right hand side of (2.24) is positive in this case, also giving a contradiction and establishing the claim that $w(x, t) > 0$ for all $x \in \mathbb{R}$ and $t \geqq 0$.

If $\hat{c} = 0$, assume that $\hat{u}_n$, $n \in \mathbb{Z}$, is the corresponding stationary wave solution. Let

$$w_n(t) = u(n + z - ct + B(1 - e^{-\alpha t})) + \mu e^{-\alpha t} - \hat{u}_n, \quad n \in \mathbb{Z}.$$

We then use the same estimates as in (2.24) to get $w_n(t) > 0$ for all $n \in \mathbb{Z}$ and $t \geqq 0$, the only difference being that instead of $J_\delta * w$ we now have to use the operator $(J * w)_n$, $n \in \mathbb{Z}$, as defined before.

Now fix $\bar{x}$ such that $\hat{u}(\bar{x}) > -1$ and use the observation that $u(P(\bar{x}, t)) \to -1$ as $t \to \infty$ (because $c > \hat{c}$) to contradict the positivity of $w$.

In the case $c < \hat{c}$, define

$$w(x, t) = -u(x - z + (\hat{c} - c)t - B(1 - e^{\alpha t})) + \mu e^{-\alpha t} + \hat{u}(x),$$

where $z$ is chosen so that $w(x, 0) > 0$. The same analysis as before leads to a contradiction in this case too, proving the uniqueness of $c$.

**Remark.** In the above proof we replaced $\lambda f(u(P) + \mu e^{-\alpha t_0}) - \lambda f(u(P))$ by $\lambda f'(z_0)\mu e^{-\alpha t_0}$ for some $z_0 \in (u(P), u(P) + \mu e^{-\alpha t_0})$ and had chosen $\mu$ small so that $z_0$ lies in an interval close to 1 and $-1$, thereby ensuring $\lambda f'(z_0) > \alpha$. We could instead first choose $\mu > 0$ with $\mu < \mu_0 \equiv \min\{1 - a, a + 1\}$ ($a$ is the middle zero of $f$). Now choose $\alpha, d > 0$ so that for $u \in (1 - d, 1)$ we have

$\lambda f(u) - \lambda f(u - z) > \alpha z$ and for $u \in (-1, -1 + d)$, $\lambda f(u + z) - \lambda f(u) > \alpha z$ for all $z \in (0, \mu)$. Such choices are clearly possible since $f$ is bistable with zeros only at $-1$, $a$ and 1. Now $M > 0$ is chosen so that $||u(x)| - 1| < d$ when $|x| \geqq M$ and $u'(x) > d$ when $|x| \leqq M$. The point of this is that $\mu$ need not be small. In fact, we can usually do better than the bound $\mu_0$ when we are considering one-sided estimates. For instance, if we are squeezing from above we can take any $\mu < a + 1$.

These considerations, similar to the proof of the previous lemma yield a proof for the following 'stability' result.

Let $u_n(t),\ n \in \mathbb{Z}$, be the solution to the initial value problem

$$\dot{u}_n = (J * u)_n - u_n - \lambda f(u_n),\ n \in \mathbb{Z},$$
$$u_n(0) = u_n^0.$$

**Proposition 2.2.** *Let* $(u_\delta, c_\delta)$ *be the solution given by Theorem 2.1, and let* $c_\delta \neq 0$. *Assume* $-2 + a < u_n^0 < 2 + a$ *for all* $n \in \mathbb{Z}$, $\liminf_{n\to\infty} u_n^0 > a$ *and* $\limsup_{n\to-\infty} u_n^0 < a$. *Then there exist constants* $s_1$, $s_2$ *and* $\mu_1, \mu_2, \alpha > 0$, *such that*

$$u_\delta(n - s_1 - c_\delta t) - \mu_1 e^{-\alpha t} \leqq u_n(t) \leqq u_\delta(n - s_2 - c_\delta t) + \mu_2 e^{-\alpha t}$$

*for all* $n \in \mathbb{Z}$ *and* $t > 0$.

We now return to the proof of Theorem 2.2, and show that, up to a translation, $\hat{u} = u$.

