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Existence of Weak Solutions for the Motion of Rigid Bodies in a Viscous Fluid

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Abstract

We study the evolution of a finite number of rigid bodies within a viscous incompressible fluid in a bounded domain of $\mathbb{R}^d$ ($d = 2$ or 3) with Dirichlet boundary conditions. By introducing an appropriate weak formulation for the complete problem, we prove existence of solutions for initial velocities in $H_0^1(\Omega)$. In the absence of collisions, solutions exist for all time in dimension 2, whereas in dimension 3 the lifespan of solutions is infinite only for small enough data.

1. Introduction

Let $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) be a $C^{1,1}$ domain occupied by a viscous incompressible fluid surrounding k rigid bodies $1 \leqq i \leqq k$ of masses $m_1, \ldots, m_k$, with $C^{1,1}$ regularity. The fluid has density $\bar{\rho}_F > 0$, viscosity $\mu > 0$, pressure p, velocity $\boldsymbol{v}$ and is governed by the Navier-Stokes equations for incompressible fluids:

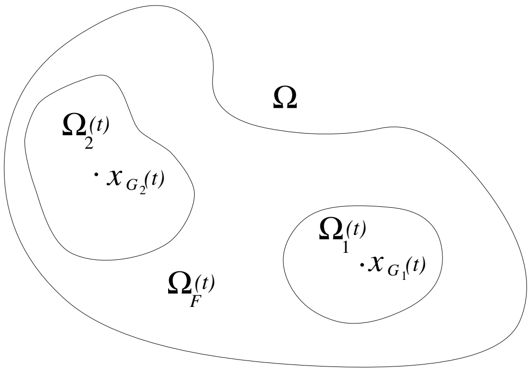
$$\bar{\rho}_F\,(\partial_t \boldsymbol{v} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v})) + \nabla p - \mu \Delta \boldsymbol{v} = \bar{\rho}_F \boldsymbol{f} \;\text{ in }\; \mathscr{D}'(Q_T)^d, \tag{1}$$

$$\operatorname{div} \boldsymbol{v} = 0 \;\text{ in }\; Q_T, \quad \boldsymbol{v}_{|t=0} = \boldsymbol{v}_0 \;\text{ in }\; \Omega_F(0), \tag{2}$$

where $\Omega_F(t) \subset \Omega$ denotes the fluid domain at time t,

$$Q_T = \{(t, \boldsymbol{x}) \,/\, t \in (0, T),\ \boldsymbol{x} \in \Omega_F(t)\},$$

and $\boldsymbol{f} \in L^2((0,T) \times \Omega)^d$ is a homogeneous external bulk force, for instance gravity. Let $\Omega_i(t)$ be the bounded open connected subdomain of Ω representing the i^{th} body at time t. For each body, we define the density $\bar{\rho}_i > 0$, the center of gravity $\boldsymbol{x}_{G_i}(t)$ and its velocity $\boldsymbol{w}_{G_i}(t)$, the velocity field $\boldsymbol{w}_i$, the rotation vector $\boldsymbol{R}_i(t)$ and the symmetric inertial matrix $J_i \in \mathscr{M}(\mathbb{R}^d)$ by



$$\bar{\rho}_i = \frac{m_i}{|\Omega_i(0)|}, \quad \boldsymbol{x}_{G_i}(t) = \frac{1}{|\Omega_i(t)|}\int_{\Omega_i(t)} \boldsymbol{x}\, d\boldsymbol{x}, \quad \boldsymbol{w}_{G_i}(t) = \frac{d\boldsymbol{x}_{G_i}(t)}{dt},$$

$$\boldsymbol{w}_i(t,\boldsymbol{x}) = \boldsymbol{w}_{G_i}(t) + \boldsymbol{R}_i(t) \times (\boldsymbol{x} - \boldsymbol{x}_{G_i}(t)) \text{ for } \boldsymbol{x} \in \Omega_i(t), \tag{3}$$

$$^t\boldsymbol{y} J_i \boldsymbol{y} = \bar{\rho}_i \int_{\Omega_i(0)} |\boldsymbol{y} \times (\boldsymbol{x} - \boldsymbol{x}_{G_i}(0))|^2\, d\boldsymbol{x} \text{ for all } \boldsymbol{y} \in \mathbb{R}^d.$$

The evolution law of the i^{th} body is given by

$$m_i \frac{d\boldsymbol{w}_{G_i}}{dt} = \int_{\partial\Omega_i(t)} \sigma \cdot \boldsymbol{n}\, d\tau + \int_{\Omega_i(t)} \bar{\rho}_i \boldsymbol{f}\, d\boldsymbol{x}, \tag{4}$$

$$J_i \frac{d\boldsymbol{R}_i}{dt} = \boldsymbol{R}_i \times (J_i \cdot \boldsymbol{R}_i) + \int_{\partial\Omega_i(t)} (\boldsymbol{x} - \boldsymbol{x}_{G_i}(t)) \times (\sigma \cdot \boldsymbol{n})\, d\tau$$

$$+ \int_{\Omega_i(t)} \bar{\rho}_i (\boldsymbol{x} - \boldsymbol{x}_{G_i(t)}) \times \boldsymbol{f}\, d\boldsymbol{x}, \tag{5}$$

$\sigma = 2\mu \mathrm{D}(\boldsymbol{v}) - p\boldsymbol{I}$ denoting the stress tensor of the fluid, where the strain tensor $\mathrm{D}(\boldsymbol{v})$ is defined as the symmetric part of $\nabla \boldsymbol{v}$. Indeed, the local force applied by the fluid on an elementary surface $d\tau$ of $\partial\Omega_i(t)$ with normal $\boldsymbol{n}$ pointing outside the solid is $\sigma \cdot \boldsymbol{n}\, d\tau$. Next, we write the initial condition on $\boldsymbol{w}_i$ as

$$\boldsymbol{w}_{i\,|t=0}(\boldsymbol{x}) = \boldsymbol{w}_i^0(\boldsymbol{x}) = \boldsymbol{w}_{G_i}(0) + \boldsymbol{R}_i(0) \times (\boldsymbol{x} - \boldsymbol{x}_{G_i}(0)). \tag{6}$$

At the domain boundary, we enforce homogeneous Dirichlet boundary conditions

$$\boldsymbol{v} = 0 \text{ on } \partial\Omega \cap \partial\Omega_F(t), \tag{7}$$

and at the interface between the fluid and solid bodies, we require the velocity and the stress to be continuous in the normal direction

$$\boldsymbol{w}_i \cdot \boldsymbol{n}_i = \boldsymbol{v} \cdot \boldsymbol{n}_i \text{ and } \sigma \cdot \boldsymbol{n}_i = T_i \text{ on } \partial\Omega_i(t), \quad t \geqq 0, \tag{8}$$

where $-T_i$ is the force applied by the i^{th} body on the fluid. Notice that it can be expressed in terms of the Cauchy stress tensor Σ_i in the solid as $T_i = \Sigma_i \cdot \boldsymbol{n}_i$.

