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The Euler Limit of the Navier-Stokes Equations, and Rotating Fluids with Boundary

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Abstract

In this paper we study the convergence of weak solutions of the Navier-Stokes equations in some particular domains, with different horizontal and vertical viscosities, when they go to zero with different speeds. The difficulty here comes from the Dirichlet boundary conditions. Precisely we show that if the ratio of the vertical viscosity to the horizontal viscosity also goes to zero, then the solutions converge to the solution of the Euler system. We study the same limit for rotating fluids with Rossby number also going to zero.

1. Introduction

The zero-viscosity limit for the Navier-Stokes equations for incompressible fluids in a bounded domain, with Dirichlet boundary conditions, is a challenging problem due to the formation of a boundary layer satisfying the Prandtl equations, which seem to be ill-posed. The case of the whole space was treated by several authors; we can refer, for instance, to SWANN [13] and KATO [7].

In another work, KATO [6] gives some equivalent formulations of this problem in the case of bounded domains, showing that the convergence to the Euler system is equivalent to the fact that the L^2 strength of the boundary layer goes to 0. Recently, CAFLISCH & SAMMARTINO [3] solved the problem for analytic solutions on a half space by solving the Prandtl equations.

In this paper, we show without using the Prandtl equations, and in some particular domains such as the half-space, that if the ratio of vertical viscosity to horizontal velocity also goes to zero, then all the weak solutions of the Navier-Stokes equations converge to the expected limit, namely, the solution of the Euler system. In general, the anisotropy of the viscosity coefficients is not natural. However, this is a classical situation in geophysical fluids. In fact, instead of putting the classical viscosity $-\tilde{\nu}\Delta_{x,y,z}$ of the fluid in the equations, meteorologists often model turbulent diffusion by using a viscosity of the form $-A_H\Delta_{x,y} - A_V\partial^2_{zz}$, where A_H and A_V are empirical constants, and where A_V is usually much smaller than A_H (for instance, in the ocean, A_V ranges from 1 to 10^3 cm^2/sec whereas A_H ranges from 10^5 to 10^8 cm^2/sec. We recall that the viscosity of the water is of order 10^{-2} cm^2/sec). We refer to the book of PEDLOVSKY [12, Chapter 4], for a more complete discussion.

In the second part we show that weak solutions of the Navier-Stokes equations with a large Coriolis term converge to the Euler system with a damping term as the Rossby number and the horizontal and vertical viscosities go to zero, by using the Ekman layer [4] in the case of appropriate initial data. The case of general initial data, possibly depending on z, will be investigated in a forthcoming paper. This system was studied by GRENIER & MASMOUDI [5] in the case of constant horizontal viscosity. The energy estimates here are different from those of [5] since $|\nabla_{xy}u|$ is no longer bounded here. We refer to the introduction of that paper for a physical discussion of this model.

1.1. Statement of Theorems

1.1.1. The Euler Limit of the Navier-Stokes Equations

We consider the system of equations

$$\partial_t u^{\eta,\nu} + \nabla(u^{\eta,\nu}\otimes u^{\eta,\nu}) - \nu\partial_z^2 u^{\eta,\nu} - \eta\Delta_{x,y}u^{\eta,\nu} = -\nabla p + F^{\eta,\nu} \quad \text{in } \Omega, \tag{1}$$

$$\nabla\cdot u^{\eta,\nu} = 0 \quad \text{in } \Omega, \tag{2}$$

$$u^{\eta,\nu} = 0 \quad \text{in } \partial\Omega, \tag{3}$$

$$u^{\eta,\nu}(0) = u_0^{\eta,\nu} \quad \text{with } \nabla\cdot u_0^{\eta,\nu} = 0 \tag{4}$$

where $\Omega = \omega\times[0,h]$, or $\Omega = \omega\times[0,\infty[$, and $\omega = \mathbb{T}^2$, or $[\mathbb{R}^2$, $\Delta_{x,y}$ denotes the two-dimensional Laplace operator in the variables x and y, $F^{\eta,\nu}$ is an exterior force which converges to F^0 in $L^1(0,T,L^2(\Omega)^3)$, where $T \leqq \infty$, and $u_0^{\eta,\nu}$ is a divergence-free initial condition converging strongly in $L^2(\Omega)$ to a function u_0, as η, ν go to 0.

Let us recall a result concerning the Euler system in Ω:

$$\begin{gathered}\partial_t w + \nabla(w\otimes w) = -\nabla p + F^0 \quad \text{in } \Omega,\\ \nabla\cdot w = 0 \quad \text{in } \Omega,\\ w\cdot n = \pm w_3 = 0 \quad \text{on } \partial\Omega,\\ w(t=0) = w^0.\end{gathered} \tag{5}$$

It is known that if s is given, $s > \frac{5}{2}$, and

$$\begin{gathered}w^0 \in H^s(\Omega)^3, \quad \nabla\cdot w^0 = 0, \quad w^0\cdot n = 0 \text{ on } \partial\Omega,\\ F^0 \in L^1(0,T;H^s(\Omega)^3),\end{gathered}$$

then there exists T^*, and a unique solution of (5), defined on $[0, T^*]$ and satisfying

$$w \in L^\infty(0, T^*; H^s(\Omega)^3).$$

We refer to [1] and [15] for a proof of this classical result.

Theorem 1.1. *Let $u^{\eta,\nu}(0)$ converge in $L^2(\Omega)$ to w^0, $F^{\eta\nu}$ converge in $L^1(0, T, L^2(\Omega))$ to F^0, and ν/η converge to* 0. *Then, there exist global weak solutions $u^{\eta,\nu}$ of* (1)–(4) *such that*

$$u^{\eta,\nu} - w \to 0 \quad \textit{in } L^\infty(0, T^*, L^2(\Omega)),$$
$$\sqrt{\eta}\nabla_{x,y}u^{\eta,\nu}, \sqrt{\nu}\partial_z u^{\eta,\nu} \to 0 \quad \textit{in } L^2(0, T^*, L^2(\Omega)).$$

1.1.2. Rotating Fluids

We consider this system of equations

$$\begin{aligned} &\partial_t u^{\varepsilon,\nu,\eta} + \nabla(u^{\varepsilon,\nu,\eta} \otimes u^{\varepsilon,\nu,\eta}) - \nu\partial_z^2 u^{\varepsilon,\nu,\eta} - \eta\Delta_{x,y}u^{\varepsilon,\nu,\eta} \\ &\qquad + \frac{e_3 \times u^{\varepsilon,\nu,\eta}}{\varepsilon} = -\frac{\nabla p}{\varepsilon} + F^{\varepsilon,\nu,\eta} \quad \text{in } \Omega, \end{aligned} \tag{6}$$

$$\nabla \cdot u^{\varepsilon,\nu,\eta} = 0 \quad \text{in } \Omega, \tag{7}$$

$$u^{\varepsilon,\nu,\eta} = 0 \quad \text{on } \partial\Omega, \tag{8}$$

$$u^{\varepsilon,\nu,\eta}(0) = u_0^{\varepsilon,\nu,\eta} \quad \text{with } \nabla \cdot u_0^{\varepsilon,\nu,\eta} = 0, \tag{9}$$

where $\Omega = \mathbb{T}^2 \times [0, h]$ and e_3 is the unit vertical vector. In the sequel we omit the parameters ε, ν, η if no ambiguity can occur.

These equations describe the evolution of an incompressible three-dimensional fluid in a rotating frame, $\varepsilon^{-1}e_3 \times u^{\varepsilon,\nu,\eta}$ being the Coriolis force created by the rotation at high frequency ε^{-1}.

We assume that $u^{\varepsilon,\nu,\eta}(0)$ converges in $L^2(\Omega)$ to w^0, where $w^0 \in H^s$, $s > 2$, $w_3^0 = 0$, $\int w^0 = 0$, and w^0 does not depend on z, and that $F^{\varepsilon,\nu,\eta}$ converges in $L^1(0, T, L^2(\Omega))$ to F^0, where $F^0 \in L^2(0, T, H^s)$, $F_3^0 = 0$, and F^0 does not depend on z. We also assume that ε, ν, η go to zero.