The same analysis which yielded $w > 0$ for $w$ defined in (2.23) can be carried out for $\hat{c} = c$. Taking the limit $t \to \infty$, we get

$$u(x + z + B) \geqq \hat{u}(x) \text{ for all } x \in \mathbb{R}.$$

Thus there exists a minimal $\bar{z}$ such that

$$u(x) \geqq \hat{u}(x - z) \text{ for all } z > \bar{z} \text{ for all } x \in \mathbb{R}.$$

Note that if $u(x) \not\equiv \hat{u}(x - \bar{z})$ then $u(x) > \hat{u}(x - \bar{z})$. Suppose otherwise, i.e., that for some $x_0$, $u(x_0) = \hat{u}(x_0 - \bar{z})$. Let $w(x) = u(x) - \hat{u}(x - \bar{z})$. Then at $x = x_0$ we have

$$0 \leqq J_\delta * w = -cw'(x_0) + w(x_0) + \lambda f(u(x_0)) - \lambda f(\bar{u}(x_0 - \bar{z})) = 0,$$

therefore $w(x_0 - i) = 0$ for all $i \in \mathbb{Z}$. $v_n(t) \equiv w(x_0 + n - ct)$, $n \in \mathbb{Z}$ satisfies

$$\dot{v}_n = (J * v)_n - v_n - q_n(t)v_n,$$
$$v_n(0) = 0 \tag{2.25}$$

for some function $q_n(t)$, by the Mean Value Theorem. But (2.25) has a unique solution, namely $v_n(t) \equiv 0$, and hence $w \equiv 0$, a contradiction.

For $\eta > 0$ define

$$z(\eta) = \inf\{z : u(x) \geqq \hat{u}(x - z) - \eta \text{ for all } x \in \mathbb{R}\}.$$

Note that $z(\eta) < \bar{z}$ since $u'$ is bounded and $\lim_{\eta \to 0} z(\eta) = \bar{z}$ by minimality of $\bar{z}$.

Fix $N > 0$. We claim that there exists $\eta_N > 0$ such that for all $\eta \in (0, \eta_N]$

$$u(x) > \hat{u}(x - z(\eta)) - \eta \text{ for } |x| \leqq N. \tag{2.26}$$

If not, there exist $\eta_n \searrow 0,\ x_n \to x_0 \in [-N, N]$ with

$$u(x_n) = \hat{u}(x - z(\eta_n)) - \eta_n.$$

Taking the limit as $n \to \infty$ then gives $u(x_0) = \hat{u}(x_0 - \bar{z})$, a contradiction to our previously established assertion.

Let

$$\hat{w}(x, t) \equiv u(x) - \hat{u}(x - (\bar{z} - \varepsilon)) + \mu e^{-\alpha t},$$

where $\mu < \eta_M$, $M$ is from (2.19)–(2.21), $\alpha$ is as in (2.18) and $\varepsilon > 0$ is taken so that $2\varepsilon < \bar{z} - z(\eta)$. Then $\bar{w}(x, 0) > 0$, and if for some $t_0 > 0$ and $x_0 \in \mathbb{R}$, $\hat{w}(x_0, t_0) = 0 < \hat{w}(x, t)$ for all $t < t_0$ and $x \in \mathbb{R}$, then at $(x_0, t_0)$

$$\begin{aligned} 0 &\geqq \hat{w}_t - J_\delta * \hat{w} - \hat{w} - \hat{c}\hat{w}_x \\ &= -\alpha\mu e^{-\alpha t} + \lambda f(\hat{u}) - \lambda f(u) = [\lambda f'(p) - \alpha]\mu e^{-\alpha t} \end{aligned} \tag{2.27}$$

for some $p \in (u(x_0), u(x_0) + \mu e^{-\alpha t_0})$. Since $u(x_0) = \hat{u}(x_0 - (\bar{z} - \varepsilon)) - \mu e^{-\alpha t_0}$, it follows that $z(\mu e^{-\alpha t_0}) = \bar{z} - \varepsilon$, and because $\mu e^{-\alpha t_0} < \eta_M$, (2.26) implies that $|x_0| > M$ and hence $||p| - 1| \leqq d$, by (2.19) and (2.20). Consequently, $\lambda f'(p) - \alpha > 0$ by (2.18), contradicting (2.27).

