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# *Stable Configurations in Superconductivity: Uniqueness, Multiplicity, and Vortex-Nucleation*

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**Abstract**

We find new stable solutions of the Ginzburg-Landau equation for high $\kappa$ superconductors with exterior magnetic field $h_{\rm ex}$. First, we prove the uniqueness of the Meissner-type solution. Then, we prove, in the case of a disc domain, the coexistence of branches of solutions with $n$ vortices of degree one, for any $n$ not too high and for a certain range of $h_{\rm ex}$; and describe these branches. Finally, we give an estimate on the nucleation energy barrier, to pass continuously from a vortexless configuration to a configuration with a centered vortex.

## 1. Introduction

The aim of this work is to improve the results of a previous work [S1, S2] on the gauge-invariant two-dimensional Ginzburg-Landau functionals

$$J(u, A) = \frac{1}{2}\int_\Omega |\nabla_A u|^2 + |h - h_{\rm ex}|^2 + \tfrac{1}{2}\kappa^2(1 - |u|^2)^2 - \frac{1}{2}\int_\Omega h_{\rm ex}^2, \tag{1.1}$$

which describe, according to the Ginzburg-Landau model, the energy of a superconductor put in a prescribed uniform magnetic field $h_{\rm ex}$. The stationary states of such a superconductor are the critical points of $J$, among which the stable states are the local minimizers.

Any critical configuration $(u, A)$ of this functional is a solution of the Ginzburg-Landau equations:

$$-\nabla_A^2 u = \kappa^2 u(1 - |u|^2), \quad - * dh = (iu, d_A u). \tag{G.L.}$$

Let us emphasize that this is the free Ginzburg-Landau functional (used in physics), there are no boundary conditions, and all the complexity of the solutions comes from the variations of the parameter $h_{\rm ex}$.

We recall the notations used for this problem (see [S1] for a more detailed introduction).

$*$ is the Hodge transform for 1-forms ($*dx = dy,\ *dy = -dx$).

$A$ is the vector potential, considered here as a real-valued 1-form $A_1dx_1 + A_2dx_2$ or as a vector $(A_1, A_2)$, and $h = *dA$ (or curl $A$), is the magnetic field in the superconductor.

We assume that the exterior field $h_{\text{ex}}$ is uniform, vertical, and that the superconductor is a vertical cylinder, so that the problem is reduced to a two-dimensional problem in a bounded domain $\Omega \subset \mathbb{R}^2$. We also assume that $\Omega$ is smooth and simply connected.

$u$ is a complex-valued function called the *order parameter*. The superconductivity phenomenon in the material can be described through pairs of superconducting electrons called *Cooper pairs*. Then, $|u(x)|$ represents the local density of Cooper pairs at a point $x$ of the superconductor. One has $|u| \leqq 1$ and, where $|u| \simeq 1$, the material is in its superconducting phase, whereas, where $|u| \simeq 0$, it is in its normal phase.

$\kappa = 1/\varepsilon$ is the Ginzburg-Landau parameter, depending on the material only. In physics, $\kappa = \lambda/\xi$, where $\lambda$ is the London length, and $\xi$ is the superconductor's coherence length. The superconductors we shall study are those with high $\kappa$, which are called in physics *extreme type-II superconductors*. Here, as in [S1] and [S2], we carry out an asymptotic analysis for $\kappa \to +\infty$, equivalent to $\varepsilon \to 0$. $h_{\text{ex}}$ is a function of $\varepsilon$, and the configurations $(u, A)$ we consider depend on $\varepsilon$, but we shall drop the subscripts most of the time.

Physically, at low temperatures ($T < T_c$, the critical temperature), there exists a critical magnetic field known as the first critical magnetic field ($H_{c_1}$), such that:

- For $h_{\text{ex}} < H_{c_1}$, the superconductor is in the superconducting phase everywhere ($|u| \simeq 1$), and the magnetic field does not penetrate it. This is called the Meissner phase.
- For $h_{\text{ex}} \geqq H_{c_1}$, small defects, called vortices, appear. They are small regions of size $\kappa^{-1}$ that switch to the normal phase. Each of them carries with it a quantized amount of the magnetic flux.

The most important feature is thus the appearance of zeros of $u$, and the challenge, as in all the similar problems [BBH, BR], is to study these zeros and the properly defined vortex-structure of the solutions. Mathematically, we define the vortices of size $\delta$ of a function $u$ to be points $a_i$ such that $|u| \geqq \frac{3}{4}$ outside $\cup_i B(a_i, \delta)$, associated with the integers $d_i = \deg\,(u, \partial B(a_i, \delta))$, whenever they are well-defined. In practice, $u$ vanishes in each of those $B(a_i, \delta)$.

When $h_{\text{ex}}$ is raised, these vortices become more and more numerous, and repel one another, thus forming some kind of triangular lattice, called in physics the Abrikosov lattice.

In [S1, S2], I studied $J$ for $h_{\text{ex}} \leqq C|\text{Log}\,\varepsilon|$, but the analysis that was carried out still holds if some a priori bound $h_{\text{ex}} \leqq C\varepsilon^{-\alpha}$ is valid. In addition, the main tools that we use here are on the one hand technical tools concerning vortices already introduced in [S1, S2] and inspired from [AB, BR, BBH], and on the other hand a convenient splitting of the energy $J$. These tools are supplemented with a slightly special minimization idea that was already used in [S1, S2], but which is more widely used here. This idea allows us to isolate local minimizers of $J$ having an arbitrary number of vortices. We also emphasize that this work, as well as the previous one, concerns configurations with few vortices, i.e., with a number of vortices that remains bounded as $\varepsilon \to 0$. Configurations with a possibly divergent number of vortices (i.e., global minimizers instead of local ones) and high external fields are studied in [SS1, SS2].

In [S1, S2], I proved some theorems concerning the behavior around the first critical field. We call $H_{c_1}$ the value of $h_{\text{ex}}$ for which the energy of the single-vortex configuration becomes equal to the energy of the vortexless configuration (in which $|u| \geqq \frac{3}{4}$). I proved that

$$H_{c_1} = \frac{\operatorname{Log} \kappa}{2 \max |\xi_0|} + O(1) \qquad \text{as } \kappa \to +\infty, \tag{1.2}$$

where $\xi_0$ is a smooth function, depending on the domain $\Omega$ only, that satisfies

$$\left\{ \begin{array}{ll} -\Delta^2 \xi_0 + \Delta \xi_0 = 0 & \text{in } \Omega \\ \Delta \xi_0 = 1 & \text{on } \partial\Omega \\ \xi_0 = 0 & \text{on } \partial\Omega \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{ll} -\Delta \xi_0 + \xi_0 + 1 = 0 & \text{in } \Omega \\ \xi_0 = 0 & \text{on } \partial\Omega \end{array} \right\}. \tag{1.3}$$

$J_0$ was defined to be $J(1, h_{\text{ex}} d^* \xi_0)$, the approximate minimal energy for vortexless configurations, which satisfies $J_0 + \frac{1}{2} \int_\Omega h_{\text{ex}}^2 = \frac{1}{2} h_{\text{ex}}^2 \int_\Omega |\xi_0|$. Setting $\mathscr{M} > 0$, I sought minimizers of $J$ in the domain

$$\begin{aligned} D = \Big\{ &(u, A) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2) / F(u) \\ &:= \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{1}{2\varepsilon^2} (1 - |u|^2)^2 < \mathscr{M} |\operatorname{Log} \varepsilon| \Big\}, \end{aligned} \tag{1.4}$$

(which is roughly the domain of configurations with less than $\mathscr{M}/\pi$ vortices), hoping that for $\mathscr{M}$ large enough, they were in fact global minimizers. Notice that $F$ is the functional that was studied in [BBH].

The first main result of [S1] is

**Theorem 1.1.** *There exist $k_2^\varepsilon = O_\varepsilon(1)$ and $k_3^\varepsilon = o_\varepsilon(1)$, such that*

$$H_{c_1} = k_1 |\operatorname{Log} \varepsilon| + k_2^\varepsilon, \qquad \left( k_1 = \frac{1}{2 \max |\xi_0|} \right)$$

*and $\varepsilon_0(\mathscr{M})$ such that for $0 < \varepsilon < \varepsilon_0$, the following statements hold:*

- *If $h_{\text{ex}} \leqq H_{c_1}$, then a solution of* (G.L.) *that is minimizing in $D$ exists, satisfies $\frac{1}{2} \leqq |u| \leqq 1$ and has an energy $J_0 + o_\varepsilon(1)$.*

- *If* $H_{c_1} + k_3^\varepsilon \leqq h_{\rm ex} \leqq H_{c_1} + O_\varepsilon(1)$*, then a solution of* (G.L.) *that is minimizing in* $D$ *exists, it has a bounded positive number of vortices* $a_i^\varepsilon$ *of degree* 1*, such that* dist $(a_i^\varepsilon, \Lambda) \to 0$ *where*

$$\Lambda = \{x \in \Omega / |\xi_0(x)| = \max |\xi_0|\},$$

*and there exists a* $C > 0$*, such that* dist $(a_i^\varepsilon, a_j^\varepsilon) \geqq C$ *for* $i \neq j$*, i.e., the* $a_i$*'s tend to distinct points* $\in \Lambda$.

This result shows the bifurcation behavior of the equation, i.e., that the minimizing solution is vortexless for $h_{\rm ex} \leqq H_{c_1}$, and has one vortex for $h_{\rm ex} \simeq H_{c_1}$. However, it does not prove that the physical system switches from the Meissner (i.e., vortexless) solution to one with vortices at $h_{\rm ex} = H_{c_1}$. Indeed, there exists another critical field, called the superheating field $H_{\rm sh}$, such that the Meissner solution becomes unstable for $h_{\rm ex} > H_{\rm sh}$. This bifurcation has been studied for example in [BBC], where the authors prove that (with our normalization of the quantities), $H_{\rm sh} \simeq c\kappa = c/\varepsilon$. Here, with that behavior in mind, we wish to study the domain of existence, uniqueness and stability of the Meissner solution, and we prove

**Theorem 1.** *There exist* $\alpha > 0$ *and* $\varepsilon_0$ *such that, if* $\varepsilon < \varepsilon_0$*, a stable vortexless solution of* (G.L.) *for* $h_{\rm ex} \leqq C\varepsilon^{-\alpha}$ *with* $\int_\Omega |\nabla u|^2 \leqq o(\varepsilon^\alpha)$ *is unique.*

*Let* $E_0 = \{(u, A) \in D / |u| \geqq \frac{3}{4}\}$*. For* $\varepsilon < \varepsilon_0$*, there exists a unique locally minimizing solution* $(u, A) = (u, d^*\xi)$ *of* (G.L.) *in* $E_0$ *for* $h_{\rm ex} \leqq C\varepsilon^{-\alpha}$ *that minimizes* $J$ *over* $E_0$*. Its energy is* $J_0 + o(1)$*. In addition,*

$$\inf_{\theta \in [0, 2\pi]} \|(u, \xi) - (e^{i\theta}, h_{\rm ex}\xi_0)\| \longrightarrow 0 \quad \text{as } \varepsilon \to 0,$$

*where* $\|.\|$ *is defined as*

$$\|(u, z)\|^2 = \|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2 + \|\nabla z\|_{L^2}^2 + \|\Delta z\|_{L^2}^2.$$

Thus, we prove that the branch of vortexless solutions *continues to exist* up to $h_{\rm ex} = C\varepsilon^{-\alpha}$, and we have an approximation of it: $(1, h_{\rm ex} d^*\xi_0)$, up to a gauge-transformation. Since this branch of solutions remains stable, in the process of raising $h_{\rm ex}$ vortices should not appear at $H_{c_1}$, but rather at a higher value of $h_{\rm ex}$, probably equal to $H_{\rm sh}$. E. SANDIER and I [SS1 ] proved the global minimality of the Meissner solution under $H_{c_1}$.

After this work was completed, we learned that the uniqueness of the Meissner solution (in another function space) had also been proved by A. BONNET, J. CHAPMAN & R. MONNEAU in [BCM] for $h_{\rm ex} \simeq C/\varepsilon$.

In [S2], I also obtained a result analyzing the appearance of $n$-vortices minimizing solutions in a disc (the disc is the model case of $\Lambda$ being reduced to a finite number of points, which is the case for convex $\Omega$'s for example; see [S1]), with vortices at a characteristic distance $C/\sqrt{h_{\rm ex}}$ from the center, and from one another ($C$ always denotes a positive constant). Calling $H_n$ the value of $h_{\rm ex}$ for which a minimizing solution (in $D$) with $n$ vortices appears, I proved that

$$H_n \simeq k_1(|\text{Log}\,\varepsilon| + (n-1)|\text{Log}\,|\text{Log}\,\varepsilon||).$$

The precise theorem is:

**Theorem 1.2.** *If $\Omega = B(0, R)$ (in this case $\Lambda = \{0\}$), there exists $\varepsilon_0(\mathscr{M}) > 0$ such that, for $\varepsilon < \varepsilon_0$, a minimizer $(u, A)$ of $J$ in $\bar{D}$ is a solution of* (G.L.) *and if*

$$h_{\text{ex}} = k_1(|\text{Log}\,\varepsilon| + \delta|\text{Log}\,|\text{Log}\,\varepsilon||) + O_\varepsilon(1), \quad 0 \leqq n-1 < \delta < n < \frac{\mathscr{M}}{\pi} \quad (n \in \mathbb{N}),$$

*then $u$ has $n$ vortices $a_i^\varepsilon$ of degree 1 such that $|a_i^\varepsilon| \leqq C/\sqrt{|\text{Log}\,\varepsilon|}$, for all $i$. Moreover, if $n \geqq 2$, $|a_i^\varepsilon - a_j^\varepsilon| \geqq C/\sqrt{|\text{Log}\,\varepsilon|}$ for all $i \neq j$, and if $\tilde{a}_i = a_i^\varepsilon(k_1|\text{Log}\,\varepsilon|)^{1/2}$, then the configuration of the $\tilde{a}_i$'s converges to a minimizer of $w$ as $\varepsilon \to 0$, where $w$ is defined as*

$$w(x_1, \ldots, x_n) = -\pi \sum_{i \neq j} \text{Log}\,|x_i - x_j| + \pi \xi_0''(0) \sum_{i=1}^{n} |x_i|^2.$$

We improve this result here by proving

**Theorem 2.** *Suppose $\Omega = B(0, R)$ and $h_{\text{ex}}$ is any function of $\varepsilon$ such that $h_{\text{ex}} \to \infty$ as $\varepsilon \to 0$, with $h_{\text{ex}} \leqq C\varepsilon^{-\alpha}$.*

1. *If $\varepsilon$ is sufficiently small,* (G.L.) *has a locally minimizing solution $(u, A)$ with* exactly one *vortex $a^\varepsilon$ of degree 1, satisfying*

$$|a^\varepsilon| \leqq \frac{C}{\sqrt{h_{\text{ex}}}},$$

$$J(u, A) = J_0 + \pi\left(|\text{Log}\,\varepsilon| - \frac{h_{\text{ex}}}{k_1}\right) + O(1).$$

2. *More generally, for each $n \in \mathbb{N}^*$ such that $\pi n < \mathscr{M}$, if $\varepsilon < \varepsilon_0(\mathscr{M})$, then* (G.L.) *has a locally minimizing solution $(u, A)$ with* exactly $n$ *vortices $a_i^\varepsilon$ of degree 1. In addition, if $\tilde{a}_i = a_i^\varepsilon\sqrt{h_{\text{ex}}}$, then*

$$|\tilde{a}_i| \leqq C \quad \forall i, \qquad |\tilde{a}_i - \tilde{a}_j| \geqq C \quad \forall i \neq j,$$

*and the configuration of the $\tilde{a}_i$'s converges to a minimizer of $w$.*

*Furthermore,*

$$J(u, A) = J_0 + \pi n\left(|\text{Log}\,\varepsilon| - \frac{h_{\text{ex}}}{k_1}\right) + \frac{\pi}{2}(n^2 - n)\text{Log}\,h_{\text{ex}} + w(\tilde{a}_1, \cdots, \tilde{a}_n) + Q_n + o(1)$$

where $Q_n$ is a constant depending only on $n$.

