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Application of Uniform Asymptotics to the Second Painlevé Transcendent

Andrew P. Bassom, Peter A. Clarkson,
C. K. Law & J. Bryce McLeod

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Abstract

In this work we propose a new method for investigating connection problems for the class of nonlinear second-order differential equations known as the Painlevé equations. Such problems can be characterized by the question as to how the asymptotic behaviours of solutions are related as the independent variable is allowed to pass towards infinity along different directions in the complex plane. Connection problems have been previously tackled by a variety of methods. Frequently these are based on the ideas of isomonodromic deformation and the matching of WKB solutions. However, the implementation of these methods often tends to be heuristic in nature and so the task of rigorising the process is complicated. The method we propose here develops uniform approximations to solutions. This removes the need to match solutions, is rigorous, and can lead to the solution of connection problems with minimal computational effort.

Our method relies on finding uniform approximations of differential equations of the generic form

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\eta^2} = -\xi^2 F(\eta, \xi)\phi$$

as the complex-valued parameter $\xi \to \infty$. The details of the treatment rely heavily on the locations of the zeros of the function F in this limit. If they are isolated, then a uniform approximation to solutions can be derived in terms of Airy functions of suitable argument. On the other hand, if two of the zeros of F coalesce as $|\xi| \to \infty$, then an approximation can be derived in terms of parabolic cylinder functions. In this paper we discuss both cases, but illustrate our technique in action by applying the parabolic cylinder case to the "classical" connection problem associated with the second Painlevé transcendent. Future papers will show how the technique can be applied with very little change to the other Painlevé equations, and to the wider problem of the asymptotic behaviour of the general solution to any of these equations.

1. Introduction

Asymptotic behaviour of solutions of the second Painlevé transcendent (PII),

$$q'' = 2q^3 + xq + \beta, \tag{1.1}$$

where $' \equiv \mathrm{d}/\mathrm{d}x$ and β is a complex constant, have been much studied, for example in [7, 8, 11–30, 32–35, 38–41]. In particular, connection problems have been investigated in which one attempts to relate the asymptotic behaviour in one x-direction to that in another. Some of the results are heuristic, and some rigorous. The heuristic arguments tend to use the method of isomonodromic deformations, linked with asymptotic arguments that use the WKB method and matching, and although DEIFT & ZHOU [8, 9] have given a rigorous version of this for one problem associated with the second Painlevé transcendent (1.1), the task of extending these techniques rigorously to more complicated problems, and in particular to problems associated with the higher equations, seems formidable.

Linear connection problems for ordinary differential equations have been extensively studied for over a hundred years; however, nonlinear connection problems are rare. The usual method for linear equations is to consider x as a complex variable and pass from $x \to +\infty$ to $x \to -\infty$ along a large semi-circle in the complex x-plane. Provided that the coefficients in the equation have a reasonably simple asymptotic behaviour as $x \to \pm\infty$, then it is usually possible to construct an asymptotic expansion for the solution at all points on the semi-circle, and so relate the asymptotic behaviour as $x \to +\infty$ to the asymptotic behaviour as $x \to -\infty$.

However, this method fails in general for nonlinear equations since the solutions may be very complicated as $x \to \pm\infty$; though, for nonlinear equations such as the Painlevé equations which have the Painlevé property (and in particular those equations such as (1.1) whose solutions are meromorphic in the finite complex plane), this method is feasible. Indeed BOUTROUX [5, 6] (see also [4, 17]) studied the asymptotics of the first Painlevé equation in considerable detail and remarks that his ideas can be extended to the other Painlevé equations as well. Essentially, the solutions behave asymptotically like elliptic functions, at least locally, and although not considered by BOUTROUX, the solution of the connection problem is a matter of matching different elliptic functions in different sectors on the large semi-circle in the complex plane. Whereas this method is theoretically feasible, it certainly involves considerable technical difficulties and the connection problems are solvable by a different method. Recently, JOSHI & KRUSKAL [24, 25], discuss how one can extend the ideas of BOUTROUX and use singular perturbative techniques, to obtain connection formulae for the first and second Painlevé equations.

An alternative method for determining connection formulae for the Painlevé equations is through the *isomonodromic deformation technique* (cf. ITS & NOVOKSHENOV [20]). Classically, FUCHS [14], GARNIER [15] and SCHLESINGER [39] considered the Painlevé equations as the isomonodromic conditions for suitable linear systems with rational coefficients possessing regular and irregular singular points (see also [36]). Since the development of the inverse scattering method for solving partial differential equations, there has been renewed interest in expressing the Painlevé equations as isomonodromic conditions for suitable linear systems [11, 21–23]. Subsequently, there has been considerable interest in the use of the isomonodromy method, which is a very powerful technique, to derive properties of the Painlevé equations, including for PII (1.1) [8, 9, 11–13, 18–20, 26–30, 32, 34, 35, 41].

In this paper we develop a new technique for investigating asymptotic problems for the Painlevé equations. The technique uses the method of isomonodromy, but thereafter develops a uniform approximation which dispenses with matching, is rigorous and even from a computational point of view is simpler than previous methods. We will use it in this paper to study the asymptotic behaviour of solutions of PII (1.1) when $\beta = 0$, giving the algorithm which enables one to compare asymptotic behaviour in different directions, but we emphasise that the method is certainly not restricted to PII, and we will return in later papers to its application to the other transcendents.

In particular, of course, we can solve once again the "classic" problem for PII (1.1), which for convenience and completeness we state here. Its statement depends upon the following theorem, a proof of which was given by HASTINGS & MCLEOD [16].

Theorem A. *There exists a unique solution of* (1.1) *with* $\beta = 0$ *which is asymptotic to* $a\ Ai(x)$ *as* $x \to +\infty$, *a being any positive number. If* $a < 1$, *this solution exists for all real* x *as* x *decreases to* $-\infty$, *and, as* $x \to -\infty$,

$$q(x) \sim d|x|^{-1/4} \sin\left\{\tfrac{2}{3}|x|^{3/2} - \tfrac{3}{4}d^2 \log|x| + \gamma\right\}$$

for some constants d, γ *which depend on* a.

HASTINGS & MCLEOD also proved that if $a = 1$, then $q(x)$ grows algebraically as $x \to -\infty$ according to $q \sim \left(-\tfrac{1}{2}x\right)^{1/2}$, whilst if $a > 1$, then solutions blow up at some finite value of x (which, of course, depends on a). From the statement of Theorem A it is easy to compute more detailed asymptotics which hold as $x \to +\infty$:

$$q(x) = \tfrac{1}{2}a\pi^{-1/2}x^{-1/4}\exp\big(-\tfrac{2}{3}x^{3/2}\big)\big[1 - \tfrac{5}{48}x^{-3/2} + O\big(x^{-3}\big)\big], \tag{1.2a}$$

$$r(x) = \frac{\mathrm{d}q}{\mathrm{d}x} = -\tfrac{1}{2}a\pi^{-1/2}x^{1/4}\exp\big(-\tfrac{2}{3}x^{3/2}\big)\big[1 + \tfrac{7}{48}x^{-3/2} + O\big(x^{-3}\big)\big]. \tag{1.2b}$$

The usual connection problem is the question of the specific dependence of d and γ on a, and this is given as follows:

Theorem B.

$$d^2(a) = -\pi^{-1}\log\big(1 - a^2\big), \tag{1.3a}$$

$$\gamma(a) = \tfrac{3}{4}\pi - \tfrac{3}{2}d^2\log 2 - \arg\Gamma\big(-\tfrac{1}{2}\mathrm{i}d^2\big). \tag{1.3b}$$

The amplitude connection formula (1.3a) and the phase connection formula (1.3b) were first conjectured, derived heuristically and subsequently verified numerically by ABLOWITZ & SEGUR [2] and SEGUR & ABLOWITZ [40], respectively. It was not until some years later that CLARKSON & MCLEOD [7] gave a rigorous proof of (1.3a), using a Gel'fand-Levitan-Marchenko integral equation approach, and SULEIMANOV [41] derived (1.3a) and (1.3b) using an isomonodromy approach. However, it was only in the very recent work of DEIFT & ZHOU [8,9] that the form of the phase formula (1.3b) was finally proved rigorously.

This long history of the search for a rigorous verification of Theorem B illustrates the main problem pertaining to connection formulae for the Painlevé transcendents. The second Painlevé transcendent has a simpler structure than most of the remainder (see for example [17] for a list of all six transcendents). Given the formidable technical difficulties that have had to be overcome in order to establish Theorem B, the prospect of extending the existing techniques to problems for the other transcendents is somewhat daunting.

It is with this background that we expound our new method which, as we have already mentioned, involves the concept of isomonodromy, and we now quickly review the relevant facts [11]. (Again we give the details for PII (1.1) but emphasise that comparable results are known [20] for all the other Painlevé transcendents, and indeed that there is a hierarchy of equations [3] of higher order which fit into the same general framework.) Suppose that x and λ are independent complex variables and there exists a 2×2 matrix function $\boldsymbol{\Psi}(x, \lambda)$ which satisfies both

$$\frac{\partial\boldsymbol{\Psi}}{\partial x} = (-\mathrm{i}\lambda\boldsymbol{\sigma}_3 + q\boldsymbol{\sigma}_1)\boldsymbol{\Psi}, \quad \text{i.e.,}\ D_x\boldsymbol{\Psi} = 0, \tag{1.4}$$

and

$$\frac{\partial\boldsymbol{\Psi}}{\partial\lambda} = \left\{-\mathrm{i}(4\lambda^2 + x + 2q^2)\boldsymbol{\sigma}_3 + 4\lambda q\boldsymbol{\sigma}_1 - 2r\boldsymbol{\sigma}_2 - \frac{\beta}{\lambda}\boldsymbol{\sigma}_1\right\}\boldsymbol{\Psi}, \quad \text{i.e.,}\ D_\lambda\boldsymbol{\Psi} = 0. \tag{1.5}$$

Here

$$\boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

are the standard Pauli spin matrices which, in particular, satisfy

$$\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 = \mathrm{i}\boldsymbol{\sigma}_3, \quad \boldsymbol{\sigma}_2\boldsymbol{\sigma}_3 = \mathrm{i}\boldsymbol{\sigma}_1, \quad \boldsymbol{\sigma}_3\boldsymbol{\sigma}_1 = \mathrm{i}\boldsymbol{\sigma}_2.$$

Then there is a compatibility condition

$$[D_x, D_\lambda]\boldsymbol{\Psi} = (D_x D_\lambda - D_\lambda D_x)\boldsymbol{\Psi} = 0, \tag{1.6}$$

and an easy calculation shows that (1.6) reduces to (1.1). Conversely, if $q(x)$ evolves according to (1.1), then (1.4) and (1.5) are compatible. Thus (1.1) is equivalent to compatibility and compatibility is easily seen to imply isomonodromy.

For suppose that we have two fundamental solutions $\boldsymbol{\Psi}^{(1)}$, $\boldsymbol{\Psi}^{(2)}$ of (1.5) in two different but overlapping sectors in the λ-plane. (The equation has an irregular singularity at $\lambda = \infty$ and a regular singularity at $\lambda = 0$ but, as far as monodromy is concerned, we need only deal with the irregular singularity.) Since $\boldsymbol{\Psi}^{(1)}$ and $\boldsymbol{\Psi}^{(2)}$ are both fundamental solutions, there must be a matrix $\mathbf{S}$ independent of λ but in general dependent on x, such that

$$\boldsymbol{\Psi}^{(2)}(x, \lambda) = \boldsymbol{\Psi}^{(1)}(x, \lambda)\mathbf{S}(x), \tag{1.7}$$

and $\mathbf{S}$ is referred to as the monodromy matrix. (Of course, $\mathbf{S}$ depends on the particular fundamental solutions which are compared, and we return to this point later.) There is a monodromy matrix for each pair of sectors, and the assemblage of all the monodromy matrices forms the monodromy data. If we now differentiate (1.7) with respect to x and use the fact that $\boldsymbol{\Psi}^{(1)}$ and $\boldsymbol{\Psi}^{(2)}$ satisfy (1.4), we obtain immediately that $\mathbf{S}$ is independent of x, which is to say that the problem is isomonodromic in x. It should be noted that this involves care in choosing $\boldsymbol{\Psi}^{(1)}$ and $\boldsymbol{\Psi}^{(2)}$, for if we multiply $\boldsymbol{\Psi}^{(1)}(x, \lambda)$ by a function of x, it still satisfies (1.5), but no longer (1.4).

