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Gas Dynamics in Thermal Nonequilibrium and General Hyperbolic Systems with Relaxation

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Abstract

We study gas flow in vibrational nonequilibrium. The model is a 4×4 nonlinear hyperbolic system with relaxation. Under physical assumptions, properties of thermodynamic variables relevant to stability are obtained, global existence for Cauchy problems with smooth and small data is established, and large time behavior is studied in the pointwise sense. We formulate the fundamental solution in a systematic way for a general linear system with relaxation. The fundamental solution provides insights to the behavior of the nonlinear system, and is crucial to obtain our pointwise asymptotic picture for the nonequilibrium flow. We also clarify in a general setting the relation between subcharacteristic conditions and a dissipative criterion that was originally proposed for hyperbolic-parabolic systems and has now proved to be important also for hyperbolic systems with relaxation.

1. Introduction

It is well-known that the motion of a gas in local thermodynamic equilibrium is governed by the compressible Euler equations. In Lagrangian coordinates, the equations for one dimensional flow read:

$$\begin{aligned} &v_t - u_x = 0,\\ &u_t + p_x = 0,\\ &(e + \frac{1}{2}u^2)_t + (pu)_x = 0, \end{aligned} \tag{1.1}$$

where v, u, p and e are, respectively, the specific volume, velocity, pressure and internal energy of the gas. According to the thermodynamic equation

$$de = T\,ds - p\,dv, \tag{1.2}$$

only two of the thermodynamic variables v, p, e, T and s are independent, where T and s are the temperature and entropy respectively. Therefore, (1.1) are closed by the equation of state; for instance, $pv = RT$ for a perfect gas, where R is the gas constant.

We know that gases are composed of molecules and atoms whose overall behavior is described by kinetic theory and whose individual properties are governed by the laws of quantum mechanics. The above model of "equilibrium flow" actually implies that all molecular processes take place within the gas infinitely rapidly. In other words, the gas can adjust instantaneously to any change in its environment. That is a reasonable assumption for air, the commonest gas, at room temperature. The flow is an equilibrium flow of a perfect gas. If the temperature of the air is increased, however, deviations from the perfect gas behavior can be observed. This is because the vibrational mode of the molecules becomes excited, dissociation of both oxygen and nitrogen molecules occurs, nitric oxide is formed, etc. The molecular processes now take a considerable time due to the chemical reactions and the fact that the internal structure of the molecules is no longer negligible and takes time to adjust to the translational mode. The air then loses its local thermodynamic equilibrium state.

In this paper, we consider the simplest case of nonequilibrium flow. We shall allow for only one nonequilibrium process, say, vibrational nonequilibrium. Therefore, we assume that the flow is everywhere in instantaneous translational and rotational equilibrium. As above, we use u, v, p to denote the velocity, specific volume and pressure of the gas. But now we use e_1 to denote the total of the (specific) translational energy and rotational energy of the molecules. Similarly, we use T_1 to denote the common temperature of the translational and rotational modes, and s_1 to denote the total (specific) entropy of these two modes. For the vibrational mode, we assume that it has a Boltzmann distribution over its energy states, hence its temperature and entropy can be defined. We use q, T_2 and s_2 to denote the (specific) vibrational energy, vibrational temperature and (specific) vibrational entropy, respectively. Here for a nonequilibrium state it is necessary to have $T_1 \neq T_2$. We now let

$$e = e_1 + q, \quad \text{and} \quad s = s_1 + s_2$$

be, respectively, the total internal energy and total entropy.

The thermodynamic variables related to the equilibrium modes and vibrational mode obey different thermodynamic equations:

$$de_1 = T_1 ds_1 - p dv, \tag{1.3a}$$

$$dq = T_2 ds_2. \tag{1.3b}$$

Here we note that the energy of an internal mode is volume independent. Combining the two equations, we have the thermodynamic equation for the gas as a whole in vibrational nonequilibrium,

$$de = T_1 ds + (T_2 - T_1) ds_2 - p dv. \tag{1.4}$$

From (1.3) it is clear that among v, p, e_1, T_1 and s_1 any two determine the rest, while among q, T_2 and s_2 any one determines the others. Therefore, we need two thermodynamic variables to characterize the equilibrium modes, one for the nonequilibrium mode, and one for the velocity. This means that the flow needs to be described by a system of four equations for four unknowns.

Under the assumption that T_2 is slightly away from T_1, gas dynamics in vibrational nonequilibrium is governed by the following equations in Lagrangian coordinates:

$$\begin{aligned} &v_t - u_x = 0, \\ &u_t + p_x = 0, \\ &(e + \frac{1}{2}u^2)_t + (pu)_x = 0, \\ &q_t = \frac{Q - q}{\tau}, \end{aligned} \tag{1.5}$$

where $Q = Q(v, e_1)$ and $\tau = \tau(v, e_1) > 0$ are given functions. To be more specific, Q is the local equilibrium value for q. Let

$$q = \omega(T_2), \tag{1.6a}$$

where ω is an increasing function from physical consideration. Then

$$Q = \omega(T_1). \tag{1.6b}$$

Q is the value q would have if the gas were in equilibrium with local temperature $T_2 = T_1$. As for τ, it is referred to as a local relaxation time. It characterizes the time scale for the nonequilibrium mode to relax to the equilibrium state. System (1.5) is closed by appropriate equations of state.

Associated with the nonequilibrium flow there are two important limiting cases: the equilibrium flow and the frozen flow. As mentioned before, the equilibrium flow means that all internal processes (relaxation processes) take place within the gas infinitely rapidly. That is, $\tau \to 0$. In general, we expect q_t to stay finite in the limit. (Although exceptions may occur, they are beyond our interests in this paper.) Therefore, it is necessary to have $q = Q$ in the equilibrium flow, or $T_2 = T_1$ by (1.6). On the other hand, the frozen flow means that all internal processes take place infinitely slowly, i.e. $\tau \to \infty$. Hence in the frozen flow we must have $q_t = 0$. In both limiting cases, the last equation in (1.5) is replaced by an algebraic equation, either $q(x, t) = Q(x, t)$ or $q(x, t) = q(x, 0)$. System (1.5) is then reduced to the Euler equations (1.1). However, the two limiting cases are associated with different equations of state and consequently have different sound speeds. Later we will see that both the equilibrium speed of sound and the frozen speed of sound play important roles in the study of (1.5).

We now study important properties of thermodynamic variables related to the well-posedness of (1.5). First of all, we introduce the following notation:

$$\begin{aligned} &p = p(v, e_1) = \bar{p}(v, s_1) = \tilde{p}(v, T_1), \quad &&T_1 = T_1(v, e_1) = \bar{T}_1(v, s_1), \\ &s_1 = s_1(v, e_1) = \tilde{s}_1(v, T_1), &&e_1 = \bar{e}_1(v, s_1) = \tilde{e}_1(v, T_1). \end{aligned} \tag{1.7}$$

Then we make the following physical assumptions:

$$\begin{aligned} &\tilde{p}_v = \frac{\partial}{\partial v}\tilde{p}(v, T_1) < 0, && (T_1)_{e_1} = \frac{\partial}{\partial e_1} T_1(v, e_1) > 0, \\ &\omega'(T_1) > 0, && p_{e_1} = \frac{\partial}{\partial e_1} p(v, e_1) \not\equiv 0. \end{aligned} \tag{1.8}$$

By straightforward calculation using (1.7), (1.3) and (1.6), the basic assumptions (1.8) imply

$$\begin{aligned} &c_f^2 \equiv -\bar{p}_v = p p_{e_1} - p_v = -\tilde{p}_v + T_1\big(\tilde{p}_{T_1}\big)^2 (T_1)_{e_1} > 0, \\ &Q_{e_1} = \omega'(T_1)(T_1)_{e_1} > 0, \qquad a \equiv p Q_{e_1} - Q_v = \omega'(T_1) T_1 P_{e_1} \not\equiv 0, \\ &p_v - \frac{(T_1)_v p_{e_1}}{(T_1)_{e_1}} = \tilde{p}_v < 0, \\ &-(T_1)_{e_1} p_v + \big[(T_1)_{e_1} p - (T_1)_v\big] \frac{(T_1)_v}{T_1} = -(T_1)_{e_1} \tilde{p}_v > 0, \end{aligned} \tag{1.9}$$

see, e.g., [LZ1], Section 9 for details. Notice that here c_f is exactly the frozen speed of sound.

We first discuss properties of the entropy. Notice that (1.5) is in the form

$$w_t + f(w)_x = r(w), \tag{1.10}$$

where

$$w = \begin{pmatrix} v \\ u \\ e + \frac{1}{2}u^2 \\ q \end{pmatrix}, \quad f(w) = \begin{pmatrix} -u \\ p \\ pu \\ 0 \end{pmatrix}, \quad r(w) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{\chi}{\tau} \end{pmatrix}, \tag{1.11}$$

$$\chi \equiv Q - q. \tag{1.12}$$

We claim that as a function of w, $-s$ is strictly convex. In fact, the gradient and the Hessian of $-s$ with respect to w are

$$\begin{aligned} -\nabla s &= \big(-(s_1)_v, (s_1)_{e_1} u, -(s_1)_{e_1}, (s_1)_{e_1} - s_2'(q)\big) \\ &= \left(-\frac{p}{T_1}, \frac{u}{T_1}, -\frac{1}{T_1}, \frac{1}{T_1} - \frac{1}{T_2}\right) \end{aligned} \tag{1.13}$$

and

$$H \equiv -\nabla^2 s = \begin{pmatrix} -(s_1)_{vv} & u(s_1)_{e_1 v} & -(s_1)_{e_1 v} & (s_1)_{e_1 v} \\ u(s_1)_{v e_1} & (s_1)_{e_1} - u^2 (s_1)_{e_1 e_1} & u(s_1)_{e_1 e_1} & -u(s_1)_{e_1 e_1} \\ -(s_1)_{v e_1} & u(s_1)_{e_1 e_1} & -(s_1)_{e_1 e_1} & (s_1)_{e_1 e_1} \\ (s_1)_{v e_1} & -u(s_1)_{e_1 e_1} & (s_1)_{e_1 e_1} & -(s_1)_{e_1 e_1} - s_2''(q) \end{pmatrix} \tag{1.14}$$

by (1.7) and (1.3). Also with (1.6), (1.8) and (1.9), we can verify that

$$\begin{aligned} \det H &= -s_2''(q)(s_1)_{e_1}\big[(s_1)_{vv}(s_1)_{e_1 e_1} - \big((s_1)_{e_1 v}\big)^2\big] \\ &= \frac{1}{T_2^2 \omega'(T_2) T_1^4}\left[-(T_1)_{e_1} p_v + \big((T_1)_{e_1} p - (T_1)_v\big)\frac{(T_1)_v}{T_1}\right] > 0. \end{aligned}$$

Similarly, we can verify that all the principal minor determinants are positive. Hence H is positive definite.

We further discuss the role of H as a symmetrizer of the system. Recall that in the general theory of hyperbolic conservation laws, we define an entropy function U with a corresponding entropy flux F as the following: U and F are functions of the unknown vector w of the hyperbolic conservation laws

$$w_t + f(w)_x = 0, \tag{1.15}$$

such that $(\nabla U)f' = \nabla F$. This implies that the Hessian H of U, $H = \nabla^2 U$, is a symmetrizer of (1.15). That is, Hf' is symmetric, [FL]. It is well-known that for the Euler equations (1.1) describing the ideal gas, we have an entropy pair $U = -s$, $F = 0$. Therefore, regarding s as a function of the unknown vector of (1.1), $(v, u, e + u^2/2)^t$, $H = -\nabla^2 s$ is a symmetrizer of (1.1). If the effect of viscosity and (or) heat conduction is further included, we have Navier-Stokes equations in the form

$$w_t + f(w)_x = (B(w)w_x)_x, \tag{1.16}$$

which is a hyperbolic-parabolic system with a 3×3 degenerate matrix B. In this case, $H = -\nabla^2 s$ symmetrizes the Navier-Stokes equations as well, [Ka]. That is, not only Hf' is symmetric, HB is symmetric and semi-positive definite as well.

In our current study for gas dynamics in thermal nonequilibrium, (1.10) or (1.5), we have a similar result: For H given in (1.14), Hf' is symmetric. Moreover, on the equilibrium manifold $T_2 = T_1$, Hr' is symmetric and semi-negative definite. In fact, from (1.11) we have

$$f' = \begin{pmatrix} 0 & -1 & 0 & 0 \\ p_v & -p_{e_1}u & p_{e_1} & -p_{e_1} \\ p_v u & -p_{e_1}u^2 + p & p_{e_1}u & -p_{e_1}u \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad r' = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix}, \tag{1.17}$$

where

$$\begin{aligned} &r_1 = r_2 = r_3 = (0, 0, 0, 0), \\ &r_4 = \left(-\frac{\chi\tau_v}{\tau^2} + \frac{Q_v}{\tau}, \left(\frac{\chi\tau_{e_1}}{\tau^2} - \frac{Q_{e_1}}{\tau}\right)u, -\frac{\chi\tau_{e_1}}{\tau^2} + \frac{Q_{e_1}}{\tau}, \frac{\chi\tau_{e_1}}{\tau^2} - \frac{Q_{e_1}+1}{\tau}\right). \end{aligned} \tag{1.18}$$

To verify Hf' is symmetric, we may apply the result for Euler equations to conclude that the 3×3 submatrix at the upper left corner of Hf' is symmetric. All we need to do then is to verify that the fourth row of Hf' has the same entries as the fourth column. But this is trivial by (1.14), (1.17) and (1.3a). To verify the property of Hr', we notice that

$$Hr' = h_4 r_4 \tag{1.19}$$

by (1.17) and (1.18), where h_4 is the fourth column of H. By (1.3),

$$h_4 = -\frac{1}{T_1^2}\big((T_1)_v, -(T_1)_{e_1}u, (T_1)_{e_1}, -(T_1)_{e_1} + T_1^2 s_2''(q)\big)^t.$$

From (1.3b), $s_2'(q) = 1/T_2$. Thus on the equilibrium manifold $T_2 = T_1$, we can further simplify the last component of h_4 to obtain

$$h_4 = -\frac{1}{T_1^2}\left((T_1)_v, -(T_1)_{e_1}u, (T_1)_{e_1}, -(T_1)_{e_1} - \frac{1}{\omega'(T_1)}\right)^t, \tag{1.20}$$

using (1.6a). Also on the equilibrium manifold, we have $\chi = 0$, and

$$r_4 = \frac{\omega'(T_1)}{\tau}\left((T_1)_v, -(T_1)_{e_1}u, (T_1)_{e_1}, -(T_1)_{e_1} - \frac{1}{\omega'(T_1)}\right), \tag{1.21}$$

using (1.6b). Clearly, (1.19)–(1.21) and (1.8) imply that Hr' is symmetric and semi-negative definite on the equilibrium manifold.

Summarizing the above results for the entropy, we have

Theorem 1.1. *Let* (1.8) *be true. Let w be the unknown vector of* (1.5) *given by* (1.11)*. If the entropy s is regarded as a function of w, then $-s$ is strictly convex. Moreover, the Hessian H of $-s$ with respect to w is a symmetrizer of* (1.5) *in the following sense: Hf' is symmetric for all w under consideration, while Hr' is symmetric and semi-negative definite on the equilibrium manifold $T_2 = T_1$. Here f and r, also given by* (1.11)*, are respectively the flux function and relaxation vector of* (1.5).

In physics, the entropy characterizes the direction of any process that the system is undergoing. In the general theory of hyperbolic conservation laws and hyperbolic-parabolic conservation laws, the existence of a strictly convex entropy function, which is a generalization of the physical entropy, is a basic condition for the well-posedness, [FL], [Ka]. For a general hyperbolic system with relaxation, an entropy condition was introduced in [CLL], see Definition 4.1 below. However, to extend our results in this paper on global existence and large time behavior for nonequilibrium flow to such a general system, we expect that another form of entropy condition, consistent with Theorem 1.1, is needed.

There is another important condition, called the subcharacteristic condition, related to the well-posedness of relaxation systems, [Wh], [Liu], [CLL]. The condition says that the characteristic speeds of the equilibrium system are interlaced with the characteristic speeds of the full system. Especially, it was shown in [CLL] that the entropy condition defined there implies the subcharacteristic condition, see Theorem 4.2 below. For our nonequilibrium flow, we now want to verify the subcharacteristic condition under physical assumptions (1.8). Clearly from (1.17) the full system (1.5) has characteristic speeds

$$\lambda_1 = -c_f, \quad \lambda_2 = \lambda_3 = 0, \quad \lambda_4 = c_f, \tag{1.22}$$

where c_f is the frozen speed of sound given in (1.9). It is a classical result that the equilibrium system has characteristic speeds

$$\lambda_1^{(r)} = -c, \quad \lambda_2^{(r)} = 0, \quad \lambda_3^{(r)} = c, \tag{1.23}$$

where c is the equilibrium speed of sound. Here we use superscript (r) to denote the reduced system (equilibrium system). Notice that the equilibrium system is the Euler equations (1.1) with

$$e = e_1 + Q(v, e_1). \tag{1.24}$$

Under assumptions (1.8), which imply (1.9), equation (1.24) defines e_1 as a function of v and e,

$$e_1 = e_1^{(r)}(v, e),$$

with

$$\big(e_1^{(r)}\big)_v = \frac{\partial}{\partial v} e_1^{(r)}(v, e) = -\frac{Q_v}{1 + Q_{e_1}}, \quad \big(e_1^{(r)}\big)_e = \frac{\partial}{\partial e} e_1^{(r)}(v, e) = \frac{1}{1 + Q_{e_1}}.$$

Therefore, the pressure of the equilibrium flow can be expressed as

$$p = p\big(v, e_1^{(r)}(v, e)\big) \equiv p^{(r)}(v, e) \equiv \bar{p}^{(r)}(v, s).$$

Now it is straightforward to calculate c. By (1.4), which is reduced to (1.2) for $T_2 = T_1$, we have

$$c^2 \equiv -\bar{p}_v^{(r)} = p p_e^{(r)} - p_v^{(r)} = \frac{p p_{e_1}}{1 + Q_{e_1}} - p_v + \frac{p_{e_1} Q_v}{1 + Q_{e_1}} = c_f^2 - \frac{a p_{e_1}}{1 + Q_{e_1}}, \tag{1.25}$$

where a is defined in (1.9). Next we define a quantity

$$b \equiv \frac{p_{e_1}}{-p_{e_1} + c_f^2 (1 + Q_{e_1})/a}. \tag{1.26}$$

By (1.6), (1.9) and (1.8) we have

$$b = \frac{a p_{e_1}}{c_f^2 - \omega'(T_1)(T_1)_{e_1} \tilde{p}_v} > 0. \tag{1.27}$$

Using (1.26), equation (1.25) can be written as

$$c^2 = \frac{c_f^2}{1 + b}.$$

Obviously, $0 < c^2 < c_f^2$ by (1.27). Thus we have

Proposition 1.2. *Under assumptions* (1.8), *we have*

$$0 < c < c_f \tag{1.28}$$

on the equilibrium manifold $T_2 = T_1$, where c and c_f are, respectively, the equilibrium speed of sound and the frozen speed of sound.

Proposition 1.2, (1.22) and (1.23) immediately imply the subcharacteristic condition

$$\lambda_1 < \lambda_1^{(r)} < \lambda_2 = \lambda_2^{(r)} = \lambda_3 < \lambda_3^{(r)} < \lambda_4. \tag{1.29}$$

Because of the equal signs, the condition is satisfied in the non-strict sense.

In this paper, we are interested in Cauchy problems of (1.5). Assuming that the initial data are a small perturbation of a constant state that is an equilibrium state, we want to establish the global existence and study the large time behavior for the solution. A natural question to ask is: If the solution exists, does it converge to the constant state as $t \to \infty$? Notice that (1.5) is a hyperbolic system with relaxation. Generally speaking, for such a system the relaxation term induces a dissipative effect. This effect then competes with the hyperbolicity. If the dissipation is sufficiently strong to dominate the hyperbolicity, the system is dissipative, and the solution converges to the constant state. Otherwise, the dissipation and the hyperbolicity are equally important. Then we expect that only part of the perturbation diffuses. In the latter case we say that the system is of composite type.

In general, there are several ways to identify whether a hyperbolic system with relaxation is dissipative or of composite type. One way is completely parallel to the case of the hyperbolic-parabolic system, which was discussed in [Ka], [LZ1] and [LZ2]. Assume that a relaxation system in the form of (1.10) admits a strictly convex entropy that is also a symmetrizer of the whole system in the sense of Theorem 1.1. If on the equilibrium manifold, any eigenvector of f' is not in the null space of r', then we expect the system to be dissipative. Otherwise, it is a composite type system. We refer to this condition as the dissipative criterion. Further discussion on the criterion will be given in Sections 3 and 4. For our nonequilibrium flow (1.5), it is easy to see from (1.17), (1.18) and (1.21) that

$$\zeta = \big(\omega'(T_1)\big)^{-1}\big(p_{e_1}, 0, -p_v, 0\big)^t + \big(p_{e_1}(T_1)_v - p_v(T_1)_{e_1}\big)\big(0, 0, 1, 1\big)^t$$

is an eigenvector of f' and is in the null space of r' on the equilibrium manifold. Therefore, (1.5) is of composite type. The failure to satisfy the dissipative criterion in fact means the failure of the relaxation term in the linear level to have a "positive projection" on all the characteristic directions of the equilibrium system, see Section 3.

Another way to discuss this issue is to look at the subcharacteristic condition if the full system has exactly one more equation than the equilibrium system, which is our current situation. In this case, under appropriate conditions the dissipative criterion is equivalent to the subcharacteristic condition in the strict sense, Theorem 4.5. Again, (1.29) tells us that (1.5) cannot be dissipative since $\lambda_2 = \lambda_2^{(r)} = \lambda_3$.