This problem is "formally well-posed", since *a priori* bounds for finite energy solutions are easy to derive from (1)–(8). A similar observation for a somewhat different model was already made in [6, 7, 11].

In the case of one heavy enough rigid disk interacting with a two-dimensional viscous incompressible fluid, local existence of strong solutions has been proved in [10, 11]. In the case of one rigid body in the whole space under the action of gravity, see [13, 14]. See also [8] for a different problem in the whole space.

In this paper, we introduce a global weak formulation for the above problem. For initial velocities in $H_0^1(\Omega)$, we show the existence of solutions by using techniques which are related to those used to solve the multifluid Navier-Stokes problem (see [12, 1])). One of the key arguments involves the DiPerna-Lions theorem on compactness of sequences of solutions to linear transport equations [5]. Our method requires some additional regularity: $\partial_t \boldsymbol{u} \in L^2((0,T)\times\Omega)^d$. This explains why our results are limited to solutions local in time in dimension 3, since such a regularity property for global weak solutions to the homogeneous Navier-Stokes equations (without solids) is still unknown. Apart from this limitation, the lifespan can also be affected by the appearance of body-body or body-boundary collisions.

In a forthcoming work, we shall address the case of deformable bodies embedded in an incompressible fluid.

The paper is organized as follows. In Section 2, we introduce the weak formulation for the fluid-solid interaction problem. Our main theorem is stated in Section 3, and is proved in Section 5. Section 4 is devoted to the derivation of *a priori* bounds and compactness results needed in Section 5, where we sketch the approximation procedure. The construction of approximate solutions will be found with greater details and in a more general framework in [4].

2. Alternative Formulation

In order to give a more global formulation of $(1)-(8)$, we introduce the Eulerian densities $\rho_F(t,\boldsymbol{x}) = \bar{\rho}_F 1_{\Omega_F(t)}(\boldsymbol{x})$, $\rho_i(t,\boldsymbol{x}) = \bar{\rho}_i 1_{\Omega_i(t)}(\boldsymbol{x})$ and the global density of the system $\rho = \rho_F + \sum_{i=1}^k \rho_i$. We also define in Ω the global velocity $\boldsymbol{u}$ by

$$\boldsymbol{u}(t,x) = \begin{cases} \boldsymbol{v}(t,x) \ \text{ in } \ \Omega_F(t), \\ \boldsymbol{w}_i(t,x) \ \text{ in } \ \Omega_i(t), \ \ 1 \leqq i \leqq k. \end{cases} \tag{9}$$

Thus, in view of the conservation of mass, the density function ρ is the solution of the linear transport equation

$$\partial_t \rho + \operatorname{div}(\rho \boldsymbol{u}) = 0. \tag{10}$$

On the other hand, denoting the global rate-of-deformation tensor by $\mathrm{D}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + {}^t\nabla \boldsymbol{u})$, we can formulate the evolution of the momentum for the fluid as

$$\begin{aligned}\partial_t(\rho_F \boldsymbol{u}) &+ \operatorname{div}(\rho_F \boldsymbol{u} \otimes \boldsymbol{u}) \\ &= \frac{1}{\bar{\rho}_F} \operatorname{div}\left(\rho_F\left(2\mu \mathrm{D}(\boldsymbol{u}) - p\boldsymbol{I}\right)\right) + \sum_{i=1}^{k} \frac{1}{\bar{\rho}_i} \Sigma_i \cdot \nabla \rho_i + \rho_F \boldsymbol{f}.\end{aligned} \tag{11}$$

Note that $\Sigma_i \cdot \nabla \rho_i$ is supported on $\partial \Omega_i(t)$ since it is a surface force.

On the walls, we enforce homogeneous Dirichlet boundary conditions $\boldsymbol{u}_{|\partial\Omega} = 0$. Moreover, the incompressibility of the fluid, the rigidity of the structure and (8) readily imply that $\operatorname{div} \boldsymbol{u} = 0$.

In order to derive the equations for the solids, we remark that (4), (5) can be expressed in terms of Eulerian quantities as follows

$$\partial_t(\rho_i \boldsymbol{u}) + \operatorname{div}(\rho_i \boldsymbol{u} \otimes \boldsymbol{u}) = \rho_i \boldsymbol{f} + \frac{1}{\bar{\rho}_i} \operatorname{div}(\rho_i \Sigma_i) - \frac{1}{\bar{\rho}_i} \sigma \cdot \nabla \rho_i, \tag{12}$$

where the right-hand side is the sum of the external force, the internal rigidity force, and the force applied by the fluid on the surface.

Summing (11) and the k equations (12), using (8) and introducing the global stress tensor $\mathscr{T}$, we obtain the global system in $\mathscr{D}'((0,T)\times\Omega)^d$:

$$\partial_t(\rho \boldsymbol{u}) + \operatorname{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) = \operatorname{div} \mathscr{T} + \rho \boldsymbol{f}, \tag{13}$$

$$\mathscr{T} = \frac{1}{\bar{\rho}_F} \rho_F \sigma + \sum_{i=1}^{k} \frac{1}{\bar{\rho}_i} \rho_i \Sigma_i, \tag{14}$$

$$\operatorname{div} \boldsymbol{u} = 0, \quad \partial_t \rho + \operatorname{div}(\rho \boldsymbol{u}) = 0, \quad \rho_i \mathrm{D}(\boldsymbol{u}) = 0 \quad 1 \leqq i \leqq k, \tag{15}$$