We thus show that the branch of stable solutions with $n$ vortices found in [S2] continues to exist for $h_{\text{ex}} \leqq H_n$ and $h_{\text{ex}} \geqq H_{n+1}$, although it is only globally minimizing (in $D$) for $h_{\text{ex}} \in [H_n, H_{n+1}]$. Moreover, it specifies that the characteristic distance from the vortices to the center and to one another is still of the order of $1\sqrt{h_{\text{ex}}}$ and their positions are still governed by the renormalized energy $w$. As mentioned in [S1], a study of the minimizers of $w$ has been carried out in [GS]. For small values of $n$ ($n \leqq 6$), the regular polygons centered at the origin are local minimizers (and probably global minimizers) together with the regular stars centered at the origin for $4 \leqq n \leqq 10$. For higher $n$, the minimizers form certain kinds of lattices concentrated around the center. These are thus the shapes, up to rescaling, of the vortex configurations that we exhibit in Theorem 2.

This theorem also yields solutions for relatively small values of $h_{\text{ex}}$. Usually, the value of $h_{\text{ex}}$ under which any vortex solution is unstable is called the subcooling field $H_{\text{sc}}$. Since we find stable solutions for any function $h_{\text{ex}} \to +\infty$, we can conclude that there exists a positive constant $\Gamma$ (independent of $\varepsilon$) and a sequence $\varepsilon_n \to 0$, such that there is a sequence of stable vortex solutions for $h_{\text{ex}} = \Gamma$ (the contrary would contradict the theorem).

We thus obtain that $H_{\text{sc}} = O(1)$, which was already proved, using a different method by Q. DU & F. H. LIN [DL]: Actually, they proved the existence of a stable vortex solution for a constant value of $h_{\text{ex}}$, using the heat-flow. Here, as already mentioned, we get in addition a whole branch of stable solutions for $h_{\text{ex}}$ ranging from $O(1)$ to $\varepsilon^{-\alpha}$ for arbitrary numbers of vortices, and we have details on the positions of their vortices.

Superconductors thus have theoretically a hysteretic behavior: starting in a Meissner state, when raising $h_{\text{ex}}$, vortices only appear for $h_{\text{ex}} \simeq H_{\text{sh}} \simeq C/\varepsilon$, while, starting in a vortex state and decreasing $h_{\text{ex}}$, they only disappear for $h_{\text{ex}} \simeq H_{\text{sc}} = O(1)$. Yet, for quantum mechanical reasons, this is not exactly what happens in reality, and it seems that the physical systems jumps from one branch to another following the minimal curve.

The picture of the branches of solutions we find (which exist from $h_{\text{ex}} = O(1)$ to $h_{\text{ex}} \leqq C\varepsilon^{-\alpha}$) is approximately that shown in Fig. 1, where we sketch $J + \frac{1}{2}\int_\Omega h_{\text{ex}}^2$ instead of $J$. All the curves are roughly parabolas. These results totally agree with the experimental knowledge on the subject, and especially with very recent experiments in which $dJ/dh_{\text{ex}}$ is measured and gives very similar curves (with regularly placed intersections).

The study for $h_{\text{ex}} \geqq C\varepsilon^{-\alpha}$ seems to be more difficult. The first reason is technical: As we perform in the proofs a regularization of the functions $u$ at the scale $\varepsilon^\gamma$ $(0 < \gamma < 1)$, the errors do not tend to zero when $h_{\text{ex}}$ is too high compared to $\varepsilon^\gamma$. The second reason is that the order of the repulsion between the vortices then becomes equivalent to their intrinsic cost ($\pi|\text{Log}\,\varepsilon|$). Consequently, the approach used here, which consists in studying vortices individually in the spirit of the renormalized energy of [BBH], is no longer convenient. However, it is possible, as E. SANDIER and I have done in a forthcoming work [SS2], to study global minimizers of the energy. We have switched to a different approach and a different construction of the vortices. We prove, by finding equivalent lower and upper bounds, that the minimal energy for $H_{c_1} \ll h_{\text{ex}} \ll H_{c_2}$ is

$$J + \frac{1}{2}\int_\Omega h_{\text{ex}}^2 \simeq \frac{1}{2}\text{vol}(\Omega) h_{\text{ex}} \text{Log}\frac{1}{\varepsilon\sqrt{h_{\text{ex}}}},$$

and the density of vortices converges to the uniform measure equal to $h_{\text{ex}}$, whence their number diverges, and their mutual distances are still of the order of $C/\sqrt{h_{\text{ex}}}$. For further details, refer to [SS2].

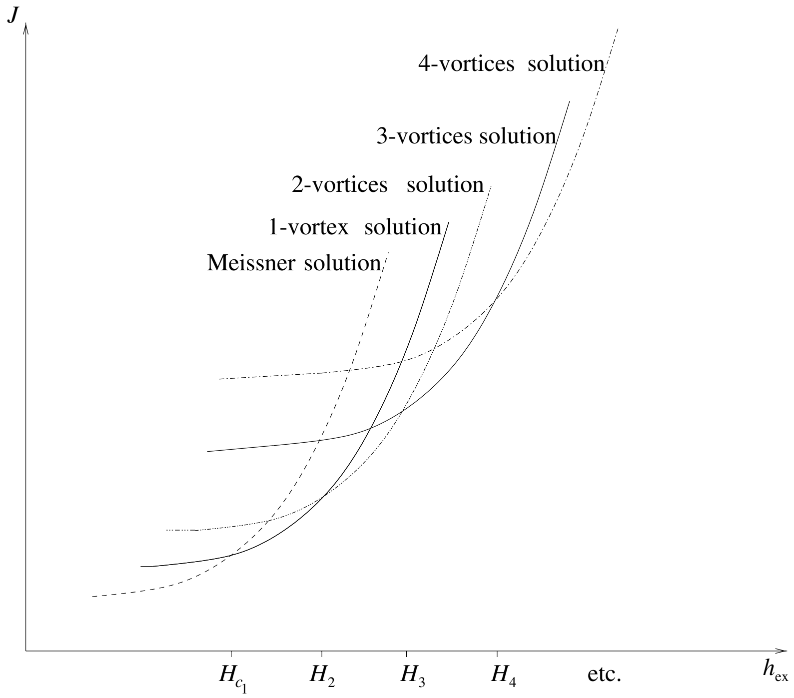


Fig. 1. The branches of solutions

Our last result is devoted to the problem of the appearance of a vortex when $h_{\text{ex}}$ is sufficiently high, considered on a more physical and heuristic viewpoint. It was meant to answer a question asked by YVES POMEAU. Physicists assume that, in order to follow a continuous path from the Meissner solution to the vortex solution, a vortex is created at the boundary of the superconductor and moves to the center. We give here a justification for this assumption and we prove a result concerning the energetic cost of such a path, or "nucleation energy barrier", by computing rigorously the function $\phi = \inf J$ over all the configurations having exactly one vortex (of degree 1) at a distance $r$ from the boundary.

**Theorem 3.** *Assume that* $\Omega = B(0, R)$. *Then*

1. $\max_{[\varepsilon^\beta, R]} \phi = J_0 + \pi|\text{Log}\,\varepsilon| - \pi\text{Log}\,h_{\text{ex}} + O(1)$.
2. $\phi$ *achieves its maximum at* $r_{\max}$, *and there is a* $C > 0$ *such that*

$$r_{\max} \leqq \frac{C}{h_{\text{ex}}}.$$

3. $$\phi(\varepsilon^\beta) \leqq J_0 + \pi(1-\beta)|\text{Log}\,\varepsilon| + \pi\text{Log}\,2 + \gamma_0 + o(1).$$

Thus, we see that the order of the energy barrier to be climbed must be $\max\phi - J_0 = \pi|\text{Log}\,\varepsilon| - \pi\text{Log}\,h_{\text{ex}} + O(1)$.

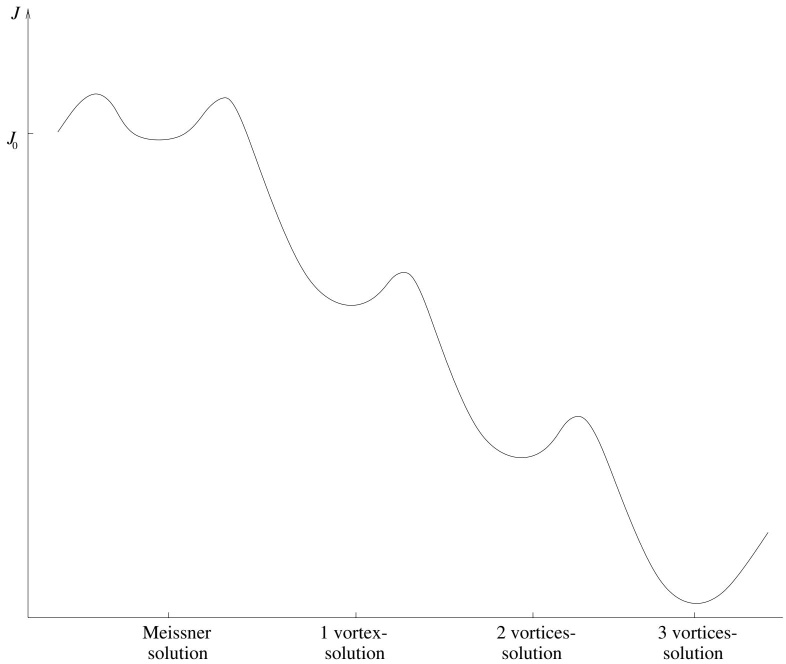


Fig. 2. Shape of the energy

If $h_{\mathrm{ex}}$ is raised, the barrier decreases, and at some point (estimated by $h_{\mathrm{ex}} \leqq H_{\mathrm{sh}} \simeq C/\varepsilon = C\kappa$), becomes so small that the system can pass over it by thermic agitation and “fall” into the vortex solution well (or by “tunnel effect”). Actually, this seems to happen very early in experiments.

In addition, we must have the same energy barrier between the single-vortex solution of our Theorem 2, and the two-vortices solution of Theorem 2 of [S2]; or between the two-vortices solution and the three-vortices solution etc. Indeed, each time, the process probably is the nucleation of one vortex migrating from the boundary to the center. Thus, I believe that we have an energy configuration of the form of Fig. 2 (where, as an example, $k_1(|\mathrm{Log}\ \varepsilon| + 2|\mathrm{Log}\ |\mathrm{Log}\ \varepsilon||) \leqq h_{\mathrm{ex}} \leqq k_1(|\mathrm{Log}\ \varepsilon| + 3|\mathrm{Log}\ |\mathrm{Log}\ \varepsilon||)$). From our existence results, we immediately infer the existence of mountain-pass solutions between the vortexless and one-vortex solutions, and between the $n$-vortex and $(n+1)$-vortex solutions in the case of a disc. We may conjecture that these solutions have a vortex at a distance $r_{\max}$ from the boundary of the domain. Also see Section 4.3 for remarks, interpretations, and open problems.

We now recall some notations and definitions from [S1]. $J(u, A)$ is invariant under $U(1)$-gauge transformations, i.e., transformations of the type

$$v = e^{i\phi}u \quad \text{for } \phi \in H^2(\Omega, \mathbb{R}), \qquad B = A + d\phi, \tag{1.5}$$

which makes the problem non-compact. Therefore, we impose the gauge condition

$$d^*A = \operatorname{div} A = 0 \quad \text{on } \Omega, \qquad A.n = 0 \quad \text{on } \partial\Omega. \tag{1.6}$$

Since we assume that $\Omega$ is simply connected, we can say that there exists $\xi \in H^2(\Omega, \mathbb{R})$ such that

$$A = d^*\xi = -\xi_{x_2} dx_1 + \xi_{x_1} dx_2, \qquad (\text{or } \nabla^\perp \xi)$$

with $d^*$ being the Hodge differential, and we obviously get

$$h = \Delta\xi.$$

Let us now turn to the method. In Section 2, we prove the uniqueness of the Meissner solution. To do so, we use an argument that is similar to a convexity argument: Assuming that there are two such solutions $(u_1, A_1)$ and $(u_2, A_2)$, we prove, through some explicit computations that

$$J\left(\frac{u_1 + u_2}{2}, \frac{A_1 + A_2}{2}\right) < \frac{J(u_1, A_1) + J(u_2, A_2)}{2}.$$

Then, we deduce that for all $t \in ]0, 1[$, $J((1-t)u_1 + tu_2, (1-t)A_1 + tA_2) \leqq \max(J(u_1, A_1), J(u_2, A_2))$. This contradicts the stability of the solutions.

After this work was completed, we learned of a paper of D. Ye & F. Zhou ([YZ]) in which a similar method was used to prove the uniqueness of the solution of the Ginzburg-Landau equation studied in [BBH2].

In Section 3, we prove Theorem 2 and Proposition 3.1, thus finding new solutions of (G.L.). The main idea is to find them as minimizers over well-chosen domains. For this purpose, we minimize $J$ over the domains

$$U_n = \Big\{(u, A) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2)/ \\ n\pi|\text{Log}\,\varepsilon| < F(u) < (n\pi + \tfrac{1}{2})|\text{Log}\,\varepsilon|\Big\}$$

$(n \in \mathbb{N})$. From the analysis of [S1] inspired from that of [BBH], $(u, A) \in U_n$ roughly means that $u$ has $n$ vortices (counted with multiplicity). A first difficulty is to prove that the minimum is achieved. We then derive (using our technical tools) qualitative properties on the vortices of $(\tilde{u}, \tilde{A})$ achieving $\min_{\bar{U}_n} J$, such as their positions and the fact that all are of degree 1. Then, as in [S1], we have to prove that $\min_{\bar{U}_n} J$ cannot be achieved on the boundary of $U_n$, and this is proved thanks to the qualitative properties previously deduced. Hence, the minimum is achieved in $\overset{\circ}{U}_n$, thus leading to a solution that is a local minimizer, and has $n$ vortices of degree 1. To sum up, roughly speaking, we can say that we find our $n$-vortex solutions by taking the minimizer of the energy over all $n$-vortex configurations, and we prove in the paper that it is exactly the case. For the vortexless solution, we just replace $n|\text{Log}\,\varepsilon|$ by $c_\varepsilon \to 0$ in the definition of the domain.