The type of system (1.5) can also be studied through Chapman-Enskog expansion. Using (1.12), we write

$$q = Q - \chi. \tag{1.30}$$

Here χ is small. For the first-order expansion we set $\chi = 0$, which leads to the equilibrium system, the Euler equations (1.1). For the second-order expansion, we need the first-order correction for χ. Substitute (1.30) into the rate equation in (1.5), and use other equations. We have

$$-au_x - \chi_t = \frac{\chi}{\tau}(1 + Q_{e_1}), \tag{1.31}$$

where a is defined in (1.9). Since χ_t is of a higher order, the first-order correction for χ is

$$\chi \approx -\frac{a\tau}{1 + Q_{e_1}} u_x.$$

Therefore, the total energy up to the first-order correction is

$$e = e_1 + Q(v, e_1) + \left(\frac{a\tau}{1 + Q_{e_1}}\right)(v, e_1)u_x. \tag{1.32}$$

As in the equilibrium system, we define $e_1^{(r)}(v, e)$ by

$$e = e_1^{(r)} + Q(v, e_1^{(r)}). \tag{1.33}$$

Subtract (1.33) from (1.32). Up to the first-order correction we have

$$e_1 - e_1^{(r)} + Q_{e_1}(v, e_1^{(r)})(e_1 - e_1^{(r)}) \approx -\left(\frac{a\tau}{1 + Q_{e_1}}\right)(v, e_1)u_x,$$

or

$$e_1 - e_1^{(r)} \approx -\left(\frac{a\tau}{(1 + Q_{e_1})^2}\right)(v, e_1^{(r)})u_x.$$

Therefore, up to the same accuracy, we can express the pressure as

$$p = p(v, e_1) = p(v, e_1^{(r)}) - \left(\frac{p_{e_1}a\tau}{(1 + Q_{e_1})^2}\right)(v, e_1^{(r)})u_x. \tag{1.34}$$

Substitute (1.34) into the first three equations in (1.5). We obtain the second-order Chapman-Enskog expansion

$$\begin{aligned} &v_t - u_x = 0, \\ &u_t + p_x^{(r)} = (\mu u_x)_x, \\ &(e + \frac{1}{2}u^2)_t + (p^{(r)}u)_x = (\mu u u_x)_x, \end{aligned} \tag{1.35a}$$

where

$$\begin{aligned} &p^{(r)}(v, e) \equiv p\big(v, e_1^{(r)}(v, e)\big), \\ &\mu(v, e) \equiv \left(\frac{p_{e_1}a\tau}{(1 + Q_{e_1})^2}\right)\big(v, e_1^{(r)}(v, e)\big) > 0 \end{aligned} \tag{1.35b}$$

by (1.8), (1.9). System (1.35) is the Navier-Stokes equations with zero heat conductivity, where μ plays the role of viscosity. System (1.35) has been studied in [LZ2] and is known as a composite type system. Hence we also expect the original system (1.5) is of composite type.

The fact that (1.5) is not fully dissipative can be easily seen through a special solution as well. That is, $u =$ constant, $v = v(x)$, $p =$ positive constant, $q =$

$\omega(T_1(x))$, where T_1 is determined by v and p. If $v(x)$ and $q(x)$ are perturbations of constants, the perturbations stay for all time. Clearly, the solution is a frozen flow and an equilibrium flow at the same time.

To study the large time behavior, or even only to establish the global existence by an energy method, it is necessary to separate the solution of (1.5) into two parts: a dissipative one and a nondissipative one. Therefore, we need to choose four appropriate unknowns to represent these two parts. Our choice is based on the detailed study of the Green's function of the linearized system, given in Section 3. The result there tells us that there are three waves in the leading term, represented by a δ-function and two heat kernels. Consequently, we need to choose one unknown for the nondissipative part, and two for the leading term of the dissipative one. The fourth unknown then needs to go to the next order (higher order). The above-mentioned special solution immediately suggests that u and p are the leading term of the dissipative part, while v or q can be used for the nondissipative one. However, we prefer to use s instead since it gives the simplest equation. The unknown for the higher order term can be chosen as χ.

Now we derive the equations for the unknowns so chosen. From (1.5) and (1.9) we have

$$p_t = p_v v_t + p_{e_1}(e_t - q_t) = -c_f^2 u_x - p_{e_1}\frac{\chi}{\tau}.$$

To derive the equation for s, we need to use (1.4) and (1.3) as well:

$$s_t = \frac{1}{T_1}e_t + \left(\frac{1}{T_2} - \frac{1}{T_1}\right)q_t + \frac{p}{T_1}v_t = \left(\frac{1}{T_2} - \frac{1}{T_1}\right)\frac{\chi}{\tau}.$$

Thus with (1.31), the complete system in p, u, χ, s is

$$\begin{aligned} &p_t + c_f^2 u_x = -p_{e_1}\frac{\chi}{\tau}, \\ &u_t + p_x = 0, \\ &\chi_t + a u_x = -(1 + Q_{e_1})\frac{\chi}{\tau}, \\ &s_t = \left(\frac{1}{T_2} - \frac{1}{T_1}\right)\frac{\chi}{\tau}. \end{aligned} \tag{1.36}$$

Systems (1.5) and (1.36) are equivalent. We will use both at our convenience.

We are now ready to state our problems and results for the nonequilibrium flow. Consider the Cauchy problem of (1.5) with initial data

$$(v, u, e_1, q)(x, 0) = (v_0, u_0, e_{1,0}, q_0)(x). \tag{1.37}$$

Assume that the given initial function $(v_0, u_0, e_{1,0}, q_0)$ is a small perturbation of a constant state (v^*, u^*, e_1^*, q^*), where $v^* > 0$, $e_1^* > 0$, $q^* > 0$ and without loss of generality $u^* = 0$. Further assume that the constant state is an equilibrium state. That is,

$$q^* = Q^*, \qquad \text{or} \qquad T_2^* = T_1^*. \tag{1.38}$$

Here we use "*" to denote the constant state. Thus $Q^* = Q(v^*, e_1^*), T_2^* = \omega^{-1}(q^*)$, etc. If the initial perturbation is small, we first establish the global existence, then study the large time behavior. To simplify our notation, let

$$\|\cdot\|_m \equiv \|\cdot\|_{H^m}, \qquad \|\cdot\| \equiv \|\cdot\|_{L^2}, \tag{1.39}$$

where the norms are with respect to the space variable x.

Theorem 1.3. *Let* (1.8) *be true, and* v^*, e_1^* *and* q^* *be positive constants such that* (1.38) *holds. Let* $m \geqq 2$ *be an integer. Then there exist positive constants* ε *and* C, *such that if*

$$\|(v_0 - v^*, u_0, e_{1,0} - e_1^*, q_0 - q^*)\|_m \leqq \varepsilon,$$

system (1.5) *with initial condition* (1.37) *has a unique global solution*

$$(v, u, e_1, q)(x, t),$$

satisfying

$$\begin{aligned}&(v - v^*, u, e_1 - e_1^*, q - q^*) \in C^0([0, \infty); H^m) \cap C^1([0, \infty); H^{m-1}),\\ &p_x,\ u_x \in L^2([0, \infty); H^{m-1}), \quad \chi \in L^2([0, \infty); H^m),\end{aligned} \tag{1.40}$$

and the following energy inequality

$$\begin{aligned}&\sup_{t \geqq 0} \|(v - v^*, u, e_1 - e_1^*, q - q^*)\|_m^2(t)\\ &\qquad + \int_0^\infty \big(\|p_x\|_{m-1}^2(t) + \|u_x\|_{m-1}^2(t) + \|\chi\|_m^2(t)\big)\, dt\\ &\quad \leqq C\|(v_0 - v^*, u_0, e_{1,0} - e_1^*, q_0 - q^*)\|_m^2.\end{aligned} \tag{1.41}$$

Theorem 1.4. *Let* (1.8) *be true, and* v^*, e_1^* *and* q^* *be positive constants such that* (1.38) *holds. Let the initial data* $(v_0, u_0, e_{1,0}, q_0)$ *be a perturbation of the constant state* $(v^*, 0, e_1^*, q^*)$, *satisfying*

$$\begin{aligned}&(v_0 - v^*, u_0, e_{1,0} - e_1^*, q_0 - q^*) \in H^6(\mathbb{R}),\\ &(v_0 - v^*, u_0, e_{1,0} - e_1^*, q_0 - q^*)(x) = O(1)(x^2 + 1)^{-\frac{3}{4}},\\ &(v_0', u_0', e_{1,0}', q_0')(x) = O(1)(x^2 + 1)^{-\frac{3}{4}},\\ &s_0''(x) = O(1)(x^2 + 1)^{-\frac{1}{4}},\\ &\|(v_0 - v^*, u_0, e_{1,0} - e_1^*, q_0 - q^*)\|_6\\ &\quad + \sup_{x \in \mathbb{R}} \Big\{(x^2 + 1)^{\frac{3}{4}}(|v_0 - v^*| + |u_0| + |e_{1,0} - e_1^*| + |q_0 - q^*|\\ &\quad + |v_0'| + |u_0'| + |e_{1,0}'| + |q_0'|)(x) + (x^2 + 1)^{\frac{1}{4}}|s_0''|(x)\Big\} \equiv \varepsilon_0 \ll 1,\end{aligned} \tag{1.42}$$

where s_0 *is the initial entropy. Then for all* $x \in \mathbb{R}$, $t \geqq 0$, *the solution of* (1.5), (1.37) *has the following property:*

$$\begin{aligned}
(p-p^*,u)(x,t) &= O(1)\varepsilon_0\Big\{(t+1)^{-\frac{1}{2}}\Big(e^{-\frac{(x+c^*(t+1))^2}{\nu(t+1)}}+e^{-\frac{(x-c^*(t+1))^2}{\nu(t+1)}}\Big)\\
&\quad+[(x+c^*(t+1))^2+t+1]^{-\frac{3}{4}}+[(x-c^*(t+1))^2+t+1]^{-\frac{3}{4}}\Big\},\\
(\chi,p_x,u_x)(x,t) &= O(1)\varepsilon_0(t+1)^{-\frac{1}{2}}\Big\{(t+1)^{-\frac{1}{2}}\Big(e^{-\frac{(x+c^*(t+1))^2}{\nu(t+1)}}+e^{-\frac{(x-c^*(t+1))^2}{\nu(t+1)}}\Big)\\
&\quad+[(x+c^*(t+1))^2+t+1]^{-\frac{3}{4}}+[(x-c^*(t+1))^2+t+1]^{-\frac{3}{4}}\Big\},\\
\|(\chi_x,p_{xx},u_{xx})(\cdot,t)\|_{L^\infty} &= O(1)\varepsilon_0(t+1)^{-\frac{3}{2}},\\
s(x,t)-s^* = O(1)\varepsilon_0(x^2+1)^{-\frac{3}{4}},&\qquad s_x(x,t)=O(1)\varepsilon_0(x^2+1)^{-\frac{3}{4}},
\end{aligned} \tag{1.43}$$

where p^*, c^*, s^* *are the pressure, equilibrium speed of sound and entropy evaluated at the constant state, and* $\nu > 0$ *is a constant depending only on the constant state.*

In this section we have given an introduction to the relaxation model that describes gas flow in thermal nonequilibrium. Important thermodynamic properties related to the well-posedness have been given in Theorem 1.1 and Proposition 1.2.

In Section 2 we prove Theorem 1.3 to establish the global existence of the solution to (1.5). The proof makes crucial use of the existence of a strictly convex entropy function and other thermodynamic properties. The energy method employed there applies to more general relaxation systems if there exists a strictly convex entropy function whose Hessian is a symmetrizer in the sense of Theorem 1.1.

In Section 3 we obtain another main result of this paper, the fundamental solution for a general class of linear hyperbolic systems with relaxation, Theorems 3.6 and 3.9. Since the fundamental solution of a composite type system is a sum of nondecaying δ-functions and the fundamental solution of a fully dissipative system, all we need is to formulate the fundamental solution for a general dissipative system. Theorem 3.6 gives detailed information on the Green's function for such a system: The leading term consists of heat kernels along the characteristic directions of the equilibrium system; the singular part consists of exponentially decaying δ-functions along the characteristic directions of the full system; and the remainder is a higher order of those heat kernels by a factor $(t+1)^{-1/2}$. The leading term and the singular part are both determined explicitly by the two coefficient matrices. The proof of Theorem 3.6 is similar to the proof of a corresponding result for a general linear hyperbolic-parabolic system obtained in our previous work, [LZ1]. They are based on the spectral representation of the matrix occurring in the Fourier transform. In fact in Section 3 we make extensive use of the spectral properties obtained in [LZ1], with the switching on the leading term and the singular part. However, there is an important difference between a hyperbolic-parabolic system and a hyperbolic system with relaxation. For the latter, the heat kernels in the leading term are along the characteristic directions of the equilibrium system, which is in reduced size. This means that the number of principal waves is alway less than the number of unknowns. Therefore, part of the Green's function must be identically zero in the leading term. For this part, it is necessary to obtain a precise expression in the next order for our purpose of studying the large time behavior of a nonlinear system. Such a refinement is then given in Theorem 3.9: The part of the Green's function that is identically zero in the leading term can be expressed as the derivative of the heat kernels occurring in the leading term, plus a higher order of the derivative by a factor $(t+1)^{-1/2}$ and corresponding singular part. At the end of Section 3, the general result is applied to our nonequilibrium flow. The Green's function of the linearized system consists of a nondecaying δ-function along the particle path, two heat kernels along the equilibrium acoustic directions, a higher order term of these heat kernels, and three exponentially decaying δ-functions along the frozen acoustic directions and the particle path. The part of the Green's function that is identically zero in the leading term is refined as the derivative of the two heat kernels, plus higher order and exponentially decaying singular terms. This part corresponds to the unknown χ.

Section 4 is an extension of Section 3 on the discussion of a general system. We are particularly interested in the relation between the subcharacteristic condition and the dissipative criterion. Our purpose is to investigate whether certain forms of subcharacteristic conditions can be used to distinguish composite type systems from fully dissipative ones. Under a weaker version of the entropy condition, Assumption 4.1, we obtain a sufficient condition and a necessary one for the dissipative criterion to hold, in terms of subcharacteristic conditions, Theorem 4.5. It turns out that if the full system has exactly one more equation than the equilibrium system, then the dissipative criterion is equivalent to the subcharacteristic condition in the strict sense. In the general case, however, there is a gap between the sufficient condition and the necessary one. Examples satisfying or not satisfying the dissipative criterion can both fall into this gap. Therefore, in the general case subcharacteristic conditions are not appropriate for characterizing dissipation.

Section 5 is on the evolution of elementary waves. Our elementary waves include heat kernels and waves of algebraic type, of which the solution consists at large time, see (1.43). Notice that heat kernels have already occurred in the Green's function of the linear system. Therefore, they are natural in the ansatz for the nonlinear system. On the other hand, waves of algebraic type are exclusively related to the nonlinear system. They come from the nonlinear coupling of different families in the system, [Liu2].

In Section 6 we will prove Theorem 1.4, which describes the large time behavior of the nonequilibrium flow. The approach is a combination of pointwise approach, weighted energy estimate and energy estimate. The pointwise approach is necessary to obtain pointwise estimate (1.43). The procedure is to write (1.36) as a linear system with nonlinear source, to use Duhamel's principle to write the solution as convolutions of the Green's function with the initial data and with the nonlinear source, and to perform an a priori estimate. The two main components of such an approach, precise estimates on the Green's function and on the evolution of elementary waves, have been covered in Sections 3 and 5 respectively. However, the pointwise approach cannot be closed by itself. This is true for any system whose Green's function contains δ-functions (even exponentially decaying ones). Examples are hyperbolic systems with relaxation and hyperbolic-parabolic systems. The δ-functions prevent the derivatives in the nonlinear source from being removed by integration by parts. Therefore, we need to continue the estimate to the derivatives. A strategy to close the analysis is to reduce the decay rate for higher derivatives, although they in fact decay faster. This is possible because they only occur in products (nonlinear source), and another factor gives a contribution to the decay rate. After several steps of reduction, we then only need derivatives of a particular order to be bounded. With the help of the Sobolev inequality, the analysis is then closed by the energy estimate. If the system is of composite type, such a combination of pointwise approach and energy estimate is not sufficient. This is because a part of the solution no longer decays, and its contribution to the decay rate of the nonlinear source is no longer trivial. In this case we need to further incorporate the fact that the nondecaying wave and the decaying ones are along different directions, hence their products decay. This needs to be done by a weighted energy method.

In the stability analysis carried out in Section 6, it is important to identify different time intervals for integration. In a certain time interval, the decay rate of the integral comes from the decay rate of the Green's function. In this case the lower order term in the nonlinear source needs to be converted into derivatives using the rate equation. Then the derivatives are moved to the Green's function by integration by parts. In this way, the decay rate of the Green's function is increased. In another time interval, the decay rate of the integral comes from the decay rate of the nonlinear source. In this case we need to choose an appropriate order for the derivatives in the source such that it gives the best rate. The order can be increased or reduced by converting the lower order term or by integration by parts. The order to be chosen depends on the part of the Green's function and the part of the source so involved. It also depends on the ansatz that we choose. In the last type of time interval, the singularity in the Green's function becomes important. The lower order term in the source needs to stay as lower order to avoid difficulties in the closing.

To finish this section, we make a final remark on Theorems 1.3 and 1.4. As we can see from the Green's function in Section 3, the relaxation term in (1.5) induces a dissipative effect in the equilibrium acoustic directions. These are the two genuinely nonlinear fields. The dissipation then competes with the compression. For sufficiently small data, the dissipation is able to prevent any formation of shocks. On the other hand, the relaxation term has "zero projection" in the particle path direction. But this is a linearly degenerate field. If the initial data are smooth, we do not expect singularities in this field. Therefore, for smooth and small data, system (1.5) has a global classical solution as stated in Theorem 1.3. As for Theorem 1.4, it gives a complete picture of the large time behavior. The entropy wave stays as expected. The perturbation of pressure and velocity are represented by two heat kernels, consistent with the linear theory, and waves of algebraic type, coming from nonlinear coupling. The departure of vibrational energy from its local equilibrium value is a lower order term. It decays faster and behaves like derivatives of pressure and velocity, as predicted by the linear theory in Section 3. The pointwise estimate (1.43) immediately gives decay rates in L^p, $1 \leqq p \leqq \infty$. It also allows us to see different rates along or away from the equilibrium acoustic directions. All the rates obtained are optimal.

Throughout the paper we will use C to denote a universal positive constant, and ∂_x^j to denote $\partial^j/\partial x^j$. Readers are referred to [VK] for detailed discussions on establishing models for gas dynamics in thermal nonequilibrium. For related works in classical solutions and large time behavior for relaxation systems, see [Ze], [Ch], [Yo] and references therein. The topic of existence of weak solutions is a difficult one. It started with [DH], and so far global existence can only be established for certain 2×2 systems.

2. Global Existence

In this section we prove Theorem 1.3. We establish the global existence through the energy method. Notice that (1.5) is a symmetrizable hyperbolic system by Theorem 1.1. Here the symmetrization is regarded without the relaxation term. Since local existence and uniqueness are standard for such systems, e.g., see [Ma] and references therein, it is sufficient to prove the following a priori estimate.

Proposition 2.1. *Assume that* (1.8) *is true, and* v^*, e_1^* *and* q^* *are positive constants such that* (1.38) *holds. Let* $m \geqq 2$ *be an integer and* $t_0 > 0$ *be a constant. Let* $(v, u, e_1, q)(x, t)$ *be a solution to* (1.5), (1.37) *satisfying* (1.40) *with* $[0, \infty)$ *replaced by* $[0, t_0]$. *Set*

$$N_m^2(t) = \sup_{0 \leqq t' \leqq t} \left\| \left(v - v^*, u, e_1 - e_1^*, q - q^*\right) \right\|_m^2 (t') + \int_0^t \left(\|p_x\|_{m-1}^2 + \|u_x\|_{m-1}^2 + \|\chi\|_m^2 \right)(t')\, dt', \tag{2.1}$$

and especially,

$$N_m(0) = \left\| \left(v_0 - v^*, u_0, e_{1,0} - e_1^*, q_0 - q^*\right) \right\|_m .$$

Then there exist positive constants ε *and* C, *independent of* t_0, *such that if* $N_m(t_0) \leqq \varepsilon$, *then*

$$N_m(t_0) \leqq C N_m(0). \tag{2.2}$$

Proof. We use C to denote a universal positive constant independent of t_0. Set

$$\mathscr{S}(w) = -s + s^* + \left(\nabla s\right)^* \cdot (w - w^*),$$

where w is the unknown vector in (1.5) as given in (1.11), and w^* is the corresponding constant state,

$$w^* = \left(v^*, 0, e_1^* + q^*, q^*\right).$$

From Theorem 1.1 we know that $-s$ is strictly convex with respect to w. Thus $\mathscr{S}$ is equivalent to $\left|w - w^*\right|^2$, or $\left| \left(v - v^*, u, e_1 - e_1^*, q - q^*\right) \right|^2$, for $N_m(t_0) \leqq \varepsilon$, where ε is small and independent of t_0. From (1.10)–(1.13) and (1.38) we have

$$\mathscr{S}(w)_t = -s_t + \left(\nabla s\right)^* \cdot w_t = -s_t - \left(\nabla s\right)^* \cdot \left(f(w) - f(w^*)\right)_x .$$

Integrate this equality over $\mathbb{R}\times[0,t]$ for $0 \leqq t \leqq t_0$. Use the last equation in (1.36) and notice that

$$\left(\frac{1}{T_2}-\frac{1}{T_1}\right)\frac{\chi}{\tau}=\frac{\omega^{-1}(Q)-\omega^{-1}(q)}{T_1T_2}\frac{\chi}{\tau}\geqq\frac{\chi^2}{C}$$

by (1.6), (1.8) and the smallness of $N_m(t_0)$. We have

$$\left\|\left(v-v^*,u,e_1-e_1^*,q-q^*\right)\right\|^2(t)+\int_0^t\|\chi\|^2(t')\,dt'\leqq CN_0^2(0). \tag{2.3}$$

Next we take derivatives of (1.36) and replace the third equation by a combination of the first and the third ones. Using (1.6) we have

$$\begin{aligned}
&\left(\partial_x^l p\right)_t+\partial_x^l\left(c_f^2u_x\right)=-\partial_x^l\left(\frac{p_{e_1}}{\tau}\chi\right),\\
&\left(\partial_x^l u\right)_t+\partial_x^{l+1}p=0,\\
&\left(\partial_x^l p-\frac{c_f^2}{a}\partial_x^l\chi\right)_t=-\partial_x^l\left(\frac{p_{e_1}}{\tau}\chi\right)+\frac{c_f^2}{a}\partial_x^l\left(\frac{1+Q_{e_1}}{\tau}\chi+au_x\right)\\
&\qquad\qquad -\partial_x^l\left(c_f^2u_x\right)-\left(\frac{c_f^2}{a}\right)_t\partial_x^l\chi,\\
&\left(\partial_x^l s\right)_t=\partial_x^l\left[\left(\omega^{-1}(Q)-\omega^{-1}(q)\right)\frac{\chi}{\tau T_1T_2}\right].
\end{aligned}\tag{2.4}$$

Multiply the first to the fourth equations by $\partial_x^l p$, $c_f^2\partial_x^l u$, $b\left(\partial_x^l p-\frac{c_f^2}{a}\partial_x^l\chi\right)$ and $\partial_x^l s$ respectively, where $b>0$ is defined by (1.26). Add them up and use (1.5) to convert the derivatives of v and e_1 with respect to t into derivatives with respect to x. We have