Thus $\hat{w}(x, t) > 0$ for all $t > 0$ and $x \in \mathbb{R}$. Taking the limit as $t \to \infty$ gives

$$u(x) \geqq \hat{u}(x - (\bar{z} - \varepsilon)) \text{ for all } x \in \mathbb{R},$$

contradicting the minimality of $\bar{z}$ and proving that $u \equiv \hat{u}$. □

## 3. Stationary solutions, in the case $\lambda$ large, general $J$

In this section, we construct stationary solutions to (1.1), i.e., solutions to the equation

$$(J * u)_n - u_n - \lambda f(u_n) = 0, \tag{3.1}$$

where now $n \in \mathbb{Z}^d$, $d \geqq 1$ and $(J * u)_n \equiv \sum_{|i| \neq 0} J(i) u_{n-i}$. Compared to Section 2, we relax the assumptions on $J$. First, we assume that $f$ has at least linear growth outside $[-1, 1]$. Let

$$\begin{aligned} \lambda f(u) &\leqq r(u + 1) \text{ for } u \leqq -1, \\ \lambda f(u) &\geqq r(u - 1) \text{ for } u \geqq 1 \end{aligned} \tag{3.2}$$

for some $r > 0$. Define

$$P \equiv \{i \in \mathbb{Z}^d \backslash \{0\} : \ J(i) > 0\},$$
$$N \equiv \{i \in \mathbb{Z}^d \backslash \{0\} : \ J(i) < 0\}.$$

We now allow $J$ to change sign, but with the restriction that

$$r > 2 \sum_{i \in N} |J(i)|. \tag{3.3}$$

Note that we do not require $J$ to be even or have a finite first moment. First we show that (3.2) and (3.3) imply some 'a priori' bounds for solutions of (3.1). These estimates will be needed in a later argument.

**Proposition 3.1.** *Assume* (3.3) *holds. Then any solution $u$ to* (3.1) *satisfies*

$$\frac{-r}{r - 2\sum_{i \in N} |J(i)|} \leqq u_n \leqq \frac{r}{r - 2\sum_{i \in N} |J(i)|} \text{ for all } n \in \mathbb{Z}^d. \tag{3.4}$$

**Proof.** Set $M_1 \equiv \inf_{n \in \mathbb{Z}^d} u_n$, $M_2 \equiv \sup_{n \in \mathbb{Z}^d} u_n$. Let $\{n_k^1\}$ be a sequence in $\mathbb{Z}^d$ such that $u_{n_k^1} \to M_1$ as $k \to \infty$, and $\{n_k^2\}$ a sequence such that $u_{n_k^2} \to M_2$ as $k \to \infty$. If $M_1$ or $M_2$ are achieved at some points $n^1$ or $n^2 \in \mathbb{Z}^d$, then the corresponding sequence $\{n_k^1\}$ or $\{n_k^2\}$ is defined as $n_k^1 \equiv n^1$ or $n_k^2 \equiv n^2$. We have

$$\begin{aligned} &\lambda f(u_{n_k^2}) - M_2 \sum_{i \in P} J(i) - M_1 \sum_{i \in N} J(i) \\ &= \sum_{i \in P} J(i)(u_{n_k^2 - i} - M_2) + \sum_{i \in N} J(i)(u_{n_k^2 - i} - M_1) - u_{n_k^2} \leqq -u_{n_k^2}. \end{aligned}$$

Passing to the limit as $k \to \infty$, we get

$$\lambda f(M_2) \leqq (M_2 - M_1) \sum_{i \in N} |J(i)|. \tag{3.5}$$

A similar argument shows that

$$\lambda f(M_1) \geqq (M_1 - M_2) \sum_{i \in N} |J(i)|. \tag{3.6}$$

If $M_1 \geqq -1$ and $M_2 \leqq 1$, then obviously (3.4) is satisfied. Let us assume otherwise.

First, suppose that $M_1 < -1$ and $M_2 > 1$. Applying (3.2) to (3.5) and (3.6), we get

$$r(M_2 - 1) \leqq (M_2 - M_1) \sum_{i \in N} |J(i)| \tag{3.7}$$

and

$$r(M_1 + 1) \geqq (M_1 - M_2) \sum_{i \in N} |J(i)|. \tag{3.8}$$

From (3.8) and (3.3) we obtain

$$-M_1 \leqq \frac{r + M_2 \sum_{i\in N} |J(i)|}{r - \sum_{i\in N} |J(i)|}.$$

Substituting this into (3.7), we get

$$M_2\Big(r - \sum_{i\in N} |J(i)| - \frac{(\sum_{i\in N} |J(i)|)^2}{r - \sum_{i\in N} |J(i)|}\Big) \leqq r + \frac{r \sum_{i\in N} |J(i)|}{r - \sum_{i\in N} |J(i)|}.$$

