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Axisymmetric Solutions of the Euler Equations for Polytropic Gases

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Abstract

We construct rigorously a three-parameter family of self-similar, globally bounded, and continuous weak solutions in two space dimensions for all positive time to the Euler equations with axisymmetry for polytropic gases with a quadratic pressure-density law. We use the axisymmetry and self-similarity assumptions to reduce the equations to a system of three ordinary differential equations, from which we obtain detailed structures of solutions besides their existence. These solutions exhibit familiar structures seen in hurricanes and tornadoes. They all have finite local energy and vorticity with well-defined initial and boundary values. These solutions include the one-parameter family of explicit solutions reported in a recent article of ours.

1. Introduction

We are interested in finding some solutions to the initial-value problem for the two-dimensional Euler equations for compressible fluids. Our approach is to generalize to two dimensions some of the results on Riemann problems for the one-dimensional Euler equations for compressible fluids. One natural generalization is to consider initial data which consist of four constant states, or any finite number of constant states [10–12]. The well-known configurations of regular and Mach reflections are special cases of such a generalization. However, no rigorous proofs of existence of any nontrivial solutions to these problems have been established.

A more complete generalization is to consider initial data which depend only on the polar angle in the two-dimensional space of positions. This generalization certainly looks more formidable, if not impossible. However, it now contains a special three-parameter family of data, namely, the axisymmetric initial data, which allows us to reduce the initial-value problem of the partial differential equations to a boundary-value problem for a system of ordinary differential equations. With this simplification we were able to construct in [13] a two-parameter family of solutions corresponding to pure rotational initial data.

Our main task here is to study rigorously the resulting boundary-value problem for the non-autonomous system of three ordinary differential equations, in the hope of *rigorously* constructing all solutions for the three-parameter data. Singularity points of the system consist of two-dimensional manifolds in the four-dimensional phase space. These singularity points correspond to surfaces of characteristics in physical space-time, where solutions may not be differentiable. A typical global solution may consist of as many as three nontrivial connecting orbits chained together continuously. The complete construction of the three-parameter family of solutions is described in the Conclusions at the end of the paper.

We find that our solutions capture some typical properties of swirling flows such as hurricanes and tornadoes. For example, there are the eye structure near the center with low rotational speeds and the wall region with high speeds (see Figure 8.1). We refer the reader to our paper [13] for calculations of explicit particle trajectories which clearly exhibit spiral structures.

We mention that our initial data in this paper will be restricted to the set of data with nonnegative radial velocities (swirling outward), in addition to axisymmetry and radial symmetry. Shock waves may arise if the initial radial velocities are allowed to be negative (swirling inward). Also, in this paper we study only the special case where the pressure is a quadratic function of density. For partial results on non-quadratic pressure-density cases, we refer the reader to our paper [13] or article [14]. Rigorous proofs for solutions constructed in [13] and more complete constructions of the three-parameter family of solutions for non-square pressure-density cases are forthcoming.

There is a great deal of related work, from which we mention only the most pertinent. For general existence of weak solutions with axisymmetry for the two-dimensional Euler equations for compressible fluids outside a core region, we refer the reader to the recent work of CHEN & GLIMM [4]. For some explicit solutions of such Euler equations with spherical symmetry but without swirls, see COURANT & FRIEDRICHS [3]. For swirling motions of viscous fluids, we refer the reader to BELLAMY-KNIGHTS [1]; COLONIUS, LELE & MOIN [2]; MACK [6]; POWELL [8] and SERRIN [9].

2. The Problem

We consider the two-dimensional Euler equations for a compressible and polytropic gas:

$$\begin{aligned} \rho_t + (\rho u)_x + (\rho v)_y &= 0, \\ (\rho u)_t + (\rho u^2 + p)_x + (\rho u v)_y &= 0, \\ (\rho v)_t + (\rho u v)_x + (\rho v^2 + p)_y &= 0, \end{aligned} \tag{2.1}$$

where $p = p(\rho)$ is a given increasing function of ρ. Global existence of weak solutions to its initial-value problem is open. Attempts have been made through considering special situations such as the diffraction of a planar shock at a wedge or generalized Riemann problems with four different initial constant states. Here we consider a situation which involves swirling motions.

We impose axisymmetry to the system. That is, we assume that our solutions (u, v, ρ) have the property

$$\begin{aligned} \rho(t,r,\theta) &= \rho(t,r,0), \\ \begin{pmatrix} u(t,r,\theta) \\ v(t,r,\theta) \end{pmatrix} &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} u(t,r,0) \\ v(t,r,0) \end{pmatrix} \end{aligned} \tag{2.2}$$

for all $t \geqq 0$, $\theta \in \mathbb{R}$ and $r > 0$, where (r, θ) are the polar coordinates of the (x, y)-plane. With this symmetry, system (2.1) can be reduced for continuous solutions to

$$\begin{aligned} \rho_t + (\rho u)_r + \frac{\rho u}{r} &= 0, \\ u_t + u u_r + \frac{p_r}{\rho} - \frac{v^2}{r} &= 0, \\ v_t + u v_r + \frac{u v}{r} &= 0, \end{aligned} \tag{2.3}$$

where $\rho = \rho(t, r, 0)$, etc. Notice now that u and v in (2.3) represent the radial and pure rotational velocities in the flow, respectively.

We limit ourselves to Riemann-type initial data; that is, we require the initial data to be independent of the radial variable $r > 0$:

$$(\rho(0, r, \theta), u(0, r, \theta), v(0, r, \theta)) = (\rho_0(\theta), u_0(\theta), v_0(\theta)). \tag{2.4}$$

We remark in passing that the initial-value problem of system (2.1) with the type of data in (2.4) may justly be called the two-dimensional Riemann problem for (2.1). It degenerates to the classical Riemann problem for the one-dimensional case; it is simple and yet general enough to contain important waves such as swirling motions as well as shock and rarefaction waves and slip lines (surfaces).

When the axisymmetry condition (2.2) is imposed onto (2.4), we find that our data are limited to

$$\begin{aligned} u(0,r,\theta) &= u_0 \cos\theta - v_0 \sin\theta, \\ v(0,r,\theta) &= u_0 \sin\theta + v_0 \cos\theta, \\ \rho(0,r,\theta) &= \rho_0, \end{aligned} \tag{2.5}$$

where $\rho_0 > 0$, $(u_0, v_0) \in \mathbb{R}^2$ are arbitrary constants. Hence our data for system (2.3) are

$$\rho(0, r, 0) = \rho_0, \quad u(0, r, 0) = u_0, \quad v(0, r, 0) = v_0. \tag{2.6}$$

Since the problem (2.3), (2.6) is invariant under self-similar transformations, we look for self-similar solutions (ρ, u, v) which depend only on $\xi = r/t$. We thus have the following boundary-value problem for a system of ordinary differential equations:

$$\rho_r = \frac{\rho}{r}\frac{\Theta}{\Delta}, \quad u_r = \frac{1}{r}\frac{\Sigma}{\Delta}, \quad v_r = \frac{uv}{r(r-u)}, \tag{2.7}$$

$$\lim_{r\to+\infty}(\rho, u, v) = (\rho_0, u_0, v_0) \tag{2.8}$$

where

$$\begin{aligned} \Delta &\equiv p'(\rho) - (u-r)^2, \\ \Theta &\equiv v^2 - u(r-u), \\ \Sigma &\equiv (r-u)\Theta - u\Delta = v^2(r-u) - up'(\rho), \end{aligned}$$

and r is used in place of ξ for notational convenience.

We construct global continuous solutions or establish their existence for problem (2.7), (2.8) for any $\rho_0 > 0$, $u_0 \geqq 0$, $v_0 \in \mathbb{R}$ and $p(\rho) = A_2\rho^2$ where $A_2 > 0$ is a constant. We can assume that $v_0 \geqq 0$ because the case $v_0 < 0$ can be transformed by $v \to -v$ to the case $v_0 > 0$. It can be verified that these solutions are also global continuous solutions to the original Euler equations.

3. Far-Field Solutions

We show that problem (2.7), (2.8) has a local solution near $r = +\infty$ for any datum (ρ_0, u_0, v_0) with $\rho_0 > 0$. Let $s = 1/r$. Then (2.7), (2.8) can be written as

$$\begin{aligned} \frac{d\rho}{ds} &= \frac{\rho[u(1-us) - sv^2]}{s^2p'(\rho) - (1-us)^2}, \\ \frac{du}{ds} &= \frac{sup'(\rho) - v^2(1-us)}{s^2p'(\rho) - (1-us)^2}, \\ \frac{dv}{ds} &= -\frac{uv}{1-us}, \end{aligned} \tag{3.1}$$

$$(\rho, u, v)|_{s=0} = (\rho_0, u_0, v_0). \tag{3.2}$$

Problem (3.1), (3.2) is a classically well-posed problem which has a unique local solution for any initial datum with $\rho_0 > 0$.

We find that $v = 0$ and

$$sv^2 - u(1-us) = 0 \tag{3.3}$$

are invariant surfaces in the four-dimensional (ρ, u, v, s) space. It can be verified that

$$\frac{d}{ds}[sv^2 - u(1-us)] = \frac{(1-3su)sp' + u(1-su)^2}{(1-su)[s^2p' - (1-su)^2]}[sv^2 - u(1-us)].$$

Explicit Solutions. In the special case $u_0 = 0$, we find from the invariant surface (3.3) a set of explicit solutions near $r = +\infty$:

$$\rho = \rho_0, \quad u = \frac{v_0^2}{r}, \quad v = \frac{v_0}{r}\sqrt{r^2 - v_0^2}, \qquad r \geqq r^* \tag{3.4}$$

where

$$r^* \equiv \frac{1}{2}\left(\sqrt{p'(\rho_0) + 4v_0^2} + \sqrt{p'(\rho_0)}\right). \tag{3.5}$$

The functions in (3.4) are defined for $r \geqq v_0$, but we cannot use them up to v_0 with absolute certainty since (3.1) has a singularity at the point $r = r^*$ on the curve (3.4). In fact, we find that the position r, the radial velocity u, and the sound speed $\sqrt{p'(\rho)}$ at r^* along (3.4) have the relation

$$r^* = u(r^*) + \sqrt{p'(\rho_0)},$$

which is to say that r^* is the radial characteristic speed; that is, the distance that a small disturbance generated from the origin at time zero can travel radially in time $t = 1$.