supplemented with the initial conditions

$$\boldsymbol{u}_{|t=0}(\boldsymbol{x}) = \begin{cases} \boldsymbol{v}_0(\boldsymbol{x}) \text{ in } \Omega_F(0), \\ \boldsymbol{w}_i^0(\boldsymbol{x}) = \boldsymbol{w}_{G_i}(0) + \boldsymbol{R}_i(0) \times (\boldsymbol{x} - \boldsymbol{x}_{G_i}(0)) \text{ in } \Omega_i(0), \ i = 1, \ldots, k, \end{cases} \tag{16}$$

$$\rho_{i,0}(\boldsymbol{x}) = \bar{\rho}_i 1_{\Omega_i(0)}(\boldsymbol{x}), \quad 1 \leqq i \leqq k, \quad \rho_{F,0}(\boldsymbol{x}) = \bar{\rho}_F 1_{\Omega_F(0)}(\boldsymbol{x}),$$

$$\rho_{|t=0} = \rho_0 := \rho_{F,0} + \sum_{i=1}^{k} \rho_{i,0}, \tag{17}$$

where $(\Omega_F(0), \Omega_1(0), \ldots, \Omega_k(0))$ is a given open partition of Ω. More precisely, we assume that $\Omega_i(0)$ are $C^{1,1}$ open domains such that

$$\Omega_i(0) \subset \Omega, \quad \Omega_i(0) \cap \Omega_j(0) = \emptyset \quad \text{for } i \neq j, \quad \Omega_F(0) = \Omega \Big\backslash \bigcup_{i=1}^{k} \overline{\Omega_i(0)}. \tag{18}$$

Taking the inner product of (13) with $\boldsymbol{u}$ and integrating the product by parts, we obtain the total energy conservation of the system, i.e., $\frac{dE}{dt} = 0$, where $E = E_k + E_d + E_p$, E_k is the kinetic energy:

$$E_k = \int_\Omega \tfrac{1}{2}\rho|\boldsymbol{u}|^2 d\boldsymbol{x} = \int_\Omega \tfrac{1}{2}\rho_F|\boldsymbol{u}|^2 d\boldsymbol{x} + \sum_{i=1}^{k} \tfrac{1}{2}\left(m_i|\boldsymbol{u}_{G_i}|^2 + {}^t\boldsymbol{R}_i J_i \boldsymbol{R}_i\right), \quad (19)$$

E_d is the viscous dissipation:

$$E_d = \frac{1}{\bar{\rho}_F}\int_0^t\int_\Omega \mu\rho_F\, \mathrm{D}(\boldsymbol{u}) : \mathrm{D}(\boldsymbol{u})\, d\boldsymbol{x}\, ds, \quad (20)$$

and E_p the total work of the external forces $\boldsymbol{f}$

$$E_p = -\int_0^t \int_\Omega \rho\, \boldsymbol{f}\cdot\boldsymbol{u}\, d\boldsymbol{x}\, ds. \quad (21)$$

Thus, formally, the energy satisfies the *a priori* estimate

$$\int_\Omega \tfrac{1}{2}\rho|\boldsymbol{u}|^2\, d\boldsymbol{x} + \mu\int_0^t\int_\Omega |\nabla\boldsymbol{u}|^2\, d\boldsymbol{x}ds \leqq \int_\Omega \tfrac{1}{2}\rho_0|\boldsymbol{u}_0|^2 d\boldsymbol{x} + \int_0^t\int_\Omega \rho\, \boldsymbol{f}\cdot\boldsymbol{u}\, d\boldsymbol{x}\, ds \quad (22)$$

Notice that in the case of curl-free time-independent bulk forces $\boldsymbol{f} = \nabla W$ (for instance gravity for which $W(\boldsymbol{x}) = -\gamma x_d$), we have

$$\int_0^t\int_\Omega \rho\, \boldsymbol{f}\cdot\boldsymbol{u}\, d\boldsymbol{x}\, ds = \int_\Omega W(\boldsymbol{x})\rho(t,\boldsymbol{x})\, d\boldsymbol{x} - \int_\Omega W(\boldsymbol{x})\rho_0(\boldsymbol{x})\, d\boldsymbol{x}, \quad (23)$$

Let us now make precise the notion of weak solutions of the above physical system.

We say that $(\rho, \boldsymbol{u})$ is a weak solution of (13)–(17) in $(0, T)$ if it satisfies the *a priori* energy bounds

$$\rho \in L^\infty((0,T)\times\Omega), \quad \boldsymbol{u} \in L^\infty(0,T; L^2(\Omega))^d \cap L^2(0,T; H_0^1(\Omega))^d,$$

and if for all $\phi \in \mathscr{V}$ and for almost every $t \in (0, T)$,

$$\int_0^t\int_\Omega \big(\rho\boldsymbol{u}\cdot\partial_t\phi + \rho\boldsymbol{u}\otimes\boldsymbol{u} : \mathrm{D}(\phi) - \mu\mathrm{D}(\boldsymbol{u}) : \mathrm{D}(\phi) + \rho\boldsymbol{f}\cdot\phi\big)\, d\boldsymbol{x}\, ds + \int_\Omega \rho_0\boldsymbol{u}_0\cdot\phi(0)\, d\boldsymbol{x} = \Big(\int_\Omega \rho\boldsymbol{u}\cdot\phi\, d\boldsymbol{x}\Big)(t), \quad (24)$$

$$\partial_t\rho + \mathrm{div}(\rho\boldsymbol{u}) = 0 \ \text{ in } \ \mathscr{D}'((0,T)\times\Omega), \quad (25)$$

$$\mathrm{div}\,\boldsymbol{u} = 0, \ \ \rho_i\mathrm{D}(\boldsymbol{u}) = 0, \ \ 1 \leqq i \leqq k, \quad (26)$$

$$\boldsymbol{u}_{|\partial\Omega} = 0, \ \ \rho_0 \in L^\infty(\Omega) \ \text{ satisfies (17)}, \ \ \boldsymbol{u}_0 \in L^2(\Omega)^d, \quad (27)$$

where $\mathscr{V}$ is defined by

$$\mathscr{V} = \left\{\phi \in H^1((0,T)\times\Omega)^d \ / \ \phi(t) \in V(t) \ \ \forall t \in (0,T)\right\}, \quad (28)$$

and

$$V(t) = \left\{ \phi \in H_0^1(\Omega)^d \,/\, \operatorname{div} \phi = 0, \quad \rho_i(t)\mathrm{D}(\phi) = 0, \quad 1 \leqq i \leqq k \right\}. \tag{29}$$

In this formulation, the Lagrange multipliers of the problem, namely the pressure p and the Cauchy stress tensors Σ_i of the solids, no longer appear, since the corresponding constraints are taken into account by the choice of test functions.