We make the remark also that we shall be able to arrange that the monodromy matrix takes the form of a triangular matrix with 1 as the principal diagonal. Thus the monodromy data reduces to the one remaining entry in the matrix, the so-called Stokes multiplier.

Given isomonodromy, we can now prove Theorem B as follows. We work out the monodromy data for (1.5) as $x \to +\infty$, using the known asymptotic dependence of q on x, and then the monodromy data as $x \to -\infty$, and equate them to give the required result. The way in which this has so far been carried out is to compute the fundamental solutions in different sectors and use (1.7) to obtain $\mathbf{S}$. This means that we have to compute the solutions (or at least their asymptotic behaviours) as $|\lambda| \to \infty$ and also as $|x| \to \infty$. This uses WKB asymptotics, and also matching, since the form of the asymptotics depends on the relative values of λ and x, and we have to match different forms in different regions. The procedure can be complicated and rigorising it difficult.

The procedure would be much simplified if one could find approximations to solutions which are uniformly valid for all relevant large $|\lambda|$, $|x|$. This we can in fact do, and in a general form which is certainly applicable to more than just PII (1.1). Once it is done, there is no further rigorous analysis required; it is merely a matter of computing the monodromy data by relating it to the (known) data for the approximations.

In Section 2 we describe, in the context of PII (1.1), the heuristic reasoning which leads to the uniform approximation. Then in Sections 3, 4 we state and prove two theorems on uniform approximations, which we believe to be the only such theorems necessary for the discussion of any of the Painlevé equations. In the final sections of the paper we use these theorems to compute monodromy data both in a general setting and in the particular case of PII, and finally as an application prove Theorem B.

It should be remarked that for the purposes of Theorem B only the first of the two approximation theorems (that relating to double turning points) is required. For more general solutions of PII, and for a general discussion of the other Painlevé equations, the second theorem is also required. We intend to return to such developments in later papers.

2. Deriving a Uniform Approximation

To see the nature of the uniform approximation, we turn (1.5) into a single second-order equation. We first make the scaling

$$\xi = x^{3/2}, \quad \eta = x^{-1/2}\lambda,$$

so that (1.5) becomes

$$\frac{\mathrm{d}\Psi}{\mathrm{d}\eta} = \xi\left\{-\mathrm{i}\left(4\eta^2 + 1 + \frac{2q^2}{x}\right)\boldsymbol{\sigma}_3 + \left(\frac{4\eta q}{\sqrt{x}} - \frac{\beta}{\eta\xi}\right)\boldsymbol{\sigma}_1 - \frac{2r}{x}\boldsymbol{\sigma}_2\right\}\Psi,$$

which, with

$$\Psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix},$$

is equivalent to

$$\frac{\mathrm{d}\psi_1}{\mathrm{d}\eta} = \xi\left\{-\mathrm{i}\left(4\eta^2 + 1 + \frac{2q^2}{x}\right)\psi_1 + \left(\frac{4\eta q}{\sqrt{x}} - \frac{\beta}{\eta\xi} + \frac{2\mathrm{i}r}{x}\right)\psi_2\right\}, \tag{2.1a}$$

$$\frac{\mathrm{d}\psi_2}{\mathrm{d}\eta} = \xi\left\{\mathrm{i}\left(4\eta^2 + 1 + \frac{2q^2}{x}\right)\psi_2 + \left(\frac{4\eta q}{\sqrt{x}} - \frac{\beta}{\eta\xi} - \frac{2\mathrm{i}r}{x}\right)\psi_1\right\}. \tag{2.1b}$$

Eliminating ψ_1, we obtain

$$\begin{aligned}\frac{\mathrm{d}^2\psi_2}{\mathrm{d}\eta^2} = \xi\Bigg\{&\mathrm{i}\left(4\eta^2 + 1 + \frac{2q^2}{x}\right)\frac{\mathrm{d}\psi_2}{\mathrm{d}\eta} + 8\mathrm{i}\eta\psi_2 + \left(\frac{4q}{\sqrt{x}} + \frac{\beta}{\eta^2\xi}\right)\psi_1\\ &- \mathrm{i}\xi\left(\frac{4\eta q}{\sqrt{x}} - \frac{\beta}{\eta\xi} - \frac{2\mathrm{i}r}{x}\right)\left(4\eta^2 + 1 + \frac{2q^2}{x}\right)\psi_1\\ &+ \xi\left(\frac{4\eta q}{\sqrt{x}} - \frac{\beta}{\eta\xi} + \frac{2\mathrm{i}r}{x}\right)\left(\frac{4\eta q}{\sqrt{x}} - \frac{\beta}{\eta\xi} - \frac{2\mathrm{i}r}{x}\right)\psi_2\Bigg\}\end{aligned}$$

$$= \xi \left\{ -\xi \left(4\eta^2 + 1 + \frac{2q^2}{x}\right)^2 \psi_2 + 8\mathrm{i}\eta\psi_2 + \xi \left[\left(\frac{4\eta q}{\sqrt{x}} - \frac{\beta}{\eta\xi} \right)^2 + \frac{4r^2}{x^2} \right] \psi_2 \right.$$

$$+ \frac{1}{\xi}\left(1 + \frac{\beta}{4\eta^2 qx}\right)\left(\eta - \frac{\mathrm{i}r}{2q\sqrt{x}} - \frac{\beta}{4\eta qx}\right)^{-1}$$

$$\left. \times \left[\frac{\mathrm{d}\psi_2}{\mathrm{d}\eta} - \mathrm{i}\xi\left(4\eta^2 + 1 + \frac{2q^2}{x}\right)\psi_2 \right] \right\}.$$

The term $\mathrm{d}\psi_2/\mathrm{d}\eta$ can be removed by setting

$$\phi = \left(\eta - \frac{\mathrm{i}r}{2q\sqrt{x}} - \frac{\beta}{4\eta qx}\right)^{-1/2} \psi_2,$$

whence

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\eta^2} = \xi^2\phi \left\{ -(4\eta^2+1)^2 + \frac{8\mathrm{i}\eta}{\xi} + \left(\frac{4r^2}{x^2} - \frac{4q^2}{x} - \frac{4q^4}{x^2} - \frac{8q\beta}{x^2} + \frac{\beta^2}{\eta^2 x^3} \right) \right.$$

$$- \frac{\mathrm{i}}{\xi}\left(1 + \frac{\beta}{4\eta^2 qx}\right)\left(4\eta^2 + 1 + \frac{2q^2}{x}\right)\left(\eta - \frac{\mathrm{i}r}{2q\sqrt{x}} - \frac{\beta}{4\eta qx}\right)^{-1}$$

$$+ \frac{1}{\xi^2}\frac{\beta}{4\eta^3 qx}\left(\eta - \frac{\mathrm{i}r}{2q\sqrt{x}} - \frac{\beta}{4\eta qx}\right)^{-1}$$

$$\left. + \frac{3}{4\xi^2}\left(1 + \frac{\beta}{4\eta^2 qx}\right)^2\left(\eta - \frac{\mathrm{i}r}{2q\sqrt{x}} - \frac{\beta}{4\eta qx}\right)^{-2} \right\}. \tag{2.2}$$

In (2.2), attention should be drawn to the terms

$$\frac{4r^2}{x^2} - \frac{4q^2}{x} - \frac{4q^4}{x^2} - \frac{8q\beta}{x^2} = M(\xi), \tag{2.3}$$

say, which depend only on x or ξ, and not on η. How $M(\xi)$ behaves for large ξ (which is always our interest) depends on the asymptotics of the functions $q(x)$, $r(x)$ as $|x| \to \infty$, and therefore on the particular solution of PII. For the remainder of this heuristic discussion we will consider the case where $M(\xi) \to 0$, since in the case of Theorem B this is certainly true from the given asymptotics both as $x \to +\infty$ and as $x \to -\infty$. But it is not true for a general solution that $M(\xi) \to 0$, and our methods do not need it, and we will point out where the essential difference lies.

Assuming then that $M(\xi) \to 0$ as $|\xi| \to \infty$, we expect from the form of (2.2) that, as $|\eta| \to \infty$ with $|\xi|$ large, the dominant term on the right-hand side is

$$-\xi^2(4\eta^2+1)^2\phi,$$

so that, from the usual WKB approximation, the solution should be asymptotically of the form

$$\eta^{-1}\exp\left\{\pm \mathrm{i}\xi\int^{\eta}(4\sigma^2+1)\,\mathrm{d}\sigma\right\} = \eta^{-1}\exp\left\{\pm \mathrm{i}\xi\left(\frac{4}{3}\eta^3+\eta\right)\right\}.$$

The two exponentials are thus equipollent in directions

$$\arg\left(\xi\eta^3\right) = 0,\ \pm\pi,\ \pm 2\pi, \ldots,$$

i.e.,

$$\arg\eta = -\tfrac{1}{3}\arg\xi \pm \tfrac{1}{3}k\pi, \quad k = 0, 1, 2, \ldots,$$

and these are the so-called Stokes directions. We can determine the Stokes multipliers by relating the asymptotic behaviour of a solution in one Stokes direction to its asymptotic behaviour in the next, since it is in Stokes directions (and only in Stokes directions) that the full asymptotics appear and solutions can be defined by their asymptotics.

However, in order to connect the behaviours as $|\eta| \to \infty$ on, say, $\arg\eta = -\frac{1}{3}\arg\xi$ and $\arg\eta = -\frac{1}{3}\arg\xi + \frac{1}{3}\pi$, we need to follow the solution along a curve for which $\mathrm{Re}\left\{\mathrm{i}\xi\int^{\eta}(4\sigma^2+1)\mathrm{d}\sigma\right\} = 0$, for if we depart significantly from such curves (so-called Stokes curves), we lose equipollence, and so the effect of exponentially small solutions and therefore the Stokes multiplier. Now there is some choice of Stokes curve depending on the initial point of integration, but to obtain a uniform approximation we consider Stokes curves which pass through turning points of equation (2.2); by a turning point we mean a value of η which is a zero of the right-hand side of (2.2) although we will slightly adapt this definition later.

The idea of uniform asymptotics through turning points was first proposed by LANGER [31] and TITCHMARSH [42] in work on the distribution of eigenvalues for the Schrödinger equation (see also [37]). They dealt with the equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}z^2} + [\mu - q(z)]y = 0, \qquad -\infty < z < \infty, \tag{2.4}$$

where, for example, we may think of μ as a large positive parameter and $q(z) \to \infty$ as $z \to \infty$. If q is strictly monotonic, then there is a simple turning point at $q(z) = \mu$. LANGER pointed out that the prototypical case of this is $q(z) = z$, so that the equation becomes

$$\frac{\mathrm{d}^2y}{\mathrm{d}z^2} + (\mu - z)y = 0,$$

whose general solution is a linear combination of $\mathrm{Ai}(z-\mu)$ and $\mathrm{Bi}(z-\mu)$, where Ai, Bi are the usual Airy functions. He then went on to show that one could obtain a uniform approximation to solutions of (2.4), valid for large μ and $z \to \pm\infty$, by introducing Airy functions of a suitable argument.

We need to modify the idea further, because LANGER's approximation relates to situations where the turning point is simple, whereas in our case (2.2) there are two turning points which, for large ξ, are close to $\eta = \frac{1}{2}\mathrm{i}$ (and two others close to $\eta = -\frac{1}{2}\mathrm{i}$). (This of course is a consequence of our assumption that $M(\xi) \to 0$. If $M(\xi) \not\to 0$, then the turning points are simple, and it is then a matter of adapting

LANGER's approximation using Airy functions.) In our present situation, therefore, it seems that the parabolic cylinder equation

$$\frac{d^2y}{dz^2} = \left[\tfrac{1}{4}z^2 - \left(\nu + \tfrac{1}{2}\right)\right]y, \tag{2.5}$$

with linearly independent solutions $D_\nu(z)$ and $D_{-\nu-1}(-iz)$, is an appropriate one for coping with coalescing turning-points, and in fact this possibility has already been explored by OLVER [37] and DUNSTER [10], primarily for real values of z. With our particular applications in mind, it will be better to consider (2.5) in the form

$$\frac{d^2y}{dz^2} = -\xi^2\left(z^2 - \frac{2\nu+1}{i\xi}\right)y \tag{2.6}$$

$$= -\xi^2(z^2-\alpha^2)y, \tag{2.7}$$

where $i\xi\alpha^2 = 2\nu+1$, which has solutions $D_\nu(e^{\pi i/4}\sqrt{2\xi}z)$ and $D_{-\nu-1}(e^{-\pi i/4}\sqrt{2\xi}z)$.