In Section 4, we prove Theorem 3 concerning the nucleation energy for $\Omega = B(0, R)$. We use a family of sets defined in [S1]:

$$E_a = \{(u, A) \in \bar{D} / \text{ the regularized } u \text{ has a unique vortex of degree 1 centered near } a\},$$

and we compute precisely

$$\phi(r) = \inf_{\cup\{E_a, \text{dist}(a,\partial\Omega)=r\}} J,$$

which is the minimal energy for configurations having a vortex at a distance $r$ from the boundary. Using the computations already made in [S1, S2] and using the splitting of $J$, we compare $\phi$ with two auxiliary functions, and obtain the desired estimates.

## 2. Uniqueness of the Meissner Solution

### *2.1. Preliminary Results*

We recall some of the notations and definitions of [S1]:

$$F(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1 - |u|^2)^2.$$

We define the following regularization of $u$, introduced in [AB]: Given any $0 < \gamma < 1$, for any $(u, A) \in D$ such that $|u| \leqq 1$, $u^\gamma$ is defined as a minimizer for

$$\inf_{H^1(\Omega,\mathbb{C})} \int_\Omega \frac{1}{2}|\nabla v|^2 + \frac{1}{4\varepsilon^2}(1 - |v|^2)^2 + \frac{|v - u|^2}{2\varepsilon^{2\gamma}}.$$

We denote by $(a_i, d_i)_{i \in \mathscr{J}'}$ the vortices of $u^\gamma$ of characteristic size $\rho$ as in Proposition 3.2 of [S1]. There exist some $0 < \gamma < \bar{\mu} < \mu < 1$ such that $\varepsilon^\mu \leqq \rho \leqq \varepsilon^{\bar{\mu}}$ (see [S1] for the definition of $\mu$ and $\bar{\mu}$). The $a_i$'s are the centers of the vortices, and the $d_i$'s are the degrees. More precisely, these vortices are defined to be such that $|u^\gamma| \geqq \frac{3}{4}$ on $\Omega \backslash \cup_{i \in \mathscr{J}'} B(a_i, \rho)$, and $\deg(u^\gamma/|u^\gamma|, \partial B(a_i, \rho)) = d_i$. Moreover, $d_i \neq 0$, for all $i \in \mathscr{J}'$ (see Section 3 in [S1]), so that $u^\gamma$ has to vanish somewhere in each $B(a_i, \rho)$. In [S1, S2], we chose $\frac{1}{2}$ instead of $\frac{3}{4}$ in the definition, but it is very easy to see that both are equivalent as $\varepsilon$ tends to 0. Let us also make precise that the vortices of $u^\gamma$ are well-defined, which is not the case for an arbitrary $u$. Through this regularization, we are thus able to define a vortex-structure (with a bounded number of vortices) for any $u \in D$. This regularization, as seen in [S1, S2], introduces many technical difficulties, but allows us to deal with local minimizers.

We also recall the definitions

$$V(\xi) = \frac{1}{2}\int_\Omega |\nabla\xi|^2 + |\Delta\xi|^2 + 2\pi \sum_{i \in \mathscr{J}'} d_i \xi(a_i) - h_{\text{ex}} \int_\Omega \Delta\xi,$$

$$\tilde{V}(\zeta) = \frac{1}{2}\int_{\Omega} |\nabla\zeta|^2 + |\Delta\zeta|^2 + 2\pi \sum_{i\in\mathcal{J}'} d_i \zeta(a_i).$$

In the sequel, $C$ will always denote a positive constant. Moreover, most of the quantities will depend on $\varepsilon$, but we shall drop the subscript as soon as no confusion is possible.

We now state some preliminary results, which are improvements of some results of [S1, S2]. As in [S1], $D$ is any domain of the form defined in (1.4).

**Lemma 2.1.** *There exists $\alpha \in ]0, \frac{1}{4}[$, such that if $h_{\text{ex}} \leqq C\varepsilon^{-\alpha}$, then*

1. *For all $(\tilde{u}, \tilde{A}) \in \bar{D}$ with $J(\tilde{u}, \tilde{A}) \leqq Ch_{\text{ex}}^2$, there exists $(u, A) \in \bar{D}$ such that*

$$u = T(\tilde{u}) = \left\{ \begin{array}{l} \tilde{u} \text{ if } |\tilde{u}| \leqq 1 \\ \dfrac{\tilde{u}}{|\tilde{u}|} \text{ if } |\tilde{u}| \geqq 1 \end{array} \right\}, \qquad |u| \leqq 1, \qquad F(u) \leqq F(\tilde{u}),$$

   *$(u, A)$ satisfies the second equation of* (G.L.)*:*

$$- * dh = (iu, d_A u),$$

   *and*

$$J(u, A) \leqq J(\tilde{u}, \tilde{A}) + o(\varepsilon^{\alpha}).$$

2. *The energy can be split as*

$$\begin{aligned} J(u, A) &= F(u) + V(\xi) + o(\varepsilon^{\alpha}) \\ &= J_0 + F(u) + 2\pi h_{\text{ex}} \sum_{i\in\mathcal{J}'} d_i \xi_0(a_i) + \tilde{V}(\zeta) + o(\varepsilon^{\alpha}), \end{aligned}$$

   *where the $(a_i, d_i)$'s denote the vortices of $u^{\gamma}$.*
3. *If $(\tilde{u}, \tilde{A})$ minimizes $J$ over $\bar{D}$, then $F(u) = F(\tilde{u}) + o(\varepsilon^{\alpha})$ and $J(u, A) = J(\tilde{u}, \tilde{A}) + o(\varepsilon^{\alpha}) = J(u, \tilde{A}) + o(\varepsilon^{\alpha})$.*

**Proof.** Just follow the proof of Lemma 2.2 and Section 4 of [S1], replacing the assumption $h_{\text{ex}} \leqq C|\text{Log}\,\varepsilon|$ by $h_{\text{ex}} \leqq C\varepsilon^{-\alpha}$. □

**Lemma 2.2.** *If $(u, A)$ is a solution of* (G.L.)*, and if $J(u, A) \leqq Ch_{\text{ex}}^2$ with $h_{\text{ex}} \leqq C\varepsilon^{-\alpha}$, then*

$$\|u\|_{L^\infty(\Omega)} \leqq 1, \qquad \|\nabla u\|_{L^\infty(\Omega)} \leqq \frac{C}{\varepsilon}.$$

*In addition, if $(u, A)$ is energy-minimizing in D, then for all $\beta > 0$, there exists $\varepsilon_0$ such that $|u| \geqq \frac{3}{4}$ on $\{x \in \Omega/\text{dist}(x, \partial\Omega) \leqq \varepsilon^{\beta}\}$ for all $\varepsilon < \varepsilon_0$.*

**Proof.** Follow the proof of Proposition 6.2 of [S1], replacing $\frac{1}{2}$ by $\frac{3}{4}$. □

If $(u, A)$ is a solution of (G.L.), then $|\nabla u| \leqq C/\varepsilon$ is satisfied. Hence, following [BBH], we are able to define vortices of $u$ of size $\lambda\varepsilon$. ($\lambda$ is some constant; refer to [S1, Section 2.2.]). Then, as in Proposition 3.2 of [S1], we can define its vortices of size $\rho$, exactly as for $u^{\gamma}$. The following proposition relates the vortices (of size $\rho$) of $u$ and $u^{\gamma}$.

**Proposition 2.1.** *If* $(u, A)$ *is a solution of* (G.L.) *such that* $u^\gamma$ *has no vortex* (*i.e.,* $|u^\gamma| \geqq \frac{3}{4}$) *and that* $J(u, A) \leqq J_0$*, then* $u$ *has no vortex on* $\Omega$*. If* $(u, A)$ *is a solution of* (G.L.) *given by Theorem 1 or 2 of* [S1] *or* [S2]*, then its vortices of size* $\rho$ *satisfy the same conclusions as those of* $u^\gamma$*. In addition, if the* $(a_i)_{i\in\mathscr{J}'}$ *are the vortices of* $u^\gamma$ *of degree* 1*, then the vortices* $b_i$ *of* $u$ *are of degree* 1 *and*

$$d(a, b) := \inf_{\sigma\in S_{\mathrm{card}\ \mathscr{J}'}} \sum_{i=1}^{\mathrm{card}\ \mathscr{J}'} |a_i - b_{\sigma(i)}| \leqq C\varepsilon^\gamma |\mathrm{Log}\ \varepsilon|.$$

**Proof.** Again, refer to the proof of Proposition 6.2 of [S1]. □

### *2.2. Proof of the Uniqueness*

We assume here that $h_{\mathrm{ex}} \leqq C\varepsilon^{-\alpha}$. We prove that if a Meissner (i.e., a vortexless) solution $(u, A)$ exists and is stable, then it is unique among the solutions satisfying $\|\nabla u\|^2_{L^2(\Omega)} \leqq o(\varepsilon^\alpha)$. In particular, a solution $(u, A)$ that is energy-minimizing among vortexless solutions is unique. The existence of such a solution for $h_{\mathrm{ex}} \leqq H_{c_1}$ is justified by Theorem 1 of [S1] combined with Proposition 2.1. Its global minimality is proved in [SS1]. Furthermore, we shall prove the existence of such a solution for $h_{\mathrm{ex}} \geqq H_{c_1}$ in the next section.

To prove the uniqueness, we assume, for contradiction, that there are two distinct stable solutions $(u_1, A_1)$ and $(u_2, A_2)$ of (G.L.) with the choice of gauge $\operatorname{div} A_j = 0$ $(j = 1, 2)$, such that $\int_\Omega |\nabla u_j|^2 \leqq o(\varepsilon^\alpha)$. We assume that

$$J(u_1, A_1) \leqq J(u_2, A_2).$$

We denote $\eta_j = |u_j|$ as in [S1].

**Lemma 2.3.** *For all* $j \in [1, 2]$, $(u_j, A_j)$ *is gauge-equivalent to* $(\eta_j, A'_j)$*, with*

$$\operatorname{div}(\eta_j^2 A'_j) = 0, \tag{2.1}$$

(2.2)

$$J(u_j, A_j) = \frac{1}{2}\int_\Omega \eta_j^2|A'_j|^2 + |\nabla\eta_j|^2 + \frac{1}{2\varepsilon^2}(1-\eta_j^2)^2 + |dA'_j - h_{\mathrm{ex}}|^2 - h_{\mathrm{ex}}^2.$$

**Proof.** Since $\eta_j \geqq \frac{3}{4}$, we can write

$$u_j = \eta_j e^{i\phi_j}$$

globally on $\Omega$. Then, $(u_j, A_j)$ is gauge-equivalent to

$$(u_j e^{-i\phi_j}, A_j - d\phi_j) = (\eta_j, A_j - d\phi_j).$$

We write $A'_j = A_j - d\phi_j$. Thus, since $\int_\Omega |\nabla_A u|^2$ is invariant under gauge-transformations,

$$\int_\Omega |\nabla_A u|^2 = \int_\Omega |\nabla_{A'_j}\eta_j|^2 = \int_\Omega |\nabla\eta_j - iA'_j\eta_j|^2 = \int_\Omega |\nabla\eta_j|^2 + \eta_j^2|A'_j|^2.$$

The expression (2.2) follows.

For (2.1), notice that the second (G.L.) equation gives

$$-*dh = (iu_j, d_{A_j}u_j) = (i\eta_j, d_{A_j'}\eta_j) = -\eta_j^2 A_j',$$

which means that $\eta_j^2 A_j' = \nabla^\perp h$, hence $\operatorname{div}(\eta_j^2 A_j') = 0$. □

**Lemma 2.4.** *Under the same hypotheses, if $h_{\rm ex} \leqq C\varepsilon^{-\alpha}$, then*

$$\|A_j'\|_{L^\infty(\Omega)} \leqq o\left(\frac{1}{\varepsilon}\right) \quad \text{as } \varepsilon \to 0.$$

**Proof.** As in Lemma 2.2 of [S1], since $(u_j, A_j)$ is a solution of (G.L.), we have $\|A_j\|_{L^\infty(\Omega)} \leqq Ch_{\rm ex} \leqq C\varepsilon^{-\alpha}$. Then, if $(u_j, A_j)$ is energy-minimizing among vortexless solutions,

$$J(u_j, A_j) \leqq J(1, h_{\rm ex} d^*\xi_0) = J_0. \tag{2.3}$$

Writing as usual $\xi = h_{\rm ex}\xi_0 + \zeta$, and dropping the subscript $j$, we obtain

$$\begin{aligned}
& J_0 \geqq J(u_j, A_j),\\
& J_0 \geqq \frac{1}{2}\int_\Omega |\nabla u|^2 + |\nabla \xi|^2\\
&\qquad + \frac{1}{2\varepsilon^2}(1-|u|^2)^2 + \int_\Omega |\Delta\xi|^2 - h_{\rm ex}\int_\Omega \Delta\xi + o(\varepsilon^\alpha),\\
\Leftrightarrow\quad & J_0 + o(\varepsilon^\alpha) \geqq J_0 + F(u) + \frac{1}{2}\int_\Omega |\Delta\zeta|^2 + |\nabla\zeta|^2,\\
\Leftrightarrow\quad & o(\varepsilon^\alpha) \geqq F(u) + \frac{1}{2}\int_\Omega |\Delta\zeta|^2 + |\nabla\zeta|^2.
\end{aligned}$$

Therefore,

$$\int_\Omega |\nabla u|^2 = \int_\Omega |\nabla\eta|^2 + \eta^2|\nabla\phi|^2 \leqq o(\varepsilon^\alpha). \tag{2.4}$$

We now assume that this condition is satisfied. We then return to the estimate on $A'$. We have

$$\|A'\|_{L^\infty} \leqq \|A\|_{L^\infty} + \|\nabla\phi\|_{L^\infty} \leqq \frac{C}{\varepsilon}.$$

By interpolation, for any $p > 1$,

$$\begin{aligned}
\|\nabla\eta\|_{L^p} &\leqq C\|\nabla\eta\|_{L^\infty}^{1-\frac{2}{p}}\|\nabla\eta\|_{L^2}^{\frac{2}{p}}\\
&\leqq C\varepsilon^{-1+\frac{2}{p}}\varepsilon^{\frac{\alpha}{p}} \qquad \text{(by Lemma 2.2 and (2.4))}\\
&\leqq C\varepsilon^\delta
\end{aligned}$$

for some $\delta > 0$, provided that $p < \alpha + 2$.

