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Heteroclinic Networks in Coupled Cell Systems

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Abstract

We give an intrinsic definition of a heteroclinic network as a flow-invariant set that is indecomposable but not recurrent. Our definition covers many previously discussed examples of heteroclinic behavior. In addition, it provides a natural framework for discussing cycles between invariant sets more complicated than equilibria or limit cycles. We allow for cycles that connect chaotic sets (cycling chaos) or heteroclinic cycles (cycling cycles). Both phenomena can occur robustly in systems with symmetry.

We analyze the structure of a heteroclinic network as well as dynamics on and near the network. In particular, we introduce a notion of 'depth' for a heteroclinic network (simple cycles between equilibria have depth 1), characterize the connections and discuss issues of attraction, robustness and asymptotic behavior near a network.

We consider in detail a system of nine coupled cells where one can find a variety of complicated, yet robust, dynamics in simple polynomial vector fields that possess symmetries. For this model system, we find and prove the existence of depth-2 networks involving connections between heteroclinic cycles and equilibria, and study bifurcations of such structures.

Contents

1. Introduction

Dynamical systems that commute with a group of symmetries often display complicated robust dynamics that results from the presence of symmetry. In symmetric (equivariant) dynamical systems one can find robust attractors that (a) are not recurrent, and (b) do not display ergodic behavior. The simplest examples of this phenomenon are heteroclinic cycles of the type made famous by GUCKENHEIMER & HOLMES [23]. Slight variations on these examples can lead to rather complicated, but apparently robust, attractors. While the presence of symmetry can lead to complexity, the assumption of symmetry yields a range of new tools, algebraic and geometric, that can often make the complicated dynamics analytically tractable.

Up until now, there has been no general definition that covers all examples of 'heteroclinic'-type attractors. In this paper, we aim to give a usable definition that includes all known examples of heteroclinic cycles and networks. Our definition is nonetheless strong enough that we can prove structural results about heteroclinic networks. To this end, we concentrate on the problem of describing the dynamics on the network. That is, the dynamics *intrinsic* to the network, rather than dynamics *near* the network. We then examine the consequences for dynamics near the network.

The paper is organized in the following way. In Sections 2–4, we present definitions and theoretical results. The remaining sections discuss specific models and examples.

In Section 2, we start by discussing recurrence properties of flows, in particular, topological and chain recurrence. Next we give definitions for *homoclinic* and *heteroclinic cycles*. Roughly speaking, these are cyclic chains of connections between recurrent invariant sets. This leads up to our intrinsic definition of a *heteroclinic network* in Section 2.3. A heteroclinic network is a continuous flow on a compact metric space that is indecomposable and such that the set of recurrent points (the set of nodes) satisfies certain regularity conditions.

We refer to the set of non-recurrent points as the *set of connections*. Associated to every network we define a positive integer invariant that we call the *depth* of the network. Heteroclinic cycles have depth 1, more complicated networks have depth greater than 1. A quantity related to depth appears in the construction of the 'Birkhoff center' of a dynamical system. We prove some basic results on the structure of heteroclinic networks and discuss a number of examples.

In Section 3, we discuss the dynamics and asymptotics near a heteroclinic network that is embedded as an invariant set of a dynamical system on $\mathbb{R}^n$. We discuss various conditions that imply that an embedded heteroclinic network is an attractor for nearby trajectories and we characterize the behavior of observables from trajectories converging to such networks.

In Section 4 we tackle the problem of how to decide whether a given network is robust in a given equivariant setting. At least for networks of depth greater than 1, structural stability does not appear to form the basis of a good definition of robustness. We formulate a weak, though verifiable, definition of stability that relates the asymptotics on the network to the orbit structure of the group action.

In the remaining sections we focus on specific examples of equivariant systems in $\mathbb{R}^n$. Let $\Delta_n = (\mathbb{Z}_2)^n$ be the group generated by the set of reflections in all coordinate hyperplanes and Γ be a finite group of linear symmetries of $\mathbb{R}^n$ containing Δ_n. It has been known for some time that the presence of the symmetries Δ_n can lead to robust attracting heteroclinic cycles in flows with symmetry Γ.

In Sections 5 and 6, we study a model system of nine coupled identical one-degree-of-freedom cells with $\mathbb{Z}_3 \times \mathbb{Z}_3$ global permutation symmetry and 'internal' $\mathbb{Z}_2$ symmetry. We do this both from a theoretical point of view and also numerically. We find, among other phenomena,

- Existence of depth-2 heteroclinic networks between equilibria with depth-1 subnetworks that are attracting.
- Bifurcation of such networks to create networks between periodic orbits.
- Existence of cycles between 'synchronized' states that are 'essentially asymptotically stable'.

All of this behavior is robust to perturbations preserving the symmetry. We also briefly discuss some generic one-parameter bifurcations of these generalized networks.

Finally, in Section 7, we consider the correspondence between our model equivariant systems and Lotka-Volterra-type equations that arise in game dynamics [33]. (This correspondence arises because equations that are symmetric under Δ_n restricted to an invariant sphere are equivalent to a game system on an $(n-1)$-simplex.) We also discuss some other consequences of the observed behavior and implications for cycles between more complicated invariant sets, for example *cycling chaos* [12, 20].

2. Heteroclinic Networks

In this section our emphasis is on describing the intrinsic properties of a class of compact flow-invariant sets. Although this class naturally arises in the study of smooth flows on $\mathbb{R}^n$, it is helpful at first to formulate our definitions in an abstract setting. Consequently, throughout this section, we shall work with continuous flows on a compact metric space. In Sections 3 and 4, this metric space is embedded into $\mathbb{R}^n$ and the flow is the restriction of a smooth flow on $\mathbb{R}^n$.

2.1. Preliminaries

Suppose Σ is a compact connected metric space with metric ρ. We consider the continuous flow

$$\phi_t : \Sigma \to \Sigma, \ t \in \mathbb{R}. \tag{1}$$

In the sequel, we sometimes write (Σ, ϕ) to denote the set Σ together with the flow ϕ. If the flow is clear from the context, we usually just write Σ.

For $x \in \Sigma$, let $\omega(x)$ and $\alpha(x)$ denote the set of limit points of the trajectory passing through x as $t \to \infty$ and $-\infty$, respectively. Recall that $\omega(x)$ and $\alpha(x)$ are compact, connected flow-invariant subsets of Σ.

Definition 2.1. Given $x, y \in \Sigma$ and $\varepsilon, T > 0$, we say there is an (ε, T)*-pseudo orbit* joining x to y if we can find a finite subset $\{x = x_0, y_0, x_1, \ldots, x_n, y_n = y\}$ of Σ and $t_i \geqq T$, $0 \leqq i < n$ such that for all $0 \leqq i < n$ we have

$$\begin{aligned} \rho(x_i, y_i) &< \varepsilon, \\ x_{i+1} &= \phi_{t_i}(y_i). \end{aligned}$$

Remark 2.2. Our definition of (ε, T)-pseudo orbit follows that given in SHUB [41, page 18] and is slightly different from that originally used by CONLEY [11]. It has the advantage that an (ε, T)-pseudo orbit joining x to y is automatically an (ε, T)-pseudo orbit joining y to x for the time reversed flow.

Following [11], we define a relation $\sim$ on Σ^2 by requiring that $x \sim y$ if and only if for all $\varepsilon, T > 0$, there exists an (ε, T)-pseudo orbit joining x to y. Let $P(\Sigma) = \{(x, y) \in \Sigma^2 \mid x \sim y\}$. Just as in [11, Chapter III, §6], it may be shown that $\sim$ is transitive and that $P(\Sigma) \subset \Sigma^2$ is closed and invariant with respect to the diagonal flow on Σ^2.

Definition 2.3. The *chain recurrent* set $\mathscr{R}_{\mathrm{ch}}(\Sigma)$ of Σ is defined to be the subset of Σ consisting of all x such that $x \sim x$.

Since we may identify $\mathscr{R}_{\mathrm{ch}}(\Sigma)$ with the intersection of $P(\Sigma)$ and the diagonal of Σ^2, it follows that $\mathscr{R}_{\mathrm{ch}}(\Sigma)$ is a closed flow-invariant subset of Σ.

We recall that

$$\mathscr{R}_{\mathrm{ch}}(\mathscr{R}_{\mathrm{ch}}(\Sigma)) = \mathscr{R}_{\mathrm{ch}}(\Sigma), \tag{2}$$

$$\mathscr{R}_{\mathrm{ch}}(\omega(x)) = \omega(x), \quad (x \in \Sigma), \tag{3}$$

$$\mathscr{R}_{\mathrm{ch}}(\alpha(x)) = \alpha(x), \quad (x \in \Sigma). \tag{4}$$

Definition 2.4 (cf. Guckenheimer & Holmes [22, Defn. 5.2.5]). We say that (Σ, ϕ) is *indecomposable* if $x \sim y$ for all $x, y \in \Sigma$.

Since we are assuming that Σ is compact and connected, it follows that $\mathscr{R}_{\text{ch}}(\Sigma) = \Sigma$ if and only if Σ is indecomposable.

Remarks 2.5. (1) The definition of indecomposability given by Guckenheimer & Holmes is weaker than ours in that they require points to be connected by $(\varepsilon, 1)$-pseudo orbits, $\varepsilon > 0$.
(2) It follows from (4) that $\omega(x)$ and $\alpha(x)$ are always indecomposable.

Definition 2.6. A ϕ-invariant subset $S \subset \Sigma$ is *recurrent* if there is an $x \in S$ such that $\omega(x) = \alpha(x) = S$.

Remarks 2.7. (1) A recurrent subset of Σ is always connected and compact.
(2) Obviously, if S is recurrent, then S is topologically transitive. Conversely, if there exists an $x \in S$ such that $\omega(x) = S$, then there is a residual subset of S consisting of recurrent points (see, for example, Mañé [34]).
(3) If there exists $x \in S$ such that $S = \alpha(x) \cup \omega(x)$ and $x \in \alpha(x) \cap \omega(x)$, then S is recurrent.

If S is an invariant subset of Σ, we define

$$R(S) = \{x \in S \mid x \in \omega(x) \cap \alpha(x)\}.$$

We call $R(S)$ the set of recurrent points (of S).

Remarks 2.8. (1) Even if S is closed, $R(S)$ may not be a closed subset of S. In the literature, $R(S)$ is often defined to be the *closure* of the set of recurrent points. In our applications we shall primarily be interested in $R(\Sigma)$ and as part of our regularity hypotheses, we shall require that $R(\Sigma)$ is closed.
(2) If $S = R(S)$, then S is a union of recurrent sets – $S = \cup_{x \in S} \alpha(x) \cap \omega(x)$.

Lemma 2.9. *Suppose that X is a closed invariant connected subset of Σ and that X is a union of recurrent sets. Then X is indecomposable.*

Proof. Suppose that $X = \cup_{i \in I} X_i$, where the X_i are recurrent sets. If $x, y \in X$ and $\varepsilon, T > 0$, we may choose $\{x = x_0, x_1, \ldots, x_n\} \subset X$ such that $\rho(x_i, x_{i+1}) < \frac{1}{2}\varepsilon$, $0 \leqq i < n$. Since recurrent sets are indecomposable (Remarks 2.5(2)), we have $x_i \sim x_i$. Hence there is an $(\frac{1}{2}\varepsilon, T)$-pseudo orbit O_i joining x_i to x_i, $0 \leqq i < n$. Concatenate the pseudo orbits $O_0, \ldots, O_{n-1}$ to obtain a (ε, T)-pseudo orbit joining x to y. □

2.2. Heteroclinic Cycles

Before we give our definition of a heteroclinic network, we briefly review the definition and basic properties of a heteroclinic cycle. We start by giving an intrinsic definition of a heteroclinic cycle (that is, without reference to the phase space in which it may be embedded). Suppose that ϕ_t has equilibria

$$\mathscr{A} = \{p_0, \ldots, p_{k-1}, p_k = p_0\}.$$

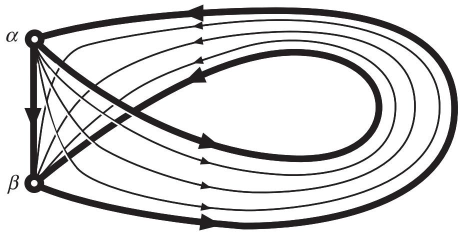


Fig. 1. Sketch of a flow on a Möbius band that is a heteroclinic cycle from α to β and back. Observe that the closure of the set of connections from α to β contains the single connection from β to α.

Definition 2.10 (cf. KRUPA & MELBOURNE [32, Definition 2.1]). We say that Σ is a *heteroclinic cycle* connecting the equilibria $\mathscr{A}$ if for all $x \in \Sigma$ there is a j with $0 \leqq j < k$ such that either $x = p_j$, or

$$\alpha(x) = p_j \text{ and } \omega(x) = p_{j+1}.$$

Remarks 2.11. (1) According to Definition 2.10 there can be infinitely many connections between adjacent equilibria p_j, p_{j+1} of the cycle. (This is also possible with KRUPA & MELBOURNE's definition.)
(2) It is possible to construct examples for which the closure of the set of all connections between just one pair of adjacent equilibria is equal to all of Σ. For example, see Figure 1.

Example 2.12. One of the simplest (and best known) examples of a heteroclinic cycle that occurs in equivariant dynamics is the cycle described by GUCKENHEIMER & HOLMES [23] (see Figure 3(b)). In this case, Σ is one-dimensional and consists of three equilibria connected by three trajectories. Since Σ is a heteroclinic cycle of a $\Delta_3 \rtimes \mathbb{Z}_3$-equivariant vector field on $\mathbb{R}^3$, it follows by equivariance that $\gamma\Sigma$ is also a heteroclinic cycle for all $\gamma \in \Delta_3 \rtimes \mathbb{Z}_3$ (see also [20, §4.2]).

There are extensions of Definition 2.10 to allow for heteroclinic cycles connecting limit cycles or even chaotic sets (see [36, 20]). We describe one such generalization formulated in terms of stable and unstable sets.