Thus, because of (3.3), we have

$$M_2 \leqq \frac{r}{r - 2\sum_{i\in N} |J(i)|}$$

and

$$M_1 \geqq \frac{-r}{r - 2\sum_{i\in N} |J(i)|}.$$

Finally, if $M_2 \leqq 1$ and $M_1 < -1$, (3.8) and (3.3) imply that

$$M_1 \geqq \frac{-r - \sum_{i\in N} |J(i)|}{r - \sum_{i\in N} |J(i)|} \geqq \frac{-r}{r - 2\sum_{i\in N} |J(i)|}.$$

A similar argument shows that $M_2 > 1$ and $M_1 \geqq -1$ implies

$$M_2 \leqq \frac{r}{r - 2\sum_{i\in N} |J(i)|},$$

which completes the proof. □

**Remark.** Note that if $J \geqq 0$ and $u$ is a nonconstant solution of (3.1), then (3.4) implies that

$$-1 \leqq u_n \leqq 1 \quad \text{for all} \quad n \in \mathbb{Z}^d,$$

but one can then see from (3.1) that $-1 < u_n < 1$ for all $n \in \mathbb{Z}^d$ for such nonconstant solutions.

Define

$$b \equiv \frac{r}{r - 2\sum_{i\in N} |J(i)|} \sum_{i\neq 0} |J(i)|.$$

Then, obviously (3.4) implies that $|J * u|_\infty \leqq b$. We now assume that $\lambda$ is large enough that

$$|1 + \lambda f'(u)| > \sum_{i\neq 0} |J(i)| \text{ whenever } |u + \lambda f(u)| \leqq b. \tag{3.9}$$

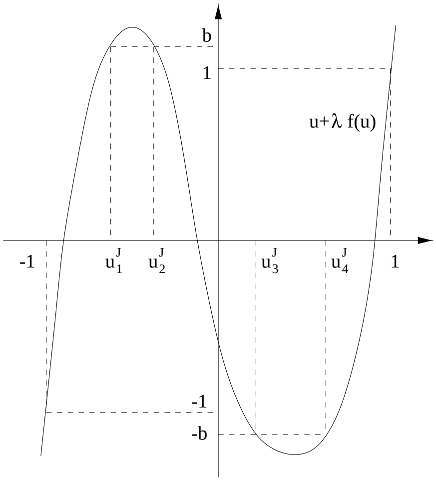


**Fig. 3.1.** $g(u) = u + \lambda f(u)$.

Let $u_1^J$, $u_2^J$ be the zeros of $u + \lambda f(u) - b$ such that $-1 < u_1^J < u_2^J < 1$ and $u_3^J$, $u_4^J$ the two zeros of $u + \lambda f(u) + b$ such that $-1 < u_3^J < u_4^J < 1$ (see Figure 3.1). Define

$$
\begin{aligned}
I_1^J &\equiv \left[\frac{-r}{r - 2\sum_{i\in N}|J(i)|}, u_1^J\right],\\
I_2^J &\equiv \left[u_2^J, u_3^J\right],\\
I_3^J &\equiv \left[u_4^J, \frac{r}{r - 2\sum_{i\in N}|J(i)|}\right].
\end{aligned}
$$

We note that (3.9) and (3.4) enable us to improve the 'a priori' bounds (3.4) for a solution of (3.1).

**Proposition 3.2.** *Assume* (3.3) *and* (3.9) *hold. Then any solution $u$ to* (3.1) *satisfies*

$$u_n \in I_1^J \cup I_2^J \cup I_3^J \text{ for all } n \in \mathbb{Z}^d. \tag{3.10}$$

The proof easily follows from the definition of $b$.

We now state our theorem.

**Theorem 3.1 (**Existence of stationary solutions**).** *Let $\lambda_*$ be the infimum of $\lambda$'s for which* (3.9) *holds. Fix $\lambda \geqq \lambda_*$. All solutions of* (3.1) *can be characterized as follows.*

*Let $S_1$ and $S_2$ be any two disjoint nontrivial subsets of $\mathbb{Z}^d$. Then there exists a solution $\hat{u}$ of* (3.1)*, such that $\hat{u}_n \in I_1^J$ for $n \in S_1$, $\hat{u}_n \in I_2^J$ for $n \in S_2$ and $\hat{u}_n \in I_3^J$ for $n \in \mathbb{Z}^d \backslash (S_1 \cup S_2)$. $\hat{u}$ is the unique such solution, if* (3.9) *holds.*