4. Intermediate Field Equations

We can simplify system (3.1) by assuming the relation

$$p(\rho) = A_2\rho^\gamma \tag{4.1}$$

for some $A_2 > 0$ and $\gamma > 1$, and introducing the variables

$$I = su, \quad J = sv, \quad K = s\sqrt{p'(\rho)}. \tag{4.2}$$

Then system (3.1) can be put into the form

$$\begin{aligned} s\frac{dI}{ds} &= \frac{2IK^2 - (1-I)[J^2 + I(1-I)]}{K^2 - (1-I)^2}, \\ s\frac{dJ}{ds} &= J\frac{1-2I}{1-I}, \\ s\frac{dK}{ds} &= \frac{K}{2}\frac{2K^2 - 2(1-I)^2 - (\gamma-1)[J^2 - I(1-I)]}{K^2 - (1-I)^2}. \end{aligned} \tag{4.3}$$

Corresponding to the initial data (3.2), we look for solutions of (4.3) with the initial condition

$$(I, J, K) \sim s(u_0, v_0, \sqrt{p'(\rho_0)}) \tag{4.4}$$

as $s \to 0+$. We note that (4.3) is now autonomous for I, J, K with respect to the new variable $s' = \ln s$.

The invariant surfaces of (4.3) are the surface $J = 0$, the surface $K = 0$, and the surface

$$H \equiv J^2 - I(1-I) = 0 \tag{4.5}$$

since

$$s\frac{d}{ds}H = \frac{(1-2I)[2K^2 - 3(1-I)^2]}{(1-I)[K^2 - (1-I)^2]}H.$$

Scaling Symmetry. System (4.3) is invariant under the coordinate transformation $s \to \alpha s$ for any constant $\alpha > 0$. In particular, we can take $\alpha = 1/\sqrt{p'(\rho_0)}$. Thus we may assume that $\rho_0 > 0$ is such that $\sqrt{p'(\rho_0)} = 1$. Hence the structure of any solution of problem (4.3), (4.4) depends only on the ratios $u_0/\sqrt{p'(\rho_0)}$ and $v_0/\sqrt{p'(\rho_0)}$.

5. Solutions Without Swirls

Let us first determine the distribution of integral curves on the invariant surface $J = 0$.

Assume that $v_0 = 0$. We look for solutions to problem (4.3), (4.4) with $J = 0$. Hence we have a subsystem for (I, K):

$$s\frac{dI}{ds} = I\frac{2K^2 - (1-I)^2}{K^2 - (1-I)^2}, \tag{5.1}$$

$$s\frac{dK}{ds} = K\frac{K^2 - (1-I)^2 + \frac{\gamma-1}{2}I(1-I)}{K^2 - (1-I)^2}. \tag{5.2}$$

Introducing a new parameter τ, we can rewrite (5.1), (5.2) as

$$\frac{dI}{d\tau} = I\left[(1-I)^2 - 2K^2\right], \tag{5.3}$$

$$\frac{dK}{d\tau} = K\left[(1-I)^2 - K^2 - \frac{\gamma-1}{2}I(1-I)\right], \tag{5.4}$$

$$\frac{ds}{d\tau} = s\left[(1-I)^2 - K^2\right]. \tag{5.5}$$

Note that (5.3), (5.4) form an autonomous subsystem. If u_0 also vanishes, then we have a trivial solution $\rho = \rho_0,\ u = v = 0$.

Suppose the $u_0 > 0$. It can be verified that our far-field solutions starting at $s = 0+$ enter the region $\Omega \subset \mathbb{R}^2$ in the (I, K)-phase space given by

$$\Omega : \begin{cases} I > 0,\ K > 0, & \\ a \equiv (1-I)^2 - K^2 - \frac{\gamma-1}{2}I(1-I) > 0 & \text{in } 0 < I \leqq \frac{1}{\gamma}, \\ b \equiv (1-I)^2 - 2K^2 > 0 & \text{in } \frac{1}{\gamma} \leqq I < 1. \end{cases}$$

See Figure 5.1.

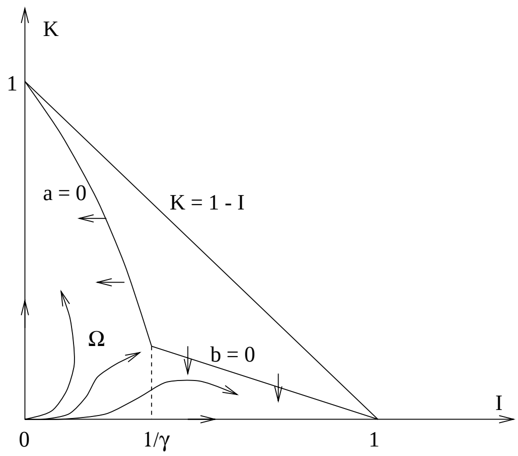

Figure 5.1. Phase portrait of solutions without swirls

We show that solutions starting in the closure $\bar{\Omega}$ do not leave $\bar{\Omega}$ as s increases. Notice first that $s > 0$ is an increasing function of τ in Ω by equation (5.5) so we show that solutions of the two equations (5.3), (5.4) starting in the closure $\bar{\Omega}$ do not leave $\bar{\Omega}$ as τ increases. The stationary points of (5.3),(5.4) in $\bar{\Omega}$ are the points $(I, K) = (0,0), (0,1), (1,0), \left(\frac{1}{\gamma}, \frac{1}{\sqrt{2}}\left(1-\frac{1}{\gamma}\right)\right)$. The axis $K = 0$ in $0 < I < 1$, and the axis $I = 0$ in $0 < K < 1$ are trivial solutions. On the boundary $b = 0$, i.e.,

$$K = \frac{1}{\sqrt{2}}(1-I), \quad \frac{1}{\gamma} < I < 1, \tag{5.6}$$

we find that $\frac{dI}{d\tau} = 0$, $\frac{dK}{d\tau} < 0$. So solutions enter Ω on (5.6). Finally on the boundary $a = 0$, i.e.,

$$K^2 = (1-I)^2 - \frac{\gamma-1}{2} I(1-I), \qquad 0 < I < \frac{1}{\gamma}, \tag{5.7}$$

we find that $\frac{dI}{d\tau} < 0$, and $\frac{dK}{d\tau} = 0$. So solutions enter Ω on (5.7) also.

Without showing further details we conclude that some solutions in Ω go to the point $(1, 0)$, while others go to the point $(0, 1)$, with exactly one integral curve going to the point $\left(\frac{1}{\gamma}, \frac{1}{\sqrt{2}}\left(1-\frac{1}{\gamma}\right)\right)$.

To determine what data (u_0, ρ_0) yield the transitional solution leading to the stationary point $\left(\frac{1}{\gamma}, \frac{1}{\sqrt{2}}\left(1-\frac{1}{\gamma}\right)\right)$, we first eliminate τ from (5.3), (5.4) by division and introduce $L \equiv (K/I)^2$ to find

$$\frac{dL}{dI} = -L\frac{(\gamma-1)(1-I) - 2IL}{(1-I)^2 - 2I^2 L}, \quad 0 < I < 1, \tag{5.8}$$

$$L = M_0^{-2} \ \text{ at } \ I = 0 \tag{5.9}$$

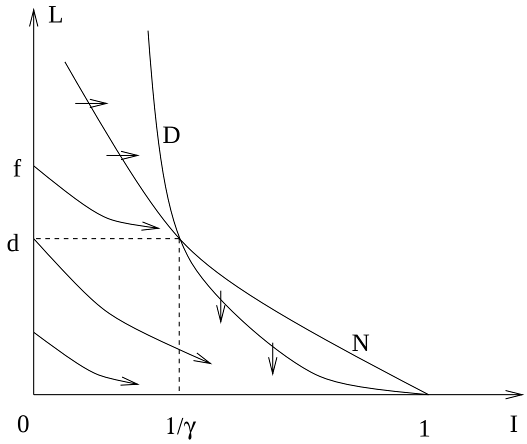

Figure 5.2. Estimate of the transitional Mach number $M(\gamma) < \frac{\sqrt{2}}{\gamma-1}$

where

$$M_0 \equiv u_0/\sqrt{p'(\rho_0)}$$

denotes the Mach number of the initial states $(u_0, 0, \rho_0)$. Problem (5.8), (5.9) is well-posed for any $M_0 > 0$. The transitional solution goes from the point $(I, L) = (0, M_0^{-2})$ to $\left(\frac{1}{\gamma}, \frac{1}{2}(\gamma-1)^2\right)$. However, there does not seem to be an explicit formula for the value M_0 which yields the transitional solution. We give an estimate instead. For convenience we let $M(\gamma)$ denote the initial Mach number for the transitional solution. It can be seen that $M_0 = \frac{\sqrt{2}}{\gamma-1}$ is an upper bound for the transitional Mach number $M(\gamma)$. In fact, any solution $L(I)$ of (5.8), (5.9) with datum $M_0 \geqq \frac{\sqrt{2}}{\gamma-1}$ starts as a decreasing function of $I \geqq 0$ till $I = \frac{1}{\gamma}$. In the interval $I \in \left[\frac{1}{\gamma}, 1\right]$, the solution remains below the two curves on which the numerator and denominator of the right-hand side of (5.8) vanish, respectively, and therefore remains decreasing until the final stationary point $(I, L) = (1, 0)$; see Figure 5.2, where $d = \frac{1}{2}(\gamma-1)^2$, $f = M^{-2}(\gamma)$, and N and D are where the numerator and denominator of the right-hand side of (5.8) respectively vanish.