By assumption, the initial data do not depend on z. This is linked to the Taylor-Proudman theorem [14] which says that $e_3 \times u$ is a gradient if and only if u is independent of z.

Let us recall a result concerning the Euler system with a damping term in the two-dimensional case ($\omega = \mathbb{T}^2$):

$$\begin{aligned} \partial_t w^{\varepsilon,\nu} + \nabla(w^{\varepsilon,\nu} \otimes w^{\varepsilon,\nu}) + \sqrt{\nu/\varepsilon}\frac{\sqrt{2}}{h}w^{\varepsilon,\nu} &= -\nabla p + F^0, \\ \nabla \cdot w^{\varepsilon,\nu} &= 0, \\ w^{\varepsilon,\nu}(t = 0) &= w^0. \end{aligned} \tag{10}$$

If s is given, $s > 2$, and

$$w^0 \in H^s(\mathbb{T}^2)^2, \quad \nabla \cdot w^0 = 0, \quad \int w^0 = 0,$$

$$F^0 \in L^1(0, T; H^s(\mathbb{T}^2)^2),$$

then there exists a unique strong solution w of (10) such that

$$w \in L^\infty(0, T'; H^s(\mathbb{T}^2)^2) \quad \text{for any } T' < T,$$
$$w \in L^\infty(0, T; H^s(\mathbb{T}^2)^2) \quad \text{if } T < \infty.$$

This can be proved by using the classical results of [8, 11]. We only notice that since $\sqrt{\nu/\varepsilon}\frac{\sqrt{2}}{h}w^{\varepsilon,\nu}$ is a damping term, the bounds we get on w are uniform in ε, ν. We use $|w|_{L^\infty(0,T',H^s)}$ to denote this bound.

Theorem 1.2. *Let $u^{\varepsilon,\nu,\eta}(0)$ strongly converge in $L^2(\Omega)$ to w^0, and $F^{\varepsilon,\nu,\eta}$ strongly converge in $L^1(0, T, L^2(\Omega))$ to F^0. Then there exist global weak solutions $u^{\varepsilon,\nu,\eta}$ of* (6)–(9), *and a constant C(h), such that if*

$$\eta > C(h)\varepsilon|w|^2_{L^\infty},$$

then

$$u^{\varepsilon,\nu,\eta} - w^{\varepsilon,\nu} \to 0 \quad \text{in } L^\infty(0, T', L^2(\Omega)),$$
$$\sqrt{\eta}\nabla_{x,y}u^{\varepsilon,\nu,\eta}, \sqrt{\nu}\partial_z u^{\varepsilon,\nu,\eta} \to 0 \quad \text{in } L^2(0, T', L^2(\Omega)),$$

for any $T' < T$.

1.2. Preliminary Results and Definitions

Let us denote by V^0 the subspace of $L^2(\Omega)^3$ consisting of divergence-free vectors (div $u = 0$) that are tangent to $\partial\Omega$ so that $u_3(0) = 0$, and $u_3(h) = 0$ if $\Omega = \omega \times [0, h]$. For $m \geqq 0$ we also set

$$V^m = H^m(\Omega)^3 \cap V^0 = \{u \in H^m(\Omega)^3, \ \text{div}\, u = 0, u \cdot n = 0 \quad \text{on } \partial\Omega\}.$$

Let $\mathscr{P}$ be the orthogonal projection of $L^2(\Omega)^3$ onto V^0. We recall that $\mathscr{P}$ is also a linear continuous operator from $H^m(\Omega)^3$ into itself; hence

$$|\mathscr{P}(u)|_{H^m} \leqq C|u|_{H^m}.$$

We use some classical results concerning the Sobolev spaces $H^s(R^d)$, namely, if $s > \frac{1}{2}d$, $t \in R^+$, then

$$|u|_{L^\infty} \leqq C|u|_{H^s}, \quad |u\, v|_{H^t} \leqq C|u|_{H^s}|v|_{H^t},$$

where $u \in H^s$, and $v \in H^t$.

We also need a trace theorem: for $t > \frac{1}{2}$,

$$|u_{|\partial\Omega}|_{H^{t-1/2}(\partial\Omega)} \leqq C|u|_{H^t}.$$

We recall the Hardy-Littlewood inequality

$$\left|\frac{1}{z}\int_0^z f(t)dt\right|_{L^2(R^+)} \leqq C|f|_{L^2(R^+)};$$

we refer for instance to [9] for a proof of these two classical results. Finally we show a lemma of Gronwall's type.

Lemma 1.1. *Let $\alpha > 0$ and let a, b, c be nonnegative functions in $L^1(0, T)$ satisfying*

$$\int_0^T a(t)dt \leqq C,$$

$$\int_0^T b(t)dt \leqq C\alpha, \quad \int_0^T c(t)dt \leqq C\alpha^2.$$

If a nonnegative function f satisfies

$$\partial_t(f^2) \leqq a(t)f^2 + b(t)f + c(t), \quad f(0) \leqq C\alpha,$$

then $|f(t)| \leqq M\alpha$, for all $t \geqq 0$, where M depends only on C.

Proof. We have

$$b(t)f = \frac{b(t)}{\alpha}\alpha f \leqq \frac{b(t)}{\alpha} f^2 + \frac{b(t)}{\alpha}\alpha^2.$$

Hence denoting

$$\tilde{a} = a + \frac{b(t)}{\alpha}, \quad \tilde{c} = c + b(t)\alpha,$$

we have

$$\int_0^T a(t)dt \leqq C', \quad \int_0^T c(t)dt \leqq C'\alpha^2, \quad \partial_t(f^2) \leqq \tilde{a}(t)f^2 + \tilde{c}(t).$$

The standard Gronwall Lemma implies that

$$|f^2(t)| \leqq \exp\left(\int_0^t \tilde{a}(s)ds\right)\left[f^2(0) + \int_0^t \tilde{c}(s)ds\right];$$

hence

$$|f(t)| \leqq M\alpha.$$

Notice that this bound does not depend on T. This concludes the proof of the lemma. □

2. The Euler Limits of the Navier-Stokes equations

If we try to use energy estimates to show that $u^{\eta,\nu} - w$ remains small, we see that the integrations by parts introduce terms that we cannot control, since $u^{\eta,\nu} - w$ does not vanish at the boundary.

2.1. Construction of the Boundary Layer

We construct a boundary layer which allows us to recover the Dirichlet boundary conditions. Therefore, $\mathscr{B}$ is a corrector with small L^2 norm, localized near $\partial\Omega$. We assume that $\Omega = \omega \times [0, \infty[$, and construct the boundary layer near $z = 0$. In the case $\Omega = \omega \times [0, h]$, the construction is the same near $z = h$. So $\mathscr{B}$ has to satisfy

$$\mathscr{B}(z = 0) + w(z = 0) = 0, \quad \mathscr{B}(z = \infty) = 0. \tag{11}$$

Usually, boundary layers are due to two or many terms of the same order. Here, we add an extra term to obtain a simple expression. This is not a problem because the boundary layer has a vanishing L^2 norm.

$$-\nu\partial_z^2\mathscr{B}^1 = \frac{1}{\theta}\mathscr{B}^1, \quad \mathscr{B}^1(z = 0) = -w(z = 0), \quad \mathscr{B}^1(z = \infty) = 0, \tag{12}$$

where θ is a parameter that will be chosen later on. Obviously, we have

$$\mathscr{B}_1^1 = -w_1(0)e^{-z/\sqrt{\theta\nu}}, \quad \mathscr{B}_2^1 = -w_2(0)e^{-z/\sqrt{\theta\nu}}. \tag{13}$$

In order to respect the divergence-free condition, we must add $\mathscr{B}_3^1$ so that

$$\partial_z\mathscr{B}_3^1 = -(\partial_x\mathscr{B}_1^1 + \partial_y\mathscr{B}_2^1).$$

Hence, we have $\partial_z\mathscr{B}_3^1 = \partial_z w_3(0)e^{-z/\sqrt{\theta\nu}}$. Integrating this, and using the condition $\mathscr{B}_3^1(z = \infty) = 0$, we obtain

$$\mathscr{B}_3^1(z) = \sqrt{\theta\nu}\partial_z w_3(0)e^{-z/\sqrt{\theta\nu}}.$$

Then, we see that $\mathscr{B}_3^1$ does not satisfy the correct boundary condition at $z = 0$.