On the other hand, thanks to (2.1), we have

$$\eta^2 \text{div } A' = -2\eta \nabla \eta . A',$$

and since $A' = d^*\xi - d\phi$, this transforms into

$$-\Delta \phi = -\frac{2}{\eta} \nabla \eta . A'.$$

We deduce that

$$\|\Delta \phi\|_{L^p} \leqq C \|A'\|_{L^\infty} \|\nabla \eta\|_{L^p}.$$

Thus, taking $2 < p < \alpha + 2$, we have

$$\|\Delta \phi\|_{L^p} \leqq C \|A'\|_{L^\infty} \|\nabla \eta\|_{L^p} \leqq C \frac{\varepsilon^\delta}{\varepsilon} \leqq o\left(\frac{1}{\varepsilon}\right).$$

In addition, we have a Neumann boundary condition for $u$: $\frac{\partial u}{\partial n} = 0$, implying $\frac{\partial \phi}{\partial n} = 0$ on $\partial \Omega$. Thus, by an elliptic estimate and by Sobolev embedding, since $p > 2$, we obtain

$$\|\nabla \phi\|_{L^\infty} \leqq o\left(\frac{1}{\varepsilon}\right),$$

implying that

$$\|A'\|_{L^\infty} \leqq o\left(\frac{1}{\varepsilon}\right). \qquad \square$$

From now on, we assume that the change of gauge has been performed, and write $A_j$ instead of $A'_j$. We are going to prove that

$$J\left(\frac{\eta_1 + \eta_2}{2}, \frac{A_1 + A_2}{2}\right) < \frac{J(\eta_1, A_1) + J(\eta_2, A_2)}{2} \leqq J(\eta_2, A_2),$$

using some kinds of convexity arguments, and thus getting a contradiction. The main idea we follow is that we consider separately the expanded terms of $\frac{1}{2}(J(\eta_1, A_1) + J(\eta_2, A_2)) - J\left(\frac{\eta_1+\eta_2}{2}, \frac{A_1+A_2}{2}\right)$ and try to exhibit positive expressions like $(\eta_1 - \eta_2)^2$ and $(A_1 - A_2)^2$. Indeed, this is inspired by the simplest case of convexity:

$$\frac{a^2 + b^2}{2} - \left(\frac{a + b}{2}\right)^2 = \frac{(a - b)^2}{4}.$$

The main difficulty is that, for the term $\int_\Omega \eta^2 |A|^2$, expressions combine $\eta$ and $A$. We display expressions using the differences $(\eta_1 - \eta_2)$ and $(A_1 - A_2)$. Still, this is not enough, and we have to use combinations of the term $\int_\Omega \eta^2 |A|^2$ with $\frac{1}{\varepsilon^2} \int_\Omega (1 - \eta^2)^2$, which is dominant thanks to $\frac{1}{\varepsilon^2}$ and to our previous estimate $\|A\|_{L^\infty} \leqq o(\varepsilon^{-1})$.

**Lemma 2.5.** *If* $(\eta_1, A_1) \neq (\eta_2, A_2)$, *then*

$$\int_\Omega \left(\frac{\eta_1+\eta_2}{2}\right)^2 \left|\frac{A_1+A_2}{2}\right|^2 + \frac{1}{2\varepsilon^2}\left(1-\left(\frac{\eta_1+\eta_2}{2}\right)^2\right)^2$$
$$< \frac{1}{2}\int_\Omega \eta_1^2|A_1|^2 + \frac{1}{2\varepsilon^2}(1-\eta_1^2)^2 + \eta_2^2|A_2|^2 + \frac{1}{2\varepsilon^2}(1-\eta_2^2)^2.$$

**Proof.** We compute $X = X_1 + X_2$, where

$$X_1 = \frac{1}{2}\int_\Omega \eta_1^2|A_1|^2 + \eta_2^2|A_2|^2 - \int_\Omega \left(\frac{\eta_1+\eta_2}{2}\right)^2 \left|\frac{A_1+A_2}{2}\right|^2, \tag{2.5}$$

$$X_2 = \frac{1}{2}\left(\frac{1}{4\varepsilon^2}\int_\Omega (1-\eta_1^2)^2 + (1-\eta_2^2)^2\right) - \frac{1}{4\varepsilon^2}\int_\Omega \left(1-\left(\frac{\eta_1+\eta_2}{2}\right)^2\right)^2. \tag{2.6}$$

First, we expand all the terms in $X_2$:

$$\begin{aligned}
X_2 &= \frac{1}{64\varepsilon^2}\int_\Omega 7\eta_1^4 + 7\eta_2^4 - 8\eta_1^2 - 8\eta_2^2 - 6\eta_1^2\eta_2^2 + 16\eta_1\eta_2 - 4\eta_1^3\eta_2 - 4\eta_2^3\eta_1 \\
&= \frac{1}{64\varepsilon^2}\int_\Omega 7(\eta_1^2-\eta_2^2)^2 + 8\eta_1^2\eta_2^2 - 8(\eta_1-\eta_2)^2 - 4\eta_1\eta_2(\eta_1^2+\eta_2^2) \\
&= \frac{1}{64\varepsilon^2}\int_\Omega 7(\eta_1^2-\eta_2^2)^2 - 4\eta_1\eta_2(\eta_1^2+\eta_2^2-2\eta_1\eta_2) - 8(\eta_1-\eta_2)^2 \\
&= \frac{1}{64\varepsilon^2}\int_\Omega 7(\eta_1^2-\eta_2^2)^2 - 4\eta_1\eta_2(\eta_1-\eta_2)^2 - 8(\eta_1-\eta_2)^2 \\
&= \frac{1}{64\varepsilon^2}\int_\Omega (\eta_1-\eta_2)^2(7(\eta_1+\eta_2)^2 - 4\eta_1\eta_2 - 8).
\end{aligned}$$

Now, since $u_1$ and $u_2$ are vortexless solutions, we know that $\frac{3}{4} \leqq \eta_1 \leqq 1$ and $\frac{3}{4} \leqq \eta_2 \leqq 1$, which guarantees that they are in the domain of convexity of the function $(1-x^2)^2$. Thus $7(\eta_1+\eta_2)^2 - 4\eta_1\eta_2 - 8 \geqq 7\frac{9}{4} - 12 \geqq 3$. We conclude that

$$X_2 \geqq \frac{3}{64\varepsilon^2}\int_\Omega (\eta_1-\eta_2)^2. \tag{2.7}$$

Next, we turn to the computation of $X_1$:

$$\begin{aligned}
X_1 &= \frac{1}{16}\int_\Omega \big(7\eta_1^2|A_1|^2 + 7\eta_2^2|A_2|^2 - \eta_1^2|A_2|^2 - \eta_2^2|A_1|^2 \\
&\quad - 2\eta_1\eta_2(|A_1|^2 + |A_2|^2 + 2A_1.A_2) - 2(\eta_1^2+\eta_2^2)A_1.A_2\big) \\
&= \frac{1}{16}\int_\Omega \big( - 2A_1.A_2(\eta_1+\eta_2)^2 + |A_1|^2(7\eta_1^2 - 2\eta_1\eta_2 - \eta_2^2) \\
&\quad + |A_2|^2(7\eta_2^2 - 2\eta_1\eta_2 - \eta_1^2)\big) \\
&= \frac{1}{16}\int_\Omega \big( - 2A_1.A_2(\eta_1+\eta_2)^2 + 2|A_1|^2(\eta_1-\eta_2)^2 + 4\eta_1^2|A_1|^2 \\
&\quad + 4\eta_2^2|A_2|^2 + (|A_2|^2 - |A_1|^2)(3\eta_2^2 - \eta_1^2 - 2\eta_1\eta_2)\big) \\
&= \frac{1}{16}\int_\Omega \big( - 2A_1.A_2(\eta_1^2+\eta_2^2 + 2\eta_1\eta_2 - 4\eta_1\eta_2) + 2|A_1|^2(\eta_1-\eta_2)^2 \\
&\quad + 4|\eta_1A_1 - \eta_2A_2|^2 + (A_2 - A_1).(A_2+A_1)((\eta_2-\eta_1)^2 \\
&\quad + 2(\eta_2-\eta_1)(\eta_2+\eta_1))\big) \\
&= \frac{1}{16}\int_\Omega \big( - 2A_1.A_2(\eta_1-\eta_2)^2 + 2|A_1|^2(\eta_1-\eta_2)^2 + 4|\eta_1A_1 - \eta_2A_2|^2 \\
&\quad + (A_2 - A_1).(A_2+A_1)((\eta_2-\eta_1)^2 + 2(\eta_2-\eta_1)(\eta_2+\eta_1))\big).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
|\eta_1A_1 - \eta_2A_2|^2 &= \eta_1^2|A_1 - A_2|^2 + |A_2|^2(\eta_2^2 - \eta_1^2) + 2(\eta_1^2 - \eta_1\eta_2)A_1.A_2 \\
&= \eta_1^2|A_1 - A_2|^2 + (\eta_2-\eta_1)(|A_2|^2(\eta_2+\eta_1) - 2\eta_1A_1.A_2) \\
&= \eta_1^2|A_1 - A_2|^2 + (\eta_2-\eta_1)(A_2.(A_2-A_1)(\eta_1+\eta_2) \\
&\qquad + A_1.A_2(-2\eta_1 + \eta_1 + \eta_2)) \\
&= \eta_1^2|A_1 - A_2|^2 + (\eta_2-\eta_1)(A_2.(A_2-A_1)(\eta_1+\eta_2)) \\
&\qquad + (\eta_2-\eta_1)^2A_1.A_2.
\end{aligned}$$

Hence

$$\begin{aligned}
X_1 &= \frac{1}{16}\int_\Omega \big(2A_1.A_2(\eta_1-\eta_2)^2 + 2|A_1|^2(\eta_1-\eta_2)^2 + 4\eta_1^2|A_1 - A_2|^2 \\
&\quad + 4A_2.(A_2-A_1)(\eta_2-\eta_1)(\eta_1+\eta_2) \\
&\quad + (A_1+A_2).(A_2-A_1)((\eta_2-\eta_1)^2 + 2(\eta_2-\eta_1)(\eta_2+\eta_1))\big) \\
&= \frac{1}{16}\int_\Omega \big((\eta_1-\eta_2)^2(2|A_1|^2 + 2A_1.A_2) + 4\eta_1^2|A_1-A_2|^2 \\
&\quad + (\eta_2-\eta_1)(A_2-A_1).(-4A_2(\eta_1+\eta_2) + (A_2+A_1)(-\eta_1 - 3\eta_2))\big) \qquad (2.8) \\
&= \frac{1}{16}\int_\Omega \big((\eta_1-\eta_2)^2|A_1+A_2|^2 + 4\eta_1^2|A_1-A_2|^2 \\
&\quad + (\eta_2-\eta_1)(A_2-A_1).(A_1(-2\eta_1 - 4\eta_2) + A_2(-6\eta_1 - 8\eta_2))\big).
\end{aligned}$$

Now, let us assume for contradiction that $X \leqq 0$; by combining (2.7) and (2.8) this would lead to

$$\frac{1}{16}\int_\Omega (\eta_1-\eta_2)^2\left(|A_1+A_2|^2+\frac{3}{4\varepsilon^2}\right)+4\eta_1^2|A_1-A_2|^2$$
$$\leqq \int_\Omega |\eta_1-\eta_2||A_2-A_1|(6|A_1|+14|A_2|)$$
$$\leqq C(\|A_1\|_{L^\infty}+\|A_2\|_{L^\infty})\|\eta_1-\eta_2\|_{L^2}\|A_1-A_2\,|_{L^2}.$$

On the other hand,

$$\frac{1}{16}\int_\Omega (\eta_1-\eta_2)^2\left(|A_1+A_2|^2+\frac{3}{4\varepsilon^2}\right)+4\eta_1^2|A_1-A_2|^2$$
$$\geqq \frac{2}{16}\left(\frac{3}{4\varepsilon^2}\right)^{1/2}\|\eta_1-\eta_2\|_{L^2}\|A_1-A_2\|_{L^2}.$$

We would thus obtain

$$\|\eta_1-\eta_2\|_{L^2}\|A_1-A_2\|_{L^2} \leqq C\varepsilon(\|A_1\|_{L^\infty}+\|A_2\|_{L^\infty})\|\eta_1-\eta_2\|_{L^2}\|A_1-A_2\|_{L^2}$$
$$\leqq o(1)\|\eta_1-\eta_2\|_{L^2}\|A_1-A_2\|_{L^2}$$

from Lemma 2.4, which implies $\eta_1=\eta_2$ or $A_1=A_2$.

If $\eta_1=\eta_2$, a simple convexity argument proves that

$$\int_\Omega \eta_1^2\left|\frac{A_1+A_2}{2}\right|^2 < \frac{1}{2}\int_\Omega \eta_1^2|A_1|^2+\eta_2^2|A_2|^2,$$

and thus $X>0$. ($A_1$ cannot then be equal to $A_2$.)

If $A_1=A_2$, again by convexity

$$\int_\Omega\left(\frac{\eta_1+\eta_2}{2}\right)^2|A_1|^2+\frac{1}{4\varepsilon^2}\int_\Omega\left(1-\left(\frac{\eta_1+\eta_2}{2}\right)^2\right)^2$$
$$<\frac{1}{2}\int_\Omega \eta_1^2|A_1|^2+\frac{1}{4\varepsilon^2}(1-\eta_1^2)^2+\eta_2^2|A_2|^2+\frac{1}{4\varepsilon^2}(1-\eta_2^2)^2,$$

and $X>0$. ($\eta_1$ cannot then be equal to $\eta_2$.)

We are led to a contradiction in all cases; therefore $X>0$, which proves the lemma. □

**Proposition 2.2.** *There exists $\varepsilon_0$ such that, if $\varepsilon<\varepsilon_0$, a stable vortexless solution of* (G.L.) *for $h_{\mathrm{ex}} \leqq C\varepsilon^{-\alpha}$ with $\int_\Omega |\nabla u|^2 \leqq o(\varepsilon^\alpha)$ is unique.*

*Let $E_0=\{(u,A)\in D/|u|\geqq \frac{3}{4}\}$. For $\varepsilon<\varepsilon_0$, if there exists a solution of* (G.L.) *in $E_0$ for $h_{\mathrm{ex}}\leqq C\varepsilon^{-\alpha}$ that minimizes $J$ over $E_0$, then it is unique.*

**Proof.** We continue the proof of our lemma. By convexity,

$$\int_\Omega \left|\nabla \frac{\eta_1+\eta_2}{2}\right|^2 \leqq \frac{1}{2}\int_\Omega |\nabla \eta_1|^2 + \frac{1}{2}\int_\Omega |\nabla \eta_2|^2,$$

$$\int_\Omega \left|d\left(\frac{A_1+A_2}{2}\right) - h_{\mathrm{ex}}\right|^2 \leqq \frac{1}{2}\int_\Omega |dA_1 - h_{\mathrm{ex}}|^2 + \frac{1}{2}\int_\Omega |dA_2 - h_{\mathrm{ex}}|^2,$$

with strict inequalities respectively if $\eta_1 \neq \eta_2$ and $A_1 \neq A_2$. Combining these with the result of Lemma 2.5 and the expression (2.2) for the energy, we obtain

$$J\left(\frac{\eta_1+\eta_2}{2}, \frac{A_1+A_2}{2}\right) < \frac{1}{2}(J(\eta_1, A_1) + J(\eta_2, A_2)) \leqq J(\eta_2, A_2).$$

Now, by a standard argument, this is true on the whole segment $[(\eta_1, A_1), (\eta_2, A_2)]$, i.e., for all $t \in ]0, 1[$,

$$J((1-t)\eta_1 + t\eta_2, (1-t)A_1 + tA_2) < J(\eta_2, A_2),$$

thus contradicting the stability of $(\eta_2, A_2)$. Hence $(\eta_1, A_1) = (\eta_2, A_2)$. □

By Theorem 1 of [S1] (combined with Proposition 2.1), we know that, for $\varepsilon < \varepsilon_0$ and for $h_{\mathrm{ex}} \leqq H_{c_1}$, a minimizing solution in $D$ exists and satisfies $|u| \geqq \frac{3}{4}$ on $\Omega$. This justifies the existence of a solution satisfying the hypotheses of this proposition. In addition, with this proposition, we have proved that this solution was unique. We are going to show a stronger existence result in the next section.