This transitional solution yields a one-parameter family of smooth solutions in terms of (r, u, v, ρ). We see from equation (5.5) that $\ln s$ approaches infinity as the solutions approach the point $(I, K) = \left(\frac{1}{\gamma}, \frac{1}{\sqrt{2}}\left(1 - \frac{1}{\gamma}\right)\right)$, since $(1-I)^2 - K^2 \neq 0$ at the point. So $s \to +\infty$ and thus $r \to 0+$. Also the solutions have the asymptotics

$$u(r) = \frac{1}{\gamma} r, \qquad p'(\rho)(r) = \frac{1}{2}\left(1 - \frac{1}{\gamma}\right)^2 r^2$$

as $r \to 0+$. These global transitional solutions are similar; one is sketched in Figure 5.3.

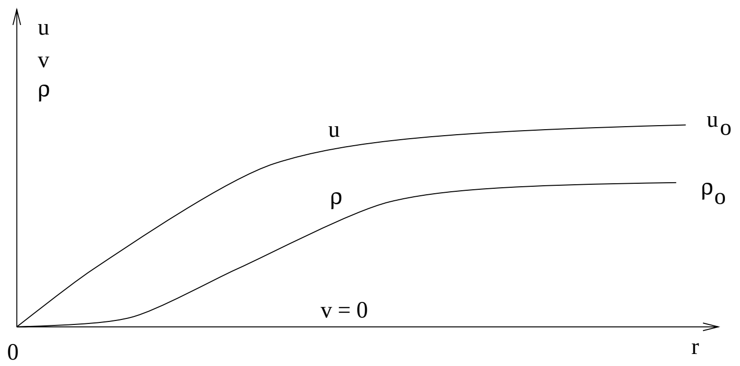

Figure 5.3. A zero-swirl transitional solution

We next show that the parameter s approaches finite numbers when solutions of (5.3)–(5.5) approach either the points $(I, K) = (1, 0)$ or $(0, 1)$.

We can linearize the two equations (5.3), (5.4) at $(I, K) = (0, 1)$ to find

$$\frac{dI}{d\tau} = -I$$
$$\frac{d(K-1)}{d\tau} = -\frac{\gamma+3}{2} I - 2(K-1). \tag{5.10}$$

The eigenvalues of (5.10) are $\lambda_1 = -1$, $\lambda_2 = -2$. So solutions of (5.3), (5.4) near $(0, 1)$ approach $(0, 1)$ exponentially as $\tau \to +\infty$. From equation (5.5), we find

$$\ln \frac{s}{s_0} = \int_{\tau_0}^{\tau} \left[(1-I)^2 - K^2\right] d\tau$$

for some constants $s_0 > 0$ and τ_0. So s approaches a finite number as $\tau \to \infty$ since $(1-I)^2 - K^2$ approaches zero exponentially.

Linearization at the point $(I, K) = (1, 0)$ of the two equations (5.3), (5.4) yield the trivial system with zero right-hand sides. We need a different approach. We show that solutions near $(1, 0)$ enter $(1, 0)$ in the sector bounded by $K = 0$ and the line

$$K = \alpha(1-I) \tag{5.11}$$

for some $0 < \alpha < \frac{1}{\sqrt{2}}$, such that $\alpha^2 + \frac{\gamma-3}{4} > 0$. See Figure 5.4.

In fact, a vector in the normal direction of (5.11) is $(\alpha, 1)$. We calculate the inner product of the vector field of (5.3), (5.4) with the direction $(\alpha, 1)$ to find

$$\frac{d}{d\tau}(I, K)\cdot(\alpha, 1) = -\alpha\left[2I\left(\alpha^2 + \frac{\gamma-3}{4}\right) - (1-\alpha^2)(1-I)\right](1-I)^2. \tag{5.12}$$

The expression (5.12) is negative if I is close to 1 and α is such that $\alpha^2 + \frac{\gamma-3}{4} > 0$. So we conclude that every solution that ends at $(1, 0)$ is such that

$$K < \alpha(1-I) \tag{5.13}$$

near $I = 1$ for some $\alpha < \frac{1}{\sqrt{2}}$, since $\gamma > 1$.

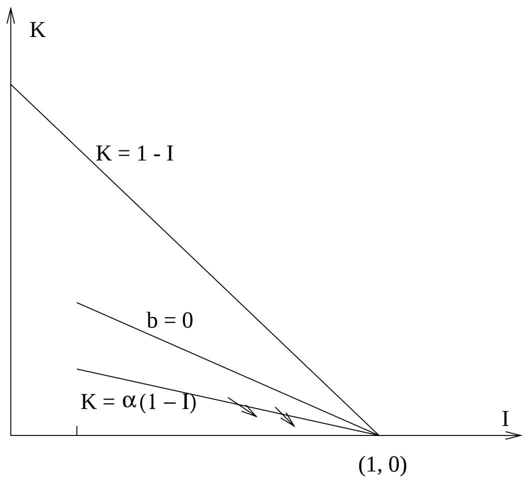

Figure 5.4. The parameter s is finite near $(1, 0)$

Now we look at the equation (5.1) and use (5.13) to find that

$$s\frac{dI}{ds} > I\frac{1-2\alpha^2}{1-\alpha^2}$$

when I is close to 1. Thus

$$s\frac{dI}{ds} > \text{a positive constant}$$

when I is near 1. Therefore s is finite at the end $(1, 0)$ since I is bounded and the geometric integral $\int_1^{+\infty} \frac{1}{s}\,ds$ diverges.

We are now ready to construct global solutions for (5.1), (5.2). For each solution ending at $(I, K) = (0, 1)$, we continue the solution by the constant state $(u, v, \rho) = (0, 0, \rho^{**})$ where ρ^{**} is the value of ρ at the ending point. These are continuous extensions. The relation between the density ρ^{**} and the terminal value s^{**} is

$$p'(\rho^{**}) = s^{**-2},$$

or equivalently

$$p'(\rho^{**}) = r^{**2}$$

when r^{**} is the radius of the circle of the constant state in the physical plane $t = 1$. We therefore have constructed global solutions in this case.

For each solution ending at the point $(I, K) = (1, 0)$, we continue the solution by the vacuum state $\rho = 0$. We do not need to specify the functions u or v in the vacuum since the Euler equations have ρ as a factor in every term. Each vacuum occupies a circular region of radius r^{**} determined by $r^{**} = 1/s^{**} = u^{**}$ in the physical plane at $t = 1$, where u^{**} is the terminal radial velocity of the fluid at the edge of the vacuum. In all, we have constructed global solutions to the reduced system (5.1), (5.2), the special case of (4.3), (4.4) with zero swirl.

6. General Solutions in the Intermediate Field

Now consider the case $v_0 > 0$ and $u_0 \geqq 0$ for system (4.3). Let $\Omega_3 \subset \mathbb{R}^3$ be the set of points (I, J, K) satisfying $0 < I < 1,\ J > 0,\ K > 0$,

$$H = J^2 - I(1-I) < 0,$$
$$B \equiv (1-I)[J^2 + I(1-I)] - 2IK^2 > 0 \ \text{ if } \ \frac{1}{\gamma} \leqq I < 1,$$
$$A \equiv 2(1-I)^2 + (\gamma - 1)[J^2 - I(1-I)] - 2K^2 > 0 \ \text{ if } \ 0 < I < \frac{1}{\gamma}.$$

See Figure 6.1. It can be verified that all far-field solutions with $u_0 \neq 0,\ v_0 > 0,\ \rho_0 > 0$ enter the region Ω_3 in $s > 0$. Far-field solutions with $u_0 = 0$ enter the side $H = 0$ of Ω_3. We omit these tedious verifications.

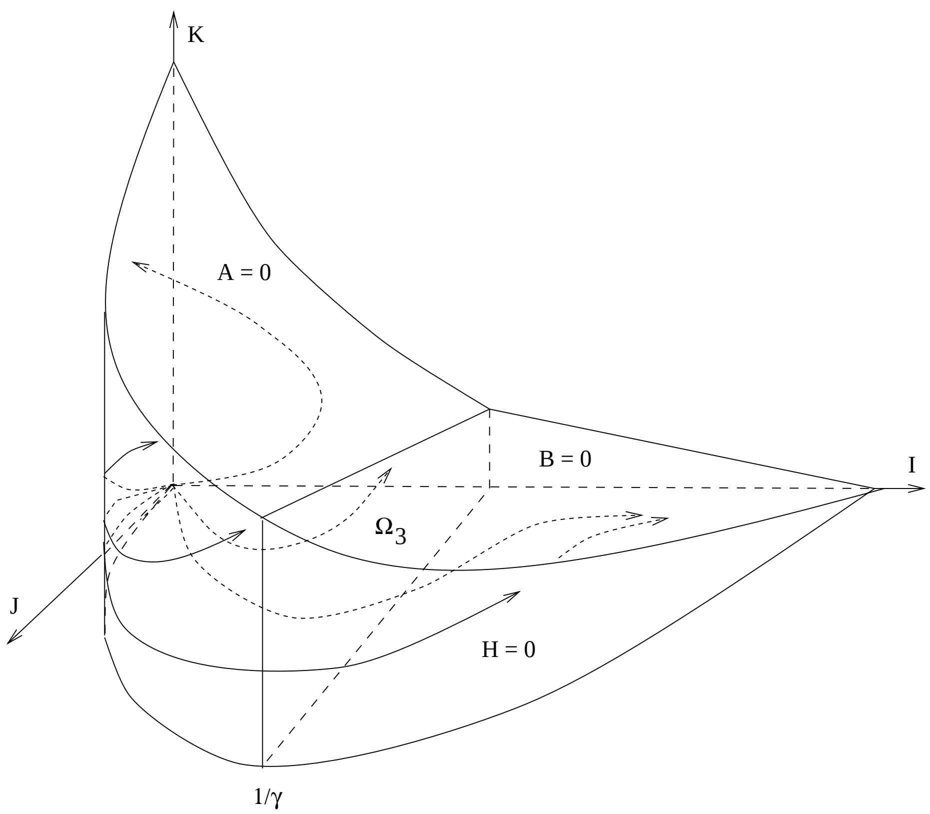

Figure 6.1. The region Ω_3

We find that it is convenient to introduce a new variable τ, as in Section 5, to write the system (4.3) in the form

$$\frac{dI}{d\tau} = (1-I)B, \tag{6.1}$$

$$\frac{dJ}{d\tau} = J(1-2I)[(1-I)^2 - K^2], \tag{6.2}$$

$$\frac{dK}{d\tau} = \frac{1}{2}K(1-I)A, \tag{6.3}$$

$$\frac{ds}{d\tau} = s(1-I)[(1-I)^2 - K^2]. \tag{6.4}$$

This is an autonomous system for (I, J, K, s), and the first three equations (6.1)–(6.3) form an autonomous subsystem for (I, J, K).