All this suggests taking $\mathscr{B}$ of the form

$$\begin{aligned} \mathscr{B}_1 &= -w_1(0)\phi\left(z/\sqrt{\theta\nu}\right), \\ \mathscr{B}_2 &= -w_2(0)\phi\left(z/\sqrt{\theta\nu}\right), \\ \mathscr{B}_3 &= -\sqrt{\theta\nu}\partial w_3(0)\psi\left(z/\sqrt{\theta\nu}\right) \end{aligned} \tag{14}$$

where ϕ and ψ are smooth and satisfy

$$\begin{gathered} \psi' = \phi, \\ \phi(0) = 1, \quad \psi(0) = 0, \\ \phi(z) = \psi(z) = 0 \quad \text{for } z > 1. \end{gathered} \tag{15}$$

In conclusion, notice that $\mathscr{B}$ is a linear combination of $w_1(0)$, $w_2(0)$, $\partial_z w_3(0)$ and can be written as

$$\mathscr{B}(z) = \mathscr{M}(z)A(t, x, y)$$

where $A = (w_1(0), w_2(0), \partial_z w_3(0))$, and $\mathscr{M}$ is a (3,3)-matrix. Notice that $\mathscr{M}$ is composed of two blocks, namely, M a (2,2)-matrix and m a (1,1)-matrix and that we have

$$|\mathscr{M}(z)|_{L^2_z} \leqq C\sqrt{\sqrt{\theta\nu}}, \quad |m(z)|_{L^\infty_z} \leqq C\sqrt{\theta\nu},$$

$$|z\partial_z M(z)|_{L^2_z} \leqq C\sqrt{\sqrt{\theta\nu}}, \quad |\partial_z M(z)|_{L^2_z} \leqq C/\sqrt{\sqrt{\theta\nu}},$$

$$|z^2\partial_z M(z)|_{L^\infty_z} \leqq C\sqrt{\theta\nu}.$$

In fact, m satisfies a better estimate, namely, $|m(z)|_{L^2_z} \leqq C\sqrt{\theta\nu}\sqrt{\sqrt{\theta\nu}}$, but we do not use this refinement since we want to use the same proof for the case of rotating fluids.

2.2. Energy Estimates

We now state a theorem more precise than Theorem 1.1. We recall that $u^{\eta,\nu}(0)$ converges in $L^2(\Omega)$ to w^0, and $F^{\eta\nu}$ converges in $L^1(0,T,L^2(\Omega))$ to F^0, with $w^0 \in V^s$ and $F^0 \in L^1(0,T,H^s)$.

Theorem 2.1. *Let*

$$|u^{\eta,\nu}(0) - w^0|_{L^2} \leqq C\beta(\eta,\nu),$$

$$|F^{\eta\nu} - F^0|_{L^1(0,T,L^2(\Omega))} \leqq C\beta(\eta,\nu)$$

with $\beta(\eta,\nu) \geqq \sqrt{\eta + \sqrt{\nu/\eta}}$, *and converging to* 0. *Then there exist global weak solutions* $u^{\eta,\nu}$ *of* (1)– (4) *satisfying*

$$|u^{\eta,\nu} - w|_{L^\infty(0,T^*,L^2(\Omega))} \leqq C(T^*, w)\beta,$$

$$\eta|\nabla_{x,y}u^{\eta,\nu}|^2_{L^2(0,T^*,L^2(\Omega))} + \nu|\partial_z u^{\eta,\nu}|^2_{L^2(0,T^*,L^2(\Omega))} \leqq C(T^*, w)\beta^2,$$

where $T^* \leqq T$ *and* w *is the strong solution of the Euler system* (5) *in* $L^\infty(0, T^*, H^s)$.

Proof. We prove Theorem 2.1, when $\Omega = \omega \times [0, \infty[$. The case of $\Omega = \omega \times [0, h]$ can be handled similarly by considering a second boundary layer near $z = h$.

For any η, ν the existence of a weak solution of (1) – (4) can be shown by using the Galerkin method. We refer here to [8] and to [16]. We find a sequence $u_n^{\eta,\nu}$ of smooth approximate solutions that converges weakly to $u^{\eta,\nu}$ in $L^\infty(0,T,L^2)$ and in $L^2(0,T,H^1)$. Let $v^{\eta,\nu} = u^{\eta,\nu} - w^{\eta,\nu} - \mathscr{B}$, where $\mathscr{B}$ is the boundary layer constructed in the last section. Replacing $u^{\eta,\nu}$ by $w^{\eta,\nu} + v^{\eta,\nu} + \mathscr{B}$ in (1) and subtracting (5) yields

$$\begin{aligned}
&\partial_t\mathscr{B} + \partial_t v + u.\nabla v + w.\nabla\mathscr{B} + \mathscr{B}.\nabla\mathscr{B} + v.\nabla\mathscr{B} + \mathscr{B}.\nabla w + v.\nabla w \\
&\quad - \eta\Delta_{x,y}\mathscr{B} - \eta\Delta_{x,y}v - \eta\Delta_{x,y}w - \nu\partial_z^2\mathscr{B} - \nu\partial_z^2 v - \nu\partial_z^2 w \qquad (16)\\
&\quad = -\nabla q' + F - F^0.
\end{aligned}$$

We take the scalar product of (16) with v yields to get

$$\begin{aligned}
&\tfrac{1}{2}\partial_t \int |v|^2 + \int \eta |\nabla_{x,y} v|^2 + \nu \int |\partial_z v|^2 \\
&\quad = -\int \partial_t \mathscr{B} v - \int u \cdot \nabla v v - \int w \cdot \nabla \mathscr{B} v - \int \mathscr{B} \cdot \nabla \mathscr{B} v \\
&\qquad - \textstyle\int v \cdot \nabla \mathscr{B} v - \displaystyle\int \mathscr{B} \cdot \nabla w v - \int v \cdot \nabla w v + \eta \int \Delta_{x,y} \mathscr{B} v \\
&\qquad + \eta \textstyle\int \Delta_{x,y} w v + \nu \displaystyle\int \partial_z^2 \mathscr{B} v + \nu \int \partial_z^2 w v + \int (F - F^0) v
\end{aligned} \tag{17}$$

where we use the fact that $\int \nabla q' v = -\int q' \nabla . v = 0$. (In fact, we should carry out all these computations on the approximate sequence $u_n^{\eta,\nu}$, and then take the limit.) In the sequel, we denote the first and second components of v respectively by V_1 and V_2, and those of $\mathscr{B}$ by B_1 and B_2, and so on.

We now have to bound each of the twelve terms of the right-hand side of (17). In the sequel, C denotes any constant depending only on h, and c denotes a small constant which will be chosen later.

1) First of all, we notice that

$$\partial_t \mathscr{B} = \mathscr{M}(z) \partial_t A,$$

and since $\partial_t w = \mathscr{P}(-w \cdot \nabla w + F^0)$, we have

$$|\partial_t w|_{H^{s-1}} \leqq C(h)(|w \cdot \nabla w|_{H^{s-1}} + |F^0|_{H^{s-1}}) \leqq C|w|_{H^s}^2 + C|F^0|_{H^{s-1}}.$$

Hence we have

$$\begin{aligned}
\left| \int \partial_t \mathscr{B} v \right| &\leqq |\partial_t \mathscr{B}|_{L^2} |v|_{L^2} \\
&\leqq |\mathscr{M}(z)|_{L_z^2} |\partial_t A|_{L_\omega^2} |v|_{L^2} \\
&\leqq C \sqrt{\sqrt{\theta \nu}} |\partial_t w(0)|_{H^1} |v|_{L^2} \\
&\leqq C \sqrt{\sqrt{\theta \nu}} |\partial_t w|_{H^{s-1}} |v|_{L^2} \\
&\leqq C \sqrt{\sqrt{\theta \nu}} (|w|_{H^s}^2 + |F^0|_{H^{s-1}}) |v|_{L^2}
\end{aligned}$$

since $s - 1 > \frac{3}{2}$.