3. Main Theorem

Let us define the minimal distance $\delta(t)$ by

$$\delta(t) = \min \left\{ \mathrm{d}(\Omega_i(t), \Omega_j(t)), \mathrm{d}(\Omega_i(t), \partial\Omega), \; i, j = 1, \ldots, k \; ; \; i \neq j \right\},$$

and make the following assumptions on the data:

$$\boldsymbol{u}_0 \in H_0^1(\Omega)^d, \quad \operatorname{div} \boldsymbol{u}_0 = 0, \quad \rho_{i,0}\mathrm{D}(\boldsymbol{u}_0) = 0, \quad 1 \leqq i \leqq k, \tag{30}$$

$$\boldsymbol{f} \in L^2((0, T) \times \Omega)^d \quad \text{for all} \quad T > 0, \tag{31}$$

$$\delta(0) > 0. \tag{32}$$

We now state our main result:

Theorem 1. *Under assumptions* (30)–(32)*, there exist* $T^* \in (0, +\infty]$ *and a solution* $(\rho, \boldsymbol{u})$ *of* (24)–(27) *such that*

- $\beta(\rho) \in C([0, T]; L^p(\Omega)) \cap L^\infty((0, \infty) \times \Omega)$ *for all* $T < T^*$*,* $p < \infty$ *and* $\beta \in C^1(\mathbb{R}; \mathbb{R})$.
- $\boldsymbol{u} \in L^\infty(0, T; H_0^1(\Omega))^d$ *and* $\partial_t \boldsymbol{u} \in L^2((0, T) \times \Omega)^d$ *for all* $T < T^*$.
- *If* $d = 2$*, then* $T^* = \min\{ t \,/\, \delta(t) = 0\}$*, and* $\boldsymbol{u} \in L^2(0, T; W_0^{1,p}(\Omega))^2$ *for all* $p < +\infty$*,* $T < T^*$.
- *If* $d = 3$*, then* $\boldsymbol{u} \in L^2(0, T; W_0^{1,6}(\Omega))^3$ *for all* $T < T^*$ *and* $T^* = +\infty$ *only if* $|\boldsymbol{f}|_{L^2((0,+\infty)\times\Omega)} + |\nabla \boldsymbol{u}_0|_{L^2(\Omega)}$ *is small enough and* $\delta(t) > 0$ *for all* $t > 0$.

Moreover, the energy inequality (22) *holds for all time* $t \in (0, T^*)$.

Remark. In dimension $d = 2$, a more precise regularity result is available for the velocity $\boldsymbol{u}$: There exists $\gamma \in L^2_{\mathrm{loc}}([0, T^*))$ such that for all $p \in [2, \infty)$

$$|\nabla \boldsymbol{u}(t, .)|_{L^p(\Omega)} \leqq C\sqrt{p}\,\gamma(t) \quad \text{for all} \quad t < T^*. \tag{33}$$

4. Auxiliary Results

4.1. Elliptic Estimates in Time-Dependent Domains

Let $t \in (0, \infty)$ be such that $\delta(t) > 0$ and consider the Stokes problem satisfied by $\boldsymbol{u}$ in the fluid domain

$$-\mu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{g}, \ \operatorname{div}\boldsymbol{u} = 0 \quad \text{in} \ \ \Omega_F(t), \ \boldsymbol{u}_{|\partial\Omega} = 0, \quad \boldsymbol{u}_{|\partial\Omega_i(t)} = \boldsymbol{w}_{i\,|\partial\Omega_i(t)}.$$

That $\boldsymbol{u}$ and $\boldsymbol{w}_i$, and not only their normal components, coincide on $\partial\Omega_i(t)$ derives from the viscosity term in the equation which yields H^1 *a priori* bounds on $\boldsymbol{u}$ (see (22) and the beginning of the next section). By using *ad hoc* cut-off functions and using the rigid structure of the velocity in $\Omega_i(t)$, we deduce in dimension 2 that for all $p \geqq 2$,

$$\begin{aligned}|\nabla\boldsymbol{u}|_{L^p(\Omega)} \leqq\ & C\sqrt{p}\,|\boldsymbol{g}|^{1-2/p}_{L^2(\Omega_F(t))}\Big(|\nabla\boldsymbol{u}|_{L^2(\Omega)} + \delta^{-1}|\boldsymbol{u}|_{L^2(\Omega)}\Big)^{2/p}\\ &+ C\sqrt{p}\Big(|\nabla\boldsymbol{u}|_{L^2(\Omega)} + \delta^{-1}|\boldsymbol{u}|_{L^2(\Omega)}\Big)\delta^{(2/p)-1},\end{aligned}$$

whereas in dimension 3, we have

$$|\nabla\boldsymbol{u}|_{L^6(\Omega)} \leqq C\big(|\boldsymbol{g}|_{L^2(\Omega_F(t))} + \delta^{-1}|\nabla\boldsymbol{u}|_{L^2(\Omega)} + \delta^{-2}|\boldsymbol{u}|_{L^2(\Omega)}\big),$$

so that in particular

$$\begin{aligned}|\nabla\boldsymbol{u}|_{L^{2d/(d-1)}(\Omega)} \leqq\ & C\big(|\nabla\boldsymbol{u}|_{L^2(\Omega)} + (3-d)\delta^{-1}|\boldsymbol{u}|_{L^2(\Omega)}\big)^{1/2}|\boldsymbol{g}|^{1/2}_{L^2(\Omega_F(t))}\\ &+ C\delta^{-1/2}|\nabla\boldsymbol{u}|_{L^2(\Omega)} + C\delta^{-3/2}|\boldsymbol{u}|_{L^2(\Omega)}. \qquad (34)\end{aligned}$$