If $A \subset \Sigma$ is a compact flow-invariant set, we define the stable and unstable sets of A by

$$\mathscr{W}^{\mathrm{u}}(A) = \{y \in \Sigma \mid \lim_{t\to-\infty} \rho(\phi_t(y), A) = 0\},$$
$$\mathscr{W}^{\mathrm{s}}(A) = \{y \in \Sigma \mid \lim_{t\to\infty} \rho(\phi_t(y), A) = 0\}.$$

Obviously, $\mathscr{W}^{\mathrm{u}}(A), \mathscr{W}^{\mathrm{s}}(A)$ are flow-invariant subsets of Σ and $\mathscr{W}^{\mathrm{u}}(A), \mathscr{W}^{\mathrm{s}}(A) \supset A$. Observe that $\omega(x) \subset A$ if and only if $x \in \mathscr{W}^{\mathrm{s}}(A)$. There is a similar relation between $\alpha(x)$ and $\mathscr{W}^{\mathrm{u}}(A)$.

Suppose that $\mathscr{N} = \{N_i \mid i = 0, \ldots, k-1\}$ is a finite set of mutually disjoint compact flow-invariant subsets of Σ. For notational convenience, we define $N_k = N_0$.

Definition 2.13. We say that Σ is a *heteroclinic cycle* with *node set* $\mathscr{N}$ if

(a) $\mathscr{W}^{\mathrm{u}}(N_i) \cap \mathscr{W}^{\mathrm{s}}(N_j) \neq \emptyset$ if and only if $j = i + 1$ or $j = k$, $i = 0$, and
(b) $\cup_i \mathscr{W}^{\mathrm{u}}(N_i) = \cup_i \mathscr{W}^{\mathrm{s}}(N_i) = \Sigma$.

Remarks 2.14. (1) If $\mathscr{N}$ consists of equilibria, then Definition 2.13 is equivalent to Definition 2.10.
(2) Although we have connections between *nodes* of the cycle, there may be few, if any, connections between proper invariant subsets of nodes.

It is worthwhile singling out a class of particularly well behaved heteroclinic cycles, also called *closed* cycles in [4].

Definition 2.15. We say that Σ is a *regular* heteroclinic cycle if $\mathscr{W}^{\mathrm{u}}(N_j) \cup \{N_{j+1}\}$ is closed, $j \geqq 0$.

Remarks 2.16. (1) If Σ is regular, then the behavior described in Remarks 2.11(2) does not occur. For example, the Guckenheimer-Holmes cycle is regular.
(2) An equivalent definition of regularity is to require that $\{N_j\} \cup \mathscr{W}^{\mathrm{s}}(N_{j+1})$ is closed, $j \geqq 0$.
(3) If the stable (or unstable) sets of the nodes are all one-dimensional, then Σ is regular.

A characteristic feature of a heteroclinic cycle is that the asymptotic dynamics of points within the cycle is supported on the nodes of the cycle. That is, for all $x \in \Sigma$, we have

$$\omega(x) \cup \alpha(x) \subset \cup_i N_i.$$

In the literature, there have been several attempts to extend the concept of a heteroclinic cyclic to allow for more complicated connections between equilibria or other invariant sets (see, for example, [31], [16, §15], [20], [4]). The resulting constructions are typically referred to as 'heteroclinic networks' or 'heteroclinic complexes'.

In the next section, we present a definition of a heteroclinic network that generalizes these definitions. Roughly speaking, we require that asymptotic dynamics on the network is supported on the nodes of the network. The nodes may, for example, consist of equilibria or more generally compact topologically transitive sets. Our definition includes the examples of 'cycling chaos' found in [20, 12, 1] but not the 'Shilnikov' network discussed in [20, Appendix]. It also includes the type of cycling chaos discussed in [5] where there are connections to fixed points contained *within* chaotic attractors. Although the dynamics on the cycle is relatively simple, we emphasize that the dynamics in a neighborhood of an *embedded* cycle is typically rich and complex.

2.3. Intrinsic Definition of a Heteroclinic Network

We define

$$C(\Sigma) = \Sigma \setminus R(\Sigma).$$

In the sequel, we refer to $C(\Sigma)$ as the set of *connections* of Σ.

Lemma 2.17. *If X is a compact invariant subset of Σ, then $X \cap R(\Sigma) \neq \emptyset$.*

Proof. Since $R(X) = \cup_{x\in X}\alpha(x)\cap\omega(x) \subset \cup_{x\in\Sigma}\alpha(x)\cap\omega(x) = R(\Sigma)$, it follows that $R(X) \subset R(\Sigma)\cap X$. Since X is compact and invariant, $R(X) \neq \emptyset$ (see [30, Chapter 3]). □

Definition 2.18. We say that (Σ, ϕ) has a *finite nodal set* if we can write $R(\Sigma)$ as a finite union of disjoint, compact, connected flow-invariant subsets. The set $\mathcal{N}$ of such subsets is referred to as the *nodal set* of (Σ, ϕ). Elements of $\mathcal{N}$ are referred to as the *nodes*.

Remarks 2.19. (1) If (Σ, ϕ) admits a finite nodal set, then necessarily $R(\Sigma)$ is *closed* and $C(\Sigma)$ is open. Moreover, if it is finite then the nodal set is unique (up to re-ordering).
(2) Since a node is connected and is a union of recurrent sets, it follows from Lemma 2.9 that nodes are indecomposable.

Examples 2.20. (1) Suppose that Σ is a heteroclinic cycle between the equilibria $p_0, \dots, p_k = p_0$. Then Σ has finite nodal set $\mathcal{N} = \{p_0, \dots, p_{k-1}\}$.
(2) Suppose that Γ is a non-finite connected compact Lie group acting continuously on Σ and ϕ is Γ-equivariant. Suppose that (Σ, ϕ) has nodal set $\mathcal{N} = \{N_i \mid 0 \leqq i \leqq k-1\}$. It follows by the Γ-equivariance of ϕ and the connectedness of Γ that each N_i is Γ-invariant. Let $\tilde{\phi}_t$ denote the flow induced by ϕ_t on the orbit space $\tilde{\Sigma} = \Sigma/\Gamma$. Then $\tilde{\mathcal{N}} = \{N_i/\Gamma \mid 0 \leqq i \leqq k-1\}$ is a finite nodal set for $(\tilde{\Sigma}, \tilde{\phi})$. In this context, it is natural to assume that each orbit space N_i/Γ is recurrent for $\tilde{\phi}$. This would be the situation, for example, if the sets N_i were relative equilibria. In particular, Σ is a heteroclinic cycle between relative equilibria $N_0, \dots, N_k = N_0$ if and only if $\tilde{\Sigma}$ is a heteroclinic cycle between equilibria $N_0/\Gamma, \dots, N_k/\Gamma = N_0/\Gamma$.

Suppose that X is a compact subset of Σ. Define

$$\lambda_+(X) = \overline{\cup_{x\in X}\omega(x)},$$
$$\lambda_-(X) = \overline{\cup_{x\in X}\alpha(x)},$$
$$\lambda^1(X) = \lambda(X) = \lambda_-(X) \cup \lambda_+(X).$$

Remark 2.21. If X is closed but not finite, neither $\cup_{x\in X}\omega(x)$ nor $\cup_{x\in X}\alpha(x)$ need be closed.

For $n > 1$ we define $\lambda^n(X) = \lambda(\lambda^{n-1}(X)))$ inductively. Taking $X = \Sigma$, we have the sequence of inclusions

$$\Sigma = \lambda^0(\Sigma) \supseteq \lambda^1(\Sigma) \supseteq \cdots \supseteq \lambda^n(\Sigma) \supseteq \cdots.$$

Set $\Sigma_n = \lambda^n(\Sigma)$, $n \geqq 0$. We call $\{\Sigma_0, \Sigma_1, \dots\}$ the *asymptotic filtration* of (Σ, ϕ). Obviously, $\Sigma_n \supset R(\Sigma)$, $n \geqq 0$. Moreover, since Σ_n is a compact flow-invariant set, each connected component of Σ_n has non-empty intersection with $R(\Sigma)$.

Definition 2.22. We say that (Σ, ϕ) has *depth N* if

(a) $\Sigma_N = R(\Sigma)$.
(b) $\Sigma_n \supsetneq R(\Sigma), n < N$.

Remarks 2.23. (1) If depth$(\Sigma) = N$, then $\Sigma_n = R(\Sigma)$, $n \geqq N$. In this case, we regard $\{\Sigma_0, \dots, \Sigma_N\}$ as the asymptotic filtration.
(2) depth$(\Sigma) = 0$ if and only if $R(\Sigma) = \Sigma$.

Example 2.24. Let Σ be a heteroclinic cycle connecting equilibria $p_0, \dots, p_k = p_0$. Then depth$(\Sigma) = 1$ and the asymptotic filtration of Σ is given by $\Sigma_0 = \Sigma$, i.e., the whole cycle, and $\Sigma_1 = \{p_0, \dots, p_{k-1}\}$, the set of equilibria. For example, if Σ is the Guckenheimer-Holmes cycle, then Σ_1 consists of three equilibria.

Lemma 2.25. *Suppose that* depth$(\Sigma) = N$ *and that X is a compact connected invariant subset of Σ. There exists a unique n, $0 \leqq n \leqq N$, and a connected component Σ_n^0 of Σ_n such that*

(a) $X \subset \Sigma_n^0$.
(b) $X \not\subset \Sigma_n \setminus \Sigma_{n+1}$.

Proof. Obviously there exists a largest $n \leqq N$ such that $X \subset \Sigma_n$. Since X is connected, X is a subset of a unique connected component Σ_n^0 of Σ_n. If $X \subset \Sigma_n$, then $\lambda(X) \subset \Sigma_{n+1}$. Since $X \supset \lambda(X)$, it follows that $X \not\subset \Sigma_n \setminus \Sigma_{n+1}$. □

Definition 2.26. We say that (Σ, ϕ) is a *heteroclinic network* if

(a) Σ is indecomposable.
(b) Σ has a finite nodal set.
(c) Σ has finite depth.

Remarks 2.27. (1) If depth$(\Sigma) > 0$, or equivalently if the set of connections $C(\Sigma)$ is non-empty, we say that the heteroclinic network is *non-trivial*; otherwise we say the network is *trivial*.
(2) Since each connected component of Σ_n contains at least one node, it follows that Σ_n has finitely many connected components, $n > 0$.

Definition 2.28. If Σ is a heteroclinic network and $R(\Sigma)$ is a finite set of equilibria, then we say Σ is a *heteroclinic network* between equilibria in $R(\Sigma)$.

Lemma 2.29. *Suppose that Σ is a heteroclinic network between finitely many equilibria and that* depth$(\Sigma) = 1$. *Then Σ is a (possibly infinite) union of heteroclinic cycles.*

Proof. We are given that depth$(\Sigma) = 1$ and $R(\Sigma)$ is a finite set of equilibria. Hence, if $x \in C(\Sigma)$, we can find $p, q \in R(\Sigma)$ such that $\alpha(x) = p$, $\omega(x) = q$. Define X_1 to consist of the union of $\mathscr{W}^s(p)$ and those equilibria which are the α-limit points of trajectories in $\mathscr{W}^s(p)$. Iterating in the obvious way, we obtain an increasing sequence of subsets (X_n) of Σ. Since there are only finitely many equilibria, it follows that there exists $N \geqq 1$ such that $X_n = X_N$, $n \geqq N$. If

$X_N = \Sigma$, we are done since then $q \in X_N$ and so there is a sequence of connections of distinct equilibria joining q to p. The required cycle is obtained by adding the connection from p to q. We assert that X_N is a closed connected flow-invariant subset of Σ. Under this assertion, it follows that $X_N = \Sigma$, since if $X_N \neq \Sigma$, then Σ cannot be indecomposable (X_N is a repellor). To prove the assertion, let $z \in \overline{X}_N \setminus R(\Sigma)$. Necessarily, the trajectory through z must also lie in the closure of X_N. But this implies that $\lambda(z) \subset X_N$ (otherwise depth$(\Sigma) > 1$). Hence $z \in X_N$. □

Since a heteroclinic network has an associated asymptotic filtration, it is natural to ask whether each component of the filtration has the structure of a heteroclinic network.

Theorem 2.30. *Let (Σ, ϕ) be a heteroclinic network of depth N. For $N \geqq n > 0$, Σ_n is a finite union of heteroclinic networks each with depth less than or equal to $N - n$.*

Proof. We use induction on N. If $N = 1$, the result is trivial since $\lambda(\Sigma)$ is a finite union of trivial heteroclinic networks. Suppose that the result is proved for $N-1$ and that depth$(\Sigma) = N$. Recall that $\lambda(\Sigma) \supseteq R(\Sigma)$ and that each connected component of $\lambda(\Sigma)$ intersects $R(\Sigma)$. Since $R(\Sigma)$ has finitely many connected components, we can conclude that $\lambda(\Sigma)$ has finitely many connected components. Therefore we restrict our attention to one connected component Λ of $\lambda(\Sigma)$. Observe that if a connected component of $R(\Sigma)$ intersects Λ, then it is contained within Λ and so $R(\Lambda)$ has a nodal set that is a union of connected components of $R(\Sigma)$. Moreover, depth$(\Lambda) < N$, and so it only remains to prove that Λ is indecomposable.