The stationary points of the system (6.1)–(6.3) contained in the closure $\bar{\Omega}_3$ are given by the edge

$$K = 1 - I,\ J^2 = I(1-I), \quad \forall I \in [0, 1], \tag{6.5}$$

and the point

$$(I, J, K) = \left(\frac{1}{\gamma}, 0, \frac{1}{\sqrt{2}}\left(1 - \frac{1}{\gamma}\right)\right) \tag{6.6}$$

in the case $\gamma \neq 2$. For the case $\gamma = 2$, the stationary points of (6.1)–(6.3) are given by edge (6.5) and the curve

$$I = \tfrac{1}{2}, \quad K^2 = \tfrac{1}{2}\left(J^2 + \tfrac{1}{4}\right), \quad 0 \leqq J < \tfrac{1}{2}, \tag{6.7}$$

which is the intersection of $A = 0$ with $B = 0$. Hence there is no stationary point in the open region Ω_3; all the stationary points are on the boundary of Ω_3.

Assertion. *Solutions inside Ω_3 do not leave Ω_3 from its sides (excluding possibly edges or corners) as s increases when $\gamma \geqq 2$.*

Proof. First note that the sides of Ω_3 in the surfaces $K = 0$, or $J = 0$ or $H = 0$ are invariant regions. We need only to prove that no solution leaves Ω_3 from the top two adjoining sides $A = 0$ and $B = 0$ when $\gamma \geqq 2$.

Consider first the top side given by

$$B = I(1-I)^2 + (1-I)J^2 - 2IK^2 = 0 \quad \text{in } \frac{1}{\gamma} < I < 1. \tag{6.8}$$

In the coordinate order (I, J, K) an outward normal is given by

$$(2K^2 - (1-I)^2 + 2I(1-I) + J^2,\ -2(1-I)J,\ 4IK) \equiv \vec{n}_B.$$

We calculate the inner product of the normal $\vec{n}_B$ with the tangent vector of an integral curve of (6.1)–(6.3) on the surface (6.8) to yield

$$\vec{n}_B \cdot \frac{d}{d\tau}(I, J, K) = -(1-I)^2[(1-I)I - J^2][(\gamma-2)J^2 + (\gamma I - 1)(1-I)] < 0$$

when $\frac{1}{\gamma} < I < 1$ and $\gamma \geqq 2$. Notice that $\frac{ds}{d\tau} > 0$ in Ω_3. Thus no solution leaves Ω_3 from this side as s increases.

Now consider the top side given by

$$A = 2(1-I)^2 + (\gamma-1)[J^2 - I(1-I)] - 2K^2 = 0, \quad 0 < I < \frac{1}{\gamma}.$$

An outward normal is given by

$$(4(1-I) + (\gamma-1)(1-2I),\ -2(\gamma-1)J,\ 4K) \equiv \vec{n}_A$$

in the order (I, J, K). We similarly calculate the inner product of the normal $\vec{n}_A$ with the tangent of the integral curves on this surface to yield

$$\begin{aligned}\vec{n}_A \cdot &\frac{d}{d\tau}(I, J, K)\\ &= H\{[\gamma + 3 - 2(\gamma+1)I](1-I)(1-\gamma I) + (\gamma-1)^2 J^2 (1-2I)\} < 0\end{aligned}$$

in $0 < I < \frac{1}{\gamma}$ because $J^2 < I(1-I)$ and $\gamma + 3 - 2(\gamma+1)I > 0$ and $\gamma \geqq 2$. Hence no solution leaves Ω_3 from this side either. □

We need to study the local structure of solutions near stationary points given in formula (6.5). It is helpful to first describe solutions of (6.1)–(6.4) with data $u_0 = 0,\ v_0 > 0,\ \rho_0 > 0$. Our far-field solutions (3.4), (3.5) can be written in terms of I, J, K as

$$\begin{aligned} I &= s^2 v_0^2,\\ J &= s v_0 \sqrt{1 - s^2 v_0^2},\\ K &= s\sqrt{p'(\rho_0)}, \end{aligned} \tag{6.9}$$

valid for $s \in [0, s^*]$ where $s^* = 1/r^*$. These solutions are all in the surface $H = 0$, and they all start from the origin $(I, J, K) = (0, 0, 0)$ and end at points of the stationary edge given by (6.5). As the initial Mach number

$$M_0 \equiv \frac{|v_0|}{\sqrt{p'(\rho_0)}}$$

varies in $(0, \infty)$, the ending points of the solutions cover all the interior points of the stationary edge (6.5) exactly once. See Figure 6.1.

Now we study the structure of solutions near the stationary edge (6.5). More precisely, let E denote the open set

$$E \equiv \{(I, J, K) \subset \mathbb{R}^3 | J = \sqrt{I(1-I)},\ K = 1 - I,\ 0 < I < 1\} \tag{6.10}$$

which contains the interior points of the curve (6.5). Apparently the solutions of (6.9) give one set of directions of integral curves for (6.1)–(6.3). The second set of integral directions is given by the tangent directions of the stationary edge E (6.10). By linearization, we find the third set of directions to be

$$\vec{n}_3 \equiv \big(2(1-I)[2(\gamma-1)I-1], -2(\gamma+1)J(2I-1), \\ -(1-I)[2(3\gamma-1)I-(\gamma+3)]\big) \tag{6.11}$$

where $J=\sqrt{I(1-I)}$ and $0<I<1$. Along this direction (6.11), we calculate

$$\vec{n}_3 \cdot \vec{n}_A \,\big|_E = 2(1-I)[4(3\gamma-1)I^2-4(2\gamma+1)I+\gamma+3], \tag{6.12}$$

$$\vec{n}_3 \cdot \vec{n}_B \,\big|_E = -2(1-I)^2[4(\gamma-1)I^2-2\gamma I+1], \tag{6.13}$$

$$\vec{n}_3 \cdot \vec{n}_H \,\big|_E = 2(1-I)(1-2I)(1+4I), \tag{6.14}$$

where $\vec{n}_H$ is used to denote an outward normal to the surface $J^2=I(1-I)$.

From now on we restrict ourselves to the case $\gamma=2$, and leave the other cases $\gamma \ne 2$ to a forthcoming paper. The formulae (6.11)–(6.13) can be simplified to

$$\vec{n}_3 = (2I-1)(2(1-I), -6J, -5(1-I)), \tag{6.15}$$

$$\vec{n}_3 \cdot \vec{n}_A \,\big|_E = 10(1-I)(2I-1)^2, \tag{6.16}$$

$$\vec{n}_3 \cdot \vec{n}_B \,\big|_E = -2(1-I)^2(2I-1)^2. \tag{6.17}$$

So we find that $\vec{n}_3$ points into Ω_3 for $I \in \left(\frac{1}{2},1\right)$, and $-\vec{n}_3$ points into Ω_3 for $I \in \left(0,\frac{1}{2}\right)$.

Now we can depict the integral curves inside Ω_3; see Figure 6.1. First we observe that s is an increasing function of τ inside Ω_3, while J is an increasing function of τ if $0<I<\frac{1}{2}$, but changes to decreasing when $I \in \left(\frac{1}{2},1\right)$. There are three kinds of integral curves: the first kind consists of integral curves which go to the stationary point $(1,0,0)$. Each of the second kind goes to a stationary point on the curve given in (6.7). Each of the third kind goes to a stationary point on the curve E with $I \in \left(0,\frac{1}{2}\right)$ given in (6.10). We mention in particular that no integral curve from inside Ω_3 goes to a point of E between $\frac{1}{2}<I<1$ because I is an increasing function of τ and $\vec{n}_3$ is pointing towards $(1,0,0)$ for $\frac{1}{2}<I<1$. See the proof of Lemma A.1 of the Appendix for a complete proof. Also there is no integral curve from inside Ω_3 which goes to the point $(0,0,1)$ because J is an increasing function of τ for $I \in \left(0,\frac{1}{2}\right)$.

We now show how to continue each of the first and second kinds of solutions further to construct global solutions for all $s \in (0,\infty)$.

First we consider the second kind of integral curves inside Ω_3 which end on the stationary curve (6.7). Since $K<1-I$ on (6.7), we find that the right hand side of (6.4) does not vanish on (6.7). Thus $s \to \infty$ as the integral curves approach (6.7) with $\tau \to \infty$. Hence these integral curves are already solutions in the entire domain $s \in (0,\infty)$, i.e., $r \in (0,\infty)$.

For the first kind of integral curves which end at $(1,0,0)$, we show that s approaches finite values although $\tau \to \infty$. Since the proof is long and tedious, we put it in the Appendix to avoid interruption of our construction. We use the natural continuation of vacuum $\rho = 0$ to extend our solutions till $s = \infty$. The values of u, v do not need to be specified in the vacuum.