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\left[\left(\partial_x^l p\right)^2\right]+\frac{\partial}{\partial t}\left[\frac{c_f^2}{2}\left(\partial_x^l u\right)^2\right]+\frac{\partial}{\partial t}\left[\frac{b}{2}\left(\partial_x^l p-\frac{c_f^2}{a}\partial_x^l\chi\right)^2\right]\\
&\quad+\frac{1}{2}\frac{\partial}{\partial t}\left[\left(\partial_x^l s\right)^2\right]+\bar b\left(\partial_x^l\chi\right)^2=-\left(c_f^2\partial_x^l u\,\partial_x^l p\right)_x+\partial_x^l p\left[\left(c_f^2\partial_x^l u\right)_x\right.\\
&\quad\left.-\partial_x^l\left(c_f^2u_x\right)\right]-\partial_x^l p\left[\partial_x^l\left(\frac{p_{e_1}}{\tau}\chi\right)-\frac{p_{e_1}}{\tau}\partial_x^l\chi\right]\\
&\quad+b\left(\partial_x^l p-\frac{c_f^2}{a}\partial_x^l\chi\right)\left\{\left[\frac{p_{e_1}}{\tau}\partial_x^l\chi-\partial_x^l\left(\frac{p_{e_1}}{\tau}\chi\right)\right]\right.\\
&\quad\left.+\left[\frac{c_f^2}{a}\partial_x^l\left(\frac{1+Q_{e_1}}{\tau}\chi\right)-\frac{c_f^2(1+Q_{e_1})}{\tau a}\partial_x^l\chi\right]+\left[\frac{c_f^2}{a}\partial_x^l(au_x)-\partial_x^l\left(c_f^2u_x\right)\right]\right\}\\
&\quad+\partial_x^l s\,\partial_x^l\left[\left(\omega^{-1}(Q)-\omega^{-1}(q)\right)\frac{\chi}{\tau T_1T_2}\right]\\
&\quad-b\left(\partial_x^l p-\frac{c_f^2}{a}\partial_x^l\chi\right)\partial_x^l\chi\left\{\left[\left(\frac{c_f^2}{a}\right)_v-p\left(\frac{c_f^2}{a}\right)_{e_1}\right]u_x-\left(\frac{c_f^2}{a}\right)_{e_1}\frac{\chi}{\tau}\right\}
\end{aligned}\tag{2.5}$$

$$+ c_f\left(\partial_x^l u\right)^2\left\{\left[(c_f)_v - p(c_f)_{e_1}\right]u_x - \frac{(c_f)_{e_1}}{\tau}\chi\right\}$$
$$+ \frac{1}{2}\left(\partial_x^l p - \frac{c_f^2}{a}\partial_x^l \chi\right)^2\left[(b_v - pb_{e_1})u_x - \frac{b_{e_1}}{\tau}\chi\right],$$

where

$$\bar{b} \equiv \frac{c_f^2 p_{e_1}}{\tau a} > 0 \tag{2.6}$$

by (1.9) and (1.8). Integrate this equation over $\mathbb{R} \times [0, t]$ for $0 \leqq t \leqq t_0$. For $1 \leqq l \leqq m$ and small $N_m(t_0)$, we have

$$\|\left(\partial_x^l p, \partial_x^l u, \partial_x^l \chi, \partial_x^l s\right)\|^2(t) + \int_0^t \|\partial_x^l \chi\|^2(t')\,dt' \leqq C\left\{N_l^2(0) + N_m^3(t_0)\right\}. \tag{2.7}$$

Next we apply ∂_x^{l-1} to the third equation in (1.36) and multiply the result by $\frac{1}{a}\partial_x^l u$ to obtain

$$\begin{aligned}\left(\partial_x^l u\right)^2 = &-\left(\frac{1}{a}\partial_x^l u \partial_x^{l-1}\chi\right)_t + \partial_x^l u \partial_x^{l-1}\chi\left\{\left[\left(\frac{1}{a}\right)_v - \left(\frac{1}{a}\right)_{e_1} p\right]u_x - \left(\frac{1}{a}\right)_{e_1}\frac{\chi}{\tau}\right\}\\ &-\left(\frac{1}{a}\partial_x^l p \partial_x^{l-1}\chi\right)_x + \left(\frac{1}{a}\partial_x^{l-1}\chi\right)_x \partial_x^l p\\ &-\left[\frac{1}{a}\partial_x^l u \partial_x^{l-1}(a u_x) - \left(\partial_x^l u\right)^2\right] - \frac{1}{a}\partial_x^l u \partial_x^{l-1}\left(\frac{1 + Q_{e_1}}{\tau}\chi\right),\end{aligned} \tag{2.8}$$

where we have used (1.5). Integrate this equation over $\mathbb{R} \times [0, t]$ for $0 \leqq t \leqq t_0$. Then similarly, for $1 \leqq l \leqq m$ and small $N_m(t_0)$, we have

$$\begin{aligned}\int_0^t \|\partial_x^l u\|^2(t')\,dt' \leqq\; &C\left\{\|\partial_x^l u\|(t)\,\|\partial_x^{l-1}\chi\|(t) + N_l^2(0) + N_m^3(t_0)\right\}\\ &+ C\int_0^t\int_{-\infty}^{\infty}\left|\partial_x^l \chi\,\partial_x^l p\right|(x, t')\,dx\,dt' + C\int_0^t\int_{-\infty}^{\infty}\left|\partial_x^l u \partial_x^{l-1}\chi\right|(x, t')\,dx\,dt'.\end{aligned}$$

Use (2.3) and (2.7) on the right-hand side. After simplifying, we obtain

$$\begin{aligned}\int_0^t \|\partial_x^l u\|^2(t')\,dt' \leqq\; &C\left\{N_l^2(0) + N_m^3(t_0)\right\}\\ &+ C\int_0^t\int_{-\infty}^{\infty}\left|\partial_x^l \chi\,\partial_x^l p\right|(x, t')\,dx\,dt'.\end{aligned} \tag{2.9}$$

Then we apply ∂_x^{l-1} to the second equation in (1.36) and multiply the result by $\partial_x^l p$ to obtain

$$\begin{aligned}\left(\partial_x^l p\right)^2 = &-\left(\partial_x^l p \partial_x^{l-1} u\right)_t + \left(\partial_x^{l-1} p_t \partial_x^{l-1} u\right)_x\\ &+ \partial_x^{l-1}\left(c_f^2 u_x\right)\partial_x^l u + \partial_x^{l-1}\left(\frac{p_{e_1}}{\tau}\chi\right)\partial_x^l u,\end{aligned} \tag{2.10}$$

where we have used the first equation in (2.4). Now integrate this equation over $\mathbb{R} \times [0, t]$ for $0 \leqq t \leqq t_0$. Again, for $1 \leqq l \leqq m$ and small $N_m(t_0)$, we have

$$\int_0^t \|\partial_x^l p\|^2(t')\,dt' \leqq C\big\{\|\partial_x^l p\|(t)\,\|\partial_x^{l-1} u\|(t) + N_l^2(0) + N_m^3(t_0)\big\} + C\int_0^t \|\partial_x^l u\|^2(t')\,dt' + C\int_0^t \|\partial_x^{l-1}\chi\|^2(t')\,dt'.$$

Using (2.3), (2.7) and (2.9) to bound the right-hand side, we arrive at

$$\begin{aligned}\int_0^t \|\partial_x^l p\|^2(t')\,dt' &\leqq C\big\{N_l^2(0) + N_m^3(t_0)\big\} + C\int_0^t \int_{-\infty}^{\infty} \big|\partial_x^l \chi\,\partial_x^l p\big|(x, t')\,dx\,dt' \\ &\leqq C\big\{N_l^2(0) + N_m^3(t_0)\big\} + \frac{1}{2}\int_0^t \|\partial_x^l p\|^2(t')\,dt' \\ &\quad + C\int_0^t \|\partial_x^l \chi\|^2(t')\,dt',\end{aligned}$$

which is further simplified to

$$\int_0^t \|\partial_x^l p\|^2(t')\,dt' \leqq C\big\{N_l^2(0) + N_m^3(t_0)\big\}. \tag{2.11}$$

Using (2.7) and (2.11) to bound the right-hand side of (2.9), we also have

$$\int_0^t \|\partial_x^l u\|^2(t')\,dt' \leqq C\big\{N_l^2(0) + N_m^3(t_0)\big\}. \tag{2.12}$$

Add up (2.7), (2.11), (2.12) for $1 \leqq l \leqq m$ and (2.3). We then obtain

$$\big\|\big(v - v^*, u, e_1 - e_1^*, q - q^*\big)\big\|_m^2(t) + \int_0^t \Big(\|p_x\|_{m-1}^2 + \|u_x\|_{m-1}^2 + \|\chi\|_m^2\Big)(t')\,dt' \leqq C\big\{N_m^2(0) + N_m^3(t_0)\big\}.$$

By (2.1), this gives us

$$N_m^2(t_0) \leqq C\big\{N_m^2(0) + N_m^3(t_0)\big\}.$$

Therefore, for a small ε independent of t_0, $N_m(t_0) \leqq \varepsilon$ implies

$$N_m(t_0) \leqq C N_m(0). \qquad \square$$

3. Fundamental Solutions

In this section we discuss the fundamental solutions of linear systems. The linearized system of (1.36) around the constant state “*” is

$$\begin{aligned} p_t + c_f^{*2} u_x &= -\frac{p_{e_1}^*}{\tau^*}\chi, \\ u_t + p_x &= 0, \\ \chi_t + a^* u_x &= -\frac{1+Q_{e_1}^*}{\tau^*}\chi, \\ s_t &= 0. \end{aligned} \tag{3.1}$$

Here the entropy equation is completely decoupled from the others, and its fundamental solution is simply a Dirac δ-function along the particle path. On the other hand, the other three equations compose a fully dissipative subsystem. (We will verify that it satisfies the dissipative criterion defined in Assumption 3.2 below.) At this point we see that at the linear level, a composite type system is decoupled into two parts: a fully dissipative subsystem and a homogeneous hyperbolic one. That is, although the relaxation gives rise to a dissipative effect, for a certain part of the nonlinear system such an effect is of a higher order. Here for our nonequilibrium flow (1.36), the effect on the entropy equation is second order,

$$s_t = O(\chi^2).$$

For our purpose of constructing a fundamental solution, all we need is to construct one for the dissipative subsystem. The homogeneous part is trivial. In this section, we first formulate the fundamental solution for a general dissipative relaxation system, then apply the result to (3.1). A similar result has been obtained for a viscoelastic model with fading memory, which can be written as a relaxation system, [SZ]. The formulation for a general relaxation system in fact follows the same approach as for a general hyperbolic-parabolic system, which has been studied in a systematic way in [LZ1]. However, there is an important difference between the two. The leading term of the fundamental solution in both cases is the fundamental solution of a diagonalizable uniformly parabolic system. But for a relaxation system, this corresponding parabolic system is reduced in size. Therefore, part of the leading term of the fundamental solution must be identically zero. For this part, it is necessary to find out further details of the next order. This is demanded by the stability analysis for the nonlinear system, carried out in Section 6. In this section we will only outline the key steps. Relevant results will be cited from [LZ1].

Consider a general linear system in the form

$$w_t + Aw_x = Bw, \quad -\infty < x < \infty, \quad t > 0, \tag{3.2}$$

where $w = w(x,t)$ is an n-vector, while A and B are $n \times n$ constant matrices. We make the following basic assumptions for (3.2):

Assumption 3.1. There exists a symmetric positive definite matrix A_0, such that $A_0 A$ is symmetric, and $A_0 B$ is symmetric semi-negative definite.

Assumption 3.2. Any eigenvector of A is not in the null space of B.

Assumption 3.1 is a linear version of the entropy condition consistent with Theorem 1.1, that is, A_0, A, B can be taken, respectively, as the Hessian of the entropy function and Jacobi matrices of the flux function and relaxation vector at the constant state. Assumption 3.2 is the dissipative criterion mentioned in Section 1. The criterion and various equivalent forms were introduced in [SK], and played a crucial role in the study of hyperbolic-parabolic systems, [SK], [Ka], [LZ1]. Assumptions 3.1 and 3.2 are the same assumptions given in Section 6 of [LZ1], which studies the fundamental solution of hyperbolic-parabolic system, except that for a relaxation system it is necessary to change $A_0 B$ from semi-positive definite to semi-negative definite. Note that here we allow A to have multiple eigenvalues, although Assumption 3.1 implies that A has a complete set of eigenvectors, that is, (3.2) is completely hyperbolic, not necessarily strictly hyperbolic.

Let $G(x,t)$ be the Green's function of (3.2). That is, it is an $n \times n$ matrix satisfying

$$\begin{aligned} &G_t + AG_x = BG, \\ &G(x,0) = \delta(x) I, \end{aligned} \tag{3.3}$$

where δ is the Dirac δ-function. The fundamental solution is then $G(x-y, t-t')$. Perform the Fourier transform to (3.3) with respect to x. Use $\hat{G}$ to denote the Fourier transform of G and ξ to denote the Fourier variable. We have

$$\begin{aligned} &\hat{G}_t = (-i\xi A + B)\hat{G}, \\ &\hat{G}(\xi, 0) = I. \end{aligned}$$

Thus

$$\hat{G}(\xi, t) = e^{(-i\xi A + B)t}. \tag{3.4}$$

To obtain the leading term and singular part of G, we need to study the spectral representation of

$$\bar{E}(z) = -zA + B \tag{3.5}$$

as $z \to 0$ and $z \to \infty$ respectively. Notice that

$$\bar{E}(z) = zE\left(-\frac{1}{z}\right), \tag{3.6}$$

where

$$E(z) = -A - zB. \tag{3.7}$$

Since $E(z)$ is exactly the same as defined by (6.11) in [LZ1] with B replaced by $-B$, we have the spectral representation

$$\begin{aligned} &E(z) = \sum_{j=1}^{N} \tilde{\lambda}_j(z) \tilde{P}_j(z), \\ &\tilde{P}_j \tilde{P}_k = \delta_{jk} \tilde{P}_j, \qquad j,k = 1, \dots, N, \qquad \sum_{j=1}^{N} \tilde{P}_j = I, \end{aligned} \tag{3.8}$$

in the whole complex plane except for a finite number of exceptional points z. See (6.24) in [LZ1]. Here the $\tilde{\lambda}_j$ are the distinct eigenvalues of $E(z)$ and the $\tilde{P}_j$ the corresponding eigenprojections. The exceptional points are points where the eigenvalues split. Away from those points the number N of distinct eigenvalues is a constant. Equations (3.4)–(3.6) and (3.8) imply

$$\hat{G}(\xi, t) = \sum_{j=1}^{N} e^{i\xi\tilde{\lambda}_j(-\frac{1}{i\xi})t}\tilde{P}_j\Big(-\frac{1}{i\xi}\Big). \tag{3.9}$$

Lemma 3.1. *Under Assumptions* 3.1 *and* 3.2, *for small* z *we have*

$$\begin{aligned} &e^{z\tilde{\lambda}_j(-1/z)t}\tilde{P}_j(-1/z) \\ &\quad = \begin{cases} e^{z(-\lambda_j^{(r)}+z\mu_j^{(r)}+O(z^2))t}\big(P_j^{(r)}+O(z)\big), & 1 \leqq j \leqq m', \\ O(1)e^{(-c_{j,-1}+O(z))t}, & m' < j \leqq N, \end{cases} \end{aligned} \tag{3.10}$$

where $m' \leqq m$, *and* m *is the multiplicity of the zero eigenvalue of* B. *For* $1 \leqq j \leqq m'$, $\lambda_j^{(r)}$ *are real constants,* $\mu_j^{(r)}$ *are positive constants and* $P_j^{(r)}$ *are constant projections, satisfying*

$$\begin{aligned} &\sum_{j=1}^{m'} \lambda_j^{(r)} P_j^{(r)} = Q_0 A Q_0, \\ &P_j^{(r)} P_k^{(r)} = \delta_{jk} P_j^{(r)}, \qquad j, k = 1, \dots, m', \\ &\sum_{j=1}^{m'} P_j^{(r)} = Q_0, \end{aligned} \tag{3.11}$$

where Q_0 *is the eigenprojection of* B *corresponding to the zero eigenvalue. For* $m' < j \leqq N$, $c_{j,-1}$ *are positive constants. Here we have arranged* $\tilde{\lambda}_j(z)$, $j = 1, \dots, N$, *in an appropriate order.*

Proof. The proof is basically the same as the proof of Lemma 6.12 in [LZ1]. The readers are referred to it for details. Here we only cite some facts from it so we can see clearly where m', $\lambda_j^{(r)}$, $\mu_j^{(r)}$ and $P_j^{(r)}$ are from. These constants and constant projections will represent the leading term in G.

Under Assumptions 3.1 and 3.2, we have the following expansions as $z \to \infty$,

$$\begin{aligned} &\tilde{\lambda}_j(z) = c_{j,-1}z + c_{j,0} + c_{j,1}\frac{1}{z} + \cdots, \\ &\tilde{P}_j(z) = P_{j,0} + P_{j,1}\frac{1}{z} + \cdots, \quad j = 1, \dots, N, \end{aligned} \tag{3.12}$$

see (6.34) in [LZ1]. Here all the coefficients are real, and $c_{j,-1}$, $j = 1, \dots, N$, are eigenvalues of $-B$ with eigenprojections $P_{j,0}$. After rearranging the $\tilde{\lambda}_j(z)$, we have $c_{j,-1} = 0$ for $1 \leqq j \leqq m'$, and $c_{j,-1} > 0$ for $m'+1 \leqq j \leqq N$, with some integer $m' \leqq m$. Introduce the following notation

$$\lambda_j^{(r)} = -c_{j,0}, \quad \mu_j^{(r)} = -c_{j,1}, \quad P_j^{(r)} = P_{j,0}, \quad 1 \leqq j \leqq m', \tag{3.13}$$

where $\mu_j^{(r)} > 0$ by an equivalent form of the dissipative criterion. Equation (3.10) is then straightforward by (3.12) and (3.13). The last two equations in (3.11) are trivial by the property of $P_{j,0}$, while the first one can be obtained by comparing the coefficients for z^0 on both sides of

$$\left(\sum_{j=1}^{m'} \tilde{P}_j(z)\right)(-A - zB)\left(\sum_{j=1}^{m'} \tilde{P}_j(z)\right) = \sum_{j=1}^{m'} \tilde{\lambda}_j(z)\tilde{P}_j(z). \qquad \square$$

The coefficients in (3.12) are completely determined by A and B through a reduction process, see [Kt]. For our purpose of computing m', $\lambda_j^{(r)}$, $\mu_j^{(r)}$ and $P_j^{(r)}$, we cite the result from [LZ1]. There, in the hyperbolic-parabolic case, these constants and constant projections represent the singular part of G instead. Comparing our definition (3.13) with (6.36) in [LZ1], we cite Remark 6.4 in [LZ1] as the following.

Procedure 3.2. Let $l_1^0, \dots, l_m^0$ and $r_1^0, \dots, r_m^0$, respectively, be the left eigenvectors and right eigenvectors of B associated with the zero eigenvalue, satisfying $l_j^0 r_k^0 = \delta_{jk}$, $j, k = 1, \dots, m$. Then the $\lambda_j^{(r)}$ take all the distinct eigenvalues of $L^0 A R^0$, where

$$L^0 = \begin{pmatrix} l_1^0 \\ \vdots \\ l_m^0 \end{pmatrix}, \qquad R^0 = (r_1^0, \dots, r_m^0). \tag{3.14}$$

Let $\nu_1, \dots, \nu_\rho$ be all the nonzero (hence negative) eigenvalues of B with corresponding eigenprojections $Q_1, \dots, Q_\rho$. Set

$$S = \sum_{j=1}^{\rho} \frac{1}{\nu_j} Q_j, \tag{3.15}$$

which is the value at zero of the reduced resolvent of B with respect to the zero eigenvalue.

According to (3.11) the $\lambda_j^{(r)}$ are the eigenvalues of $Q_0 A Q_0$ when restricted to the range of Q_0. Here Q_0 is the eigenprojection of B corresponding to the zero eigenvalue. Then associated with each distinct $\lambda_j^{(r)}$, $Q_0 A Q_0$ has the eigenprojection $\mathcal{Q}_j$. Here if $\lambda_j^{(r)}$ happens to be zero, $\mathcal{Q}_j$ is taken as the subprojection $\mathcal{Q}_j Q_0$. Corresponding to this $\lambda_j^{(r)}$ there may be several $\mu_j^{(r)}$, which are all the nonzero distinct eigenvalues of $-\mathcal{Q}_j ASA\mathcal{Q}_j$. Each $\mu_j^{(r)}$ then associates with an eigenprojection of $-\mathcal{Q}_j ASA\mathcal{Q}_j$, which is the corresponding $P_j^{(r)}$. Lastly m' is the total number of the $\mu_j^{(r)}$ so constructed for all the distinct $\lambda_j^{(r)}$.