**Proof.** Let $S_1$ and $S_2$ be any two nontrivial disjoint subsets of $\mathbb{Z}^d$. Define

$$F(u, \tfrac{1}{\lambda})_n = \tfrac{1}{\lambda}[(J * u)_n - u_n] - f(u_n), \quad n \in \mathbb{Z}^d$$

and

$$u_n^0 = \begin{cases} -1, & n \in S_1, \\ a, & n \in S_2, \\ 1, & n \in \mathbb{Z}^d \backslash (S_1 \cup S_2). \end{cases} \tag{3.11}$$

Note that $F(u^0, 0)_n = 0$, $n \in \mathbb{Z}^d$. Let $L^0$ be the Frechet derivative of $F$ at $(u^0, 0)$, i.e., $L^0 v = \frac{\partial F}{\partial u}(u^0, 0)$. It is easily seen that

$$(L^0 v)_n = -f'(u_n^0) v_n, \ n \in \mathbb{Z}^d. \tag{3.12}$$

Since $L^0$ is invertible in $l^\infty(\mathbb{Z}^d)$, by the Implicit Function Theorem, there is some $\frac{1}{\lambda_0} > 0$ such that there exists a locally unique solution $u^\lambda$ of $F(u, \frac{1}{\lambda}) = 0$ for $\lambda \geqq \lambda_0$. We continue this solution of (3.1), with $\lambda \geqq \lambda_0$, to the interval $\lambda \geqq \lambda_*$, in the following way.

By a change of notation, $F(u, \frac{1}{\lambda}) = 0$ is equivalent to

$$G(u, \lambda)_n \equiv (J * u)_n - u_n - \lambda f(u_n) = 0, \quad n \in \mathbb{Z}^d. \tag{3.13}$$

Clearly, $G$ is $C^1$ on $l^\infty(\mathbb{Z}^d) \times \mathbb{R}$. When $\lambda = \lambda_0$, (3.13) has the solution $u^{\lambda_0}$. We use the Implicit Function Theorem to obtain the same conclusion for all $\lambda \in [\lambda_*, \lambda_0]$.

Let $u^\lambda$ be a solution of $G(u, \lambda) = 0$, and $L_\lambda$ be the linear operator defined in $l^\infty(\mathbb{Z}^d)$ by

$$(L_\lambda v)_n = (J * v)_n - [1 + \lambda f'(u_n^\lambda)] v_n, \quad n \in \mathbb{Z}^d. \tag{3.14}$$

Note that $L_\lambda \equiv \frac{\partial G}{\partial u}(u^\lambda, \lambda)$.

Let $\lambda_1 \in (\lambda_*, \lambda_0]$ be such that a solution, $u^{\lambda_1}$, exists to the equation $G(u, \lambda_1) = 0$. First, we show that there exists $\varepsilon > 0$ such that for $\lambda \in (\lambda_1 - \varepsilon, \lambda_1]$, (3.13) has a solution.

By the Implicit Function Theorem, it suffices to show that $L_{\lambda_1}$ is invertible. Recall the notation $g(u) \equiv u + \lambda f(u)$. The defining equation (3.14) can be rewritten as

$$(L_{\lambda_1} v)_n = g'(u_n^{\lambda_1}) \Big[ \frac{1}{g'(u_n^{\lambda_1})} (J * v)_n - v_n \Big], \quad n \in \mathbb{Z}^d.$$

Since (3.10) implies that $|g'(u_n^{\lambda_1})| > \sum_{|i| \neq 0} |J(i)|$, it follows that $L_{\lambda_1}$ is invertible.

To show that we can continue the solution branch to $\lambda \in [\lambda_*, \lambda_0]$, we argue by contradiction. Suppose that there is some $\bar{\lambda} \geqq \lambda_*$ such that a solution exists for $\lambda \in (\bar{\lambda}, \lambda_0]$, but not for $\lambda = \bar{\lambda}$. Choose a sequence $\lambda_k \to \bar{\lambda}$, as $k \to \infty$. By a diagonal argument, there exists a subsequence, which we also denote by $\lambda_k$, such that $u_n^{\lambda_k} \to u_n^{\bar{\lambda}}$ for each $n \in \mathbb{Z}^d$, as $k \to \infty$. Continuity and the Dominated Convergence Theorem imply that $u^{\bar{\lambda}}$ is a solution of $G(u, \bar{\lambda}) = 0$. This completes the existence proof.