## 3. Some New Stable Solutions

### *3.1. Existence of a Vortexless Solution*

In this section, we show that for $h_{\mathrm{ex}}$ higher than the critical field (up to $C\varepsilon^{-\alpha}$), the vortexless solution of (G.L.) continues to exist and to be locally minimizing. From now on, we use the notations and conventions of [S1] and [S2], i.e., we assume that $A = d^*\xi$, $\xi = h_{\mathrm{ex}}\xi_0 + \zeta$, and consider $u^\gamma$ as defined previously.

**Proposition 3.1.** *There exist $\alpha > 0$ and $\varepsilon_0$ such that, if $\varepsilon < \varepsilon_0$ and $h_{\mathrm{ex}} \leqq C\varepsilon^{-\alpha}$, there exists a unique solution $(u, A) = (u, d^*\xi)$ of* (G.L.) *with $|u| \geqq \frac{3}{4}$, that is a local minimizer for $J$. In addition,*

$$\inf_{\theta\in[0,2\pi]} \|(u,\xi) - (e^{i\theta}, h_{\mathrm{ex}}\xi_0)\| \longrightarrow 0 \quad \text{as } \varepsilon \to 0,$$

*where $\|.\|$ is defined as*

$$\|(u,z)\|^2 = \|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2 + \|\nabla z\|_{L^2}^2 + \|\Delta z\|_{L^2}^2.$$

**Proof.** *Step 1.* We define the open domain

$$U = \left\{ (u, A) \in D /\ F(u) + \frac{1}{2} \int_\Omega |\nabla \zeta|^2 + |\Delta \zeta|^2 < \varepsilon^{\frac{\alpha}{2}} \right\},$$

where $\alpha$ is given by Lemma 2.1. Assume that $(\tilde{u}, \tilde{A})$ achieves $\min_{\bar{U}} J$. Such a configuration exists by the same argument as in [S1], i.e., by following the proof of Theorem 1 of [BR] and using the lower semi-continuity of $F$. By applying Lemma 2.1, we derive a configuration $(u, A)$ that satisfies

$$F(u^\gamma) \leqq F(u) \leqq F(\tilde{u}) < \varepsilon^{\alpha/2} = o(1).$$

Denoting by $(a_i, d_i)_{i \in \mathscr{J}'}$ the vortices of $u^\gamma$ (given by Proposition 3.2 of [S1]), we may assert, using Lemma 5.1 of [S1], that

$$\begin{aligned} F(u^\gamma) &\geqq \pi \sum_{i \in \mathscr{J}'} |d_i| \left| \operatorname{Log} \frac{\varepsilon}{\rho} \right| + O(1) \\ &\geqq \pi(1-\mu) \sum_{i \in \mathscr{J}'} |d_i| |\operatorname{Log} \varepsilon| + O(1), \end{aligned}$$

because $\rho \geqq \varepsilon^\mu$. This implies that $\mathscr{J}' = \varnothing$, i.e., $u^\gamma$ has no vortex (recall that $d_i \neq 0$). Therefore, by Lemma 2.1,

$$\begin{aligned} J(u, A) &= F(u) + V(\xi) + o(\varepsilon^\alpha) \\ &= J_0 + F(u) + \frac{1}{2} \int_\Omega |\Delta \zeta|^2 + |\nabla \zeta|^2 + o(\varepsilon^\alpha). \end{aligned} \tag{3.1}$$

On the other hand, $(1, h_{\mathrm{ex}} d^* \xi_0)$ is a comparison map that belongs to $U$; hence by the minimality of $(\tilde{u}, \tilde{A})$, we obtain

$$J(\tilde{u}, \tilde{A}) \leqq J_0.$$

Then, by Lemma 2.1,

$$J(u, A) \leqq J(\tilde{u}, \tilde{A}) + o(\varepsilon^\alpha) \leqq J_0 + o(\varepsilon^\alpha).$$

Therefore, in view of (3.1),

$$F(u) + \frac{1}{2} \int_\Omega |\Delta \zeta|^2 + |\nabla \zeta|^2 \leqq o(\varepsilon^\alpha).$$

This guarantees that $(u, A) \in \bar{U}$ for $\varepsilon$ sufficiently small. Hence, by minimality again,

$$J(\tilde{u}, \tilde{A}) \leqq J(u, A) \leqq J(u, \tilde{A}) \leqq J(\tilde{u}, \tilde{A}) + o(\varepsilon^\alpha).$$

Arguing by contradiction, let us assume that $(\tilde{u}, \tilde{A}) \in \partial U$. Then,

$$F(\tilde{u}) + \frac{1}{2} \int_\Omega |\nabla \tilde{\zeta}|^2 + |\Delta \tilde{\zeta}|^2 = \varepsilon^{\alpha/2}.$$

Using assertion 3 of Lemma 2.1, in which we can replace the domain $D$ by $U$, we find that $F(u) = F(\tilde{u}) + o(\varepsilon^\alpha)$ and $J(\tilde{u}, \tilde{A}) = J(u, \tilde{A}) + o(\varepsilon^\alpha)$. In addition, since $\tilde{\xi} = h_{\mathrm{ex}}\xi_0 + \tilde{\zeta}$ and since we have $\|\tilde{\zeta}\|_{H^2} \leqq \varepsilon^{\alpha/4}$, the splitting of the energy is also true for $(u, \tilde{A})$:

$$J(u, \tilde{A}) = J_0 + F(u) + \frac{1}{2}\int_\Omega |\nabla\tilde{\zeta}|^2 + |\Delta\tilde{\zeta}|^2 + o(\varepsilon^\alpha).$$

Hence,

$$\begin{aligned} J(\tilde{u}, \tilde{A}) &= J(u, \tilde{A}) + o(\varepsilon^\alpha) \\ &= J_0 + F(u) + \frac{1}{2}\int_\Omega |\Delta\tilde{\zeta}|^2 + |\nabla\tilde{\zeta}|^2 + o(\varepsilon^\alpha) \\ &= J_0 + F(\tilde{u}) + \frac{1}{2}\int_\Omega |\Delta\tilde{\zeta}|^2 + |\nabla\tilde{\zeta}|^2 + o(\varepsilon^\alpha) \\ &= J_0 + \varepsilon^{\alpha/2} + o(\varepsilon^\alpha) \\ &> J_0 \end{aligned}$$

if $\varepsilon$ is sufficiently small. But this contradicts the minimality of $(\tilde{u}, \tilde{A})$. We conclude that there is some $(\tilde{u}, \tilde{A}) \in \overset{\circ}{U}$ such that $J(\tilde{u}, \tilde{A}) = \min_{\bar{U}} J$, and that $(\tilde{u}, \tilde{A})$ is a local minimizer for $J$. In addition, $(\tilde{u}, \tilde{A})$ is a stable solution of (G.L.), and is vortexless by Proposition 2.1. Thus $(\tilde{u}, \tilde{A}) = (u, A)$ in view of the construction of $(u, A)$. By Proposition 2.2, such a local minimizer is unique.

*Step 2.* We prove the second assertion of the proposition. Defining the norm

$$\|(u, z)\|^2 = \|\nabla u\|^2_{L^2} + \|u\|^2_{L^2} + \|\nabla z\|^2_{L^2} + \|\Delta z\|^2_{L^2},$$

($\|\nabla z\|^2_{L^2} + \|\Delta z\|^2_{L^2}$ is equivalent to the $H^2$ norm on $\{\xi / \xi = 0 \text{ on } \partial\Omega\}$), we suppose that there is a $C > 0$ such that there exists a sequence $\varepsilon_n \to 0$, for which

$$\inf_{\theta\in[0,2\pi]} \|(u_{\varepsilon_n}, \xi_{\varepsilon_n}) - (e^{i\theta}, h_{\mathrm{ex}}\xi_0)\| \geqq \sqrt{C}. \tag{3.2}$$

We have

(3.3)
$$\|(u, \xi) - (e^{i\theta}, h_{\mathrm{ex}}\xi_0)\|^2 = \int_\Omega |\nabla u|^2 + \int_\Omega |u - e^{i\theta}|^2 + \int_\Omega |\nabla\zeta|^2 + \int_\Omega |\Delta\zeta|^2.$$

Since $(u, A) \in U$, we have

$$\int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(1 - |u|^2)^2 + \frac{1}{2}\int_\Omega |\nabla\zeta|^2 + |\Delta\zeta|^2 \leqq o(1). \tag{3.4}$$

Combining (3.3) and (3.4), we get

$$\forall n,\ \forall\theta \in [0, 2\pi], \quad \int_\Omega |u_{\varepsilon_n} - e^{i\theta}|^2 \geqq C - o(1) \geqq \frac{C}{2}. \tag{3.5}$$

On the other hand, using the Poincaré-Wirtinger inequality, we have

$$\text{(3.6)} \qquad \forall\, \varepsilon < \varepsilon_0, \quad \int_\Omega |u - \bar{u}|^2 \leqq C \int_\Omega |\nabla u|^2 \leqq o(1),$$

(by (3.4)), where

$$\bar{u} = \frac{1}{|\Omega|} \int_\Omega u.$$

In addition,

$$\int_\Omega \big|\, |u| - |\bar{u}|\, \big|^2 \leqq \int_\Omega |u - \bar{u}|^2 \leqq o(1),$$

i.e., $\| |u| - |\bar{u}| \|_{L^2} \leqq o(1)$. But we also have

$$\|1 - |u|\|_{L^2} \leqq \|1 - |u|^2\|_{L^2} \leqq o(1).$$

This implies

$$\|\, |\bar{u}| - 1\|_{L^2} \leqq o(1).$$

Since $\bar{u}$ is a constant function, we can thus write

$$\forall\, \varepsilon < \varepsilon_0, \quad \bar{u} = e^{i\theta_\varepsilon} + o(1),$$

for some $\theta_\varepsilon \in [0, 2\pi]$. Inequalities (3.6) now transform into

$$\forall\, \varepsilon < \varepsilon_0, \quad \int_\Omega |u - e^{i\theta_\varepsilon} + o(1)|^2 \leqq o(1),$$

and

$$\forall\, \varepsilon < \varepsilon_0, \quad \exists\, \theta_\varepsilon \in [0, 2\pi] \quad \text{such that} \int_\Omega |u - e^{i\theta_\varepsilon}|^2 \leqq o(1),$$

in contradiction to (3.5). Inequality (3.2) was hence false, and we are able to conclude that

$$\inf_{\theta \in [0,2\pi]} \|(u, \xi) - (e^{i\theta}, h_{\text{ex}}\xi_0)\| \underset{\varepsilon \to 0}{\longrightarrow} 0. \qquad \square$$

Notice that all the $(e^{i\theta}, h_{\text{ex}} d^*\xi_0)$ are gauge-equivalent to $(1, h_{\text{ex}} d^*\xi_0)$. Thus, we have shown that, up to a gauge-equivalence, our solution gets closer and closer to $(1, h_{\text{ex}} d^*\xi_0)$, which is a good approximate solution, although it is not an exact solution of (G.L.). This result is similar to that of Theorem 3 of [S2].

Theorem 1 now results from the combination of Propositions 2.2 and 3.1.

**Theorem 1.** *There exist $\alpha > 0$ and $\varepsilon_0$ such that, if $\varepsilon < \varepsilon_0$, a stable vortexless solution of* (G.L.) *for $h_{\text{ex}} \leqq C\varepsilon^{-\alpha}$ with $\int_\Omega |\nabla u|^2 \leqq o(\varepsilon^\alpha)$ is unique.*

*Let $E_0 = \{(u, A) \in D / |u| \geqq \frac{3}{4}\}$. For $\varepsilon < \varepsilon_0$, there exists a unique locally minimizing solution $(u, A) = (u, d^*\xi)$ of* (G.L.) *in $E_0$ for $h_{\text{ex}} \leqq C\varepsilon^{-\alpha}$ that minimizes $J$ over $E_0$. In addition,*

$$\inf_{\theta \in [0,2\pi]} \|(u, \xi) - (e^{i\theta}, h_{\text{ex}}\xi_0)\| \longrightarrow 0, \quad \varepsilon \to 0,$$

*where $\|.\|$ is defined as*

$$\|(u, z)\|^2 = \|\nabla u\|^2_{L^2} + \|u\|^2_{L^2} + \|\nabla z\|^2_{L^2} + \|\Delta z\|^2_{L^2}.$$

### *3.2. Survival of Vortex Solutions*

We now set $\Omega = B(0,R)$, and recall the definition of the renormalized energy (depending on the number $d$ of points) introduced in [S1]:

$$w(x_1, \cdots, x_d) = -\pi \sum_{i \neq j} \operatorname{Log} |x_i - x_j| + \pi \xi_0''(0) \sum_{i=1}^{d} |x_i|^2.$$

In [S2], I have proved Theorem 1.2 stated in the Introduction. Our new result is:

**Theorem 2.** *Suppose that $\Omega = B(0,R)$ and that $h_{\mathrm{ex}}$ is any function of $\varepsilon$ such that $h_{\mathrm{ex}} \to +\infty$ as $\varepsilon \to 0$, with $h_{\mathrm{ex}} \leqq C\varepsilon^{-\alpha}$.*

1. *If $\varepsilon$ is sufficiently small,* (G.L.) *has a locally minimizing solution $(u, A)$ with exactly one vortex $a^\varepsilon$ of degree 1, satisfying*

$$|a^\varepsilon| \leqq \frac{C}{\sqrt{h_{\mathrm{ex}}}},$$

$$J(u, A) = J_0 + \pi \left( |\operatorname{Log} \varepsilon| - \frac{h_{\mathrm{ex}}}{k_1} \right) + O(1).$$

2. *More generally, for each $n \in \mathbb{N}^*$, such that $\pi n < \mathscr{M}$, if $\varepsilon < \varepsilon_0(\mathscr{M})$,* (G.L.) *has a locally minimizing solution $(u, A)$ with exactly $n$ vortices $a_i^\varepsilon$ of degree 1. In addition, if $\tilde{a}_i = a_i^\varepsilon \sqrt{h_{\mathrm{ex}}}$, then*

$$|\tilde{a}_i| \leqq C \quad \forall i, \qquad |\tilde{a}_i - \tilde{a}_j| \geqq C \quad \forall i \neq j,$$

*and the configuration of the $\tilde{a}_i$'s converges to a minimizer of $w$. Furthermore,*

$$J(u, A) = J_0 + \pi n \left( |\operatorname{Log} \varepsilon| - \frac{h_{\mathrm{ex}}}{k_1} \right) + \frac{\pi}{2}(n^2 - n) \operatorname{Log} h_{\mathrm{ex}} + w(\tilde{a}_1, \ldots, \tilde{a}_n) + Q_n + o(1).$$

*($Q_n$ is a constant depending only on $n$).*

To prove this theorem, we use the same method as in Section 3.1. As in [S1] and [S2], we define

$$E_d = \left\{ (\tilde{u}, \tilde{A}) \in \bar{D} / u^\gamma \text{ has } d \text{ vortices of degree } 1, \text{ and } \exists\, c > 0,\ \operatorname{dist}(a_i, \partial\Omega) \geqq c \right\}.$$

Then, for $n$, a given positive integer $< \mathscr{M}/\pi$, we define, as stated in the Introduction,

$$U_n = \left\{ (\tilde{u}, \tilde{A}) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2) / \pi n |\operatorname{Log} \varepsilon| + B < F(T(\tilde{u})) < \left(n + \tfrac{1}{2}\right) \pi |\operatorname{Log} \varepsilon| \right\}.$$

$B$ is a constant, to be defined later, and $T$ is the continuous mapping

$$\tilde{u} \in H^1(\Omega, \mathbb{C}) \longmapsto u = T(\tilde{u}) = \begin{cases} \tilde{u} & \text{if } |\tilde{u}| \leqq 1, \\ \dfrac{\tilde{u}}{|\tilde{u}|} & \text{if } |\tilde{u}| \geqq 1 \end{cases}$$

that was used in Lemma 2.1. $U_n$, being the inverse image of an open set by the continuous mapping $F \circ T$, is an open set in $H^1 \times H^1$.