In the special case that all the $\lambda_j^{(r)}$ are simple, we have

$$m' = m, \qquad P_j^{(r)} \text{ are determined by (3.11)},$$
$$\mu_j^{(r)} = \operatorname{tr}\big(-ASAP_j^{(r)}\big), \qquad j = 1, \ldots, m. \tag{3.16}$$

To study the singular part of G, we introduce the following notation. Assumption 3.1 implies that all the eigenvalues of A are real, and that A has a complete set of eigenvectors. Denote all the distinct eigenvalues of A as $\lambda_1, \lambda_2, \ldots, \lambda_{n'}$, with multiplicities $m_1, m_2, \ldots, m_{n'}$, $m_1 + m_2 + \cdots + m_{n'} = n$. Denote the left eigenvectors and right eigenvectors associated with λ_j as $l_k^{(j)}$ and $r_k^{(j)}$, respectively, $k = 1, \ldots, m_j$, satisfying

$$Ar_k^{(j)} = \lambda_j r_k^{(j)}, \qquad l_k^{(j)} A = \lambda_j l_k^{(j)}, \qquad l_k^{(j)} r_{k'}^{(j')} = \delta_{jj'}\delta_{kk'},$$
$$j, j' = 1, \ldots, n', \qquad k = 1, \ldots, m_j, \qquad k' = 1, \ldots, m_{j'}. \tag{3.17}$$

Set

$$l^{(j)} = \begin{pmatrix} l_1^{(j)} \\ \vdots \\ l_{m_j}^{(j)} \end{pmatrix}, \qquad r^{(j)} = \big(r_1^{(j)}, \ldots, r_{m_j}^{(j)}\big), \qquad j = 1, \ldots, n',$$
$$L = \begin{pmatrix} l^{(1)} \\ \vdots \\ l^{(n')} \end{pmatrix}, \qquad R = \big(r^{(1)}, \ldots, r^{(n')}\big). \tag{3.18}$$

Clearly

$$LR = I.$$

Since B in general is degenerate, LBR is degenerate as well. Assumptions 3.1 and 3.2, however, guarantee that each diagonal block $l^{(j)}Br^{(j)}$ is similar to an $m_j \times m_j$ symmetric negative definite matrix, $j = 1, \ldots, n'$. See Lemma 2.1 in [LZ1]. Therefore, we may choose the $l_k^{(j)}$ and $r_k^{(j)}$ appropriately such that

$$l^{(j)}Br^{(j)} = -\operatorname{diag}(\mu_{j1}, \ldots, \mu_{jm_j}),$$
$$\mu_{jk} > 0, \quad k = 1, \ldots, m_j, \quad j = 1, \ldots, n'. \tag{3.19}$$

For $l_k^{(j)}$ and $r_k^{(j)}$ so chosen, define

$$\bar{B} = R \operatorname{diag}\big(l^{(1)}Br^{(1)}, \ldots, l^{(n')}Br^{(n')}\big)L. \tag{3.20}$$

By (3.17)–(3.20), we have

$$A = \sum_{j=1}^{n'}\sum_{k=1}^{m_j} \lambda_j r_k^{(j)} l_k^{(j)},$$
$$\bar{B} = -\sum_{j=1}^{n'}\sum_{k=1}^{m_j} \mu_{jk} r_k^{(j)} l_k^{(j)}. \tag{3.21}$$

On the other hand, by Lemmas 6.8 and 6.10 in [LZ1], $\tilde{\lambda}_j(z)$ and $\tilde{P}_j(z)$ are holomorphic at the origin under Assumption 3.1, satisfying

$$\begin{aligned} A &= -\sum_{j=1}^{N} \tilde{\lambda}_j(0)\tilde{P}_j(0), \\ \bar{B} &= -\sum_{j=1}^{N} \tilde{\lambda}'_j(0)\tilde{P}_j(0). \end{aligned} \tag{3.22}$$

Therefore,

$$\begin{aligned} -zA + \bar{B} &= \sum_{j=1}^{N} \big(\tilde{\lambda}_j(0)z - \tilde{\lambda}'_j(0)\big)\tilde{P}_j(0) \\ &= \sum_{j=1}^{n'}\sum_{k=1}^{m_j} (-\lambda_j z - \mu_{jk}) r_k^{(j)} l_k^{(j)}. \end{aligned}$$

Note that by (3.8),

$$\tilde{P}_j(0)\tilde{P}_k(0) = \delta_{jk}\tilde{P}_j(0), \qquad j,k = 1,\ldots,N, \qquad \sum_{j=1}^{N}\tilde{P}_j(0) = I.$$

Also note the property of $r_k^{(j)}$ and $l_k^{(j)}$ in (3.17). We have

$$\begin{aligned} e^{(-zA+\bar{B})t} &= \sum_{j=1}^{N} e^{\big(\tilde{\lambda}_j(0)z - \tilde{\lambda}'_j(0)\big)t}\tilde{P}_j(0) \\ &= \sum_{j=1}^{n'}\sum_{k=1}^{m_j} e^{(-\lambda_j z - \mu_{jk})t} r_k^{(j)} l_k^{(j)}. \end{aligned} \tag{3.23}$$

Lemma 3.3. *Under Assumptions 3.1 and 3.2, for large z we have*

$$\begin{aligned} \sum_{j=1}^{N} e^{z\tilde{\lambda}_j(-1/z)t}\tilde{P}_j(-1/z) &= \sum_{j=1}^{n'}\sum_{k=1}^{m_j} e^{-(\lambda_j z + \mu_{jk})t} r_k^{(j)} l_k^{(j)} \\ &+ \sum_{j=1}^{N} e^{\big(\tilde{\lambda}_j(0)z - \tilde{\lambda}'_j(0)\big)t}\big[z^{-1}c_j + O(z^{-2})(1+t+t^2e^{O(tz^{-1})})\big], \end{aligned} \tag{3.24}$$

where λ_j, $r_k^{(j)}$ and $l_k^{(j)}$ are, respectively, the eigenvalues, right eigenvectors and left eigenvectors of A satisfying (3.17)*; $l_k^{(j)}$, $r_k^{(j)}$ and $\mu_{jk} > 0$ satisfy* (3.19) *with $l^{(j)}$ and $r^{(j)}$ defined in* (3.18)*; for $j = 1\ldots, N$, $\tilde{\lambda}_j(0)$ are real constants, $\tilde{\lambda}'_j(0)$ are positive constants, and c_j are polynomial matrices in t with degrees not more than 1.*

Proof. Since $\tilde{\lambda}_j(z)$ and $\tilde{P}_j(z)$ are holomorphic at the origin, for large z we have

$$\begin{aligned}
&\sum_{j=1}^{N} e^{z\tilde{\lambda}_j(-1/z)t}\tilde{P}_j(-1/z)\\
&\quad=\sum_{j=1}^{N} e^{\left(\tilde{\lambda}_j(0)z-\tilde{\lambda}_j'(0)+\frac{1}{2}\tilde{\lambda}_j''(0)z^{-1}+O(z^{-2})\right)t}\left[\tilde{P}_j(0)-\tilde{P}_j'(0)z^{-1}+O(z^{-2})\right]\\
&\quad=\sum_{j=1}^{N} e^{\left(\tilde{\lambda}_j(0)z-\tilde{\lambda}_j'(0)\right)t}\tilde{P}_j(0)+\sum_{j=1}^{N} e^{\left(\tilde{\lambda}_j(0)z-\tilde{\lambda}_j'(0)\right)t}\left\{z^{-1}\left[\tfrac{1}{2}\tilde{\lambda}_j''(0)t\tilde{P}_j(0)-\tilde{P}_j'(0)\right]\right.\\
&\qquad\left.+O(z^{-2})(1+t+t^2e^{O(tz^{-1})})\right\}.
\end{aligned}$$

Applying (3.23) to the first summation on the right gives us (3.24). Equations (3.21) and (3.22) imply that for $l=1,\ldots,N$, each $\tilde{\lambda}_l(0)$ is one of the $-\lambda_j$, and each $\tilde{\lambda}_l'(0)$ is one of the μ_{jk}. Hence they have the property stated in the Lemma. □

With Lemmas 3.1 and 3.3, we subtract the leading term and the singular part from G and set

$$\begin{aligned}
\mathscr{R}(x,t)=G(x,t)&-\sum_{j=1}^{m'}\frac{1}{\sqrt{4\pi\mu_j^{(r)}t}}e^{-\frac{\left(x-\lambda_j^{(r)}t\right)^2}{4\mu_j^{(r)}t}}P_j^{(r)}\\
&-\sum_{j=1}^{n'}\sum_{k=1}^{m_j}e^{-\mu_{jk}t}\delta(x-\lambda_jt)r_k^{(j)}l_k^{(j)}.
\end{aligned}\tag{3.25}$$

By (3.9),

$$\begin{aligned}
\mathscr{R}(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}&\left\{\sum_{j=1}^{N}e^{i\xi\tilde{\lambda}_j(-\frac{1}{i\xi})t}\tilde{P}_j\left(-\frac{1}{i\xi}\right)\right.\\
&\left.-\sum_{j=1}^{m'}e^{i\xi\left(-\lambda_j^{(r)}+i\xi\mu_j^{(r)}\right)t}P_j^{(r)}-\sum_{j=1}^{n'}\sum_{k=1}^{m_j}e^{-(\lambda_ji\xi+\mu_{jk})t}r_k^{(j)}l_k^{(j)}\right\}e^{ix\xi}\,d\xi.
\end{aligned}\tag{3.26}$$

Under Assumption 3.1, Assumption 3.2 is equivalent to

$$\mathrm{Re}\left\{i\xi\tilde{\lambda}_j\left(-\frac{1}{i\xi}\right)\right\}<0,\quad j=1,\ldots,N,$$

for any real $\xi \neq 0$, [SK]. Also, it is easy to see by (3.4) that $\hat{G}(\xi,t)$ is an entire function of ξ. Using these facts and Lemmas 3.1 and 3.3, we can estimate the right-hand side of (3.26) in the same way as in [LZ1] and obtain the following lemmas parallel to Lemmas 6.13 and 6.14 in that reference.

Lemma 3.4. *Under Assumptions 3.1 and 3.2, for all $-\infty < x < \infty$, $t > 0$, we have*

$$|\mathscr{R}(x,t)| \leqq C(t+1)^{-1}\sum_{j=1}^{m'} e^{-\frac{\left(x-\lambda_j^{(r)}t\right)^2}{Ct}} + Ct^{-1/2}e^{-t/C} + Ce^{-t/C}\sum_{j=1}^{n'}|x-\lambda_j t|. \tag{3.27}$$

Lemma 3.5. *Let $K > 0$ be large. Under Assumptions 3.1 and 3.2, for all $-\infty < x < \infty$, $t > 0$, if $|x|/t \geqq K$, then*

$$|\mathscr{R}(x,t)| \leqq Ct^{-\frac{1}{2}}\left(e^{-\frac{x^2}{Ct}} + \sum_{j=1}^{m'} e^{-\frac{\left(x-\lambda_j^{(r)}t\right)^2}{Ct}}\right). \tag{3.28}$$

Equation (3.25) and Lemmas 3.4 and 3.5 immediately give us the following main theorem of this section for the case $l = 0$. The case $l \geqq 1$ can be proved in a similar way.

Theorem 3.6. *Suppose that Assumptions 3.1 and 3.2 are satisfied, and that B has the zero eigenvalue of multiplicity m, $0 \leqq m < n$. Let the integer m', constants $\lambda_j^{(r)}$, positive constants $\mu_j^{(r)}$, and constant projections $P_j^{(r)}$, $1 \leqq j \leqq m'$, be computed from A and B through Procedure 3.2. Let λ_j, $1 \leqq j \leqq n'$, be all the distinct eigenvalues of A with multiplicities m_j, $\sum_{j=1}^{n'} m_j = n$. For each λ_j, let $l_k^{(j)}$ and $r_k^{(j)}$, $1 \leqq k \leqq m_j$, be the left eigenvectors and the right eigenvectors, respectively, satisfying* (3.17). *Moreover, let $l_k^{(j)}$ and $r_k^{(j)}$ be so chosen such that* (3.19) *is true, where $l^{(j)}$ and $r^{(j)}$ are defined in* (3.18). *Then for $-\infty < x < \infty$, $t \geqq 0$, the Green's function G of system (3.2) has the property*

$$\begin{aligned}\frac{\partial^l}{\partial x^l}G(x,t) &= \frac{\partial^l}{\partial x^l}\left[\sum_{j=1}^{m'}\frac{1}{\sqrt{4\pi\mu_j^{(r)}t}}e^{-\frac{\left(x-\lambda_j^{(r)}t\right)^2}{4\mu_j^{(r)}t}}P_j^{(r)}\right] \\ &\quad + O(1)(t+1)^{-\frac{1}{2}}t^{-\frac{l+1}{2}}\sum_{j=1}^{m'}e^{-\frac{\left(x-\lambda_j^{(r)}t\right)^2}{Ct}} \\ &\quad + \sum_{j=1}^{n'}\sum_{k=1}^{m_j}e^{-\mu_{jk}t}\sum_{i=0}^{l}\delta^{(l-i)}(x-\lambda_j t)P_{jk}^{(i)}(t),\end{aligned} \tag{3.29}$$

where $l \geqq 0$ is any integer; $C > 0$ is a constant; δ is the Dirac δ-function; and for $1 \leqq j \leqq n'$, $1 \leqq k \leqq m_j$, $\mu_{jk} > 0$ are given by (3.19), *while $P_{jk}^{(i)}(t)$, $0 \leqq i \leqq l$, are $n \times n$ polynomial matrices in t with degrees not more than i, in particular, $P_{jk}^{(0)}(t) = r_k^{(j)} l_k^{(j)}$.*

Remark 3.7. It is not surprising that there are n δ-functions in $G(x,t)$ since (3.2) is hyperbolic under Assumption 3.1. These δ-functions describe how an initial singularity will propagate. For a smooth solution, however, they are important only when B has full rank. In that case $m' = m = 0$ and the heat kernels in (3.29) disappear. Otherwise, if B is degenerate, we always have $m' \geqq 1$. The heat kernels in (3.29) then represent the leading term in the solution since the δ-functions decay exponentially.

Remark 3.8. Replace B by $\frac{1}{\tau}B$ in (3.2), where the constant $\tau > 0$ is considered as the relaxation time. Denote the corresponding Green's function as $G(x,t;\tau)$. Then by (3.3) it is easy to see that $G(x,t;\tau) = \frac{1}{\tau}G(x/\tau, t/\tau; 1)$. In Section 1 we know that the limiting case as $\tau \to 0$ corresponds to the equilibrium flow. If we let $\tau \to 0$, the δ-functions in (3.29) disappear since $\mu_{jk}/\tau \to +\infty$. The second summation goes to zero as well, while the first one becomes a combination of (nondecaying) δ-functions along $\lambda_j^{(r)}$ directions, $1 \leqq j \leqq m'$. In fact, the $\lambda_j^{(r)}$ are the characteristic speeds of the equilibrium system as we will see in a more general case in the next section. As $\tau \to 0$, $G(x,t;\tau)$ becomes the Green's function of the equilibrium system.

Our next step is to refine the part of G that becomes zero in the leading term, as mentioned at the beginning of this section. Let η be any row vector in $\mathbb{R}^n$ such that $\eta Q_0 = 0$, where Q_0 is the eigenprojection of B corresponding to the zero eigenvalue. From (3.11) it is easy to see that

$$\eta P_j^{(r)} = \eta Q_0 P_j^{(r)} = 0, \quad j = 1, \ldots, m'.$$

Thus for $\eta G(x,t)$, the first summation in (3.29) becomes zero. We now want to find out the next order term in detail. First we refine Lemma 3.1 as the following: For small z,

$$\eta e^{z\tilde{\lambda}_j(-1/z)t}\tilde{P}_j(-1/z) = \begin{cases} e^{z\left(-\lambda_j^{(r)} + z\mu_j^{(r)} + O(z^2)\right)t}\left(\eta_j z + O(z^2)\right), & 1 \leqq j \leqq m', \\ O(1)e^{(-c_{j,-1}+O(z))t}, & m' < j \leqq N, \end{cases}$$

where $\eta_j = -\eta P_{j,1}$ is a constant row vector, $P_{j,1}$ defined in (3.12), $1 \leqq j \leqq m'$. With this we can then prove the following theorem similar to Theorem 3.6.

Theorem 3.9. *Let η be any constant row vector in $\mathbb{R}^n$ such that $\eta Q_0 = 0$, where Q_0 is the eigenprojection of B corresponding to the zero eigenvalue. Then under the same assumptions and same notation as in Theorem* 3.6, *we have*

$$\begin{aligned}\frac{\partial^l}{\partial x^l}(\eta G)(x,t) =& \frac{\partial^{l+1}}{\partial x^{l+1}}\left[\sum_{j=1}^{m'} \frac{1}{\sqrt{4\pi\mu_j^{(r)}t}} e^{-\frac{\left(x-\lambda_j^{(r)}t\right)^2}{4\mu_j^{(r)}t}} \eta_j\right] \\ &+ O(1)(t+1)^{-\frac{1}{2}} t^{-\frac{l}{2}-1} \sum_{j=1}^{m'} e^{-\frac{\left(x-\lambda_j^{(r)}t\right)^2}{Ct}} \\ &+ \sum_{j=1}^{n'}\sum_{k=1}^{m_j} e^{-\mu_{jk}t} \sum_{i=0}^{l} \delta^{(l-i)}(x-\lambda_j t)\eta P_{jk}^{(i)}(t)\end{aligned} \tag{3.30}$$

for $l \geqq 0$, where η_j, $1 \leqq j \leqq m'$, are constant row vectors.

To finish this section, we apply Theorem 3.6 to gas dynamics in thermal nonequilibrium. Write the first three equations in (3.1) in the form (3.2),

$$w_t + Aw_x = Bw, \tag{3.31a}$$

where

$$w = (p, u, \chi)^t,$$

$$A = \begin{pmatrix} 0 & c_f^{*2} & 0 \\ 1 & 0 & 0 \\ 0 & a^* & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & -\frac{p_{e_1}^*}{\tau^*} \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1+Q_{e_1}^*}{\tau^*} \end{pmatrix}. \tag{3.31b}$$

Let

$$A_0 = \begin{pmatrix} 1+b^* & 0 & -b^*c_f^{*2}/a^* \\ 0 & c_f^{*2} & 0 \\ -b^*c_f^{*2}/a^* & 0 & b^*c_f^{*4}/a^* \end{pmatrix},$$

where b is defined by (1.26). Clearly A_0 is symmetric and positive definite by (1.27) and (1.9). It is easy to verify that A_0A is symmetric, and A_0B is symmetric and semi-negative definite. Therefore, Assumption 3.1 is satisfied.

From (3.31*b*), A has eigenvalues

$$\lambda_1 = -c_f^*, \qquad \lambda_2 = 0, \qquad \lambda_3 = c_f^*,$$

which are simple. The left eigenvectors are

$$l^{(1)} = (1, -c_f^*, 0), \qquad l^{(2)} = (-a^*/c_f^{*2}, 0, 1), \qquad l^{(3)} = (1, c_f^*, 0), \tag{3.32a}$$

right eigenvectors are

$$r^{(1)} = -\frac{1}{2c_f^*}\begin{pmatrix} -c_f^* \\ 1 \\ -a^*/c_f^* \end{pmatrix}, \quad r^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad r^{(3)} = \frac{1}{2c_f^*}\begin{pmatrix} c_f^* \\ 1 \\ a^*/c_f^* \end{pmatrix}, \tag{3.32b}$$

satisfying $l^{(j)}r^{(k)} = \delta_{jk}$, $1 \leqq j, k \leqq 3$. Obviously, none of $r^{(j)}$ is in the null space of B. Assumption 3.2 is satisfied as well. For

$$\mu_j = -l^{(j)}Br^{(j)}, \quad 1 \leqq j \leqq 3,$$

we have

$$\mu_1 = \mu_3 = \frac{p^*_{e_1} a^*}{2\tau^* c_f^{*2}} > 0, \quad \mu_2 = \frac{p^*_{e_1} a^*}{b^* \tau^* c_f^{*2}} > 0. \tag{3.33}$$

To compute the leading term of G, notice that B has a zero eigenvalue of multiplicity $m = 2$. Following Procedure 3.2, we have

$$L^0 = \begin{pmatrix} 1 & 0 & -\frac{p^*_{e_1}}{1+Q^*_{e_1}} \\ 0 & 1 & 0 \end{pmatrix}, \quad R^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Thus by (1.25),

$$L^0 A R^0 = \begin{pmatrix} 0 & c^{*2} \\ 1 & 0 \end{pmatrix}.$$

This gives

$$\lambda_1^{(r)} = -c^*, \quad \lambda_2^{(r)} = c^*,$$

which are simple. Consequently, $m' = 2$, and $P_1^{(r)}$ and $P_2^{(r)}$ are determined by (3.11). That is, they are the eigenprojections of $R^0 L^0 A R^0 L^0$ when restricted to the range of $R^0 L^0$:

$$P_1^{(r)} = \begin{pmatrix} \frac{1}{2} & -\frac{c^*}{2} & -\frac{p^*_{e_1}}{2(1+Q^*_{e_1})} \\ -\frac{1}{2c^*} & \frac{1}{2} & \frac{p^*_{e_1}}{2c^*(1+Q^*_{e_1})} \\ 0 & 0 & 0 \end{pmatrix}, \quad P_2^{(r)} = \begin{pmatrix} \frac{1}{2} & \frac{c^*}{2} & -\frac{p^*_{e_1}}{2(1+Q^*_{e_1})} \\ \frac{1}{2c^*} & \frac{1}{2} & -\frac{p^*_{e_1}}{2c^*(1+Q^*_{e_1})} \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.34}$$

The only nonzero eigenvalue of B is

$$\nu_1 = -\frac{1 + Q^*_{e_1}}{\tau^*},$$

with corresponding eigenprojection

$$Q_1 = \left(\frac{p^*_{e_1}}{1 + Q^*_{e_1}}, 0, 1 \right)^t (0, 0, 1).$$

The dissipative parameters in the leading term of G are

$$\mu_j^{(r)} = \operatorname{tr}\left(-\frac{1}{\nu_1} A Q_1 A P_j^{(r)} \right) = \frac{\tau^* a^* p^*_{e_1}}{2(1 + Q^*_{e_1})^2} \equiv \frac{\mu^*}{2}, \quad j = 1, 2. \tag{3.35}$$

Comparing this with (1.35b) we see that μ^* is exactly the value taken at the constant state “$*$” by the “viscosity coefficient” in the second-order Chapman-Enskog expansion. Here we notice that the constant state is an equilibrium state.

We are now ready to write down (3.29) for system (3.31): Under physical assumptions (1.8), for $-\infty < x < \infty$, $t \geqq 0$, the Green's function G of (3.31) has the property

$$\begin{aligned}\frac{\partial^l}{\partial x^l}G(x,t) = \frac{\partial^l}{\partial x^l}\left[\frac{1}{\sqrt{2\pi\mu^* t}}\left(e^{-\frac{(x+c^*t)^2}{2\mu^* t}}P_1^{(r)} + e^{-\frac{(x-c^*t)^2}{2\mu^* t}}P_2^{(r)}\right)\right] \\ + O(1)(t+1)^{-\frac{1}{2}}t^{-\frac{l+1}{2}}\left(e^{-\frac{(x+c^*t)^2}{Ct}} + e^{-\frac{(x-c^*t)^2}{Ct}}\right) \\ + \sum_{i=0}^{l}\left[e^{-\mu_1 t}\delta^{(l-i)}(x+c_f^*t)P_1^{(i)}(t) + e^{-\mu_2 t}\delta^{(l-i)}(x)P_2^{(i)}(t) \right. \\ \left. + e^{-\mu_3 t}\delta^{(l-i)}(x-c_f^*t)P_3^{(i)}(t)\right].\end{aligned} \tag{3.36}$$

Here $l \geqq 0$ is any integer. At the constant state "$*$" that is an equilibrium state, the equilibrium sound speed c^* and the frozen sound speed c_f^* satisfy $0 < c^* < c_f^*$ by Proposition 1.2. The dissipative parameters $\mu^* > 0$ and $\mu_j > 0$, $1 \leqq j \leqq 3$, are given in (3.35) and (3.33). The constant projections $P_1^{(r)}$ and $P_2^{(r)}$ are given by (3.34). $C > 0$ is a constant. And $P_j^{(i)}(t)$, $1 \leqq j \leqq 3$, $0 \leqq i \leqq l$, are 3×3 polynomial matrices in t with degrees not more than i. Especially, $P_j^{(0)}(t) = r^{(j)}l^{(j)}$, where $r^{(j)}$ and $l^{(j)}$ are given by (3.32).