To show uniqueness when (3.9) holds, assume that there are two distinct solutions $u^1$ and $u^2$ of (3.1), such that $u_n^1, u_n^2 \in I_1^J$ for $n \in S_1$, $u_n^1, u_n^2 \in I_2^J$ for $n \in S_2$ and $u_n^1, u_n^2 \in I_3^J$ for $n \in \mathbb{Z}^d \backslash (S_1 \cup S_2)$. Then

$$|u^1 - u^2|_\infty \leqq |g_i^{-1}(J * u^1) - g_i^{-1}(J * u^2)|_\infty \leqq k|u^1 - u^2|_\infty,$$

where $g_i^{-1}$, $i = 1, 2, 3$, is defined to be one of the three branches of $g^{-1}$ and $k < 1$ by (3.9). Thus $u_n^1 = u_n^2$ for all $n \in \mathbb{Z}^d$. □

We now provide a stability theorem for the solutions constructed in Theorem 3.1.

**Theorem 3.2 (**Stability of stationary solutions**).** *Let $\hat{u}$ be a solution of* (3.1), *with $\lambda$ such that* (3.9) *holds. Then*

1. *If $\hat{u}_n \in I_1^J \cup I_3^J$ for all $n \in \mathbb{Z}^d$, then $\hat{u}$ is (locally) exponentially stable in the $l^\infty(\mathbb{Z}^d)$ norm.*
2. *If $\hat{u}_n \in I_2^J$ for $n \in S$, where $S$ is a nontrivial subset of $\mathbb{Z}^d$, then $\hat{u}$ is unstable in the $l^\infty(\mathbb{Z}^d)$ norm.*

**Proof.** We investigate $\sigma(L_\lambda)$, the spectrum of $L_\lambda$, the operator defined in (3.14). Note that

$$((L_\lambda - \mu)v)_n = (J * v)_n - [g'(\hat{u}_n) + \mu]v_n,\ n \in \mathbb{Z}^d,$$

is invertible for

$$\mu \in \cap_{p \in G}\{z : |z + p| > \sum_{i \neq 0} |J(i)|\},$$

where $G \equiv \{g'(\hat{u}_n) : n \in \mathbb{Z}^d\}$, since

$$((L_\lambda - \mu)v)_n = (g'(\hat{u}_n) + \mu)\left[\frac{1}{g'(\hat{u}_n)+\mu}(J * v)_n - v_n\right],\ n \in \mathbb{Z}^d.$$

Thus,

$$\sigma(L_\lambda) \subset \cup_{p \in G}\left\{z : |z + p| \leqq \sum_{i \neq 0} |J(i)|\right\}. \tag{3.15}$$

If $\hat{u}_n \in I_1^J \cup I_3^J$ for all $n \in \mathbb{Z}^d$, then $\sigma(L_\lambda)$ lies in the left-half plane, thus by [6], $\hat{u}$ is (locally) exponentially asymptotically stable, which proves 1.

Assume on the other hand, that $\hat{u}_n \in I_2^J$ for $n \in S$, where $S$ is a nontrivial subset of $\mathbb{Z}^d$. From the construction in Theorem 3.1, this solution is continued from $u^0$ given by (3.11). Note that

$$\sigma(L^0) \subset \{-f'(-1), -f'(a), -f'(1)\},\ \text{and in particular } -f'(a) \in \sigma(L^0),$$

where $L^0$ is given by (3.12). Each of the points in $\sigma(L^0)$ is an eigenvalue of infinite multiplicity. Since our continuation is a $C^1$ deformation, by (3.15), $\sigma(L_\lambda)$ does not intersect the imaginary axis and $\sigma(L_\lambda)$ contains values in the right-half plane. Thus from [6] we conclude that $\hat{u}$ is unstable, which proves 2. □

**Remark.** In this section we considered stationary solutions on the lattice $\mathbb{Z}^d$. We could have easily considered a general multi-dimensional lattice $\Lambda$, interpreting $(J * u)_p$ appropriately to give the weighted average of values of $u$ in a neighborhood centered at $p \in \Lambda$.

*Acknowledgements.* P. W. BATES was supported in part by NSF grant DMS 9622785. A. CHMAJ was supported by a Postdoctoral Fellowship in the Department of Mathematics at Brigham Young University.

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(*Accepted March 29, 1999*)