To see how this applies to (2.2), we restrict ourselves to the particular case when $\beta = 0$. We try as a uniform approximation to a solution of (2.2) the expression

$$\phi(\eta) = \rho(\eta)D_\nu\left(e^{i\pi/4}\sqrt{2\xi}\zeta(\eta)\right) = \rho(\eta)F_\nu(\zeta(\eta)), \quad \text{say}, \tag{2.8}$$

where the functions ρ and ζ are to be determined. Substituting (2.8) in (2.2) with $\beta = 0$ we have

$$\begin{aligned}
&\rho'' F_\nu + 2\rho'\zeta' F_\nu' + \rho((\zeta')^2 F_\nu'' + \zeta'' F_\nu') \\
&\quad = \xi^2\rho F_\nu\left\{-(4\eta^2+1)^2 + \frac{8i\eta}{\xi} + \left(\frac{4r^2}{x^2} - \frac{4q^2}{x} - \frac{4q^4}{x^2}\right)\right. \\
&\qquad \left. - \frac{i}{\xi}\left(4\eta^2+1+\frac{2q^2}{x}\right)\left(\eta - \frac{ir}{2q\sqrt{x}}\right)^{-1} + \frac{3}{4\xi^2}\left(\eta - \frac{ir}{2q\sqrt{x}}\right)^{-2}\right\}.
\end{aligned} \tag{2.9}$$

Recalling that F_ν satisfies (2.6) we can compare coefficients of F_ν' and F_ν in (2.9). The vanishing of the coefficient of F_ν' gives

$$2\rho'\zeta' + \rho\zeta'' = 0,$$

so that we can take

$$\rho = (\zeta')^{-1/2}, \tag{2.10}$$

for the choice of integration constant at this point is inconsequential. The vanishing of the coefficient of F_ν gives

$$\begin{aligned}
&\xi^2(\zeta^2-\alpha^2)(\zeta')^2 \\
&\quad = \xi^2\left\{(4\eta^2+1)^2 - \frac{8i\eta}{\xi} - \left(\frac{4r^2}{x^2} - \frac{4q^2}{x} - \frac{4q^4}{x^2}\right)\right. \\
&\qquad \left. + \frac{i}{\xi}\left(4\eta^2+1+\frac{2q^2}{x}\right)\left(\eta - \frac{ir}{2q\sqrt{x}}\right)^{-1} - \frac{3}{4\xi^2}\left(\eta - \frac{ir}{2q\sqrt{x}}\right)^{-2} + \frac{\rho''}{\xi^2\rho}\right\}.
\end{aligned} \tag{2.11}$$

If we ignore the last two terms in $\{\ldots\}$ as being of smaller order (for large ξ) than the others, then we are left with

$$
\begin{aligned}
(\zeta^2-\alpha^2)(\zeta')^2 &= (4\eta^2+1)^2 - \frac{8\mathrm{i}\eta}{\xi} - \left(\frac{4r^2}{x^2} - \frac{4q^2}{x} - \frac{4q^4}{x^2}\right) \\
&\quad + \frac{\mathrm{i}}{\xi}\left(4\eta^2 + 1 + \frac{2q^2}{x}\right)\left(\eta - \frac{\mathrm{i}r}{2q\sqrt{x}}\right)^{-1} \\
&= G(\eta,\xi), \quad \text{say,}
\end{aligned}
\tag{2.12}
$$

which, apart from a constant of integration, defines ζ as a function of η once we have specified α. (We recall always that r, q, x, ξ are constants as far as η, ζ are concerned.) We note however from (2.10) that we certainly want to avoid zeros of ζ' and, from (2.12), ζ' has a zero wherever $G(\eta,\xi) = 0$; i.e., essentially at a turning point of the equation, unless we can choose α so that the zeros of $\zeta^2 - \alpha^2$ coincide with those of G. For large ξ, there are two turning points, say η_1 and η_2, close to $\frac{1}{2}\mathrm{i}$, and two close to $-\frac{1}{2}\mathrm{i}$. If we are interested in a Stokes curve which passes through (or close to) $\frac{1}{2}\mathrm{i}$, then we must choose α so that $\zeta = -\alpha$ corresponds to $\eta = \eta_1$ and $\zeta = +\alpha$ corresponds to $\eta = \eta_2$. We can ensure that one of these holds by use of the constant of integration implicit in the evaluation of ζ from (2.12). The second can be achieved by defining α so that

$$\int_{-\alpha}^{\alpha} (\zeta^2-\alpha^2)^{1/2}\,\mathrm{d}\zeta = \int_{\eta_1}^{\eta_2} G^{1/2}(\eta,\xi)\,\mathrm{d}\eta.$$

Since the left-hand side integrates easily to $\frac{1}{2}\pi\mathrm{i}\alpha^2$, we have α given by

$$\frac{1}{2}\pi\mathrm{i}\alpha^2 = \int_{\eta_1}^{\eta_2} G^{1/2}(\eta,\xi)\,\mathrm{d}\eta. \tag{2.13}$$

With α so defined, and ζ chosen according to

$$\int_{\alpha}^{\zeta} (\tau^2-\alpha^2)^{1/2}\,\mathrm{d}\tau = \int_{\eta_2}^{\eta} G^{1/2}(\sigma,\xi)\,\mathrm{d}\sigma,$$

we can hope that solutions of (2.2) are approximated, uniformly on η for large ξ, by some linear combination of

$$(\zeta')^{-1/2} D_\nu(\mathrm{e}^{\mathrm{i}\pi/4}\sqrt{2\xi}\,\zeta) \quad \text{and} \quad (\zeta')^{-1/2} D_{-\nu-1}(\mathrm{e}^{-\mathrm{i}\pi/4}\sqrt{2\xi}\,\zeta).$$

A precise statement and proof of this conjecture is given in the next section.

We remark finally that it is a consequence of this uniform approximation that the monodromy data for (2.2) as $|\xi| \to \infty$ is the same as that for the parabolic cylinder functions, which can be found in any text on special functions, modified only by some allowance for the various changes of variable involved. We work this out more precisely in Sections 5 and 6.

3. The Uniform Approximation Theorem for a Double Turning Point

We are interested in differential equations of the form

$$\frac{d^2\phi}{d\eta^2} = -\xi^2 F(\eta, \xi)\phi \tag{3.1}$$

and, guided by the heuristic discussion in Section 2, we make the following assumptions about F. Suppose that our concern is with the limit $|\xi| \to \infty$ with $\arg\xi \to \theta$; we then hypothesise that

H1. *There is a sequence of values* ξ_n, $|\xi_n| \to \infty$, $\arg\xi_n \to \theta$, *such that*

$$F(\eta, \xi_n) = F_0(\eta)\,(\eta - \eta_0)^2 - \frac{\widetilde{F}(\eta, \xi_n)}{\xi_n},$$

where

(i) $F_0(\eta)$ *is a polynomial in* η, *with* $F_0(\eta_0) \neq 0$ *and*

$$F_0(\eta) \sim A\eta^m \quad \text{as} \quad \eta \to \infty, \tag{3.2}$$

(ii) $\widetilde{F}(\eta, \xi_n)$ *is a rational function of* η, *with the location of its poles possibly dependent on* ξ_n.

(*Further assumptions on* $\widetilde{F}$ *are given in due course.*)

Remarks.

1. The assumption that F_0 is polynomial is not essential. Polynomial-like behaviour of some sort would certainly be sufficient, but in applications to the Painlevé transcendents it is always the case that F_0 is a polynomial, and since no new ideas would be involved in generalization, we do not consider this here. Similar comments hold with regard to the rational behaviour of $\widetilde{F}$.

2. The assumption that $F_0(\eta_0) \neq 0$ is crucial. It implies that (3.1) is to have a double turning point at $\eta = \eta_0$ (or, more precisely, for large $|\xi|$, two turning points close to η_0), but no other turning points close to η_0.

3. Our assumption is only about a sequence of values ξ_n since it will turn out in our applications to be a consequence of isomonodromy that behaviour as $|\xi| \to \infty$ through a sequence is sufficient to determine behaviour as $|\xi| \to \infty$ generally. However, in the present approximation theorem, which is in itself quite independent of the concept of isomonodromy, we are not involved in comparing different sequences, and so we can without confusion drop the subscript in ξ_n, and this is done henceforth.

4. The usual WKB approximation for (3.1) would suggest, in view of (3.2), that, for large η, the solutions of (3.1) are asymptotic to linear combinations of

$$\eta^{-(m+2)/4} \exp\left(\pm i\xi \int^{\eta} F_0^{1/2}(s)\,(s - \eta_0)\,ds\right),$$

and so for Stokes directions we must have

$$\arg\left(\xi A^{1/2}\eta^{(4+m)/2}\right) = 0,\ \pm\pi,\ \pm 2\pi, \ldots$$

or

$$\left(\tfrac{1}{2}m + 2\right)\arg\eta = -\arg\xi - \tfrac{1}{2}\arg A + k\pi \quad (k = 0, \pm 1, \ldots). \tag{3.3}$$

To compute monodromy data for (3.1), we need the behaviour of solutions in two successive Stokes directions, and this leads to the next hypothesis.

H2. *There exists a Stokes curve $C_{k,k+1}$, defined by*

$$\operatorname{Re}\left(\mathrm{i}\xi \int_{\eta_0}^{\eta} F^{1/2}(\sigma, \xi)\,\mathrm{d}\sigma\right) = 0,$$

which connects ∞ in two successive Stokes directions (given by $k\pi$ and $(k+1)\pi$ in (3.3) *above), passes through η_0 and is (for large ξ) bounded from any zero of F_0.*

H3. *On and in a neighbourhood of $C_{k,k+1}$, $\widetilde{F}$ has no poles, at least for large ξ, and, for all η and uniformly for large ξ,*

$$\frac{\widetilde{F}(\eta, \xi)}{F_0(\eta)} = O(|\eta| + 1),$$

while, for large η and uniformly for large ξ,

$$F'/F = O(\eta^{-1}), \quad F''/F = O(\eta^{-2}), \quad F' = dF/d\eta.$$

Remarks.

1. It is now clear from Rouché's theorem that, for ξ sufficiently large, $F(\eta, \xi)$ has, as a function of η, precisely two zeros η_1, η_2 close to η_0. In fact, if $\widetilde{F}(\eta, \xi) \to L$ as $\eta \to \eta_0, |\xi| \to \infty$, we have

$$\eta_j = \eta_0 \pm \left(\frac{L}{F_0(\eta_0)}\right)^{1/2} \xi^{-1/2}\{1 + o(1)\}, \quad j = 1, 2. \tag{3.4}$$

(We can take η_2 to correspond to the upper sign.)

2. In line with the heuristic discussion in Section 2, we define a number α by

$$\tfrac{1}{2}\pi\mathrm{i}\alpha^2 = \int_{-\alpha}^{\alpha} (\tau^2 - \alpha^2)^{1/2}\,\mathrm{d}\tau = \int_{\eta_1}^{\eta_2} F^{1/2}(\eta, \xi)\,\mathrm{d}\eta, \tag{3.5}$$

and a new variable ζ by

$$\int_{\alpha}^{\zeta} (\tau^2 - \alpha^2)^{1/2}\,\mathrm{d}\tau = \int_{\eta_2}^{\eta} F^{1/2}(s, \xi)\,\mathrm{d}s. \tag{3.6}$$

There is a choice of signs for the various square roots, but any consistent choice will do. Other choices merely lead to a permutation amongst the solutions $D_\nu(z)$, $D_\nu(-z)$, $D_{-\nu-1}(\mathrm{i}z)$ and $D_{-\nu-1}(-\mathrm{i}z)$ of (2.5) (or, of course, (2.6)) and do not therefore affect the space of approximating functions in our theorem below. We note also that F does not vanish on or near $C_{k,k+1}$ if ξ is large, except at η_1, η_2, and so there is no ambiguity in the sign of $F^{1/2}$ once some initial value has been chosen.