Notice that in (3.34), the third rows of $P_1^{(r)}$ and $P_2^{(r)}$ are zero. The only linearly independent η in Theorem 3.9 is $\eta = (0, 0, 1)$. Therefore, if we denote the third row of G as G_3, (3.30) becomes

$$\begin{aligned}\frac{\partial^l}{\partial x^l}G_3(x,t) = \frac{\partial^{l+1}}{\partial x^{l+1}}\left[\frac{1}{\sqrt{2\pi\mu^* t}}\left(e^{-\frac{(x+c^*t)^2}{2\mu^* t}}\eta_1 + e^{-\frac{(x-c^*t)^2}{2\mu^* t}}\eta_2\right)\right] \\ + O(1)(t+1)^{-\frac{1}{2}}t^{-\frac{l}{2}-1}\left(e^{-\frac{(x+c^*t)^2}{Ct}} + e^{-\frac{(x-c^*t)^2}{Ct}}\right) \\ + \sum_{i=0}^{l}\left[e^{-\mu_1 t}\delta^{(l-i)}(x+c_f^*t)P_{1,3}^{(i)}(t) + e^{-\mu_2 t}\delta^{(l-i)}(x)P_{2,3}^{(i)}(t) \right. \\ \left. + e^{-\mu_3 t}\delta^{(l-i)}(x-c_f^*t)P_{3,3}^{(i)}(t)\right]\end{aligned} \tag{3.37}$$

for $l \geqq 0$, where η_1 and η_2 are constant row vectors in $\mathbb{R}^3$, and $P_{j,3}^{(i)}$ is the third row of $P_j^{(i)}$, $1 \leqq j \leqq 3$, $0 \leqq i \leqq l$.

4. Subcharacteristic Conditions and the Dissipative Criterion

In our study of a fundamental solution of a general linear relaxation system in the last section, we made two basic assumptions. Assumption 3.1 implies the nonnegativity of the dissipative parameters in the Green's function. Assumption

3.2 further guarantees that they cannot be zero, hence must be positive. In other words, Assumption 3.1 is related to stability, while Assumption 3.2 is related to dissipativeness. This is why it is called the dissipative criterion. Our study was based on spectral properties of the matrix $\bar{E}(z) = -zA + B = zE(-\frac{1}{z})$, where $E(z) = -A - zB$, (3.5)–(3.7). Since $E(z)$ is exactly the matrix occurring in the Fourier transform for the hyperbolic-parabolic system

$$w_t + Aw_x = (-B)w_{xx},$$

we were able to make use of the spectral properties of $E(z)$ obtained in [LZ1]. In that paper, the dissipative criterion played a crucial role in several places, and an equivalence theorem from [SK] was used. The theorem says that under Assumption 3.1, Assumption 3.2 has several equivalent forms. Assumption 3.2 is the simplest one to verify. The others give a bound on the real part of eigenvalues of $zE(z)$, necessary for the decay analysis, or give rise to a good term in the energy estimate when establishing the global existence for (nonlinear) hyperbolic-parabolic systems. Clearly, under Assumption 3.1, Assumption 3.2 is also equivalent to the following: Any left eigenvector of A is not in the left null space of B. When this criterion fails, we have a nonzero row vector l such that $lA = \lambda l$ and $lB = 0$. System (3.2) then yields

$$(lw)_t + \lambda(lw)_x = 0,$$

which is a decoupled hyperbolic equation. Therefore, (3.2) is necessarily of composite type in such a situation.

As mentioned in Section 1, subcharacteristic conditions are important for stability in a relaxation context. The purpose of this section is to clarify the relation between subcharacteristic conditions and the dissipative criterion. We want to investigate whether or not a subcharacteristic condition implies dissipation. In this section, our discussion is for a general system.

Following the set-up in [CLL], we consider the general nonlinear system (1.10),

$$w_t + f(w)_x = r(w), \tag{4.1}$$

where $w, f, r \in \mathbb{R}^n$. Assume that w is in an open convex set $\mathbb{O} \in \mathbb{R}^n$, f and r are smooth, and the system is completely hyperbolic. That is, f' has only real eigenvalues and is diagonalizable. The relaxation term r is assumed to be a vector field that leaves $\mathbb{O}$ invariant under the flow

$$\frac{dw}{dt} = r(w), \tag{4.2}$$

such that it has $m < n$ independent conserved quantities. That is, there exists a constant matrix $L^0 \in \mathbb{R}^{m\times n}$ with rank m, such that

$$L^0 r(w) = 0 \tag{4.3}$$

for every $w \in \mathbb{O}$. The conserved quantities are then $w^{(r)} \equiv L^0 w$. Moreover, assume that each orbit of (4.2) has an equilibrium that is uniquely determined by the constants $w^{(r)}$. Denote the equilibrium as $\mathscr{E}(w^{(r)})$,

$$r\big(\mathscr{E}(w^{(r)})\big) = 0. \tag{4.4}$$

Then for $w^{(r)} \in L^0\mathbb{O} \subset \mathbb{R}^m$,

$$L^0\mathscr{E}(w^{(r)}) = w^{(r)}, \tag{4.5}$$

$$L^0\mathscr{E}'(w^{(r)}) = I_{m\times m}, \tag{4.6}$$

and r' has rank $n-m$. Multiplying (4.1) by L^0 from the left, we obtain m conservation laws

$$w^{(r)}{}_t + L^0 f(w)_x = 0.$$

These can be closed as a reduced system for $w^{(r)}$ if we make the local equilibrium approximation

$$w = \mathscr{E}(w^{(r)}).$$

That is, the reduced system (equilibrium system) is

$$w^{(r)}{}_t + f^{(r)}(w^{(r)})_x = 0, \tag{4.7a}$$

where

$$f^{(r)}(w^{(r)}) = L^0 f\big(\mathscr{E}(w^{(r)})\big). \tag{4.7b}$$

A subcharacteristic condition is a relation between the eigenvalues of (4.1) and of (4.7). To give a sufficient condition for such a relation, we need to introduce an entropy condition given in [CLL].

Definition 4.1. A twice-differentiable function $U : \mathbb{O} \to \mathbb{R}$ is said to be an *entropy* for system (4.1) provided

(i) $\nabla^2 U(w) f'(w)$ is symmetric;
(ii) $\nabla U(w) r(w) \leqq 0$;
(iii) the following statements are equivalent,
 (a) $r(w) = 0$,
 (b) $\nabla U(w) r(w) = 0$,
 (c) $\nabla U(w)$ is a linear combination of the row vectors of L^0.

Theorem 4.2 ([CLL]**).** *Assume the existence of a strictly convex entropy U for system* (4.1). *Then the reduced system* (4.7) *is hyperbolic with a strictly convex entropy $U(\mathscr{E}(w^{(r)}))$. Repeated with multiplicities, let the eigenvalues of $f'(w)$ be*

$$\lambda_1(w) \leqq \lambda_2(w) \leqq \cdots \leqq \lambda_n(w), \tag{4.8}$$

and the eigenvalues of $\big(f^{(r)}(w^{(r)})\big)'$ be

$$\lambda_1^{(r)}(w^{(r)}) \leqq \lambda_2^{(r)}(w^{(r)}) \leqq \cdots \leqq \lambda_m^{(r)}(w^{(r)}). \tag{4.9}$$

Then we have the following subcharacteristic condition

$$\lambda_j\big(\mathscr{E}(w^{(r)})\big) \leqq \lambda_j^{(r)}(w^{(r)}) \leqq \lambda_{j+n-m}\big(\mathscr{E}(w^{(r)})\big), \quad 1 \leqq j \leqq m. \tag{4.10}$$

Next we relate $\lambda_j^{(r)}$ in (4.9) to those defined in Procedure 3.2. From (4.3) we have

$$L^0 r'(w) = 0.$$

Since r' has rank $n - m$, the multiplicity of the zero eigenvalue of r' is m, and the row vectors of L^0 are m linearly independent left eigenvectors associated with the zero eigenvalue. From (4.4) we also have

$$r'\big(\mathscr{E}(w^{(r)})\big)\mathscr{E}'(w^{(r)}) = 0.$$

Let $R^0 \equiv \mathscr{E}'(w^{(r)})$. With (4.6), the column vectors of R^0 are then m linearly independent eigenvectors of $r'\big(\mathscr{E}(w^{(r)})\big)$ associated with the zero eigenvalue, satisfying $L^0 R^0 = I_{m\times m}$. From (4.7b) we have

$$\big(f^{(r)}(w^{(r)})\big)' = L^0 f'\big(\mathscr{E}(w^{(r)})\big)R^0. \tag{4.11}$$

This immediately implies the following proposition.

Proposition 4.3. *Let $A = f'\big(\mathscr{E}(w^{(r)})\big)$ and $B = r'\big(\mathscr{E}(w^{(r)})\big)$. Then the $\lambda_j^{(r)}$ defined in Procedure* 3.2 *are the characteristic values of the reduced system* (4.7).

Therefore, if Assumptions 3.1 and 3.2 are satisfied, the heat kernels in the Green's function of the linearization are along the characteristic directions of the reduced system, cf. (3.29).

The proof of Theorem 4.2 was by Legendre dual functions. It is also easily seen from Definition 4.1(iii) and direct calculation that

$$\nabla_{w^{(r)}} U\big(\mathscr{E}(w^{(r)})\big) = \nabla U\big(\mathscr{E}(w^{(r)})\big)R^0 = \eta L^0 R^0 = \eta$$

for some $\eta = (\eta_1, \cdots, \eta_m)$. Therefore,

$$\nabla_{w^{(r)}} U\big(\mathscr{E}(w^{(r)})\big)L^0 = \eta L^0 = \nabla U\big(\mathscr{E}(w^{(r)})\big).$$

Differentiating both sides yields

$$\nabla^2_{w^{(r)}} U\big(\mathscr{E}(w^{(r)})\big)L^0 = (R^0)^t \nabla^2 U\big(\mathscr{E}(w^{(r)})\big). \tag{4.12}$$

That is,

$$\nabla^2_{w^{(r)}} U\big(\mathscr{E}(w^{(r)})\big) = (R^0)^t \nabla^2 U\big(\mathscr{E}(w^{(r)})\big)R^0, \tag{4.13}$$

which is positive definite if $\nabla^2 U$ is positive definite. By (4.11) and (4.12), the left-hand side of (4.13) symmetrizes the reduced system:

$$\nabla^2_{w^{(r)}} U\big(\mathscr{E}(w^{(r)})\big)\big(f^{(r)}(w^{(r)})\big)' = (R^0)^t \nabla^2 U\big(\mathscr{E}(w^{(r)})\big) f'\big(\mathscr{E}(w^{(r)})\big)R^0.$$

To discuss the relation between subcharacteristic conditions and the dissipative criterion, we now turn our attention to a general linear relaxation system (3.2),

$$w_t + Aw_x = Bw, \tag{4.14}$$

where $w \in \mathbb{R}^n$, and A, $B \in \mathbb{R}^{n\times n}$ are constant matrices. For system (4.14), we impose the following assumption:

Assumption 4.1. (i) There exists a symmetric positive definite matrix A_0, such that $A_0 A$ is symmetric.

(ii) B has the zero eigenvalue with multiplicity m, $0 < m < n$, and has m left eigenvectors l_j^0 and m right eigenvectors r_j^0, satisfying

$$l_j^0 B = 0, \quad B r_j^0 = 0, \quad 1 \leqq j \leqq m,$$
$$l_j^0 r_k^0 = \delta_{jk}, \quad 1 \leqq j, k \leqq m.$$

(iii) Set

$$L^0 = \begin{pmatrix} l_1^0 \\ \vdots \\ l_m^0 \end{pmatrix}, \quad R^0 = (r_1^0, \ldots, r_m^0).$$

Then A_0 in (i) satisfies

$$(R^0)^t A_0 R^0 L^0 = (R^0)^t A_0. \tag{4.15}$$

From the above discussion, if (4.14) is taken as the linearization of (4.1) around an equilibrium state, i.e., $A = f'\big(\mathscr{E}(w^{(r)})\big)$ and $B = r'\big(\mathscr{E}(w^{(r)})\big)$ for a fixed $w^{(r)}$, and (4.1) has a strictly convex entropy U, then by (4.12) Assumption 4.1 is satisfied with $A_0 = \nabla^2 U\big(\mathscr{E}(w^{(r)})\big)$. Also, Assumption 4.1 is a weaker condition than Assumption 3.1.

Proposition 4.4. *Assumption 3.1 implies Assumption 4.1 if B has the zero eigenvalue with multiplicity m, $0 < m < n$.*

Proof. It is obvious that Assumption 3.1 implies (i) and (ii) in Assumption 4.1. As for (iii), by the symmetry of $A_0 B$ we have

$$(R^0)^t A_0 B = (A_0 B R^0)^t = 0.$$

Here the row vectors of $(R^0)^t A_0$ are left eigenvectors of B associated with the zero eigenvalue, and there exists a $K \in \mathbb{R}^{m\times m}$ such that $(R^0)^t A_0 = K L^0$. Equation (4.15) is then straightforward.

The proposition can also be shown by taking $f(w) = Aw$, $r(w) = Bw$ and $U(w) = w^t A_0 w$ in (4.1), and verifying that all the requirements for (4.1) are satisfied. $\square$

We now discuss the relation between subcharacteristic conditions and the dissipative criterion under Assumption 4.1. First of all, Assumption 4.1 implies that A_0 symmetrizes A, while $(R^0)^t A_0 R^0$ symmetrizes $L^0 A R^0$. Here both A_0 and $(R^0)^t A_0 R^0$ are symmetric and positive definite. Denote the eigenvalues of A as

$$\lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_n, \tag{4.16}$$

and the eigenvalues of $L^0 A R^0$ as

$$\lambda_1^{(r)} \leqq \lambda_2^{(r)} \leqq \cdots \leqq \lambda_m^{(r)}. \tag{4.17}$$

Theorem 4.5. (i) *Assumption 4.1 implies a subcharacteristic condition*

$$\lambda_j \leqq \lambda_j^{(r)} \leqq \lambda_{j+n-m}, \quad 1 \leqq j \leqq m. \tag{4.18}$$

(ii) *Under Assumption 4.1, a sufficient condition for the dissipative criterion, Assumption 3.2, to hold is that A and L^0AR^0 do not have any common eigenvalue.*

(iii) *Under Assumption 4.1, a necessary condition for the dissipative criterion to hold is*

$$\lambda_j < \lambda_j^{(r)} < \lambda_{j+n-m}, \quad 1 \leqq j \leqq m. \tag{4.19}$$

(iv) *Under Assumption 4.1, if $m = n-1 > 0$, then the dissipative criterion is equivalent to the subcharacteristic condition in the strict sense:*

$$\lambda_j < \lambda_j^{(r)} < \lambda_{j+1}, \quad 1 \leqq j \leqq n-1. \tag{4.20}$$

Proof. (i) Let $\mathbb{W}_j \in \mathbb{R}^n$ be any subspace with dimension j and $\mathbb{W}_j^{(r)} \in \mathbb{R}^m$ be any subspace with dimension j. Assumption 4.1 implies that

$$\lambda_j = \min_{\mathbb{W}_j} \max_{w\in\mathbb{W}_j} \frac{w^tA_0Aw}{w^tA_0w} = \max_{\mathbb{W}_{n-j+1}} \min_{w\in\mathbb{W}_{n-j+1}} \frac{w^tA_0Aw}{w^tA_0w}, \quad 1 \leqq j \leqq n,$$

and

$$\begin{aligned}\lambda_j^{(r)} &= \min_{\mathbb{W}_j^{(r)}} \max_{w^{(r)}\in\mathbb{W}_j^{(r)}} \frac{\left(R^0w^{(r)}\right)^tA_0AR^0w^{(r)}}{\left(R^0w^{(r)}\right)^tA_0R^0w^{(r)}}\\ &= \max_{\mathbb{W}_{m-j+1}^{(r)}} \min_{w^{(r)}\in\mathbb{W}_{m-j+1}^{(r)}} \frac{\left(R^0w^{(r)}\right)^tA_0AR^0w^{(r)}}{\left(R^0w^{(r)}\right)^tA_0R^0w^{(r)}}, \quad 1 \leqq j \leqq m.\end{aligned}$$

Inequality (4.18) follows immediately.

(ii) If the dissipative criterion were not true, there would be an $\eta \in \mathbb{R}^n$ satisfying

$$\eta \neq 0, \quad A\eta = \lambda\eta, \quad \eta = R^0\eta^{(r)}$$

for some $\lambda \in \mathbb{R}$ and some $\eta^{(r)} \in \mathbb{R}^m$. These would imply

$$\begin{aligned}L^0\eta &= \eta^{(r)},\\ L^0AR^0\eta^{(r)} &= L^0A\eta = \lambda L^0\eta = \lambda\eta^{(r)}.\end{aligned}$$

Therefore, A and L^0AR^0 would have a common eigenvalue λ.

(iii) Under Assumption 4.1, we are able to choose eigenvectors w_j of A and eigenvectors $w_j^{(r)}$ of L^0AR^0 such that

$$\begin{aligned}Aw_j = \lambda_jw_j, &\quad {w_j}^tA_0w_k = \delta_{jk}, \qquad 1 \leqq j,k \leqq n,\\ L^0AR^0w_j^{(r)} = \lambda_j^{(r)}w_j^{(r)}, &\quad \left(R^0w_j^{(r)}\right)^tA_0R^0w_k^{(r)} = \delta_{jk}, \qquad 1 \leqq j,k \leqq m.\end{aligned}$$

The eigenvalues can then be expressed by the Rayleigh quotient as

$$\lambda_j = \min_{w\in\{w_j,\cdots,w_n\}} \frac{w^t A_0 A w}{w^t A_0 w} = \max_{w\in\{w_1,\cdots,w_j\}} \frac{w^t A_0 A w}{w^t A_0 w}, \quad 1 \leqq j \leqq n,$$

$$\begin{aligned} \lambda_j^{(r)} &= \min_{w^{(r)}\in\{w_j^{(r)},\cdots,w_m^{(r)}\}} \frac{\left(R^0 w^{(r)}\right)^t A_0 A R^0 w^{(r)}}{\left(R^0 w^{(r)}\right)^t A_0 R^0 w^{(r)}} \\ &= \max_{w^{(r)}\in\{w_1^{(r)},\cdots,w_j^{(r)}\}} \frac{\left(R^0 w^{(r)}\right)^t A_0 A R^0 w^{(r)}}{\left(R^0 w^{(r)}\right)^t A_0 R^0 w^{(r)}}, \quad 1 \leqq j \leqq m, \end{aligned} \tag{4.21}$$

where $\{\cdots\}$ denotes the span of the vectors enclosed. If $\lambda_j = \lambda_j^{(r)}$ for some $1 \leqq j \leqq m$, then take $w \in \{w_j, \cdots, w_n\} \cap \{R^0 w_1^{(r)}, \cdots, R^0 w_j^{(r)}\}$ such that $w \neq 0$. From (4.21) we would have

$$\lambda_j \leqq \frac{w^t A_0 A w}{w^t A_0 w} \leqq \lambda_j^{(r)} = \lambda_j.$$

Consequently,

$$\lambda_j = \frac{w^t A_0 A w}{w^t A_0 w}$$

and w would have to be an eigenvector of A corresponding to λ_j. Since $w \in \{R^0 w_1^{(r)}, \cdots, R^0 w_j^{(r)}\}$, w is in the null space of B. This contradicts the dissipative criterion. Similarly, we can show that $\lambda_j^{(r)} \neq \lambda_{j+n-m}$.

(iv) When $m = n - 1 > 0$, the derivation of (4.20) by (ii) and (iii) is trivial. □

Further discussions of subcharacteristic conditions follow, based on Theorem 4.5. Condition (4.18) is known as a stability condition in the following sense: At least in the 2×2 case, if the relaxation term is given the correct sign from physical considerations, then (4.18) implies the nonnegativity of the dissipative parameter in the Chapman-Enskog expansion, [Liu]. On the other hand, the condition itself does not include any sign information for the relaxation term. This can be seen by changing B to $-B$ in Assumption 4.1, which implies (4.18). The sign information for the relaxation term needs to be given explicitly elsewhere, such as the semi-negativity of $A_0 B$ in Assumption 3.1, or (ii) in Definition 4.1 for the entropy function.

We then want to know whether a certain form of subcharacteristic condition can prevent "zero projection" of dissipation on equilibrium characteristic directions. That is, we want to see if there is an equivalence of the dissipative criterion and certain forms of subcharacteristic conditions. From (iv) of Theorem 4.5, this is true if $m = n - 1$. In this case the dissipative criterion is indeed equivalent to the subcharacteristic condition in the strict sense. In the general case $0 < m < n - 1$, however, there is a gap between the sufficient condition in (ii) and the necessary condition in (iii) of Theorem 4.5. For instance, let $n = 3$ and $m = 1$. What happens if

$$\lambda_1 < \lambda_1^{(r)} < \lambda_3 \quad \text{and } \lambda_1^{(r)} = \lambda_2, \tag{4.22}$$

i.e., the necessary condition is satisfied while the sufficient one is not?

Example. For a constant $c > 0$, let

$$A = \begin{pmatrix} -c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$\text{(a)} \quad B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \text{(b)} \quad B = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

It is easy to verify that for both choices of B, Assumption 3.1 is satisfied. Clearly, $n = 3$ and $m = 1$. Direct calculation yields $\lambda_1 = -c$, $\lambda_2 = 0$, $\lambda_3 = c$, and $\lambda_1^{(r)} = 0$. Therefore, (4.22) is true for either B. However, the dissipative criterion fails for case (a), while it is satisfied for case (b). Therefore, subcharacteristic conditions are not appropriate for characterizing dissipation unless $m = n - 1$.

5. Evolution of Elementary Waves

To obtain large time behavior of the nonequilibrium flow in the pointwise sense, we need precise estimates on the evolution of the elementary waves of which the solution is composed. This type of analysis was started in [Liu2], and greatly generalized in [LZ1], [Liu3] and [LZ2]. The lemmas in this section are mostly cited from those papers.