4.2. A Priori Bounds on the Velocity

Taking $\phi = \boldsymbol{u}$ in (24), we obtain formally

$$\begin{aligned}&\int_\Omega \tfrac{1}{2}\rho(t)|\boldsymbol{u}(t)|^2 d\boldsymbol{x} + \mu\int_0^t\int_\Omega |\nabla\boldsymbol{u}|^2 d\boldsymbol{x}\,ds\\ &\quad = \int_0^t\int_\Omega \rho\boldsymbol{f}\cdot\boldsymbol{u}\,d\boldsymbol{x}\,ds + \int_\Omega \tfrac{1}{2}\rho_0|\boldsymbol{u}_0|^2 d\boldsymbol{x}, \qquad (35)\end{aligned}$$

so that we have the natural energy bounds $\rho \in L^\infty((0,T)\times\Omega)$, $\sqrt{\rho}\boldsymbol{u} \in L^\infty(0,T; L^2(\Omega))^d$, $\boldsymbol{u} \in L^2(0,T; H^1_0(\Omega))^d$ provided that $\boldsymbol{f} \in L^2((0,T)\times\Omega)^d$. Let us point out that in the case when $\boldsymbol{f}$ reduces to gravity forces: $\boldsymbol{f} = -\gamma\boldsymbol{e}_d$ with $\gamma > 0$, the left-hand side of (35) can be estimated globally in time by

$$\int_\Omega \tfrac{1}{2}\rho_0|\boldsymbol{u}_0|^2 d\boldsymbol{x} + 2\gamma\int_\Omega \rho_0 d\boldsymbol{x}. \qquad (36)$$

Starting from $H^1_0(\Omega)$ initial velocities $\boldsymbol{u}_0$, we obtain additional bounds by taking $\phi = \partial_t\boldsymbol{u}$ in (24) and integrating by parts. Indeed, assuming that $\boldsymbol{u}$ is suitably smooth, which will be the case in the approximate problem introduced in Section 5, $\partial_t \boldsymbol{u}$ can be taken as a test function in the weak formulation, since $\rho_i \mathrm{D}\boldsymbol{u} = 0$ implies that $\rho_i \mathrm{D}(\partial_t \boldsymbol{u}) = 0$. Thus, by the Cauchy-Schwarz inequality, there exists a constant $C > 0$ depending only on μ and $|\rho_0|_{L^\infty(\Omega)}$ such that

$$\int_0^t \int_\Omega \rho |\partial_t \boldsymbol{u}|^2 d\boldsymbol{x}\, ds + \int_\Omega |\nabla \boldsymbol{u}(t)|^2 d\boldsymbol{x} \leqq C \int_\Omega |\nabla \boldsymbol{u}_0|^2 d\boldsymbol{x} + C \int_0^t |\boldsymbol{f}|^2_{L^2(\Omega)} ds + C \int_0^t |\boldsymbol{u} \cdot \nabla \boldsymbol{u}|^2_{L^2(\Omega)}\, d\boldsymbol{x} ds. \tag{37}$$

We now have to estimate $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$ in terms of the left-hand side of (37). Writing the equation for the fluid as a Stokes problem at some fixed time $t > 0$ and using the results of the preceding section and the Gagliardo-Nirenberg inequality written as

$$|\boldsymbol{u}|_{L^{2d}(\Omega)} \leqq C |\boldsymbol{u}|^{(3-d)/2}_{L^2(\Omega)} |\boldsymbol{u}|^{(d-1)/2}_{H^1_0(\Omega)},$$

we deduce that

$$\begin{aligned}
|\boldsymbol{u} \cdot \nabla \boldsymbol{u}|^2_{L^2(\Omega)} &\leqq |\boldsymbol{u}|^2_{L^{2d}(\Omega)} |\nabla \boldsymbol{u}|^2_{L^{(2d)/d-1}(\Omega)} \\
&\leqq C\delta^{-1} |\boldsymbol{u}|^{3-d}_{L^2(\Omega)} |\nabla \boldsymbol{u}|^{d+1}_{L^2(\Omega)} + C\delta^{-3} |\boldsymbol{u}|^{5-d}_{L^2(\Omega)} |\nabla \boldsymbol{u}|^{d-1}_{L^2(\Omega)} \\
&\quad + C |\boldsymbol{u}|^{3-d}_{L^2(\Omega)} |\nabla \boldsymbol{u}|^{d-1}_{L^2(\Omega)} \, |\rho \partial_t \boldsymbol{u} + \rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \rho \boldsymbol{f}|_{L^2(\Omega)} \\
&\qquad \times \big(|\nabla \boldsymbol{u}|_{L^2(\Omega)} + (3-d)\delta^{-1} |\boldsymbol{u}|_{L^2(\Omega)} \big) \qquad (38) \\
&\leqq \varepsilon \, |\rho \partial_t \boldsymbol{u} + \rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \rho \boldsymbol{f}|^2_{L^2(\Omega)} + C\varepsilon^{-1} |\boldsymbol{u}|^{2(3-d)}_{L^2(\Omega)} |\nabla \boldsymbol{u}|^{2d}_{L^2(\Omega)} \\
&\quad + C(3-d)\varepsilon^{-1}\delta^{-2} |\boldsymbol{u}|^{2(4-d)}_{L^2(\Omega)} |\nabla \boldsymbol{u}|^{2(d-1)}_{L^2(\Omega)} \\
&\quad + C\delta^{-1} |\boldsymbol{u}|^{3-d}_{L^2(\Omega)} |\nabla \boldsymbol{u}|^{d+1}_{L^2(\Omega)} + C\delta^{-3} |\boldsymbol{u}|^{5-d}_{L^2(\Omega)} |\nabla \boldsymbol{u}|^{d-1}_{L^2(\Omega)} \qquad (39)
\end{aligned}$$

for all $\varepsilon > 0$. Denote

$$A_0(t) = |\nabla \boldsymbol{u}_0|^2_{L^2(\Omega)} + \int_0^t |\boldsymbol{f}|^2_{L^2(\Omega)} ds, \quad B_0(t) = |\boldsymbol{u}_0|^2_{L^2(\Omega)} + \int_0^t |\boldsymbol{f}|^2_{L^2(\Omega)} ds.$$