If $x \in \Lambda$, there exist sequences $\{x_n\} \subset \Lambda$ and $\{y_n\} \subset \Sigma$ such that $x_n \to x$ and $x_n \in \lambda(y_n)$, $n \geqq 0$. Without loss of generality we assume that $x_n \in \omega(y_n)$, $n \geqq 0$. Since $\omega(y_n)$ is connected, it follows that $\omega(y_n) \subset \Lambda$, $n \geqq 0$. But $\omega(y_n)$ is indecomposable and so $x_n \sim x_n$ on $\omega(y_n)$ for all $n \geqq 0$. Hence $x_n \sim x_n$ on Λ and so, letting $n \to \infty$, $x \sim x$. □

Remark 2.31. Obviously, there are other possible ways of defining the 'asymptotic filtration'. For example, we could define the filtration by taking Σ_i to be the non-wandering set of $\phi|_{\Sigma_{i-1}}$. This gives a nested sequence $\Sigma_0 \supseteq \Sigma_1 \supseteq \cdots$ that in the transfinite limit defines the *Birkhoff center* of the system (see [30, pp. 129–130]). If we assume finite depth, the Birkhoff center corresponds to the nodal set. However, it is possible to construct examples for which Σ_i is not indecomposable and so Theorem 2.30 fails (for example, see [11, §6.4]).

Definition 2.32. We say that a heteroclinic network of depth N is *regular* if all connected components in Σ_n, $0 \leqq n < N$, which are of depth 1 are finite unions of regular heteroclinic cycles.

Remark 2.33. In the case of a regular heteroclinic cycle, one can represent the dynamics as a directed graph on vertices that represent the equilibria. For heteroclinic networks with depth greater than 1, this is not possible since there is an $x \in \Sigma$ whose limits are not contained within $R(\Sigma)$. For networks with depth 1, there is a

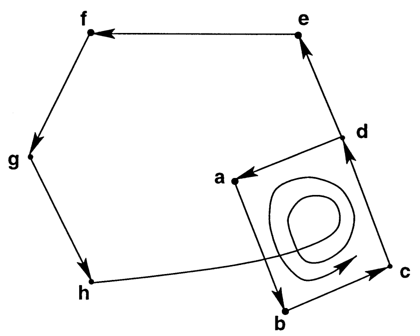


Fig. 2. The Chawanya heteroclinic network; a network with depth equal to 2.

graphical representation of the network, although this can be misleading because in principle the set of connections between one pair of equilibria may accumulate on connections between totally unrelated equilibria. This problem is related to our definition of regularity for heteroclinic cycles. See also an example of CHOSSAT, GUYARD & LAUTERBACH [9].

Example 2.34. An example due to CHAWANYA [7, 8] of a regular (and robust) heteroclinic network of depth 2 is shown in Figure 2. This network contains a connection between an equilibrium and a heteroclinic cycle. More precisely, if we let Σ' denote the heteroclinic cycle $a \to b \to c \to d \to a$, then $\omega(y) = \Sigma'$ for all points $y \neq h$ with $\alpha(y) = \{h\}$. Since the ω-limit set of any point in Σ' lies in $\{a, b, c, d\}$, it follows that the network has depth 2. The associated asymptotic filtration is given by $\Sigma_0 = \Sigma$, $\Sigma_1 = \Sigma' \cup \{e, f, g, h\}$, $\Sigma_2 = \{a, b, \ldots, h\}$. We remark that in HOMBURG [29] there is a (non-robust) example of a network where there is a connection from an equilibrium to a heteroclinic cycle.

In the next example we show how to construct a (regular) network of arbitrary depth.

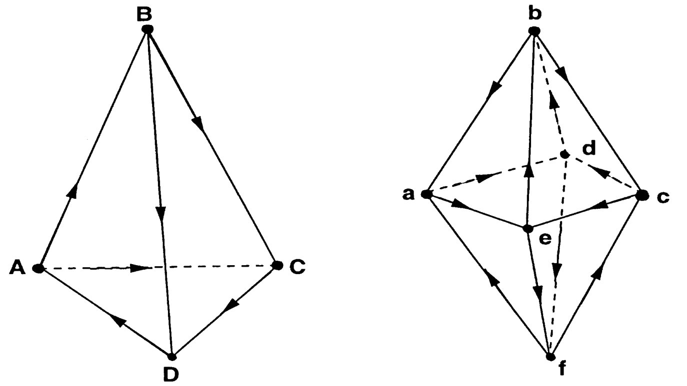
Example 2.35. Let $N \geqq 1$. We construct a smooth flow ϕ_t on the torus $\mathbb{T}^N = \mathbb{R}^N/(2\pi\mathbb{Z})^N$ such that $(\mathbb{T}^N, \phi)$ is a heteroclinic network of depth N. Define ϕ_t to be the flow of the system

$$\dot{\theta}_j = (1 - \cos\theta_j)^2 + \alpha(1 - \cos\theta_{j+1}), \quad 1 \leqq j < N,$$
$$\dot{\theta}_N = (1 - \cos\theta_N)^2,$$

where $\alpha > 0$ is a constant. Regard $\mathbb{T}^j$ as embedded in $\mathbb{T}^{j+1}$ by the inclusion $(\theta_1, \ldots, \theta_j) \mapsto (\theta_1, \ldots, \theta_j, 0)$, and $\mathbb{T}^0 \subset \mathbb{T}^N$ as the point $\theta_1 = \ldots = \theta_N = 0$.

We assert that if $x^\star \in \mathbb{T}^N \setminus \mathbb{T}^{N-1}$, then the closure of the trajectory through $x^\star$ has depth N and $\lambda(x^\star) \subset \mathbb{T}^{N-1}$ has depth $N - 1$. Given this assertion, the result follows easily.

The fact that for any $x^\star \in \mathbb{T}^N \setminus \mathbb{T}^{N-1}$ we have $[\phi_t(x^\star)]_N \to 0$ as $t \to \infty$ and $[\phi_t(x^\star)]_{N-1}$ winds arbitrarily many times around $[0, 2\pi]$ as long as $\theta_N(t) \neq 0$ proves the assertion for $N = 2$. Suppose the result is proved for all $2 \leqq n \leqq N - 1$



(a) Kirk–Silber network (b) Guckenheimer–Holmes network

Fig. 3. Examples of heteroclinic networks.

and pick any $x^\star \in \mathbb{T}^N \setminus \mathbb{T}^{N-1}$. It follows from the equation for θ'_N, that $\theta_N(t) \to 0$ as $t \to \pm\infty$. Hence $\lambda(x^\star) \subset \mathbb{T}^{N-1}$. Since $\theta_{N-1}(t)$ visits all values infinitely often, given any $0 < \xi < 2\pi$ there must be (by compactness) an accumulation point of the trajectory having $\theta_{N-1} = \xi$. Therefore there exists a $y^\star \in \lambda(x^\star) \cap (\mathbb{T}^{N-1} \setminus \mathbb{T}^{N-2})$ and so $\overline{\phi_{y^\star}(\mathbb{R})} \subset \lambda(x^\star)$. By the inductive hypothesis, $\text{depth}(\overline{\phi_{y^\star}(\mathbb{R})}) = N - 1$. Since $\overline{\phi_{x^\star}(\mathbb{R})}$ is indecomposable, it follows that $\text{depth}(\overline{\phi_{x^\star}(\mathbb{R})}) = N$.

Heteroclinic networks often occur robustly in systems with symmetry and we give some simple examples below of heteroclinic networks that occur in equivariant dynamics. All of these examples have depth 1.

Examples 2.36. In Figure 3(a), we show the one-dimensional heteroclinic network studied by KIRK & SILBER [31]. The network contains four equilibria A, B, C, D and three heteroclinic cycles: $A \to B \to D \to A$, $A \to C \to D \to A$ and $A \to B \to C \to D \to A$.

In Figure 3(b) we show the $\Delta_3 \rtimes \mathbb{Z}_3$-orbit of the GUCKENHEIMER & HOLMES heteroclinic cycle. Note that the network contains more cycles than just the translates of the original cycle by elements of $\Delta_3 \rtimes \mathbb{Z}_3$. For example, $a \to e \to b \to c \to d \to f \to a$ is a heteroclinic cycle.

In Figure 4 we show an example of an irregular two-dimensional heteroclinic network. This example occurs in a $\Delta_5 \rtimes \mathbb{Z}_5$-equivariant vector field on $\mathbb{R}^5$ [16, §15]. If we let $\mathbb{Z}_5$ act by mapping a to b, b to c, etc., then the network is the $\mathbb{Z}_5$-image of the triangle $\triangle abd$ (shaded in the figure). Just as in the previous examples, the ω-limit of any point in the network is one of the equilibria $a, \ldots, e$ and so the network has depth one.

We conclude this section with a technical lemma that gives an effective and simple way of proving the indecomposability of a heteroclinic network. We make use of this result in Section 5.

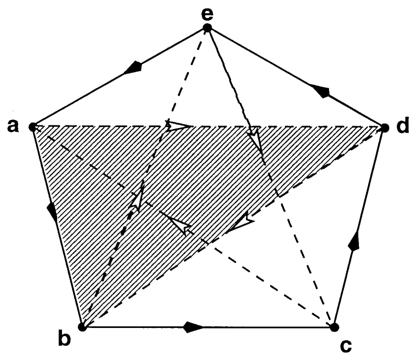


Fig. 4. A two-dimensional, depth-1 network.

Lemma 2.37. *Suppose that Φ is a continuous flow on a compact connected metric space Σ such that $\lambda(\Sigma) \subset \Sigma'$, where Σ' is compact and indecomposable. Then Σ is indecomposable.*

Proof. Suppose $x \in \Sigma$. Either $x \in \Sigma'$, and so x is chain recurrent, or $x \notin \Sigma'$. In the latter case, since $\lambda(\Sigma) \subset \Sigma'$ we pick any two points $y \in \omega(x)$ and $z \in \alpha(x)$. Given any $\varepsilon, T > 0$, we can find (ε, T)-pseudo orbits from z to x and from x to y and so $z \sim x$ and $x \sim y$. Since Σ' is indecomposable, we have $y \sim z$ and so, by transitivity, $x \sim x$. □

3. Dynamics and Asymptotics Near Embedded Heteroclinic Networks

Although the dynamics on a heteroclinic network are relatively simple to quantify, the dynamics that can occur in a neighborhood of an embedded network can be very subtle and complex. For example, the Chawanya network arose out of a study of a Lotka-Volterra type cubic five-dimensional system of differential equations restricted to a four-dimensional hyperplane. Careful numerical investigations by Chawanya indicate that there can be parameter values for which there are infinitely many attractors in a neighborhood of the cycle [7, 8].

For the remainder of this section, we suppose that F is a smooth vector field on $\mathbb{R}^n$ with flow Φ_t. Given $x, y \in \mathbb{R}^n$, let $d(x, y) = \|x - y\|$ denote Euclidean distance, and let $d(x, X) = \inf_{y \in X} d(x, y)$. If X is a compact Φ-invariant subset of $\mathbb{R}^n$, we define the stable and unstable sets of X by

$$W^u(X) = \{y \in \mathbb{R}^n \mid \lim_{t \to -\infty} d(\Phi_t(y), X) = 0\},$$
$$W^s(X) = \{y \in \mathbb{R}^n \mid \lim_{t \to \infty} d(\Phi_t(y), X) = 0\}.$$

If X has *hyperbolic structure*, then $W^{\mathrm{s}}(X)$, $W^{\mathrm{u}}(X)$ are fibered by smooth manifolds (see [30, Chapter 6]). In particular, if X is a hyperbolic equilibrium or limit cycle, then $W^{\mathrm{s}}(X)$ and $W^{\mathrm{u}}(X)$ are smoothly immersed manifolds. If X is not hyperbolic, then $W^{\mathrm{s}}(X)$, $W^{\mathrm{u}}(X)$ typically do not have smooth structure.

On occasions, we refer to $W^s(X)$ as the *basin of attraction* of X. We say that X is a (Milnor) *attractor* [38] if

$$\ell(W^s(X)) > 0,$$

where $\ell(\cdot)$ is Lebesgue measure on $\mathbb{R}^n$.

Now suppose that Σ is a compact Φ-invariant subset of $\mathbb{R}^n$ and set $\Phi_t|_\Sigma = \phi_t$. For the remainder of this section, we assume that (Σ, ϕ) is a heteroclinic network with node set $\mathcal{N} = \{N_1, \ldots, N_k\}$.

3.1. Recurrence and Attraction Near Embedded Networks

Although Σ is indecomposable (in particular, chain recurrent), Σ is not recurrent unless $\mathrm{depth}(\Sigma) = 0$. However, if Σ is embedded in $\mathbb{R}^n$, it is natural to ask about the ω-limit sets of points in the basin of attraction of Σ. In particular, under what circumstances can we find points $x \in W^s(\Sigma)$ such that $\omega(x) \supseteq R(\Sigma)$? We remark that the problem of characterizing those (Σ, ϕ) which are representable as ω-limit sets has been considered by BOWEN [6] in the context of discrete continuous dynamical systems defined on (possibly disconnected) spaces.

Example 3.1. The Guckenheimer-Holmes network Σ arises as the heteroclinic network of a $\Delta_3 \rtimes \mathbb{Z}_3$-equivariant cubic vector field F on $\mathbb{R}^3$. We recall that F depends on three real parameters (a, b, c). It may be shown that there is a non-empty open set Π of parameters for which we may represent Σ as a subset of a globally attracting flow-invariant 2-sphere $S \subset \mathbb{R}^3$. Moreover, we may choose Π so that Σ is an asymptotically stable attractor (we refer to [20, §6.2] for details). Henceforth, assume $(a, b, c) \in \Pi$. We may represent the Guckenheimer-Holmes heteroclinic *cycle* Σ_0 as the intersection of Σ with the positive octant of S. Since the coordinate planes $x_i = 0$ are flow-invariant subspaces, it follows that if $x = (x_0, x_1, x_2) \in \mathbb{R}^3$, then $\omega(x)$ is a subset of any octant containing x. Further, provided that $|x_0|$, $|x_1|$ and $|x_2|$ are not all equal and are non-zero, the octant is unique and there exists a unique $\gamma \in \Delta_3$ such that $\omega(x) = \gamma\Sigma_0$. Hence, although the Guckenheimer-Holmes network is chain recurrent, nearby trajectories do not visit all of the nodes, even if the network is asymptotically stable. Of course, if we ignore the symmetry, it is an easy exercise to embed (Σ, ϕ) in a flow $\tilde{\Phi}$ on $\mathbb{R}^3$ so that Σ is an asymptotically stable attractor for $\tilde{\Phi}$ and there exist points $x \in \mathbb{R}^3$ such that $\omega(x) = \Sigma$. Indeed, we may require that $\tilde{\Phi}$ be equal to the original equivariant flow near $R(\Sigma)$.