3. There is a certain arbitrariness in the precise choice of a Stokes curve. All that is required is that on it both WKB approximations are equipollent, so that both appear in an asymptotic expansion of a solution. With this in mind, it would be equally good to choose a curve connecting two Stokes directions on which $\mathrm{Re}(\mathrm{i}\xi \int_{\eta_0}^{\eta} F^{1/2}(\sigma, \xi)\, \mathrm{d}\sigma)$ is bounded independently of η and ξ, and we shall make use of this possibility.

Given these three assumptions concerning the problem (3.1) we can now show that solutions of this equation can be approximated uniformly by parabolic cylinder functions so long as η remains on $C_{k,k+1}$. This result can be summarised thus:

Theorem 1. *Under hypotheses H1–H3, and given any solution ϕ of* (3.1)*, there exist constants c_1, c_2 such that, uniformly for η on $C_{k,k+1}$, as $|\xi| \to \infty$,*

$$\left(\frac{\zeta^2 - \alpha^2}{F(\eta, \xi)}\right)^{-1/4} \phi(\eta, \xi)$$
$$= \left\{[c_1 + o(1)]D_\nu\left(\mathrm{e}^{\pi\mathrm{i}/4}\sqrt{2\xi}\zeta\right) + [c_2 + o(1)]D_{-\nu-1}\left(\mathrm{e}^{-\pi\mathrm{i}/4}\sqrt{2\xi}\zeta\right)\right\}.$$

Proof. We have to compare the equations

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\eta^2} = -\xi^2 F(\eta, \xi)\phi \tag{3.7}$$

and

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}\zeta^2} = -\xi^2(\zeta^2 - \alpha^2)\psi. \tag{3.8}$$

Set

$$p = \frac{\mathrm{d}\eta}{\mathrm{d}\zeta} = \left(\frac{\zeta^2 - \alpha^2}{F}\right)^{1/2}, \tag{3.9}$$

which we note is bounded both above and below on any bounded part of $C_{k,k+1}$. The only problem can occur near η_0, and there we notice that $\zeta^2 - \alpha^2$ and F have the same zeros, so that p and p^{-1} are analytic in a neighbourhood of η_0. Since trivially p and p^{-1} are bounded on some fixed small circle $|\eta - \eta_0| = k$, say, it follows from the maximum principle that p and p^{-1} are bounded inside $|\eta - \eta_0| = k$. Also as $\eta, \zeta \to \infty$, it is immediate from (3.6) that

$$\frac{1}{2}\zeta^2 \sim \frac{2A^{1/2}\eta^{2+m/2}}{4+m},$$

so that p is asymptotically some power of η (or ζ), and so, considering $p = p(\zeta)$ and $p' = dp/d\zeta$, we have, as $|\zeta| \to \infty$, from H3,

$$\frac{p'}{p} = O\left(\frac{1}{\zeta}\right), \qquad \frac{p''}{p} = O\left(\frac{1}{\zeta^2}\right),$$

and the bounds implicit in the O-terms are independent of ξ. Now

$$\frac{\mathrm{d}\phi}{\mathrm{d}\eta} = \frac{1}{p}\frac{\mathrm{d}\phi}{\mathrm{d}\zeta}, \quad \frac{\mathrm{d}^2\phi}{\mathrm{d}\eta^2} = \frac{1}{p^2}\frac{\mathrm{d}^2\phi}{\mathrm{d}\zeta^2} - \frac{p'}{p^3}\frac{\mathrm{d}\phi}{\mathrm{d}\zeta},$$

so that (3.7) becomes

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\zeta^2} = -\xi^2(\zeta^2-\alpha^2)\phi + \frac{p'}{p}\frac{\mathrm{d}\phi}{\mathrm{d}\zeta}.$$

Setting

$$\phi = p^{1/2}\Phi, \tag{3.10}$$

we have

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}\zeta^2} = -\xi^2(\zeta^2-\alpha^2)\Phi - \frac{1}{2}\left[\frac{p''}{p} - \frac{3}{2}\frac{(p')^2}{p^2}\right]\Phi. \tag{3.11}$$

Now we have already seen that linearly independent solutions of (3.8) are

$$D_\nu\left(\mathrm{e}^{\pi\mathrm{i}/4}\sqrt{2\xi}\zeta\right), \quad D_{-\nu-1}\left(\mathrm{e}^{-\pi\mathrm{i}/4}\sqrt{2\xi}\zeta\right), \tag{3.12}$$

where, by (2.7),

$$\nu = -\tfrac{1}{2} + \tfrac{1}{2}\mathrm{i}\xi\alpha^2, \tag{3.13}$$

and the asymptotics of the functions in (3.12), as $|\sqrt{2\xi}\zeta| \to \infty$, are always linear combinations of

$$\exp\left(-\tfrac{1}{2}\mathrm{i}\xi\zeta^2\right)\left(\sqrt{2\xi}\zeta\right)^{\nu}, \quad \exp\left(\tfrac{1}{2}\mathrm{i}\xi\zeta^2\right)\left(\sqrt{2\xi}\zeta\right)^{-\nu-1}.$$

(For the asymptotics of parabolic cylinder functions, one can consult, for example, [43].) We want to assert that these are bounded on $C_{k,k+1}$, which is so if

$$\mathrm{Re}\left(\tfrac{1}{2}\mathrm{i}\xi\zeta^2 - \nu\log\left(\sqrt{2\xi}\zeta\right)\right) \quad \text{is bounded.} \tag{3.14}$$

But by definition

$$\begin{aligned} \mathrm{i}\xi\int_\alpha^\zeta (\tau^2-\alpha^2)^{1/2}\mathrm{d}\tau &= \mathrm{i}\xi\int_{\eta_2}^{\eta} F^{1/2}(s,\xi)\mathrm{d}s \\ &= \mathrm{i}\xi\int_{\eta_0}^{\eta} F^{1/2}(s,\xi)\,\mathrm{d}s + \mathrm{i}\xi\int_{\eta_2}^{\eta_0} F^{1/2}(s,\xi)\,\mathrm{d}s, \end{aligned}$$

and the last term is bounded independent of ξ. (Merely set $s-\eta_0 = t\xi^{-1/2}$ in the integrand, and use the fact that $(\eta_2-\eta_0)\xi^{1/2}$ is bounded.) Thus, on $C_{k,k+1}$,

$$\mathrm{Re}\left(\mathrm{i}\xi\int_\alpha^\zeta (\tau^2-\alpha^2)^{1/2}\mathrm{d}\tau\right) \quad \text{is bounded,} \tag{3.15}$$

and it is an elementary integration that

$$\int_\alpha^\zeta (\tau^2-\alpha^2)^{1/2}\mathrm{d}\tau = \tfrac{1}{2}\left\{\zeta(\zeta^2-\alpha^2)^{1/2} - \alpha^2\log\left(\zeta+(\zeta^2-\alpha^2)^{1/2}\right) + \alpha^2\log\alpha\right\}. \tag{3.16}$$

Substituting for α from (3.13), we see easily that (3.15) implies (3.14).

We can now turn (3.11) into an integral equation in the usual way. In fact, any solution of (3.11) satisfies, for some constants c_1, c_2, the integral equation

$$\begin{aligned}\Phi(\zeta) &= c_1 D_\nu\left(\mathrm{e}^{\pi \mathrm{i}/4}\sqrt{2\xi}\zeta\right) + c_2 D_{-\nu-1}\left(\mathrm{e}^{-\pi \mathrm{i}/4}\sqrt{2\xi}\zeta\right) \\ &\quad - \frac{\mathrm{i}}{2\sqrt{2\xi}}\int_\alpha^\zeta \left\{D_\nu\left(\mathrm{e}^{\pi \mathrm{i}/4}\sqrt{2\xi}\zeta\right) D_{-\nu-1}\left(\mathrm{e}^{-\pi \mathrm{i}/4}\sqrt{2\xi}t\right)\right. \\ &\quad \left. - D_{-\nu-1}\left(\mathrm{e}^{\pi \mathrm{i}/4}\sqrt{2\xi}\zeta\right) D_\nu\left(\mathrm{e}^{-\pi \mathrm{i}/4}\sqrt{2\xi}t\right)\right\}\left[\frac{p''}{p} - \frac{3}{2}\frac{(p')^2}{p^2}\right]\Phi(t)\,\mathrm{d}t.\end{aligned} \tag{3.17}$$

In deriving (3.17) we have made use of the standard result that the Wronskian

$$W\left(D_\nu\left(\mathrm{e}^{\pi \mathrm{i}/4}\sqrt{2\xi}\zeta\right), D_{-\nu-1}\left(\mathrm{e}^{-\pi \mathrm{i}/4}\sqrt{2\xi}\zeta\right)\right) = \mathrm{i}\sqrt{2\xi}$$

and the integral is to be taken along $C_{k,k+1}$. Since D_ν, $D_{-\nu-1}$ are bounded on this curve, and

$$\frac{p''}{p} - \frac{3}{2}\frac{(p')^2}{p^2} = O\left(\frac{1}{\zeta^2}\right)$$

and so is integrable to infinity on $C_{k,k+1}$, we can solve (3.17) by iteration (see, for example [42]), to conclude that Φ is bounded on $C_{k,k+1}$. Furthermore, we deduce that

$$\Phi(\zeta) = c_1 D_\nu\left(\mathrm{e}^{\pi \mathrm{i}/4}\sqrt{2\xi}\zeta\right) + c_2 D_{-\nu-1}\left(\mathrm{e}^{-\pi \mathrm{i}/4}\sqrt{2\xi}\zeta\right) + O\left(\frac{|c_1| + |c_2|}{\sqrt{\xi}}\right)$$

and, returning to ϕ via the transformation (3.10), we see that the theorem is proved.

4. The Uniform Approximation Theorem for a Simple Turning Point

Consider differential equations of the form

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\eta^2} = -\xi^2 F(\eta, \xi)\phi, \tag{4.1}$$

where we make the following assumptions about F in the limit as $|\xi| \to \infty$, $\arg\xi = \theta$.

H1. *There is a sequence of values ξ_n, $|\xi_n| \to \infty$, $\arg\xi_n \to \theta$, such that*

$$F(\eta, \xi_n) = F_0(\eta, \xi_n)(\eta - \eta_0(\xi_n)) - \frac{\widetilde{F}(\eta, \xi_n)}{\xi_n}, \tag{4.2}$$

where

(i) $\eta_0(\xi_n) \to \eta_\infty$ *as* $\xi_n \to \infty$, η_∞ *finite,*

(ii) *$F_0(\eta, \xi_n)$ is a polynomial in η whose zeros tend to finite limits as $\xi_n \to \infty$, all distinct from η_∞, and*

$$F_0(\eta, \xi_n) \sim A\eta^m \quad \textit{as} \quad \eta \to \infty, \tag{4.3}$$

(iii) *$\widetilde{F}(\eta, \xi_n)$ is a rational function of η.*

Remarks.

1. We are allowing the possibility that the turning point η_0 may depend on ξ. (We drop the subscript n as in Section 3.) We could do this also in Theorem 1, but this does not seem relevant in the applications of Theorem 1, whereas it certainly is in applications of the present case.

2. The usual WKB approximation for (4.1) would suggest, from (4.3), that for large η, solutions of (4.1) are asymptotic to linear combinations of

$$\eta^{-(m+1)/4} \exp\left(\pm \mathrm{i}\xi \int^\eta F_0^{1/2}(s, \xi)(s - \eta_0)^{1/2} \mathrm{d}s\right)$$

and so for Stokes directions we must have

$$\arg\left(\xi A^{1/2} \eta^{(3+m)/2}\right) = 0, \quad \pm\pi, \quad \pm 2\pi, \ldots$$

or

$$\tfrac{1}{2}(m+3) \arg \eta = -\arg \xi - \tfrac{1}{2} \arg A + k\pi \qquad (k = 0, \pm 1, \ldots). \tag{4.4}$$

Monodromy data for (4.1) can be computed once the behaviours of solutions in two successive Stokes directions are known. We therefore assume:

H2. *There exists a Stokes curve $C_{k,k+1}$, defined by*

$$\mathrm{Re}\left(\mathrm{i}\xi \int_{\eta_0}^{\eta} F^{1/2}(\sigma, \xi) \mathrm{d}\sigma\right) = 0,$$

which connects ∞ in two successive Stokes directions (given by $k\pi$ and $(k+1)\pi$ in (4.4) above) and which passes through η_0 and is (for large ξ) bounded away from any zero of F_0. (We drop the explicit dependence of η_0 on ξ.)