2) Here, we write as usual

$$\int (u \cdot \nabla v) \cdot v = \int u_i \partial_i (\tfrac{1}{2} v_j^2) = -\int (\partial_i u_i) \tfrac{1}{2} v_j^2 = 0.$$

3) We split this term into two parts. On one hand,

$$\begin{aligned}\left|\int W\cdot\partial_{x,y}\mathscr{B}v\right| &\leqq |W\cdot\partial_{x,y}\mathscr{B}|_{L^2}|v|_{L^2}\\ &\leqq |W|_{L^\infty}|\mathscr{M}(z)|_{L^2_z}|\nabla_{x,y}A|_{L^2_\omega}|v|_{L^2}\\ &\leqq \sqrt{\sqrt{\theta\nu}}|w|^2_{H^s}|v|_{L^2}\end{aligned}$$

since $|\nabla_{x,y}A|_{L^2_\omega}| \leqq |w(0)|_{H^2_\omega} \leqq |w|_{H^s}$. (We recall here that W denotes the first and the second components of w.) On the other hand,

$$\int w_3\partial_z\mathscr{B}v = \int_\omega\int_0^1 w_3\partial_z\mathscr{B}v$$

since $\mathscr{B}(z)$ vanishes for $z>1$. Also

$$\begin{aligned}\left|\int w_3\partial_z\mathscr{B}v\right| &\leqq |\frac{w_3}{z}|_{L^\infty}|z\partial_z\mathscr{B}|_{L^2(\omega\times[0,1])}|v|_{L^2}\\ &\leqq |\partial_z w_3|_{L^\infty}|z\partial_z\mathscr{M}|_{L^2}|A|_{L^2_\omega}|v|_{L^2}\\ &\leqq C|w|_{H^s}\sqrt{\sqrt{\theta\nu}}|w(0)|_{H^1}|v|_{L^2}\\ &\leqq C\sqrt{\sqrt{\theta\nu}}|w|^2_{H^s}|v|_{L^2}\end{aligned}$$

since $|A|_{L^2} \leqq |\nabla_{x,y}w(0)|_{L^2} + |w|_{L^2} \leqq |w(0)|_{H^1}$.

4) We split the integral into two parts:

$$\begin{aligned}\int \mathscr{B}\cdot\nabla\mathscr{B}v &= \int B\cdot\nabla_{x,y}\mathscr{B}v + \int b\cdot\partial_z\mathscr{B}v\\ &\leqq |B\cdot\nabla\mathscr{B}|_{L^2}|v|_{L^2} + |b\cdot\partial_z\mathscr{B}|_{L^2}|v|_{L^2}.\end{aligned}$$

The first term is treated as the third term (see 3) above:

$$\begin{aligned}|B\cdot\nabla\mathscr{B}|_{L^2}|v|_{L^2} &\leqq |w(0)|_{L^\infty}|\partial_{x,y}\mathscr{B}|_{L^2}|v|_{L^2}\\ &\leqq C\sqrt{\sqrt{\theta\nu}}|w|^2_{H^s}|v|_{L^2}.\end{aligned}$$

The second one is very easily treated:

$$\begin{aligned}|b\partial_z\mathscr{B}|_{L^2}|v|_{L^2} &\leqq |m|_{L^\infty}|A|_{L^\infty}|\partial_z\mathscr{M}|_{L^2(z)}|A|_{L^2_{x,y}}|v|_{L^2}\\ &\leqq C\sqrt{\theta\nu}|w|_{H^s}\frac{1}{\sqrt{\sqrt{\theta\nu}}}|w|_{H^s}|v|_{L^2}\\ &\leqq C\sqrt{\sqrt{\theta\nu}}|w|^2_{H^s}|v|_{L^2}\end{aligned}$$

since $|\partial_z w(0)|_{L^\infty_\omega},\ |A|_{L^2_\omega} \leqq |w|_{H^s}$.

5) We deal here with the "worst" term:

$$\int v\cdot\nabla\mathscr{B}v = \int V\cdot\nabla_{x,y}BV + \int V\cdot\nabla_{x,y}bv_3 + \int v_3\partial_z\mathscr{B}v$$
$$= I_1 + I_2 + I_3.$$

We first observe that

$$|I_1| \leqq 2|V|^2_{L^2}|\nabla_{x,y}B|_{L^\infty}$$
$$\leqq |w(0)|_{H^{2+r}_\omega}|v|^2_{L^2}$$
$$C \leqq |w|_{H^s}|v|^2_{L^2},$$

where $r = s - \frac{5}{2}$.

Integrating the second term by parts, we find that

$$|I_2| = |-\int(\partial_x v_1 + \partial_y v_2)bv_3 + v_1 b\partial_x v_3 + v_2 b\partial_y v_3|$$
$$\leqq |\nabla_{x,y}v|_{L^2}|b|_{L^\infty}|v|_{L^2}$$
$$\leqq \sqrt{\theta\nu}|w|_{H^s}|\nabla_{x,y}v|_{L^2}|v|_{L^2}$$
$$\leqq c\nu\theta|\nabla_{x,y}v|^2_{L^2} + \frac{C}{c}|w|^2_{H^s}|v|^2_{L^2},$$

while

$$I_3 = I_3' + I_3'', \qquad |I_3'| = \left|\int_{x,y}\int_0 v_3\partial_z BV\right|.$$

Here, we use the fact that v vanishes at $z = 0$, and thus v_3 is small where $\partial_z B$ is large. We set $\Omega' = \omega\times[0,1]$.

$$\int_{x,y}\int_0^1 v_3\partial_z BV dz = \int_{x,y}\int_0^1 \frac{v_3}{z}z^2\partial_z B\frac{V}{z}dz$$
$$\leqq |v_3/z|_{L^2(\Omega')}|z^2\partial_z B|_{L^\infty(\Omega')}|v/z|_{L^2(\Omega')}$$
$$\leqq C|\partial_z v_3|_{L^2(\Omega')}|z^2\partial_z M|_{L^\infty}|A|_{L^\infty}|\partial_z v|_{L^2(\Omega')}$$
$$\leqq C|\partial_z v_3|_{L^2(\Omega')}\sqrt{\theta\nu}|w(0)|_{L^\infty}|\partial_z v|_{L^2(\Omega')}$$
$$\leqq C\sqrt{\theta\nu}|w|_{L^\infty}|\nabla_{x,y}v|_{L^2}|\partial_z v|_{L^2}$$
$$\leqq c\eta|\nabla_{x,y}v|^2_{L^2} + \frac{C\theta|w|^2_{L^\infty}}{c\eta}\nu|\partial_z v|^2_{L^2}.$$

Next, we have

$$|I_3''| = \left|\int v_3\partial_z bv_3\right| \leqq C|w|_{H^s}|v|^2_{L^2}.$$

6) This term is handled like the third and fourth terms:

$$\int \mathscr{B} \cdot \nabla w v \leqq |\mathscr{B}|_{L^2} |\nabla_{x,y} w|_{L^\infty} |v|_{L^2}$$

$$\leqq C\sqrt{\sqrt{\theta\nu}}|w|^2_{H^s}|v|_{L^2}.$$

7) As for the sixth term, we have

$$\left|\int v \cdot \nabla w v\right| \leqq |\nabla w|_{L^\infty}|v|^2_{L^2}$$

$$\leqq |w|_{H^s}|v|^2_{L^2}.$$

8) Integrating by parts, we find that

$$\eta\left|\int \Delta_{x,y}\mathscr{B}v\right| \leqq \frac{c\eta}{2}|\nabla_{x,y}v|^2_{L^2} + \frac{\eta}{2c}|\nabla_{x,y}\mathscr{B}|^2_{L^2}$$

$$\leqq \frac{c\eta}{2}|\nabla_{x,y}v|^2_{L^2} + \frac{C}{c}\eta\sqrt{\theta\nu}|w|^2_{H^s}.$$