We now give a number of conditions on embedded heteroclinic networks that strengthen the transitivity condition and avoid the pathology described in the previous example. Note however that the presence of codimension-one invariant subspaces may force the previous behavior to be typical.

Condition A1. There exists an $x \in \mathbb{R}^n$ such that $\omega(x) = \Sigma$.

Condition A2. There exists a positive Lebesgue measure set of $x \in \mathbb{R}^n$ such that $\omega(x) = \Sigma$.

Condition A3. There exists an open set of $x \in \mathbb{R}^n$ such that $\omega(x) = \Sigma$.

Conditions B1–B3. The same as conditions (A1)–(A3) except that $\omega(x) \supseteq R(\Sigma)$.

Conditions C1–C3. The same as conditions (A1)–(A3) except that $\omega(x) \cap N \neq \emptyset$ for each $N \in \mathcal{N}$.

We have the following diagram of implications between the conditions.

$$\begin{array}{ccccc} A3 & \Rightarrow & A2 & \Rightarrow & A1 \\ \Downarrow & & \Downarrow & & \Downarrow \\ B3 & \Rightarrow & B2 & \Rightarrow & B1 \\ \Downarrow & & \Downarrow & & \Downarrow \\ C3 & \Rightarrow & C2 & \Rightarrow & C1 \end{array}$$

In particular, every condition implies (C1). If Σ is a one-dimensional heteroclinic cycle, then conditions (An) and (Bn) are equivalent. For higher-dimensional cycles, these conditions are not equivalent. For example, ASHWIN & CHOSSAT [4] show there are cycles where (B3) holds but (A3) fails. An example of ASHWIN & RUCKLIDGE [5] shows that the (Cn) conditions generally do not give much information about dynamics near the network.

For a given embedding of a heteroclinic network, it is generally nontrivial to verify condition (B1) (or even (C1)). Nevertheless, we conjecture that in the absence of codimension-one invariant subspaces for the flow, it is possible to satisfy condition (B2) generically. (Note that if the network has at least two nodes, we can never satisfy any of the conditions with trajectories through points of Σ.)

Remarks 3.2. (1) Condition (B1) is the weakest condition we can envisage placing on an embedded heteroclinic network that excludes the pathology described in Example 3.1. Thus, the only subnetworks of the Guckenheimer-Holmes network that satisfy (B1) are the group translates of the Guckenheimer-Holmes cycle.
(2) In practice, one would hope that an embedded 'attracting' network satisfies condition (B2).
(3) Conditions (B2) and (B3) suggest the possibility of defining a symbolic dynamics associated to the cycle. Symbol sequences would be defined in terms of visitation of nodes.

Examples 3.3. (1) An illustrative example is provided by the 4-dimensional system studied by GUCKENHEIMER & WORFOLK [25] (see also WORFOLK [43]). GUCKENHEIMER & WORFOLK study a cubic vector field on $\mathbb{R}^4$ with symmetry group Γ equal to the determinant one subgroup of $\Delta_4 \rtimes \mathbb{Z}_4$. Just as for the Guckenheimer-Holmes network, there is an open region of parameter space for which this system has an asymptotically stable one-dimensional heteroclinic network Σ and this network may be represented as a Γ-invariant subset of the 3-sphere in $\mathbb{R}^4$. Let Σ_0 denote the intersection of Σ with the positive sector $\mathbb{R}^4_+ \subset \mathbb{R}^4$. Unlike the Guckenheimer-Holmes network the coordinate hyperplanes $x_i = 0$ are not flow-invariant and so there is the possibility of trajectories which are asymptotic to Σ leaving the $\mathbb{R}^4_+$ and having ω-limit equal to Σ. In this case, trajectories twist around the the one-dimensional connections and repeatedly visit all the 'octants' of $\mathbb{R}^4$. While this phenomenon has not yet been verified for the cubic form used by GUCKENHEIMER

& WORFOLK, it can be shown that condition (A2) is satisfied for generic vector fields near the cubic normal form.
(2) The example of KIRK & SILBER [31], shown in Figure 3(a), gives an example where none of the conditions (A1)–(B3) is satisfied. This is because of the presence of codimension-one invariant subspaces. KIRK & SILBER show that points are attracted to one of two possible subcycles.

3.2. Averaged Behavior Near Embedded Networks

Suppose that Σ is a heteroclinic network embedded in $\mathbb{R}^n$ and $\omega(x) \subset \Sigma$ for some $x \in \mathbb{R}^n$. If $\mathrm{depth}(\Sigma) > 0$, this will have consequences on the behavior of averages along the trajectory of x; in particular, any intersection of $\omega(x)$ with the nodes causes 'slowing-down' behavior typical of heteroclinic orbits. We now state this precisely.

For any x with $\omega(x) \subseteq \Sigma$, define $\Omega(x)$ to be the set of accumulation points of the set of measures

$$\frac{1}{T}\int_{t=0}^{T} \delta_{\Phi_t(x)}\, dt, \quad (T \in \mathbb{R}),$$

as $T \to \infty$ in the weak dual topology on the space of continuous functions on Σ.

Measures in $\Omega(x)$ are Φ_t-invariant and so we may define the *essential ω-limit set* of x (see also [42, 2])

$$\omega_{\mathrm{ess}}(x) = \overline{\cup_{\mu \in \Omega(x)} \mathrm{supp}(\mu)}.$$

The compactness of Σ implies that the set of invariant probability measures is also compact and so $\omega_{\mathrm{ess}}(x) \neq \emptyset$. Obviously, $\omega(x) \supseteq \omega_{\mathrm{ess}}(x)$ and $\omega_{\mathrm{ess}}(x)$ is flow invariant.

Remark 3.4. The set $\omega_{\mathrm{ess}}(x)$ can be thought of as the set of all limit points that contribute asymptotically to averages of observables along the trajectory of x.

Our main result about $\omega_{\mathrm{ess}}(x)$ is that it is disjoint from the set $C(\Sigma)$ of connections. This follows directly from [30, Theorem 4.1.18(1)]

Theorem 3.5. *Let (Σ, ϕ) be a heteroclinic network embedded in $\mathbb{R}^n$. Suppose that $x \in \mathbb{R}^n$ and that $\omega(x) \subseteq \Sigma$. Then*

$$\omega_{\mathrm{ess}}(x) \subseteq R(\Sigma).$$

Remark 3.6. (1) For particular examples one can say more. In particular, if $\omega_{\mathrm{ess}}(x)$ is not a single ergodic measure, then the trajectory through x behaves non-ergodically. That is, averages of observables along the orbit do not converge but subsequences can be found that converge to a continuum of values. This has been shown for specific heteroclinic cycles [21, 42].
(2) It follows from [30, Theorem 4.1.18] that $R(\Sigma)$ supports *all* flow-invariant probability measures on Σ.

It is possible for a sub-network of an embedded heteroclinic network to be a Milnor attractor [38]. In this case, (A2) can hold for the sub-network but (A3) cannot. This can occur, for example, if the sub-network is essentially asymptotically stable in the terminology of MELBOURNE [37]. Numerical results of CHAWANYA [7, 8] suggest that close to such invariant sets, there may be a countable infinity of stable periodic orbits accumulating on the set.

4. Regularity and Stability of Embedded Heteroclinic Networks

It is well-known that heteroclinic cycles can occur robustly in systems that possess families of invariant subspaces. We recall that there are several natural classes of dynamical system of this type. Differential equation models for population dynamics often have invariant subspaces (extinction is a conserved quantity) and we refer to the article by HOFBAUER [27] and book by HOFBAUER & SIGMUND [28] for examples of robust heteroclinic cycles in population models. Another important and widely studied class of examples are symmetrically coupled systems of identical cells or oscillators (see, for example, [13, 20]). More generally, if Γ is a Lie group, (V, Γ) is a Γ-representation, and $H \subset \Gamma$, then the fixed point space $V^H = \{v \in V \mid H(v) = \{v\}\}$ is an invariant linear subspace for all Γ-equivariant vector fields on V. If Γ is *finite* or *compact Abelian*, then V has finitely many fixed-point subspaces. If Γ is compact non-Abelian, then V typically has infinitely many invariant subspaces (for example, take the standard action of SO(3) on $\mathbb{R}^3$).

In this section, we restrict our attention to Γ-equivariant vector fields on a finite dimensional representation (V, Γ), Γ finite. We do this mainly to simplify our presentation. All our results and definitions extend straightforwardly to population models defined on $\mathbb{R}^n_+$ or indeed to any class of vector fields which preserve a sufficiently regular filtration of the phase space.

The first stability result on heteroclinic cycles was obtained by DOS REIS [40]. He proved a result characterizing structural stability for equivariant vector fields on compact 2-manifolds. In particular, he showed that if certain conditions on eigenvalues hold, then cycles like the Guckenheimer-Holmes cycle are (locally) structurally stable. Unfortunately, it is unrealistic to expect that local structural stability holds, even generically, for heteroclinic networks of depth greater than 1, even if the nodes are structurally stable. In the examples below, we sketch two of the ways in which structural stability can fail in networks of depth greater than one.

Examples 4.1. (1) Let $\Sigma \subset \mathbb{R}^4$ denote the one-dimensional heteroclinic network considered by GUCKENHEIMER & WORFOLK [25]. This network has depth 1 and is the group orbit of the base cycle $\Sigma' = \mathbb{R}^4_+ \cap \Sigma$. For an open region of parameters, Σ is attracting and contained in an attracting flow-invariant 3-sphere $S \subset \mathbb{R}^4$. In this example there are no codimension-1 reflection planes and so trajectories typically twist around the one-dimensional connections between equilibria and appear to visit randomly all of the cycles in the group orbit of Σ'. Although the network is robust under equivariant perturbation, the flow is never locally structurally stable in a neighborhood of the network in S. We present a brief sketch of an argument

showing the failure of local structural stability. We associate to each forward trajectory asymptotic to Σ' a symbol sequence identifying the ordered sequence of edges in Σ' visited by the trajectory (we ignore the measure-zero set of trajectories which are asymptotic to one of the equilibria in the network). The symbol sequence of a trajectory is then a conjugacy invariant of the trajectory (we may assume the conjugacy is C^0-close to the identity map). By making arbitrarily small perturbations supported near a single edge we can change the order of symbol sequences and hence structural stability fails.
(2) It is possible to modify the Guckenheimer-Holmes cycle and obtain a cycle $\Sigma \subset \mathbb{R}^6$ between three limit cycles which has two-dimensional connections and each pair of cycles lying in a four-dimensional fixed point space. We may further require that Σ be attracting and that Σ be contained in an attracting flow-invariant 5-sphere S. In this case, the flow is not structurally stable in a neighborhood of $\Sigma \subset S$ because of the appearance of *moduli of stability* [39] such as ratios of eigenvalues of linearizations near fixed points. In fact the codimension of the C^0-equivalence class of the flow is infinite.

In spite of the limited prospects of obtaining satisfactory conjugacy or structural stability results for networks, or even higher-dimensional cycles, there are still good stability questions one can ask. In particular, we would like to have some stability in the asymptotics and symmetry properties of the network. This stability should be related to the structure of the invariant subspaces of our phase space. Our aim in this section is to formulate a verifiable concept of robustness for heteroclinic networks in a symmetric system.

4.1. Orbit Strata

Henceforth, we assume that Γ is a finite group and (V, Γ) is a finite-dimensional real Γ-representation.

If $J \subset \Gamma$, we let V^J denote the fixed point set of J acting on V. Obviously, V^J is a linear subspace of V and is equal to the fixed-point subspace of the subgroup of Γ generated by J. It is known (see below) that there is an open and dense subset U of V^J such that all points in U have the same isotropy, say H, and

$$V^J = V^H.$$

It follows that the set of invariant linear subspaces of V is parametrized by the set $\mathcal{T} = \mathcal{T}(V, \Gamma)$ of isotropy groups for the action of Γ on V. Let $V^{(H)}$ denote the set of points in V^H with isotropy group equal to H. Then

$$V^{(H)} = V^H \setminus \bigcup_{J \supsetneq H, J \in \mathcal{T}} V^J,$$

and so $V^{(H)}$ is open and dense in V^H.

Let $\langle H \rangle$ denote the conjugacy class of H in Γ and define

$$V^{\langle H \rangle} = \bigcup_{J \in \langle H \rangle} V^{(J)}.$$

The collection $\{V^{\langle H\rangle} \mid H \in \mathscr{T}(V,\Gamma)\}$ defines the *stratification of* V *by isotropy type* or the *orbit stratification of* V. We call connected components of $V^{\langle H\rangle}$ *orbit strata*. Note that $V^{\langle H\rangle}$ consists of at least $|\Gamma/H|$ orbit strata. We let $\mathscr{S}(V,\Gamma)$, or just $\mathscr{S}$, denote the set of all orbit strata. If X is a smooth Γ-equivariant vector field on V, then the orbit strata are all invariant by the flow of X.