H3. *On $C_{k,k+1}$ and in a neighbourhood of it, $\widetilde{F}$ has no poles, at least for large ξ, and, for all η and uniformly for large ξ,*

$$\frac{\widetilde{F}(\eta, \xi)}{F_0(\eta, \xi)} = O(1),$$

whilst, for large η and uniformly for large ξ,

$$F'/F = O(\eta^{-1}), \quad F''/F = O(\eta^{-2}).$$

Based on the above, it follows from Rouché's theorem that, for ξ sufficiently large, $F(\eta, \xi)$ has, as a function of η, precisely one zero η^* close to η_0 and thus close to η_∞. In fact,

$$\eta^* = \eta_0 + O(\xi^{-1}). \tag{4.5}$$

Now if we define a new variable ζ by

$$\tfrac{2}{3}\zeta^{3/2} = \int_0^\zeta \tau^{1/2}\mathrm{d}\tau = \int_{\eta^*}^\eta F^{1/2}(s,\xi)\mathrm{d}s, \tag{4.6}$$

then we can obtain uniform approximations to the solution of (4.1) according to

Theorem 2. *Under hypotheses H1-H3, and given any solution* ϕ *of* (4.1), *there exist constants* c_1, c_2 *such that, uniformly for* η *on* $C_{k,k+1}$, *as* $|\xi| \to \infty$,

$$\left(\frac{\zeta}{F(\eta,\xi)}\right)^{-1/4}\phi(\eta,\xi) = \left\{[c_1 + o(1)]\mathrm{Ai}\left(\mathrm{e}^{\pi \mathrm{i}/3}\xi^{2/3}z\right) + [c_2 + o(1)]\mathrm{Bi}\left(\mathrm{e}^{\pi \mathrm{i}/3}\xi^{2/3}\zeta\right)\right\}$$

where Ai *and* Bi *are the usual Airy functions.*

Proof. We need to compare the equations

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\eta^2} = -\xi^2 F(\eta,\xi)\phi \quad \text{and} \quad \frac{\mathrm{d}^2\psi}{\mathrm{d}\zeta^2} = -\xi^2\zeta\psi, \tag{4.7a, b}$$

and do so by setting

$$p = \frac{\mathrm{d}\eta}{\mathrm{d}\zeta} = \left(\frac{\zeta}{F}\right)^{1/2}.$$

Now p is bounded both above and below on any bounded part of $C_{k,k+1}$. The only difficulty might arise near η_0 and there we note that ζ and F have the same simple zero (from definition (4.6)) so that p and p^{-1} are analytic near η_0. Since trivially p and p^{-1} are bounded on some fixed small circle $|\eta - \eta_0| = \varepsilon$, say, it is a consequence of the maximum principle that both p and p^{-1} are bounded inside $|\eta - \eta_0| = \varepsilon$. As $\eta, \zeta \to \infty$, it is obvious from (4.6) that

$$\frac{2}{3}\zeta^{3/2} \sim \frac{2A^{1/2}\eta^{(m+3)/2}}{m+3}, \tag{4.8}$$

so that p is asymptotically some power of η (or ζ). Therefore, considering $p = p(\zeta)$ and $p' \equiv \mathrm{d}p/\mathrm{d}\zeta$ we have as $|\zeta| \to \infty$, from H3, that

$$\frac{p'}{p} = O\left(\frac{1}{\zeta}\right), \quad \frac{p''}{p} = O\left(\frac{1}{\zeta^2}\right),$$

and the bounds implicit in the O-terms are independent of ξ. Since

$$\frac{\mathrm{d}\phi}{\mathrm{d}\eta} = \frac{1}{p}\frac{\mathrm{d}\phi}{\mathrm{d}\zeta}, \quad \frac{\mathrm{d}^2\phi}{\mathrm{d}\eta^2} = \frac{1}{p^2}\frac{\mathrm{d}^2\phi}{\mathrm{d}\zeta^2} - \frac{p'}{p^3}\frac{\mathrm{d}\phi}{\mathrm{d}\zeta},$$

equation (4.7a) becomes

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\zeta^2} = -\xi^2\zeta\phi + \frac{p'}{p}\frac{\mathrm{d}\phi}{\mathrm{d}\zeta},$$

and on setting

$$\phi = p^{1/2}\Phi \tag{4.9}$$

we obtain

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}\zeta^2} = -\xi^2\zeta\,\Phi - \frac{1}{2}\left[\frac{p''}{p} - \frac{3}{2}\frac{(p')^2}{p^2}\right]\Phi. \tag{4.10}$$

It is a standard result that linearly independent solutions of (4.7b) are

$$\mathrm{Ai}\left(\mathrm{e}^{\mathrm{i}\pi/3}\xi^{2/3}\zeta\right), \quad \mathrm{Bi}\left(\mathrm{e}^{\mathrm{i}\pi/3}\xi^{2/3}\zeta\right) \tag{4.11}$$

and the asymptotics of these functions as $|\xi^{2/3}\zeta| \to \infty$ are always linear combinations of

$$\xi^{-1/6}\zeta^{-1/4}\exp\left\{\pm\tfrac{2}{3}\mathrm{i}\xi\zeta^{3/2}\right\}.$$

We would like to assert that these are bounded on $C_{k,k+1}$, which is the case if $\mathrm{Re}\left(\mathrm{i}\xi\zeta^{3/2}\right)$ is bounded. However, we have from (4.6) that

$$\tfrac{2}{3}\mathrm{i}\xi\zeta^{3/2} = \mathrm{i}\xi\int_{\eta^*}^{\eta} F^{\frac{1}{2}}(s,\xi)\,\mathrm{d}s = \mathrm{i}\xi\int_{\eta^*}^{\eta_0} F^{1/2}(s,\xi)\,\mathrm{d}s + \mathrm{i}\xi\int_{\eta_0}^{\eta} F^{1/2}(s,\xi)\,\mathrm{d}s$$

so that, on $C_{k,k+1}$, $\mathrm{Re}(\mathrm{i}\xi\zeta^{3/2})$ is bounded if and only if $\mathrm{Re}\left(\mathrm{i}\xi\int_{\eta^*}^{\eta_0} F^{1/2}(s,\xi)\,\mathrm{d}s\right)$ is bounded. This latter expression is $O\left(|\xi|^{-1/2}\right)$ for large $|\xi|$ (by using (4.2) and (4.5)) so that $\mathrm{Re}\left(\mathrm{i}\xi\zeta^{3/2}\right)$ is indeed bounded on the Stokes curve.

To complete the proof of Theorem 2 we turn (4.10) into an integral equation in the usual way. It follows that any solution of (4.10) satisfies, for some constants c_1 and c_2, the equation

$$\begin{aligned}\Phi(\zeta) = {}& c_1\mathrm{Ai}\left(\mathrm{e}^{\pi\mathrm{i}/3}\xi^{2/3}\zeta\right) + c_2\mathrm{Bi}\left(\mathrm{e}^{\pi\mathrm{i}/3}\xi^{2/3}\zeta\right)\\ &- \frac{\mathrm{i}}{4\xi^{5/6}}\int_0^{\zeta}\left\{\mathrm{Ai}\left(\mathrm{e}^{\pi\mathrm{i}/3}\xi^{2/3}\zeta\right)\mathrm{Bi}\left(\mathrm{e}^{\pi\mathrm{i}/3}\xi^{2/3}t\right)\right.\\ &\left.-\mathrm{Bi}\left(\mathrm{e}^{\pi\mathrm{i}/3}\xi^{2/3}\zeta\right)\mathrm{Ai}\left(\mathrm{e}^{\pi\mathrm{i}/3}\xi^{2/3}t\right)\right\}\left[\frac{p''}{p} - \frac{3}{2}\left(\frac{p'}{p}\right)^2\right]\Phi(t)\,\mathrm{d}t.\end{aligned} \tag{4.12}$$

In deriving (4.12) we have used the standard result for Wronskians that

$$W\left(\mathrm{Ai}\left(\mathrm{e}^{\pi\mathrm{i}/3}\xi^{2/3}\zeta\right), \mathrm{Bi}\left(\mathrm{e}^{\pi\mathrm{i}/3}\xi^{2/3}\zeta\right)\right) = 2\mathrm{i}\xi^{5/6}.$$

The integral within (4.12) is taken along the Stokes curve $C_{k,k+1}$ and since Ai and Bi are bounded there and

$$\frac{p''}{p} - \frac{3}{2}\left(\frac{p'}{p}\right)^2 = O\left(\frac{1}{\zeta^2}\right)$$

and so is integrable to infinity on $C_{k,k+1}$, we can solve (4.12) by iteration to conclude that Φ is bounded on $C_{k,k+1}$. Furthermore, we have that

$$\Phi(\zeta) = c_1 \mathrm{Ai}\big(\mathrm{e}^{\pi \mathrm{i}/3}\xi^{2/3}\zeta\big) + c_2 \mathrm{Bi}\big(\mathrm{e}^{\pi \mathrm{i}/3}\xi^{2/3}\zeta\big) + O\left(\frac{|c_1| + |c_2|}{\xi^{5/6}}\right)$$

and, returning to the variable ϕ via the transformation (4.9), we conclude that the theorem is proved.

5. Monodromy Data for Parabolic Cylinder Functions

This section sets out the well-known results that we shall need concerning Stokes multipliers for the parabolic cylinder function. We shall be interested in computing the multipliers for the curve $C_{k,k+1}$; i.e., we wish to compare the asymptotic behaviours on

$$\big(\tfrac{1}{2}m + 2\big) \arg\eta + \arg\xi + \tfrac{1}{2}\arg A = \quad k\pi \quad \text{and} \quad (k+1)\pi$$

and, since $2\arg\zeta \sim \frac{1}{2}\arg A + (\frac{1}{2}m + 2)\arg\eta$ for large η, ζ, this is equivalent to comparing behaviours on

$$\arg\Big(\sqrt{2\xi}\,\zeta\Big) = \tfrac{1}{2}k\pi \quad \text{and} \quad \tfrac{1}{2}(k+1)\pi.$$

Let us set $z \equiv \mathrm{e}^{\pi \mathrm{i}/4}\sqrt{2\xi}\,\zeta$; the complete asymptotic behaviours of $D_\nu(z)$ as $|z| \to \infty$ are well known (see for example [4]) and are given by

$$D_\nu(z) \sim \begin{cases} z^\nu \exp\big(-\tfrac{1}{4}z^2\big), & \text{if } |\arg z| < \tfrac{3}{4}\pi, \\ z^\nu \exp\big(-\tfrac{1}{4}z^2\big) - \frac{\sqrt{2\pi}}{\Gamma(-\nu)}\mathrm{e}^{\mathrm{i}\pi\nu} z^{-\nu-1}\exp\big(\tfrac{1}{4}z^2\big), & \text{on } \arg z = \tfrac{3}{4}\pi, \\ \mathrm{e}^{-2\mathrm{i}\pi\nu} z^\nu \exp\big(-\tfrac{1}{4}z^2\big) - \frac{\sqrt{2\pi}}{\Gamma(-\nu)}\mathrm{e}^{\mathrm{i}\pi\nu} z^{-\nu-1}\exp\big(\tfrac{1}{4}z^2\big), & \text{on } \arg z = \tfrac{5}{4}\pi, \\ \mathrm{e}^{-2\mathrm{i}\pi\nu} z^\nu \exp\big(-\tfrac{1}{4}z^2\big), & \text{if } \tfrac{5}{4}\pi < \arg z < \tfrac{11}{4}\pi. \end{cases} \tag{5.1}$$