We define the following functions of $x \in \mathbb{R}$ and $t \geqq 0$:

$$\begin{aligned} \theta_\alpha(x, t; \lambda, \nu) &= (t+1)^{-\frac{\alpha}{2}} e^{-\frac{(x-\lambda(t+1))^2}{\nu(t+1)}}, \\ \psi(x, t; \lambda) &= [(x - \lambda(t+1))^2 + t + 1]^{-\frac{3}{4}}, \\ \tilde{\psi}(x, t; \lambda) &= \left[|x - \lambda(t+1)|^3 + (t+1)^2\right]^{-\frac{1}{2}}, \\ \bar{\psi}(x, t; \lambda) &= [(x - \lambda(t+1))^2 + 1]^{-\frac{3}{4}}, \end{aligned} \tag{5.1}$$

where α, λ are constants, and ν is a positive constant.

Lemma 5.1 ([Liu3]). *Let the constants α, α', β and ν be such that $\alpha \geqq \alpha' \geqq 0$, $\alpha - \alpha' < 3$, $\beta > 0$, $\nu > 0$, and let λ be any constant. Then for all x values in the range $-\infty < x < \infty$, $t \geqq 0$, we have*

$$\begin{aligned} &\int_0^t \int_{-\infty}^{\infty} (t-t')^{-\frac{\alpha-\alpha'}{2}} (t-t'+1)^{-\frac{\alpha'}{2}} e^{-\frac{(x-y-\lambda(t-t'))^2}{\nu(t-t')}} \theta_\beta(y, t'; \lambda, \nu)\, dy\, dt' \\ &\qquad = \begin{cases} O(1)\theta_\gamma(x, t; \lambda, \nu) \log(t+2), & \textit{if } \alpha = 3 \textit{ or } \beta = 3, \\ O(1)\theta_\gamma(x, t; \lambda, \nu), & \textit{otherwise}, \end{cases} \end{aligned} \tag{5.2}$$

where $\gamma = \min(\alpha, 3) + \min(\beta, 3) - 3$.

Denote the characteristic function of a set $\mathscr{D}$ as $\mathrm{char}\{\mathscr{D}\}$.

Lemma 5.2 ([Liu3]). *Let the constants α, α', β, ν, λ and λ' be such that $\alpha \geqq 1$, $\alpha' \geqq 0$, $0 \leqq \alpha - \alpha' < 3$, $\beta \geqq 1$, $\nu > 0$, and $\lambda \neq \lambda'$. Then for any given $\varepsilon > 0$, $K \geqq |\lambda - \lambda'|$, and all x values in the range $-\infty < x < \infty$, $t \geqq 0$, we have*

$$
\begin{aligned}
&\int_0^t \int_{-\infty}^{\infty} (t-t')^{-\frac{\alpha-\alpha'}{2}} (t-t'+1)^{-\frac{\alpha'}{2}} e^{-\frac{(x-y-\lambda(t-t'))^2}{\nu(t-t')}} \theta_\beta(y,t';\lambda',\nu)\,dy\,dt' \\
&= O(1)\big[\theta_\gamma(x,t;\lambda,\nu+\varepsilon) + \theta_{\gamma'}(x,t;\lambda',\nu+\varepsilon) \\
&\quad + |x-\lambda(t+1)|^{-\frac{\beta-1}{2}} |x-\lambda'(t+1)|^{-\frac{\alpha-1}{2}} \\
&\quad \cdot \operatorname{char}\big\{\min(\lambda,\lambda')(t+1) + K\sqrt{t+1} \leqq x \leqq \max(\lambda,\lambda')(t+1) - K\sqrt{t+1}\big\}\big] \\
&\quad + \begin{cases} O(1)\theta_\alpha(x,t;\lambda,\nu+\varepsilon)\log(t+1), & \text{if } \beta = 3 \\ 0, & \text{otherwise} \end{cases} \\
&\quad + \begin{cases} O(1)\theta_\beta(x,t;\lambda',\nu+\varepsilon)\log(t+1), & \text{if } \alpha = 3 \\ 0, & \text{otherwise}, \end{cases}
\end{aligned} \tag{5.3}
$$

where $\gamma = \alpha + \frac{1}{2}\min(\beta,3) - \frac{3}{2}$, and $\gamma' = \frac{1}{2}\min(\alpha,3) + \beta - \frac{3}{2}$.

Lemma 5.3 ([LZ2]). *Let the constants α, α', β and ν be such that $\alpha \geqq \alpha' \geqq 0$, $\alpha - \alpha' < 3$, $\beta \geqq 0$, $\nu > 0$, and let λ be any constant. Then for any given $\varepsilon > 0$, and all x values in the range $-\infty < x < \infty$, $t \geqq 0$, we have*

$$
\begin{aligned}
&\int_0^t \int_{-\infty}^{\infty} (t-t')^{-\frac{\alpha-\alpha'}{2}} (t-t'+1)^{-\frac{\alpha'}{2}} e^{-\frac{(x-y-\lambda(t-t'))^2}{\nu(t-t')}} (t'+1)^{-\frac{\beta}{2}} \psi(y,t';\lambda)\,dy\,dt' \\
&= O(1)\big[\theta_\gamma(x,t;\lambda,\nu+\varepsilon) + (t+1)^{-\frac{\sigma}{2}} \psi(x,t;\lambda)\big] \\
&\quad + \begin{cases} O(1)\theta_\alpha(x,t;\lambda,\nu+\varepsilon)\log(t+1), & \text{if } \beta = \frac{3}{2} \\ 0, & \text{otherwise} \end{cases} \\
&\quad + \begin{cases} O(1)(t+1)^{-\frac{\sigma}{2}}\psi(x,t;\lambda)\log(t+1), & \text{if } \alpha = 3 \text{ or } \beta = 2 \\ 0, & \text{otherwise}, \end{cases}
\end{aligned} \tag{5.4}
$$

where $\gamma = \alpha + \min(\beta,\frac{3}{2}) - \frac{3}{2}$, and $\sigma = \min(\alpha,3) + \min(\beta,2) - 3$.

Lemma 5.4 ([LZ2]). *Let the constants α, α', β, ν, λ and λ' be such that $\alpha \geqq 1$, $\alpha' \geqq 0$, $0 \leqq \alpha - \alpha' < 3$, $\beta \geqq 0$, $\nu > 0$, and $\lambda \neq \lambda'$. Then for any given $\varepsilon > 0$, $K > 2|\lambda - \lambda'|$, and all x values in the range $-\infty < x < \infty$, $t \geqq 0$, we have*

$$\int_0^t \int_{-\infty}^{\infty} (t-t')^{-\frac{\alpha-\alpha'}{2}} (t-t'+1)^{-\frac{\alpha'}{2}} e^{-\frac{(x-y-\lambda(t-t'))^2}{\nu(t-t')}} (t'+1)^{-\frac{\beta}{2}} \psi(y,t';\lambda')\, dy\, dt'$$
$$= O(1)\theta_\gamma(x,t;\lambda,\nu+\varepsilon) + O(1)(t+1)^{-\frac{\sigma}{2}} \left[(x-\lambda(t+1))^2 + (t+1)^{\frac{5}{3}-\frac{1}{3}\min(\beta,2)}\right]^{-\frac{3}{4}}$$
$$+ O(1)(t+1)^{-\frac{\sigma'}{2}} \psi(x,t;\lambda')^{\frac{1}{3}\min(\alpha,3)} \left[(x-\lambda(t+1))^2 + (t+1)^2\right]^{-\frac{3}{4}(1-\frac{1}{3}\min(\alpha,3))}$$
$$\cdot \begin{cases} 1, & \text{if } \alpha \neq 3 \\ 1+\log(t+1), & \text{if } \alpha = 3 \end{cases}$$
$$+ O(1)|x-\lambda(t+1)|^{-\frac{1}{2}\min(\beta,\frac{5}{2})-\frac{1}{4}} |x-\lambda'(t+1)|^{-\frac{1}{2}(\alpha-1)}$$
$$\cdot \operatorname{char}\left\{\min(\lambda,\lambda')(t+1) + K\sqrt{t+1} \leqq x \leqq \max(\lambda,\lambda')(t+1) - K\sqrt{t+1}\right\}$$
$$+ O(1) \begin{cases} \theta_\alpha(x,t;\lambda,\nu+\varepsilon)\log(t+1), & \text{if } \beta = \frac{3}{2} \\ (t+1)^{-\frac{1}{2}(\alpha-1)} \psi(x,t;\lambda)\log(t+1), & \text{if } \beta = 2 \\ 0, & \text{otherwise}, \end{cases} \tag{5.5}$$

where $\gamma = \alpha + \frac{1}{2}\min(\beta, \frac{3}{2}) - \frac{3}{4}$, $\sigma = \alpha + \min(\beta, 2) - 3$, $\sigma' = \min(\alpha, 3) + \beta - 3$.

Lemma 5.5 ([LZ2]**).** *Let the constants ν, ν', λ and λ' be such that $\nu > 0$, $\nu' > 0$, and $\lambda \neq \lambda'$. Let $k = 0, 1$. If a function $h(x,t)$ satisfies*

$$\begin{aligned} \partial_x^j h(x,t) &= O(1)\big[\theta_{2+j}(x,t;\lambda',\nu') + (t+1)^{-\frac{3}{4}}\psi(x,t;\lambda') \\ &\quad + (t+1)^{-\frac{3}{4}}\psi(x,t;\lambda)\big], \quad 0 \leqq j \leqq k, \\ h_t + \lambda' h_x - \frac{\nu}{4} h_{xx} &= O(1)\big[\theta_4(x,t;\lambda',\nu') + (t+1)^{-\frac{3}{2}} \\ &\quad \cdot \psi(x,t;\lambda') + (t+1)^{-\frac{3}{2}}\psi(x,t;\lambda)\big], \end{aligned} \tag{5.6}$$

then for any given $K > 2|\lambda - \lambda'|$, and all x values in the range $-\infty < x < \infty$, $t \geqq 0$,

$$\int_0^t \int_{-\infty}^{\infty} (t-t')^{-\frac{1}{2}} e^{-\frac{(x-y-\lambda(t-t'))^2}{\nu(t-t')}} \partial_y^{k+1} h(y,t')\, dy\, dt'$$
$$= O(1)(t+1)^{-\frac{k}{2}} \big[\psi(x,t;\lambda) + (t+1)^{-\frac{1}{4}}\psi(x,t;\lambda')$$
$$+ |x-\lambda(t+1)|^{-1}|x-\lambda'(t+1)|^{-\frac{1}{2}} \tag{5.7}$$
$$\cdot \operatorname{char}\left\{\min(\lambda,\lambda')(t+1) + K\sqrt{t+1} \leqq x \leqq \max(\lambda,\lambda')(t+1) - K\sqrt{t+1}\right\}\big].$$

Lemma 5.6 ([LZ2]**).** *Let the constants α, α', β, ν, λ and λ' be such that $\alpha \geqq 1$, $\alpha' \geqq 0$, $0 \leqq \alpha - \alpha' < 3$, $\beta \geqq 0$, $\nu > 0$ and $\lambda \neq \lambda'$. Then for any given $\varepsilon > 0$,*

$K > 2|\lambda-\lambda'|$, *and all* x *values in the range* $-\infty < x < \infty$, $t \geqq 0$, *we have*

$$\begin{aligned}
&\int_0^t \int_{-\infty}^{\infty} (t-t')^{-\frac{\alpha-\alpha'}{2}} (t-t'+1)^{-\frac{\alpha'}{2}} e^{-\frac{(x-y-\lambda(t-t'))^2}{\nu(t-t')}} (t'+1)^{-\frac{\beta}{2}} \bar{\psi}(y,t';\lambda')\,dy\,dt' \\
&= O(1)\big[\theta_\gamma(x,t;\lambda,\nu+\varepsilon)+\theta_{\gamma'}(x,t;\lambda',\nu)\big] \\
&\quad + O(1)(t+1)^{-\frac{\sigma}{2}}\big[(x-\lambda(t+1))^2+(t+1)^{\frac{5}{3}-\frac{1}{3}\min(\beta,2)}\big]^{-\frac{3}{4}} \\
&\quad + O(1)(t+1)^{-\frac{\sigma'}{2}}\psi(x,t;\lambda')^{\frac{1}{3}\min(\alpha,3)}\big[(x-\lambda(t+1))^2+(t+1)^2\big]^{-\frac{3}{4}(1-\frac{1}{3}\min(\alpha,3))} \\
&\quad + O(1)\big[|x-\lambda(t+1)|^{-\frac{1}{2}\min(\beta,3)}|x-\lambda'(t+1)|^{-\frac{\alpha-1}{2}} \qquad\qquad (5.8)\\
&\quad + |x-\lambda(t+1)|^{-\frac{\beta-1}{2}}|x-\lambda'(t+1)|^{-\frac{\alpha}{2}}\big] \\
&\quad \cdot \operatorname{char}\big\{\min(\lambda,\lambda')(t+1)+K\sqrt{t+1} \leqq x \leqq \max(\lambda,\lambda')(t+1)-K\sqrt{t+1}\big\} \\
&\quad + \begin{cases} O(1)\big[\theta_\alpha(x,t;\lambda,\nu+\varepsilon)+(t+1)^{-\frac{\alpha-1}{2}}\psi(x,t;\lambda)\big]\log(t+1), & \textit{if } \beta=2 \\ 0, & \textit{otherwise} \end{cases} \\
&\quad + \begin{cases} O(1)(t+1)^{-\frac{\beta}{2}}\psi(x,t;\lambda')\log(t+1), & \textit{if } \alpha=3 \\ 0, & \textit{otherwise,} \end{cases}
\end{aligned}$$

where $\gamma = \alpha+\frac{1}{2}\min(\beta,2)-1$, $\gamma' = \frac{1}{2}\min(\alpha,2)+\beta-1$, $\sigma = \alpha+\min(\beta,2)-3$, *and* $\sigma' = \min(\alpha,3)+\beta-3$.

Lemma 5.7 ([LZ1]**).** *Let* $\nu > 0$, σ *and* λ *be constants. Then for all* $-\infty < x < \infty$, $t \geqq 0$, *we have*

$$\int_0^t e^{-(t-t')/\nu}\psi(x-\sigma(t-t'),t';\lambda)\,dt' = O(1)\psi(x,t;\lambda), \tag{5.9}$$

$$\int_0^t e^{-(t-t')/\nu}\tilde{\psi}(x-\sigma(t-t'),t';\lambda)\,dt' = O(1)\tilde{\psi}(x,t;\lambda). \tag{5.10}$$

Lemma 5.8 ([LZ2]**).** *Let* λ *be a constant. Then for* $-\infty < x < \infty$, *we have*

$$\int_0^\infty (t'+1)^{-\frac{5}{4}}\psi(x,t';\lambda)\,dt' = O(1)(x^2+1)^{-\frac{3}{4}}. \tag{5.11}$$

Lemma 5.9. *Let* $\nu > 0$, λ *and* λ' *be constants. Then for* $-\infty < x < \infty$, $t/2 \leqq t' < t$, *we have*

$$\int_{-\infty}^{\infty} (t-t')^{-\frac{1}{2}} e^{-\frac{(x-y-\lambda(t-t'))^2}{\nu(t-t')}}\tilde{\psi}(y,t';\lambda')\,dy = O(1)e^{C(t-t')}\tilde{\psi}(x,t;\lambda'), \tag{5.12}$$

where $C > 0$ *is a constant.*

Proof. Denote the left-hand side of (5.12) as $I(x,t,t';\nu,\lambda,\lambda')$. Clearly,

$$I(x,t,t';\nu,\lambda,\lambda') = O(1)e^{C(t-t')}I(x,t,t';2\nu,\lambda',\lambda').$$

Thus we consider the case $\lambda = \lambda'$. Divide the real axis into two parts, $|x - y - \lambda(t - t')| \leqq \frac{1}{2}|x - \lambda(t + 1)|$ and its complement. Then

$$\begin{aligned} I(x, t, t'; \nu, \lambda, \lambda) = O(1) \int_{-\infty}^{\infty} &(t - t')^{-\frac{1}{2}} e^{-\frac{(x-y-\lambda(t-t'))^2}{\nu(t-t')}} \\ &\cdot \left[|x - \lambda(t + 1)|^3 + (t + 1)^2\right]^{-\frac{1}{2}} dy \\ &+ O(1)\int_{-\infty}^{\infty} (t-t')^{-\frac{1}{2}} e^{-\frac{(x-y-\lambda(t-t'))^2}{2\nu(t-t')}} e^{-\frac{(x-\lambda(t+1))^2}{8\nu(t-t')}} (t' + 1)^{-1} dy \\ = &O(1)\tilde{\psi}(x, t; \lambda) + O(1)\theta_2(x, t; \lambda, 4\nu) \\ = &O(1)\tilde{\psi}(x, t; \lambda). \qquad \square \end{aligned}$$

6. Large Time Behavior

In this section we go back to gas dynamics in thermal nonequilibrium. As the last part of the paper, we prove Theorem 1.4, which is about the large time behavior of a flow that is slightly away from an equilibrium state.

Since the solution of (1.5) or (1.36) is a small perturbation around the constant state “*”, we rewrite the first three equations in (1.36) as

$$\tilde{w}_t + A\tilde{w}_x = B\tilde{w} + \tilde{g}, \tag{6.1}$$

where

$$\tilde{w} = (p - p^*, u, \chi)^t, \tag{6.2}$$

A and B are defined in (3.31b), and

$$\begin{aligned} \tilde{g} = (\tilde{g}_1, \tilde{g}_2, \tilde{g}_3)^t = \Bigg(&\left(c_f^{*2} - c_f^2\right)u_x \\ &+ \left(\frac{p_{e_1}^*}{\tau^*} - \frac{p_{e_1}}{\tau}\right)\chi, 0, (a^* - a)u_x + \left(\frac{1 + Q_{e_1}^*}{\tau^*} - \frac{1 + Q_{e_1}}{\tau}\right)\chi\Bigg)^t. \end{aligned} \tag{6.3}$$

The Green’s function for (6.1) has been found in Section 3 as (3.36) and (3.37). Introduce a linear transform to diagonalize its leading term:

$$w = (w_1, w_2, w_3)^t = L^{(r)}\tilde{w}, \quad \tilde{w} = R^{(r)}w, \tag{6.4a}$$

where

$$L^{(r)} = \begin{pmatrix} 1 & -c^* & -\frac{p_{e_1}^*}{1+Q_{e_1}^*} \\ 1 & c^* & -\frac{p_{e_1}^*}{1+Q_{e_1}^*} \\ 0 & 0 & 1 \end{pmatrix}, \quad R^{(r)} = \left(L^{(r)}\right)^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{p_{e_1}^*}{1+Q_{e_1}^*} \\ -\frac{1}{2c^*} & \frac{1}{2c^*} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{6.4b}$$

Also let

$$w_4 = s - s^*, \tag{6.4c}$$

and

$$g = (g_1, g_2, g_3)^t = L^{(r)}\tilde{g}, \quad g_4 = \left(\frac{1}{T_2} - \frac{1}{T_1}\right)\frac{\chi}{\tau}. \tag{6.5}$$

We have

$$w_t + L^{(r)}AR^{(r)}w_x = L^{(r)}BR^{(r)}w + g, \tag{6.6a}$$

where

$$L^{(r)}AR^{(r)} = \begin{pmatrix} -c^* & 0 & -\frac{c^* p^*_{e_1}}{1+Q^*_{e_1}} \\ 0 & c^* & \frac{c^* p^*_{e_1}}{1+Q^*_{e_1}} \\ -\frac{a^*}{2c^*} & \frac{a^*}{2c^*} & 0 \end{pmatrix}, \quad L^{(r)}BR^{(r)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1+Q^*_{e_1}}{\tau^*} \end{pmatrix} \tag{6.6b}$$

by (1.25). Let the Green's function of (6.6) be G. From (3.36) we have for $-\infty < x < \infty$, $t \geqq 0$,

$$G(x,t) = D(x,t) + H(x,t), \tag{6.7a}$$

where for $l \geqq 0$,

$$\partial_x^l D(x,t) = \partial_x^l \left[\frac{1}{\sqrt{2\pi\mu^* t}} \sum_{i=1}^{2} e^{-\frac{(x-c_i t)^2}{2\mu^* t}} P_i^{(r)} \right],$$

$$\begin{aligned} \partial_x^l H(x,t) = {} & O(1)(t+1)^{-\frac{1}{2}} t^{-\frac{l+1}{2}} \sum_{i=1}^{2} e^{-\frac{(x-c_i t)^2}{C^* t}} \\ & + \sum_{j=0}^{l}\sum_{k=1}^{3} e^{-\mu_k t}\delta^{(l-j)}(x-d_k t) P_k^{(j)}(t), \end{aligned} \tag{6.7b}$$

$$c_{1,2} = \mp c^*, \quad d_{1,3} = \mp c_f^*, \quad d_2 = 0,$$

$$P_1^{(r)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_2^{(r)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

the dissipative parameters μ^* and μ_k, $1 \leqq k \leqq 3$, are positive constants given in (3.35) and (3.33), $C^* > 0$ is a constant, and $P_k^{(j)}(t)$, $0 \leqq j \leqq l$, $1 \leqq k \leqq 3$, are 3×3 polynomial matrices in t with degrees not more than j. Use G_i to denote the ith row of G, etc. Then G_3 can be refined through (3.37),

$$G_3(x,t) = H_3(x,t) = H_{3a}(x,t) + H_{3b}(x,t), \tag{6.8a}$$

$$\partial_x^l H_{3a}(x,t) = \partial_x^{l+1}\left(\frac{1}{\sqrt{2\pi\mu^* t}} \sum_{i=1}^{2} e^{-\frac{(x-c_i t)^2}{2\mu^* t}} \eta_i \right),$$

$$\begin{aligned} \partial_x^l H_{3b}(x,t) = {} & O(1)(t+1)^{-\frac{1}{2}} t^{-\frac{l}{2}-1} \sum_{i=1}^{2} e^{-\frac{(x-c_i t)^2}{C^* t}} \\ & + \sum_{j=0}^{l}\sum_{k=1}^{3} e^{-\mu_k t}\delta^{(l-j)}(x-d_k t) P_{k,3}^{(j)}(t), \end{aligned} \tag{6.8b}$$

where H_3 is the third row of H, η_1 and η_2 are constant row vectors in $\mathbb{R}^3$, and $P_{k,3}^{(j)}$ are the third row of $P_k^{(j)}$.