**Lemma 3.1.** $\inf_{E_n \cap U_n} J$ *and* $\inf_{E_n} J$ *are both equal to*

$$J_0 + \pi d \left( |\text{Log } \varepsilon| - \frac{h_{\text{ex}}}{k_1} \right) + \frac{\pi}{2}(n^2 - n)\text{Log } h_{\text{ex}} + \inf w + Q_n + o(1).$$

**Proof.** The proof is very similar to that of Lemma 3.3 of [S2]; therefore, we only state its main steps.

*Step 1.* With the usual conventions, considering $(\tilde{u}, \tilde{A}) \in E_n$ and transforming it into $(u, A)$ as in Lemma 2.1, we obtain

$$J(u, A) = J_0 + F(u) + 2\pi h_{\text{ex}} \sum_i \xi_0(a_i) + \tilde{V}(\zeta) + o(1).$$

It is proved in [S2, Lemma 2.2], that there exist positive constants $b_1$ and $b_2$ such that

$$b_1|x|^2 \leqq \xi_0(x) - \xi_0(0) \leqq b_2|x|^2.$$

As in [S2] we deduce (using $2|\xi_0| \leqq k_1^{-1}$), that

$$\begin{aligned} J(u, A) \geqq J_0 + \pi n \left( |\text{Log } \varepsilon| - \frac{h_{\text{ex}}}{k_1} \right) + 2\pi h_{\text{ex}} \sum_i (\xi_0(a_i) - \xi_0(0)) \\ - \pi \sum_{i \neq j} \text{Log } |a_i - a_j| + O(1). \end{aligned}$$

Writing $r = \max_i |a_i|$, as $|a_i - a_j| \leqq 2r$, we are led to

$$\begin{aligned} J(\tilde{u}, \tilde{A}) \geqq J_0 + \pi n \left( |\text{Log } \varepsilon| - \frac{h_{\text{ex}}}{k_1} \right) \\ - \pi(n^2 - n)\text{Log } 2r + 2\pi h_{\text{ex}} b_1 r^2 + O(1) \quad \forall (\tilde{u}, \tilde{A}) \in E_n. \end{aligned} \tag{3.7}$$

Minimizing over $r$, we find that

$$\begin{aligned} J(\tilde{u}, \tilde{A}) \geqq J_0 + \pi n \left( |\text{Log } \varepsilon| - \frac{h_{\text{ex}}}{k_1} \right) \\ + \frac{\pi}{2}(n^2 - n)\text{Log } h_{\text{ex}} + O(1) \quad \forall (\tilde{u}, \tilde{A}) \in E_n. \end{aligned}$$

Conversely, we construct a function $u$ having $n$ vortices of degree 1 on a regular polygon of radius $\left( \dfrac{d^2 - d}{2b_2 h_{\text{ex}}} \right)^{1/2}$ centered at the origin. $(u, h_{\text{ex}} d^* \xi_0)$ is easily seen to belong to $E_n$ as in [S2] (see Proposition 3.3 in [S1], compares the vortices of $u$ with those of $u^\gamma$).

Referring to [S1, Proposition 5.2], we can also construct it so that $|u| \leqq 1$ and

$$F(u) = \pi n|\mathrm{Log}\,\varepsilon| + \pi/2(n^2 - n)\mathrm{Log}\,h_{\mathrm{ex}} + O(1).$$

Hence, $(u, h_{\mathrm{ex}} d^*\xi_0) \in E_n \cap U_n$ (if $\alpha$ is chosen sufficiently small), and

$$\inf_{E_n} J \leqq \inf_{E_n \cap U_n} J \leqq J_0 + \pi n \left( |\mathrm{Log}\,\varepsilon| - \frac{h_{\mathrm{ex}}}{k_1} \right) + \frac{\pi}{2}(n^2 - n)\mathrm{Log}\,h_{\mathrm{ex}} + O(1),$$

with equality

$$\inf_{E_n \cap U_n} J = J_0 + \pi n \left( |\mathrm{Log}\,\varepsilon| - \frac{h_{\mathrm{ex}}}{k_1} \right) + \frac{\pi}{2}(n^2 - n)\mathrm{Log}\,h_{\mathrm{ex}} + O(1). \tag{3.8}$$

*Step 2*. If $(\tilde{u}, \tilde{A})$ is a minimizer (or approximates the infimum closely enough) in $E_n \cap U_n$, then we deduce that $r = \max_i |a_i| \leqq C h_{\mathrm{ex}}^{-1/2}$. Indeed, comparing (3.7) and (3.8), we necessarily obtain

$$\begin{aligned} \tfrac{\pi}{2}(n^2 - n)\mathrm{Log}\,h_{\mathrm{ex}} + O(1) &\leqq -\pi(n^2 - n)\mathrm{Log}\,r + 2\pi h_{\mathrm{ex}} b_1 r^2 \\ &\leqq \tfrac{\pi}{2}(n^2 - n)\mathrm{Log}\,h_{\mathrm{ex}} + O(1), \end{aligned}$$

so that

$$-\pi(n^2 - n)\mathrm{Log}\,(r h_{\mathrm{ex}}^{1/2}) + 2\pi b_1 (r h_{\mathrm{ex}}^{1/2})^2 = O(1).$$

Thus, $r h_{\mathrm{ex}}^{1/2}$ remains bounded and

$$|a_i^\varepsilon| \leqq C h_{\mathrm{ex}}^{-1/2} \quad \forall\, i, \text{ for small } \varepsilon.$$

*Step 3*. As in [S2], we have the more precise estimate

$$\inf_{E_n \cap U_n} J = J_0 + \inf_{E_{(a)}} F + 2\pi h_{\mathrm{ex}} \sum_{i=1}^{n} \xi_0(a_i) + \inf_{E_{(a)}} \tilde{V} + o(1) \tag{3.9}$$

for a suitable configuration $(a)$ (or sequence of configurations). Indeed, $\inf_{E_{(a)}} F = \pi n|\mathrm{Log}\,\varepsilon| + \frac{\pi}{2}(n^2 - n)\mathrm{Log}\,h_{\mathrm{ex}} + O(1)$ can be approximated by functions belonging to $U_n$ as in Step 1.

Exactly as in [S2], we have the inequalities $|a_i^\varepsilon| \leqq C h_{\mathrm{ex}}^{-\frac{1}{2}}$, to show that

$$\inf_{E_{(a)}} \tilde{V} = \pi n^2 \zeta^0(0) + o(1), \tag{3.10}$$

$$\begin{aligned} 2\pi h_{\mathrm{ex}} \sum_{i=1}^{n} \xi_0(a_i) &= -\pi n \frac{h_{\mathrm{ex}}}{k_1} + 2\pi h_{\mathrm{ex}} \sum_{i=1}^{n} (\xi_0(a_i) - \xi_0(0)) \\ &= -\pi n \frac{h_{\mathrm{ex}}}{k_1} + 2\pi h_{\mathrm{ex}} \xi_0''(0) \sum_{i=1}^{n} \frac{|a_i|^2}{2} + o(1). \end{aligned} \tag{3.11}$$

By Proposition V.2 of [S1],

$$\inf_{E_{(a)}} F = \pi d|\text{Log } \varepsilon| + W(a_1, \ldots, a_n) + d\gamma_0 + o(1), \tag{3.12}$$

where $\gamma_0$ is an absolute constant defined in [BBH], and exactly as in [S1],

$$\begin{aligned} W(a_1, \ldots, a_n) &= -\pi \sum_{i \neq j} \text{Log } |a_i - a_j| - \pi \sum_{i=1}^{d} R_0(a_i) \\ &= -\pi \sum_{i \neq j} \text{Log } |a_i - a_j| + (\pi d)^2 \text{Log } R + o(1). \end{aligned} \tag{3.13}$$

Combining (3.9)–(3.13), we get

$$\begin{aligned} \inf_{E_n \cap U_n} J = J_0 + \pi n |\text{Log } \varepsilon| - \pi \sum_{i \neq j} \text{Log } |a_i - a_j| \\ - \pi n \frac{h_{\text{ex}}}{k_1} + \pi h_{\text{ex}} \xi_0''(0) \sum_{i=1}^{n} |a_i|^2 + Q_n + o(1), \end{aligned}$$

where $Q_n = \pi n^2 \zeta^0(0) + (\pi n)^2 \text{Log } R + n\gamma_0$ is a constant depending only on $n$.

We now perform our change of variables $\tilde{a_i} = a_i \sqrt{h_{\text{ex}}}$ ($\tilde{a_i}$ is bounded by Step 2), and find that

$$\begin{aligned} \inf_{E_n} J = J_0 + \pi n \left( |\text{Log } \varepsilon| - \frac{h_{\text{ex}}}{k_1} \right) \\ + \frac{\pi}{2}(n^2 - n) \text{Log } h_{\text{ex}} + w(\tilde{a_1}, \ldots, \tilde{a_n}) + Q_n + o(1). \qquad \square \end{aligned}$$

**Proof of the theorem.** *Step 1*. We consider $(\tilde{u}, \tilde{A}) \in \bar{U}_n$ such that $J(\tilde{u}, \tilde{A}) \leqq \inf_{\bar{U}_n} J + 1$. We transform it into $(u, A)$ with Lemma 2.1, and denote as usual by $(a_i, d_i)$ the vortices of $u^\gamma$. Notice that $(u, A) \in \bar{U}_n$. We have

$$J(\tilde{u}, \tilde{A}) = J(u, A) + O(1) = J_0 + F(u) + 2\pi h_{\text{ex}} \sum_i d_i \xi_0(a_i) + \tilde{V}(\zeta) + O(1). \tag{3.14}$$

Constructing a test configuration in $U_n$ having $n$ vortices suitably located (as for (3.8)), we find that

$$\inf_{\bar{U}_n} J \leqq \inf_{E_n \cap U_n} J = J_0 + \pi n \left( |\text{Log } \varepsilon| - \frac{h_{\text{ex}}}{k_1} \right) + \frac{\pi}{2}(n^2 - n) \text{Log } h_{\text{ex}} + O(1). \tag{3.15}$$

*Step 2*. We first prove that $d_i > 0$. By Proposition 5.1 of [S1], we have

$$F(u) \geqq \pi(1 - \mu) \sum_i |d_i| |\text{Log } \varepsilon| + O(1).$$

On the other hand,

$$\pi n|\text{Log}\,\varepsilon| + B \leqq F(u) \leqq \pi(n+\tfrac{1}{2})|\text{Log}\,\varepsilon|.$$

By choosing $\mu$ small enough (which can be done in the construction of Proposition 3.2 of [S1]), we deduce that $\sum_i |d_i| \leqq n$. But the fact that $(\tilde{u}, \tilde{A}) \in \bar{U}_n$ also implies that

$$\begin{aligned} J(u,A) &= J_0 + F(u) + 2\pi h_{\text{ex}} \sum_i d_i \xi_0(a_i) + O(1) \\ &\geqq J_0 + \pi n|\text{Log}\,\varepsilon| + 2\pi h_{\text{ex}} \sum_i d_i \xi_0(a_i) + O(1). \end{aligned} \tag{3.16}$$

Comparing this to (3.15), we obtain

$$\begin{aligned} \pi n|\text{Log}\,\varepsilon| + 2\pi h_{\text{ex}} \sum_i d_i \xi_0(a_i) &\leqq \pi n|\text{Log}\,\varepsilon| \\ &\quad -\pi n \frac{h_{\text{ex}}}{k_1} + \frac{\pi}{2}(n^2-n)\text{Log}\,h_{\text{ex}} + O(1), \\ n\frac{h_{\text{ex}}}{k_1} \leqq 2h_{\text{ex}} \sum_{i/d_i>0} d_i|\xi_0(a_i)| &+ \frac{n^2-n}{2}\text{Log}\,h_{\text{ex}} + O(1). \end{aligned} \tag{3.17}$$

Dividing by $h_{\text{ex}} \to \infty$, we obtain, as $k_1 = (2\ \max|\xi_0|)^{-1}$,

$$n \leqq \sum_{i/d_i>0} d_i + o(1),$$

while $\sum_i |d_i| \leqq n$. Hence, $d_i > 0$ and $\sum_i d_i = n$ for small $\varepsilon$ and for all $i \in \mathscr{J}'$.

We now prove that $a_i \to 0$ for all $i \in \mathscr{J}'$ ($\Omega = B(0,R)$). Considering (3.17) again, we obtain

$$n\frac{h_{\text{ex}}}{k_1} \leqq 2h_{\text{ex}} \sum_i d_i|\xi_0(0)| + 2h_{\text{ex}} \sum_i d_i(\xi_0(0)-\xi_0(a_i)) + \frac{1}{2}(n^2-n)\text{Log}h_{\text{ex}} + O(1),$$

whence

$$2h_{\text{ex}} \sum_i d_i(\xi_0(a_i) - \xi_0(0)) \leqq \frac{1}{2}(n^2-n)\text{Log}\,h_{\text{ex}} + O(1),$$

$$\forall i, \qquad 0 \leqq \xi_0(a_i) - \xi_0(0) \leqq o(1).$$

Since $\xi_0$ achieves its single minimum at 0, we conclude that $a_i \to 0$ as $\varepsilon \to 0$; hence $\text{dist}(a_i, \partial\Omega)$ remains bounded below, for all $i$.