Let S be a stratum of the orbit stratification and let ∂S denote the frontier of S. Using the linearity of the Γ-action, we can show that if $H \in \mathscr{T}$, then

$$\partial V^{\langle H\rangle} \cap V^{\langle J\rangle} \neq \emptyset \iff J \supsetneq H \ \text{ and } \partial V^{\langle H\rangle} \supset V^{\langle J\rangle}.$$

Example 4.2. The symmetry group associated to the Guckenheimer-Holmes cycle is the semi-direct product $\Delta_3 \rtimes \mathbb{Z}_3$, and $\Delta_3 \rtimes \mathbb{Z}_3$ acts linearly on $\mathbb{R}^3$. Define connected subsets of $\mathbb{R}^3$ by $V_0 = \{(0,0,0)\}$, $V_1 = \{(x,0,0) \mid x > 0\}$, $V_2 = \{(x,x,x) \mid x > 0\}$, $V_3 = \{(x,y,0) \mid x,y > 0\}$, $V_4 = \{(x,y,z) \mid x,y,z > 0\} \setminus V_2$. The orbit stratification of $\mathbb{R}^3$ is given as the union of the $\Delta_3 \rtimes \mathbb{Z}_3$-orbits of $V_0,\dots,V_4$. Thus, γV_j is a connected orbit stratum for all $\gamma \in \Delta_3 \rtimes \mathbb{Z}_3$. All points in $\Delta_3 \rtimes \mathbb{Z}_3(V_j)$ have the same isotropy type, and $x, y \in \mathbb{R}^3$ have the same isotropy type if and only if $x, y \in \Delta_3 \rtimes \mathbb{Z}_3(V_j)$ for some (unique) j.

4.2. Relating the Asymptotic Filtration to the Orbit Stratification

Let $\Sigma \subset V$ be a 'robust' heteroclinic network for a Γ-equivariant flow Φ on V and suppose that $\mathrm{depth}(\Sigma) = N$. Let $x \in \Sigma$ and suppose that x lies in the orbit stratum $S \in \mathscr{S}$. Provided that $x \notin R(\Sigma)$, it is often the case that $\lambda(x) \subset \partial S$. Indeed, the existence of a robust cycle between equilibria, implies that we have at least one non-transverse saddle connection between equilibria. The only way these can persist under equivariant perturbation is if at least some of the equilibria lie in the frontier of the orbit strata containing the connections.

We restrict our attention to networks where we have the strongest relation between the asymptotic filtration of the network to the orbit stratification of V. It should be possible to weaken our requirements to allow for cycles like those constructed by Matthews et al. [35] (see below).

Suppose that Σ has asymptotic filtration $\{\Sigma = \Sigma_0, \Sigma_1, \dots, \Sigma_N\}$. Given $j \in \{0,\dots,N\}$, we recall that Σ_j can be written (uniquely) as a finite union $\Sigma_j = \cup_{i=1}^{p(j)} \Sigma_{ij}$ of heteroclinic networks, each of depth less than or equal to $N - j$. Let $\mathscr{A}(\Sigma)$ denote the set of all subnetworks of Σ derived in this way from the asymptotic filtration. Suppose that $S \in \mathscr{A}$. Denote the asymptotic filtration of S by $\{S_0,\dots,S_M\}$, where $M = \mathrm{depth}(S)$. For $0 \leqq k \leqq M$, let $\rho_k(S)$ be the minimal union of orbit strata such that

$$S_k \setminus S_{k+1} \subset \rho_k(S).$$

Set $F(S) = (\rho_0(S),\dots,\rho_{k-1}(S))$. We call $F(S)$ the *orbit flag* of S.

Definition 4.3. Let Σ, Σ' be heteroclinic networks. We say that Σ, Σ' are *isomorphic* if there is a bijection $\iota : \mathscr{A}(\Sigma) \to \mathscr{A}(\Sigma')$ such that for all $S \in \mathscr{A}(\Sigma)$, $F(S) = F(\iota(S))$.

Definition 4.4. Let $S \in \mathscr{A}(\Sigma)$. We say that S is *symmetry adapted* if

$$S_{k+1} \subset \partial\rho_k(S),\ 0 \leqq k < \text{depth}(S).$$

If all the heteroclinic subnetworks in $\mathscr{A}(\Sigma)$ are symmetry adapted, we say that Σ is symmetry adapted.

Example 4.5. The Guckenheimer-Holmes network is symmetry adapted as indeed are all the edge cycles and networks described in [20, Chapter 6]. So also is the KIRK-SILBER network [31]. However, the robust heteroclinic cycle of MATTHEWS et al. [35] is not symmetry adapted as there are connections between equilibria within a fixed orbit stratum. In other words, their cycle includes connections that limit to equilibria with the same symmetry as the points on the connecting orbit.

4.3. Robustness of Networks

Definition 4.6. Let Σ be a symmetry adapted heteroclinic network for the Γ-equivariant vector field X. We say that Σ is *geometrically robust* if for every open Γ-invariant neighborhood U of Σ, we can find an open neighborhood $\mathscr{U}$ of X in the C^1-topology such every $Y \in \mathscr{U}$ has a heteroclinic network $\Sigma_Y \subset U$ such that Σ_Y is symmetry adapted and $\mathscr{A}(\Sigma)$ is isomorphic to $\mathscr{A}(\Sigma_Y)$.

Example 4.7. The Guckenheimer-Holmes network is geometrically robust as indeed are all the edge cycles and networks described in [20, Chapter 6]. So also is the KIRK-SILBER network [31].

5. A Coupled Cell System

We think of a cell as being a low-dimensional ordinary differential equation that can be coupled to other cells. In this way we can build up a higher-dimensional dynamical system with desired symmetries that can have specifiable properties such as a heteroclinic network.

In particular, we consider a coupled-cell network consisting of nine cells, each with one degree of freedom, coupled directionally such that the network has global $\mathbb{Z}_3 \times \mathbb{Z}_3$ symmetry. To simplify notation, we write $\mathbb{Z}_3 \times \mathbb{Z}_3 = \mathbb{Z}(3,3)$. We shall assume that each of the cells has an independent internal $\mathbb{Z}_2$ symmetry. We may think of this system as a coupling of three Guckenheimer-Holmes models [23] in a ring. The symmetry group Γ of the system is $\Delta_9 \rtimes \mathbb{Z}(3,3)$ or, equivalently, the wreath-product $\mathbb{Z}_2 \wr \mathbb{Z}(3,3)$ [13]. Some of the dynamical and bifurcation theoretic consequences of wreath-product symmetries are discussed in [13] (see also the works [16, 20] which treat similar groups in semi-direct rather than wreath-product notation).

The phase space we work with is $\mathbb{R}^9$. We regard $\mathbb{R}^9$ as $(\mathbb{R}^3)^3$ and denote points in $\mathbb{R}^9$ as 3-tuples $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$, where $\boldsymbol{x} = (x_0, x_1, x_2)$, $\boldsymbol{y} = (y_0, y_1, y_2)$ and $\boldsymbol{z} = (z_0, z_1, z_2)$.

Let σ be a generator of $\mathbb{Z}_3$ and define an action of $\mathbb{Z}_3$ on $\mathbb{R}^3$ by $\sigma(x, y, z) = (y, z, x)$. We let $\kappa : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $\kappa(x, y, z) = (-x, y, z)$.

The action of Γ on $\mathbb{R}^9$ is generated by

$$\rho_1(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = (\sigma \boldsymbol{x}, \sigma \boldsymbol{y}, \sigma \boldsymbol{z}),$$
$$\rho_2(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = (\boldsymbol{y}, \boldsymbol{z}, \boldsymbol{x}),$$
$$\kappa_x^0(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = (\kappa \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}).$$

Note that

$$\mathbb{Z}_3 \times \mathbb{Z}_3 = \langle \rho_1, \rho_2 \rangle,$$

and that $\mathbb{Z}_3 \times \mathbb{Z}_3$ is a transitive subgroup of S_9. Consequently, in order to specify a Γ-equivariant differential equation on $\mathbb{R}^9$ it suffices to write down one component of the equation (see [16]).

In the sequel we use the following notational conventions. Let $\mathbf{A} = (a_0, a_1, a_2)$, $\mathbf{B} = (b_0, b_1, b_2)$, ... denote general points of $\mathbb{R}^3$. We let $\mathbf{A}_0$ denote a point $(0, a_1, a_2)$ of $\mathbb{R}^3$ with first coordinate zero. If $\boldsymbol{x} \in \mathbb{R}^3$, $\mathbf{A} \in \mathbb{R}^3$, we define

$$\mathbf{A}(\boldsymbol{x}^2) = a_0 x_0^2 + a_1 x_1^2 + a_2 x_2^2.$$

If $\mathbf{X} = (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \in \mathbb{R}^9$, we define $\|\mathbf{X}\|^2 = \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 + \|\boldsymbol{z}\|^2$, where $\| \ \ \|$ denotes the Euclidean norm.

We consider two model Γ-equivariant vector fields whose dynamics are uniquely determined by their first component:

(5) $\dot{x}_0 = x_0(1 - \|\mathbf{X}\|^2 + \mathbf{A}_0(\boldsymbol{x}^2) + \mathbf{B}(\boldsymbol{y}^2) + \mathbf{C}(\boldsymbol{z}^2)),$

(6) $\dot{x}_0 = x_0(1 - \|\mathbf{X}\|^2 + \mathbf{A}_0(\boldsymbol{x}^2) + \mathbf{B}(\boldsymbol{y}^2) + \mathbf{C}(\boldsymbol{z}^2)) + x_0(d x_1^2 x_2^2 + e y_0^2 z_0^2).$

Both vector fields consist of a general Γ-equivariant cubic polynomial. In (6), fifth-order terms with (small) coefficients d and e have been added to the second vector field in order to break a degeneracy of the third-order system; see [1]. The coefficient of the radial term $x_0\|\mathbf{X}\|^2$ is chosen to be -1 so that if $\|\mathbf{A}_0\|$, $\|\mathbf{B}\|$, $\|\mathbf{C}\|$ are sufficiently small, then the conditions of the invariant-sphere theorem hold [15]. That is, near the origin of $\mathbb{R}^9$, the dynamics of both systems are forward asymptotic to a flow-invariant 8-sphere, $S \subset \mathbb{R}^9$.

In fact, we will often discuss a system that has more symmetry. Let $\rho_3 \in S_9$ be defined by

$$\rho_3(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = (\sigma \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}),$$

where σ is a generator of $\mathbb{Z}_3$. Let $H = \langle \rho_1, \rho_2, \rho_3 \rangle$ and set $\Gamma^\star = \Delta_9 \rtimes H$. In terms of wreath products we have

$$\Gamma^\star = (\mathbb{Z}_2 \wr \mathbb{Z}_3) \wr \mathbb{Z}_3.$$

The system (5) is $\Gamma^\star$-equivariant if and only if $\mathbf{B} = (b, b, b)$ and $\mathbf{C} = (c, c, c)$, for some $b, c \in \mathbb{R}$.

We emphasize that all of the phenomena we describe below persist under perturbations with only Δ_9 symmetry.

5.1. Equilibria

In order to describe the equilibria of (6) it is easiest to work with the truncated cubic system (5). It follows from [16, §§13, 14] that there is an open and dense semialgebraic subset $\mathscr{R}$ of $\mathbb{R}^8$ such that if $(\mathbf{A}_0, \mathbf{B}, \mathbf{C}) \in \mathscr{R}$, then all equilibria of (5) are hyperbolic. Consequently, for fixed $(\mathbf{A}_0, \mathbf{B}, \mathbf{C}) \in \mathscr{R}$, the equilibria of (6) are hyperbolic for sufficiently small $|d|$, $|e|$. Moreover, the isotropy of the equilibria is the same as in the truncated system. That is, the symmetry of equilibria is unchanged if we perturb (6) by higher-order symmetric terms.

Using the results of [16, §§13, 14] enables us to give a useful 'parametrization' of the equilibria that occur for $(\mathbf{A}_0, \mathbf{B}, \mathbf{C}) \in \mathscr{R}$. First, however, we need some preliminaries. We define a fundamental domain for the action of Δ_9:

$$\mathbb{R}^9_+ = \{(x_0, \dots, z_2) \mid x_0, \dots, z_2 \geqq 0\}.$$

Since $\Delta_9 \subset \Gamma$, it is clear that if $\mathbf{X}$ is an equilibrium of (5), then we can find $\delta \in \Delta_9$ such that $\delta\mathbf{X} \in \mathbb{R}^9_+$. Consequently, to describe the set of equilibria of (5), it suffices to find the equilibria lying in $\mathbb{R}^9_+$. Since $\mathbb{R}^9_+$ is $\mathbb{Z}(3,3)$-invariant for the flow of (5), it follows that the action of $\mathbb{Z}(3,3)$ restricts to an action on $\mathbb{R}^9_+$.

Let $\mathscr{E}$ denote the $\mathbb{Z}(3,3)$-invariant subset of $\mathbb{R}^9_+$ consisting of all non-zero vectors $\mathbf{V}$ such that each component of $\mathbf{V}$ lies in $\{0, +1\}$. It is shown in [16] that if $\mathbf{X} \in \mathbb{R}^9_+$ is a hyperbolic equilibrium of (5), then there exists a unique point $\mathbf{V} \in \mathscr{E}$ such that $\Gamma_{\mathbf{X}} = \Gamma_{\mathbf{V}}$ (equivalently, $\mathbb{Z}(3,3)_{\mathbf{X}} = \mathbb{Z}(3,3)_{\mathbf{V}}$). If $\mathbf{V} \in \mathscr{E}$, then $\mathbf{V} = (\mathbf{a}, \mathbf{b}, \mathbf{c})$, where $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$. In future, we just set $\mathbf{V} = \mathbf{abc}$. If $\mathbf{a} = (0,0,0)$, we write $\mathbf{a} = \mathbf{0}$. If $\mathbf{a} = (1,0,0)$, we set $\mathbf{a} = \mathbf{1}$ and if $\mathbf{a} = (1,1,1)$, we set $\mathbf{a} = \mathbf{3}$. In the sequel, if $\mathbf{X} \in \mathbb{R}^9_+$ is an equilibrium, we usually set $X = p_{\mathbf{a}}$, where $\mathbf{a}$ is the unique point in $\mathscr{E}$ such that $\Gamma_{\mathbf{X}} = \Gamma_{\mathbf{a}}$.