We comment that solutions on the surface $J^2 = I(1 - I)$ starting from the origin and ending on a point of E with $\frac{1}{2} < I < 1$ can be continued through the direction (6.15) into Ω_3, and they go toward $(1, 0, 0)$ with finite terminal s values. The Appendix can be adapted trivially for this case and the further extension by vacuum is valid also. It is interesting to note that for the initial data (u_0, v_0, ρ_0) with $u_0 = 0$ and $M_0 > \sqrt{2}$, the corresponding solutions always end on E with $I > \frac{1}{2}$. The critical value $M_0 = \sqrt{2}$ yields a special solution that ends at the point $(I, J, K) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ with final s value given by $s^* = 1/r^*$. This solution can be continued by the constant solution $(I, J, K) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ to $s = \infty$. Thus, we find an explicit solution given by

$$u = \tfrac{1}{2}r, \quad v = \tfrac{1}{2}r, \quad \sqrt{p'(\rho)} = \tfrac{1}{2}r \quad \text{for } 0 < r < r^*, \tag{6.18}$$

for $\gamma = 2$ and data $u_0 = 0$ and $M_0 = \sqrt{2}$. It can be seen from the relation $K = 1 - I$ that a point of E with $\frac{1}{2} < I < 1$ is a state whose radial velocity u is greater than the sound speed, and may be called *radially supersonic*. Similarly, a point of E with $0 < I < \frac{1}{2}$ is a state whose radial velocity u is less than the sound speed, and may be called *radially subsonic*. We have found $M_0 = \sqrt{2}$ to be a critical value in the study of instability of vortex sheets; see MAJDA [7], and references therein.

For the third kind of integral curves which end on the upper half of E with $I \in \left(0, \frac{1}{2}\right)$, we show that s approaches finite values also. The proof is easy because we have

$$\frac{ds}{dJ} = \frac{s}{J}\frac{1 - I}{1 - 2I} \tag{6.19}$$

from the second equation of system (4.3). The right-hand side of (6.19) is nonsingular for $(I, J, K) \in E$ with $0 < I < \frac{1}{2}$. Thus s is finite around any point of E with $0 < I < \frac{1}{2}$.

So we have constructed all solutions globally except those which end on E with $0 < I < \frac{1}{2}$.

7. Inner-Field Solutions

We extend solutions that end on E with $0 < I < \frac{1}{2}$ in this section. We find that these solutions go along the directions $\vec{n}_3$ given in (6.15) into the region between $A = 0$ and $B = 0$ in $0 < I < \frac{1}{2}$ and $J^2 > I(1 - I)$, and eventually go to infinity. The scaled variables I, J, and K are not suitable for this portion of the solutions.

We restart from system (2.7) with data (u, v, ρ, r) satisfying the relations E : $K = 1 - I$ and $J^2 = I(1 - I)$ for $I \in \left(0, \frac{1}{2}\right)$. In terms of (u, v, ρ, r), these data are in the form

$$u\big|_{r=\alpha} = \beta, \quad v\big|_{r=\alpha} = \sqrt{\beta(\alpha - \beta)}, \quad \sqrt{p'(\rho)}\big|_{r=\alpha} = \alpha - \beta, \tag{7.1}$$

where $\alpha > 0$ and $\beta \in \left(0, \frac{1}{2}\alpha\right)$ are arbitrary.

We show that problem (2.7), (7.1) has solutions $(u(r), v(r), \rho(r))$ which also vanish as $r \to 0+$. An asymptotic analysis can be performed a priori to determine the orders at which (u, v, ρ) vanish as $r \to 0+$. This asymptotic analysis also suggests using the scaled variables

$$R = \frac{r}{\sqrt{p'(\rho)}}, \quad U = \frac{u}{\sqrt{p'(\rho)}}, \quad V = \frac{v}{\sqrt{p'(\rho)}}\,. \tag{7.2}$$

For polytropic gases $p(\rho) = A_2\rho^\gamma$, we can rewrite system (2.7) into a new form

$$\frac{du}{d\tau} = \frac{r-u}{w^3}\Sigma, \quad \frac{dv}{d\tau} = \frac{uv}{w^3}\Delta, \quad \frac{dr}{d\tau} = \frac{r(r-u)}{w^3}\Delta, \quad \frac{dw}{d\tau} = \frac{\gamma-1}{2}\frac{r-u}{w^2}\Theta \tag{7.3}$$

where $w \equiv \sqrt{p'(\rho)}$ and τ is a parameter. In terms of the variables in (7.2), we find that

$$\begin{aligned} \frac{dU}{d\tau} &= (R-U)\tilde{\Sigma} - \frac{\gamma-1}{2}U(R-U)\tilde{\Theta} \equiv A_1, \\ \frac{dV}{d\tau} &= UV\tilde{\Delta} - \frac{\gamma-1}{2}V(R-U)\tilde{\Theta} \equiv C_1, \\ \frac{dR}{d\tau} &= R(R-U)\tilde{\Delta} - \frac{\gamma-1}{2}R(R-U)\tilde{\Theta} \equiv B_1 \end{aligned} \tag{7.4}$$

where $\tilde{\Delta} \equiv 1 - (U-R)^2$, $\tilde{\Theta} \equiv V^2 - U(R-U)$ and $\tilde{\Sigma} \equiv (R-U)\tilde{\Theta} - U\tilde{\Delta}$. System (7.4) is autonomous for (U, V, R). We find that the last equation in (7.3) can be written as

$$\frac{dw}{wd\tau} = \frac{\gamma-1}{2}(R-U)\tilde{\Theta}. \tag{7.5}$$

So w can be integrated from (7.5) once (U, V, R) are obtained from (7.4). The corresponding data of (7.1) for (7.4) and (7.5) are given by any stationary point (U^*, V^*, R^*, w^*) satisfying

$$R^* - U^* = 1, \quad {V^*}^2 = U^*, \quad 0 < 2U^* < R^*, \tag{7.6}$$

$$w^* > 0. \tag{7.7}$$

After (7.4)–(7.7) are solved, we use the third equation in (7.3) to show that r is an increasing function of $\tau \in \mathbb{R}$ and $r \to 0$ and $\alpha \in (0, +\infty)$ as τ goes to $\mp\infty$ respectively.

We now study the problem (7.4) and (7.6). We claim that there exists a one-to-one correspondence between the data (7.6) and the set

$$F = \{(U, V, R) | \, U = 0, \quad 0 < V < \sqrt{2}, \ R = 0\}. \tag{7.8}$$

The integral curves of (7.4) that connect the two sets provide the natural correspondence. The set F consists of stationary points of (7.4) and corresponds to the point $(u, v, \sqrt{p'(\rho)}, r) = (0, 0, 0, 0)$.

We consider the domain Ω_4 of points (U, V, R) in $\mathbb{R}^3$ given by

$$C_1 < 0, \quad B_1 > 0, \quad U > 0, \quad R > 0, \quad V > 0.$$

The domain is depicted in Figure 7.1 for $\gamma = 2$. We find that Ω_4 lies in $0 < R - U < 1$ and $R > 2U$.

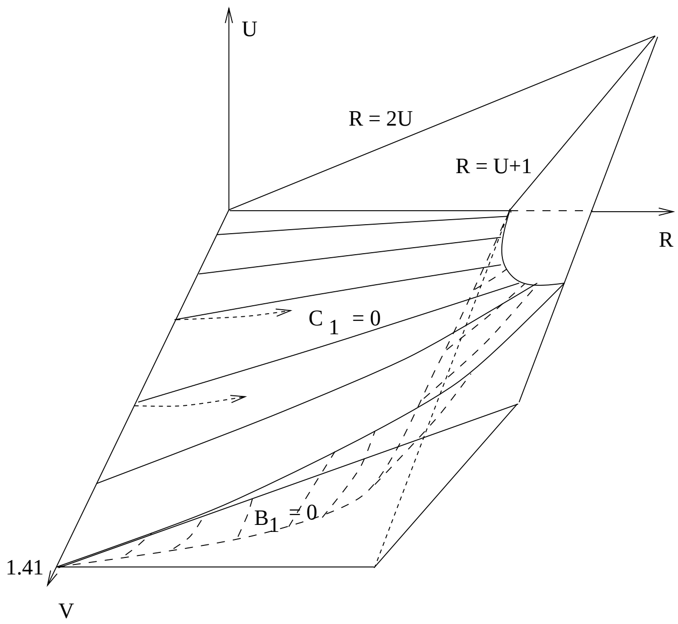

Figure 7.1. The domain Ω_4

We first find all stationary points of (7.4) in the closure $\bar{\Omega}_4$. They are given in (7.6), (7.8) and the set

$$G = \{(U, V, R) | R = 2U,\ V^2 = 2 - U^2,\ 0 < U < 1\}. \tag{7.9}$$

Then we observe that $A_1 > 0$ in Ω_4. The region Ω_4 has five edges, three of them are stationary points given in (7.6), (7.8) and (7.9). The fourth edge $U = 0,\ V = 0,\ 0 < R < 1$ is a solution to (7.4). The fifth edge is ordinary: $U = 0,\ V^2 = 2(1 - R^2),\ 0 < R < 1$.