9) Similarly, we have

$$\eta\left|\int \Delta_{x,y}wv\right| \leqq \frac{c\eta}{2}|\nabla_{x,y}v|^2_{L^2} + \frac{\eta}{2c}|\nabla_{x,y}w|^2_{L^2}$$

$$\leqq \frac{c\eta}{2}|\nabla_{x,y}v|^2_{L^2} + \frac{C}{c}\eta|w|^2_{H^s}.$$

10) Integrating by parts once more, we obtain

$$\nu\left|\int \partial_z^2\mathscr{B}v\right| \leqq \frac{\nu}{2c}|\partial_z\mathscr{B}|^2_{L^2} + \frac{c\nu}{2}|\partial_z v|^2_{L^2}$$

$$\leqq \frac{c\nu}{2}|\partial_z v|^2_{L^2} + C\nu|\partial_z\mathscr{M}|^2_{L^2}|A|^2_{L^2}$$

$$\leqq \frac{c\nu}{2}|\partial_z v|^2_{L^2} + C\nu\frac{1}{\sqrt{\theta\nu}}|w|^2_{H^s}$$

$$\leqq \frac{c\nu}{2}|\partial_z v|^2_{L^2} + \frac{C}{c}\sqrt{\frac{\nu}{\theta}}|w|^2_{H^s}.$$

11) Similarly, we obtain

$$\nu\left|\int \partial_z^2 wv\right| \leqq \frac{\nu}{2c}|\partial_z w|^2_{L^2} + \frac{c\nu}{2}|\partial_z v|^2_{L^2}$$

$$\leqq \frac{c\nu}{2}|\partial_z v|^2_{L^2} + \frac{C}{c}\nu|w|^2_{H^s}.$$

12) Using the Cauchy-Schwarz inequality, we have

$$\int (F^{\eta\nu} - F^0)v \leqq |F^{\eta\nu} - F^0|_{L^2}|v|_{L^2}.$$

Adding up the estimates for the twelve terms yields

$$\begin{aligned}
&\tfrac{1}{2}\partial_t |v|_{L^2}^2 + \eta|\nabla_{x,y}v|_{L^2}^2 + \nu|\partial_z v|_{L^2}^2 \\
&\quad\leqq c(\eta + \nu\theta)|\nabla_{x,y}v|_{L^2}^2 \\
&\qquad + \nu(c + \frac{\theta|w|_{L^\infty}^2}{c\eta})|\partial_z v|_{L^2}^2 + C\left(\eta + \sqrt{\frac{\nu}{\theta}}\right)|w|_{H^s}^2 \\
&\qquad + C|w|_{H^s}|v|_{L^2}^2 + \left(C\sqrt{\sqrt{\theta\nu}}\left(|w|_{H^s} + |w|_{H^s}^2\right)\right. \\
&\qquad \left. + |F^{\eta,\nu} - F^0|_{L^2}\right)|v|_{L^2}.
\end{aligned} \tag{18}$$

Until now, we only assumed that $\eta, \nu, \theta < 1$. In the sequel we assume that $\nu < \eta$, and $\nu < \theta$. Taking c small enough we can absorb the $|\nabla_{x,y}v|_{L^2}^2$ and $|\partial_z v|_{L^2}^2$ of the right-hand side into the left-hand side, provided that

$$\theta \leqq C(h)\frac{\eta}{|w|_{L^\infty}}. \tag{19}$$

We can suppose that (19) holds because θ is a free parameter. In fact, we take $\theta = C(h)\eta/|w|_{L^\infty}^2$. We notice then that (18) leads to

$$\tfrac{1}{2}\partial_t |v|_{L^2}^2 \leqq a(t)|v|_{L^2}^2 + b(t)|v|_{L^2} + c(t)$$

with

$$\begin{aligned}
a(t) &= C|w|_{H^s}, \\
b(t) &= C\sqrt{\sqrt{\theta\nu}}\left(|w|_{H^s} + |w|_{H^s}^2\right) + |F^{\eta,\nu} - F^0|_{L^2}, \\
c(t) &= C\left(\eta + \sqrt{\frac{\nu}{\eta}}|w|_{H^s}^2\right)|w|_{H^s}^2.
\end{aligned}$$

The conditions of Lemma 1.1 are then satisfied, with

$$\alpha = \beta \geqq C\sqrt{\eta + \sqrt{\frac{\nu}{\eta}}}.$$

This yields

$$|v^{\eta\nu}|_{L^\infty(0,T',L^2)} \leqq M\beta,$$

where M is a constant that depends only on h, and $|w|_{L^\infty(0,T',H^s)}$, while the estimates on $\mathscr{B}$ yield

$$|\mathscr{B}|_{L^\infty(0,T',L^2)} \leqq C\sqrt{\sqrt{\theta\nu}} \leqq C\beta,$$

whence

$$|u^{\eta,\nu} - w|_{L^\infty(0,T',L^2)} \leqq C(T', w)\beta.$$

We also have

$$\eta|\nabla_{x,y}v|^2_{L^2} + \nu|\partial_z v|^2_{L^2} \leqq M'\beta^2,$$

and since

$$|\nabla_{x,y}\mathscr{B}|^2_{L^2}, |\nabla w|^2_{L^2} \leqq K,$$

$$\nu|\partial_z\mathscr{B}|^2_{L^2} \leqq K\nu \times \frac{1}{\sqrt{\theta\nu}} = K'\sqrt{\frac{\nu}{\eta}},$$

we find that

$$\eta|\nabla_{x,y}u^{\eta,\nu}|^2_{L^2(0,T',L^2(\Omega))} + \nu|\partial_z u|^2_{L^2(0,T',L^2(\Omega))} \leqq C(T', w)\beta^2.$$

Then, if we assume that $\eta \to 0$ and that $\nu/\eta \to 0$, we easily conclude the proof of the theorem.

Remarks. 1) There is no need to take subsequences because we do not use compactness results here: We start with a solution of the limit system and construct an approximate solution, and then prove that there is a solution near this approximate solution.

2) Notice that the theorem does not give any information about the "real" boundary layer. Therefore it gives us more information about $u^{\eta,\nu}$ than the simple convergence to w in the L^2 norm, namely,

$$\eta|\nabla_{x,y}u^{\eta,\nu}|^2_{L^2} + \nu|\partial_z u^{\eta,\nu}|^2_{L^2} \leqq M'\beta^2. \tag{20}$$

This shows that there is no energy dissipation in Ω. In [6], when $\eta = \nu$, and for general bounded domains, KATO shows that the absence of dissipation in Ω is equivalent to the absence of dissipation in the boundary strip of width $c\nu$, and also equivalent to the convergence of $u^{\eta,\nu}$ to the solution of the Euler equation. The proof of KATO can be extended to the case where η and ν are different.

3) If $\beta = \sqrt{\eta + \sqrt{\nu/\eta}}$, and ν/η^3 is bounded, then (20) yields that $\nabla_{x,y}u^{\eta,\nu}$ is bounded in $L^\infty(0, T', L^2)$.

4) The theorem can be extended to the case of dimension d, where $d \geqq 2$, by taking $s > \frac{1}{2}(d+2)$, which allows us to have the existence for the Euler system and to bound $|A|_{L^\infty}$ by the H^s norm of w. Moreover, when $d = 2$, the Euler system has a global solution and the convergence is then global and uniform on any compact time interval.

3. Rotating Fluids

As in the second section, we now prove a theorem more precise than Theorem 1.2. We recall that $u^{\varepsilon,\nu,\eta}(0)$ converges in $L^2(\Omega)$ to w^0, and $F^{\varepsilon,\nu,\eta}$ converges in $L^1(0,T,L^2(\Omega))$ to F^0 , where $w^0 \in H^s$, $s > 2$, $w_3^0 = 0$, $\int w^0 = 0$, and w^0 does not depend on z. We also suppose that $F^0 \in L^2(0,T,H^s)$, $F_3^0 = 0$ and F^0 does not depend on z.