As an easy application of the methods in [16] (see also [20]), we have

Lemma 5.1. *An equilibrium* $\mathbf{X} \in \mathbb{R}^9_+$ *of* (6), (5) *has maximal isotropy type if and only if* $\mathbf{X}$ *corresponds to a point on the* $\mathbb{Z}(3,3)$*-orbit of* **100** *or* **111** *or* **300** *or* **333**. *All other equilibria have submaximal isotropy type. The same result also holds if equations are* $\Gamma^\star$*-equivariant* (*with* $\mathbb{Z}(3,3)$ *replaced by the subgroup of* $\Gamma^\star$ *leaving* $\mathbb{R}^9_+$ *invariant*).

If $\mathbf{V} \in \mathscr{E}$ is not maximal, (5) may have no equilibria with isotropy equal to $\Gamma_{\mathbf{V}}$. In fact, corresponding to each submaximal $\mathbf{V} \in \mathscr{E}$ it is possible to compute explicit equations and inequalities that determine the closed subset of $\mathbb{R}^8$ for which there are no equilibria with isotropy equal to $\Gamma_{\mathbf{V}}$. In practice, these computations, although quite tractable, can be complicated (see [16, §14]). The following result will suffice for our needs.

Lemma 5.2. *There is a nonempty open subset* $\mathscr{D}$ *of* $\mathscr{R}$ *such that if* $(\mathbf{A}_0, \mathbf{B}, \mathbf{C}) \in \mathscr{D}$, *then* (5) *has no equilibria with isotropy group equal to* $\Gamma_{\mathbf{ab0}}$, *where* $\mathbf{a}, \mathbf{b}$ *range over all 3-tuples for which* $\Gamma_{\mathbf{ab0}}$ *is submaximal.*

Proof. Let us start by assuming that (5) is $\Gamma^\star$-equivariant. Let $\mathscr{R}^\star \subset \mathbb{R}^4$ be the open and dense semialgebraic subset consisting of all $\mathbf{A}_0, \mathbf{B} = (b, b, b)$ and

$\mathbf{C} = (c, c, c)$ for which (5) has only hyperbolic equilibria. Let $\mathscr{D}^\star$ denote the open subset of $\mathscr{R}^\star$ corresponding to which (5) has no equilibria with submaximal isotropy equal to $\Gamma_{\mathbf{ab0}}$.

Straightforward computations show that a point $(a_1, a_2, b, c) \in \mathscr{R}^\star$ lies in $\mathscr{D}^\star$ if and only if

$$\begin{aligned} a_1 a_2 &< 0, \\ bc &< 0, \\ (3c - a_1 - a_2)b &< 0, \\ (3c - a_1 - a_2)(3b - a_1 - a_2) &< 0. \end{aligned}$$

Hence $\mathscr{D}^\star \neq \emptyset$. But now every point of $\mathscr{D}^\star$ determines an interior point of $\mathscr{D}$. □

5.2. *Stabilities of the Equilibria $p_{\mathbf{100}}$ and $p_{\mathbf{300}}$*

It is straightforward to compute the eigenvalues and eigenspaces of the linearization of (5) at the equilibria $p_{\mathbf{100}}$ and $p_{\mathbf{300}}$. The eigenvalues at $p_{\mathbf{100}}$ may be written as the 3-tuple

$$([-2, a_2, a_1], [c_0, c_1, c_2], [b_0, b_1, b_2]), \tag{7}$$

where the eigenvalue -2 corresponds (as always) to the radial direction. The triples $[-2, a_2, a_1]$, $[c_0, c_1, c_2]$, and $[b_0, b_1, b_2]$ respectively correspond to eigenvalues of the linearization in the $\boldsymbol{x}$-hyperplane, $\boldsymbol{y}$-hyperplane and $\boldsymbol{z}$-hyperplane. (Each of these subspaces is invariant by the flow of (5).) For every eigenvalue, the eigenvector can be taken to be the unit vector along the corresponding coordinate axis.

The eigenvalues at $p_{\mathbf{300}}$ are given by the triple

$$\left(\left[-2, \frac{a_1 + a_2 \pm \imath\sqrt{3}(a_1 - a_2)}{a_1 + a_2 - 3}\right], \left[\frac{c_0 + c_1 + c_2}{3 - (a_1 + a_2)}\right], \left[\frac{b_0 + b_1 + b_2}{3 - (a_1 + a_2)}\right]\right). \tag{8}$$

The single eigenvalues associated to the $\boldsymbol{y}$, and $\boldsymbol{z}$-spaces occur with multiplicity 3. The eigenvalues in the $\boldsymbol{x}$-space are exactly those that occur in the linearization analysis of the Guckenheimer-Holmes cycle (see [16, §15] or [20, Chapter 6]).

5.3. *The Invariant-Sphere Theorem*

We recall some details on the invariant-sphere theorem. Suppose that X is a smooth vector field on $\mathbb{R}^n$ of the form $X(x) = x + Q(x)$, where Q is a homogeneous cubic polynomial. If we define

$$\begin{aligned} m(Q) &= \inf\{(Q(u), u) \mid u \in \mathbb{R}^n, \|u\| = 1\}, \\ M(Q) &= \sup\{(Q(u), u) \mid u \in \mathbb{R}^n, \|u\| = 1\}, \end{aligned}$$

then for all $x \in \mathbb{R}^n$

$$m(Q)\|x\|^4 \leqq (Q(x), x) \leqq M(Q)\|x\|^4.$$

If $M(Q) < 0$, then $(Q(x), x) < 0$ for all $x \in \mathbb{R}^n$, $x \neq 0$. If this condition on Q holds, we say that Q is *contracting*. It is shown in [20, Chapter 5] that if Q is contracting, then there exists a topologically embedded flow-invariant $(n-1)$-sphere S for the flow of $\dot{x} = X(x)$ such that $\omega(x) \subset S$ for all $x \in \mathbb{R}^n$, $x \neq 0$. Moreover, the invariant sphere S is contained in the annulus $A(r, R) = \{x \mid r \leqq \|x\| \leqq R\}$, where

$$r = \sqrt{\frac{-1}{M(Q)}}, \quad R = \sqrt{\frac{-1}{m(Q)}}.$$

In general, S is not differentiably embedded.

For $a \in \mathbb{R}$, define $Q_a(x) = Q(x) + a\|x\|^2 x$, $X_a = I + Q_a$. Obviously, $M(Q_a) = M(Q) + a$, $m(Q_a) = m(Q) + a$. It follows by rescaling and the theory of normal hyperbolic sets [26] that if we fix Q, $N \in \mathbb{N}$, we can find $a_N \in \mathbb{R}$ such that if $a \leqq a_N$, then Q_a is contracting and the corresponding invariant sphere $S = S_a$ is embedded as a C^N-submanifold of $\mathbb{R}^n$.

Remark 5.3. If $a \leqq a_N$, the invariant sphere S_a is contained in the annulus $A(r, R)$, where $r = \sqrt{\frac{-1}{M(Q)+a}}$, $R = \sqrt{\frac{-1}{m(Q)+a}}$. In particular, $r \to 0$, $R/r \to 1$ as $a \to -\infty$.

It follows from the previous remark that if we are given Q and $N \in \mathbb{N}$, we can rescale so that $R < 1$, for all $a \leqq a_N$.

Suppose that Z is a smooth vector field on $\mathbb{R}^n$. Let $\|Z\|_1$ denote the uniform C^1-norm of Z restricted to the unit ball. As a straightforward consequence of the theory of normally hyperbolic sets, we have the following differentiable version of the invariant-sphere theorem.

Theorem 5.4 (cf. [15, Theorem 5.2]). *Let $N \in \mathbb{N}$ and let Q be a homogeneous cubic polynomial on $\mathbb{R}^n$. We may choose $a_N \in \mathbb{R}$ and $\delta > 0$ such that if Z is a smooth vector field on $\mathbb{R}^n$ satisfying $Z(0) = 0$, $DZ(0) = 0$, and $\|Z\|_1 < \delta$, then for $a \leqq a_N$ the vector field $Z_a(x) = x + Q_a(x) + Z(x)$ has a unique C^N flow-invariant $(n-1)$-sphere S contained in the unit ball of $\mathbb{R}^n$. Further, $\omega(x) \subset S$ for all $x \in \mathbb{R}^n$, $0 < \|x\| \leqq 1$.*

Remark 5.5. In practice, we apply Theorem 5.4 when Q is the third-order truncation of a smooth vector field on $\mathbb{R}^n$ and Z is the remainder term. Roughly speaking, the theorem implies that if $X(x) = x + Q(x) + O(\|x\|^4)$ is a smooth vector field on $\mathbb{R}^n$, then we can add a cubic term $a\|x\|^2 x$, so that if a is sufficiently negative the resulting equation has a differentiably embedded flow-invariant sphere containing the origin.

We use the invariant-sphere theorem in our study of dynamics of (6) in the following way. First of all, we consider the cubic truncation (5). Provided that the homogeneous cubic part is contracting, all non-trivial trajectories of (5) are forward asymptotic to a flow-invariant embedded 8-sphere. Further, for an open dense set of coefficients $\mathbf{A}_0$, $\mathbf{B}$, $\mathbf{C}$, all equilibria of (5) are hyperbolic. The hyperbolicity of equilibria persists if we add on a term $a\|\mathbf{X}\|^2\mathbf{X}$, $a < 0$, except at possibly finitely many values of a. For sufficiently negative values of a, the invariant sphere can be made to have any prescribed finite order of differentiability. If the invariant sphere is at least of class C^1 (in fact, C^0 in our case), we can add on small fifth- and higher-order terms without changing stability or destroying the invariant sphere. (Of course, dynamics on the invariant sphere may and do change.)

5.4. Connections

We recall that there exists a nonempty open subset $\mathscr{D}$ of $\mathbb{R}^8$ such that if $(\mathbf{A}_0, \mathbf{B}, \mathbf{C}) \in \mathscr{D}$, then (5) has no equilibria of submaximal isotropy type in $(\boldsymbol{x}, \boldsymbol{y})$-space (Lemma 5.2). In particular, all equilibria in $(\boldsymbol{x}, \boldsymbol{y})$-space are of maximal isotropy type and lie in $\{(\boldsymbol{x}, \boldsymbol{y}) \mid \boldsymbol{x} = 0 \text{ or } \boldsymbol{y} = 0\}$. Let $\mathscr{D}'$ denote the nonempty open subset of $\mathscr{D}$ consisting of coefficient values for which the conditions of the invariant-sphere theorem hold (C^0-invariant spheres suffice). We investigate connections between equilibria in $\boldsymbol{x}$-space and $\boldsymbol{y}$-space under the assumption that $(\mathbf{A}_0, \mathbf{B}, \mathbf{C}) \in \mathscr{D}'$.

First, we need some notation. Let $\mathbb{Z}_3 = \langle \rho_1 \rangle$, and recall that ρ_1 acts on the $\mathbb{R}^9$ by simultaneous cyclic permutation of coordinates in the $\boldsymbol{x}$-, $\boldsymbol{y}$- and $\boldsymbol{z}$-coordinate subspaces. We also set $\tilde{\mathbb{Z}}_3 = \langle \rho_2 \rangle$, and recall that ρ_2 cyclically permutes the coordinate subspaces.

We write $\mathbf{100} \longrightarrow \mathbf{010}$, if there exist $\rho, \delta \in \mathbb{Z}_3$ such that there is a connection from $\rho p_{\mathbf{100}} = p_{\rho\mathbf{100}}$ to $\delta p_{\mathbf{010}}$. That is, if $W^{\mathrm{u}}(\rho p_{\mathbf{100}}) \cap W^{\mathrm{s}}(\delta p_{\mathbf{010}}) \neq \emptyset$. Note that it follows by $\mathbb{Z}_3$-equivariance that if there is a connection from $\rho p_{\mathbf{100}}$ to $\delta p_{\mathbf{010}}$, then there are connections from $\rho_1^j \rho p_{\mathbf{100}}$ to $\rho_1^j \delta p_{\mathbf{010}}$, for $j = 1, 2$.

If there are connections from $\rho p_{\mathbf{100}}$ to $\delta p_{\mathbf{010}}$ for *all* $\rho, \delta \in \mathbb{Z}_3$, we write

$$\mathbf{100} \rightrightarrows \mathbf{010}.$$

We generalize this notation to allow for connections between **300** and **030** or **010**. Of course, it follows by $\mathbb{Z}_3$-equivariance that

$$\begin{aligned} \mathbf{100} &\longrightarrow \mathbf{030} \Longrightarrow \mathbf{100} \rightrightarrows \mathbf{030}, \\ \mathbf{300} &\longrightarrow \mathbf{030} \Longrightarrow \mathbf{300} \rightrightarrows \mathbf{030}, \\ \mathbf{300} &\longrightarrow \mathbf{010} \Longrightarrow \mathbf{300} \rightrightarrows \mathbf{010}. \end{aligned}$$

Lemma 5.6. *There is a nonempty open subset $\mathscr{D}^\star$ of $\mathscr{D}'$ such that if* $(\mathbf{A}_0, \mathbf{B}, \mathbf{C}) \in$ *$\mathscr{D}^\star$, then* (5) *has connections*

$$\mathbf{100} \rightrightarrows \mathbf{010}, \ \mathbf{100} \rightrightarrows \mathbf{030}, \ \mathbf{300} \rightrightarrows \mathbf{010}, \ \mathbf{300} \rightrightarrows \mathbf{030}.$$

All of these connections persist for small values of e, f.

Proof. It follows from Section 5.2, that we can choose a nonempty open subset $\mathscr{D}''$ of $\mathscr{D}'$ such that if the coefficients of (5) lie in $\mathscr{D}''$, then the signs of eigenvalues of the linearizations at $p_{\mathbf{100}}$ and $p_{\mathbf{300}}$ are given according to the following table.

	$\boldsymbol{x}$-directions	$\boldsymbol{y}$-directions	$\boldsymbol{z}$-directions
$p_{\mathbf{100}}$	$(-,+,-)$	$(+,+,+)$	$(-,-,-)$
$p_{\mathbf{300}}$	$(-,a,\bar{a})$	$(+,+,+)$	$(-,-,-)$

Note that $a, \bar{a}$ signify that there is a complex conjugate pair of eigenvalues with nonzero real part.