We show that all integral curves of (7.4) on the surface of Ω_4 enter Ω_4 as τ increases. The surface $C_1 = 0$ is given more explicitly by

$$(R - U)V^2 - U[2 - (R - U)^2] = 0. \tag{7.10}$$

An outward normal may be found to be

$$\vec{n}_{C_1} = (V^2 + 2 - (R - U)^2 + 2U(R - U),\ -2V(R - U),\ -V^2 - 2U(R - U)). \tag{7.11}$$

The components of the vector are in the order (U, V, R). We calculate the inner product of $\vec{n}_{C_1}$ with the tangent vector of any integral curve on the surface $C_1 = 0$ to yield

$$\vec{n}_{C_1} \cdot \frac{d}{d\tau}(U, V, R) = A_1[V^2 + 2 - (R-U)^2 + 2U(R-U)] + B_1[-V^2 - 2U(R-U)].$$

On $C_1 = 0$ we find

$$\begin{aligned} A_1 &= (R-U)[(R-U)\tilde{\Theta} - U\tilde{\Delta}] - \tfrac{1}{2}U\tilde{\Theta} \\ &= \tfrac{1}{2}(R-U)(R-2U)\tilde{\Theta}, \\ B_1 &= R(R-U)\big[\tilde{\Delta} - \tfrac{1}{2}\tilde{\Theta}\big] \\ &= \frac{1}{2U}R(R-U)(R-2U)\tilde{\Theta}, \end{aligned}$$

and so

$$\frac{B_1}{A_1} = \frac{R}{U} \text{ on } C_1 = 0. \tag{7.12}$$

Thus

$$\vec{n}_{C_1} \cdot \frac{d}{d\tau}(U, V, R) = -(R-U)(R-2U)\tilde{\Theta}\ (R-U)^2 < 0$$

since

$$\begin{aligned} \tilde{\Theta} &= V^2 - U(R-U) = \frac{1}{R-U}[2U - U(R-U)^2 - U(R-U)] \\ &= \frac{U}{R-U}[2 - (R-U)^2 - (R-U)] > 0 \end{aligned}$$

on $C_1 = 0$. So integral curves of (7.4) on the surface $C_1 = 0$ enter Ω_4 as τ increases.

The surface $B_1 = 0$ is given more explicitly by

$$V^2 + 2(U-R)^2 - U(R-U) - 2 = 0. \tag{7.13}$$

Its outward normal is

$$\vec{n}_{B_1} = (6U - 5R,\ 2V,\ 4R - 5U). \tag{7.14}$$

We calculate the inner product of the outward normal $\vec{n}_{B_1}$ with the tangent vector of any integral curve on the surface $B_1 = 0$:

$$\vec{n}_{B_1} \cdot \frac{d}{d\tau}(U, V, R) = A_1(6U - 5R) + 2C_1V.$$

On $B_1 = 0$, i.e., where $\tilde{\Delta} = \frac{1}{2}\tilde{\Theta}$, we find that

$$\begin{gathered} A_1 = (R-U)(R-2U)\tilde{\Theta}, \quad C_1 = V\Big(U - \frac{R}{2}\Big)\tilde{\Theta}, \\ \tilde{\Theta} = V^2 - U(R-U) = 2[1 - (R-U)^2] > 0. \end{gathered}$$

Thus

$$\begin{aligned} \vec{n}_{B_1} \cdot \frac{d}{d\tau}(U, V, R) &= (R-2U)\tilde{\Theta}[(R-U)(6U-5R) - V^2] \\ &= (R-2U)\tilde{\Theta}[-3(R-U)^2 - 2] \\ &= -(R-2U)\tilde{\Theta}[3(R-U)^2 + 2] < 0. \end{aligned}$$

So integral curves do not exit Ω_4 from $B_1 = 0$.

Integral curves on the surface $U = 0$ enter Ω_4 as τ increases since

$$A_1\big|_{U=0} = R^2V^2 > 0 \quad \text{for } R > 0,\ V > 0.$$

We have thus proved that any integral curve that starts inside Ω_4 remains in Ω_4 for all positive τ.

We now linearize the system (7.14) at its stationary points of F. For any point $(U, V, R) = (0, V_0, 0)$ of F from (7.8), we find

$$\frac{\partial(A_1, C_1, B_1)}{\partial(U, V, R)}\Big|_{(0,V_0,0)} = \begin{bmatrix} 0 & 0 & 0 \\ V_0 + \frac{1}{2}V_0^3 & 0 & -\frac{1}{2}V_0^3 \\ 0 & 0 & 0 \end{bmatrix}. \tag{7.15}$$

But (7.15) has no nonzero eigenvalue so that the Center Manifold Theorem [5] yields only the trivial conclusion. By introducing the new variables

$$X = \frac{U}{R} \tag{7.16}$$

and τ' such that

$$d\tau' = Rd\tau, \tag{7.17}$$

we can rewrite (7.4) as

$$\frac{dX}{d\tau'} = (1 - X)[V^2(1 - X) - 2X + X(1 - X)^2R^2], \tag{7.18}$$

$$\frac{dV}{d\tau'} = XV[1 - (1 - X)^2R^2] - \frac{1}{2}V(1 - X)[V^2 - X(1 - X)R^2], \tag{7.19}$$

$$\frac{dR}{d\tau'} = R(1 - X)\{1 - (1 - X)^2R^2 - \frac{1}{2}[V^2 - X(1 - X)R^2]\}. \tag{7.20}$$

We are interested only in stationary points of (7.18)–(7.20) given in the form

$$R = 0, \quad V^2(1 - X) = 2X, \tag{7.21}$$

or more explicitly:

$$R = 0, \quad X = \frac{a}{1 + a}, \quad V = \sqrt{2a} \quad (0 < a < 1), \tag{7.22}$$

which correspond to points of F from (7.8) on the V axis with directions pointing into the region Ω_4. We find that the linearization of (7.18)–(7.20) at any point $(X, V, R) = (X_0, V_0, 0)$ of (7.22) has the matrix form

$$\begin{bmatrix} -2 & 2V_0(1 - X_0)^2 & 0 \\ V_0 + \frac{1}{2}V_0^3 & -2X_0 & 0 \\ 0 & 0 & (1 - X_0)\left(1 - \dfrac{V_0^2}{2}\right) \end{bmatrix} \begin{bmatrix} X - X_0 \\ V - V_0 \\ R \end{bmatrix}. \tag{7.23}$$

We compute the eigenvalues to yield

$$\lambda_1 = 0,\ \lambda_2 = -2(1 + X_0),\ \lambda_3 = (1 - X_0)\big(1 - \tfrac{1}{2}V_0^2\big). \tag{7.24}$$

So the linearized system has a solution along the direction $(X, V, R) = (0, 0, 1)$ corresponding to the eigenvalue $\lambda_3 > 0$. We conclude that our system (7.18)–(7.20) also has a solution which goes along the direction $(X, V, R) = (0, 0, 1)$ as τ' increases from $\tau' = -\infty$, by the Center Manifold Theorem [5]. From (7.16), (7.17), we further conclude that there exists a solution from any point of (7.8) which goes along the direction $\frac{U}{R} = \frac{a}{1+a}$ on the plane $V = \sqrt{2a},\ 0 < a < 1$, as τ leaves $-\infty$. It can be verified by direct computation (which we omit) that those solutions all go into Ω_4.

Our final purpose is to show that for each point on (7.6), there exists a point on F of (7.8) from which a solution originates and goes into Ω_4 to end at the point of (7.6). We can show easily that if the point on F is close to the origin, then the solution remains close to the R-axis and ends at a point on (7.6); this is because V is a decreasing function of R.

We now show that the ratio U/R along a solution in Ω_4 is an increasing function of R. In fact we know that any solution originating from F goes into Ω_4. We can estimate the derivative

$$\frac{dU}{dR} = \frac{A_1}{B_1} \tag{7.25}$$

in Ω_4. Notice that $A_1 > 0$, $B_1 > 0$ in Ω_4, and

$$\begin{aligned} \frac{dU}{dR} &= \frac{(R-U)\tilde{\Theta} - u\tilde{\Delta} - \frac{1}{2}U\tilde{\Theta}}{R\big(\tilde{\Delta} - \frac{1}{2}\tilde{\Theta}\big)} \\ &= \frac{\frac{\tilde{\Theta}}{\tilde{\Delta}}(2R - 3U) - 2U}{R\left(2 - \frac{\tilde{\Theta}}{\tilde{\Delta}}\right)}. \end{aligned} \tag{7.26}$$

The last expression is an increasing function of $\tilde{\Theta}/\tilde{\Delta}$ for fixed R, U, such that $R \geqq 2U$. Observe that in Ω_4 we have $C_1 < 0$, i.e.,

$$\frac{\tilde{\Theta}}{\tilde{\Delta}} \geqq \frac{2U}{R-U}. \tag{7.27}$$

We use (7.27) in (7.26) to derive

$$\frac{dU}{dR} \geqq UR$$

which yields

$$\frac{d}{dR}\left(\frac{U}{R}\right) \geqq 0. \tag{7.28}$$

If initially $\frac{U}{R}\Big|_{R=0} = \frac{a}{1+a}$, and $a \to 1$, the final point of the solution cannot be on (7.6) with $R^* < 2$, since $\frac{U^*}{R^*} = \frac{R^*-1}{R^*} = \frac{1}{R^*} < \frac{1}{2}$. So the solutions from points of F (close to the end point $(U_0, V_0, R_0) = (0, \sqrt{2}, 0)$) will end up either at $(U, R^*) = (1, 1, 2)$ or a point of G. By the same token, no solutions from G can go into Ω_4 and end up on (7.6) because $\frac{U^*}{R^*} = \frac{1}{2}$ on G.

We omit the proof that solutions originating from F change continuously in Ω_4. Thus each point on (7.6) has a solution going to a point of F through Ω_4.

8. Conclusions

We summarize our results. By a weak solution to the two-dimensional Euler equations (2.1) we mean a bounded vector function (u, v, ρ) satisfying the equations in the sense of distributions. Since we deal with only continuous weak solutions in this paper, we do not mention any other requirements such as the Rankine-Hugoniot relation or entropy conditions. Set

$$M_0 \equiv \frac{\sqrt{u_0^2 + v_0^2}}{\sqrt{p'(\rho_0)}}$$

which is consistent with previous versions of M_0.