Theorem 3.1. *Let*

$$|u^{\varepsilon,\nu,\eta}(0) - w^0|_{L^2} \leqq C\beta,$$

$$|F^{\varepsilon\nu} - F^0|_{L^1(0,T,L^2(\Omega))} \leqq C\beta$$

where

$$\beta \geqq C\sqrt{\eta + \sqrt{\varepsilon\nu}}.$$

Then there exist global weak solutions $u^{\varepsilon\nu\eta}$ of (6)–(9) *and a constant $C(h)$ such that if*

$$\frac{\eta}{\varepsilon} > C(h)|w|^2_{L^\infty},$$

then

$$|u^{\varepsilon\nu\eta} - w^{\varepsilon\nu}|_{L^2} \leqq C(t,w)\beta,$$

$$\eta|\nabla_{xy}u^{\varepsilon\nu\eta}|^2_{L^2} + \nu|\partial_z u^{\varepsilon\nu\eta}|^2_{L^2} \leqq C(t,w)\beta^2.$$

Remarks. 1) The case where $\eta = \nu$ is valid if $\varepsilon < C(h)|w|^2_{L^\infty}\eta$. In fact this is the case of a rotating fluid in a container.

2) Here, we have a global convergence in time since the two-dimensional Euler system has global existence in $C(0,T,H^s)$, $s > 2$. Therefore we do not have uniform convergence on $[0,\infty[$, since the H^s norm of w can grow very fast with t.

3) Notice that the condition

$$\frac{\eta}{\varepsilon} > C(h)|w|^2_{L^\infty}$$

says that the product of the regularizing terms, namely, the horizontal viscosity and the Coriolis term has to be great enough.

3.1. *Construction of the Boundary Layer*

We can derive the limit equation and the form of the boundary layer by using a formal asymptotic expansion. We refer to [5], and to the book of PEDLOVSKY [12] for this formal expansion. In this section we only construct a boundary layer which allows us to recover the Dirichlet boundary conditions and to balance the damping term in the limit equation.

The boundary layer $\mathscr{B}$ is the sum of four terms $\mathscr{B}^1, \ldots, \mathscr{B}^4$. First of all

$$\mathscr{B}^1 = \begin{pmatrix} \tilde{w}_1 + \breve{w}_1 \\ \tilde{w}_2 + \breve{w}_2 \\ \sqrt{\varepsilon\nu}\,\mathrm{curl}(w)G(z) \end{pmatrix}$$

where

$$\tilde{w}_1 = -e^{-\theta/\sqrt{2}}\left(w_1 \cos\left(\frac{\theta}{\sqrt{2}}\right) + w_2 \sin\left(\frac{\theta}{\sqrt{2}}\right)\right),$$

$$\tilde{w}_2 = -e^{-\theta/\sqrt{2}}\left(w_2 \cos\left(\frac{\theta}{\sqrt{2}}\right) - w_1 \sin\left(\frac{\theta}{\sqrt{2}}\right)\right),$$

$$\tilde{w}_3 = -\frac{e^{-\theta/\sqrt{2}}}{\sqrt{2}}\left(\partial_x w_2 - \partial_y w_1\right)\left(\sin\left(\frac{\theta}{\sqrt{2}}\right) + \cos\left(\frac{\theta}{\sqrt{2}}\right)\right)$$

with $\theta = z/\sqrt{\varepsilon\nu}$ and

$$\breve{w}_1 = -e^{-\lambda/\sqrt{2}}\left(w_1 \cos\left(\frac{\lambda}{\sqrt{2}}\right) + w_2 \sin\left(\frac{\lambda}{\sqrt{2}}\right)\right),$$

$$\breve{w}_2 = -e^{-\lambda/\sqrt{2}}\left(w_2 \cos\left(\frac{\lambda}{\sqrt{2}}\right) - w_1 \sin\left(\frac{\lambda}{\sqrt{2}}\right)\right),$$

$$\breve{w}_3 = \frac{e^{-\lambda/\sqrt{2}}}{\sqrt{2}}(\partial_x w_2 - \partial_y w_1)\left(\sin\left(\frac{\lambda}{\sqrt{2}}\right) + \cos\left(\frac{\lambda}{\sqrt{2}}\right)\right)$$

with $\lambda = (h - z)/\sqrt{\varepsilon\nu}$, and where

$$G(z) = \left[-e^{-z/\sqrt{2\varepsilon\nu}} \sin\left(\frac{z}{\sqrt{2\varepsilon\nu}} + \frac{\pi}{4}\right) + e^{-h-z/\sqrt{2\varepsilon\nu}} \sin\left(\frac{h-z}{\sqrt{2\varepsilon\nu}} + \frac{\pi}{4}\right)\right].$$

In fact, $\tilde{w}$ satisfies the equations of the boundary layer near $z = 0$ at the leading order, namely,

$$\begin{aligned} \nu\partial_z^2 \tilde{w}_1 &= -1/\varepsilon \tilde{w}_2, \\ \nu\partial_z^2 \tilde{w}_2 &= +1/\varepsilon \tilde{w}_1, \\ \tilde{w}_1(z=0) &= -w_1, \quad \lim_{z\to\infty} \tilde{w}_1 = 0, \\ \tilde{w}_2(z=0) &= -w_2, \quad \lim_{z\to\infty} \tilde{w}_2 = 0 \end{aligned}$$

and $\breve{w}$ satisfies the same equation near $z = h$. The third component of $\mathscr{B}^1$ allows us to satisfy the divergence-free condition.

Next, we add $\mathscr{B}^2$ to recover at the order $\sqrt{\varepsilon\nu}$ the appropriate boundary condition for the third component, since $G(0) = -G(h) \simeq -\sqrt{2}/2$,

$$\mathscr{B}^2 = \begin{pmatrix} \sqrt{\varepsilon\nu}\ \frac{\sqrt{2}}{h} w_2 \\ -\sqrt{\varepsilon\nu}\ \frac{\sqrt{2}}{h} w_1 \\ \sqrt{\varepsilon\nu}\ \ \mathrm{curl}(w)\frac{\sqrt{2}}{2}(1-\frac{2z}{h}) \end{pmatrix}.$$

Notice here that we cannot use the same trick as in the construction of the last section, since the boundary layer has to satisfy a precise equation.

Now, the boundary values of $\mathscr{B}^1 + \mathscr{B}^2 + w^{\varepsilon\nu}$ at $z = 0$ and $z = h$ are

$$\begin{pmatrix} w_1 e^{-\frac{h}{\sqrt{2\varepsilon\nu}}} \cos(\frac{h}{\sqrt{2\varepsilon\nu}}) - w_2 e^{-\frac{h}{\sqrt{2\varepsilon\nu}}} \sin(\frac{h}{\sqrt{2\varepsilon\nu}}) + \sqrt{\varepsilon\nu}\ \frac{\sqrt{2}}{h} w_2 \\ w_2 e^{-\frac{h}{\sqrt{2\varepsilon\nu}}} \cos(\frac{h}{\sqrt{2\varepsilon\nu}}) + w_1 e^{-\frac{h}{\sqrt{2\varepsilon\nu}}} \sin(\frac{h}{\sqrt{2\varepsilon\nu}}) - \sqrt{\varepsilon\nu}\ \frac{\sqrt{2}}{h} w_1 \\ \pm\sqrt{\varepsilon\nu}\ \ \mathrm{curl}(w) e^{-\frac{h}{\sqrt{2\varepsilon\nu}}} \sin(\frac{h}{\sqrt{2\varepsilon\nu}} + \frac{\pi}{4}) \end{pmatrix}$$

where the $+$ is taken at $z = 0$ and the $-$ at $z = h$.