Then, on $\arg z = \pm\frac{1}{4}\pi + 2l\pi$, with l integral,

$$D_\nu(z\mathrm{e}^{-2\mathrm{i}l\pi}) \sim (z\mathrm{e}^{-2\mathrm{i}l\pi})^\nu \exp\big(-\tfrac{1}{4}z^2\big)$$

and so, since $D_\nu(z)$ is single-valued,

$$D_\nu(z) \sim \exp\big(-\tfrac{1}{4}z^2\big) z^\nu \mathrm{e}^{-2l\pi\mathrm{i}\nu}. \tag{5.2}$$

Similarly, on $\arg z = \frac{3}{4}\pi + 2l\pi$, we have

$$D_\nu(z) \sim \exp\big(-\tfrac{1}{4}z^2\big) z^\nu \mathrm{e}^{-2l\pi\mathrm{i}\nu} - \frac{\sqrt{2\pi}}{\Gamma(-\nu)}\mathrm{e}^{\nu\pi\mathrm{i}}\exp\big(\tfrac{1}{4}z^2\big) z^{-\nu-1}\mathrm{e}^{2l\pi\mathrm{i}(\nu+1)}. \tag{5.3}$$

To evaluate the Stokes multiplier we proceed as follows. In any sector between two adjacent Stokes directions there is (modulo multiplication by a constant) a unique solution f_1 which is asymptotic to the small exponential. All other solutions are necessarily asymptotic to some multiple of the large exponential, but if we take such a solution on the first Stokes line, then we find that on the second Stokes line its asymptotics have added a multiple of f_1. That multiple is the Stokes multiplier. Thus, relative to the asymptotic forms $\exp(-\frac{1}{4}z^2)z^{\nu}$ and $\exp(\frac{1}{4}z^2)z^{-\nu-1}$, the Stokes multiplier for the sector $\frac{1}{4}\pi+2l\pi$ to $\frac{3}{4}\pi+2l\pi$, in which $\exp(-\frac{1}{4}z^2)z^{\nu}$ is dominant, can be immediately deduced from (5.2) and (5.3). Consequently,

$$\mathrm{SM}\left(\frac{1}{4}\pi+2l\pi,\,\frac{3}{4}\pi+2l\pi\right)=-\frac{\sqrt{2\pi}}{\Gamma(-\nu)}\,\mathrm{e}^{\nu\pi\mathrm{i}}\mathrm{e}^{4l\pi\mathrm{i}\nu}. \tag{5.4}$$

Similar calculations for each of the other pairs of sectors yields the complete monodromy data in the form

$$\begin{aligned}\mathrm{SM}\left(\frac{3}{4}\pi+2l\pi,\,\frac{5}{4}\pi+2l\pi\right)&=\frac{\Gamma(-\nu)}{\sqrt{2\pi}}\,\mathrm{e}^{-\nu\pi\mathrm{i}}\mathrm{e}^{-4l\pi\mathrm{i}\nu}(1-\mathrm{e}^{-2\pi\mathrm{i}\nu})\\&=\mathrm{i}\sqrt{\frac{2}{\pi}}\,\Gamma(-\nu)\mathrm{e}^{-(4l+2)\pi\mathrm{i}\nu}\sin\pi\nu,\end{aligned} \tag{5.5}$$

$$\mathrm{SM}\left(-\frac{3}{4}\pi+2l\pi,-\frac{1}{4}\pi+2l\pi\right)=\frac{\sqrt{2\pi}}{\Gamma(-\nu)}\,\mathrm{e}^{-\nu\pi\mathrm{i}}\mathrm{e}^{4l\pi\mathrm{i}\nu}, \tag{5.6}$$

$$\mathrm{SM}\left(-\frac{1}{4}\pi+2l\pi,\;\frac{1}{4}\pi+2l\pi\right)=-\mathrm{i}\sqrt{\frac{2}{\pi}}\,\Gamma(-\nu)\mathrm{e}^{-4l\pi\mathrm{i}\nu}\sin\pi\nu. \tag{5.7}$$

To obtain (5.7) we need the asymptotics of $D_{-\nu-1}$, which are that

$$D_{-\nu-1}(\mathrm{i}z)\sim\begin{cases}\mathrm{e}^{-\pi\mathrm{i}(\nu+1)/2}z^{-\nu-1}\exp\left(\dfrac{1}{4}z^2\right), & \text{on }\arg z=-\frac{1}{4}\pi,\\ -\dfrac{\sqrt{2\pi}}{\Gamma(\nu+1)}\,\mathrm{e}^{-\pi\mathrm{i}(\nu+2)/2}z^{\nu}\exp\left(-\dfrac{1}{4}z^2\right), & \text{on }\arg z=+\frac{1}{4}\pi.\end{cases}$$

6. Monodromy Data for (3.1)

Although one might expect the double turning-point case to be more complicated than the simple case (and in some sense it is), yet in the double turning-point case one can work out the monodromy data quite explicitly, in terms of the coefficients of the monodromy equations, even for a general form of equation. It is this that leads to the wealth of explicit connection formulae given, for example, in [20]. They are explicit because they are connecting directions where the behaviour of the solution of the Painlevé equation leads to a double turning point in the isomonodromy equations.

In the present section, we show how this monodromy data can be calculated. To do this, we add to Hypotheses H1–H3 in Theorem 1 the following additional hypothesis.

H4. *Suppose that in H1 we can express $\widetilde{F}$ in the form*

$$\widetilde{F}(\eta,\xi) = F_1(\eta,\xi) + \xi^{-\gamma} F_2(\eta,\xi)$$

for some $\gamma > 0$, where F_1 and F_2 are rational in η. Suppose also that F_0 is a perfect square, so that $F_1(\eta,\xi)/\{F_0^{1/2}(\eta)(\eta-\eta_0)\}$ is rational in η with partial fraction decomposition

$$\frac{F_1(\eta,\xi)}{F_0^{1/2}(\eta)(\eta-\eta_0)} = \sum_{i=1}^{N} \frac{A_i}{\eta - s_i}, \quad \text{with} \quad s_1 = \eta_0. \tag{6.1}$$

Finally, suppose that, on $C_{k,k+1}$ and in a neighbourhood of it, $F_2/F_0^{1/2}$ is bounded uniformly in ξ.

Remarks.

1. The quantities A_i, s_i in general depend on ξ, but we suppress that dependence. We, however, assume that they tend to finite limits as $|\xi| \to \infty$.

2. It is obvious that

$$A_1 = \frac{F_1(\eta_0,\xi)}{F_0^{1/2}(\eta_0)}, \tag{6.2a}$$

and we set

$$B = \sum_{i=1}^{N} A_i. \tag{6.2b}$$

3. To compute the monodromy data, we need the relation between ζ and η for large ξ. This is the content of the next theorem.

Theorem 3. *Under Hypothesis H4, and Hypotheses H1–H3 of Theorem 1, for large ξ and η,*

$$\begin{aligned} &\zeta^2 - \alpha^2 \log\zeta + \frac{1}{4}\alpha^2 \log F_0(\eta_0) + o(\xi^{-1}) \\ &\quad = 2\int_{\eta_0}^{\eta} F_0^{1/2}(s)(s-\eta_0)\,\mathrm{d}s - \frac{B}{\xi}\log\eta + \frac{1}{\xi}\sum_{i=2}^{N} A_i \log(\eta_0 - s_i). \end{aligned} \tag{6.3}$$

Proof. From the definition of ζ and (3.16), we have

$$\tfrac{1}{4}\left\{2\zeta^2 - 2\alpha^2\log(2\zeta) + 2\alpha^2\log\alpha - \alpha^2 + O(\alpha^4\zeta^{-2})\right\} = \int_{\eta_2}^{\eta} F^{1/2}(s,\xi)\,\mathrm{d}s. \tag{6.4}$$

In calculating the right-hand side, we replace $F(\eta,\xi)$ by

$$\widehat{F}(\eta,\xi) = F_0(\eta)(\eta-\eta_0)^2 - \frac{F_1(\eta,\xi)}{\xi},$$

i.e., we ignore F_2. This is justifiable because we will find that even the F_1 term contributes only a term of size $O(\xi^{-1})$, which is all that we are interested in. The term F_2, if we included it, would similarly contribute a term of only $O(\xi^{-1-\gamma})$. Thus

$$\int_{\eta_2}^{\eta} \widehat{F}^{1/2}(s,\xi)\,\mathrm{d}s = \left(\int_{\eta_2}^{\eta^*} + \int_{\eta^*}^{\eta}\right) \widehat{F}^{1/2}(s,\xi)\,\mathrm{d}s = I_1 + I_2, \tag{6.5}$$

say, where

$$\eta^* = \eta_0 + T\xi^{-1/2}$$

and T is a large positive number to be specified more precisely later. In I_1 we make the change

$$s - \eta_0 = t\xi^{-1/2},$$

and then

$$I_1 = \frac{1}{\xi}\int_{\{F_1(\eta_0)/F_0(\eta_0)\}^{1/2}}^{T} \{F_0(s)t^2 - F_1(s)\}^{1/2}\,\mathrm{d}t.$$

Since $s - \eta_0 = O(\xi^{-1/2})$ and we are only concerned with evaluating (6.5) correct to $O(\xi^{-1})$, we can safely replace s by η_0 in I_1 which, on integration using (3.16), gives

$$\begin{aligned} I_1 = \frac{F_0^{1/2}(\eta_0)}{4\xi}\biggl\{ 2T^2 - \frac{2F_1(\eta_0)}{F_0(\eta_0)}\log(2T) + \frac{F_1(\eta_0)}{F_0(\eta_0)}\log\left(\frac{F_1(\eta_0)}{F_0(\eta_0)}\right) \\ - \frac{F_1(\eta_0)}{F_0(\eta_0)}\biggr\} + O(\xi^{-1}T^{-2}) + o(\xi^{-1}). \end{aligned} \tag{6.6}$$

Taking $T = -(F_1(\eta_0)/F_0(\eta_0))^{1/2}$, we can compute I_1 explicitly (with $s = \eta_0$) and so conclude from (3.5) that

$$\frac{1}{2}\mathrm{i}\pi\alpha^2 = \frac{F_0^{1/2}(\eta_0)}{4\xi}\left[2\mathrm{i}\pi\frac{F_1(\eta_0)}{F_0(\eta_0)}\right] + o(\xi^{-1}),$$

or

$$\alpha^2 = \frac{F_1(\eta_0)}{\xi F_0^{1/2}(\eta_0)} + o(\xi^{-1}). \tag{6.7}$$

Also, by the binomial expansion, the integral I_2 in (6.5) is given by

$$\begin{aligned} I_2 = \int_{\eta^*}^{\eta_0} F_0^{1/2}(s)(s-\eta_0)\left[1 - \frac{F_1(s)}{2\xi F_0(s)(s-\eta_0)^2}\right]\mathrm{d}s \\ + O\left(\int_{\eta^*}^{\eta}\left|\frac{F_1^2(s)\mathrm{d}s}{\xi^2 F_0^{3/2}(s)(s-\eta_0)^3}\right|\right). \end{aligned} \tag{6.8}$$

According to Hypothesis H4, F_1 is bounded by $F_0^{1/2}$ and so the final term in this expression is of size

$$O\left(\frac{1}{\xi^2}\int_{\eta^*}^{\eta}\frac{|\mathrm{d}s|}{|F_0^{1/2}(s)||s-\eta_0|^3}\right) = O\left(\frac{1}{\xi^2}\frac{1}{|\eta^*-\eta_0|^2}\right) = O(\xi^{-1}T^{-2}).$$

Using the proposed form for $F_1(\eta,\xi)$ as given by (6.1), we find that the second term in I_2 is equal to

$$-\frac{1}{2\xi}\sum_{i=1}^{N}\int_{\eta^*}^{\eta}\frac{A_i}{s-s_i}\mathrm{d}s = -\frac{B}{2\xi}\log\eta + O(\eta^{-1}\xi^{-1}) + o(\xi^{-1}) \qquad (6.9a)$$
$$+\frac{1}{2\xi}\log\prod_{j=2}^{N}(\eta_0-s_j)^{A_j} + \frac{F_1(\eta_0)}{4\xi[F_0(\eta_0)]^{1/2}}\log\left(\frac{T^2}{\xi}\right),$$

whilst the first term may be written as

$$\int_{\eta^*}^{\eta}F_0^{1/2}(s)(s-\eta_0)\,\mathrm{d}s$$
$$=\int_{\eta_0}^{\eta}F_0^{1/2}(s)(s-\eta_0)\,\mathrm{d}s - \frac{1}{2}F_0^{1/2}(\eta_0)T^2\xi^{-1} + o(\xi^{-1}). \qquad (6.9b)$$

Combining (6.5), (6.6), (6.9a) and (6.9b) yields

$$\int_{\eta^*}^{\eta}F^{1/2}(s)\,\mathrm{d}s = \frac{F_1(\eta_0)}{4\xi F_0^{1/2}(\eta_0)}\left\{-2\log 2 - \log\xi + \log\left(\frac{F_1(\eta_0)}{F_0(\eta_0)}-1\right)\right\}$$
$$-\frac{B}{2\xi}\log\eta + \int_{\eta_0}^{\eta}F_0^{1/2}(s)(s-\eta_0)\,\mathrm{d}s$$
$$+\frac{1}{2\xi}\log\prod_{j=2}^{N}(\eta_0-s_j)^{A_j} + O(\xi^{-1}T^{-2}) + o(\xi^{-1}),$$

and so, using (6.4) and (6.7) and making a choice of T large, we see that Theorem 3 is verified.