By Duhamel's principle and the last equation in (1.36), we have for $l \geqq 0$,

$$\begin{aligned}\partial_x^l w_i(x,t) &= \int_{-\infty}^{\infty} G_i(x-y,t)\partial_y^l w(y,0)\,dy \\ &\quad + \int_0^t \int_{-\infty}^{\infty} G_i(x-y,t-t')\partial_y^l g(y,t')\,dy\,dt', \quad 1 \leqq i \leqq 3, \\ \partial_x^l w_4(x,t) &= \partial_x^l w_4(x,0) + \int_0^t \partial_x^l g_4(x,t')\,dt'.\end{aligned} \tag{6.9}$$

Next we define the ansatz of the solution. Let $\theta_\alpha, \psi, \tilde{\psi}, \bar{\psi}$ be defined as in (5.1). Let ν be any fixed number such that

$$\nu > \max\{2\mu^*, C^*\},$$

where C^* is the one in (6.7b) and (6.8b). Set

$$\begin{aligned}&\phi_i(x,t) = \theta_1(x,t;c_i,\nu) + \psi(x,t;c_i) + \tilde{\psi}(x,t;c_j), \quad i,j = 1,2 \quad \text{and } j \neq i; \\ &\phi_4(x) = \bar{\psi}(x,t;0) = (x^2+1)^{-\frac{3}{4}}.\end{aligned} \tag{6.10}$$

Let

$$\begin{aligned} M(t) = \sup_{0\leqq t'\leqq t} \max_{i=1,2} \Bigg\{ &\|(w_i\phi_i^{-1})(\cdot,t')\|_{L^\infty} + \|(w_{ix}\phi_i^{-1})(\cdot,t')\|_{L^\infty}(t'+1)^{\frac{1}{2}} \\ &+ \sum_{l=2}^{4} \|\partial_x^l w_i(\cdot,t')\|_{L^\infty}(t'+1)^{\frac{5-l}{2}} \Bigg\} \\ &+ \sup_{0\leqq t'\leqq t} \Bigg\{ \left\|\left(w_3(\phi_1+\phi_2)^{-1}\right)(\cdot,t')\right\|_{L^\infty}(t'+1)^{\frac{1}{2}} \\ &+ \left\| w_{3x}(\cdot,t')\left(\sum_{i=1}^{2}\tilde{\psi}(\cdot,t';c_i)\right)^{-1}\right\|_{L^\infty}(t'+1)^{\frac{1}{2}} \\ &+ \sum_{l=2}^{4}\left\|\partial_x^l w_3(\cdot,t')\right\|_{L^\infty}(t'+1)^{\frac{5-l}{2}} + \left\|(w_4\phi_4^{-1})(\cdot,t')\right\|_{L^\infty} \\ &+ \left\|(w_{4x}\phi_4^{-1})(\cdot,t')\right\|_{L^\infty} + \left\|(w_{4xx}\phi_4^{-\frac{1}{3}})(\cdot,t')\right\|_{L^\infty} \Bigg\}. \end{aligned} \tag{6.11}$$

Clearly, for $-\infty < x < \infty$, $t \geqq 0$, we have

$$\begin{aligned}&|w_i(x,t)| \leqq M(t)\phi_i(x,t), \quad |w_{ix}(x,t)| \leqq M(t)(t+1)^{-\frac{1}{2}}\phi_i(x,t), \\ &\qquad |\partial_x^l w_i(x,t)| \leqq M(t)(t+1)^{-\frac{5-l}{2}}, \quad 2 \leqq l \leqq 4, \quad i = 1,2,\end{aligned} \tag{6.12a}$$

and

$$
\begin{aligned}
&|w_3(x,t)| \leqq M(t)(t+1)^{-\frac{1}{2}} \sum_{i=1}^{2} \phi_i(x,t), \\
&|w_{3x}(x,t)| \leqq M(t)(t+1)^{-\frac{1}{2}} \sum_{i=1}^{2} \tilde{\psi}(x,t;c_i), \\
&\left|\partial_x^l w_3(x,t)\right| \leqq M(t)(t+1)^{-\frac{5-l}{2}}, \quad 2 \leqq l \leqq 4, \\
&|w_4(x,t)|,\ |w_{4x}(x,t)| \leqq M(t)\phi_4(x), \quad |w_{4xx}(x,t)| \leqq M(t)\phi_4^{\frac{1}{3}}(x).
\end{aligned} \tag{6.12b}
$$

Before using (6.9) to perform a pointwise a priori estimate, we need to obtain decay rates for higher derivatives of u and χ in the neighborhood of the particle path by using a weighted energy method. These rates are necessary for closing the pointwise analysis. For $l \geqq 1$, set

$$
\begin{aligned}
E_l(t;\varepsilon) = &\int_{-\varepsilon t}^{\varepsilon t} (\varepsilon^2 x^2+1)^{-\frac{3}{2}} \left[\left(\partial_x^l u\right)^2 + \left(\partial_x^l p\right)^2 + \left(\partial_x^l \chi\right)^2\right](x,t)\,dx \\
&+ \int_{t/2}^{t}\int_{-\varepsilon t}^{\varepsilon t} e^{-\varepsilon(t-t')}(\varepsilon^2x^2+1)^{-\frac{3}{2}} \\
&\left\{\sum_{j=1}^{l}\left[\left(\partial_x^j u\right)^2 + \left(\partial_x^j p\right)^2\right] + \sum_{j=0}^{l}\left(\partial_x^j \chi\right)^2\right\}(x,t')\,dx\,dt'.
\end{aligned} \tag{6.13}
$$

Lemma 6.1. *Under the assumptions of Theorem* 1.4*, we have, for* $t \geqq 0$, $2 \leqq l \leqq 5$*, and* $\varepsilon > 0$ *small, the following recursive relation:*

$$
\begin{aligned}
E_l(t;\varepsilon) = &O(1)\left(M^2(t)+\varepsilon_0^2\right) \\
&\cdot \varepsilon^{-1}\left[\varepsilon^{-1}(t+1)^{-4} + (t+1)^{-\max\{\frac{9}{2}-l,0\}}(\varepsilon^4t^2+1)^{-\frac{3}{2}} + e^{-\varepsilon t/2}\right] \\
&+ O(1)E_{l-1}(t;\varepsilon),
\end{aligned} \tag{6.14}
$$

where ε_0 *is the one in* (1.42).

Proof. In a method similar to that used for the energy estimate in Section 2, we multiply the first three equations in (2.4) by $\partial_x^l p$, $c_f^2\partial_x^l u$ and $b(\partial_x^l p - (c_f^2/a)\partial_x^l \chi)$ respectively, where $b > 0$ is defined by (1.26). Sum the resulting equations, and use (1.5) to convert the derivatives with respect to t into those with respect to x. These give us (2.5) without the two terms involving s. Multiply the result by the weight $e^{-\varepsilon(t-t')}(\varepsilon^2x^2+1)^{-\frac{3}{2}}$, then integrate it over $[-\varepsilon t, \varepsilon t] \times [t/2, t]$. After integration by parts and applying (6.2), (6.4), (6.12) and (1.41), we arrive at

$$\begin{aligned}
&\int_{-\varepsilon t}^{\varepsilon t}(\varepsilon^2x^2+1)^{-\frac{3}{2}}\left[\left(\partial_x^l u\right)^2+\left(\partial_x^l p\right)^2+\left(\partial_x^l\chi\right)^2\right](x,t)\,dx\\
&+\int_{t/2}^{t}\int_{-\varepsilon t}^{\varepsilon t}e^{-\varepsilon(t-t')}(\varepsilon^2x^2+1)^{-\frac{3}{2}}\left(\partial_x^l\chi\right)^2(x,t')\,dx\,dt'\\
&=O(1)\varepsilon^{-1}e^{-\varepsilon t/2}\varepsilon_0^2+O(1)(\varepsilon+\varepsilon_0)\\
&\cdot\int_{t/2}^{t}\int_{-\varepsilon t}^{\varepsilon t}e^{-\varepsilon(t-t')}(\varepsilon^2x^2+1)^{-\frac{3}{2}}\left[\left(\partial_x^l u\right)^2+\left(\partial_x^l p\right)^2\right](x,t')\,dx\,dt'\\
&+O(1)\varepsilon^{-1}\left(\varepsilon^4t^2+1\right)^{-\frac{3}{2}}\left(M^2(t)+\varepsilon_0^2\right)(t+1)^{-\max\{\frac{9}{2}-l,0\}}\\
&+O(1)\varepsilon_0\int_{t/2}^{t}\int_{-\varepsilon t}^{\varepsilon t}e^{-\varepsilon(t-t')}(\varepsilon^2x^2+1)^{-\frac{3}{2}}\left[\sum_{j=1}^{l-1}\left(\partial_x^j u\right)^2+\sum_{j=0}^{l-1}\left(\partial_x^j\chi\right)^2\right](x,t')\,dx\,dt'\\
&+O(1)\varepsilon^{-2}\varepsilon_0M^2(t)(t+1)^{-4}.
\end{aligned}\tag{6.15}$$

Similarly, multiplying (2.8) and (2.10) by the weight and integrating, we have

$$\begin{aligned}
&\int_{t/2}^{t}\int_{-\varepsilon t}^{\varepsilon t}e^{-\varepsilon(t-t')}(\varepsilon^2x^2+1)^{-\frac{3}{2}}\left(\partial_x^l u\right)^2(x,t')\,dx\,dt'\\
&=\ O(1)\int_{-\varepsilon t}^{\varepsilon t}(\varepsilon^2x^2+1)^{-\frac{3}{2}}\left[\left(\partial_x^l u\right)^2+\left(\partial_x^{l-1}\chi\right)^2\right](x,t)\,dx+O(1)\varepsilon^{-1}\varepsilon_0^2e^{-\varepsilon t/2}\\
&+O(1)\varepsilon^{-2}M(t)\varepsilon_0^2(t+1)^{-4}\\
&+O(1)\varepsilon^{-1}\left(\varepsilon_0^2+M^2(t)\right)(t+1)^{-\max\{\frac{9}{2}-l,0\}}(\varepsilon^4t^2+1)^{-\frac{3}{2}}\\
&+O(1)\int_{t/2}^{t}\int_{-\varepsilon t}^{\varepsilon t}e^{-\varepsilon(t-t')}(\varepsilon^2x^2+1)^{-\frac{3}{2}}\left[\sum_{j=0}^{l-1}\left(\partial_x^j\chi\right)^2+\sum_{j=1}^{l-1}\left(\partial_x^j u\right)^2\right](x,t')\,dx\,dt'\\
&+O(1)(\varepsilon+\varepsilon_0)\int_{t/2}^{t}\int_{-\varepsilon t}^{\varepsilon t}e^{-\varepsilon(t-t')}(\varepsilon^2x^2+1)^{-\frac{3}{2}}\left(\partial_x^l p\right)^2(x,t')\,dx\,dt'\\
&+O(1)\int_{t/2}^{t}\int_{-\varepsilon t}^{\varepsilon t}e^{-\varepsilon(t-t')}(\varepsilon^2x^2+1)^{-\frac{3}{2}}\left|\partial_x^l p\,\partial_x^l\chi\right|(x,t')\,dx\,dt'
\end{aligned}\tag{6.16}$$

and

$$\begin{aligned}
&\int_{t/2}^{t}\int_{-\varepsilon t}^{\varepsilon t}e^{-\varepsilon(t-t')}(\varepsilon^2x^2+1)^{-\frac{3}{2}}\left(\partial_x^l p\right)^2(x,t')\,dx\,dt'\\
&=O(1)\int_{-\varepsilon t}^{\varepsilon t}(\varepsilon^2x^2+1)^{-\frac{3}{2}}\left[\left(\partial_x^l p\right)^2+\left(\partial_x^{l-1}u\right)^2\right](x,t)\,dx+O(1)\varepsilon^{-1}\varepsilon_0^2e^{-\varepsilon t/2}\\
&+O(1)\varepsilon^{-1}\left(\varepsilon_0^2+M^2(t)\right)(t+1)^{-\frac{11}{2}+l}(\varepsilon^4t^2+1)^{-\frac{3}{2}}\\
&+O(1)\int_{t/2}^{t}\int_{-\varepsilon t}^{\varepsilon t}e^{-\varepsilon(t-t')}(\varepsilon^2x^2+1)^{-\frac{3}{2}}\left[\sum_{j=1}^{l-1}\left(\partial_x^j u\right)^2+\sum_{j=0}^{l-1}\left(\partial_x^j\chi\right)^2\right](x,t')\,dx\,dt'
\end{aligned}$$

$$+ O(1)\int_{t/2}^{t}\int_{-\varepsilon t}^{\varepsilon t} e^{-\varepsilon(t-t')}(\varepsilon^2x^2+1)^{-\frac{3}{2}}\big(\partial_x^l u\big)^2(x,t')\,dx\,dt'. \tag{6.17}$$

Substitute (6.16) into (6.17) and simplify. We then have

$$\begin{aligned}
&\int_{t/2}^{t}\int_{-\varepsilon t}^{\varepsilon t} e^{-\varepsilon(t-t')}(\varepsilon^2x^2+1)^{-\frac{3}{2}}\big(\partial_x^l p\big)^2(x,t')\,dx\,dt' \\
&= O(1)\int_{-\varepsilon t}^{\varepsilon t}(\varepsilon^2x^2+1)^{-\frac{3}{2}}\big[\big(\partial_x^l u\big)^2+\big(\partial_x^l p\big)^2\big](x,t)\,dx + O(1)\varepsilon^{-1}\varepsilon_0^2 e^{-\varepsilon t/2} \\
&+ O(1)\varepsilon^{-2}M(t)\varepsilon_0^2(t+1)^{-4} \\
&+ O(1)\varepsilon^{-1}\big(\varepsilon_0^2+M^2(t)\big)(t+1)^{-\max\{\frac{9}{2}-l,0\}}(\varepsilon^4t^2+1)^{-\frac{3}{2}} \\
&+ O(1)E_{l-1}(t;\varepsilon) + O(1)\int_{t/2}^{t}\int_{-\varepsilon t}^{\varepsilon t} e^{-\varepsilon(t-t')}(\varepsilon^2x^2+1)^{-\frac{3}{2}}\big(\partial_x^l \chi\big)^2(x,t')\,dx\,dt'.
\end{aligned} \tag{6.18}$$

Substitute (6.18) into (6.16). The left-hand side of (6.16) is then equal to the right-hand side of (6.18). The result and (6.18) further simplify the right-hand side of (6.15), which is then replaced by the right-hand side of (6.14). Therefore,

$$\begin{aligned}
&\int_{-\varepsilon t}^{\varepsilon t}(\varepsilon^2x^2+1)^{-\frac{3}{2}}\big[\big(\partial_x^l u\big)^2+\big(\partial_x^l p\big)^2+\big(\partial_x^l \chi\big)^2\big](x,t)\,dx \\
&+\int_{t/2}^{t}\int_{-\varepsilon t}^{\varepsilon t} e^{-\varepsilon(t-t')}(\varepsilon^2x^2+1)^{-\frac{3}{2}}\big[\big(\partial_x^l u\big)^2+\big(\partial_x^l p\big)^2+\big(\partial_x^l \chi\big)^2\big](x,t')\,dx\,dt'
\end{aligned}$$

is equal to the right-hand side of (6.14). This immediately gives (6.14). □

Lemma 6.2. *Under the assumptions of Theorem* 1.4*, for* $t \geqq 0$ *we have*

$$\big\|\big(w_4\partial_x^l w_i\big)(\cdot,t)\big\|_{L^\infty} = O(1)M(t)\big(M(t)+\varepsilon_0\big)(t+1)^{-\sigma}, \tag{6.19a}$$

where $1 \leqq i \leqq 3$*, and*

$$\sigma = \begin{cases} 2, & \text{if } l = 2, \\ \frac{15}{8}, & \text{if } l = 3, \\ \frac{13}{8}, & \text{if } l = 4, \\ \frac{3}{4}, & \text{if } l = 5. \end{cases} \tag{6.19b}$$

Proof. Take a small $\varepsilon > 0$. For $|x| \geqq \varepsilon t$, use (6.12) to bound $\big(w_4\partial_x^l w_i\big)(x,t)$. For $|x| \leqq \varepsilon t$,

$$\begin{aligned}
\big|\big(w_4\partial_x^l w_i\big)(x,t)\big| = O(1)\Big\{&\big|\big(w_4\partial_x^l w_i\big)(-\varepsilon t,t)\big| \\
&+ M(t)E_l^{\frac{1}{4}}(t,\varepsilon)\big(E_l^{\frac{1}{2}}(t,\varepsilon)+E_{l+1}^{\frac{1}{2}}(t,\varepsilon)\big)^{\frac{1}{2}}\Big\}.
\end{aligned}$$

Then use (6.14) and (6.12) to bound the right-hand side. □

We now perform a pointwise a priori estimate through (6.9) and (6.12).

Lemma 6.3. *Under the assumptions of Theorem* 1.4, *we have for* $-\infty < x < \infty$, $t \geqq 0$,

$$\int_{-\infty}^{\infty} G_i(x-y,t)\partial_y^l w(y,0)\,dy = \begin{cases} O(1)\varepsilon_0(t+1)^{-\frac{l}{2}}\phi_i(x,t), & l=0,1, \\ O(1)\varepsilon_0(t+1)^{-\frac{5-l}{2}}, & 2 \leqq l \leqq 4, \end{cases} \tag{6.20}$$

where $i = 1, 2$.

Proof. Denote the left-hand side of (6.20) as I. From (6.7),

$$I = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\mu^* t}} e^{-\frac{(x-y-c_i t)^2}{2\mu^* t}} \partial_y^l w_i(y,0)\,dy + \int_{-\infty}^{\infty} H_i(x-y,t)\partial_y^l w(y,0)\,dy,$$

where H_i is the ith row of H given in (6.7b). Using (6.7b) and (1.42), we can prove (6.20) in a similar way as Lemmas 3.3–3.6 in [LZ2]. □

Similarly, using (6.8) we can prove the following lemma.

Lemma 6.4. *Under the assumptions of Theorem* 1.4, *we have for* $-\infty < x < \infty$, $t \geqq 0$,

$$\begin{aligned} &\int_{-\infty}^{\infty} G_3(x-y,t)\partial_y^l w(y,0)\,dy \\ &= \begin{cases} O(1)\varepsilon_0(t+1)^{-\frac{1}{2}} \sum_{i=1}^2 \phi_i(x,t), & l=0, \\ O(1)\varepsilon_0(t+1)^{-\frac{1}{2}} \sum_{i=1}^2 \tilde{\psi}(x,t;c_i), & l=1, \\ O(1)\varepsilon_0(t+1)^{-\frac{5-l}{2}}, & 2 \leqq l \leqq 4. \end{cases} \end{aligned} \tag{6.21}$$

The next lemma is straightforward by (6.4*c*), (1.42) and (6.10).

Lemma 6.5. *Under the assumptions of Theorem* 1.4, *we have for* $-\infty < x < \infty$,

$$\partial_x^l w_4(x,0) = \begin{cases} O(1)\varepsilon_0\phi_4(x), & l=0,1, \\ O(1)\varepsilon_0\phi_4^{\frac{1}{3}}(x), & l=2. \end{cases} \tag{6.22}$$

For the contribution from nonlinear sources we have the following lemmas.

Lemma 6.6. *Under the assumptions of Theorem* 1.4, *we have for* $-\infty < x < \infty$, $t \geqq 0$,

$$\int_0^t \int_{-\infty}^{\infty} G_i(x-y,t-t')\partial_y^l g(y,t')\,dy\,dt' = O(1)\big(M^2(t)+\varepsilon_0^2\big)(t+1)^{-\frac{l}{2}}\phi_i(x,t), \tag{6.23}$$

where $i = 1, 2$, $l = 0, 1$.

Proof. From (6.5) and (6.3) we have

$$g = \big(\zeta_1(v,e_1) - \zeta_1^*\big)u_x + \big(\zeta_2(v,e_1) - \zeta_2^*\big)\chi, \tag{6.24a}$$

where $\zeta_i \in \mathbb{R}^3$ and $\zeta_i^* = \zeta_i(v^*, e_1^*)$, $i = 1, 2$. Substituting the third equation in (1.36) into (6.24a), we also have

$$g = \bar{g} - \left[\frac{\tau}{1 + Q_{e_1}}(\zeta_2 - \zeta_2^*)\chi\right]_t, \tag{6.24b}$$

where

$$\bar{g} \equiv \left[\zeta_1 - \zeta_1^* - \frac{\tau a}{1 + Q_{e_1}}(\zeta_2 - \zeta_2^*)\right]u_x + \left[\frac{\tau}{1 + Q_{e_1}}(\zeta_2 - \zeta_2^*)\right]_t \chi. \tag{6.25}$$

By Taylor expansion, (6.2), (6.4), (6.12), (6.10), (5.1) and (1.36),

$$\bar{g}(x,t) = \bar{\zeta}\left(w_1^2\right)_x - \bar{\zeta}\left(w_2^2\right)_x + O(1)M^2(t)(t+1)^{-\frac{5}{4}} \sum_{j=1}^{2} \psi(x,t;c_j) + O(1)M^2(t)(t+1)^{-2}\bar{\psi}(x,t;0), \tag{6.26a}$$

where $\bar{\zeta} \in \mathbb{R}^3$ is a constant vector. Similarly, together with (6.19), we have

$$\left[\bar{g} - \bar{\zeta}\left(w_1^2\right)_x + \bar{\zeta}\left(w_2^2\right)_x\right]_x(x,t) = O(1)M^2(t)(t+1)^{-\frac{5}{4}} \sum_{j=1}^{2} \psi(x,t;c_j) + O(1)\left(M^2(t) + \varepsilon_0^2\right) \min\left\{(t+1)^{-2}, (t+1)^{-\frac{3}{2}}\bar{\psi}(x,t;0)\right\}. \tag{6.26b}$$

Also, for the second term in (6.24b) we have

$$\partial_x^l \left[\frac{\tau}{1 + Q_{e_1}}(\zeta_2 - \zeta_2^*)\chi\right] = \begin{cases} O(1)M^2(t)\left[(t+1)^{-\frac{3}{4}} \sum_{j=1}^{2} \psi(x,t;c_j) + (t+1)^{-2}\bar{\psi}(x,t;0)\right], & l = 0, \\ O(1)M^2(t)\left[(t+1)^{-\frac{5}{4}} \sum_{j=1}^{2} \psi(x,t;c_j) + (t+1)^{-2}\bar{\psi}(x,t;0)\right], & l = 1, \\ O(1)\left(M^2(t) + \varepsilon_0^2\right)(t+1)^{-\frac{1}{2}} \sum_{j=1}^{2} \tilde{\psi}(x,t;c_j), & l = 2. \end{cases} \tag{6.27}$$

Denote the left-hand side of (6.23) as I. Then by (6.24b),

$$\begin{aligned} I = &\int_0^t \int_{-\infty}^{\infty} G_i(x-y, t-t')\partial_y^l \bar{g}(y,t')\,dy\,dt' \\ &+ \int_0^t \int_{-\infty}^{\infty} G_i(x-y, t-t')\partial_y^l \left[-\frac{\tau}{1 + Q_{e_1}}(\zeta_2 - \zeta_2^*)\chi\right]_{t'}(y,t')\,dy\,dt' \\ \equiv &\, I_1 + I_2. \end{aligned} \tag{6.28}$$