We next prove that $d_i = 1$, for all $i \in \mathscr{J}'$. As in [S1, Section 6], the two previous results imply that $W(a_i) \geqq O(1)$; hence by Lemma 5.1 of [S1], we obtain

$$F(u) \geqq F(u^\gamma) \geqq \pi \sum_i d_i^2|\text{Log}\,\rho| + \pi \sum_i |d_i| \left|\text{Log}\,\frac{\varepsilon}{\rho}\right| + O(1).$$

Since $\sum d_i = n$, we have

$$(3.18)\quad \begin{aligned} F(u) &\geqq \pi n|\mathrm{Log}\,\varepsilon| + \pi \sum(d_i^2 - d_i)|\mathrm{Log}\,\rho| + O(1),\\ F(u) &\geqq \pi n|\mathrm{Log}\,\varepsilon| + \bar{\mu}\pi \sum(d_i^2 - d_i)|\mathrm{Log}\,\varepsilon| + O(1),\end{aligned}$$

while, by (3.15),

$$(3.19)\quad \begin{aligned} F(u) + 2\pi h_{\mathrm{ex}} \sum_i d_i \xi_0(a_i) &\leqq \pi n \left(|\mathrm{Log}\,\varepsilon| - \frac{h_{\mathrm{ex}}}{k_1}\right)\\ &\quad + \frac{\pi}{2}(n^2 - n)\mathrm{Log}\, h_{\mathrm{ex}} + O(1),\\ F(u) &\leqq \pi n|\mathrm{Log}\,\varepsilon| + \frac{\pi}{2}(n^2 - n)\mathrm{Log}\, h_{\mathrm{ex}} + O(1).\end{aligned}$$

Inequalities (3.18), (3.19), and $h_{\mathrm{ex}} \leqq C\varepsilon^{-\alpha}$ yield

$$\bar{\mu}\pi \sum(d_i^2 - d_i)|\mathrm{Log}\,\varepsilon| \leqq \frac{\pi}{2}(n^2 - n)\mathrm{Log}\, h_{\mathrm{ex}} + O(1),$$

$$\bar{\mu}\pi \sum(d_i^2 - d_i) \leqq \frac{n^2 - n}{2}\alpha + o(1).$$

If $\alpha$ is chosen sufficiently small (according to $\mathscr{M}$ and $\bar{\mu}$), this allows us to conclude that $d_i = 1$, for all $i$. Thus, $u^\gamma$ has exactly $n$ vortices of degree 1 tending to 0. This proves that $(u, A) \in E_n \cap \bar{U}_n$.

*Step 3.*

**Lemma 3.2.** $\inf_{\bar{U}_n} J$ *is achieved.*

**Proof.** Let $(\tilde{u_k}, \tilde{A_k})$ be a minimizing sequence. As in the proof of Theorem 1 of [BR], we easily have (extracting a subsequence if necessary)

$$\tilde{u_k} \rightharpoonup \tilde{u} \quad \text{in } H^1, \quad \tilde{A_k} \rightharpoonup \tilde{A} \quad \text{in } H^1, \quad J(\tilde{u}, \tilde{A}) \leqq \inf_{\bar{U}_n} J.$$

We need to show that $(\tilde{u}, \tilde{A}) \in \bar{U}_n$. First, $u_k = T(\tilde{u}_k) \rightharpoonup u = T(\tilde{u})$ in $H^1$. Indeed, it is a bounded sequence in $H^1$, and so has a weak limit up to a subsequence, and its limit must be its distributional limit $u$. Hence, by lower semi-continuity,

$$F(T(\tilde{u})) \leqq \liminf F(T(\tilde{u}_k)) \leqq \left(n + \tfrac{1}{2}\right)\pi|\mathrm{Log}\,\varepsilon|.$$

The difficulty is to derive the lower bound on $F(u)$. The key argument is that weak $H^1$ convergence preserves the vortices, in some way.

Let us denote by $u^\gamma$ and $u_k^\gamma$ the corresponding regularized maps and by $(b_i, q_i)$, $(a_i^k, d_i^k)$ their respective vortices. We observe that

$$\begin{aligned}&\int_\Omega (iu_k, (\xi_0)_{x_2}(u_k)_{x_1} - (\xi_0)_{x_1}(u_k)_{x_2}) - (iu, (\xi_0)_{x_2}u_{x_1} - (\xi_0)_{x_1}u_{x_2})\\ &\qquad = \int_\Omega (i(u_k - u), d\xi_0 \wedge du_k) + (iu, d\xi_0 \wedge (du_k - du)).\end{aligned}$$

For the first term, we can write

$$\left|\int_\Omega (i(u_k-u), d\xi_0\wedge du_k)\right| \leqq \|\nabla\xi_0\|_{L^\infty}\|u_k-u\|_{L^2}\|\nabla u_k\|_{L^2}$$
$$\leqq o(1) \qquad \text{as } k\to\infty$$

because since $u_k \rightharpoonup u$ in $H^1$, $\|\nabla u_k\|_{L^2}$ is bounded and $u_k\to u$ in $L^2$. For the second term, we integrate by parts to obtain

$$\int_\Omega (iu, d\xi_0\wedge(du_k-du)) = \int_\Omega d(iu, d\xi_0(u_k-u)) + \int_\Omega d\xi_0\wedge(idu, u_k-u)$$
$$= \int_{\partial\Omega} d\xi_0(iu, u_k-u) + \int_\Omega d\xi_0\wedge(idu, u_k-u).$$

The first integral vanishes because $\xi_0\equiv 0$ on $\partial\Omega$. On the other hand,

$$\left|\int_\Omega d\xi_0\wedge(idu, u_k-u)\right| \leqq C\|\nabla u\|_{L^2}\|u_k-u\|_{L^2} = o(1).$$

We conclude that

(3.20)
$$\int_\Omega (iu_k, (\xi_0)_{x_2}(u_k)_{x_1} - (\xi_0)_{x_1}(u_k)_{x_2}) \underset{k\to\infty}{\longrightarrow} \int_\Omega (iu, (\xi_0)_{x_2}u_{x_1} - (\xi_0)_{x_1}u_{x_2}).$$

On the other hand, as in [S1], it is easy to obtain that

$$\begin{aligned}&\int_\Omega (iu_k, (\xi_0)_{x_2}(u_k)_{x_1} - (\xi_0)_{x_1}(u_k)_{x_2})\\ &= \int_\Omega (iu_k^\gamma, (\xi_0)_{x_2}(u_k^\gamma)_{x_1} - (\xi_0)_{x_1}(u_k^\gamma)_{x_2}) + o(1)\\ &= 2\pi\sum_i d_i^k\xi_0(a_i^k) + o(1) \quad \text{as } \varepsilon\to 0,\end{aligned} \tag{3.21}$$

while,

$$\begin{aligned}\int_\Omega (iu, (\xi_0)_{x_2}u_{x_1} - (\xi_0)_{x_1}u_{x_2}) &= \int_\Omega (iu^\gamma, (\xi_0)_{x_2}(u^\gamma)_{x_1} - (\xi_0)_{x_1}(u^\gamma)_{x_2}) + o(1)\\ &= 2\pi\sum_i q_i\xi_0(b_i) + o(1).\end{aligned} \tag{3.22}$$

Since $\tilde u_k$ is a minimizing sequence in $\bar U_n$, the argument of Steps 1 and 2 implies that $\tilde u_k\in E_n$; hence $d_i^k = 1$, $\sum_i d_i^k = n$, and by (3.17),

$$n\frac{h_{\text{ex}}}{k_1} \leqq 2h_{\text{ex}}\sum_i |\xi_0(a_i^k)| + \frac{1}{2}(n^2-n)\text{Log } h_{\text{ex}} + O(1).$$

Thus,

$$\frac{n}{k_1} \leqq 2\sum_i |\xi_0(a_i^k)| + o(1). \tag{3.23}$$

On the other hand, combining (3.20)–(3.22), we have

$$2\pi \sum_i \xi_0(a_i^k) = 2\pi \sum_i q_i \xi_0(b_i) + o_k(1) + o_\varepsilon(1). \tag{3.24}$$

By (3.23), this yields

$$\Big(\sum_{i/q_i>0} q_i\Big)\frac{1}{k_1} \geqq -2\sum_i q_i \xi_0(b_i) \geqq \frac{n}{k_1} + o_k(1) + o_\varepsilon(1),$$

and we are able to conclude that

$$\sum_{i/q_i>0} q_i \geqq n.$$

Conversely, we know that

$$\pi(1-\mu)\sum_i |q_i||\mathrm{Log}\,\varepsilon| \leqq F(u^\gamma) \leqq F(u) \leqq (n+\tfrac{1}{2})|\mathrm{Log}\,\varepsilon|,$$

and as usual we get that

$$\sum_i |q_i| \leqq n.$$

Hence $q_i > 0$ for all $i$, and $\sum_i q_i = n$. Thus, the vortices of $u^\gamma$ satisfy the same results as those mentioned in Step 2 and $\tilde{u} \in E_n$. Therefore,

$$F(u) \geqq F(u^\gamma) \geqq n\pi|\mathrm{Log}\,\varepsilon| + O(1),$$

and $\tilde{u} \in \bar{U_n}$ if $B$ is chosen small enough, so $J(\tilde{u}, \tilde{A}) = \min_{\bar{U}_n} J$. □

*Step 4*. It remains to show that $(\tilde{u}, \tilde{A})$ achieving the infimum does not belong to $\partial U$. Indeed, suppose that is does. Then $F(u) = n\pi|\mathrm{Log}\,\varepsilon| + B$, or $F(u) = (n+\frac{1}{2})\pi|\mathrm{Log}\,\varepsilon|$. Let us deal with the second case: We would have

$$J(u, A) \geqq J_0 + (n+\tfrac{1}{2})\pi|\mathrm{Log}\,\varepsilon| + 2\pi h_{\mathrm{ex}} \sum_i d_i \xi_0(a_i) + O(1).$$

Hence, by minimality and (3.15),

$$\tfrac{\pi}{2}|\mathrm{Log}\,\varepsilon| + 2\pi h_{\mathrm{ex}} \sum_i d_i \xi_0(a_i) \leqq -\pi n \frac{h_{\mathrm{ex}}}{k_1} + \tfrac{\pi}{2}(n^2-n)\mathrm{Log}\,h_{\mathrm{ex}} + O(1),$$

$$\tfrac{\pi}{2}|\mathrm{Log}\,\varepsilon| \leqq \tfrac{\pi}{2}(n^2-n)\mathrm{Log}\,h_{\mathrm{ex}} + O(1) \leqq \tfrac{\pi}{2}\alpha(n^2-n)|\mathrm{Log}\,\varepsilon| + O(1),$$

which is impossible if we choose $\alpha > 0$ small enough, and $\pi n < \mathscr{M}$.

On the other hand, since $(\tilde{u}, \tilde{A})$ achieves $\min_{E_n \cap U_n} J$, it is stated in the proof of Lemma 3.1 that $|a_i| \leqq C h_{\mathrm{ex}}^{-1/2}$. We thus have

$$\begin{aligned} F(u) &\geqq \pi n|\mathrm{Log}\,\varepsilon| - \pi \sum_{i \neq j} \mathrm{Log}\,|a_i - a_j| + O(1) \\ &\geqq \pi n|\mathrm{Log}\,\varepsilon| + \frac{\pi}{2}(n^2-n)\mathrm{Log}\,h_{\mathrm{ex}} + O(1). \end{aligned}$$

Hence the first case is impossible if $n > 1$, since $h_{\mathrm{ex}} \to \infty$. If $n = 1$, it suffices to choose $B$ small enough to get a contradiction. We conclude that $(\tilde{u}, \tilde{A}) \in \overset{\circ}{U}_n$, and minimizes $J$ locally. It is hence a stable solution of (G.L.), and $(\tilde{u}, \tilde{A}) = (u, A)$. From Lemma 3.1, we conclude that the $\tilde{a}_i$'s tend to minimize $w$; from the expression for $w$, this implies that the $|\tilde{a}_i - \tilde{a}_j|$'s remain bounded below, and the proof is complete. □

*Remark.* In addition, $J(u, A)$ is also equal to $\min_{E_n} J$ because the last terms of any sequence of minimizers of $J$ in $E_n$ are easily seen to belong to $U_n$. This proves that $\min_{E_n} J$ is achieved.

## 4. Estimate of the Energy of Vortex-Nucleation

In this section, we restrict our attention to $\Omega = B(0, R)$. In Section 3, we proved, for $h_{\mathrm{ex}} \geqq H_{c_1}$, the existence of two locally minimizing solutions of (G.L.), one without any vortex (the Meissner solution), one with one vortex near the center, having a lower energy than the previous one. We wish to estimate the value of the energy barrier around the Meissner solution. Indeed, when this value gets low, even before reaching $H_{\mathrm{sh}}$, the Meissner solution probably becomes physically unstable, and the system then moves to the vortex solution. In order to achieve this, we define a function of the location of the vortex, describing the path between the two solutions. The evaluation of this function is based on the method of the "image vortex" or reflected vortex.

### *4.1. Definition of the Function $\phi$*

We recall a definition from [S1]:

$$E_a = \{(u, A) \in \bar{D} / u^\gamma \text{ has a unique vortex of degree1 centered in } B(a, \varepsilon^{\frac{\gamma}{2}})\}.$$

$\phi$ is defined on $[0, R]$ as

$$\phi(r) = \inf_{\cup\{E_a, \mathrm{dist}(a, \partial\Omega) = r\}} J \tag{4.1}$$

By symmetry, it is clear that $inf_{E_a} J$ remains constant when $\mathrm{dist}(a, \partial\Omega)$ remains constant.

Actually, since the vortices of $u^\gamma$ are defined to have a characteristic size $\varepsilon^\mu \leqq \rho \leqq \varepsilon^{\bar{\mu}}$, (see Proposition 3.2 of [S1]), it does not really make sense to define $\phi$ for $r \leqq \varepsilon^{\bar{\mu}}$. In addition, their location is known only up to $\varepsilon^{\gamma/2}$. Hence, we choose $\beta = \frac{1}{2}\gamma$, and we shall study $\phi$ on $[\varepsilon^\beta, R]$.

*4.2. Computation of $\phi$*

We use the notations in Section 5 of [S1]. Since we consider $u^\gamma$ with only one vortex, we can assume that this vortex is at $a$ and $W(a) = -\pi R_0(a)$, where $R_0$ is defined by

$$\begin{aligned} \Delta R_0 &= 0 \qquad \text{in } B(0,R), \\ R_0 &= -\pi \operatorname{Log}|x-a| \quad \text{on } \partial B(0,R). \end{aligned} \tag{4.2}$$

We have

$$R_0 \,|_{\partial B(0,R)} \geqq -\operatorname{Log} R.$$

Therefore, by the maximum principle, $R_0 \geqq -\operatorname{Log} R$ on $B(0,R)$. Hence $W(a) \leqq \pi \operatorname{Log} R \leqq o(|\operatorname{Log} \varepsilon|)$. We can then apply Proposition 5.2 of [S1], which asserts that

(4.3)
$$\inf_{E_a} F = \pi|\operatorname{Log}\varepsilon| + W(a) + d\gamma_0 + o(1) = \pi|\operatorname{Log}\varepsilon| - \pi R_0(a) + d\gamma_0 + o(1).$$

We deduce

**Lemma 4.1.**

$$\pi|\operatorname{Log}\varepsilon| + \pi\operatorname{Log} 2r + \max\left(-C, \pi\operatorname{Log}\left(1-\frac{r}{R}\right)\right) + \gamma_0 + o(1) \leqq \inf_{E_a} F \leqq \pi|\operatorname{Log}\varepsilon| + \pi\operatorname{Log} 2r + \gamma_0 + o(1).$$

**Proof.** As in [BBH] and [S1], we use the function $\Phi_0$, satisfying

$$\begin{aligned} \Delta\Phi_0 &= 2\pi\delta_a \quad \text{in } B(0,R), \\ \Phi_0 &= 0 \qquad \text{on } \partial B(0,R). \end{aligned}$$

Then $R_0 = \Phi_0 - \operatorname{Log}|x-a|$. We set

$$v(x) = \operatorname{Log}|x-a| - \operatorname{Log}|x-a^*|, \tag{4.4}$$

where $a^*$ is the image of $a$ under the reflection relative to $\partial B(0,R)$, i.e.,

$$a^* = \left(\frac{2R}{|a|} - 1\right) \qquad 0^* = 2R.$$

$v - \Phi_0$ then satisfies

$$\Delta(v - \Phi_0) = 0 \qquad \text{in } B(0,R). \tag{4.5}$$

**Lemma 4.2.** *There exists a $C > 0$ such that*

$$\max\left(\operatorname{Log}\left(1-\frac{r}{R}\right), -C\right) \leqq v\,|_{\partial B(0,R)} \leqq 0.$$