We have shown that linearly independent solutions of (3.8) are $D_\nu(\mathrm{e}^{\pi\mathrm{i}/4}\sqrt{2\xi}\zeta)$ and $D_{-\nu-1}(\mathrm{e}^{-\pi\mathrm{i}/4}\sqrt{2\xi}\zeta)$ and the asymptotic behaviours of these functions in various sectors have been noted in (5.1). Using the result of Theorem 3 and recalling that $\mathrm{i}\xi\alpha^2 = 2\nu+1$, we find that as $\xi,\zeta\to\infty$,

$$\zeta^{1/2}\mathrm{e}^{-\mathrm{i}\xi\zeta^2/2}\left(\mathrm{e}^{\pi\mathrm{i}/4}\sqrt{2\xi}\zeta\right)^{\nu} \qquad (6.10a)$$
$$\sim \mathrm{e}^{-\mathrm{i}\xi\mathscr{F}(\eta)}\eta^{\mathrm{i}B/2}\left\{\mathrm{e}^{\mathrm{i}\xi\mathscr{F}(\eta_0)}\xi^{\nu/2}\prod_{j=2}^{N}(\eta_0-s_j)^{-\mathrm{i}A_j/2}F_{00}^{(2\nu+1)/8}2^{\nu/2}\mathrm{e}^{\mathrm{i}\pi\nu/4}\right\},$$

$$\zeta^{1/2}\mathrm{e}^{\mathrm{i}\xi\zeta^2/2}\left(\mathrm{e}^{\pi\mathrm{i}/4}\sqrt{2\xi}\zeta\right)^{-\nu-1}$$
$$\sim \mathrm{e}^{\mathrm{i}\xi\mathscr{F}(\eta)}\eta^{-\mathrm{i}B/2}\left\{\mathrm{e}^{-\mathrm{i}\xi\mathscr{F}(\eta_0)}\xi^{-(1+\nu)/2}\prod_{j=2}^{N}(\eta_0-s_j)^{\mathrm{i}A_j/2}F_{00}^{-(2\nu+1)/8} \qquad (6.10b)$$
$$2^{-(1+\nu)/2}\mathrm{e}^{-\mathrm{i}\pi(\nu+1)/4}\right\},$$

where $\mathscr{F}(\eta) \equiv \int_0^\eta F_0^{1/2}(s)(s-\eta_0)\mathrm{d}s$ and F_{00} denotes the value $F_0(\eta_0)$. These relations, together with the Stokes multipliers (5.4)–(5.7) for the parabolic cylinder function, enable us to write down the complete monodromy data for (3.1). With $z \equiv \mathrm{e}^{\pi\mathrm{i}/4}\sqrt{2\xi}\,\zeta$, the Stokes multipliers relative to the solutions

$$\mathrm{e}^{-\mathrm{i}\xi\mathscr{F}(\eta)}\eta^{\mathrm{i}B/2} \quad \text{and} \quad \mathrm{e}^{\mathrm{i}\xi\mathscr{F}(\eta)}\eta^{-\mathrm{i}B/2}$$

are

(i) from $\arg z = \frac{1}{4}\pi + 2l\pi$ to $\frac{3}{4}\pi + 2l\pi$ (where $\mathrm{e}^{-\mathrm{i}\xi\mathscr{F}(\eta)}\eta^{\mathrm{i}B/2}$ is the dominant solution)

$$-\frac{\sqrt{2\pi}}{\Gamma(-\nu)}\mathrm{e}^{\mathrm{i}\pi(4l+1)\nu-2\mathrm{i}\xi\mathscr{F}(\eta_0)}\xi^{-\nu-1/2} \times\prod_{j=2}^{N}(\eta_0-s_j)^{\mathrm{i}A_j}F_{00}^{-(2\nu+1)/4}2^{-\nu-1/2}\mathrm{e}^{-\mathrm{i}\pi(2\nu+1)/4}; \tag{6.11a}$$

(ii) from $\arg z = \frac{3}{4}\pi + 2l\pi$ to $\frac{5}{4}\pi + 2l\pi$ ($\mathrm{e}^{\mathrm{i}\xi\mathscr{F}(\eta)}\eta^{-\mathrm{i}B/2}$ dominant)

$$\mathrm{i}\sqrt{\frac{2}{\pi}}\Gamma(-\nu)\mathrm{e}^{-\mathrm{i}\pi(4l+2)\nu+2\mathrm{i}\xi\mathscr{F}(\eta_0)}\xi^{\nu+1/2} \times\prod_{j=2}^{N}(\eta_0-s_j)^{-\mathrm{i}A_j}F_{00}^{(2\nu+1)/4}2^{\nu+1/2}\mathrm{e}^{\mathrm{i}\pi(2\nu+1)/4}\sin\pi\nu; \tag{6.11b}$$

(iii) from $\arg z = -\frac{3}{4}\pi + 2l\pi$ to $-\frac{1}{4}\pi + 2l\pi$ ($\mathrm{e}^{-\mathrm{i}\xi\mathscr{F}(\eta)}\eta^{\mathrm{i}B/2}$ dominant)

$$\frac{\sqrt{2\pi}}{\Gamma(-\nu)}\mathrm{e}^{\mathrm{i}\pi(4l-1)\nu-2\mathrm{i}\xi\mathscr{F}(\eta_0)}\xi^{-\nu-1/2} \times\prod_{j=2}^{N}(\eta_0-s_j)^{\mathrm{i}A_j}F_{00}^{-(2\nu+1)/4}2^{-\nu-1/2}\mathrm{e}^{-\mathrm{i}\pi(2\nu+1)/4}; \tag{6.11c}$$

(iv) from $\arg z = -\frac{1}{4}\pi + 2l\pi$ to $\frac{1}{4}\pi + 2l\pi$ ($\mathrm{e}^{\mathrm{i}\xi\mathscr{F}(\eta)}\eta^{-\mathrm{i}B/2}$ dominant)

$$-\mathrm{i}\sqrt{\frac{2}{\pi}}\Gamma(-\nu)\mathrm{e}^{-4\mathrm{i}\pi l\nu+2\mathrm{i}\xi\mathscr{F}(\eta_0)}\xi^{\nu+1/2} \times\prod_{j=2}^{N}(\eta_0-s_j)^{-\mathrm{i}A_j}F_{00}^{(2\nu+1)/4}2^{\nu+1/2}\mathrm{e}^{\mathrm{i}\pi(2\nu+1)/4}\sin\pi\nu. \tag{6.11d}$$

We remark finally that, from Theorem 1 and (6.10), the asymptotic forms of ϕ are

$$F^{-1/4}\mathrm{e}^{-\mathrm{i}\xi\mathscr{F}(\eta)}\eta^{\mathrm{i}B/2} \quad \text{and} \quad F^{-1/4}\mathrm{e}^{\mathrm{i}\xi\mathscr{F}(\eta)}\eta^{-\mathrm{i}B/2}. \tag{6.12}$$

7. Application to the Painlevé Equations

Suppose that equation (3.1) arises after scaling from the monodromy equation of some Painlevé equation. (It is our contention that all such monodromy equations reduce to the form (3.1) with simple or double turning points.) In the preceding sections we have evaluated the monodromy data with respect to the usual WKB solutions

$$F^{-1/4}\exp\left\{\pm \mathrm{i}\xi\int_{\eta_0}^{\eta} F^{1/2}(t)\mathrm{d}t\right\}.$$

(These solutions are given in terms of the variable η but when expressed in terms of the original variable λ they are the usual WKB forms.) The theory of the Painlevé equations tells us that the monodromy data are independent of ξ provided that the λ-sector in which the Stokes multipliers are being calculated remains fixed. Since ξ, λ, η are inter-related (in the case of (2.2) $\eta = \xi^{-1/3}\lambda$), the condition that the λ-sector remains fixed means that the η-sector changes with ξ, or at least with $\arg\xi$, and also the turning point η_0 depends in general on ξ. Indeed, it may change from simple to double as $\arg\xi$ changes. (In the case of (2.2), this is a question of the behaviour of $M(\xi)$ as $|\xi| \to \infty$ in a specific direction; $M(\xi) \to 0$ gives double turning points.)

Thus, the monodromy data depend on ξ in various ways, but so long as the λ-sector remains fixed, these various dependences must cancel. This leads, for example, to relations between $M(\xi)$ and ξ for large $|\xi|$, and so to statements about the possible asymptotic behaviours of the Painlevé functions, which we will pursue in later papers.

We now turn to examine the application of our method to the specific case of PII (1.1) with $\beta = 0$ with the aim of using it to establish Theorem B. The relevant version of the generic equation (3.1) is

$$\begin{aligned}\frac{\mathrm{d}^2\phi}{\mathrm{d}\eta} = \xi^2\phi\Bigg\{ &-(4\eta^2+1)^2 + \frac{8\mathrm{i}\eta}{\xi} - \frac{\mathrm{i}}{\xi}(4\eta^2+1)\left(\eta - \frac{\mathrm{i}r}{2q\sqrt{x}}\right)^{-1} \\ &+\frac{4r^2}{x^2} - \frac{4q^2}{x} - \frac{4q^4}{x^2} - \frac{2\mathrm{i}q^2\sqrt{x}}{\xi^2}\left(\eta - \frac{\mathrm{i}r}{2q\sqrt{x}}\right)^{-1} \\ &+\frac{3}{4\xi^2}\left(\eta - \frac{\mathrm{i}r}{2q\sqrt{x}}\right)^{-2}\Bigg\},\end{aligned} \tag{7.1}$$

and this form follows directly from (2.2). If either $x \to +\infty$ or $x \to -\infty$, we have a double turning point $\eta_0 = \pm\frac{1}{2}\mathrm{i}$, and although we can choose, say, $\eta_0 = \frac{1}{2}\mathrm{i}$ when we are considering $x \to +\infty$, the turning point that we have to use when $x \to -\infty$ is then fixed. Thus we have

$$F_0(\eta) = 16\left(\eta + \tfrac{1}{2}\mathrm{i}\right)^2 \text{ if } \eta_0 = \tfrac{1}{2}\mathrm{i}, \quad F_0(\eta) = 16\left(\eta - \tfrac{1}{2}\mathrm{i}\right)^2 \text{ if } \eta_0 = -\tfrac{1}{2}\mathrm{i}. \tag{7.2}$$

In view of the behaviours of the solution of PII as $x \to \pm\infty$ (given by Theorem A and (1.2)), we will write

$$q = x^{-1/4}Q(\xi) = \xi^{-1/6}Q(\xi), \tag{7.3a}$$

so that

$$r = \frac{\mathrm{d}q}{\mathrm{d}x} = \frac{3}{2}\xi^{1/6}\left(Q'(\xi) - \frac{1}{6\xi}Q\right). \tag{7.3b}$$

With these definitions we assume that there exists a sequence $\xi_n \to \infty e^{i\theta}$ with

$$\frac{ir}{2q\sqrt{x}} = \frac{3}{4}i\left(\frac{Q'}{Q} - \frac{1}{6\xi}\right) \to l_1, \tag{7.4a}$$

$$\xi\left(\frac{4r^2}{x^2} - \frac{4q^4}{x^2} - \frac{4q^2}{x}\right) \equiv 9\left[(Q')^2 - \frac{1}{3\xi}QQ' + \frac{Q^2}{36\xi^2}\right] - \frac{4Q^4}{\xi} - 4Q^2 \to l_2. \tag{7.4b}$$