To get the appropriate boundary condition $\mathscr{B} + w^{\eta,\nu} = 0$, we add

$$\mathscr{B}^3 = e^{-h/\sqrt{2\varepsilon\nu}} \cos(\frac{h}{\sqrt{2\varepsilon\nu}}) \begin{pmatrix} -w_1 \\ -w_2 \\ 0 \end{pmatrix}$$

which is exponentially small, and

$$\mathscr{B}^4 = f(z) \begin{pmatrix} w_2 \\ -w_1 \\ 0 \end{pmatrix} + g(z) \begin{pmatrix} 0 \\ 0 \\ \mathrm{curl}(w) \end{pmatrix}$$

with $f(z) + g'(z) = 0$ and the boundary conditions

$$\begin{aligned} f(0) &= f(h) = \sqrt{\varepsilon\nu}\frac{\sqrt{2}}{h} + e^{-h/\sqrt{2\varepsilon\nu}} \sin(\frac{h}{\sqrt{2\varepsilon\nu}}), \\ g(0) &= -g(h) = -\sqrt{\varepsilon\nu}\ e^{-h/\sqrt{2\varepsilon\nu}} \sin(\frac{h}{\sqrt{2\varepsilon\nu}} + \frac{\pi}{4}). \end{aligned}$$

We can take, for instance,

$$\begin{aligned} f(z) &= a(e^{-z/\sqrt{2\varepsilon\nu}} + e^{-h-z/\sqrt{2\varepsilon\nu}}) + b, \\ g(z) &= -\sqrt{\theta\nu}\ e^{-h/\sqrt{2\varepsilon\nu}} \sin\left(\frac{h}{\sqrt{2\varepsilon\nu}} + \frac{\pi}{4}\right) - \textstyle\int_0^z f(s)ds \end{aligned}$$

where a and b satisfy $a < C(h)\sqrt{\varepsilon\nu}$, and $b < C(h)\varepsilon\nu$. We refer to [5] for a proof of this choice.

In the sequel, we denote the first and second components of $\mathscr{B}$ by B_1, B_2 and the third one by b, and we denote the first and second components of $\mathscr{B}^i$ by $\mathscr{B}^i_1$, $\mathscr{B}^i_2$. Let

$$\mathscr{B} = \mathscr{B}^1 + \mathscr{B}^2 + \mathscr{B}^3 + \mathscr{B}^4.$$

We have

$$\mathrm{div}\,\mathscr{B} = 0, \quad \text{and} \quad \mathscr{B} + w^{\eta,\nu} = 0 \quad \text{on } \partial\Omega.$$

We notice here that the boundary layer $\mathscr{B}$ satisfies an equation of the form

$$\mathscr{B} = \mathscr{M}(z)A(t,x,y),$$

where $\mathscr{M}$ is a (3×3)-matrix and $A = (w_1, w_2, \text{curl}(w))$. We notice that $\mathscr{M}$ is formed by two blocks: M a (2×2)-matrix and m a (1×1)-matrix and that

$$|\mathscr{M}(z)|_{L^2_z} \leqq C\sqrt{\sqrt{\varepsilon\nu}}, \quad |m(z)|_{L^\infty_z} \leqq C\sqrt{\varepsilon\nu},$$

$$|z\partial_z M(z)|_{L^2_z} \leqq C\sqrt{\sqrt{\varepsilon\nu}}, \quad |z\partial_z M(z)|_{L^2_z} \leqq C/\sqrt{\sqrt{\varepsilon\nu}},$$

$$|z^2\partial_z M(z)|_{L^\infty_z} \leqq C\sqrt{\varepsilon\nu}.$$

These are the only estimates we used in our computations of the first part.

3.2. *The Energy Estimates*

Proof of Theorem 3.1. The existence of the weak solutions can be proved by using the Galerkin method as in the last part. Let

$$v^{\varepsilon\nu\eta} = u^{\varepsilon,\nu,\eta} - w^{\varepsilon\nu} - \mathscr{B}.$$

Replacing $u^{\varepsilon\nu\eta}$ by $w + v^{\varepsilon\nu\eta} + \mathscr{B}$ in (6) and subtracting (10), we get

$$\begin{aligned}
&\partial_t\mathscr{B} + \partial_t v + u\cdot\nabla v + w\cdot\nabla\mathscr{B} + \mathscr{B}\cdot\nabla\mathscr{B} + v\cdot\nabla\mathscr{B} + \mathscr{B}\cdot\nabla w + v\cdot\nabla w\\
&\quad - \eta\Delta_{x,y}\mathscr{B} - \eta\Delta_{x,y}v - \eta\Delta_{x,y}w - \nu\partial_z^2\mathscr{B} - \nu\partial_z^2 v \qquad (21)\\
&\quad + \frac{e_3\times\mathscr{B}}{\varepsilon} + \frac{e_3\times v}{\varepsilon} - \sqrt{\frac{\nu}{\varepsilon}}\frac{\sqrt{2}}{h}w = -\nabla q' + F - F^0
\end{aligned}$$

where we used the fact that w does not depend on z.

Taking the scalar product of (21) with v, and integrating by parts we obtain

$$\begin{aligned}
&\tfrac{1}{2}\partial_t\int|v|^2 + \eta\int|\nabla_{x,y}v|^2 + \nu\int|\partial_z v|^2\\
&\quad = -\int\partial_t\mathscr{B}v - \int u\cdot\nabla vv - \int w\cdot\nabla\mathscr{B}v - \int\mathscr{B}\cdot\nabla\mathscr{B}v\\
&\quad\quad - \textstyle\int v\cdot\nabla\mathscr{B}v - \displaystyle\int\mathscr{B}\cdot\nabla wv - \int v\cdot\nabla wv + \eta\int\Delta_{x,y}\mathscr{B}v \qquad (22)\\
&\quad\quad + \eta\textstyle\int\Delta_{x,y}wv + \displaystyle\int Lv + \int(F - F^0)v
\end{aligned}$$

where

$$L = \nu\partial_z^2\mathscr{B} - \frac{e_3\times\mathscr{B}}{\varepsilon} + \sqrt{\frac{\nu}{\varepsilon}}\frac{\sqrt{2}}{h}w$$

and where we use the equalities $(e_3\times v)v = 0$ and $\int\nabla q' v = -\int q'\nabla\cdot v = 0$. (All these computations should be done for the approximate sequence $u_n^{\varepsilon\nu\eta}$, and then the limit should be taken.)

We must now bound the eleven terms of the right-hand side of (22). The nine first terms and the eleventh are treated exactly as in the first part, by replacing θ by ε. We no longer assume that $\mathscr{B}(z) = 0$ for $z > 1$, but this is not a problem and we replace the $\int_0^1$ by $\int_0^{h/2}$ for the boundary layer near $z = 0$ and by $\int_{h/2}^h$ for the boundary layer near $z = h$. We also notice that θ is no longer a free parameter since $\theta = \varepsilon$, and that since w does not depend on z, we need not use trace theorems to control the boundary layer by w. Hence we have a $\frac{1}{2}$-dimensional gain, and only need that $s > 2$ for the energy estimates, which is the condition imposed by the existence theorem of the Euler equation in the two-dimensional case. The only term that needs a specific treatment is the tenth. In fact, here we need the exact value of the boundary layer.

10) The expression for the boundary layer yields

$$\nu \partial_z^2 B^1 - \frac{e_3 \times B^1}{\varepsilon} = 0,$$

$$\nu \partial_z^2 B^2 - \frac{e_3 \times B^2}{\varepsilon} + \sqrt{\frac{\nu}{\varepsilon}} \frac{\sqrt{2}}{h} w = 0,$$

$$\partial_z^2 B^3 = 0.$$

We recall here that $\mathscr{B} = \mathscr{B}^1 + \mathscr{B}^2 + \mathscr{B}^3 + \mathscr{B}^4$ and that

$$\mathscr{B} = \begin{pmatrix} B_1 \\ B_2 \\ b \end{pmatrix}, \qquad \mathscr{B}^i = \begin{pmatrix} B_1^i \\ B_2^i \\ b^i \end{pmatrix}.$$