Theorem. *For any datum* (u_0, v_0, ρ_0) *with* $u_0 \geqq 0$ *and* $\rho_0 > 0$*, there exists a weak solution* (u, v, ρ) *to the initial-value problem* (2.1) *and* (2.5) *when* $p(\rho) = A_2\rho^2$ *for any constant* $A_2 > 0$*. The solution is continuous for* $t > 0$ *and is self-similar and axisymmetric. It takes on initial data almost everywhere and in* L^q_{loc} *for any* $q > 1$ *as* $t \to 0+$*. Furthermore,*

(i) *If* $v_0 = 0$*, then the solution exists even for the more general pressure-density relation* $p(\rho) = A_2\rho^\gamma$ *for any* $\gamma > 1$ *and is such that* $v = 0$ *for all time* $t > 0$*. There exists a critical value* $M(\gamma)$ *such that the solution is* C^1*-smooth for all* $t > 0$ *if* $M_0 = M(\gamma)$*; see Figure 5.3.*

If $M_0 > M(\gamma)$*, then the density* ρ *and the radial component* u *of the velocity are increasing functions of the spatial radius* r *for fixed time* $t > 0$*. The solution has a vacuum region* $\rho = 0$ *near the spatial origin for all* $t > 0$*. It is smooth beyond the vacuum and continuous everywhere. The edge of the cone of the vacuum is given by* $r = u^*t$ *where* u^* *is the speed of radial velocity at the edge and* $u^* < u_0$*.*

If $M_0 < M(\gamma)$*, then the density* ρ *and the radial component* u *of the velocity are also increasing functions of the spatial radius* r *for any fixed time* $t > 0$*. The solution has a cone near the spatial origin where the density* $\rho = \rho^* = constant$ *and* $u = 0$*. The side of the cone is given by* $r = \sqrt{p'(\rho^*)}t$*. The solution is smooth except at the edge of the cone, where it is only continuous.*

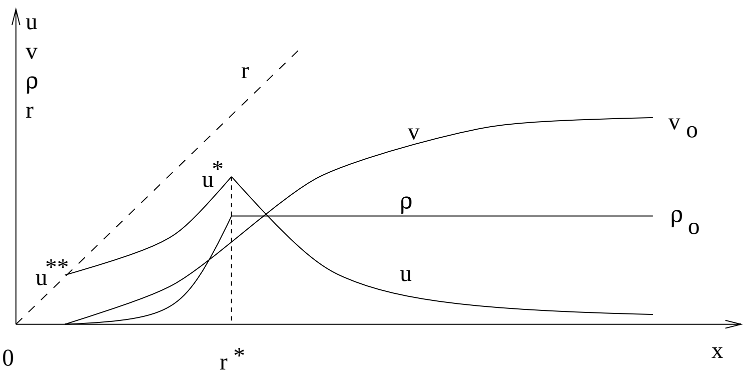

Figure 8.1. A solution (u, v, ρ) vs. the x-axis at time $t = 1$ when $u_0 = 0$ and $M_0 > \sqrt{2}$.

The critical value $M(\gamma)$ has an upper bound

$$M(\gamma) < \frac{\sqrt{2}}{\gamma - 1}.$$

(ii) *If* $u_0 = 0$, *the first piece of the solution is given explicitly by the formula* (3.4) *near* $r = \infty$. *If* $M_0 > \sqrt{2}$, *the second piece is given by the ordinary differential equation* (4.3) *with direction* (6.11)*; see Figure 8.1. The radial and pure rotational components and the density function of the solution are increasing functions of the spatial radius* r *in the region* $u^{**}t < r < r^*t$ *where* r^* *is given by* (3.5) *and* $u^{**} > 0$ *is the radial velocity at the inner end of the second piece. The third piece is the vacuum* $\rho = 0$ *with domain* $0 \leqq r < u^{**}t$. *There is no pure rotation at the edge of the vacuum.*

If $M_0 = \sqrt{2}$, *the second piece is given explicitly by formula* (6.18)*; see Figure 8.2.*

If $M_0 < \sqrt{2}$, *the second piece is given by the ordinary differential equation* (7.4), (7.5) *with two-point boundary values* (7.6) *and* (7.8)*; see Figure 8.3.*

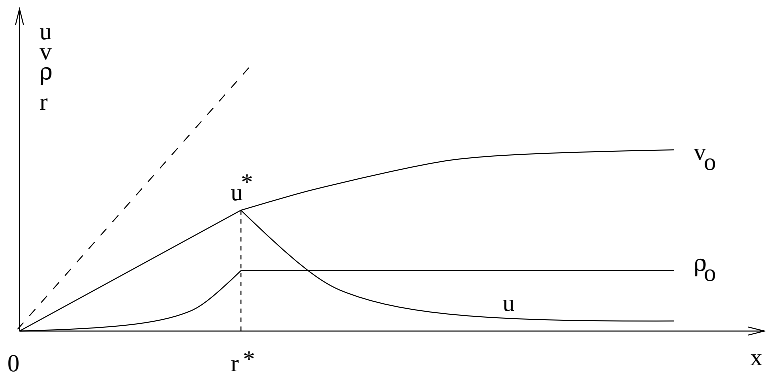

Figure 8.2. A solution (u, v, ρ) vs. the x-axis at time $t = 1$, when $M_0 = \sqrt{2}$, $u_0 = 0$. It is an explicit solution.

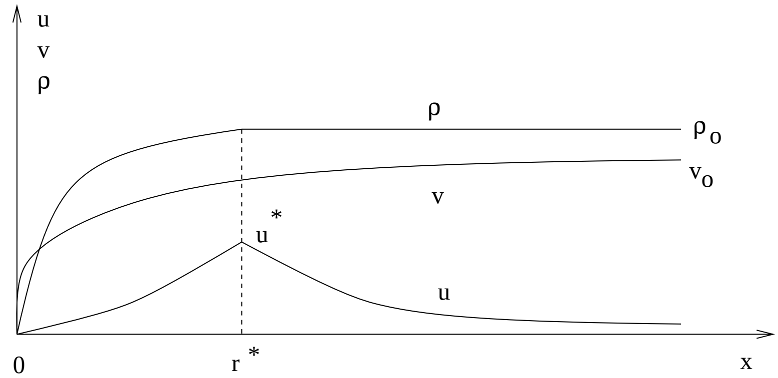

Figure 8.3. A solution (u, v, ρ) vs. the x-axis at time $t = 1$ when $u_0 = 0$ and $M_0 < \sqrt{2}$.

(iii) *For the case* $u_0 > 0,\ \rho_0 > 0$ *and* $v_0 \neq 0$, *depending on the parameters* $\left(\frac{u_0}{\sqrt{p'(\rho_0)}}, \frac{v_0}{\sqrt{p'(\rho_0)}}\right)$, *the solution can be globally smooth, contain a region of vacuum as in case* (ii) *with* $M_0 > \sqrt{2}$, *or contain an inner piece of solutions of* (ii) *with* $M_0 < \sqrt{2}$.

Appendix. Finiteness of the Parameter s at Point $(1, 0, 0)$

We show that the parameter s approaches a finite value as an integral curve approaches the stationary point $(1, 0, 0)$ in Ω_3 defined in Section 6.

Lemma A.1. *For any integral curve that ends at* $(1, 0, 0)$ *of the system* (6.1)–(6.4) *from inside the domain* Ω_3, *there exists an* $\varepsilon \in (0, 1)$ *such that the integral curve is inside the cylinder*

$$C_\varepsilon \equiv J^2 - \varepsilon I(1 - I) < 0 \tag{A.1}$$

when its I *component is in* $\left(\frac{1}{2}, 1\right)$.

Proof. We choose $\varepsilon \in (0, 1)$ such that the integral curve is inside the cylinder $J^2 < \varepsilon I(1 - I)$ at the special point $I = \frac{1}{2}$. For this ε, we show that integral curves on the surface of the cylinder $J^2 = \varepsilon I(1 - I)$ inside Ω_3 in the portion $I \in \left(\frac{1}{2}, 1\right)$ are all going into the cylinder $J^2 < \varepsilon I(1-I)$. We first calculate an outward normal to the cylinder $J^2 = \varepsilon I(1 - I)$ to be

$$\vec{n}_\varepsilon = (-\varepsilon(1 - 2I), 2J, 0).$$

We then calculate the inner product of the normal $\vec{n}_\varepsilon$ with the tangent direction of any integral curve on the cylinder $J^2 = \varepsilon I(1 - I)$

$$\begin{aligned}
&\vec{n}_\varepsilon \cdot \frac{d}{d\tau}(I, J, K)\big|_{J^2=\varepsilon I(1-I)} \\
&\quad = -\varepsilon(1 - 2I)(1 - I)[I(1 - I)^2(1 + \varepsilon) - 2IK^2] \\
&\qquad + 2\varepsilon I(1 - I)(1 - 2I) \cdot [(1 - I)^2 - K^2] \\
&\quad = \varepsilon I(1 - 2I)(1 - I)\{-(1 + \varepsilon)(1 - I)^2 + 2K^2 + 2(1 - I)^2 - 2K^2\} \\
&\quad = \varepsilon(1 - \varepsilon)I(1 - I)^3(1 - 2I) < 0 \text{ for } I \in \left(\tfrac{1}{2}, 1\right).
\end{aligned}$$

Hence the chosen integral curve remains inside the cylinder $J^2 < \varepsilon I(1 - I)$ in $I \in \left(\frac{1}{2}, 1\right)$. This completes the proof of Lemma A.1.

Lemma A.2. *For any integral curve that ends at* $(1, 0, 0)$ *of the system* (6.1)–(6.4) *in the domain* Ω_3, *there are three numbers* $\varepsilon \in (0, 1),\ \beta \in (2, \infty)$, *and* $\tilde{I} \in \left(\frac{1}{2}, 1\right)$ *such that the integral curve is inside the cylinder* $C_\varepsilon < 0$ *and below the surface*

$$B_\beta \equiv (1 - I)J^2 + I(1 - I)^2 - \beta IK^2 = 0 \tag{A.2}$$

when $I \in (\tilde{I}, 1)$.