(This assumption is certainly justified if $\theta = 0, \frac{3}{2}\pi$, and q is the solution given by Theorem A. Of course, l_1 and l_2 depend on θ.) Then, with the notation of Section 6, as $|\xi_n| \to \infty$,

$$F_1(\eta, \xi_n) \to 8i\eta - \frac{i(4\eta^2+1)}{\eta - l_1} + l_2, \tag{7.5}$$

and, from (3.13) and (6.7), in the limit as $|\xi_n| \to \infty$,

$$2\nu + 1 - i\frac{F_1(\eta_0, \xi_n)}{F_0^{1/2}(\eta_0)} \to 0, \tag{7.6a}$$

so that

$$\nu + 1 \to \frac{il_2}{16\eta_0}. \tag{7.6b}$$

(Recall that we may have $\eta_0 = \frac{1}{2}i$ or $\eta_0 = -\frac{1}{2}i$.) Furthermore, $F_1/F^{1/2}$ has poles at $\eta = \pm\eta_0$ and $\eta = l_1$ and

$$\frac{F_1}{F^{1/2}} = -\frac{i(2\nu+1)}{\eta - \eta_0} + \frac{i(2\nu+3)}{\eta+\eta_0} - \frac{i}{\eta - l_1}, \tag{7.7}$$

so that $s_2 = -\eta_0, \quad s_3 = l_1, \quad A_2 = i(2\nu+3), \quad A_3 = -i$ and

$$B = \sum_{j=1}^{3} A_j = i.$$

We are now in a position to write down the respective monodromy data by appealing to formulae (6.11). However, rather than expressing the data relative to the solutions (see (6.12))

$$\phi \sim F^{-1/4}e^{-i\xi\mathscr{F}(\eta)}\eta^{iB/2}, \quad \phi \sim F^{-1/4}e^{i\xi\mathscr{F}\eta)}\eta^{-iB/2}, \tag{7.8}$$

it is more convenient to use modified reference solutions. We must use ψ rather than ϕ (see (2.1)), since it is in terms of ψ and λ that the monodromy data are independent of ξ. As $\psi_2 = (\eta - l_1)^{1/2}\phi$ and since $B = i$, linearly independent asymptotic solutions for ψ_2 are

$$\psi_2^{(1)} \sim \eta^{-1} e^{-i\xi \mathscr{F}(\eta)}, \quad \psi_2^{(2)} \sim e^{i\xi \mathscr{F}(\eta)}.$$

In order that Ψ should satisfy (1.4), where the matrix has zero trace, we need the component $\psi_1^{(1)} \sim \exp(-i\xi\mathscr{F})$. It is then immediate from (2.1) that $\psi_2^{(1)} \sim \frac{1}{2} q i \xi^{-1/3}\eta^{-1}\exp(-i\xi\mathscr{F})$, and so we choose to establish monodromy data with respect to

$$\tfrac{1}{2} q i\xi^{-1/3}\eta^{-1} e^{-i\xi\mathscr{F}(\eta)} \quad \text{and} \quad e^{i\xi\mathscr{F}(\eta)},$$

whence, from (6.11), for $\arg z \equiv \arg\left(e^{i\pi/4}\sqrt{2\xi}\zeta\right) = \frac{1}{4}\pi + 2l\pi$ to $\frac{3}{4}\pi + 2l\pi$, the Stokes multiplier is

$$\begin{aligned} -\frac{i\sqrt{2\pi}}{\Gamma(-\nu)} e^{(4l+1)i\pi\nu - 2i\xi\mathscr{F}(\eta_0)} \xi^{-\nu-1/2} \left[\prod_{j=2}^{3} (\eta_0 - s_j)^{iA_j}\right] \\ \times F_{00}^{-(\nu+1/2)/2} 2^{-\nu-3/2} e^{-i\pi(2\nu+1)/4} q\xi^{-1/3}, \end{aligned} \tag{7.9a}$$

whilst for the sector from $\arg z = \frac{3}{4}\pi + 2l\pi$ to $\frac{5}{4}\pi + 2l\pi$ it is

$$\begin{aligned} \sqrt{\frac{2}{\pi}}\Gamma(-\nu) e^{-(4l+2)i\pi\nu + 2i\xi\mathscr{F}(\eta_0)} \xi^{\nu+1/2} \left[\prod_{j=2}^{3} (\eta_0 - s_j)^{-iA_j}\right] \\ \times F_{00}^{(\nu+1/2)/2} 2^{\nu+3/2} e^{i\pi(2\nu+1)/4} q^{-1}\xi^{1/3} \sin\pi\nu. \end{aligned} \tag{7.9b}$$

8. Monodromy Data as $x \to +\infty$

Here we take $\eta_0 = \frac{1}{2}i$, and the Stokes curves through $\eta_0 = \frac{1}{2}i$ are given by

$$\operatorname{Re}\left\{ i\xi \int_{i/2}^{\eta} \left(4\sigma^2 + 1\right) d\sigma \right\} = \operatorname{Re}\left[i\xi \left(\tfrac{4}{3}\eta^3 + \eta - \tfrac{1}{3}i\right)\right] = 0,$$

which are asymptotic to the directions (with $\arg\xi = 0$) $\arg\eta = \frac{1}{3}j\pi$, for integral j. We choose the sector bounded by $\arg\eta = 0$ and $\arg\eta = \frac{1}{3}\pi$, which corresponds to the λ-sector $0 \leqq \arg\lambda \leqq \frac{1}{3}\pi$. This λ-sector must then be the same when we consider $x \to -\infty$.

The asymptotics in Theorem A tell us that, as $x \to +\infty$,

$$\frac{ir}{2q\sqrt{x}} \to -\frac{1}{2}i,$$

so that, in the notation of Section 7,

$$l_1 = -\tfrac{1}{2}i, \quad l_2 = 0, \quad \nu = -1, \quad s_2 = -\tfrac{1}{2}i, \quad s_3 = -\tfrac{1}{2}i, \quad A_2 = i,$$
$$A_3 = -i, \quad B = i.$$

Also, from (6.10),

$$\mathscr{F}(\eta_0) = \tfrac{1}{3}i, \quad F_{00} = -16.$$

Thus from (7.9) the Stokes multiplier for the relevant z-sector $\left(\frac{1}{4}\pi \leqq \arg z \leqq \frac{3}{4}\pi\right)$ is given by

$$\mathrm{SM}_\infty = -a. \tag{8.1}$$

9. Monodromy Data as $x \to -\infty$

Since $\eta = x^{-1/2}\lambda$ and we now have $\arg x = \pi$, the requirement that the λ-sector be fixed now demands that

$$-\tfrac{1}{2}\pi \leqq \arg\eta \leqq -\tfrac{1}{6}\pi. \tag{9.1}$$

We assert that the relevant turning point must now be $-\frac{1}{2}\mathrm{i}$. For if we suppose for contradiction that it is still $+\frac{1}{2}\mathrm{i}$, then we note that the Stokes curve from $\frac{1}{2}\mathrm{i}$ to $\infty\mathrm{e}^{-\mathrm{i}\pi/2}$ passes through $-\frac{1}{2}\mathrm{i}$, since

$$\mathrm{Re}\left\{\mathrm{i}\xi\int_{\mathrm{i}/2}^{-\mathrm{i}/2}\left(4\sigma^2+1\right)\,\mathrm{d}\sigma\right\}=0.$$

(Recall that $\arg\xi = \frac{3}{2}\pi$.) Thus also the Stokes curve associated with $-\frac{1}{2}\mathrm{i}$ and the sector (9.1) must pass through $+\frac{1}{2}\mathrm{i}$, and this is impossible since the real direction from $-\frac{1}{2}\mathrm{i}$ is also a direction for which

$$\mathrm{Re}\left\{\mathrm{i}\xi\int_{-\mathrm{i}/2}\left(4\sigma^2+1\right)\mathrm{d}\sigma\right\}=0.$$

(Set $\sigma + \frac{1}{2}\mathrm{i} = \tau$ for small real τ.)

The asymptotics in Theorem A tell us that, as $x \to -\infty$,

$$\frac{\mathrm{i}r}{2q\sqrt{x}} \sim -\tfrac{1}{2}\cot\left(\tfrac{2}{3}|x|^{3/2} - \tfrac{3}{4}d^2\log|x| + \gamma\right),$$

so that

$$l_1 = -\tfrac{1}{2}\lim_{n\to\infty}\left\{\cot\left(\tfrac{2}{3}|x_n|^{3/2} - \tfrac{3}{4}d^2\log|x_n| + \gamma\right)\right\},$$

where the sequence $\{x_n\}$ (or $\{\xi_n\}$) has to be chosen so that the limit exits. Also

$$l_2 = 4\mathrm{e}^{3\pi\mathrm{i}/2}d^2, \quad \nu = -1 + \tfrac{1}{2}\mathrm{i}d^2, \quad s_2 = \tfrac{1}{2}\mathrm{i},$$

$$s_3 = l_1, \quad A_2 = \mathrm{i}(2\nu+3), \quad A_3 = -\mathrm{i},$$

$$\mathscr{F}(\eta_0) = -\tfrac{1}{3}\mathrm{i}, \qquad F_{00} = -16.$$

Note also that since, from Theorem 3, $\arg\zeta = \frac{3}{2}\arg\eta$ for large $|\eta|$, and since

$$\begin{aligned}\arg z &= \tfrac{1}{4}\pi + \tfrac{1}{2}\arg\xi + \arg\zeta\\ &= \tfrac{1}{4}\pi + \tfrac{3}{4}\arg x + \tfrac{3}{2}\arg\eta\\ &= \tfrac{1}{4}\pi + \tfrac{3}{2}\arg\lambda,\end{aligned}$$

we see that keeping the λ-sector fixed also fixes the z-sector and so we have that the relevant z-sector is again $\frac{1}{4}\pi \leqq \arg z \leqq \frac{3}{4}\pi$. Thus from (7.9) the Stokes multiplier is

$$\frac{\mathrm{i}\sqrt{2\pi}}{\Gamma(1-\frac{1}{2}\mathrm{i}d^2)} \mathrm{e}^{\pi d^2/2} \mathrm{e}^{-2\xi/3} \xi^{(1-\mathrm{i}d^2)/2} (-\mathrm{i})^{-(2\nu+3)} \left(-\frac{1}{2}\mathrm{i} - l_1\right) \times (-16)^{(1-\mathrm{i}d^2)/4} 2^{-(1+\mathrm{i}d^2)/2} \mathrm{e}^{\mathrm{i}\pi/4+\pi d^2/4} q \xi^{-1/3}. \tag{9.2}$$

Since

$$\tfrac{1}{2}\mathrm{i} + l_1 = -\tfrac{1}{2} \lim_{n\to\infty} \left\{ \frac{\exp\left\{-\mathrm{i}\left(\frac{2}{3}|\xi_n| - \frac{1}{2}d^2 \log|\xi_n| + \gamma\right)\right\}}{\sin\left(\frac{2}{3}|\xi_n| - \frac{1}{2}d^2 \log|\xi_n| + \gamma\right)} \right\}$$

we see that (9.2) reduces to

$$\mathrm{SM}_{-\infty} = \frac{2\sqrt{\pi}}{d\Gamma(-\frac{1}{2}\mathrm{i}d^2)} \mathrm{e}^{-\pi\mathrm{i}/4} \mathrm{e}^{-\mathrm{i}\gamma} 2^{-3\mathrm{i}d^2/2} \mathrm{e}^{-\pi d^2/4}. \tag{9.3}$$

Since the Stokes multiplier must be independent of the x-direction, comparison of (8.1) and (9.3) gives (1.3) and proves Theorem B.

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School of Mathematical Sciences
University of Exeter
North Park Road
Exeter EX4 4QE, United Kingdom

Institute of Mathematics & Statistics
University of Kent at Canterbury
Canterbury CT2 7NF, United Kingdom

Department of Applied Mathematics
National Sun Yat-sen University
Kaohsiung, Taiwan, Republic of China

and

Department of Mathematics & Statistics
University of Pittsburgh
Pittsburgh, Pennsylvania 15260

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