By (6.7), (6.12), (6.10), (6.26),

$$\begin{aligned} I_1 = &\int_0^{t/2} \int_{-\infty}^{\infty} \partial_x^{l+1}\left[G_i(x-y, t-t')\right]\bar{\zeta}\left(w_1^2 - w_2^2\right)(y,t')\,dy\,dt' \\ &+ \int_{t/2}^t \int_{-\infty}^{\infty} G_i(x-y, t-t')\bar{\zeta}\,\partial_y^{l+1}\left(w_1^2 - w_2^2\right)(y,t')\,dy\,dt' \end{aligned}$$

$$
\begin{aligned}
&+ \int_0^t \int_{-\infty}^{\infty} \partial_x^l \big[G_i(x-y,t-t')\big]\big[\bar{g} - \bar{\zeta}\big(w_1^2\big)_x + \bar{\zeta}\big(w_2^2\big)_x\big](y,t')\,dy\,dt' \\
&= \int_0^{t/2} \int_{-\infty}^{\infty} \partial_x^{l+1} \left[\frac{1}{\sqrt{2\pi\mu^*(t-t')}} e^{-\frac{(x-y-c_i(t-t'))^2}{2\mu^*(t-t')}}\right] \bar{\zeta}_i\big(w_1^2 - w_2^2\big)(y,t')\,dy\,dt' \\
&+ \int_{t/2}^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\mu^*(t-t')}} e^{-\frac{(x-y-c_i(t-t'))^2}{2\mu^*(t-t')}} \bar{\zeta}_i\, \partial_y^{l+1}\big(w_1^2 - w_2^2\big)(y,t')\,dy\,dt' \\
&+ O(1)M^2(t)(t+1)^{-\frac{l+1}{2}} \int_0^t \int_{-\infty}^{\infty} (t-t')^{-\frac{1}{2}}(t-t'+1)^{-\frac{1}{2}} \left(\sum_{j=1}^{2} e^{-\frac{(x-y-c_j(t-t'))^2}{C^*(t-t')}}\right) \\
&\cdot \left(\sum_{j=1}^{2} \theta_2(y,t';c_j,\nu) + (t'+1)^{-\frac{1}{2}} \sum_{j=1}^{2} \psi(y,t';c_j)\right) dy\,dt' \\
&+ O(1)M^2(t) \int_0^{t/2} e^{-(t-t')/C} \sum_{k=1}^{3} \left[(t'+1)^{-\frac{1}{4}} \sum_{j=1}^{2} \psi(x-d_k(t-t'),t';c_j)\right] dt' \\
&+ O(1)M^2(t) \int_0^{t} e^{-(t-t')/C} \sum_{k=1}^{3} \left[(t'+1)^{-\frac{3}{4}-\frac{l}{2}} \sum_{j=1}^{2} \psi(x-d_k(t-t'),t';c_j)\right] dt' \\
&+ O(1)M^2(t) \int_0^t \int_{-\infty}^{\infty} \partial_x^l \left[\frac{1}{\sqrt{2\pi\mu^*(t-t')}} e^{-\frac{(x-y-c_i(t-t'))^2}{2\mu^*(t-t')}}\right] \\
&\cdot \left[(t'+1)^{-\frac{5}{4}} \sum_{j=1}^{2} \psi(y,t';c_j) + (t'+1)^{-2}\bar{\psi}(y,t';0)\right] dy\,dt' \qquad (6.29) \\
&+ O(1)M^2(t) \int_0^t \int_{-\infty}^{\infty} (t-t')^{-\frac{l+1}{2}}(t-t'+1)^{-\frac{1}{2}} \left(\sum_{j=1}^{2} e^{-\frac{(x-y-c_j(t-t'))^2}{C^*(t-t')}}\right) \\
&\cdot \left[(t'+1)^{-\frac{5}{4}} \sum_{j=1}^{2} \psi(y,t';c_j) + (t'+1)^{-2}\bar{\psi}(y,t';0)\right] dy\,dt' \\
&+ O(1)\big(M^2(t) + \varepsilon_0^2\big) \int_0^t e^{-(t-t')/C} \sum_{k=1}^{3} (t'+1)^{-\frac{1}{2}} \tilde{\psi}(x-d_k(t-t'),t';c_i)\,dt',
\end{aligned}
$$

where $\bar{\zeta}_i$ is the ith component of $\bar{\zeta}$. The first two terms on the right-hand side of (6.29), say, for $i = 1$, are

$$
\begin{aligned}
O(1)M^2(t)(t+1)^{-\frac{l}{2}} \int_0^t \int_{-\infty}^{\infty} (t-t')^{-1} e^{-\frac{(x-y-c_1(t-t'))^2}{2(\mu^*+\varepsilon)(t-t')}} \Big[\theta_2\big(y,t';c_1,\nu/2\big) \\
+ (t'+1)^{-\frac{3}{4}}\psi(y,t';c_1) + (t'+1)^{-1}\psi(y,t';c_2)\Big] dy\,dt' \\
- \frac{\bar{\zeta}_i}{\sqrt{2\pi\mu^*}} \int_0^t \int_{-\infty}^{\infty} (t-t')^{-\frac{1}{2}} e^{-\frac{(x-y-c_1(t-t'))^2}{2\mu^*(t-t')}} \partial_y^{l+1} w_2^2(y,t')\,dy\,dt'
\end{aligned}
$$

for a small $\varepsilon > 0$. Apply Lemmas 5.1, 5.3, 5.4 to the first integral and Lemma 5.5 to the second one, where when checking condition (5.6) for $h = w_2^2$ we use (6.4), (6.1), (3.31b), (1.25), (6.7b), (6.2), (6.3), (6.12), (5.1) and (6.10). The above integrals are bounded by the right-hand side of (6.23). The other terms on the right-hand side of (6.29) can be settled by Lemmas 5.1–5.4, 5.7 and 5.6. They are also bounded by the right-hand side of (6.23).

To estimate I_2 in (6.28), we integrate by parts with respect to t' and apply (3.3), (6.27), (1.42), (6.20), (6.6), (6.7) to yield

$$\begin{aligned} I_2 &= O(1)M^2(t)(t+1)^{-\frac{1}{2}}\phi_i(x,t) + O(1)\varepsilon_0^2(t+1)^{-\frac{l}{2}}\phi_i(x,t) \\ &\quad + \int_0^t\int_{-\infty}^{\infty}\left[\frac{c_i}{\sqrt{2\pi\mu^*(t-t')}}e^{-\frac{(x-y-c_i(t-t'))^2}{2\mu^*(t-t')}}\right]_x \\ &\quad \cdot \partial_y^l\left[\frac{\tau}{1+Q_{e_1}}(\zeta_2-\zeta_2^*)_i\chi\right](y,t')\,dy\,dt' \\ &\quad + O(1)\int_0^t\int_{-\infty}^{\infty}(t-t')^{-1}(t-t'+1)^{-\frac{1}{2}}\sum_{j=1}^{2}e^{-\frac{(x-y-c_j(t-t'))^2}{C^*(t-t')}} \\ &\quad \cdot \partial_y^l\left[\frac{\tau}{1+Q_{e_1}}(\zeta_2-\zeta_2^*)\chi\right](y,t')\,dy\,dt' \\ &\quad + O(1)\int_0^t e^{-(t-t')/C}\sum_{k=1}^{3}\sum_{j=l}^{l+1}\partial_x^j\left[\frac{\tau}{1+Q_{e_1}}(\zeta_2-\zeta_2^*)\chi\right](x-d_k(t-t'),t')\,dt', \end{aligned}$$

where $(\zeta_2-\zeta_2^*)_i$ is the ith component of $(\zeta_2-\zeta_2^*)$. By integration by parts and the use of (6.27) with $l=1$, the first integral above can be shown to be the same as the sixth integral on the right-hand side of (6.29). The other two integrals are the same as the seventh, the fifth and the last integrals on the right-hand side of (6.29), or can be handled similarly. Therefore, I_2 is also bounded by the right-hand side of (6.23). □

Lemma 6.7. *Under the assumptions of Theorem* 1.4, *we have for* $-\infty < x < \infty$, $t \geqq 0$,

$$\begin{aligned} &\int_0^t\int_{-\infty}^{\infty}G_3(x-y,t-t')\partial_y^l g(y,t')\,dy\,dt' \\ &\qquad = O(1)\big(M^2(t)+\varepsilon_0^2\big)(t+1)^{-\frac{1}{2}}\begin{cases}\sum_{j=1}^2\phi_j(x,t), & l=0,\\ \sum_{j=1}^2\tilde{\psi}(x,t;c_j), & l=1.\end{cases} \end{aligned} \tag{6.30}$$

Proof. Denote the left-hand side of (6.30) as I. By (6.8), (6.24b) and integration by parts we have

$$
\begin{aligned}
I &= \int_0^{t-1}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi\mu^*(t-t')}}\sum_{i=1}^{2} e^{-\frac{(x-y-c_i(t-t'))^2}{2\mu^*(t-t')}}\eta_i\right]_x \partial_y^l \bar{g}(y,t')\,dy\,dt' \\
&- \int_0^{t-1}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi\mu^*(t-t')}}\sum_{i=1}^{2} e^{-\frac{(x-y-c_i(t-t'))^2}{2\mu^*(t-t')}}\eta_i\right]_x \\
&\cdot \partial_y^l\left[\frac{\tau}{1+Q_{e_1}}(\zeta_2-\zeta_2^*)\chi\right]_{t'}(y,t')\,dy\,dt' \\
&+ \int_0^{t-1}\int_{-\infty}^{\infty} O(1)(t-t')^{-1-\frac{l}{2}}(t-t'+1)^{-\frac{1}{2}}\sum_{i=1}^{2} e^{-\frac{(x-y-c_i(t-t'))^2}{C^*(t-t')}} g(y,t')\,dy\,dt' \\
&+ \int_0^{t-1} e^{-(t-t')/C}\sum_{j=0}^{l}\sum_{k=1}^{3}\partial_x^{l-j} g(x-d_k(t-t'),t')\,dt' \qquad (6.31)\\
&+ \int_{t-1}^{t}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi\mu^*(t-t')}}\sum_{i=1}^{2} e^{-\frac{(x-y-c_i(t-t'))^2}{2\mu^*(t-t')}}\eta_i\right]_x \partial_y^l g(y,t')\,dy\,dt' \\
&+ \int_{t-1}^{t}\int_{-\infty}^{\infty} O(1)(t-t')^{-1}(t-t'+1)^{-\frac{1}{2}}\sum_{i=1}^{2} e^{-\frac{(x-y-c_i(t-t'))^2}{C^*(t-t')}}\partial_y^l g(y,t')\,dy\,dt' \\
&+ \int_{t-1}^{t} e^{-(t-t')/C}\sum_{k=1}^{3}\partial_x^l g(x-d_k(t-t'),t')\,dt' \\
&\equiv \sum_{i=1}^{7} I_i,
\end{aligned}
$$

where η_i, $i = 1, 2$, are constant row vectors in $\mathbb{R}^3$. We substitute (6.26a) into the expression for I_1. The case where $l = 0$ has been handled in Lemma 6.6 and the first two terms and the sixth term on the right-hand side of (6.29) are $O(1)M^2(t)(t+1)^{-\frac{1}{2}}\sum_{i=1}^{2}\phi_i(x,t)$. If $l = 1$, by integration by parts we have

$$
\begin{aligned}
I_1 = \sum_{i=1}^{2} O(1)M^2(t)&\int_0^{t/2}\int_{-\infty}^{\infty}(t-t')^{-2}e^{-\frac{(x-y-c_i(t-t'))^2}{\nu(t-t')}}\sum_{j=1}^{2}\big[\theta_2\big(y,t';c_j,\nu/2\big) \\
&+ (t'+1)^{-\frac{3}{4}}\psi(y,t';c_j)\big]\,dy\,dt' + \sum_{i=1}^{2} O(1)M^2(t) \\
&\cdot \int_{t/2}^{t-1}\int_{-\infty}^{\infty}(t-t')^{-1}e^{-\frac{(x-y-c_i(t-t'))^2}{\nu(t-t')}}(t'+1)^{-\frac{5}{4}}\sum_{j=1}^{2}\psi(y,t';c_j)\,dy\,dt' \\
&+ \sum_{i=1}^{2} O(1)M^2(t)\int_0^{t-1}\int_{-\infty}^{\infty}(t-t')^{-\frac{3}{2}}e^{-\frac{(x-y-c_i(t-t'))^2}{\nu(t-t')}}
\end{aligned}
$$

$$\cdot \left[(t'+1)^{-\frac{5}{4}} \sum_{j=1}^{2} \psi(y,t';c_j) + (t'+1)^{-2}\bar{\psi}(y,t';0)\right] dy\,dt'. \tag{6.32}$$

Applying Lemmas 5.1–5.4 and 5.6 gives

$$I_1 = O(1)M^2(t)(t+1)^{-\frac{1}{2}} \sum_{i=1}^{2} \tilde{\psi}(x,t;c_i).$$

These settle I_1. For I_i, $3 \leqq i \leqq 7$, we use (6.24a) instead of (6.24b) for g. Using a method similar to that used to get (6.27), we can find

$$\partial_x^l g(x,t) = O(1)\big(M^2(t)+\varepsilon_0^2\big) \tag{6.33}$$
$$\cdot \begin{cases} (t+1)^{-\frac{3}{4}} \sum_{j=1}^{2} \psi(x,t;c_j) + (t+1)^{-2}\bar{\psi}(x,t;0), & l=0, \\ (t+1)^{-\frac{5}{4}} \sum_{j=1}^{2} \psi(x,t;c_j) + \min\big\{(t+1)^{-2}, (t+1)^{-\frac{3}{2}}\bar{\psi}(x,t;0)\big\}, & l=1, \end{cases}$$

when we use (6.19). Substitute (6.33) into I_i, $3 \leqq i \leqq 7$, in (6.31), and apply Lemmas 5.3, 5.4, 5.6, 5.7. These yield the right-hand side of (6.30). We now have only one more term, I_2, to estimate. By integration by parts and (6.27),

$$\begin{aligned} I_2 = &-\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\mu^*}} \left(\sum_{i=1}^{2} e^{-\frac{(x-y-c_i)^2}{2\mu^*}} \eta_i\right) \partial_y^{l+1}\left[\frac{\tau}{1+Q_{e_1}}(\zeta_2-\zeta_2^*)\chi\right](y,t-1)\,dy \\ &+\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\mu^* t}} \left(\sum_{i=1}^{2} e^{-\frac{(x-y-c_i t)^2}{2\mu^* t}} \eta_i\right) \partial_y^{l+1}\left[\frac{\tau}{1+Q_{e_1}}(\zeta_2-\zeta_2^*)\chi\right](y,0)\,dy \\ &+\int_0^{t-1}\int_{-\infty}^{\infty} O(1)(t-t')^{-\frac{3}{2}} \left(\sum_{i=1}^{2} e^{-\frac{(x-y-c_i(t-t'))^2}{\nu(t-t')}}\right) \\ &\cdot \left|\partial_y^l\left[\frac{\tau}{1+Q_{e_1}}(\zeta_2-\zeta_2^*)\chi\right]\right|(y,t')\,dy\,dt' \\ = &\;O(1)\big(M^2(t)+\varepsilon_0^2\big)(t+1)^{-\frac{1}{2}} \sum_{i=1}^{2} \int_{-\infty}^{\infty} e^{-\frac{(x-y-c_i)^2}{2\mu^*}} \sum_{j=1}^{2} \tilde{\psi}(y,t-1;c_j)\,dy \\ &+O(1)\varepsilon_0^2(t+1)^{-\frac{l}{2}-1} \sum_{i=1}^{2} \int_{-\infty}^{\infty} e^{-\frac{(x-y-c_i t)^2}{\nu t}} \big(y^2+1\big)^{-\frac{3}{4}}\,dy \\ &+O(1)M^2(t)\int_0^{t-1}\int_{-\infty}^{\infty} (t-t')^{-\frac{3}{2}} \left(\sum_{i=1}^{2} e^{-\frac{(x-y-c_i(t-t'))^2}{\nu(t-t')}}\right) \\ &\cdot \left[(t'+1)^{-\frac{3}{4}-\frac{l}{2}} \sum_{j=1}^{2} \psi(y,t';c_j) + (t'+1)^{-2}\bar{\psi}(y,t';0)\right] dy\,dt'. \end{aligned}$$

Apply Lemma 5.9 with $t'=t-1$ to the first term on the right-hand side. This yields $O(1)\big(M^2(t)+\varepsilon_0^2\big)(t+1)^{-\frac{1}{2}} \sum_{i=1}^{2} \tilde{\psi}(x,t;c_i)$. The second term has been estimated in Lemma 6.3. The last term is the same as I_3 in (6.31) if $l = 0$, and the same as the last term in (6.32) if $l = 1$. Together, these give the right-hand side of (6.30). □

Lemma 6.8. *Under the assumptions of Theorem* 1.4, *we have for* $-\infty < x < \infty$, $t \geqq 0$,

$$\int_0^t \int_{-\infty}^{\infty} G_i(x-y, t-t')\partial_y^l g(y,t')\,dy\,dt' = O(1)\big(M^2(t) + \varepsilon_0^2\big)(t+1)^{-\frac{5-l}{2}}, \tag{6.34}$$

where $1 \leqq i \leqq 3$, $2 \leqq l \leqq 4$.

Proof. Denote the left-hand side of (6.34) as $I^{(i)}$. For $i = 1, 2$, by (6.7) and integration by parts,

$$\begin{aligned}
I^{(i)} &= \int_0^{t-1}\int_{-\infty}^{\infty} \partial_x^l \Big[\frac{1}{\sqrt{2\pi\mu^*(t-t')}} e^{-\frac{(x-y-c_i(t-t'))^2}{2\mu^*(t-t')}}\Big] g_i(y,t')\,dy\,dt' \\
&+ O(1)\int_0^{t-1}\int_{-\infty}^{\infty} (t-t')^{-\frac{l+1}{2}}(t-t'+1)^{-\frac12}\Big(\sum_{j=1}^{2} e^{-\frac{(x-y-c_j(t-t'))^2}{C^*(t-t')}}\Big) g(y,t')\,dy\,dt' \\
&+ \int_0^{t-1} e^{-(t-t')/C}\sum_{j=0}^{l}\sum_{k=1}^{3} \partial_x^{l-j} g(x-d_k(t-t'),t')\,dt' \qquad (6.35)\\
&+ \int_{t-1}^{t}\int_{-\infty}^{\infty} \Big[\frac{1}{\sqrt{2\pi\mu^*(t-t')}} e^{-\frac{(x-y-c_i(t-t'))^2}{2\mu^*(t-t')}}\Big]_x \partial_y^{l-1} g_i(y,t')\,dy\,dt' \\
&+ O(1)\int_{t-1}^{t}\int_{-\infty}^{\infty} (t-t')^{-1}(t-t'+1)^{-\frac12}\Big(\sum_{j=1}^{2} e^{-\frac{(x-y-c_j(t-t'))^2}{C^*(t-t')}}\Big)\partial_y^{l-1} g(y,t')\,dy\,dt' \\
&+ \int_{t-1}^{t} e^{-(t-t')/C}\sum_{k=1}^{3}\big[\partial_x^l g(x-d_k(t-t'),t') + \partial_x^{l-1} g(x-d_k(t-t'),t')\big]\,dt'.
\end{aligned}$$

With the exception of the first integral with $l = 2$, we use (6.24a) for g. Then besides (6.33), we have

$$\partial_x^l g(x,t) = O(1)\big(M^2(t)+\varepsilon_0^2\big)(t+1)^{-\frac{5-l}{2}}, \quad 2 \leqq l \leqq 4,$$

by (6.2), (6.4), (6.12), (6.10), (5.1) and (6.19). The last four integrals in (6.35) are obviously equal to the right-hand side of (6.34). The first integral with $l > 2$ and the second one can be estimated using Lemmas 5.3, 5.4 and 5.6. As for the first integral with $l = 2$, it is the same as $I_1 + I_2$ in (6.31) with $l = 1$. These settle $I^{(i)}$ for $i = 1, 2$. $I^{(3)}$ is easier and can be estimated in a similar way. □

Lemma 6.9. *Under the assumptions of Theorem* 1.4, *we have for* $-\infty < x < \infty$, $t \geqq 0$,

$$\int_0^t \partial_x^l g_4(x,t')\,dt' = O(1)M^2(t)\phi_4(x), \tag{6.36}$$

where $0 \leqq l \leqq 2$.

Proof. By (6.5), (1.6), (1.12), (6.2), (6.4), (6.12), (6.10) and (5.1) we have

$$\partial_x^l g_4(x,t) = O(1)M^2(t)(t+1)^{-\frac{5}{4}} \sum_{j=1}^{2} \psi(x,t;c_j), \quad 0 \leqq l \leqq 2.$$

Therefore, (6.36) is true by Lemma 5.8. □

We now close the stability analysis. Under the assumptions of Theorem 1.4, equation (6.9) and Lemmas 6.3–6.9 imply that

$$\begin{aligned}
&|\partial_x^l w_i(x,t)| \leqq C\big[\varepsilon_0 + M^2(t)\big](t+1)^{-\frac{l}{2}}\phi_i(x,t), \quad i=1,2, \quad l=0,1,\\
&|\partial_x^l w_3(x,t)| \leqq C\big[\varepsilon_0 + M^2(t)\big](t+1)^{-\frac{1}{2}} \begin{cases} \sum_{i=1}^2 \phi_i(x,t), & l=0,\\ \sum_{i=1}^2 \tilde{\psi}(x,t;c_i), & l=1,\end{cases}\\
&|\partial_x^l w_i(x,t)| \leqq C\big[\varepsilon_0 + M^2(t)\big](t+1)^{-\frac{5-l}{2}}, \quad 1 \leqq i \leqq 3, \quad 2 \leqq l \leqq 4,\\
&|\partial_x^l w_4(x,t)| \leqq C\big[\varepsilon_0 + M^2(t)\big]\phi_4(x), \quad l=0,1,\\
&|\partial_x^2 w_4(x,t)| \leqq C\big[\varepsilon_0 + M^2(t)\big]\phi_4^{\frac{1}{3}}(x).
\end{aligned}$$

These inequalities and (6.11) then imply that

$$M(t) \leqq C\big[\varepsilon_0 + M^2(t)\big].$$

If ε_0 is sufficiently small, we have

$$M(t) \leqq C\varepsilon_0 \tag{6.37}$$

for all $t \geqq 0$. Equations (6.2), (6.4), (6.12), (6.37), (6.10), (5.1) and (6.7b) give us (1.43).

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