In dimension 2, for ε small enough, (37), (39) and Poincaré's inequality yield

$$\begin{aligned}
&\int_0^t |\partial_t \boldsymbol{u}|^2_{L^2(\Omega)} ds + |\nabla \boldsymbol{u}(t)|^2_{L^2(\Omega)} \\
&\quad \leqq C \Big(A_0(t) + \int_0^t \Big(\big((1 + B_0(t)^2) \delta(s)^{-2} + B_0(t) \delta(s)^{-3} \big) |\nabla \boldsymbol{u}|^2_{L^2(\Omega)} \qquad (40) \\
&\quad + B_0(t) |\nabla \boldsymbol{u}|^4_{L^2(\Omega)} \Big) \, ds \Big),
\end{aligned}$$

whereas in dimension $d = 3$, we obtain

$$\int_0^t |\partial_t \boldsymbol{u}|^2_{L^2(\Omega)} ds + |\nabla \boldsymbol{u}(t)|^2_{L^2(\Omega)} \leqq C\Big(A_0(t) + \int_0^t \Big(\Big(\delta(s)^{-2} + B_0(t)\delta(s)^{-3}\Big) |\nabla \boldsymbol{u}|^2_{L^2(\Omega)} + |\nabla \boldsymbol{u}|^6_{L^2(\Omega)}\Big) ds\Big). \tag{41}$$

Estimates (40), (41) enable us to conclude our proof by looking separately at the 2- and 3-dimensional cases. In dimension $d = 2$, since $\int_0^t |\nabla \boldsymbol{u}|^2_{L^2(\Omega)} ds \leqq C B_0(t)$, Gronwall's lemma allows us to prove that

$$\int_0^t |\partial_t \boldsymbol{u}|^2_{L^2(\Omega)} ds + |\nabla \boldsymbol{u}(t)|^2_{L^2(\Omega)} \leqq C A_0(t) \exp\Big(C B_0(t)^2 + C \int_0^t \Big((1 + B_0(t)^2)\delta(s)^{-2} + B_0(t)\delta(s)^{-3}\Big) ds\Big). \tag{42}$$

In dimension $d = 3$, there exists $\eta > 0$ such that

$$\int_0^t |\partial_t \boldsymbol{u}|^2_{L^2(\Omega)} ds + |\nabla \boldsymbol{u}(t)|^2_{L^2(\Omega)} \leqq C A_0(t) H(t), \tag{43}$$

where

$$H(t) = \exp\left(C \int_0^t \Big(\delta(s)^{-2} + B_0(t)\delta(s)^{-3}\Big) ds\right),$$

provided that either the time t is small enough:

$$t A_0(t)^2 H(t)^3 \leqq \eta, \tag{44}$$

or the data are not too large:

$$B_0(t) A_0(t) H(t)^2 \leqq \eta. \tag{45}$$

The preceding inequalities show that if $\delta(0) > 0$, $\delta(t)$ remains positive for $t > 0$ small enough. Indeed,

$$\delta(t) \geqq \delta(0) - \int_0^t |\boldsymbol{u}(s)|_{L^\infty(\Omega)} ds, \tag{46}$$

and in dimensions 2 and 3, $W_0^{1,4}(\Omega)$ is embedded in $L^\infty(\Omega)$.

Remark: The above estimates show that smooth enough solutions of (24)–(27) satisfy $\boldsymbol{u} \in L^2(0, T; H_0^1(\Omega))^d \cap L^\infty(0, T; L^2(\Omega))^d$ as long as they exist. Moreover, *a priori* bounds for $\partial_t \boldsymbol{u}$ in $L^2((0, T) \times \Omega)^d$ and for $\boldsymbol{u}$ in $L^2(0, T; W^{1,4}(\Omega))^d$ are available whenever there is no collision, and the above criteria on small time or small data are met in dimension 3. Note finally that $\partial_t \boldsymbol{u} \in L^2((0, T) \times \Omega)^d$ and $\boldsymbol{u} \in L^2(0, T; W^{1,4}(\Omega))^d$ imply for instance that $\boldsymbol{u} \in C^{0,\frac{1}{2}}([0, T]; H_0^1(\Omega))^d$.

4.3. Compactness Results

Let $(\rho^n, \boldsymbol{u}^n)$ be a sequence of weak solutions of (24)–(27) in $(0, T)$ such that ρ^n is bounded in $L^\infty((0,T)\times\Omega)$ uniformly in n, $\boldsymbol{u}^n$ is bounded in $L^2(0,T; H_0^1(\Omega)\cap W^{1,4}(\Omega))^d$, $\partial_t \boldsymbol{u}^n$ is bounded in $L^2((0,T)\times\Omega)^d$ uniformly in n, ρ_0^n converges to ρ_0 in $L^2(\Omega)$, $\boldsymbol{u}_0^n$ converges to $\boldsymbol{u}^0$ in $L^2(\Omega)^d$, for all $\phi^n \in \mathscr{V}_n$,

$$\left(\int_\Omega \rho^n \boldsymbol{u}^n \cdot \phi^n d\boldsymbol{x}\right)(t) = \int_\Omega \rho_n^0 \boldsymbol{u}_n^0 \cdot \phi^n(0,.)\, d\boldsymbol{x} + \int_0^t \int_\Omega \Big(\rho^n \boldsymbol{u}^n \cdot \partial_t \phi^n + \rho^n \boldsymbol{u}^n \otimes \boldsymbol{u}^n : \mathrm{D}(\phi^n) - \mu \mathrm{D}(\phi^n):\mathrm{D}(\boldsymbol{u}^n) + \rho^n \boldsymbol{f}\cdot\phi^n\Big) d\boldsymbol{x}\, ds, \tag{47}$$

$$\partial_t \rho^n + \operatorname{div}(\rho^n \boldsymbol{u}^n) = 0, \quad \operatorname{div} \boldsymbol{u}^n = 0, \quad \rho_i^n \mathrm{D}(\boldsymbol{u}^n) = 0, \quad 1 \leqq i \leqq k, \tag{48}$$

$$\alpha = \inf\left\{\delta_n(t),\ t \in [0,T],\ n \geqq 0\right\} > 0, \tag{49}$$

where δ_n is defined as δ in Section 3 with $\boldsymbol{u}$ replaced by $\boldsymbol{u}^n$.