Now, we must bound the remaining terms:

$$\begin{aligned} \left| \int \nu \partial_z^2 b v_3 \right| &\leqq \nu |\partial_z b|_{L^2} |\partial_z v_3|_{L^2} \\ &\leqq \nu |\nabla_{x,y} B|_{L^2}^2 + \nu |\nabla_{x,y} v|_{L^2}^2 \\ &\leqq \frac{\nu}{c} \sqrt{\varepsilon \nu} |w|_{H^1}^2 + c\nu |\nabla_{x,y} v|_{L^2}^2, \end{aligned}$$

$$\int \frac{e_3 \times B^3}{\varepsilon} V \leqq \sqrt{\sqrt{\varepsilon \nu}} \frac{C(h)}{\varepsilon \sqrt{\varepsilon \nu}} e^{-h/\sqrt{\varepsilon \nu}} |w|_{L^2}, |v|_{L^2},$$

$$\begin{aligned} \left| \int \nu \partial_z^2 B^4 V \right| &\leqq \int \nu |f''(z)| \, |w| \, |V| \\ &\leqq \frac{\nu}{\sqrt{\varepsilon \nu}} \int_{xy} \left(\int_z |e^{-z/\sqrt{2\varepsilon \nu}} + e^{-h-z/\sqrt{2\varepsilon \nu}}| \, |V| dz \right) |w| \, dx \, dy \\ &\leqq \frac{\nu}{\sqrt{\varepsilon \nu}} \left(\int (|e^{-z/\sqrt{2\varepsilon \nu}} + e^{-h-z/\sqrt{2\varepsilon \nu}}|)^2 \right)^{1/2} \int_{xy} |V|_{L_z^2} |w| \\ &\leqq \sqrt{\sqrt{\varepsilon \nu}} \sqrt{\frac{\nu}{\varepsilon}} |w|_{L^2} |v|_{L^2}. \end{aligned}$$

We recall here that $f = a(e^{-z/\sqrt{2\varepsilon \nu}} + e^{-h-z/\sqrt{2\varepsilon \nu}}) + b$, where a and b satisfy $a < C(h)\sqrt{\varepsilon \nu}$, and $b < C(h)\varepsilon \nu$. Finally, we have

$$\int \frac{e_3 \times B^4}{\varepsilon} V \leqq \frac{\sqrt{\varepsilon\nu}}{\varepsilon} \left(\int (|e^{-z/\sqrt{2\varepsilon\nu}} + e^{-h-z/\sqrt{2\varepsilon\nu}}|)^2 \right)^{1/2} |w|_{L^2} |V|_{L^2}$$
$$+ \nu |w|_{L^2} |V|_{L^2}$$
$$\leqq \sqrt{\sqrt{\varepsilon\nu}} \sqrt{\frac{\nu}{\varepsilon}} |w|_{L^2} |v|_{L^2}.$$

Notice that $\sqrt{\frac{\nu}{\varepsilon}} w$ is bounded in $L^1(0, T, L^2)$, since

$$\partial_t |w|_{L^2}^2 = \frac{\nu}{\varepsilon} |w|^2.$$

Combining the twelve terms yields

$$\begin{aligned} &\tfrac{1}{2} \partial_t |v|_{L^2}^2 + \eta |\nabla_{x,y} v|_{L^2}^2 + \nu |\partial_z v|_{L^2}^2 \\ &\quad \leqq c(\eta + \nu\varepsilon) |\nabla_{x,y} v|_{L^2}^2 + \nu \left(c + \frac{\varepsilon |w|_{H^s}^2}{c\eta} \right) |\partial_z v|_{L^2}^2 \\ &\qquad + C\eta |w|_{H^s}^2 + C |w|_{H^s} |v|_{L^2}^2 \\ &\qquad + \left[C\sqrt{\sqrt{\varepsilon\nu}} \left(|w|_{H^s} + |w|_{H^s}^2 + \sqrt{\frac{\nu}{\varepsilon}} |w| \right) + |F^{\varepsilon,\nu,\eta} - F^0|_{L^2} \right] |v|_{L^2}. \end{aligned} \tag{23}$$

Until now, we did not make any assumption on η, ν, ε. We assume now that $\eta, \nu, \varepsilon < 1$, $\nu < \eta$, and $\varepsilon \leqq C(h)\eta/|w|_{L^\infty}^2$. Taking c small enough we can absorb the $|\nabla_{x,y} v|_{L^2}^2$ and $|\partial_z v|_{L^2}^2$ of the right-hand side into the left-hand side.

We notice then that (23) leads to

$$\tfrac{1}{2} \partial_t |v|_{L^2}^2 \leqq a(t) |v|_{L^2}^2 + b(t) |v|_{L^2} + c(t)$$

with

$$a(t) = C |w|_{H^s},$$
$$b(t) = C\sqrt{\sqrt{\varepsilon\nu}} \left(|w|_{H^s} + |w|_{H^s}^2 + \sqrt{\frac{\nu}{\varepsilon}} |w|_{L^2} \right) + |F^{\eta,\nu} - F^0|_{L^2},$$
$$c(t) = C\eta |w|_{H^s}^2.$$

The conditions of Lemma (1.1) are then satisfied, with

$$\alpha = \beta \geqq C\sqrt{\eta + \sqrt{\varepsilon\nu}}.$$

This yields

$$|v^{\varepsilon,\nu,\eta}|_{L^\infty(0,T,L^2)} \leqq M\beta.$$

We also have

$$\eta |\nabla_{x,y} v|_{L^2}^2 + \nu |\partial_z v|_{L^2}^2 \leqq M' \beta^2.$$

Since these two estimates also hold for $\mathscr{B}$, we get

$$|u^{\varepsilon\nu\eta} - w^{\varepsilon\nu}|_{L^2} \leqq C(t, w)\beta,$$
$$\eta|\nabla_{xy} u^{\varepsilon\nu\eta}|^2_{L^2} + \nu|\partial_z u^{\varepsilon\nu\eta}|^2_{L^2} \leqq C(t, w)\beta^2.$$

Note added in proof. We can see easily that both convergence theorems given in this paper apply for any sequence of solutions satisfying the energy inequality.

References

1. J. P. BOURGUIGNON & H. BREZIS: Remarks on the Euler equation, *J. Func. Anal.* **15** (1974), 341–363.
2. T. COLIN & P. FABRIE: Rotating fluid at high Rossby number driven by a surface stress: existence and convergence, preprint, 1996.
3. R. E. CAFLISCH & M. SAMMARTINO: Zero viscosity limit for analytic solutions of the Navier-Stokes equations on a half-space, preprint, 1996.
4. V. W. EKMAN: On the influence of the earth's rotation on ocean currents, *Arkiv Matem., Astr. Fysik* **2** (1905).
5. E. GRENIER & N. MASMOUDI: Ekman layers of rotating fluids, the case of well prepared initial data, to appear in *Comm. Partial Diff. Eqs.*
6. T. KATO: Remarks on zero viscosity limit for non-stationary Navier-Stokes flows with boundary, in Seminar on Nonlinear PDE, ed. S. S. CHERN, MSRI, 1984.
7. T. KATO: Non-stationary flows of viscous and ideal fluids in R^3, *J. Func. Anal.* **9** (1972), 296–305.
8. J.-L. LIONS: Quelques méthodes de résolution des problèmes aux limites non linéaires, *Dunod*, Gauthier-Villars, 1969.
9. J.-L. LIONS & E. MAGENES: Problèmes aux limites non homogènes, Vol. 1, Dunod, 1968.
10. J.-L. LIONS, R. TEMAM & S. WANG: Modèles et analyse mathématiques du système Océan/Atmosphère, *C.R.Acad. Sci.* **316** (1993), 113–119, 211–215; *C.R. Acad. Sci.* **318** (1994), 1165–1171.
11. P.-L. LIONS: *Mathematical topics in fluid mechanics*, Vol. 1 Incompressible Models, *Oxford University Press* (1996).
12. J. PEDLOVSKY: *Geophysical fluid dynamics*, Springer, 1979.
13. H. SWANN, The convergence with vanishing viscosity of non-stationary Navier-Stokes flow to ideal flow in R^3, *Trans. Amer. Math. Soc.* **157** (1971), 373–397.
14. G. I. TAYLOR: Experiments on the motion of solid bodies in rotating fluids, *Proc. Roy. Soc. A* **104** (1923), 213–218.
15. R. TEMAM: On the Euler equations of incompressible perfect fluids, *J. Func. Anal.* **20** (1975), 3243.
16. R. TEMAM: *Navier-Stokes Equations*, North-Holland, 1983.

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