Let P denote the 2-plane in $(\boldsymbol{x}\,\boldsymbol{y})$-space defined by $P = \{(x, 0, 0, y, 0, 0) \mid x, y \in \mathbb{R}\}$. Observe that P is a fixed-point subspace of $\mathbb{R}^9$-space and hence P is invariant by the flow of (5). If we set $P^+ = \mathbb{R}^6_+ \cap P$, then P^+ is also invariant by the flow of (5). The intersection of P^+ with the invariant sphere is a flow-invariant arc joining $p_{\mathbf{100}}$ to $p_{\mathbf{010}}$. Since the only equilibria on the arc are $p_{\mathbf{100}}$ and $p_{\mathbf{010}}$, it follows that there is a connection from $p_{\mathbf{100}}$ to $p_{\mathbf{010}}$. Similarly, we may show that there is a connection from $\rho p_{\mathbf{100}}$ to $\tau p_{\mathbf{010}}$ for all $\rho, \tau \in \langle \rho_1 \rangle$. Hence $\mathbf{100} \rightrightarrows \mathbf{010}$. A similar argument proves that $\mathbf{300} \rightrightarrows \mathbf{030}$

For the remaining cases, we start by working with the group $\Gamma^\star$. Each pair of equilibria that we wish to prove connected are then contained in a two-dimensional fixed point space of $\Gamma^\star$ and intersection with the invariant sphere then gives a connecting flow-invariant arc. Connections persist when we break symmetry from $\Gamma^\star$ to Γ and hence we obtain the required open subset $\mathscr{D}^\star$ of $\mathscr{D}'$.

Finally, these connections persist if we allow e, d to be nonzero but small. □

6. Heteroclinic Cycles

In this section, we describe a variety of nontrivial robust heteroclinic networks present in the model systems (5) and (6). We present numerical simulations illustrating the asymptotic behavior of trajectories near these networks. For simplicity, we work entirely within the flow-invariant region $\mathbb{R}^9_+$.

6.1. A Depth-1 Heteroclinic Network

We assume that the coefficients of (5) lie in the region $\mathscr{D}^\star$ described by Lemma 5.6. In particular,

(a) The system (5) has an attracting invariant sphere S.
(b) The signs of the eigenvalues of the linearization of (5) at $p_{\mathbf{100}}$ and $p_{\mathbf{300}}$ are given by

$$(-, +, -, +, +, +, -, -, -) \text{ and } (-, a, \bar{a}, +, +, +, -, -, -),$$

respectively.
(c) There are no submaximal equilibria in $(\boldsymbol{x}, \boldsymbol{y})$-space.

These conditions imply that there is a 'Guckenheimer-Holmes' cycle $\Sigma_{\boldsymbol{x}}$ contained in $\boldsymbol{x}$-space. Specifically, the cycle determined by the $\mathbb{Z}_3$-orbit of the connection $p_{100\mathbf{00}} \to p_{010\mathbf{00}}$:

$$\begin{array}{ccccc} & & 100\mathbf{00} & & \\ & \nearrow & & \searrow & \\ 001\mathbf{00} & & \leftarrow & & 010\mathbf{00} \end{array} \tag{9}$$

We let $\Sigma_{\boldsymbol{y}} = \rho_2 \Sigma_{\boldsymbol{x}}$ and $\Sigma_{\boldsymbol{z}} = \rho_2^2 \Sigma_{\boldsymbol{x}}$ denote the corresponding cycles in $\boldsymbol{y}$- and $\boldsymbol{z}$-space.

It follows from Lemma 5.6 that $\mathbf{100} \rightrightarrows \mathbf{010}$. In particular, there is a cycle

$$\begin{array}{ccc} & \mathbf{100} & \\ \nearrow & & \searrow \\ \mathbf{001} & \leftarrow & \mathbf{010} \end{array} \tag{10}$$

Since $\mathbf{100} \rightrightarrows \mathbf{010}$, it follows that for all $\alpha, \beta, \gamma \in \mathbb{Z}_3$, there is a cycle

$$\begin{array}{ccc} & \alpha\mathbf{100} & \\ \nearrow & & \searrow \\ \gamma\mathbf{001} & \leftarrow & \beta\mathbf{010} \end{array} \tag{11}$$

As an immediate consequence of Lemma 5.6, we see that there is also a cycle

$$\begin{array}{ccc} & \mathbf{300} & \\ \nearrow & & \searrow \\ \mathbf{030} & \leftarrow & \mathbf{003} \end{array} \tag{12}$$

We refer to this cycle as a cycle between *synchronized* states. Finally, yet another application of Lemma 5.6 yields a plethora of cycles that switch between synchronized and single states. For example, there is a cycle

$$\begin{array}{ccc} & \mathbf{300} & \\ \nearrow & & \searrow \\ \mathbf{010} & \leftarrow & \mathbf{001} \end{array} \tag{13}$$

It follows from (11) that for coefficients in $\mathscr{D}^\star$, there exists a depth-1 heteroclinic network linking all equilibria with isotropy conjugate to that of **100**. The connections are shown schematically in Figure 5. Note that, for clarity, not all links are shown.

Remark 6.1. The connections obtained in the proof of Lemma 5.6 are one-dimensional and came by looking at invariant subspaces. It follows that there is a natural 'symmetry-determined' one-dimensional regular heteroclinic network between equilibria of symmetry type **100**. In general, of course, there may be infinitely many connections between equilibria of symmetry type **100**.

6.2. A Depth-2 Heteroclinic Network

Under certain circumstances, the depth-1 networks described above are part of a depth-2 network. More precisely, let $\mathscr{D}_2^\star$ be the nonempty open subset of $\mathscr{D}_2^\star$ for which the complex eigenvalues of the linearization of (5) have strictly positive real part. That is, we require $\mathbf{A}_0 = (a_1, a_2)$ to satisfy $a_2 < 0$ and

$$-a_2 > a_1 > 0. \tag{14}$$

If these conditions hold, then the cycle $\Sigma_{\boldsymbol{x}}$ is a (globally) attracting heteroclinic cycle in $\boldsymbol{x}$-space and the equilibrium **300** is repelling in $\boldsymbol{x}$-space.

For sufficiently small values of d and e, the cycle $\Sigma_{\boldsymbol{x}}$ persists and attracts all nonzero and nonsynchronized trajectories in some (preassigned) neighborhood of the origin in $\boldsymbol{x}$-space.

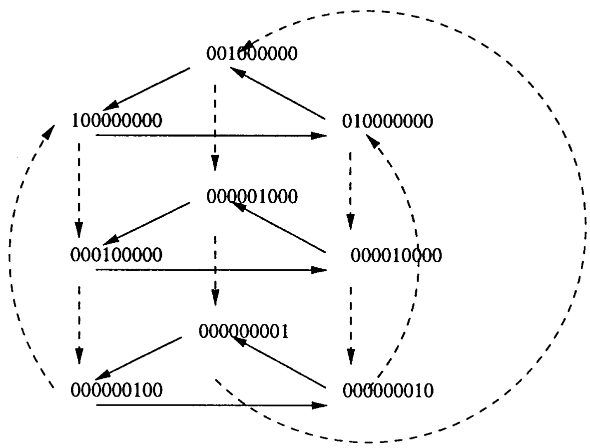


Fig. 5. Schematic partial diagram of connections between equilibria in the 'one-cell' equilibria.

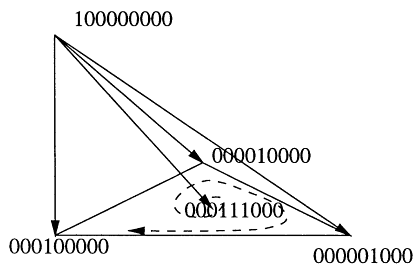


Fig. 6. Schematic diagram showing the types of connection making up the depth two heteroclinic network in the nine coupled cell system.

In particular, we obtain a robust connection $p_{\mathbf{300}} \longrightarrow \Sigma_{\boldsymbol{y}}$ from the 'synchronized' state $p_{\mathbf{300}}$ to $\Sigma_{\boldsymbol{y}}$. Combining this observation with our previous results on connections between equilibria, we see that we may construct a robust heteroclinic network of depth-2 that includes the depth-1 networks constructed above as subnetworks. The network contains new connections shown schematically in Figure 6. More precisely, if we define

$$\Sigma^H = \cup_{j=0}^{2} \rho_2^j \overline{W^{\mathrm{u}}(p_{\mathbf{300}}) \cap \mathbb{R}_+^9},$$

then Σ^H is a flow-invariant compact subset of $\mathbb{R}^9$ which is contained in the intersection of the invariant sphere S with the union of the $(\boldsymbol{x}, \boldsymbol{y})$-, $(\boldsymbol{y}, z)$- and $(z, \boldsymbol{x})$-coordinate hyperplanes.

Proposition 6.2. *The set Σ^H is a geometrically robust heteroclinic network of depth 2.*

Proof. We start by proving that depth$(\Sigma^H) = 2$. Suppose $\mathbf{X} = (\boldsymbol{x}, \boldsymbol{y}, \mathbf{0}) \in \Sigma^H$. If $\boldsymbol{y} = \mathbf{0}$, then $\omega(\mathbf{X}) \subset \Sigma_{\boldsymbol{x}}$ or $\omega(\mathbf{X}) = \{\mathbf{300}\}$. In the first case, $\omega(\mathbf{X})$ is either an equilibrium in the $\mathbb{Z}_3$-orbit of $p_{\mathbf{100}}$ or is all of $\Sigma_{\boldsymbol{x}}$. If $\boldsymbol{x}, \boldsymbol{y} \neq 0$, then either $\omega(\mathbf{X}) = \{p_{\mathbf{030}}\}$ or it is one of the equilibria in the $\mathbb{Z}_3$-orbit of $p_{\mathbf{010}}$ or it is all of $\Sigma_{\boldsymbol{y}}$. Similar results hold when we look at α-limit points, except that $\Sigma_{\boldsymbol{x}}$ no longer occurs as an α-limit set. It follows that Σ_1^H is the union of the cycles $\Sigma_{\boldsymbol{x}}$, $\Sigma_{\boldsymbol{y}}$ and $\Sigma_{\boldsymbol{z}}$ together with the equilibria $p_{\mathbf{300}}$, $p_{\mathbf{030}}$, $p_{\mathbf{003}}$. Obviously, Σ_2^H is the set of equilibria in Σ^H. Hence depth$(\Sigma^H) = 2$. The indecomposability of Σ^H follows from Lemma 2.37.

It follows immediately from our construction of Σ^H that Σ^H is symmetry adapted and dynamically coherent. Since equilibria on Σ^H are all hyperbolic, it follows easily that symmetry adaptation and dynamic coherence persist under Γ-equivariant perturbation. From this it follows that Σ^H is weakly geometrically robust. In order to complete the proof of geometric robustness, it is easiest to work in $\mathbb{R}^9$ rather than $\mathbb{R}^9_+$. If we let $S_{\boldsymbol{x}}$ denote the intersection of the invariant sphere $S \subset \mathbb{R}^9$ with $\boldsymbol{x}$-space, then Σ^H is the $\tilde{\mathbb{Z}}_3$-orbit of $\mathscr{W}^{\mathrm{u}}(S_{\boldsymbol{x}})$. Let $D_{\boldsymbol{y}}$ denote an open unit disk, centered at the origin, in $\boldsymbol{y}$-space. We may choose an open $\mathbb{Z}_3 \rtimes \Delta_9$-invariant neighborhood U of S such that $\mathscr{W}^{\mathrm{u}}(S_{\boldsymbol{x}}) \cap U$ is homeomorphic to $S_{\boldsymbol{x}} \times D_{\boldsymbol{y}}$. It follows that the homeomorphism type of Σ^H near $S_{\boldsymbol{x}}$ is constant under equivariant perturbations. Now it is easy to patch the local stability near $S_{\boldsymbol{x}}$, $S_{\boldsymbol{y}}$, $S_{\boldsymbol{z}}$ with the stability on the complement of $\Sigma^H \setminus \tilde{\mathbb{Z}}_3 U$ to obtain the required global stability result. □

Remarks 6.3. (1) For simplicity, we have worked entirely within $S \cap \mathbb{R}^9_+ \subset \mathbb{R}^9_+$. Of course, there is a completely analogous result if instead we work in $S \subset \mathbb{R}^9$.
(2) It is not unreasonable to ask whether the flow on network Σ^H is *structurally stable*. Structural stability does not follow from the arguments of the proof of Proposition 6.2. Indeed, a necessary condition for structural stability is that the invariant manifolds of the equilibria in Σ^H are stratumwise transverse (equivalently, Γ-transverse, see [20]). We have not addressed this point and indeed suspect that it is generally difficult to find conditions on the coefficients of (5) and (6) that yield stratumwise transversality of the invariant manifolds.