Proof. We first compute an outward normal $\vec{n}_\beta$ to the surface $B_\beta = 0$:

$$\vec{n}_\beta = (\beta K^2 - (1-I)^2 + 2I(1-I) + J^2,\ -2(1-I)J,\ 2\beta IK).$$

We calculate the inner product of this $\vec{n}_\beta$ with the tangent vector of any integral curve on the surface $B_\beta = 0$:

$$\begin{aligned}
&\vec{n}_\beta \cdot \frac{d}{d\tau}(I, J, K)\big|_{B_{\beta=0}} \\
&= [\beta K^2 - (1-I)^2 + 2I(1-I) + J^2] \\
&\quad \cdot (1-I)\{(1-I)^2 I + (1-I)^2 - 2IK^2\} \\
&\quad - 2(1-I)J^2(1-2I)[(1-I)^2 - K^2] + \beta IK^2(1-I) \\
&\quad \cdot [(\gamma-1)J^2 - (\gamma-1)I(1-I) + 2(1-I)^2 - 2K^2] \\
&= \left[2I(1-I) + \frac{1}{I}J^2\right](1-I)^2\left(1-\frac{2}{\beta}\right)[J^2 + I(1-I)] \\
&\quad + 2(1-I)^2(1-2I)J^2\frac{1}{\beta I}[J^2 - (\beta-1)I(1-I)] \\
&\quad + (1-I)^2[J^2 + I(1-I)]\frac{1}{I} \\
&\quad \cdot \left\{\left[(\gamma-1)I - \frac{2}{\beta}(1-I)\right]J^2 - (\gamma-1)I^2(1-I)\right. \\
&\quad \left. + 2\left(1-\frac{1}{\beta}\right)I(1-I)^2\right\}.
\end{aligned}$$

Rewriting the last expression in the form of a polynomial of J, we obtain

$$\begin{aligned}
&\vec{n}_\beta \cdot \frac{d}{d\tau}(I, J, K)\big|_{B_{\beta=0}} \\
&= J^4\left\{\frac{1}{I}(1-I)^2\left(1-\frac{2}{\beta}\right) + \frac{2}{\beta I}(1-I)^2(1-2I)\right. \\
&\quad \left. + \frac{(1-I)^2}{I}\left[(\gamma-1)I - \frac{2}{\beta}(1-I)\right]\right\} \\
&\quad + J^2\left\{2I(1-I)^3\left(1-\frac{2}{\beta}\right) + (1-I)^3\left(1-\frac{2}{\beta}\right)\right. \\
&\quad - 2\frac{\beta-1}{\beta}(1-I)^3(1-2I)(1-I)^3\left[(\gamma-1)I - \frac{2}{\beta}(1-I)\right] \\
&\quad \left. + 2\left(1-\frac{1}{\beta}\right)(1-I)^4 - (\gamma-1)I(1-I)^3\right\} +
\end{aligned}$$

$$+2I^2(1-I)^4\left(1-\frac{2}{\beta}\right)-(\gamma-1)I^2(1-I)^4+2\left(1-\frac{1}{\beta}\right)I(1-I)^5$$

$$\begin{aligned}&= J^4\frac{(1-I)^2}{\beta I}\{\beta-2+[\beta(\gamma-1)-2]I\}\\&\quad+J^2(1-I)^3\left[4I\left(1-\frac{1}{\beta}\right)+1-\frac{4}{\beta}\right]\\&\quad+I(1-I)^4\left[2I\left(1-\frac{2}{\beta}\right)-(\gamma-1)I+2\left(1-\frac{1}{\beta}\right)(1-I)\right].\end{aligned}$$

We find that both coefficients of J^4 and J^2 are positive when $\beta > 2,\ \gamma \geqq 2$, and $I \in \left(\frac{1}{2}, 1\right)$. Hence in the region $C_\varepsilon < 0$, we find that

$$\begin{aligned}&\vec{n}_\beta\cdot\frac{d}{d\tau}(I,J,K)\Big|_{B_{\beta=0},C_\varepsilon<0}\\&\quad<\frac{\varepsilon^2}{\beta}I(1-I)^4\{\beta-2+[\beta(\gamma-1)-2]I\}\\&\qquad+\varepsilon I(1-I)^4\left[4I\left(1-\frac{1}{\beta}\right)+1-\frac{4}{\beta}\right]\\&\qquad+I(1-I)^4\left[-\frac{2I}{\beta}-(\gamma-1)I+2\left(1-\frac{1}{\beta}\right)\right]\\&\quad=I(1-I)^4\left\{\varepsilon^2\left[1-\frac{2}{\beta}+\left(\gamma-1-\frac{2}{\beta}\right)I\right]+4\varepsilon I\left(1-\frac{1}{\beta}\right)+\varepsilon-\frac{4\varepsilon}{\beta}\right.\\&\qquad\left.-\left(\frac{2}{\beta}+\gamma-1\right)I+2\left(1-\frac{1}{\beta}\right)\right\}\\&\quad\equiv I(1-I)^4F(\varepsilon,\beta,I,\gamma)\end{aligned}$$

where F denotes the expression in the braces.

We need F to be negative near $I = 1$ for some $\varepsilon > 0,\ \beta > 2$, and $\gamma = 2$. For any fixed integral curve that goes to the point $(1, 0, 0)$, we first choose $\varepsilon \in (0, 1)$ so that the integral curve lies inside $C_\varepsilon < 0$ for $I > \frac{1}{2}$. Fix this ε. We calculate the value of F at the extreme point $\beta = 2$:

$$\begin{aligned}F(\varepsilon,2,I,2)&=(2I-1)\varepsilon+1-2I\\&=(2I-1)(\varepsilon-1)<0\quad\text{for } I\in\left(\tfrac{1}{2},1\right).\end{aligned}$$

Since F is a continuous function of (β, I) near the point $(2, 1)$, we conclude that there exists an $\tilde{I} \in \left(\frac{1}{2}, 1\right)$ such that $F(\varepsilon, \beta, I, 2) < 0$ for all $I \in [\tilde{I}, 1)$ when β is close to 2. Since $B_\beta = 0$ is close to $B = 0$, we can choose a $\beta > 2$ such that the integral curve lies below $B_\beta = 0$ at $I = \tilde{I}$. This integral curve remains under the surface $B_\beta = 0$ for all $I > \tilde{I}$ because of the sign $F < 0$. This completes the proof of Lemma A.2.

Theorem A.3. *The parameter s is finite for any integral curve of* (6.1)–(6.4) *that goes to the point* $(1, 0, 0)$ *inside* Ω_3.

Proof. We find from the first equation of (4.3) that

$$s\frac{dI}{ds} = 2I - (1-I)\frac{I(1-I)-J^2}{(1-I)^2-K^2}. \tag{A.3}$$

The right-hand side of (A.3) is a decreasing function of K^2 in Ω_3. Any integral curve that goes to $(1, 0, 0)$ remains in $C_\varepsilon < 0$ and $B_\beta > 0$ for I close to 1. So for I close to 1 we find that

$$\begin{aligned} s\frac{dI}{ds} &\geqq 2I - (1-I)I\frac{I(1-I)-J^2}{\left(1-\frac{1}{\beta}\right)I(1-I)^2 - \frac{1}{\beta}(1-I)J^2} \\ &= 2I - I\beta\frac{I(1-I)-J^2}{(\beta-1)I(1-I)-J^2} = 2I - \beta I + \frac{\beta(\beta-2)I^2(1-I)}{(\beta-1)I(1-I)-J^2}. \end{aligned} \tag{A.4}$$

The very last expression of (A.4) is an increasing function of J^2, so we further find by using $J^2 \geqq 0$ that

$$s\frac{dI}{ds} \geqq (2-\beta)I + \frac{\beta(\beta-2)}{\beta-1}I = \frac{\beta-2}{\beta-1}I.$$

Thus it can only take a finite amount of s for I to reach 1.

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Note added in proof. For polytropic gases with non-quadratic pressure-density laws, shock wave solutions as well as continuous solutions have since been constructed; see the book *Two-dimensional Riemann Problems for Systems of Conservation Laws* by YUXI ZHENG to be published by Birkhäuser in 1999 or 2000.

References

1. BELLAMY-KNIGHTS, P. G.: Viscous compressible heat conducting spiralling flow, *Q. J. Mech. Appl. Math.*, **33** (1980), 321–336
2. COLONIUS, T., LELE, S. K., & MOIN, P.: The free compressible viscous vortex, *J. Fluid Mech.*, **230** (1991), 45–73.
3. COURANT, R. & FRIEDRICHS, K. O.: *Supersonic Flow and Shock Waves*, Interscience, Springer 1948.
4. CHEN, G.-Q. & GLIMM, J.: Global solutions to the compressible Euler equations with geometrical structure, *Comm. Math. Phys.*, **180** (1996), 153–193.
5. KELLEY, A.: The stable, center stable, center, center unstable and unstable manifolds, *J. Diff. Eqs.*, **3** (1967), 546–570.

6. Mack, L. M.: The compressible viscous heat-conducting vortex, *J. Fluid Mech.,* **8** (1960), 284–292.
7. Majda, A.: Vorticity, turbulence, and acoustics in fluid flow, *SIAM Review,* **33** (1991), 349–388.
8. Powell, A.: Lecture series at the von Kármán Institute, 1990.
9. Serrin, J.: The swirling vortex, *Phil. Trans. Roy. Soc. London,* A **271** (1972), 325–360.
10. Tan, Dechun & Zhang, Tong: Two-dimensional Riemann problem for a 2×2 hyperbolic system of nonlinear conservation laws (I) and (II), *J. Diff. Eqs.,* **111** (1994), 203–282.
11. Wagner, D.: The Riemann problem in two space dimensions for a single conservation law, *SIAM J. Math. Anal.,* **14** (1983), 534–559.
12. Zhang, Tong & Zheng, Yuxi: Conjecture on structure of solutions of Riemann problem for 2-D gasdynamic systems, *SIAM J. Math. Anal.,* **21** (1990), 593–630.
13. Zhang, Tong & Zheng, Yuxi: Exact spiral solutions of the two dimensional compressible Euler equations, *Discrete and Continuous Dynamical Systems* **3** (1997), 117–133.
14. Zhang, Tong & Zheng, Yuxi: *Explicit and exact swirling solutions of the 2-D Euler squations, Z. angew. Math. Mech.* **76** (1996), Special issue 2, Applied Analysis, 145–148.

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