The bounds derived in Sections 4.1, 4.2 allow us to use the DIPERNA-LIONS stability results [5] for linear transport equations: There exists $(\rho, \boldsymbol{u})$ such that up to the extraction of a subsequence, $\beta(\rho^n)$ converges to $\beta(\rho)$ in $L^\infty((0,T)\times\Omega)$ weak $*$ and in $C([0,T]; L^p(\Omega))$ for all $p < +\infty$ and all $\beta \in C^1(\mathbb{R})$, and $\boldsymbol{u}^n$ converges to $\boldsymbol{u}$ in $C([0,T]; H_0^s(\Omega))^d$ for all $s < 1$. Moreover, $\boldsymbol{u} \in L^2(0,T; W^{1,4}(\Omega))^d$, $\partial_t \boldsymbol{u} \in L^2((0,T)\times\Omega)^d$, $\operatorname{div} \boldsymbol{u} = 0$, $\rho_i\ \mathrm{D}(\boldsymbol{u}) = 0$ for $1 \leqq i \leqq k$, and $\partial_t \rho + \operatorname{div}(\rho \boldsymbol{u}) = 0$. We now have to prove that (47) holds for all given $\phi \in \mathscr{V}$. First, for $1 \leqq i \leqq k$ we introduce the invertible affine transformation $T_i^n(t)$ defined by $\Omega_i^n(t) = T_i^n(t)\Omega_i^n(0)$, and similarly we introduce T_i, and we observe that for n large enough

$$\sup_{1\leqq i\leqq k,\ t\in(0,T)} \left(|T_i^n(t) - T_i(t)| + \left|\dot{T}_i^n(t) - \dot{T}_i(t)\right|\right) \leqq C_T \sup_{1\leqq i\leqq k} |\rho_i^n \boldsymbol{u}^n - \rho_i \boldsymbol{u}|_{L^\infty(0,T;L^2(\Omega))} \leqq \frac{\alpha}{4}, \tag{50}$$

since $\rho_i^n \boldsymbol{u}^n$ converges to $\rho_i \boldsymbol{u}$ in $C([0,T]; L^2(\Omega))^d$. Next considering a cut-off function $\chi \in C^\infty(\mathbb{R}^+)$ such that $\chi \equiv 1$ in $[0, \frac{1}{2}]$ and $\chi \equiv 0$ in $[1, \infty)$, we define ϕ_i as

$$\phi_i(t, \boldsymbol{x}) = \operatorname{curl}\left(\chi\left(\frac{2\mathrm{d}(\boldsymbol{x}, \Omega_i(t))}{\alpha}\right) \boldsymbol{A}(t, \boldsymbol{x})\right),$$

where $\boldsymbol{A}$ is a current function for ϕ in Ω, i.e., $\phi = \operatorname{curl} \boldsymbol{A}$ (curl denotes $\nabla\times$ when $d = 3$, and denotes $\nabla^\perp$ when $d = 2$). We now define ψ as

$$\psi = \phi - \sum_{i=1}^k \phi_i.$$

Hence, we construct a new test function $\phi^n \in \mathscr{V}_n$ as

$$\phi^n(t, \boldsymbol{x}) = \psi(t, \boldsymbol{x}) + \sum_{i=1}^{k} \phi_i \left(t, T_i(t) \circ T_i^n(t)^{-1} \cdot \boldsymbol{x} \right),$$

for which we know that (47) holds. The preceding bounds (50) imply that ϕ^n converges to ϕ in $C([0,T]; L^2(\Omega))^d$. Moreover, $\mathrm{D}(\phi^n)$ and $\partial_t \phi^n$ converge respectively to $\mathrm{D}(\phi)$ and $\partial_t \phi$ in $L^2((0,T) \times \Omega)^d$, so that all the quantities in (47) pass to the limit.

5. Proof of Theorem 1

For any $\varepsilon > 0$ such that $\varepsilon < \frac{1}{4}\delta(0)$, let $\rho_0^\varepsilon \in C^\infty(\Omega)$ be a regularization of ρ_0 defined as

$$\rho_{i,0}^\varepsilon = 0 \text{ in } \Omega_F(0),\ \rho_{i,0}^\varepsilon(\boldsymbol{x}) = \bar{\rho}_i \text{ if } \mathrm{d}(\boldsymbol{x}, \Omega_F(0)) \geqq 2\varepsilon,\ 0 < \rho_{i,0}^\varepsilon < \bar{\rho}_i \text{ otherwise,}$$

$$\rho_{F,0}^\varepsilon = \bar{\rho}_F \left(1 - \sum_{i=1}^{k} \frac{\rho_{i,0}^\varepsilon}{\bar{\rho}_i} \right), \quad \rho_0^\varepsilon := \rho_{F,0}^\varepsilon + \sum_{i=1}^{k} \rho_{i,0}^\varepsilon,$$

$$\rho_{i,0}^\varepsilon \text{ converges to } \rho_{i,0} \text{ in } L^p(\Omega)\ (1 \leqq p < +\infty,\ 1 \leqq i \leqq k).$$

Also, we introduce a regularizing operator r_ε which maps $L^2(0,T; H_0^1(\Omega))^d$ into $L^2(0,T; C_0^k(\Omega))^d$ for all $k \geqq 0$, such that $\operatorname{div} r_\varepsilon(\boldsymbol{u}) = 0$ for all $\boldsymbol{u}$ for which $\operatorname{div} \boldsymbol{u} = 0$. Then, by following the same steps as in the proof of Theorem 2.6 in [12], we find that $\rho^\varepsilon \in L^\infty((0,T) \times \Omega)$, $\boldsymbol{u}^\varepsilon \in L^\infty(0,T; L^2(\Omega))^d \cap L^2(0,T; H_0^1(\Omega))^d$ are solutions of

$$\int_0^t \int_\Omega \left(\rho^\varepsilon \boldsymbol{u}^\varepsilon \cdot \partial_t \phi + \rho^\varepsilon r_\varepsilon(\boldsymbol{u}^\varepsilon) \otimes \boldsymbol{u}^\varepsilon : \mathrm{D}(\phi) - \mu \mathrm{D}(\boldsymbol{u}^\varepsilon) : \mathrm{D}(\phi) + \rho^\varepsilon \boldsymbol{f} \cdot \phi \right) d\boldsymbol{x}\, dt$$
$$+ \int_\Omega \rho_0^\varepsilon r_\varepsilon(\boldsymbol{u}_0) \cdot \phi(0,.)\, d\boldsymbol{x} = \left(\int_\Omega \rho^\varepsilon \boldsymbol{u}^\varepsilon \cdot \phi\, d\boldsymbol{x} \right)(t), \tag{51}$$