6.3. Numerical Simulations

To investigate these networks further, simulations were carried out using the dynamical systems package `dstool` [24] with variable step Runge-Kutta inte-

Table 1. Parameter values for the simulations of equation (6) shown in Figures 7–10. For all these parameter values, there is an attracting sphere on which the asymptotic dynamics takes place.

	a_1	a_2	b_0	b_1	b_2	c_0	c_1	c_2	d	e
(a)	0.9	−0.87	1	1	1	−0.99	−0.99	−0.99	−0.1	−0.1
(b)	0.9	−0.89	1	1	1	−1.1	−1.1	−1.1	−0.1	−0.1
(c)	0.9	−0.91	1	1	1	−1.1	−1.1	−1.1	−0.1	−0.1
(d)	0.9	−0.87	1	1	1	−1.1	−1.1	−1.1	−0.1	−0.1

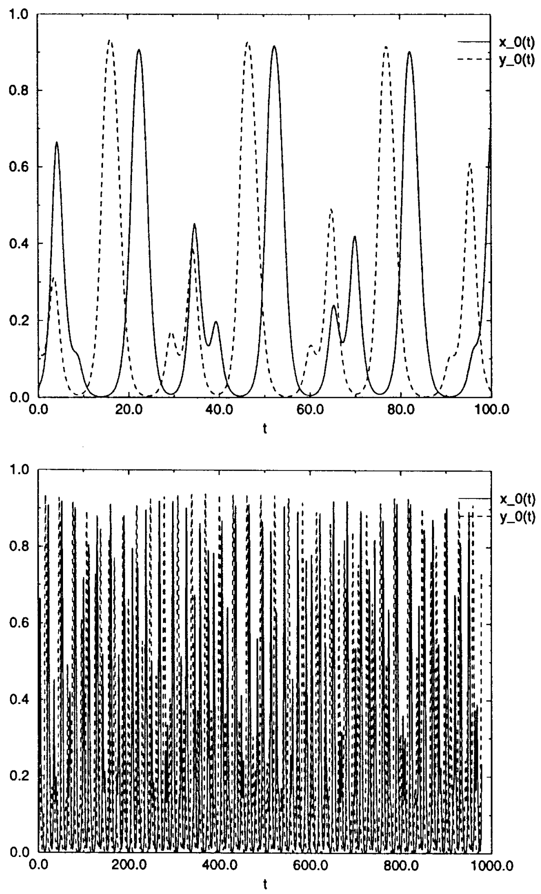


Fig. 7. Time series for parameter value Table 1(a) for components x_0 and y_0. This exhibits attracting two-frequency quasi-periodicity.

grator and error tolerance 10^{-8}. An initial condition was chosen away from any symmetry-forced invariant subspaces, and the trajectory was computed for several thousand time-units. Typical parameter values for the simulations are shown in Table 1. Illustrations of time series at these parameter values are shown in Figures 7–10; note that the parameters d and e are non-zero to break the degeneracy of the bifurcation where the cycle (9) changes stability within the subspaces with isotropy **a00**.

Figure 7 shows a two-frequency quasi-periodic attractor; this can be thought of as having come from an interaction between periodic orbits that have bifurcated from the cycles (9), (10). Note that the parameter values are such that all three of these cycles are unstable.

Figure 8 shows an attracting heteroclinic network between periodic orbits; in this case the cycle (10) is stable within the space spanned by its vertices, but it is unstable in other directions. Because d and e are non-zero, there are periodic orbits

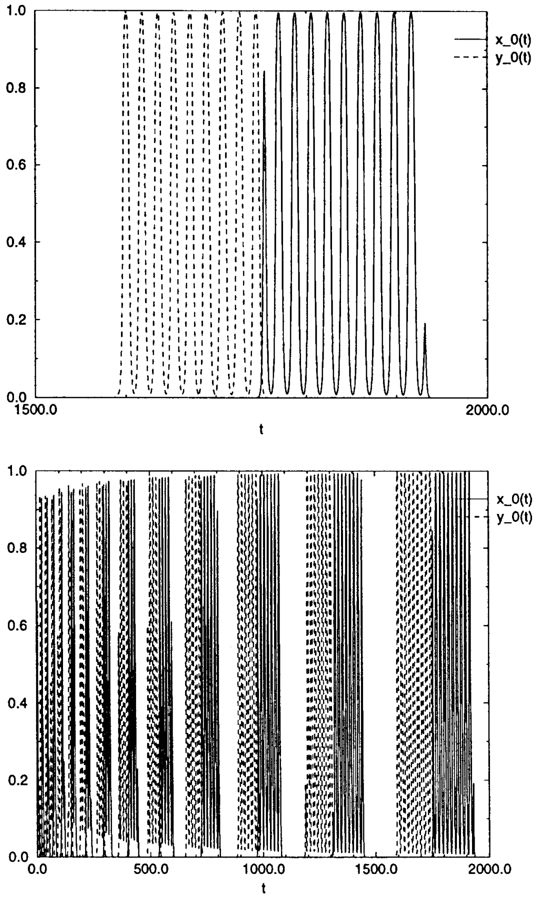


Fig. 8. Time series for parameter value Table 1(b) for components x_0 and y_0. This exhibits a heteroclinic cycle to a periodic orbit; note how the amplitude of the periodic orbit is almost constant except for the starting and finishing peaks in each 'burst'.

close to the cycle (9) that are joined by connecting orbits to form a heteroclinic network between periodic orbits.

Figure 9 shows an attracting heteroclinic network between the 'one-cell' equilibria. For these parameter values, the cycles (9), (10) are attracting within the subspaces spanned by their nodes. Simulations indicate that the depth-1 network is an attractor. Nevertheless, the attracting depth-1 network is embedded within an asymptotically stable depth-2 network which contains, for example, connections between $p_{\mathbf{100}}$ and $\Sigma_{\mathbf{y}}$. The numerics we have done suggest that the depth-1 subnetwork is essentially asymptotically stable; almost all trajectories eventually avoid connections of the form $p_{\mathbf{100}} \rightarrow \Sigma_{\mathbf{y}}$. Such 'hidden' connections are likely to generate many high-period periodic orbits if we break the asymptotic stability of the underlying depth-2 network.

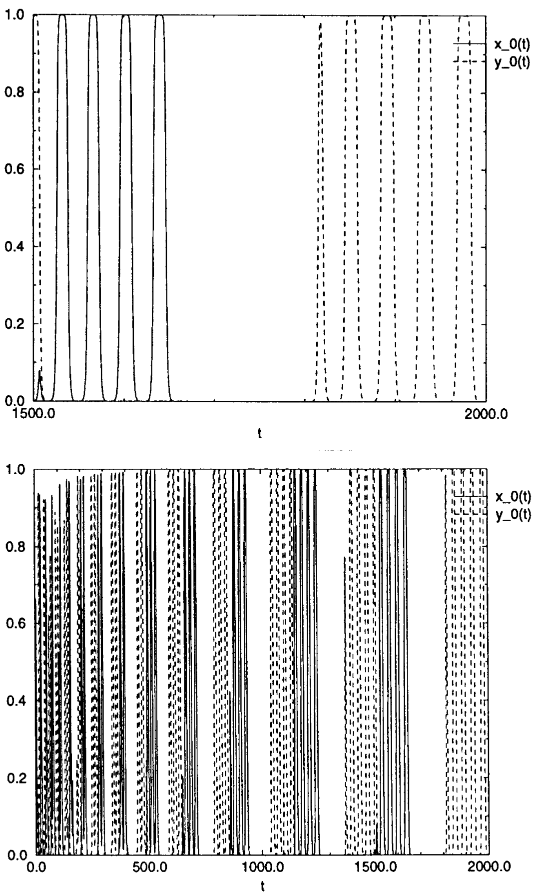


Fig. 9. Time series for parameter value Table 1(c) for components x_0 and y_0. This exhibits a heteroclinic cycle to a heteroclinic cycle; note that the two rates of slowing down are rather different.

Figure 10 shows an attracting heteroclinic network between 'synchronized 3-cell' equilibria; again, this is part of a depth-2 network, as can be seen by the presence of decaying oscillations after each switching. However, typical trajectories avoid connections of the form $p_{\mathbf{300}} \longrightarrow p_{\mathbf{010}}$.

We also investigated some parameter values where submaximal equilibria exist and can become part of the heteroclinic network; however this leaves many new possibilities open and a classification is much more difficult.

6.4. Bifurcation Behavior

The network (6) displays a number of bifurcations that are generic in this context. Notably, fix all parameters except for a_1 and increase this through $|a_1| = |a_2|$; at this point the inequality in (14) is broken and the cycle (9) loses stability within its

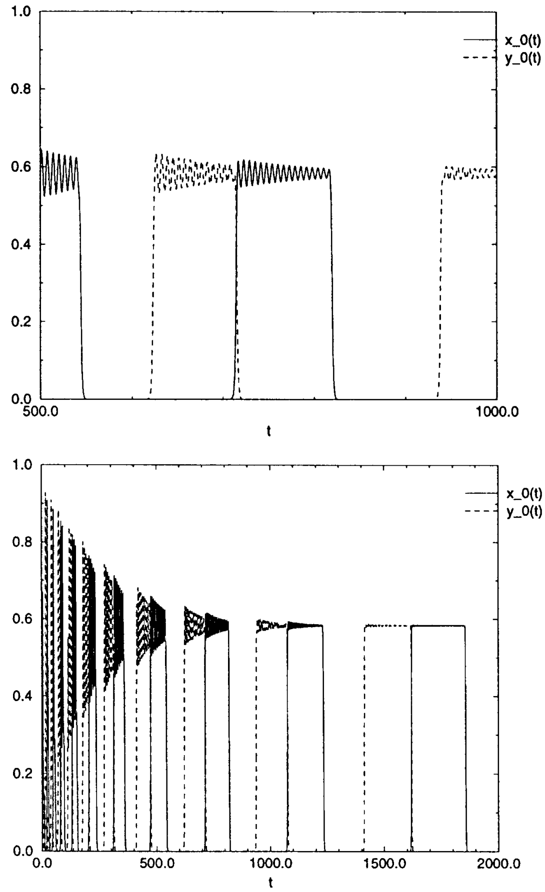


Fig. 10. Time series for parameter value Table 1(d) for components x_0 and y_0. In this case the attractor is a heteroclinic cycle between synchronized states; observe the decaying oscillations and the slowing down between each approach to a synchronized state.

invariant subspace. For $d > 0$ this leads to a bifurcation of a large-period periodic orbit from the cycle in a resonance bifurcation [10]; see [1] for resonance bifurcation of a heteroclinic cycle between chaotic invariant sets. Increasing a_1 further causes the periodic orbit to disappear at the **300** solution via a Hopf bifurcation and consequently creates a depth-2 connection from the cycle (9) to the equilibrium $p_{\mathbf{300}}$. A detailed classification of possible generic bifurcations in this system is beyond the scope of this work; we merely wish to indicate an example of how bifurcations can alter the structure of a depth two heteroclinic network.

6.5. Extensions and Generalizations

Using the results on edge cycles in [20, Chapter 6], we could easily construct one-dimensional $\mathbb{Z}_2 \wr \mathbb{Z}_p$-invariant attracting depth-1 heteroclinic networks in $\mathbb{R}^p$

for all $p \geqq 3$. Just as we did in our construction of the $\mathbb{Z}(3, 3)$-invariant network, it is then straightforward to show that for all $p, q \geqq 3$ it is possible to construct a depth-2 geometrically robust attracting heteroclinic network in $\mathbb{R}^{pq}$ with symmetry group $(\mathbb{Z}_2 \wr \mathbb{Z}_p) \wr \mathbb{Z}_q$ (or $\mathbb{Z}_2 \times (\mathbb{Z}_p \wr \mathbb{Z}_q)$.

We believe that it is possible to extend our methods so as to construct geometrically robust networks of arbitrary depth in systems of symmetrically coupled cells. As details of this work in progress will appear elsewhere, we limit ourselves to a few brief remarks and comments.

One might guess that the construction of our depth-2, $\mathbb{Z}(3, 3)$-invariant network could be iterated $N-1$ times to form a geometrically robust $\mathbb{Z}(3, 3, \ldots, 3)$-invariant network of depth N. However, this approach turns out to be too simplistic as it ignores a crucial feature of the dynamics that leads to the $\mathbb{Z}(3, 3)$-invariant network having depth 2. Specifically, depth 2 follows from the existence of points $x \in \Sigma$ such that $\alpha(x)$ is a synchronized state **100** and $\omega(x)$ is a Guckenheimer-Holmes cycle. In order to obtain higher depths, we need *cycling* between synchronized states and 'Guckenheimer-Holmes' cycles. One way to achieve this is to alternate the stabilities of the synchronized states and heteroclinic cycles in a symmetrically coupled ring consisting an even number of heteroclinic cycles. Just as before, this leads to a depth-2 heteroclinic network. An appropriate coupling of three such networks should then lead to a depth-3 heteroclinic network. We believe that iteration of this procedure would lead to geometrically robust heteroclinic networks of arbitrary depth.

A particularly interesting feature of networks of this type (including iterated $\mathbb{Z}(3, 3)$-invariant networks) is that if the network is asymptotically stable and there is a bifurcation-breaking asymptotic stability or, alternatively, a forced symmetry breaking, then there are periodic orbits near the cycle that exhibit multiple time scales corresponding to their tracking of cycles in the original network.

7. Discussion

We have proposed a definition of a heteroclinic network that encompasses many previous definitions, but also allows such cycles as that discovered by CHAWANYA [7, 8] where connections may limit on to other connections. In doing so we show that the concept of 'depth' of a flow on an invariant set (previously regarded as having little direct application to generic systems) has real relevance to structurally stable attractors in symmetric systems.

We have shown that such networks (which may contain robust continua of connections and/or chaotic invariant sets) have a hierarchical structure that can be characterized by their depth. We have given sufficient conditions for their embeddings that they can appear as ω-limit sets of nearby points, as well as sufficient conditions that they are robustly embedded in symmetric systems. We emphasize that these are only sufficient, and in many cases would be hard to verify. It is an open problem to obtain improved results for stability and robustness. This is likely to be difficult due to the appearance of essentially asymptotic stable subnetworks, and possible existence of sets of stable periodic orbits or other invariant sets accumulating on the network [8].

To address some of the questions about stability, we have examined a model system on $\mathbb{R}^9$ that has a number of relatively easily analyzable but nontrivial networks; we can show their robust existence and numerically find attracting networks.

As there is an equivalence between differential equations on $\mathbb{R}^n$ with Δ_n symmetry and game dynamics differential equations on the $(n-1)$-simplex [33], these results apply in both settings.

Trajectories that are asymptotic to 'cycling chaos' networks show a slowing-down series of switchings between shadowing of different types of recurrent behavior characterized by possibly chaotic nodal sets. If the nodes are not uniquely ergodic, then there is also a question of which of these ergodic measures contributes to averages of observables along trajectories that approach C. Chaotic sets raise a number of further questions that we leave to future research.

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