**Proof.** The point $a$ being set, we define the vector $l = \frac{1}{2}(a - a^*)$, and assume that it is small. Its norm is $r$. Using the polar parametrization of the circle: $\rho = 2R\sin\theta$ for $\theta \in [0, \pi]$, with base vector orthogonal to $l$, we can compute the function $v$ on the circle:

$$\begin{aligned} v(\theta) &= \operatorname{Log}(r^2 + \rho^2 - 4rR\sin^2\theta)^{1/2} - \operatorname{Log}(r^2 + \rho^2 + 4rR\sin^2\theta)^{1/2} \\ &= \frac{1}{2}\operatorname{Log}\left(1 - \frac{4rR\sin^2\theta}{r^2 + 4R^2\sin^2\theta}\right) - \frac{1}{2}\operatorname{Log}\left(1 + \frac{4rR\sin^2\theta}{r^2 + 4R^2\sin^2\theta}\right) \\ &= -\sum_{k=0}^{+\infty}\frac{1}{2k+1}\left(\frac{4Rr\sin^2\theta}{4R^2\sin^2\theta + r^2}\right)^{2k+1} \\ &\geqq -\sum_{k=0}^{+\infty}\frac{1}{2k+1}\left(\frac{r}{R}\right)^{2k+1} \\ &\geqq \operatorname{Log}\left(1 - \frac{r}{R}\right). \end{aligned}$$

Hence, for small $r$, we have

$$\operatorname{Log}\left(1 - \frac{r}{R}\right) \leqq v\,|_{\partial B(0,R)} \leqq 0.$$

Otherwise, in the general case, $v$ is also bounded from below by an absolute constant. Hence, we deduce the expression stated. □

By the maximum principle, we deduce from this lemma that

$$\max\left(\operatorname{Log}\left(1 - \frac{r}{R}\right), -C\right) \leqq v - \Phi_0 \leqq 0 \qquad \text{in } B(0, R).$$

By (4.4), this yields

$$\max\left(\operatorname{Log}\left(1 - \frac{r}{R}\right), -C\right) \leqq -R_0(x) - \operatorname{Log}|x - a^*| \leqq 0.$$

In addition, since $|a - a^*| = 2r$, we have

$$\max\left(\operatorname{Log}\left(1 - \frac{r}{R}\right), -C\right) + \operatorname{Log} 2r \leqq -R_0(a) \leqq \operatorname{Log} 2r.$$

By (4.3), the lemma is proved. □

We are now in a position to prove:

**Lemma 4.3.** *Let* $h_{\text{ex}} \leqq C\varepsilon^{-\alpha}$ *and define*

$$\begin{aligned} \phi_1(r) &= J_0 + \pi|\operatorname{Log}\varepsilon| + \pi\operatorname{Log} 2r + \max\left(-C, \pi\operatorname{Log}\left(1 - \frac{r}{R}\right)\right) \\ &\qquad + \gamma_0 + 2\pi h_{\text{ex}}\xi_0(R - r) + o(1), \\ \phi_2(r) &= J_0 + \pi|\operatorname{Log}\varepsilon| + \pi\operatorname{Log} 2r + \gamma_0 + 2\pi h_{\text{ex}}\xi_0(R - r) + Cr^{1/2} + o(1). \end{aligned}$$

*Then*

$$\phi_1(r) \leqq \phi(r) \leqq \phi_2(r) \qquad \forall\, r \in [\varepsilon^\beta, R].$$

**Proof.** From the analysis of [S1] and [S2], if $h_{\text{ex}} \leqq C\varepsilon^{-\alpha}$, then

$$\inf_{E_a} J = J_0 + \inf_{E_a} F + 2\pi h_{\text{ex}}\xi_0(a) + \tilde{V}(\zeta) + o(1), \tag{4.6}$$

as seen before, and

$$\inf_{E_a} \tilde{V}(\zeta) = \pi\zeta^a(a) + o(1),$$

where $\zeta$ is the solution of

$$\begin{aligned} -\Delta^2\zeta^a + \Delta\zeta^a &= 2\pi\delta_a && \text{in } B(0,R),\\ \Delta\zeta^a &= 0 && \text{on } \partial B(0,R),\\ \zeta^a &= 0 && \text{on } \partial B(0,R) \end{aligned}$$

(see [S1, Section 4.2]). In addition, by Lemma 4.6 of [S1],

$$\|\zeta^a\|_{L^\infty(B(0,R))} \leqq C\text{dist }(a, \partial B(0,R))^k \quad \forall k < 1.$$

Thus $\inf \tilde{V} \leqq Cr^k$ for all $k < 1$.

Combining this with (4.6) and Lemma 4.1, we conclude that

$$\phi_1(r) \leqq \phi(r) \leqq \phi_2(r). \qquad \square$$

Now, we see that $\phi$ is a function that passes through a maximum for some $r \in [\varepsilon^\beta, R]$. When a vortex appears in the superconductor, it is physically assumed that it is created near the boundary and then moves to the center. The energy gap to be overcome in this process must be of order $\max\phi - J_0$. This is why we want to get an estimate on $\max\phi$. This is done through the following theorem:

**Theorem 3.**

1. $$\max_{[\varepsilon^\beta, R]} \phi = J_0 + \pi|\text{Log}\,\varepsilon| - \pi\text{Log}\,h_{\text{ex}} + O(1).$$
2. *$\phi$ achieves its maximum at $r_{\max}$, and there exists a $C > 0$ such that*
$$r_{\max} \leqq \frac{C}{h_{\text{ex}}}.$$
3. $$\phi(\varepsilon^\beta) \leqq J_0 + \pi(1-\beta)|\text{Log}\,\varepsilon| + \pi\text{Log}\,2 + \gamma_0 + o(1).$$

**Proof.** *Step 1*. We begin with

$$\phi(r) \leqq \phi_2(r) \leqq J_0 + \pi|\text{Log}\,\varepsilon| + \pi\text{Log}\,2r + 2\pi h_{\text{ex}}\xi_0(R-r) + C.$$

From [S2], we know that

$$\xi_0(r) = \frac{Z(r)}{Z(R)} - 1, \qquad Z(r) = \sum_{n=0}^{+\infty} \frac{r^{2n}}{2^{2n}(n!)^2}. \tag{4.7}$$

$\xi_0$ is a convex function; hence $\xi_0(r-R)$ is convex on $[0, R]$ and its graph is located under its chord, i.e.,

$$\xi_0(R-r) \leqq \frac{\xi_0(0)r}{R}. \tag{4.8}$$

The function

$$r \mapsto \pi \operatorname{Log} 2r + 2\pi h_{\text{ex}} \frac{\xi_0(0)r}{R}$$

achieves its maximum for $r = -\frac{R}{2h_{\text{ex}}\xi_0(0)}$, and the maximum is $C - \pi \operatorname{Log} h_{\text{ex}} - \pi$, where $C$ is an absolute constant, independent of $h_{\text{ex}}$. We deduce that

$$\max_{[\varepsilon^\beta, R]} \phi \leqq J_0 + \pi |\operatorname{Log} \varepsilon| - \pi \operatorname{Log} h_{\text{ex}} + C. \tag{4.9}$$

On the other hand, $\max_{[\varepsilon^\beta, R]} \phi \geqq \max_{[\varepsilon^\beta, R]} \phi_1$. Since $\xi_0'$ is nondecreasing on $[0, R]$, it follows that $\xi_0'(r) \leqq \xi_0'(R)$ and

$$\xi_0(R-r) \geqq -r\xi_0'(R),$$

$$\begin{aligned} \max\left(\pi \operatorname{Log} 2r + 2\pi h_{\text{ex}} \xi_0(R-r)\right) &\geqq \max\left(\pi \operatorname{Log} 2r - 2\pi h_{\text{ex}} r \xi_0'(R)\right) \\ &= -\pi \operatorname{Log} h_{\text{ex}} + C. \end{aligned}$$

Hence $\max_{[\varepsilon^\beta, R]} \phi_1 \geqq J_0 + \pi |\operatorname{Log} \varepsilon| - \pi \operatorname{Log} h_{\text{ex}} - C$.

We finally conclude that

$$\max_{[\varepsilon^\beta, R]} \phi = J_0 + \pi |\operatorname{Log} \varepsilon| - \pi \operatorname{Log} h_{\text{ex}} + O(1).$$

*Step 2* . Since the function $\pi \operatorname{Log} 2r + 2\pi h_{\text{ex}} \frac{\xi_0(0)r}{R}$ is concave for $r \geqq \frac{R}{2h_{\text{ex}}|\xi_0(0)|}$, it is nonincreasing. Suppose now that $r \geqq \frac{B}{h_{\text{ex}}}$ for some $B > \frac{R}{2|\xi_0(0)|}$; then

$$C + \pi \operatorname{Log} 2r + 2\pi h_{\text{ex}} \frac{\xi_0(0)r}{R} \leqq C + \pi \operatorname{Log} 2B - \pi \operatorname{Log} h_{\text{ex}} + \frac{2\pi \xi_0(0) B}{R},$$

implying that

$$\phi(r) \leqq \phi_2(r) < \max \phi,$$

if $B$ is chosen sufficiently large. Therefore, $\phi$ cannot achieve its maximum at $r \geqq \dfrac{B}{h_{\text{ex}}}$. We conclude that $\phi$ achieves its maximum at $r_{\max}$ with $r_{\max} \leqq \frac{B}{h_{\text{ex}}}$, which tends to 0 as $h_{\text{ex}} \to +\infty$.

*Step 3*. From Lemma 4.1, we have the estimate

$$\begin{aligned} \phi(\varepsilon^\beta) &\leqq J_0 + \pi |\operatorname{Log} \varepsilon| + \pi \operatorname{Log} 2\varepsilon^\beta + C\varepsilon^{\beta/2} + 2\pi h_{\text{ex}} \xi_0(R - \varepsilon^\beta) + \gamma_0 + o(1) \\ &\leqq J_0 + \pi(1-\beta)|\operatorname{Log} \varepsilon| + \pi \operatorname{Log} 2 - 2\pi \varepsilon^\beta \xi_0'(R) h_{\text{ex}} + \gamma_0 + o(1) \\ &\leqq J_0 + \pi(1-\beta)|\operatorname{Log} \varepsilon| + \pi \operatorname{Log} 2 + \gamma_0 + o(1), \end{aligned}$$

which concludes our proof. □

We now give the justification in favor of the assumption that when a vortex appears, it is created at the boundary and moves to the center. Indeed, another reasonable assumption would be to say that a pair of vortices (one of degree 1, one of degree $-1$) is created at some point of $B(0, R)$, and that one moves out of $B(0, R)$, while the other moves to the center. We are going to prove that at some point this always costs more energy than $\max \phi$.

**Proposition 4.1.** *The minimal energy over such paths of configurations is higher than* $\max_{[\varepsilon^\beta, R]} \phi$.

**Proof.** Denote by $a_+$ the vortex of degree 1, and by $a_-$ the vortex of degree $-1$. Following Proposition 5.1 of [S1], we are going to compute the minimal energy of a configuration $(u, A)$ having these vortices when their mutual distance is equal to $\frac{1}{h_{\text{ex}}}$. Since $\frac{1}{h_{\text{ex}}} \gg \varepsilon^{\gamma/2}$, Lemma 5.1 of [S1] ensures that

$$F(u^\gamma) \geqq 2\pi |\text{Log}\, \rho| + 2\pi \left|\text{Log}\, \frac{\varepsilon}{\rho}\right| + W((a_+, 1), (a_-, -1)) + O(1), \tag{4.10}$$

where

$$W((a_+, 1), (a_-, -1)) = \pi \text{Log}|a_+ - a_-| - \pi (R_0(a_+) - R_0(a_-)). \tag{4.11}$$

But, following the same arguments as in the proof of Proposition 5.1 in [S1], we have

$$\|\nabla R_0\|_{L^\infty([a_+, a_-])} \leqq C|\text{Log}\, \varepsilon|,$$

and thus

$$|R_0(a_+) - R_0(a_-)| \leqq \frac{C|\text{Log}\, \varepsilon|}{h_{\text{ex}}} \leqq C.$$

(4.10) hence becomes

$$F(u^\gamma) \geqq 2\pi |\text{Log}\, \varepsilon| - \pi \text{Log}\, h_{\text{ex}} + O(1). \tag{4.12}$$

Now, as usual,

$$J(u, A) \geqq J_0 + F(u) + 2\pi h_{\text{ex}} (\xi_0(a_+) - \xi_0(a_-)) + O(1). \tag{4.13}$$

In addition,

$$h_{\text{ex}} (\xi_0(a_+) - \xi_0(a_-)) \leqq C h_{\text{ex}} \|\xi_0'\|_{L^\infty} \frac{1}{h_{\text{ex}}} \leqq C.$$

Combining (4.12) and (4.13), we obtain

$$J(u, A) \geqq J_0 + 2\pi |\text{Log}\, \varepsilon| - \pi \text{Log}\, h_{\text{ex}} + O(1).$$

Then, $J(u, A) > max\ \phi$, and the proposition is proved. □

### *4.3. Remarks and Interpretations*

1. If we take $\beta < 1$ as close to 1 as we can, we see that the energetical cost of a vortex at the boundary of $\Omega$ is much less than $\max \phi$.

2. $\max \phi - J_0$ roughly decreases as $h_{\text{ex}}$ increases; thus we can guess that below fields of order $\varepsilon^{-1} \simeq H_{\text{sh}}$, the gap of energy becomes very small (while the point of maximum tends to the boundary).

3. Since there are a locally minimizing vortexless solution and a solution with one vortex near the center with a lower energy, we deduce the existence of an unstable mountain-pass solution. We may reasonably think, by Theorem 3, that this solution has a vortex of degree 1 located at $r_{\max}$ from the boundary, and has an energy $\simeq \max \phi = J_0 + \pi |\text{Log}\, \varepsilon| - \pi \text{Log}\, h_{\text{ex}} + O(1)$. It is an open problem to determine if this is true.

4. Some authors do not have the same normalization of the fields and of the constants: They consider an energy functional $\frac{1}{2} \int_\Omega |(\frac{\nabla}{\kappa} - iA)u|^2 + |h - h_{\text{ex}}|^2 + \frac{1}{2}(1 - |u|^2)^2$; then their fields are $\frac{1}{\kappa}$ times our fields, and their energies are $\frac{1}{\kappa^2}$ times our energies. In this case, our results become

$$H_{c_1} = \frac{\text{Log}\, \kappa}{2 \max |\xi_0|\, \kappa} + O\left(\frac{1}{\kappa}\right),$$

$$\max \phi - J_0 \simeq \frac{\pi \text{Log}\, \kappa}{\kappa^2} - \frac{\pi \text{Log}\, (\kappa h_{\text{ex}})}{\kappa^2} \simeq -\frac{\pi \text{Log}\, h_{\text{ex}}}{\kappa^2},$$

with $h_{\text{ex}} \ll 1$.

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