$$\operatorname{div} \boldsymbol{u}^\varepsilon = 0,\ \ \rho_i^\varepsilon\, \mathrm{D}(\boldsymbol{u}^\varepsilon) = 0,\ \ 1 \leqq i \leqq k,\ \text{ in } \mathscr{D}'((0,T) \times \Omega),\ \ \boldsymbol{u}^\varepsilon_{|\partial\Omega} = 0, \tag{52}$$

$$\partial_t \rho_i^\varepsilon + \operatorname{div}(r_\varepsilon(\boldsymbol{u}^\varepsilon) \rho_i^\varepsilon) = 0,\ \ \rho^\varepsilon_{i\ |t=0} = \rho_{i,0}^\varepsilon\ \ 1 \leqq i \leqq k, \tag{53}$$

$$\partial_t \rho^\varepsilon + \operatorname{div}(r_\varepsilon(\boldsymbol{u}^\varepsilon) \rho^\varepsilon) = 0,\ \ \rho^\varepsilon_{|t=0} = \rho_0^\varepsilon \tag{54}$$

for almost every $t \in (0,T)$ and for ϕ in the space of test functions defined by

$$\mathscr{V}^\varepsilon = \left\{ \phi \in C^\infty((0,T) \times \Omega)^d \ /\ \phi(t) \in V^\varepsilon(t)\ \forall\, t \in (0,T) \right\}, \tag{55}$$

with

$$V^\varepsilon(t) = \left\{ \phi \in C_0^\infty(\Omega)^d \,/\, \operatorname{div} \phi = 0, \text{ and } \rho_i^\varepsilon(t) \mathrm{D}(\phi) = 0,\ 1 \leqq i \leqq k \right\}. \tag{56}$$

Solving this problem means solving first an appropriate linear problem, then applying a fixed-point theorem. This is made possible by the *a priori* estimates of Section 4.2 and the compactness results of Section 4.3. For all details on this procedure in a different but related situation, see [9].

By the estimates derived in Sections 4.1 and 4.2, we obtain uniform bounds for $(\rho^\varepsilon, \boldsymbol{u}^\varepsilon)$ in $L^\infty((0,T)\times\Omega)\times L^2(0,T;H_0^1(\Omega))^d$. Moreover, $\partial_t \boldsymbol{u}^\varepsilon$ is also uniformly bounded in $L^2((0,T)\times\Omega)^d$ when $d=2$, or when $d=3$ and either T or the initial data are small enough (see (44) and (45)). This essentially follows from the fact that for every $\phi \in \mathscr{V}^\varepsilon$, $\partial_t\phi$ belongs to $\mathscr{V}^\varepsilon$, since by (54), we have $|\mathrm{supp}(\partial_t \rho_i^\varepsilon) \cap \mathrm{supp}(\mathrm{D}(\phi))| = 0$ and so, we can take $\partial_t \boldsymbol{u}^\varepsilon$ as a test function. Let us next take $\tau \in (0,T)$ such that

$$\limsup_{\varepsilon\to 0}\, (\inf\{\delta_\varepsilon(t) \;/\; t\in[0,\tau]\}) > 0. \tag{57}$$

A straightforward modification of the compactness results described in Section 4.3 gives the existence of $(\rho, \boldsymbol{u})$ such that up to the extraction of a subsequence, for all $\beta \in C^1(\mathbb{R})$, $\beta(\rho^\varepsilon)$ converges to $\beta(\rho)$ in $L^\infty((0,\tau)\times\Omega)$ weak $*$ and in $C([0,\tau];L^p(\Omega))$ for all $p<+\infty$, and $\boldsymbol{u}^\varepsilon$ converges to $\boldsymbol{u}$ in $C([0,\tau];H_0^s(\Omega))^d$ for all $s<1$. In addition, $\boldsymbol{u}\in L^2(0,\tau;W^{1,4}(\Omega))^d$, $\partial_t\boldsymbol{u}\in L^2((0,\tau)\times\Omega)^d$, $\mathrm{div}\,\boldsymbol{u}=0$, $\rho_i\,\mathrm{D}(\boldsymbol{u})=0$ for $1\leqq i\leqq k$, and $\partial_t\rho+\mathrm{div}(\rho\boldsymbol{u})=0$. It follows that $(\rho,\boldsymbol{u})$ is a solution of (24)–(27). □

6. Comments

Let us first mention that some unbounded domains can also be treated by similar techniques (see for instance the regularity results in [12] and the approximating process in [1]).

Notice also that the case of N incompressible immiscible fluids interacting with k rigid bodies can be handled similarly, by adapting our weak formulation in a straightforward manner.

Finally, in dimension $d=2$, the $L^2(0,T;W^{1,p}(\Omega)\cap H_0^1(\Omega))^2$ bounds (33) on $\boldsymbol{u}$, for all p, enable us to show that

$$|\boldsymbol{u}(t,\boldsymbol{x})-\boldsymbol{u}(t,\boldsymbol{y})| \leqq C\gamma(t)|\boldsymbol{x}-\boldsymbol{y}|\sqrt{\left|\log\left(|\boldsymbol{x}-\boldsymbol{y}|\wedge\tfrac{1}{2}\right)\right|}$$

for all $(\boldsymbol{x},\boldsymbol{y})\in\Omega^2$ and $t<T^*$ (see [3]). Hence, we can prove by an easy Gronwall-type lemma that for all $h>0$ and for all $(\boldsymbol{x},\boldsymbol{y})\in\Omega^2$ and $t<T^*$,

$$|X(t,\boldsymbol{x})-X(t,\boldsymbol{y})| \leqq |\boldsymbol{x}-\boldsymbol{y}|^{1/(1+h)} \exp\left(\frac{C}{h}\left(\int_0^t \gamma(s)ds\right)^2\right),$$

where X denotes the Lagrangian flow of $\boldsymbol{u}$ defined by $\dot{X}=\boldsymbol{u}(t,X)$ and $X(0,\boldsymbol{x})=\boldsymbol{x}$. In particular, $X\in C^{0,\alpha}([0,T^*)\times\overline{\Omega})^2$ for all $\alpha\in(0,1)$.

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